ACTIONS ON SYMMETRIC CURVES FROM THE ARITHMETIC TOPOLOGY VIEWPOINT

ARISTIDES KONTOGEORGIS AND PANAGIOTIS PARAMANTZOGLOU

ABSTRACT. We give an explanation of the MKR dictionary in Arithmetic topology using Ihara's theory of profinite braid groups. Motivated by the analogy we perform explicit computations for representations of both braid groups and the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ for cyclic covers of the projective line and generalized Fermat curves.

1. INTRODUCTION

Arithmetic topology is concerned with the similarities between several notions and theorems in algebraic number theory and the theory of 3-manifolds. In this theory there is the Mazur-Kapranov-Reznikov dictionary (MKR-dictionary for short) and a short part of it is displayed at the next table:

Number Theory	Topology
prime ideals	knots
ideals	links
Number Fields	3-manifolds
class group	$H_1(M,\mathbb{Z})$
Riemann's ζ-function	Selberg's ζ-function
Algebraic extensions	Ramified Topological covers
Galois groups, $\pi_1^{\mathrm{et}}(X)$	$\pi_1(X, x_0)$

For a nice introduction and a detailed explanation see [35]. Motivated by arithmetic topology we would like to provide and expand a common framework, so that representation of Braid groups and Galois representations fit together. As far as the authors know there is no explanation of the existence of MKR-dictionary. We will attempt such an explanation in section 2 by interpreting both primes and knots as certain conjugation classes.

Knot theorists study braid groups representations, in order to provide invariants of knots (after Markov equivalence, see 2.2) and number theorists study Galois representations in order to understand the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Of course understanding braid groups and knots has the advantage that also 3-manifolds can be understood, either by surgery representations of 3-manifolds or by presenting them as ramified extensions of S^3 with ramification locus a link by Alexander theorem, [38, th. 22.3].

Y. Ihara in a series of articles [23], [24] proposed a method to treat elements in the automorphism group of the profinite free group as "profinite braids" and in this

Date: April 20, 2018.

way he got a series of Galois representations similar to classical Braid representations. The main focus of Ihara was the understanding of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and for this aim the theory of coverings of $\mathbb{P}^1_{\bar{\mathbb{Q}}} - \{0, 1, \infty\}$ was enough by Belyi's theorem. Here we mainly focus on the similarity of equivalence classes of Braids (i.e. topological links) and primes (equivalence classes of Frobenious elements) so we have to extend our point of view to ramified covers of the projective line minus s-points removed, $s \geq 3$. In other words the braid group B_2 which acts on covers of $\mathbb{P}^1_{\bar{\mathbb{Q}}} - \{0, 1, \infty\}$ is not a very interesting braid group. Notice that when the number s of points we remove is $s > 3$, then we expect that their configuration might also affect our study. So in this article we study in an explicit way representations of both Braid groups and the absolute Galois groups on homology groups of highly symmetrical curves. More precisely:

In section 2 we focus on Arithmetic Topology and motivated by Ihara's theory we give an explanation for the similar behaviour of knots and braids. They can both be realized as conjugacy classes of elements in the automorphism group of the profinite free group. From this point of view the absolute Galois group is an arithmetic analogon of the mapping class group as it is explained in 3.6.

In section 3 we introduce Ihara's ideas on representations on automorphisms groups of pro-ℓ groups following [27]. We pursue this analogy further: Actions of the absolute Galois group on vector spaces coming out naturally from algebraic curves, for example (co)homology groups, is a very active topic of Arithmetic Geometry. The same holds for actions of mapping class groups. Algebraic curves over the field of complex numbers are equivalent to Riemann surfaces, which in turn can be described as ramified covers of the projective line or - if their genus $g \geq 2$ - as quotients of the hyperbolic plane by a discrete subgroup.

Finding the fundamental group of such a Riemann surface involves the Reidemeister - Schreier method and this will be explained in section 4. We will be able to compute this group for covers of the projective line ramified above s-points. If the fundamental group of a surface is known, then the action of both the braid group B_{s-1} and the absolute Galois group can be computed. A tool which unifies these actions in terms of module action of certain commutative rings was developed by R. Crowell [11] for the discrete case and has also a pro-ℓ analogon [27, sec. 9.4].

Section 5 is devoted to examples and explicit computations. As we explain in 4.1.3 if we know how a group acts on homology we can pass to the dual space of holomorphic differentials. In this way we explain how we can recover certain results of C. McMullen on cyclic covers of the projective line by computing explicitly the fundamental group of both open and closed curves using the Reidemeister -Schreier method. In 5.3.1 we give an arithmetic analogon of the fact that the braid action on the homology of the curve respects the canonical intersection form.

In 5.4 we study the Fermat curve, this curve gives rise to one of the most interesting Diophantine equations but it is clear that it also plays a significant role within the theory of abelian covers of the projective line minus three points as Ihara proved, see [23, part II, p.63]. This curve fits within the theory of cyclic covers of the projective line but it has much more algebraic automorphisms, see [41]. The action on homology of Fermat curves is recently studied in [12], [1], [3]. We then proceed to the case of generalized Fermat curves, which will play the role of the Fermat curves in the setting of covers of the projective line minus s points, $s \geq 3$. These curves are highly symmetric and the group of their algebraic automorphisms was recently studied, see [17], [21]. Also holomorphic differentials and the Jacobian are interesting objects of study, see [20], [9].

The Tate module for the generalized Fermat curves is also studied in terms of the corresponding Alexander module $\mathscr{A}_\psi$, which can be seen as an $\mathscr{A}$-module over a complete polynomial ring introduced by Ihara, see definition 6 and section 4.1. The situation is different in comparison to the classical Fermat curves. More precisely, the Tate module is not a free $\mathscr{A}$-module as in the case of Fermat curves but it contains a free $\mathscr{A}$-submodule $\mathbb{T}^{\mathrm{prim}}$ of rank $s-1$. We therefore have a representation

$$F : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_{\ell^\infty})) \to \mathrm{GL}_{s-1}(\mathscr{A}),$$

which is an interpolation of the analogous Braid representation

$$\rho : B_{s-1} \to \mathrm{GL}_{s-1}(A),$$

where $A = k[x_1, \ldots, x_s]$. We then show how the computations for the generalized Fermat curves fit together with the theory of cylic covers of the projective line.

As Ihara observed in [24] the profinite braid action corresponding to $\mathfrak{F}_{s-1}/\mathfrak{F}'_{s-1}$ gives rise to the analogon of the profinite Galois Gassner representation and similarly on the action on the Tate module of the Jacobian. The reduction to the case of cyclic covers is the passage from the Gassner to the Burau representation. We believe this is a fascinating similarity illustrating the unity of Mathematics.

In conclusion, it seems that geometric group theory, i.e. understanding free subgroups of a free group (both in the classical and pro-ℓ) cases, seems to be a tool to handle arithmetic questions concerning the Tate module, the homology and group actions on them.

2. Primes and Braids

In order to give an explanation for MKR-dictionary we will give some alternative but equivalent definitions for primes and knots.

2.1. **What is a prime?** Consider a number field K, i.e. an algebraic extension of the rational field $\mathbb{Q}$. Let $\mathscr{O}_K$ denote the ring of algebraic integers of K. It is a Dedekind ring, so every ideal is uniquelly decomposed into a finite product of prime ideals. In scheme theory the set of primes $\mathrm{Spec}(\mathscr{O}_K)$, which equals to the set of prime ideals of $\mathscr{O}_K$, gives rise to a geometric point of view of primes of K.

On the other hand every such prime ideal gives rise to an non-archimedean valuation and the set of non-archimedean valuations is completed by adding the "infinite primes" to the set of all valuation of the field archimedean and non-archimedean.

Consider a prime ideal $p\mathbb{Z}$ of $\mathbb{Z} = \mathscr{O}_{\mathbb{Q}}$. This ideal can be extended to an ideal $p\mathscr{O}_K$ and this ideal is decomposed into prime ideals $P_1, \ldots P_r$, so that

$$p\mathscr{O}_K = P_1^{e_1} \cdots P_r^{e_r}.$$

Choose a P among the prime ideals P_i, $i = 1, \ldots, r$. The extension $\mathrm{Gal}\left(\frac{\mathscr{O}_K}{P}\big/\frac{\mathbb{Z}}{p\mathbb{Z}}\right)$ is a cyclic extension of finite fields generated by the Frobenius map $\sigma_{P,p} : x \mapsto x^p$. Assume now that $K/\mathbb{Q}$ is a Galois extension with Galois group G. We have the following short exact sequence of groups:

$$1 \to I(P) \to G(P) \to \mathrm{Gal}\left(\frac{\mathscr{O}_K}{P}\Big/\frac{\mathbb{Z}}{p\mathbb{Z}}\right) \to 1,$$

where

$$\begin{aligned} G(P) &= \{\sigma \in G : \sigma(P) = P\} \\ I(P) &= \{\sigma \in G(P) : \sigma(x) - x \in P\}. \end{aligned}$$

The generator of the cyclic group $\sigma_{P,p}$ lifts to an element F_P of $G(P) \subset G$. This element depends on the choice of the prime P which lies above $p\mathbb{Z}$. We know that if P, P' are both prime ideals lying above $p\mathbb{Z}$, then the elements F_P, F'_P are conjugate.

By taking the inverse limit in all Galois groups of all Galois extensions of $\mathbb{Q}$ we can define Frobenius elements in $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Such a selection corresponds to a selection of a prime ideal P_K in all number fields K, which are Galois extensions of $\mathbb{Q}$. But the conjugate class of F_P in $G_{\mathbb{Q}}$ depends only on the prime $p\mathbb{Z}$.

Therefore, it seems natural to identify the set of primes to the set of conjugate classes of Frobenius elemens in $G_{\mathbb{Q}}$. So we can see a "geometric element" $p \in \mathrm{Spec}(\mathbb{Z})$ as a conjugation class inside a group. We will see that the same situation holds for knots, which will be mapped as conjugation classes of braid groups.

Definition 2.1

A prime is a conjugation class of Frobenius elements in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

2.2. **What is a knot?** It is known that every knot in the three-dimensional sphere S_3 can be obtained as the closure of a braid in the braid group B_s for some $s \in \mathbb{N}$. Moreover by Markov theorem two braids give equivalent knots if one corresponds to the other by a sequence of Markov moves. One Markov move involves adding an extra strand on the braid and the other move is conjugation. For more information on the representation of knots and links as closure of braids we refer to [42, III.6.7].

Definition 2.2

A knot in S_3 is a Markov equivalence class of braids B_n. If two knots have the same number of strands then they are Markov equivalent if and only if the corresponding braids are in the same conjugation class.

To summarize, we have the following table which illustrates how geometric objects (elements in Spec$\mathscr{O}$ or knots) become algebraic objects in certain groups (the braid group or the absolute Galois group) but at the cost of having to consider conjugation classes.

	Number Theory	Topology
Geometric Object	prime ideals	knots
Algebraic Object	conjugacy classes in absolute Galois groups	conjugacy classes in Braid groups

We will explain one more similarity in 3.6, but first we have to introduce the pro-ℓ braids.

3. Profinite actions

In this paragraph we review the theory of Ihara's representation following the exposition of [27].

3.1. **Artin representation.** The group of braids can be identified with the subgroup of automorphisms of the free group F_{s-1} in terms of the Artin representation. More precisely the group B_{s-1} can be defined as the subgroup of $\mathrm{Aut}(F_{s-1})$ generated by the elements σ_i for $1 \leq i \leq s-1$, given by

$$\sigma_i(x_k) = \begin{cases} x_k & \text{if } k \neq i, i+1, \\ x_i x_{i+1} x_i^{-1} & \text{if } k = i, \\ x_i & \text{if } k = i+1. \end{cases}$$

The free group F_{s-1} is the fundamental group of X_s defined as

$$X_s = \mathbb{P}^1 - \{P_1, \dots, P_{s-1}, \infty\}. \tag{1}$$

In this setting the group F_{s-1} is given as:

$$F_{s-1} = \langle x_1, \dots, x_s | x_1 x_2 \cdots x_s = 1 \rangle, \tag{2}$$

the elements x_i correspond to homotopy classes of loop circling once clockwise around each removed point P_i, and distinguish the homotopy class $y = x_s$ of the loop circling around infinity.

3.2. **Pro-ℓ case.** Let ℓ be a prime number. Let S be a set of s, $s \geq 3$, points in $\bar{\mathbb{Q}}$ on the projective line $\mathbb{P}^1_{\bar{\mathbb{Q}}}$. Consider the field $K := \mathbb{Q}(S \backslash \{\infty\})$ generated by the coordinates of the points in $S \backslash \{\infty\}$. The set S is assumed to be acted on by the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, i.e.

$$\prod_{P \in S - \infty} (x - P) \in \mathbb{Q}[x].$$

The group $\pi_1^{\text{pro}-\ell}(\mathbb{P}^1_{\bar{\mathbb{Q}}} \backslash S) \cong \mathfrak{F}_{s-1}$. If the elements in $S - \{\infty\}$ do not satisfy any algebraic relations then the induced action on S is just the symmetric group S_{s-1}, otherwise it is a subgroup of S_{s-1}.

Assumption: We will assume from now on that $P \in \mathbb{Q}$ for all $P \in S - \{\infty\}$, so that the action on S is the trivial one. In this way the absolute Galois group corresponds to "pure braids". Ihara in [23] introduced the monodromy representation

$$\mathrm{Ih}_S : \mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}(\mathfrak{F}_{s-1}).$$

Here the group $\mathfrak{F}_{s-1}$ admits a presentation, similar to eq. (2),

$$\mathfrak{F}_{s-1} = \langle x_1, \dots, x_s | x_1 x_2 \cdots x_s = 1 \rangle. \tag{3}$$

Ihara was mainly interested for the case $S = \{0, 1, \infty\}$ and $k = \mathbb{Q}$, since by Belyi's theorem [5] the branched covers of $\mathbb{P}^1 - \{0, 1, \infty\}$ are exactly the curves defined over $\bar{\mathbb{Q}}$. The image of the Ihara representation is inside the group

$$\tilde{P}(\mathfrak{F}_{s-1}) := \left\{ \phi \in \mathrm{Aut}(\mathfrak{F}_{s-1}) | \phi(x_i) \sim x_i^{N(\phi)} (1 \leq i \leq s) \text{ for some } N(\phi) \in \mathbb{Z}_\ell^* \right\},$$

where $\sim$ denotes the conjugation equivalence. This group is the arithmetic analogon of the Artin representation of ordinary braid groups inside $\mathrm{Aut}(F_{s-1})$. Notice that the exponent $\sigma(x_i) \cong x_i^a$ depends only on σ and not on x_i. Moreover the map

$$N : \tilde{P}(\mathfrak{F}_{s-1}) \to \mathbb{Z}_\ell^*$$

is a group homomorphism and $N \circ \mathrm{Ih}_S : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^*$ coincides with the cyclotomic character.

Remark 1. The relation $x_1 \cdots x_{s-1} x_s = 1$ implies that $\tilde{P}(\mathfrak{F}_{s-1})$ also acts on the free group $\mathfrak{F}_s$ since $x_s = (x_1 \cdots x_{s-1})^{-1}$.

3.3. **The outer Galois representation.** Consider the free group F_{s-1} on $s-1$ letters $x_1,\dots,x_{s-1}$ which admits a presentation given in eq. (2) and $\mathfrak{F}_{s-1}$ the pro-ℓ completion of F_{s-1}. Any abstract automorphism of $\mathfrak{F}_{s-1}$ is bicontinuous [13, cor. 1.22] and $\mathrm{Aut}(\mathfrak{F}_{s-1})$ is virtually a pro-ℓ group [13, th. 5.6].

Consider the group $H:=\mathfrak{F}_{s-1}/[\mathfrak{F}_{s-1},\mathfrak{F}_{s-1}]$. We will denote by $[f]$ the image of $f\in\mathfrak{F}_{s-1}$ in the abelianization H. We will write H as an additive $\mathbb{Z}_\ell$-module. So H is the free $\mathbb{Z}_\ell$-module generated by $[x_1],\dots,[x_{s-1}]$ with the relation:

$$[x_1]+\cdots+[x_{s-1}]+[x_s]=0.$$

Every automorphism $\phi\in\mathrm{Aut}(\mathfrak{F}_{s-1})$ gives rise to a linear automorphism of the free $\mathbb{Z}_\ell$-module H and we will denote it by $[\phi]\in\mathrm{GL}(H)$. The effect of changing a base point of $\mathbb{P}^1_{\mathbb{Q}}\backslash S$ is given by an inner automorphism of $\mathfrak{F}_{s-1}$. We finally arrive at a continuous outer representation

$$\Phi_S:\mathrm{Gal}_k\to\mathrm{Out}(\mathfrak{F}_{s-1}):=\mathrm{Aut}(\mathfrak{F}_{s-1})/\mathrm{Inn}(\mathfrak{F}_{s-1}).$$

3.4. **Belyi's lifts.** Consider the subgroups $P(\mathfrak{F}_{s-1})<\tilde{P}(\mathfrak{F}_{s-1})$ of $\mathrm{Aut}(\mathfrak{F}_{s-1})$ defined by

$$P(\mathfrak{F}_{s-1})=\left\{\phi\in\mathrm{Aut}(\mathfrak{F}_{s-1})\;\middle|\;\begin{array}{l}\phi(x_i)\sim x_i^{N(\phi)},(1\le i\le s-2),\phi(x_{s-1})\approx x_{s-1}^{N(\phi)}\\ \phi(x_s)=x_s^{N(\phi)},\ \text{for some } N(\phi)\in\mathbb{Z}_\ell^\times\end{array}\right\},\tag{4}$$

where $\approx$ denotes conjugacy by an element of the subgroup of $\mathfrak{F}_s$ generated by the commutator $\mathfrak{F}_s'$ and $x_1,\dots,x_{s-3}$. We also denote by $P^1(\mathfrak{F}_{s-1})$ the kernel of N restricted to the subgroup $P(\mathfrak{F}_{s-1})$.

Proposition 2. *The natural homomorphism* $\mathrm{Aut}(\mathfrak{F}_{s-1})\to\mathrm{Aut}(\mathfrak{F}_{s-1})/\mathrm{Inn}(\mathfrak{F}_{s-1})$ *induces an isomorphism* $P(\mathfrak{F}_{s-1})\cong\tilde{P}(\mathfrak{F}_{r-1})/\mathrm{Inn}(\mathfrak{F}_{s-1})$.

Proof. See [23, prop.3 p.55], [27, prop. 2.2.2]. □

We can describe the above construction in terms of Galois theory as follows: Consider the maximal pro-ℓ extension $\mathscr{M}$ of $\mathbb{Q}(t)$ unramified outside the set S. The Galois group $\mathrm{Gal}(\mathscr{M}/\bar{\mathbb{Q}}(t))$ is known to be the pro-ℓ free group $\mathfrak{F}_{s-1}$ of rank $s-1$. A selection of generators $x_1,\dots,x_{s-1}$ corresponds to an isomorphism $i:\mathfrak{F}_{s-1}\to\mathrm{Gal}(\mathscr{M}/\bar{\mathbb{Q}}(t))$, such that $i(x_\nu)$ $(1\le\nu\le s)$ generates the inertia group of some place ξ_ν of $\mathscr{M}$ extending the place P_i of $\bar{\mathbb{Q}}(t)$.

We have the following exact equence:

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Gal}(\mathscr{M}/\bar{\mathbb{Q}}(t)) \arrow[r] \arrow[d, equal] & \mathrm{Gal}(\mathscr{M}/\mathbb{Q}(t)) \arrow[r] & \mathrm{Gal}(\bar{\mathbb{Q}}(t)/\mathbb{Q}(t)) \arrow[r] \arrow[d, "\cong"] & 1 \\ & \mathfrak{F}_{s-1} & & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & \end{tikzcd}\tag{5}$$

Every element $\rho\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ gives rise to an element $\rho^*\in\mathrm{Gal}(\mathscr{M}/\mathbb{Q}(t))$, and to an automorphism $x\mapsto\rho^*x\rho^{-1}\in\tilde{P}(\mathfrak{F}_{s-1})$, as it is proved by Ihara [22, p.52] . Using a Möbious transformation we can assume that the set S consists of the elements $0,1,\lambda_1,\cdots,\lambda_{s-3},\infty$.

3.5. The profinite representations.

3.5.1. *Magnus embedding.* We will explain now the Magnus embedding following [27]. This is given by the map

$$\mathfrak{F}_{s-1} \to \mathbb{Z}_\ell[[u_1, u_2, \ldots, u_{s-1}]]_{\mathrm{nc}}$$

of $\mathfrak{F}_{s-1}$ into the "non-commutative" formal power series algebra ($x_i \to 1 + u_i$ for $1 \leq i \leq s-1$). Let $\mathfrak{H}$ denote the abelianization of $\mathfrak{F}_{s-1}$, and H the abelianization of F_{s-1}

$$H := \mathrm{gr}_1(F_{s-1}) = H_1(F_{s-1}, \mathbb{Z}) \qquad \mathfrak{H} =: \mathrm{gr}_1(\mathfrak{F}_{s-1}) = H_1(\mathfrak{F}_{s-1}, \mathbb{Z}_\ell) = H \otimes_{\mathbb{Z}} \mathbb{Z}_\ell.$$

The term gr_1 above has its origin on the graded Lie algebra corresponding to a (pro-ℓ) free group, see [23, p. 58] and [31]. Following [27], [35] we consider the tensor algebras

$$T(H) = \bigoplus_{n \geq 0} H^{\otimes n}, \qquad T(\mathfrak{H}) = \bigoplus_{n \geq 0} \mathfrak{H}^{\otimes n},$$

where $\mathfrak{H}^0 = \mathbb{Z}_\ell$ and $\mathfrak{H}^{\otimes n} := \mathfrak{H} \otimes_{\mathbb{Z}_\ell} \cdots \otimes_{\mathbb{Z}_\ell} \mathfrak{H}$ (n-times) (resp. $H^0 = \mathbb{Z}$, $H^{\otimes n} = H \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} H)$). If $u_0, \ldots, u_{s-1}$ is a $\mathbb{Z}_\ell$ basis of the free $\mathbb{Z}_\ell$-module $\mathfrak{H}$, then

$$T(\mathfrak{H}) = \mathbb{Z}_\ell \langle u_1, \ldots, u_{s-1} \rangle,$$

is the non-commutative polynomial algebra over $\mathbb{Z}_\ell$.

We will denote by $\widehat{T}(\mathfrak{H})$ the completion of $T(\mathfrak{H})$ with respect to the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the two sided ideal generated by $u_1, \ldots, u_{s-1}$ and ℓ. This algebra is the algebra of non-commutative formal power series over $\mathbb{Z}_\ell$ with variables $u_1, \ldots, u_{s-1}$:

$$\widehat{T}(\mathfrak{H}) = \prod_{n \geq 0} H^{\otimes n} = \mathbb{Z}_\ell \langle\langle u_1, \ldots, u_{s-1} \rangle\rangle.$$

Let $\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]$ be the complete group algebra of $\mathfrak{F}_{s-1}$ over $\mathbb{Z}_\ell$, and let

$$\epsilon_{\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]} : \mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]] \to \mathbb{Z}_\ell$$

be the augmentation homomorphism. Denote by $I_{\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]} := \ker \epsilon_{\mathbb{Z}_\ell[[\mathfrak{F}]]}$ the augmentation ideal. The correspodence $x_i \mapsto 1 + u_i$ for $1 \leq i \leq s-1$ induces an isomorphism of topological $\mathbb{Z}_\ell$-algebras, the *pro-ℓ Magnus isomorphism.*

$$\Theta : \mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]] \xrightarrow{\cong} \widehat{T}(\mathfrak{H}).$$

For an multiindex $I = (i_1, \ldots, i_{s-1})$ we set $x_I = x_{i_1} \cdots x_{i_{s-1}}$. The coefficient of x_I in $\Theta(\alpha)$ is called the Magnus coefficient of α and it is denoted by

$$\Theta(\alpha) = \varepsilon_{\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]}(\alpha) + \sum_{|I| \geq 1} \mu(I, \alpha) x_I.$$

For certain properties of the Magnus embedding and a fascinating application to ℓ-adic Milnor invariants we refer to [35, chap. 8], [27, sec. 3.2]. Consider the Magnus powerseries of $w_i(g)$

$$w_i(g) = 1 + \sum_{\substack{I=(i_1,\ldots,i_t) \\ 1 \leq i_1,\ldots,i_t \leq s-1}} \mu(g, I)[x]^I, \tag{6}$$

the coefficients $\mu(g, I)$ are called the ℓ-adic Milnor numbers of $g \in \mathrm{Gal}(\bar{k}/k)$.

The group $P^1(\mathfrak{F}_{s-1})$ can be fully described in terms of the map:

$$D : P^1(\mathfrak{F}_{s-1}) \ni \sigma \mapsto D(\sigma) = (w_1(\sigma), w_2(\sigma), \ldots, w_{s-1}(\sigma)), \tag{7}$$

with the additional condition

$$\sigma(x_1 \cdots x_{s-1}) = x_1 \cdots x_{s-1}. \tag{8}$$

Remark 3. Observe that each $w_i(\sigma)$ satisfies the derivation condition:

$$w_i(\sigma\tau) = \sigma w_i(\tau) w_i(\sigma). \tag{9}$$

3.5.2. *The commutative Magnus ring.* In this article we will consider actions of $\mathrm{Aut}(F_{s-1})$ or $\mathrm{Aut}(\mathfrak{F}_{s-1})$ on certain $\mathbb{Z}$-modules ($\mathbb{Z}_\ell$-modules) M defined as quotients of subgroups of the (pro-ℓ) free group. For example on F_{s-1}^{ab} or on $\mathfrak{F}_{s-1}^{\mathrm{ab}}$. Since M is an abelian group (we also choose to write M additively) we have that $M = R/R'$, where $R < \mathfrak{F}_{s-1}$ (or $R < F_{s-1}$).

Remark 4. For us the commutator is given by $[a,b] = aba^{-1}b^{-1}$.

By considering the action of $u_i \in T(H)$ on F_{s-1} (resp, the action of $u_i \in \hat{T}(\mathfrak{H})$ on $\mathfrak{F}_{s-1}$) by conjugation by the element $x_i = 1 + u_i$ we can see that the conjugation action of the free group is equivalent to a $T(H)$-module (resp. $\hat{T}(\mathfrak{H})$-module) structure on M. We would like to have the following property:

$$ab \cdot m = ba \cdot m, \text{ for all } a, b \in T(H) \text{ and } m \in M. \tag{10}$$

Lemma 5. *If $M = R/R'$ and $\mathfrak{F}'_{s-1} \subset R$ (resp. $F'_{s-1} \subset R$). then the induced conjugation action on M satisfies eq. (10).*

Proof. We compute

$$abrb^{-1}a^{-1} = ba[a^{-1}, b^{-1}]r[a^{-1}, b^{-1}]^{-1}a^{-1}b^{-1}.$$

So a necessary condition for eq. (10) to hold is $[R, \mathfrak{F}'_{s-1}] \subset R''$ (resp. $[R, F'_{s-1}] \subset R''$). This condition is satisfied if $\mathfrak{F}'_{s-1} \subset R$ (resp. $F_{s-1} \subset R$) then eq. (10) holds. □

Therefore, if the assumption of lemma 5 holds, instead of considering the action of the non-commutative ring $T(H)$ (resp. $\hat{T}(\mathfrak{H})$) it makes sense to consider the action of the corresponding abelianized ring.

Definition 6. Consider the commutative $\mathbb{Z}_\ell$-algebra of formal power series

$$\begin{aligned} \mathscr{A} &= \mathbb{Z}_\ell[[u_i : 1 \leq i \leq s]] / \langle (1+u_1)(1+u_2)\cdots(1+u_s) - 1 \rangle \\ &\cong \mathbb{Z}_\ell[[u_i : 1 \leq i \leq s-1]]. \end{aligned} \tag{11}$$

The algebra $\mathscr{A}$ is the symmetric algebra of $\mathfrak{H}$ over $\mathbb{Z}_\ell$, and there is a natural map $\hat{T}(\mathfrak{H}) \to \mathrm{Sym}(\mathfrak{H}) = \mathscr{A}$.

Remark 7. As we noticed already the action of $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ can be described in terms of the cocycles $w_1(\sigma), \ldots, w_{s-1}(\sigma)$. But then we can find elements

$$\varpi_1(\sigma), \ldots, \varpi_{s-1}(\sigma) \in \mathscr{A}$$

such that

$$\sigma(x_i) = \varpi_i(\sigma) \cdot x_i^{\ell(\sigma)}. \tag{12}$$

Therefore, in order to understand the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on M (keep in mind that M is a quotient of $\mathfrak{F}_{s-1}$) it makes sense to consider the $\mathscr{A}$-module structure of M.

If M is not a quotient of $\mathfrak{F}_{s-1}$ but a quotient of a subgroup of $R \lhd \mathfrak{F}_{s-1}$, fitting in a short exact sequence

$$1 \to R \to \mathfrak{F}_{s-1} \to \Gamma \to 1$$

then we will consider instead of the ring $\mathscr{A}$ the ring $\mathscr{A}^{R,\Gamma} = \mathbb{Z}_\ell[[R/\Gamma]]$, see section 4.1. Notice also that the ring $\mathscr{A}^{R,\Gamma}$ is commutative if and only if R/Γ is abelian.

Remark 8. We can also consider the ring

$$\begin{aligned} A &= \mathbb{Z}[u_i : 1 \leq i \leq s]/\langle (1+u_1)(1+u_2)\cdots(1+u_s) - 1\rangle \\ &\cong \mathbb{Z}[u_i : 1 \leq i \leq s-1]. \end{aligned} \tag{13}$$

The later ring can play the role of $\mathscr{A}$ for the case of mapping class group actions on surfaces. The inclusion $A \subset \mathscr{A}$ is in the formal closure with respect to the zero ideal, [19, II.9].

Remark 9. In the classical theory of Magnus representations we consider automorphisms a of the free group F_{s-1} and a map $\phi : F_{s-1} \to A$ (resp. $\mathfrak{F}_{s-1} \to \mathscr{A}$), such that a keeps invariant the fibres of ϕ, see [6, p. 115]. This restriction implies that the multipication interpretation of the braid and absolute Galois group gives rise to a linear representation on the module M, i.e. the ring A (resp. $\mathscr{A}$) is left invariant under the action of braid (resp. absolute Galois group). For instance the action as given in eq. (12) is not given by a module action, unless $\ell(\sigma) = 1$. In particular such an action does not give representations in general linear groups but cocycles.

Following Ihara we will analyse several Braid and Galois representations using the ring $\mathscr{A}$ as we will explain in section 5.6.1.

3.6. **Absolute Galois group as a mapping class group.** Consider a surface S (connected, closed, orientable), and let $\mathrm{Homeo}^+(S)$ be the group of orientation preserving homeomorphisms of S and let $\mathrm{Homeo}^0(S)$ be the connected component of the identity in the compact-open topology. The mapping class group $\mathrm{Mod}(S)$ is the quotient

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S)/\mathrm{Homeo}^0(S).$$

Let D_{s-1} denote the disc with $s-1$ marked points and $\bar{D}_{s-1}$ be the disc with n-holes seen as boundary components.

The mapping class group $\mathrm{Mod}(\bar{D}_{s-1})$ is the pure braid group, while the mapping class group $\mathrm{Mod}(D_{s-1})$ is the framed pure braid group $\mathscr{H}_{s-1} = \mathbb{Z}^{s-1} \rtimes P_{s-1}$, [7], [42, th. 7.6], [14, chap. 2 p. 45].

Remark 10. The spaces D_{s-1} and $\bar{D}_{s-1}$ have the same fundamental group, but different mapping class groups.

Remark 11. We would like to point out that the framed pure braid group $\mathscr{H}_{s-1}$ is a subgroup of the frammed braid group $\mathbb{Z}^{s-1} \rtimes B_{s-1}$. The quotients $\mathbb{Z}/d\mathbb{Z} \rtimes B_{s-1}$ were proposed by Kapranov and Smirnov [25] as analoga of $\mathrm{GL}_{s-1}(\mathbb{F}_{1^d}[t])$, where $\mathbb{F}_1$ is the mythical "field with one element". In this sense $\mathbb{Z}_\ell \rtimes B_{s-1}$ is the analogon of $\mathrm{GL}_{s-1}(\bar{\mathbb{F}}_1)$. We believe that arithmetic topology should provide intuition for a definition of $\mathbb{F}_1$.

We can now provide one more similarity between knots and braids: Both are isomorphism classes of group elements in either B_{s-1} or $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Moreover elements in the braid group are acting like elements in the mapping class group of the punctured disk i.e. on the projective line minus s-points. The braid group acts like the symmetric group on the set of removed points Σ and acts like a complicated homeomorphism on the complement D_{s-1} of the $s-1$ points. In exactly the same way, the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts by permutations on the set of the $s-1$-points (infinity is invariant) according to the action of the galois group $\mathrm{Gal}(K/\mathbb{Q})$ and on the complement $\mathbb{P}^1_{\bar{\mathbb{Q}}} - \Sigma$ acts in a quite mysterious way.

Remark 12. The pro-ℓ free group and the corresponding automorphism group is an interpolation of the discrete group, i.e. $F_{s-1} \hookrightarrow \mathfrak{F}_{s-1}$. The situation is analogous to rigid or formal geometry, [19, chap. 9], [16].

4. Uniformization of ramified covers of $\mathbb{P}^1$

Consider a curve Y which can be seen as Galois ramified cover $\pi : Y \to \mathbb{P}^1$ with $\Sigma = \{P_1, \ldots, P_s\}$ ramified points and with Galois group $\mathrm{Gal}(Y/\mathbb{P}^1) = G$. The open curve $Y_0 = Y - \pi^{-1}(\Sigma)$ is a topological cover of $X_s = \mathbb{P}^1 - \Sigma$ and can be seen as a quotient of the universal covering space $\tilde{X}_s$ by a free subgroup R_0 of the free group $\pi_1(X_s, x_0) = F_{s-1}$, where $s = \#\Sigma$.

The free group R_0 can be effectively computed using the Reidemeister-Schreier method [33, sec. 2.3 th. 2.7]. Notice that there is also a profinite version of the Reidemeister-Schreir method, see [39, th. 3.6.1]. In this way we arrive at a presentation of the group R_0. We also know that the group R_0 admits a presentation

$$R_0 = \langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_s | \gamma_1\gamma_2 \cdots \gamma_s \cdot [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle,$$

where g is the genus of Y. By the short exact sequence

$$1 \to R_0 \to F_{s-1} \to G \to 1,$$

we see that there is an action of G on R_0 modulo inner automorphisms of R_0 and in particular to a well defined action of G on $R_0/R_0' = H_1(Y_0, \mathbb{Z})$. Therefore the space $H_1(Y_0, \mathbb{Z})$ can be seen as a direct sum of indecomposable $\mathbb{Z}[G]$-modules. But $H_1(Y_0, \mathbb{Z})$ is not just a $\mathbb{Z}[G]$-module. It is also equipped with an action of the braid group. For example the Burau representation appears in this way.

We will use two methods in order to see the group R_0^{ab} as a G-module. The first one is to compute in an explicit form R_0 as a subgroup of the free group using the Reidemeister-Schreier method and the other one is using the Crowell exact sequence.

The completed curve Y has a fundamental group which admits a presentation of the form

$$\begin{aligned} R &= \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g | [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle \\ &= \frac{R_0}{\langle \gamma_1, \ldots, \gamma_s \rangle}. \end{aligned}$$

There is the following short exact sequence relating the two homology groups:

$$\begin{tikzcd} 0 \arrow[r] & \langle \gamma_1, \ldots, \gamma_s \rangle \arrow[r] & H_1(Y_0, \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H_1(Y, \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & 0 \\ & & R_0/R_0' \arrow[r] & R/R' = R_0/R_0'\langle \gamma_1, \ldots, \gamma_s \rangle & \end{tikzcd} \tag{14}$$

We are also interested for the action of the group G, and the action of the braid group and the action of the absolute galois group on the homology of the complete curve Y.

Convention 13. Given $e_1, \ldots, e_t$ elements inside a group E we will denote by $\langle e_1, \ldots, e_t \rangle$ the closed normal group generated by these elements. In the case of usual groups the extra "closed" condition is automatically satisfied, since these groups have the discrete topology. This condition has an meaning only in the pro-ℓ case.

The elements $\gamma_1, \ldots \gamma_s$ are fixed by some $1 \neq g \in G$. Indeed, every such element circles around a branch point so it is fixed by an element of the Galois group G. This is in accordance to the fact that the action of the automorphism group of a compact Riemann surface on the homology is faithful [15, V.3 p.269], so the fixed elements have to be factored out. More precisely we can compute the group Γ as follows: The elements $x_1, \ldots, x_s$ correspond to small paths circling around each point in S. Therefore we can take $\gamma_i = x_i^{e_i}$, where e_i is the ramification index in the branched cover $Y \to \mathbb{P}^1$. Consider the group

$$\Gamma = \text{ the smallest closed normal subgroup of } \mathfrak{F}_{s-1}$$

containing all γ_i for $1 \leq i \leq s$. Conversely, the group R_0 is given by $R \cdot \langle \gamma_1, \ldots, \gamma_s \rangle$. Consider the space $X_s = \mathbb{P}^1 - \{P_1, \ldots, P_s\}$ with fundamental group

$$\pi_1(\mathbb{P}^1 - \{P_1, \ldots, P_s\}) = \langle x_1, \ldots, x_s | x_1 \cdot x_2 \cdots x_s = 1 \rangle \cong F_{s-1}.$$

The open curve is a quotient of the universal covering space $\tilde{X}_s$ by R_0 and the closed curve is given as the quotient $\mathbb{H}/R$,

$$\begin{tikzcd} \tilde{X}_s \arrow[dr, "R_0"'] & & \mathbb{H} \arrow[dl, "R"] \\ & Y & \end{tikzcd}$$

For the sake of simplicity we will write only the pro-ℓ case and the case of usual discrete groups can be treated in a similar way, one has to replace all $\mathbb{Z}_\ell$ with $\mathbb{Z}$ in the discrete case.

4.1. **Crowell exact sequence.** The group $\mathfrak{F}_{s-1}$ is isomorphic to $\mathfrak{F}_s / \langle x_1 \cdots x_s \rangle$. We have the short exact sequence

$$1 \to R_0/\Gamma = R \to \mathfrak{F}_{s-1}/\Gamma \xrightarrow{\psi} H \to 1. \tag{15}$$

If we also assume that $\mathfrak{F}'_{s-1} \subset R_0$, then the group $G = \mathfrak{F}_{s-1}/\Gamma$ admits the presentation:

$$G = \langle x_1, \ldots, x_s | x_1^{e_1} = \cdots = x_s^{e_s} = x_1 \cdots x_s = 1 \rangle, \tag{16}$$

therefore the group H is isomorphic to a quotient the abelian group $\mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_{s-1}\mathbb{Z}$.

We will use the Crowell Exact sequence [35, sec. 9.2, sec. 94],

$$0 \to (R)^{\mathrm{ab}} = R/R' \xrightarrow{\theta_1} \mathscr{A}_\psi^{R,\Gamma} \xrightarrow{\theta_2} \mathscr{A}^{R,\Gamma} \xrightarrow{\varepsilon_{\mathscr{A}}} \mathbb{Z}_\ell \to 0, \tag{17}$$

where

$$\mathscr{A}^{R,\Gamma} = \mathbb{Z}_\ell[[H]],$$

and $\mathscr{A}_\psi^{R,\Gamma}$ is the Alexander module, a free $\mathbb{Z}_\ell$-module

$$\mathscr{A}_\psi^{R,\Gamma} = \left(\bigoplus_{g \in \mathfrak{F}_n} \mathscr{A}^{R,\Gamma} dg \right) / \left\langle d(g_1 g_2) - dg_1 - \psi(g_1) dg_2 : g_1, g_2 \in \mathfrak{F}_{s-1} \right\rangle_{\mathscr{A}^{R,\Gamma}}.$$

The map $\theta_1 : R^{\mathrm{ab}} \to \mathscr{A}_{\psi}^{R,\Gamma}$ is given by

$$R^{\mathrm{ab}} \ni n \mapsto dn. \tag{18}$$

Proposition 14. *The module $\mathscr{A}_{\psi}^{R,\Gamma}$ admits the following free resolution as an $\mathscr{A}^{R,\Gamma}$-module:*

$$\left(\mathscr{A}^{R,\Gamma}\right)^{s+1} \xrightarrow{Q} \left(\mathscr{A}^{R,\Gamma}\right)^{s} \longrightarrow \mathscr{A}_{\psi}^{R,\Gamma} \longrightarrow 0 \tag{19}$$

where s is the number of generators of G, given in eq. (16) and $s+1$ is the number of relations. Let $\beta_1, \ldots, \beta_{s+1} \in \mathscr{A}^{R,\Gamma}$. The map Q is expressed in form of Fox derivatives [6, sec. 3.1],[35, chap. 8]*, as follows*

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{s+1} \end{pmatrix} \mapsto \begin{pmatrix} \psi\pi\left(\frac{\partial x_1^{e_1}}{\partial x_1}\right) & \psi\pi\left(\frac{\partial x_2^{e_2}}{\partial x_1}\right) & \cdots \psi\pi\left(\frac{\partial x_s^{e_s}}{\partial x_1}\right) & \psi\pi\left(\frac{\partial x_1\cdots x_s}{\partial x_1}\right) \\ \psi\pi\left(\frac{\partial x_1^{e_1}}{\partial x_2}\right) & \psi\pi\left(\frac{\partial x_2^{e_2}}{\partial x_2}\right) & \cdots \psi\pi\left(\frac{\partial x_s^{e_s}}{\partial x_2}\right) & \psi\pi\left(\frac{\partial x_1\cdots x_s}{\partial x_2}\right) \\ \vdots & \vdots & & \vdots \\ \psi\pi\left(\frac{\partial x_1^{e_1}}{\partial x_s}\right) & \psi\pi\left(\frac{\partial x_2^{e_2}}{\partial x_s}\right) & \cdots \psi\pi\left(\frac{\partial x_s^{e_s}}{\partial x_s}\right) & \psi\pi\left(\frac{\partial x_1\cdots x_s}{\partial x_s}\right) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{s+1} \end{pmatrix},$$

where π is the natural epimorphism $\mathfrak{F}_s \to G$ defined by the presentation given in eq. (16).

Proof. See [35, cor. 9.6]. □

If in eq. (17) $R = \mathfrak{F}'_{s-1}$ and $\Gamma = \{1\}$, then $\mathscr{A}^{\mathfrak{F}'_{s-1},\{1\}} = \mathbb{Z}_\ell[[u_1, \ldots, u_{s-1}]] = \mathscr{A}$, as defined in eq. (11).

To summarize, the Alexander module $\mathscr{A}_{\psi}^{R,\Gamma}$ can be computed as a cokernel of the function Q:

$$\mathscr{A}_{\psi}^{R,\Gamma} = \mathrm{coker} Q, \qquad \left(\mathscr{A}^{R,\Gamma}\right)^{s+1} = \mathbb{Z}_\ell[[H]]^{s+1} \xrightarrow{Q} \mathbb{Z}_\ell[[H]]^{s} = \left(\mathscr{A}^{R,\Gamma}\right)^{s}. \tag{20}$$

The exponents in the above formula reflect the fact that the group G is generated by $s+1$-relations over s-free variables.

Proposition 15. *If $\Gamma = \{1\}$ in eq. (15) the Crowell exact sequence gives the Blanchfield-Lyndon exact sequence:*

$$0 \to (R)^{\mathrm{ab}} \to \left(\mathscr{A}^{R,\{1\}}\right)^{s-1} \xrightarrow{d_1} \mathscr{A}^{R,\{1\}} \xrightarrow{e_{\mathscr{A}}^{R,\{1\}}} \mathbb{Z}_\ell \to 0.$$

Proof. See [35, p.118] for the discrete case and the pro-ℓ case follows similarly. □

Example 16. Assume that in eq. (15) the group $H = (\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}$. Let $\mathfrak{R}_k$ be the smallest closed normal subgroup of $\mathfrak{F}_{s-1}$ generated by $x_1^{\ell^k}, \ldots, x_{s-1}^{\ell^k}$. The group $G = \mathfrak{F}_{s-1,k} = \mathfrak{F}_{s-1}/\mathfrak{R}_k$ admits the presentation:

$$\mathfrak{F}_{s-1,k} = \left\langle x_1, \ldots, x_s | x_1^{\ell^k} = \cdots = x_s^{\ell^k} = x_1 \cdots x_s = 1 \right\rangle.$$

Denote the images of the elements x_i in H by $\bar{x}_i$. It is clear that $\mathscr{A}_{\psi}^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$ is a free $\mathbb{Z}_\ell$-module of rank $s(\ell^k)^{(s-1)} - \mathrm{rank}(Q)$. Observe that $\mathscr{A}^{\mathfrak{F}_{s-1},\mathfrak{R}_k} \cong \mathbb{Z}_\ell[[H]]$ is a free $\mathbb{Z}_\ell$-module of rank $(\ell^k)^{s-1}$ since it contains elements

$$\sum_{i_1,\ldots,i_{s-1}=0}^{\ell^k-1} a_{i_1,\ldots,i_{s-1}} \bar{x}_1^{i_1} \cdots \bar{x}_{s-1}^{i_{s-1}}, \text{ where } a_{i_1,\ldots,i_{s-1}} \in \mathbb{Z}_\ell. \tag{21}$$

By induction for $1 \leq j \leq s$ can prove

$$\frac{\partial x_i^{\ell^K}}{\partial x_j} = \delta_{ij}(1 + x_i + x_i^2 + \cdots + x_i^{\ell^k-1})$$

$$\frac{\partial x_1 x_2 \cdots x_s}{\partial x_j} = x_1 \cdots x_{j-1} \tag{22}$$

Set $\Sigma_i = 1 + x_i + \cdots + x_i^{\ell^k-1}$. The map Q in eq. (20) is given by the matrix [35, cor. 9.6]

$$\begin{pmatrix} \Sigma_1 & 0 & \cdots & 0 & 1 \\ 0 & \Sigma_2 & \ddots & \vdots & x_1 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \Sigma_s & x_1 x_2 \cdots x_{s-1} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{s+1} \end{pmatrix} = \begin{pmatrix} \Sigma_1\beta_1 + \beta_{s+1} \\ \Sigma_2\beta_2 + x_1\beta_{s+1} \\ \vdots \\ \Sigma_s\beta_s + x_1 \cdots x_{s-1}\beta_{s+1} \end{pmatrix} \tag{23}$$

where $\beta_i \in \mathscr{A}^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$ for $1 \leq i \leq s$. Let $\bar{x}_i$ denote the image of x_i in the group H. Observe that

$$(1 + \bar{x}_i + \bar{x}_i^2 + \cdots + \bar{x}_i^{\ell^k-1})\bar{x}_i^{\nu} = 1 + \bar{x}_i + \cdots + \bar{x}_i^{\ell^k-1}$$

If β_i is expressed by eq. (21), then the product $\beta_i\Sigma_i$ is given by

$$\begin{aligned} \beta_i\Sigma_i &= \sum_{i_1,\ldots,\hat{i},\ldots,i_{s-1}=0}^{\ell^k-1} \bar{x}_1^{i_1} \cdots \widehat{\bar{x}_i^{i_i}} \cdots \bar{x}_{s-1}^{i_{s-1}} \sum_{\nu=1}^{\ell^k-1} \Sigma_i \bar{x}_i^{\nu} a_{i_1,\ldots,i_{s-1}}. \\ &= \sum_{i_1,\ldots,\hat{i},\ldots,i_{s-1}=0}^{\ell^k-1} \bar{x}_1^{i_1} \cdots \widehat{\bar{x}_i^{i_i}} \cdots \bar{x}_{s-1}^{i_{s-1}} \sum_{\nu=1}^{\ell^k-1} \Sigma_i a_{i_1,\ldots,i_{s-1}}. \end{aligned}$$

In the above product the $\widehat{\cdot}$ symbol denotes omitting the corresponding factor. Therefore, the space of the above quantities depends on $(\ell^k)^{s-2}$ parameters and is an element of the module

$$\Sigma_i \cdot \mathbb{Z}_\ell[[(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}]] \cong \mathbb{Z}_\ell[[(\mathbb{Z}/\ell^k\mathbb{Z})^{s-2}]].$$

The image of the map Q equals to the space generated by elements

$$\begin{pmatrix} \Sigma_1\beta_1 \\ \Sigma_2\beta_2 \\ \vdots \\ \Sigma_s\beta_s \end{pmatrix} + \beta_{s+1} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_1 \cdots x_{s-1} \end{pmatrix}.$$

The first summand is a free $\mathbb{Z}_\ell$-module of rank $s(\ell^k)^{s-2}$ and the second summand is a free $\mathbb{Z}_\ell$-module of rank $(\ell^k)^{s-1}$. Also their intersection is just $\mathbb{Z}_\ell$.

We therefore arrive at the computation of the rank of $\mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$:

$$\begin{aligned} \operatorname{rank}_{\mathbb{Z}_\ell} \mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k} &= s(\ell^k)^{s-1} - s(\ell^k)^{s-2} - (\ell^k)^{s-1} + 1 \\ &= (s-1)(\ell^k)^{s-1} - s(\ell^k)^{s-2} + 1. \end{aligned} \tag{24}$$

From the exact sequence given in eq. (17) and the rank computation given in eq. (24) in example 16 we have:

$$\begin{aligned} \operatorname{rank} \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} &= \operatorname{rank}_{\mathbb{Z}_\ell} \mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k} - \operatorname{rank}_{\mathbb{Z}_\ell} \mathscr{A}^{\mathfrak{F}_{s-1},\mathfrak{R}_k} + 1 \\ &= (s-2)(\ell^k)^{s-1} + 2 - s(\ell^k)^{s-2}. \end{aligned} \tag{25}$$

This result coincides with the formula given in eq. (42).

Example 17. Let us now compute $\mathscr{A}_\psi^{R_{\ell^k},\mathfrak{R}_k}$ and R_{ℓ^k} is the the pro-ℓ completion of the group generated by

$$\{x_1^i x_j x_1^{-i-1} : 2 \leq j \leq s-1, 0 \leq i \leq \ell^k - 1\}.$$

These groups will be studied in detail in section 5.3. We have the short exact sequence

$$1 \to R_{\ell^k} \to \mathfrak{F}_{s-1}/\mathfrak{R}_k \to \mathbb{Z}/\ell^k\mathbb{Z} \to 0.$$

We compute $\mathscr{A}^{R_{\ell^k},\mathfrak{R}_k} = \mathbb{Z}_\ell[\mathbb{Z}/\ell^k\mathbb{Z}]$, which is an $\mathbb{Z}_\ell$-module of rank ℓ^k. On the other hand observe that the $\mathbb{Z}_\ell$-module $\mathscr{A}_\psi^{R_{\ell^k},\mathfrak{R}_k}$ is given by exactly the same cokernel as the module $\mathscr{A}^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$. The only difference is that $\mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$ is a $\mathbb{Z}_\ell[[(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}]]$-module while $\mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$ is a $\mathbb{Z}_\ell[[\mathbb{Z}/\ell^k\mathbb{Z}]]$-module.

So following exactly the same method as in example 16 we conclude that

$$\mathrm{rank}_{\mathbb{Z}_\ell} \mathscr{A}_\psi^{R_{\ell^k},\mathfrak{R}_k} = s \cdot \ell^k - s - \ell^k + 1 = (s-1)\ell^k - s + 1.$$

Also, as in the previous example exact sequence given in eq. (17) and the rank computation given in eq. (24) in example 16 we have:

$$\mathrm{rank} R_{\ell^k}/R'_{\ell^k} = \mathrm{rank}_{\mathbb{Z}_\ell} \mathscr{A}_\psi^{R_{\ell^k},\mathfrak{R}_k} - \mathrm{rank}_{\mathbb{Z}_\ell} \mathscr{A}^{R_{\ell^k},\mathfrak{R}_k} + 1 = (s-2)(\ell^k - 1).$$

This result gives twice the genus of the curve corresponding to R_{ℓ^k} and is in accordance with the formula given in eq. (32).

4.1.1. *Alexander modules after quotients.* Let Γ be a normal closed subgroup of $\mathfrak{F}_{s-1}$. Consider the diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathfrak{F}'_{s-1} \arrow[r] \arrow[d] & \mathfrak{F}_{s-1} \arrow[r] \arrow[d] & \mathfrak{F}_{s-1}/\mathfrak{F}'_{s-1} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \frac{\mathfrak{F}'_{s-1}\cdot\Gamma}{\Gamma} \arrow[r] & \frac{\mathfrak{F}_{s-1}}{\Gamma} \arrow[r] & \frac{\mathfrak{F}_{s-1}}{\mathfrak{F}'_{s-1}\cdot\Gamma} \arrow[r] & 1
\end{tikzcd}$$

We have

$$\mathscr{A}^{\mathfrak{F}'_{s-1},\Gamma} = \mathbb{Z}_\ell[[\mathfrak{F}_{s-1}/\mathfrak{F}'_{s-1}\cdot\Gamma]]$$

and

$$\mathscr{A}^{\mathfrak{F}'_{s-1},\{1\}} = \mathbb{Z}_\ell[[\mathfrak{F}_{s-1}^{\mathrm{ab}}]] \cong \mathbb{Z}_\ell[[u_1,\ldots,u_{s-1}]] = \mathscr{A}.$$

We also consider the free resolution of Alexander modules

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "Q_1"] & \mathscr{A}^s \arrow[r, "\psi_1"] \arrow[d, "\phi_2"] & \mathscr{A}^{s-1} \arrow[r] \arrow[d, "\phi_3"] & 0 \\
\left(\mathscr{A}^{\mathfrak{F}'_{s-1},\Gamma}\right)^{s+1} \arrow[r, "Q_2"] & \left(\mathscr{A}^{\mathfrak{F}'_{s-1},\Gamma}\right)^{s} \arrow[r, "\psi_2"] & \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\Gamma} \arrow[r] & 0
\end{tikzcd}$$

where Q_1, Q_2 are the maps appearing in eq. (19). In particular the map Q_1 sends

$$\mathscr{A} \ni \beta \mapsto \beta \cdot (1, x_1, \ldots, x_1 \cdot x_2 \cdots x_{s-1})$$

according to the computation done in eq. (22). The vertical map ϕ_2 is onto. The image $\phi_3(a)$ for $a \in \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\{1\}}$ is defined by selecting $b \in \mathscr{A}^s$ such that $\psi_1(b) = a$, and then $\phi_3(a) = \psi_2 \circ \phi_2(a)$. This definition is independent from the selection of b. The vertical maps are reductions modulo Γ. We have

$$\ker(\phi_3) = \psi_1(\phi_2^{-1}\mathrm{Im}(Q_2)). \tag{26}$$

The corresponding Crowell sequences are given:
(27)

$$\begin{tikzcd}
0 \arrow[r] & \frac{\mathfrak{F}'_{s-1}}{\mathfrak{F}''_{s-1}} \arrow[r] \arrow[d, "\phi"] & \mathscr{A}^{s-1} \arrow[r] \arrow[d, "\phi_3"] & \mathscr{A} \arrow[r] \arrow[d, "\mathrm{mod}\Gamma"] & \mathbb{Z}_\ell \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \left(\frac{\mathfrak{F}'_{s-1}\cdot\Gamma}{\Gamma}\right)^{\mathrm{ab}} = \frac{\mathfrak{F}'_{s-1}\cdot\Gamma}{\mathfrak{F}''_{s-1}\cdot\Gamma} \arrow[r] & \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\Gamma} \arrow[r] & \mathscr{A}^{\mathfrak{F}'_{s-1},\Gamma} \arrow[r] & \mathbb{Z}_\ell \arrow[r] & 0
\end{tikzcd}$$

We have

$$\ker\phi = \frac{\mathfrak{F}''_{s-1}\cdot\Gamma \cap \mathfrak{F}'_{s-1}}{\mathfrak{F}''_{s-1}}.$$

4.1.2. *Relating Alexander Modules.* Assume that $\mathfrak{F}'_{s-1} \subset R \subset \mathfrak{F}_{s-1}$. To the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & R \arrow[r] & \mathfrak{F}_{s-1}/\Gamma \arrow[r, "\psi"] & \mathfrak{F}_{s-1}/R\cdot\Gamma \arrow[r] & 1 \\
1 \arrow[r] & \mathfrak{F}'_{s-1} \arrow[r] \arrow[u, hook] & \mathfrak{F}_{s-1}/\Gamma \arrow[r, "\psi"] \arrow[u, equal] & \mathfrak{F}_{s-1}/\mathfrak{F}'_{s-1}\cdot\Gamma \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

we can attach two related Crowell sequences:

$$\begin{tikzcd}
0 \arrow[r] & R^{\mathrm{ab}} \arrow[r] & \mathscr{A}_\psi^{R,\Gamma} \arrow[r] & \mathscr{A}^{R,\Gamma} \arrow[r] & \mathbb{Z}_\ell \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} \arrow[r] \arrow[u, hook, "\theta_1"] & \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\Gamma} \arrow[r] \arrow[u, "\theta_2"] & \mathscr{A}^{\mathfrak{F}'_{s-1},\Gamma} \arrow[r] \arrow[u] & \mathbb{Z}_\ell \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The map θ_1 is well defined with kernel $R'/\mathfrak{F}''_{s-1}$. The map θ_2 is defined from the two corresponding free resolutions:

$$\begin{tikzcd}
\mathbb{Z}_\ell[[R/\Gamma]]^{s+1} \arrow[r, "Q_1"] & \mathbb{Z}_\ell[[R/\Gamma]]^{s} \arrow[r, "\pi_1"] & \mathscr{A}_\psi^{R,\Gamma} \arrow[r] & 0 \\
\mathbb{Z}_\ell[[\mathfrak{F}'_{s-1}/\Gamma]]^{s+1} \arrow[r, "Q_2"] \arrow[u, "\phi_1"] & \mathbb{Z}_\ell[[\mathfrak{F}'_{s-1}/\Gamma]]^{s} \arrow[r, "\pi_2"] \arrow[u, "\phi_2"] & \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\Gamma} \arrow[r] \arrow[u, "\theta_2"] & 0
\end{tikzcd}$$

Indeed, for $a \in \mathscr{A}^{\mathfrak{F}'_{s-1},\Gamma}$ we select any $b \in \mathbb{Z}_\ell[[\mathfrak{F}'_{s-1}/\Gamma]]^s$ and we set

$$\theta_2(a) = \pi_1 \circ \phi_2(b).$$

Then θ_2 is well defined, i.e. independent from the selection of b. The kernel of θ_2 is

$$\ker(\theta_2) = \pi_2\phi_2^{-1}(\mathrm{Im}(Q_1)).$$

The map θ_2 is onto. Indeed, for a $y \in \mathscr{A}_\psi^{R,\Gamma}$ we choose an element $b \in \pi_1^{-1}(\{y\}) \subset \mathbb{Z}_\ell[[R/\Gamma]]^s$ and since ϕ_2 is onto and element $b' \in \phi_2^{-1}(b)$. Then, $\theta_2(\pi_2(b')) = y$.

Observe also that both $\mathscr{A}_{\psi}^{R,\Gamma}$ and $\mathscr{A}^{\mathfrak{F}'_{s-1},\Gamma}$ are the cokernel of the same set of equations since the matrix Q depends only on the quotient $\mathfrak{F}_{s-1}/\Gamma$. The difference is that they are modules over different rings.

4.1.3. *Holomorphic Differentials.* We can pass to representations to spaces of holomorphic differentials by dualizing:

$$H_1(Y,\mathbb{Z}_\ell) = R^{\mathrm{ab}} = (R_0/\Gamma)^{\mathrm{ab}} = R_0/R_0'\Gamma = H_1(Y_0,\mathbb{Z}_\ell)/\Gamma.$$

Short exact sequence in eq. (17) can be dualized to an exact sequence as

$$\begin{array}{c}0 \to \mathrm{Hom}_{\mathbb{Q}_\ell}(\ker(\varepsilon_{\mathscr{A}}),\mathbb{Q}_\ell) \to \mathrm{Hom}_{\mathbb{Q}_\ell}(\mathscr{A}_\psi,\mathbb{Q}_\ell) \to \mathrm{Hom}_{\mathbb{Q}_\ell}\left(R^{\mathrm{ab}}\otimes_{\mathbb{Z}}\mathbb{Q}_\ell,\mathbb{Q}_\ell\right) \to 0 \\ \hspace{18em}\downarrow\cong \\ \hspace{18em}H^0(Y,\Omega_Y)\end{array}$$

5. Applications to cyclic covers of the projective line

In this section we will perform explicit computations for the case of cyclic ramified covers of the projective line. The first author considered the automorphism group of these in [28] and also considered the field of moduli versus field of definition in [4], [29].

There is a recent interest for these curves with respect to the action of mapping class groups. In [34] C. McMullen considered unitary representations of the braid group acting on global sections of differentials of cyclic covers of the projective line. Also W. Chen in [10] motivated by the fibration of cyclic groups studied the homology $H_*(B_s,V_s)$, where $V_s=\mathbb{C}[t,t^{-1}]^{s-2}$, where B_s acts on $\mathbb{C}[t,t^{-1}]$ in terms of the Burau representation.

Here we will see a more natural approach using the action on homology and Ihara's philosophy. This aproach was also mentioned in the article of McMullen on page 914 after th. 5.5. We believe that the details of this computation are worth studying.

Let S be a compact Riemann-surface of genus g. Consider the first homology group $H_1(S,\mathbb{Z})$ which is a free $\mathbb{Z}$-module of rank $2g$. Let $H^0(S,\Omega_S)$ be the space of holomorphic differentials which is a $\mathbb{C}$-vector space of dimension g. The function

$$\begin{array}{rcl} H_1(S,\mathbb{Z})\times H^0(S,\Omega_S) & \to & \mathbb{R} \\ \gamma,\omega & \mapsto & \langle\gamma,\omega\rangle = \mathrm{Re}\displaystyle\int_\gamma \omega \end{array}$$

induces a duality $H_1(S,\mathbb{Z})\otimes\mathbb{R}$ to $H^0(S,\Omega_S)^*$, see [30, th. 5.6], [18, sec. 2.2 p. 224]. Therefore an action of a group element on $H_1(S,\mathbb{Z})$ gives rise to the contragredient action on holomorphic differentials, see also [15, p. 271].

Recall that X_s denotes the projective line minus s-points and $\tilde{X}_s$ denotes the universal covering space. Let Y be the covering coresponding to the commutator group F'_{s-1}, therefore $\mathrm{Gal}(Y/X_s)\cong F_{s-1}/F'_{s-1} = H_1(S,\mathbb{Z})$. The braid group $B_{s-1}\subset \mathrm{Aut}(F_{s-1})$ acts on $H_1(X_s,\mathbb{Z})$ and leaves a rank 1 subspace I invariant.

In this section we choose to work with the discrete groups F_{s-1}. The pro-ℓ case follows similarly.

Definition 18. Consider the projection

$$0\to I\to H_1(X_s,\mathbb{Z}) \xrightarrow{w} \mathbb{Z}\to 0$$

and let C_s be the curve given as quotient Y/I, so that $\mathrm{Gal}(C_s/X_s) = \mathbb{Z}$. The map w is the winding number map which can be defined both at the fundamental group and its abelianization by: $(1 \leq i_1, \ldots, i_t \leq s, \ell_{i_1}, \ldots, \ell_{i_t} \in \mathbb{Z})$

$$w : \pi_1(X_s) \longrightarrow \mathbb{Z} \qquad x_{i_1}^{\ell_{i_1}} x_{i_2}^{\ell_{i_2}} \cdots x_{i_t}^{\ell_{i_t}} \mapsto \sum_{\mu=1}^{t} \ell_{i_\mu}.$$

We have the following commutative diagram:

$$\begin{tikzcd}
\tilde{X}_s \arrow[drr, "F'_{s-1}"] \arrow[ddr, "R_0"] \arrow[ddd, "F_{s-1}"'] & & \\
& & Y \arrow[dl, "I"'] \arrow[ddll, bend left, "H_1(X_s,\mathbb{Z})"] \\
& C_s \arrow[dl, "\mathbb{Z}"'] & \\
X_s & &
\end{tikzcd}$$

Then $H_1(C_s, \mathbb{Z}) = R_0/R_0'$, where $R_0 = \pi_1(C_s)$ is the free subgroup of F_{s-1} corresponding to C_s. Moreover $H_1(C_s, \mathbb{Z})$ is a $\mathbb{Z}[\mathbb{Z}]$-module acted on also by B_{s-1} giving rise to the so called Burau representation:

$$\rho : B_{s-1} \to \mathrm{GL}(s-1, \mathbb{Z}[t, t^{-1}]).$$

It is known that the space $H_1(C_s, \mathbb{Z})$ is a free $\mathbb{Z}[\mathbb{Z}]$-module of rank $s-2$. Keep in mind that $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$. In what follows will give a proof using the Reidemeister-Schreier algorithm.

Lemma 19. *The group R_0 (is an infinite rank group) is given by*

$$R_0 = \{x_1^i x_j x_1^{-i+1} : i \in \mathbb{Z}, j \in 2, \ldots, s-1\}. \tag{28}$$

Proof. Consider the epimorphisms

$$\begin{tikzcd}
F_{s-1} \arrow[r, "p'"] \arrow[rr, bend right, "w"'] & F_{s-1}/F'_{s-1} \arrow[r, "p''"] & \mathbb{Z} = H_1(Y, X_s)/I.
\end{tikzcd}$$

Set $w = p'' \circ p'$ Let y be an element in $w^{-1}(1_{\mathbb{Z}})$. By the properties of the winding number we can take as $y = x_1$. Moreover $w(x_j) = y$ for all $1 \leq j \leq s-1$ since the automorphism $x_i \leftrightarrow x_j$ is compatible with I and therefore introduces an automorphism of $\mathbb{Z}$, so $w(x_j) = y^{\pm 1}$, and we rename the generators x_i to x_i^{-1} if necessary.

Let $T := \{y^i : i \in \mathbb{Z}\} \subset F_{s-1}$ be a set of representatives of classes in $F_{s-1}/R_0 \cong \mathbb{Z}$. For every $x \in F_{s-1}$ we will denote by $\bar{x}$ the representative in T. Moreover for all $i \in \mathbb{Z}$ and $1 \leq j \leq s-1$ we have $\overline{y^i x_j} = y^{i+1}$ and by the Reidemeister-Schreier algorithm [33, sec. 2.3 th. 2.7] we see that

$$\{y^i x_j \left(\overline{y^i x_j}\right)^{-1} = y^i x_j y^{-i-1}\} = \{x_1^i x_j x_1^{-i-1} : i \in \mathbb{Z}, j \in 2, \ldots, s-1\}.$$

$\square$

Remark 20. The action of y on R_0/R_0' is given by conjugation. A generating set for $H_1(C_s, \mathbb{Z})$ is given by the $s-2$ elements $\beta_j := x_j x_1^{-1}$. Moreover the action is given by

$$t^n \cdot x_i x_1^{-1} = x_1^n x_i x_1^{-n-1},$$

i.e. that $H_1(C_s, \mathbb{Z})$ is a free $\mathbb{Z}[\mathbb{Z}]$-module of rank $s-2$.

Let us now consider a finite cyclic cover Y_n of X_s which is covered by C_s, i.e. we have the diagram on the right bellow:

$$\begin{tikzcd}
\tilde{X}_s \arrow[rr] \arrow[dr, "R_0"] \arrow[ddd] & & Y \arrow[dl, "I"'] \arrow[dddll, bend left, "H_1(X_s,\mathbb{Z})"] \\
& C_s \arrow[d] \arrow[ddl, "\mathbb{Z}"'] & \\
& Y_n \arrow[dl, "\mathbb{Z}/n\mathbb{Z}"] & \\
X_s & &
\end{tikzcd}$$

Lemma 21. *The group $R_n = \pi_1(Y_n) \supset R_0$ is the kernel of the map w_n*

$$\begin{tikzcd}
\pi_1(X) \arrow[r, "w"] \arrow[rr, bend left, "w_n"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/n\mathbb{Z}.
\end{tikzcd}$$

Proof. This is clear from the explicit description of the group R_0 given in eq. (28). □

Lemma 22. *The group R_n is generated by*

$$R_n = \{x_1^i x_j x_1^{-i-1} : 0 \leq i \leq n-2, 2 \leq j \leq s-1\} \cup \{x_1^{n-1} x_j : 1 \leq j \leq s-1\}.$$

which is a free group on $r = (s-2)n + 1$ generators.

Proof. In this case the transversal set equals $T = \{y^i : 0 \leq i \leq n-1\}$. Moreover

$$\overline{y^i x_j} = \begin{cases} y^{i+1} & \text{if } i < n-1 \\ 1 & \text{if } i = n-1. \end{cases}$$

The desired result follows. □

Remark 23. The above computation is compatible with the Schreier index formula [8, cor. 8.5 p.66] which asserts that

$$r - 1 = n(s-2). \tag{29}$$

Remark 24. The space $H_1(Y_n, \mathbb{Z})$ is a free $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$-module of rank $s-2$ with basis $\beta_j = x_j x_1^{-1}$.

Proposition 25. *The action of the braid group on $H_1(Y_n, \mathbb{Z})$ is given by specialization of the Burau representation to $t \mapsto \zeta_n$.*

Proof. Consider the short exact sequences for the free groups R_0, R_n corresponding to the curves C_s and Y_n:

$$\begin{tikzcd}
1 \arrow[r] & R_0 \arrow[r] \arrow[d, hook] & F_{s-1} \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & R_n \arrow[r] & F_{s-1} \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 1
\end{tikzcd}$$

which gives rise to the sequence of Alexander modules

$$\begin{tikzcd}
0 \arrow[r] & H_1(C_s, \mathbb{Z}) = R_0^{\mathrm{ab}} \arrow[r, "d_0"] \arrow[d] & \mathbb{Z}[\mathbb{Z}]^{s-1} \arrow[r] \arrow[d, "\theta_1"] & \mathbb{Z}[\mathbb{Z}] \arrow[r] \arrow[d, "\theta_2"] & \mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & H_1(Y_n, \mathbb{Z}) = R_n^{\mathrm{ab}} \arrow[r, "d_n"] & \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]^{s-1} \arrow[r] & \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

The image of the elements $\beta_j = x_j x_1^{-1}$ under d_0 and d_n are given by the vectors $(-1, 0, \ldots, 0, 1, 0, \ldots, 0)$ where the "1" is at the j-th position. The vertical maps θ_1, θ_2 are both reduction mod n sending an element $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ to the corresponding element in $\mathbb{Z}[\zeta_n]$ by evaluating $t \mapsto \zeta_n$. □

5.1. **The Burau Representation.** Consider the action of a generator σ_i of B_s seen as an automorphism of the free group, given for $1 \leq i, j \leq s-1$ as

$$\sigma_i(x_j) = \begin{cases} x_j & \text{if } j \neq i, i+1 \\ x_i & \text{if } j = i+1 \\ x_i x_{i+1} x_i^{-1} & \text{if } j = i \end{cases}$$

Therefore the conjugation action on the generators $\beta_j = x_j x_1^{-1}$ of R, seen as a $\mathbb{Z}[\mathbb{Z}]$-module, is given for $j \geq 2$ by:

$$\sigma_j(\beta_{j+1}) = \sigma_j(x_{j+1}x_1^{-1}) = x_j x_1^{-1} = \beta_j,$$

$$\begin{aligned} \sigma_j(\beta_j) = \sigma_j(x_j x_1^{-1}) &= x_j \cdot x_{j+1} \cdot x_j^{-1} \cdot x_1^{-1} = x_j x_1^{-1} \cdot x_1 x_{j+1} x_1^{-2} x_1^2 x_j^{-1} \cdot x_1^{-1} \\ &= \beta_j x_1 \beta_{j+1} x_1^{-1} x_1 \beta_j^{-1} x_1^{-1} = \beta_j \beta_{j+1}^t \beta_j^{-t} \end{aligned}$$

Also in the special case where $j = 1$ we compute:

$$\sigma_1(\beta_2) = \sigma_1(x_2 x_1^{-1}) = x_1 \cdot x_1 x_2^{-1} x_1^{-1} = \beta_2^{-t},$$

and if $i > 2$

$$\sigma_1(\beta_i) = \sigma_1(x_i x_1^{-1}) = x_i \cdot x_1 x_2^{-1} x_1^{-1} = x_i x_1^{-1} \cdot x_1 x_1 x_2^{-1} x_1^{-1} = \beta_i \beta_2^{-t}$$

We now compute the action on the $\mathbb{Z}$-module R/R', so the β_i, β_j are commuting and we arrive at the matrix of the action with respect to the basis $\{\beta_2, \ldots, \beta_{s-1}\}$:

$$\sigma_j \mapsto \begin{pmatrix} \mathrm{Id} & & & \\ & 1-t & 1 & \\ & t & 0 & \\ & & & \mathrm{Id} \end{pmatrix}, \text{if} \quad j \neq 1 \text{ and } \sigma_1 \mapsto \begin{pmatrix} 1-t & -t & & -t \\ 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Lemma 26. *The action of t on β_i commutes with the action of the braid group.*

Proof. It is obvious that for σ_j $j \geq 2$ and $a \in R_0$ we have

$$\sigma_j(a^t) = \sigma_j(x_1 a x_1^{-1}) = x_1 \sigma_j(a) x_1^{-1} = (\sigma_j(a))^t.$$

For σ_1 we observe that

$$\begin{aligned} \sigma_j(a^t) = \sigma_1(x_1 a x_1^{-1}) &= x_1 x_2 x_1^{-1} \sigma_1(a) x_1 x_2^{-1} x_1^{-1} = x_1 \beta_2 \sigma_1(a) \beta_2^{-1} x_1^{-1} \\ &= x_1 \sigma_1(a) x_1^{-1} = (\sigma_1(a))^t, \end{aligned}$$

since $\sigma_1(a)$ is expressed as product of β_ν and the elements β_i commute modulo R_0'. □

5.2. **The profinite Burau representation.** Let C_s be the curve given in definition 18. The fundamental group R_0 of the curve C_s is computed in lemma 28 and fits in the small exact sequence

$$1 \to R_0 \to \mathfrak{F}_{s-1} \to \mathbb{Z}_\ell \to 0.$$

We will now employ the Blanchfield-Lyndon exact sequence given in proposition 15:

$$0 \to (R_0)^{\mathrm{ab}} \xrightarrow{\theta_1} (\mathbb{Z}_\ell[[\mathbb{Z}_\ell]])^{s-1} \xrightarrow{d_1} \mathbb{Z}_\ell[[\mathbb{Z}_\ell]] \xrightarrow{e_{\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]}} \mathbb{Z}_\ell \to 0,$$

since $\mathscr{A}^{R_0,\{1\}} = \mathbb{Z}_\ell[[\mathbb{Z}_\ell]]$. The image of R_0^{ab} in $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]$, according to eq. (18) is generated by the images of the elements, for $2 \leq j \leq s-1$

$$d(x_j \cdot x_1^{-1}) = dx_j - \psi(x_j) dx_1^{-1} = dx_j - dx_1.$$

The above elements form a basis of the image of θ_1 in the free module $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]^{s-1}$. In this way we see the profinite Burau representation as a linear representation:

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_{s-2}(\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]).$$

Remark 27. The $\mathbb{Z}_\ell$-algebra $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]$ is a ring containing all formal infinite expressions of the monomials t^α, $\alpha \in \mathbb{Z}_\ell$. It contains the $\mathbb{Z}$-algebra $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$ which appears in the discrete topological Burau representation.

Lemma 28. *We have*

$$x_k^n x_1^{-n} = \beta_k \cdot \beta_k^t \cdot \beta_k^{t^2} \cdots \beta_k^{t^{n-1}}. \tag{30}$$

Proof. Indeed, for $n = 1$ the result is trivial while by induction

$$x_k^n x_1^{-n} = x_k \beta_k \cdots \beta_k^{t^{n-2}} x_1^{-1} = x_k x_1^{-1} x_1 \beta_k \cdots \beta_k^{t^{n-2}} x_1^{-1} = \beta_k \cdot \beta_k^t \cdot \beta_k^{t^2} \cdots \beta_k^{t^{n-1}}$$

□

We would like to study the dependence of the matrix of ρ with respect to the Magnus powerseries in eq. (6). We have that $\sigma(x_i) = w_i(\sigma) x_i^{N(\sigma)} w_i(\sigma)^{-1}$. Let us analyse $w_i(\sigma) \in \mathfrak{F}_{s-1}$: Assume that $w_i(\sigma) = \lim_{n\to\infty} w_i^{(n)}$, with $w_i^{(n)} \in F_{s-1}$. We write

$$w_i^{(n)} = x_1^{a_{1,i}} x_{i_1}^{a_{i_1,i}} \cdots x_{i_t}^{a_{i_t,i}}$$

Let us introduce the following notation: For a word

$$w = x_1^{a_{1,i}} x_{i_1}^{a_{i_1,i}} \cdots x_{i_\mu}^{a_{i_\mu,i}} \cdots x_{i_t}^{a_{i_t,i}}$$

denote the contribution to exponents corresponding to the generator x_{i_μ} by

$$v_{i_\mu}(w) = \sum_{i_j=1}^{i_\mu} a_{i_j,i}$$

and the total contribution

$$\bar{v}_i(w) = \sum_{i_\mu = i} t^{v_{i_{\mu-1}}} = \sum_{i_\mu = i} t^{\sum_{i_j=1}^{i_{\mu-1}} a_{i_j,i}}.$$

We have assumed that $w_i(\sigma)$ starts with a power of x_1 with possibly a zero exponent. We write

$$\begin{aligned} w_i^{(n)} &= x_1^{a_{1,i}} x_{i_1}^{a_{i_1,i}} x_1^{-a_{i_1,i}-a_{1,i}} x_1^{a_{1,i}+a_{i_1,i}} \cdots x_1^{\sum_{i_j=1}^{i_{t-1}} a_{i_j,i}} x_{i_t}^{a_{i_t,i}} x_1^{-\sum_{i_j=1}^{i_t} a_{i_j,i}} x_1^{\sum_{i_j=1}^{i_t} a_{i_j,i}} \\ &= (\beta_{i_1} \cdot \beta_{i_1}^t \cdots \beta_{i_1}^{t^{a_{i_1,i}-1}})^{t^{a_{1,i}}} \cdots (\beta_{i_t} \cdot \beta_{i_t}^t \cdots \beta_{i_t}^{t^{a_{i_t,i}-1}})^{t^{\sum_{i_j=1}^{i_{t-1}} a_{i_j,i}}} \cdot x_1^{\sum_{i_j=1}^{i_t} a_{i_j,i}} \end{aligned}$$

Therefore

$$\begin{aligned} \beta_i = x_i x_1^{-1} \xmapsto{\sigma} w_i^{(n)} x_i^{N(\sigma)} \left(w_i^{(n)}\right)^{-1} x_1^{-N(\sigma)} &= (\beta_{i_1} \cdot \beta_{i_1}^t \cdots \beta_{i_1}^{t^{a_{i_1,i}-1}})^{t^{a_{1,i}}} \cdots \\ \cdots (\beta_{i_t} \cdot \beta_{i_t}^t \cdots \beta_{i_t}^{t^{a_{i_t,i}-1}})^{t^{\sum_{i_j=1}^{i_{t-1}} a_{i_j,i}}} &\cdot (\beta_i \cdot \beta_i^t \cdots \beta_i^{t^{N(\sigma)-1}})^{t^{\sum_{i_j=1}^{i_t} a_{i_j,i}}}. \end{aligned}$$

$$(\beta_{i_t} \cdot \beta_{i_t}^t \cdots \beta_{i_t}^{t^{a_{i_t,i}-1}})^{-t^{N(\sigma)+\sum_{i_j=1}^{i_{t-1}} a_{i_j,i}}} \cdots (\beta_{i_1} \cdot \beta_{i_1}^t \cdots \beta_{i_1}^{t^{a_{i_1,i}-1}})^{-t^{N(\sigma)+a_{1,i}}} \cdot x_1^{N(\sigma)} \cdot x_1^{-N(\sigma)}$$

$$= \beta_2^{\bar{v}_2(w_i^{(n)})(1-t^{N(\sigma)}) \sum_{\nu=0}^{a_{2,i}-1} t^\nu} \cdots \beta_{s-1}^{\bar{v}_{s-1}(w_i^{(n)})(1-t^{N(\sigma)}) \sum_{\nu=0}^{a_{(s-1),i}-1} t^\nu} \cdot \beta_i^{t^{v_{i_t}(w_i^{(n)})} \sum_{\nu=0}^{N(\sigma)-1} t^\nu}$$

We have proved the following:

Proposition 29. *The matrix of the representation of σ acting on $H_1(C_s,\mathbb{Z})$ is given by:*

$$\left(1-t^{N(\sigma)}\right)\cdot \lim_{n\to\infty} A_n + \sum_{\nu=0}^{N(\sigma)-1} t^{\nu}\cdot \mathrm{diag}\left(t^{v_{2t}}, t^{v_{3t}},\ldots,t^{v_{it}},\ldots,t^{v_{(s-1)t}}\right),$$

where .

$$A_n=\begin{pmatrix} \bar{v}_2(w_2^{(n)})\sum_{\nu=0}^{a_{2,2}-1} t^{\nu} & \bar{v}_3(w_2^{(n)})\sum_{\nu=0}^{a_{2,3}-1} t^{\nu} & \cdots & \bar{v}_{s-1}(w_2^{(n)})\sum_{\nu=0}^{a_{2,s-1}-1} t^{\nu} \\ \vdots & \vdots & & \vdots \\ \bar{v}_2(w_i^{(n)})\sum_{\nu=0}^{a_{i,2}-1} t^{\nu} & \bar{v}_3(w_i^{(n)})\sum_{\nu=0}^{a_{i,3}-1} t^{\nu} & \cdots & \bar{v}_{s-1}(w_i^{(n)})\sum_{\nu=0}^{a_{i,s-1}-1} t^{\nu} \\ \vdots & \vdots & & \vdots \\ \bar{v}_2(w_{s-1}^{(n)})\sum_{\nu=0}^{a_{s-1,2}-1} t^{\nu} & \bar{v}_3(w_{s-1}^{(n)})\sum_{\nu=0}^{a_{s-1,3}-1} t^{\nu} & \cdots & \bar{v}_{s-1}(w_{s-1}^{(n)})\sum_{\nu=0}^{a_{s-1,s-1}-1} t^{\nu} \end{pmatrix}$$

5.3. **The compactification of cyclic covers.** Consider the complex compact Riemann surface corresponding to the cyclic cover of the projective line given by:

$$y^n=\prod_{i=1}^{s}(x-b_i)^{d_i}, \qquad (d_i,n)=1 \tag{31}$$

where $\sum_{i=1}^{s} d_i \equiv 0 \mod n$, so that there is no ramification at infinity.

Riemann-Hurwitz theorem implies that

$$g=\frac{(n-1)(s-2)}{2}, \tag{32}$$

which is compatible with the computation of $r=2g+s-1$ given in eq. (29) and also with the results in example 17.

This curve can be uniformized as a quotient $\mathbb{H}/\Gamma$ of the hyperbolic space modulo a discrete free subgroup of genus g, which admits a presentation

$$\Gamma=\langle a_1,b_1,a_2,b_2,\ldots,a_g,b_g | [a_1,b_1][a_2,b_2]\cdots[a_g,b_g]=1\rangle.$$

On the other hand side, when we remove the s-branch points we obtain a topological cover of the space X_s defined in the previous section. This topological cover corresponds to the free subgroup of $R_n<F_{s-1}$ given by

$$R_n=\langle a_1,b_1,a_2,b_2,\ldots,a_g,b_g,\gamma_1,\ldots,\gamma_s | \gamma_1\gamma_2\cdots\gamma_s\cdot[a_1,b_1]\cdots[a_g,b_g]=1\rangle$$

Proposition 30. *The $\mathbb{Z}$-module R_n/R_n' as $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$-module is isomorphic to*

$$R_n/R_n'=\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]^{s-2}\bigoplus\mathbb{Z}.$$

Proof. Set $\beta_j=x_jx_1^{-1}$ for $2\le j\le s-1$. Then the action of $\mathbb{Z}/n\mathbb{Z}=\langle g\rangle$ on β_j is given by

$$\beta_j^{g^{\ell}}=x_1^{\ell}x_jx_1^{-\ell-1}.$$

It is clear that for each fixed j, $2\le j\le s-1$, the elements $\beta_j^{g^{\ell}}$ generate a copy of the group algebra $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$. By the explicit form of the basis generators given in lemma 22 we have the alternative basis given by

$$\{x_1^ix_jx_1^{-i-1}:2\le j\le s-1, 0\le i\le n-1\}\cup\{x_1^n\}. \tag{33}$$

The result follows. ∎

Lemma 31. *The $\mathbb{Z}/n\mathbb{Z}$-invariant elements of R_n/R_n' are given by multiples of*

$$\{x_i^n : 1 \leq i \leq s-1\}.$$

Proof. Observe that an element in the group algebra $\mathbb{Z}[\langle g \rangle]$ is g-invariant if and only if it is of the form $\sum_{i=0}^{n-1} a g^i$ for some $a \in \mathbb{Z}$. Hence the invariant elements are multiples (powers in the multiplicative notation) by

$$\beta_j \beta_j^g \beta_j^{g^2} \cdots \beta_j^{g^{n-1}} = x_j^n x_1^{-n}.$$

Since x_1^n is invariant the result follows. ∎

The elements γ_i are lifts of the loops x_i around each hole in the projective line. Thus γ_i are $\mathbb{Z}/n\mathbb{Z}$-invariant. Set $\gamma_i = x_i^n$. The quotient $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]/\langle \sum_{i=0}^{n-1} g^i \rangle$ is the augmentation representation.

Lemma 32. *We have*

$$x_k^n x_i x_k^{-n} x_1^{-1} = \beta_k \cdot \beta_k^t \cdot \beta_k^{t^2} \cdots \beta_k^{t^{n-1}} \cdot \beta_i^{t^n} \cdot \beta_k^{-t^n} \cdot \beta_k^{-t^{n-1}} \cdots \beta_k^{-t^2} \cdot \beta_k^{-t}$$

Moreover in the abelian group R/R' we have

$$x_k^n x_i x_k^{-n} x_1^{-1} = \beta_i^{t^n} \beta_k^{1-t^n}.$$

Proof. Write

$$\begin{aligned} x_k^n x_i x_k^{-n} x_1^{-1} &= x_k^n x_1^{-n} \cdot x_1^n x_i x_1^{-1} x_1^{-n} x_1^{n+1} x_k^{-n} x_1^{-1} \\ &= \beta_k \cdot \beta_k^t \cdot \beta_k^{t^2} \cdots \beta_k^{t^{n-1}} \cdot x_1^n \beta_i x_1^{-n} x_1 \left(\beta_k \cdot \beta_k^t \cdot \beta_k^{t^2} \cdots \beta_k^{t^{n-1}} \right)^{-1} x_1^{-1} \\ &= \beta_k \cdot \beta_k^t \cdot \beta_k^{t^2} \cdots \beta_k^{t^{n-1}} \cdot \beta_i^{t^n} \cdot \beta_k^{-t^n} \cdot \beta_k^{-t^{n-1}} \cdots \beta_k^{-t^2} \cdot \beta_k^{-t} \end{aligned}$$

∎

Lemma 33. *The subgroup of R_n/R_n' generated by $\mathbb{Z}/n\mathbb{Z}$-invariant elements*

$$\{x_1^n, x_j^n x_1^{-n} : 2 \leq j \leq s-1\}$$

is invariant under the action of the braid group.

Proof. By lemma 28 we have

$$\begin{aligned} \sigma_1(x_1^n) &= (x_1 x_2 x_1^{-1})^n = x_1 \cdot x_2^n \cdot x_1^{-1} = x_1 \cdot x_2^n x_1^{-n} \cdot x_1^{n-1} \\ &= x_1 \cdot \beta_2 \cdot \beta_2^t \cdot \beta_2^{t^2} \cdots \beta_2^{t^{n-1}} \cdot x_1^{-1} \cdot x_1^n = \beta_2^t \cdot \beta_2^{t^2} \cdots \beta_2^{t^n} \cdot x_1^n \\ &= \beta_2 \cdot \beta_2^t \cdots \beta_2^{t^{n-1}} \cdot x_1^n = x_2^n x_1^{-n} \cdot x_1^n = x_2^n \\ \sigma_1(x_2^n) &= x_1^n, \sigma_1(x_i^n) = x_i^n \ (i > 2). \end{aligned}$$

For $j \geq 2$:
$$\begin{aligned} \sigma_j(x_j^n x_1^{-n}) &= (x_j x_{j+1} x_j^{-1})^n x_1^{-n} = x_j \cdot x_{j+1}^n \cdot x_j^{-1} \cdot x_1^{-n} \\ &= x_j x_1^{-1} \cdot x_1 (x_{j+1}^n x_1^{-n}) x_1^{-1} \cdot x_1^n \cdot x_1 x_j^{-1} \cdot x_1^{-n} \\ &= x_{j+1}^n x_1^{-n} \\ \sigma_j(x_j^n) &= \sigma_j(x_j^n x_1^{-n}) \sigma_j(x_1^n) = x_{j+1}^n. \end{aligned}$$

∎

Consider now the space

$$H_1(\bar{Y}_n,\mathbb{Z}) = \frac{R_n}{R_n' \cdot \langle \gamma_1,\dots,\gamma_s\rangle} = \frac{R_n}{R_n' \cdot \langle x_1^n,\dots,x_s^n\rangle}.$$

Observe that $R_n/R_n' \cdot \langle x_1\rangle = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]^{s-2}$. Since $\langle \gamma_1,\dots,\gamma_s\rangle$ is both $\mathbb{Z}/n\mathbb{Z}$ and B_s stable we have a natural defined action of B_s on the quotient. We compute now the action of the braid group on $\beta_j^{g^i} = x_1^i x_j x_1^{-i-1}$. We can pick as a basis of the $\mathbb{Z}$-module $H_1(\bar{Y}_n,\mathbb{Z})$ the elements

$$\{\beta_j^{t^i} = x_1^i x_j x_1^{-1-i} : 2 \le j \le s-1, 0 \le i \le n-2\}$$

and equation (30) written additively implies that $\beta_j^{t^{n-1}} = -\sum_{\nu=0}^{n-2} \beta_j^{t^\nu}$.

Consider the augmentation map

$$\begin{aligned} \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] &\longrightarrow \mathbb{Z} \\ \textstyle\sum_{i=0}^{n-1} a_i g^i &\longmapsto \textstyle\sum_{i=0}^{n-1} a_i \end{aligned}$$

which has kernel the augmentation ideal $I_{\mathbb{Z}/n\mathbb{Z}}$ generated as a $\mathbb{Z}$-module by elements $\langle g^\nu - 1 : 1 \le \nu \le n-1\rangle$. Observe that $\beta_j^{t^\nu - 1} = [x_1^\nu, x_j]$. It is well known (see, [36, Prop. 1.2]) that $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] = I_{\mathbb{Z}/n\mathbb{Z}} \oplus \mathbb{Z}$. Therefore $H_1(\bar{Y}_n,\mathbb{Z}) = I_{\mathbb{Z}/n\mathbb{Z}}^{s-2}$. Notice that the above $\mathbb{Z}$-module has the correct rank $2g = (n-1)(s-2)$. The direct sum is in the category of $\mathbb{Z}$-modules not in the category of B_s-modules. Also on the augmentation module $I_{\mathbb{Z}/n\mathbb{Z}}$ the generator of the $\mathbb{Z}/n\mathbb{Z}$ is represented by the matrix:

$$A := \begin{pmatrix} 0 & \cdots & 0 & -1 \\ 1 & \ddots & \vdots & \vdots \\ 0 & \ddots & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{34}$$

which is the companion matrix of the polynomial $x^{n-1} + \cdots + x + 1$. One way to represent $I_{\mathbb{Z}/n\mathbb{Z}}$ is in terms of the $\mathbb{Z}$-module $\mathbb{Z}[\zeta]$, where ζ is a primitive n-th root of unity, i.e.

$$\mathbb{Z}[\zeta] = \bigoplus_{\nu=0}^{n-1} \zeta^\nu \mathbb{Z},$$

and the $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$-module structure is given by multiplication by ζ. This means that each $\beta_j^{t^i}$ for $2 \le j \le s-1$ and $0 \le i \le n-1$ is mapped to ζ^i.

Since the $\mathbb{Z}/n\mathbb{Z}$-action and the braid action are commuting we have a decomposition (notice that 1 does not appear in the eigenspace decomposition below)

$$H_1(\bar{Y}_n,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{\nu=1}^{n-1} V_\nu$$

where V_ν is the eigenspace of the ζ^ν-eigenvalue. Each V_ν is a B_s-module of dimension $s-2$. In order to compute the spaces V_ν we have to diagonalize the matrix given in eq. (34). Consider the Vandermonde matrix given by:

$$P = \begin{pmatrix} 1 & \zeta_1 & \zeta_1^2 & \cdots & \zeta_1^{n-1} \\ 1 & \zeta_2 & \zeta_2^2 & \cdots & \zeta_2^{n-1} \\ \vdots & \vdots & & \vdots & \\ 1 & \zeta_{n-1} & \zeta_1^2 & \cdots & \zeta_{n-1}^{n-1} \end{pmatrix},$$

where $\{\zeta_1, \ldots, \zeta_{n-1}\}$ are all n-th roots of unity different than 1. Observe that

$$P \cdot A = \mathrm{diag}(\zeta_1, \zeta_2, \ldots, \zeta_n) \cdot P.$$

Thus the action of the braid group on the eigenspace V_ν of the eigenvalue ζ^ν can be computed by a base change as follows: Consider the initial base $\beta_j, \beta_j^t, \ldots, \beta_j^{t^{n-1}}$ for $2 \leq j \leq s-1$. The eigenspace of the ζ^ν eigenvalue has as basis the k-element of the $1 \times (n-1)$ matrix

$$\left(\beta_j, \beta_j^t, \ldots, \beta_j^{t^{n-1}}\right) \cdot P^{-1}$$

for all j such that $2 \leq j \leq n-2$. These elements are $\mathbb{C}$-linear combinations of the elements β_j and the action of the braid generators on them can be easily computed.

Remark 34. A cyclic cover given X in eq. (31) might have a bigger automorphism group then the cyclic group of order n, if the roots b_i form a special configuration. Notice also that if the number s of branched points satisfies $s > 2n$ then the automorphism group G fits in a short exact sequence

$$1 \to \mathbb{Z}/n\mathbb{Z} \to G \to H \to 1 \tag{35}$$

where H is a subgroup of $\mathrm{PGL}(2, k)$ [28, prop. 1]. The first author in [28] classified all such extensions.

Observe that the action of the mapping class group of homology is of topological nature and hence independent of the special configuration of the roots b_i. If these roots have a special configuration then elements of the mapping class group become automorphisms of the curve. This phenomenon is briefly explained on page 895 of [34].

Similarly, the action of elements of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which keeps invariant the set of branch points $\{b_i : 1 \leq i \leq s\}$ on homology is the same for all cyclic covers. For certain configurations of the branch points elements of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ can be seen as automorphism of the curve.

If the branch locus $\{b_i : 1 \leq i \leq s\}$ is invariant under the group H then $H_1(X, \mathbb{Z})$ is a $\mathbb{Z}[G]$ module, where G is an extension of H with kernel $\mathbb{Z}/n\mathbb{Z}$ given by eq. (35). In next section we will treat such highly symmetrical curves.

C. Mc Mullen in [34, sec. 3] considered the Hodge decomposition of the DeRham cohomology as

$$H^1(X) = \mathrm{Hom}_{\mathbb{C}}(H_1(X, \mathbb{Z}), \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \cong \Omega(X) \oplus \bar{\Omega}(X).$$

Of course this decomposition takes place in the dual space of holomorphic differentials, and is based on the intersection form

$$\langle \alpha, \beta \rangle = i/2 \int_X \alpha \wedge \bar{\beta}. \tag{36}$$

In this article we use the group theory approach and we focus around the homology group $H_1(X,\mathbb{Z})$. Homology group is equipped with an intersection form and a canonical symplectic basis $a_1,\ldots,a_g,b_1,\ldots,b_g$ such that

$$\langle a_i,b_j\rangle=\delta_{ij},\qquad \langle a_i,a_j\rangle=\langle b_i,b_j\rangle=0.$$

Every two homology classes γ,γ' can be written as $\mathbb{Z}$-linear combinations of the canonical basis

$$\gamma=\sum_{i=1}^{g}(\lambda_i a_i+\mu_i b_i)\qquad \gamma'=\sum_{i=1}^{g}(\lambda_i' a_i+\mu_i' b_i)$$

and the intersection is given by

$$\langle\gamma,\gamma'\rangle=(\lambda_1,\ldots,\lambda_g,\mu_1,\ldots,\mu_g)\begin{pmatrix}0 & \mathbb{I}_g\\ -\mathbb{I}_g & 0\end{pmatrix}(\lambda_1',\ldots,\lambda_g',\mu_1',\ldots,\mu_g')^t$$

This gives rise to a representation

$$\rho: B_{s-1}\to \mathrm{Sp}(2g,\mathbb{Z})$$

since $\langle\sigma(\gamma),\sigma(\gamma')\rangle=\langle\gamma,\gamma'\rangle$. Indeed, a topological homeomorphism keeps the intersection multiplicity of two curves. The relation to the unitary representation on holomorphic differentials (and the signature computations) is given by using the diagonalization of

$$\begin{pmatrix}0 & \mathbb{I}_g\\ -\mathbb{I}_g & 0\end{pmatrix}=P\cdot\mathrm{diag}(\underbrace{i,\ldots,i}_{g},\underbrace{-i,\ldots,-i}_{g})\cdot P^{-1},$$

and the extra "i" put in front of eq. (36).

5.3.1. *Arithmetic analogon:*

Lemma 35 (Galois Descent)**.** *There is a canonical basis $\{a_i,b_i,1\leq i\leq g\}$ such that for all $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and for γ,γ' in the canonical basis we have*

$$\langle\sigma\gamma,\sigma\gamma'\rangle=\langle\gamma,\gamma'\rangle.$$

Proof. According to [26, 1.7.3] we have trivial Galois cohomology

$$H^1\big(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),\mathrm{Sp}_{2g}(\bar{\mathbb{Q}})\big)=\{1\}.$$

The desired result follows by Galois descent. □

Let us now consider the elements γ,γ' given by

$$\gamma=\sum_{i=1}^{g}(\lambda_i a_i+\mu_i\beta_i),\qquad \lambda_i,\mu_i\in\hat{\mathbb{Z}}$$
$$\gamma'=\sum_{i=1}^{g}(\lambda_i' a_i+\mu_i'\beta_i),\qquad \lambda_i',\mu_i'\in\hat{\mathbb{Z}}$$

in a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-invariant symplectic canonical basis. The action of $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the intersection form is given by:

$$\langle\sigma\gamma,\sigma\gamma'\rangle=\sum_{i=1}^{g}\sigma(\lambda_i\mu_i'-\mu_i\lambda_i')\langle a_i,b_i\rangle=\langle\gamma,\gamma'\rangle^{\sigma}.$$

5.4. **Fermat Curves.** These curves are ramified curves over $\mathbb{P}^1 - \{0,1,\infty\}$ with deck group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. We have $\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, x_0) = F_2 = \langle a, b\rangle$. A transversal set T for $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ in F_2 is given by $a^i b^j$, $0 \leq i, j \leq n-1$. We also compute:

$$\overline{a^i b^j b} = \begin{cases} a^i b^{j+1} & \text{if } j < n-1 \\ a^i & \text{if } j = n-1 \end{cases}$$

and

$$\overline{a^i b^j a} = \begin{cases} a^{i+1} b^j & \text{if } i < n-1 \\ b^j & \text{if } i = n-1 \end{cases}$$

Thus

$$\begin{aligned} a^i b^j b \left(\overline{a^i b^j b}\right)^{-1} &= \begin{cases} a^i b^j b b^{-j-1} a^{-i} = 1 & \text{if } j < n-1 \\ a^i b^n a^{-i} & \text{if } j = n-1 \end{cases} \\ a^i b^j a \left(\overline{a^i b^j a}\right)^{-1} &= \begin{cases} a^i b^j a b^{-j} a^{-i-1} & \text{if } i < n-1, j \neq 0 \\ 1 & \text{if } i < n-1, j = 0, \\ a^{n-1} b^j a b^{-j} & \text{if } i = n-1 \end{cases} \end{aligned}$$

Consider the generators α, β of the cyclic group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Let R_F be the fundamental group of the Fermat curve, seen as a subgroup of the free group F_2. Observe that there is a well defined action of α (resp. β) on R_F/R_F' given by conjugation, i.e.

$$x^\alpha = x^a = axa^{-1} \qquad x^\beta = x^b = bxb^{-1}$$

for all $x \in R_F/R_F'$. Notice that this is indeed an action which implies that

$$(x^\alpha)^\beta = x^{\alpha\beta} = x^{\beta\alpha} = (x^\beta)^\alpha$$

i.e. the actions of α and β commute. We can consider the conjugation action of $\mathbb{Z}/n\mathbb{Z} = \langle\alpha\rangle$ and then we have the following sets of generators, of the free group R_F:

$$\begin{aligned} A_1 &= \left\{(b^n)^{a^i} : 0 \leq i \leq n-1\right\}, & \#A_1 &= n \\ A_2 &= \left\{[b^j, a]^{a^i} : 1 \leq j \leq n-1, 0 \leq i \leq n-2\right\} & \#A_2 &= (n-1)^2 \\ A_3 &= \{a^n[a^{-1}, b^j] : 0 \leq j \leq n-1\} & \#A_3 &= n \end{aligned}$$

We finally arrive at n^2+1 generators as predicted by Schreier index formula since $\#A_1 + \#A_2 + \#A_3 = n + (n-1)^2 + n = n^2 + 1$.

Lemma 36. *For any two elements of a group and any positive integer j we have*

(1) $[x^j, y] = [x,y]^{x^{j-1}} \cdot [x,y]^{x^{j-2}} \cdots [x,y]^x \cdot [x,y]$
(2) $[x, y^j] = [x,y] \cdot [x,y]^y \cdot [x,y]^{y^{j-1}}$.

Proof. See [13, 0.1 p.1]. ∎

Fix $0 \leq i \leq n-2$. We will prove that the $\mathbb{Z}$-module generated by the elements $\Sigma_1 := \{[b^j, a]^{\alpha^i}, 1 \leq j \leq n-1\}$ is the same with the $\mathbb{Z}$-module generated by the elements $\Sigma_2 := \{[b,a]^{\alpha^i \beta^j} 1 \leq j \leq n-2\}$. Indeed by lemma 36 (1) we have the identities (written aditively):

$$[b,a]^{\alpha^i} = [b,a]^{\alpha^i}$$

$$\begin{aligned}[b^2,a]^{\alpha^i} &= [b,a]^{\beta\alpha^i} + [b,a]^{\alpha^i}\\ \cdots &= \cdots \\ [b^{n-1},a] &= [b,a]^{\beta^{n-2}\alpha^i} + \cdots + [b,a]^{\beta\alpha^i} + [b,a]^{\alpha^i},\end{aligned}$$

i.e. for $1 \le j \le n-1$ and $0 \le i \le n-2$ we have

$$[b^j,a]^{\alpha^i} = [b,a]^{(\beta^{j-1}+\beta^{j-2}+\beta+1)a^i}$$

This proves that the elements of the set Σ_1 are transofrmed to the elements of the set Σ_2 in terms of the invertible $n \times n$ matrix with entries in $\mathbb{Z}$:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix}$$

Therefore Σ_1 and Σ_2 generate the same $\mathbb{Z}$-module.

Notice also that

$$\begin{aligned}(a^n)^{\beta^j} &= b^j a^{n-1} b^{-j} a^{-n+1} \cdot a^{n-1} b^j a b^{-j} = [b^j, a^{n-1}] + \underbrace{a^{n-1}b^j a b^{-j}}_{\in A_3} \\ &= [b^j,a]^{\alpha^{n-2}+\alpha^{n-3}+\cdots\alpha+1} + a^{n-1}b^j a b^{-j}.\end{aligned}$$

The above computation shows that $(a^n)^{\beta^j}$ can be written as a $\mathbb{Z}$-linear combination of elements of Σ_2 (which generate A_2) and A_3. Moreover

$$a^n[a^{-1},b^j] = (a^n)^{\beta^j} - [b,a]^{\left(\sum_{k=0}^{j-1}\beta^k\right)\left(\sum_{\lambda=0}^{n-2}\alpha^\lambda\right)}.$$

We have shown that

Lemma 37. *The free $\mathbb{Z}$-module R_F/R_F' can be generated by the n^2+1 elements*

$$(a^n)^{\beta^i}, (b^n)^{\alpha^i}, 0 \le i \le n-1 \text{ and } [a,b]^{\alpha^i\cdot\beta^j}, 0 \le i,j \le n-2.$$

We will now prove that in R_F/R_F' there are exacty $3n$ elements which are fixed by an element of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

The $2n$ elements are $(a^n)^{\beta^i}$ (resp. $(b^n)^{\alpha^i}$) which are fixed by $\langle\alpha\rangle$ (resp. $\langle\beta\rangle$).

The other n-elements are the elements $((ab)^n)^{\alpha^i}$ which are fixed by $\langle ab\rangle$. It is a simple computation to verify that we can write the elements $((ab)^n)^{\alpha^i}$ as follows:

$$(ab)^n = [b,a]^{\alpha^{n-1}\left(\sum_{\nu=0}^{n-2}\beta^\nu\right)+\cdots+\alpha^2(\beta+1)+\alpha}a^n b^n$$

Hence

$$((ab)^n)^{\alpha^i} = [b,a]^{\alpha^{n-1+i}\left(\sum_{\nu=0}^{n-2}\beta^\nu\right)+\cdots+\alpha^{2+i}(\beta+1)+\alpha^i} a^n\,(b^n)^{\alpha_i}.$$

We can now consider the homology group as the rank n^2+1 free $\mathbb{Z}$-module R_F/R_F'. Since R_F is a characteristic subgroup the group $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} = \langle a\rangle \times \langle b\rangle$ acts on R_F/R_F' by conjugation making R_F/R_F' a G-module.

So far we have computed the open Fermat curve addmitting a presentetation

$$R_F = \langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_{3n} | \gamma_1\gamma_2\cdots\gamma_{3n}\cdot[a_1,b_1][a_2,b_2]\cdots[a_g,b_g] = 1\rangle,$$

where g is the genous of the closed Fermat curve which equals to $(n-1)(n-2)/2$. Every ramification point of the Fermat curve is surrounded by a path γ_i and there

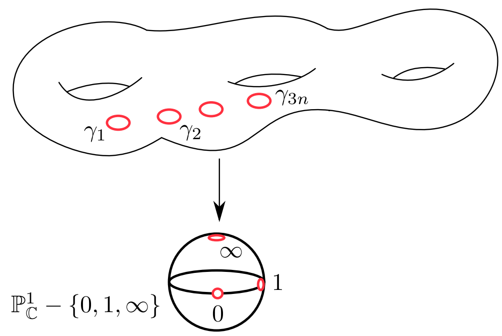

Figure 1. Open Fermat curve as cover of the projective line

are $3n$ such paths. For the computation of the genus of the closed curve, i.e. we added the $3n$ missing points, we observe

$$2g + 3n - 1 = n^2 + 1 \Rightarrow g = \frac{n^2 + 2 - 3n}{2} = \frac{(n-1)(n-2)}{2}.$$

We can see that the trasformation matrix from elements $[b^j, a]^{\alpha^i}$ to elements of the form $[b, a]^{\beta^j \alpha^i}$ is invertible. This allows us to prove that the elements in the sets A_2 and A_3 can be written as linear combinations of elements of the form $[b, a]^{\alpha^i \beta^j}$ and $(a^n)^{\beta^j}$ for $1 \leq j \leq n-1$, $0 \leq i \leq n-2$. It is clear that the elements $(a^n)^{\beta^i}, (b^n)^{\alpha^i}, ((ab)^n)^{\alpha^i}$ as given in the table below are fixed by the cyclic group mentioned in the third collumn. The elements γ_i are the n-elements $(b^n)^{\alpha^i}$ fixed by β, the n-elements $(a^n)^{\beta^i}$ fixed by α and the n invariant elements $((ab)^n)^{\alpha^i}$ in the module generated by commutators. In the following table we enumerate the fixed elements γ_i:

Invariant element γ_i	Index	Fixed by
$(a^n)^{\beta^i}$	$1 \leq i \leq n$	$\langle \alpha \rangle$
$(b^n)^{\alpha^i}$	$n+1 \leq i \leq 2n$	$\langle \beta \rangle$
$((ab)^n)^{\alpha^i}$	$2n+1 \leq i \leq 3n$	$\langle ab \rangle$

We have that

$$H_1(X, \mathbb{Z}) = \frac{R_F / R'_F}{\langle \gamma_1, \ldots, \gamma_{3n} \rangle}.$$

In order to compute the quotient we change the basis of R_F/R'_F by replacing each one of the elements $[b, a]^{\alpha^{n-1+i} \beta^{n-2}}$ by $((ab)^n)^{\alpha^i}$ for all $0 \leq i \leq n-1$. In this way we form the following basis:

Proposition 38. *A basis for the $\mathbb{Z}$-module $H_1(X, \mathbb{Z})$ consists of the set:*

$$\{[b, a]^{\alpha^i \beta^j} \mod \Gamma : 0 \leq i \leq n-2, 0 \leq j \leq n-3\},$$

where Γ is the free $\mathbb{Z}$-module generated by $\langle \gamma_1, \ldots, \gamma_{3n} \rangle$.

5.4.1. *Braid group action.* We will now consider the action of the Braid group B_3 on $H_1(X,\mathbb{Z})$ of the closed Fermat surface. We recall first that the braid group in three generators is generated by elements σ_1,σ_2 where

$$\sigma_1(a)=aba^{-1} \qquad \sigma_2(a)=a \qquad \sigma_1(b)=a \qquad \sigma_2(b)=a^{-1}b^{-1}$$

Notice that the above two automorphism in the abelianized free group with two generators acts like the matrices

$$\bar{\sigma}_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \bar{\sigma}_2=\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix},$$

in $\mathrm{GL}(2,\mathbb{Z})$, reflecting the fact that $B_3/Z(B_3)\cong \mathrm{PSL}(2,\mathbb{Z})$. Therefore,

$$\begin{aligned}\sigma_1[a,b]&=[aba^{-1},a] &&=[b,a]^{\alpha}=-[a,b]^{\alpha}\\ \sigma_2[a,b]&=[a,a^{-1}b^{-1}]&&=[b^{-1},a^{-1}].\end{aligned}$$

and more generally

$$\sigma_1([b,a]^{\alpha^i\beta^j})=[b,a]^{\alpha^{i+1}\beta^j} \qquad \sigma_2([b,a]^{\alpha^i\beta^j})=[b^{-1},a^{-1}]^{\alpha^i\beta^j}$$

We also compute

$$\begin{aligned}\sigma_1((b^n)^{\alpha^i})&=(a^n)^{\beta^i} &\qquad \sigma_1((a^n)^{\beta^j})&=(b^n)^{\alpha^{j+1}}\\ \sigma_2((b^n)^{\alpha^i})&=((ba)^n)^{-\alpha^i} &\qquad \sigma_2((a^n)^{\beta^j})&=(a^n)^{(\beta\alpha)^{-j}}.\end{aligned}$$

5.5. **The Generalized Fermat Curve.** A generalized Fermat curve of type $(k,s-1)$, where $k,s-1\geq 2$ are integers, is a non-singular irreducible projective algebraic curve $F_{k,s-1}$ defined over K admitting a group of automorphisms $H\cong(\mathbb{Z}/k\mathbb{Z})^{s-1}$ so that $F_{k,s-1}/H$ is the projective line with exactly s branch points, each one of order k. Such a group H is called a generalized Fermat group of type $(k,s-1)$. Let us consider a branched regular covering $\pi:F_{k,s-1}\to\mathbb{P}^1$, whose deck group is H. By composing by a suitable Möbius transformation (that is, an element of $\mathrm{PSL}_2(K)$) at the left of π, we may assume that the branch values of π are given by the points

$$\infty,0,1,\lambda_1,\ldots,\lambda_{s-3},$$

where $\lambda_i\in K-\{0,1\}$ are pairwise different. A generalized Fermat curve of type $(k,s-1)$ can be seen as a complete intersection in a projective space defined by the following set of equations

$$C^k(\lambda_1,\ldots,\lambda_{s-3}):=\left\{\begin{array}{rcl} x_0^k+x_1^k+x_2^k &=& 0\\ \lambda_1x_0^k+x_1^k+x_3^k &=& 0\\ \vdots & \vdots & \vdots\\ \lambda_{n-2}x_0^k+x_1^k+x_{s-1}^k &=& 0\end{array}\right\}\subset\mathbb{P}^{s-1}. \tag{37}$$

The genus of $F_{k,n}$ can be computed using the Riemann-Hurwitz formula:

$$g_{(k,s-1)}=1+\frac{k^{s-2}}{2}((s-2)(k-1)-2). \tag{38}$$

It is known [17] that generalized Fermat curves, have the orbifold uniformization $\mathbb{H}/\Gamma$ in terms of the Fuchsian group

$$\Gamma=\langle x_1,x_2,\ldots,x_s\mid x_1^k=\cdots=x_s^k=x_1x_2\cdots x_s=1\rangle. \tag{39}$$

The surface group is given [17], [32] as $F_{s-1} \cdot \langle x_1^k, \ldots, x_{s-1}^k, (x_1 \cdots x_{s-1})^k \rangle$. We will compute the genus of the generalized Fermat curves by two more different methods in eq. 42 and in section 5.6.3.

5.5.1. *Application of the Reidemeister-Schreier method.*

Consider the curve $X = \mathbb{P}^1 - \{0, 1, \infty, \lambda_1, \ldots, \lambda_{s-3}\}$ with fundamental group $F_{s-1} = \langle x_1, \ldots, x_{s-1} \rangle$ and univeral covering space $\tilde{X}$. We compute $H_1(X, \mathbb{Z}) = \mathbb{Z}^{s-1}$. We have the picture on the right. We will now employ the Reidemeister-Schreier method in order to compute the free subgroup $R_{k,s-1} \subset F_{s-1}$. A transversal set is given by

$$T = \left\{ x_1^{i_1} x_2^{i_2} \cdots x_{s-1}^{i_{s-1}}, 0 \leq i_j \leq k-1, 1 \leq j \leq s-1 \right\}.$$

$$\begin{tikzcd}
\tilde{X} \arrow[rr, bend left, "F'_{s-1}"] \arrow[dr, "R_{k,s-1}"] \arrow[dd, bend right, "F_{s-1}"'] & & Y \arrow[dl, "I"'] \arrow[ddll, bend left, "H_1(X,\mathbb{Z})"] \\
& F_{k,s-1} \arrow[dl, "H_{k,s-1}"'] & \\
X & &
\end{tikzcd}$$

For given $1 \leq \nu \leq s-1$ we have

$$\overline{x_1^{i_1} x_2^{i_2} \cdots x_{s-1}^{i_{s-1}} \cdot x_\nu} = \begin{cases} x_1^{i_1} x_2^{i_2} \cdots x_\nu^{i_\nu + 1} \cdots x_{s-1}^{i_{s-1}} & \text{if } i_\nu < k-1 \\ x_1^{i_1} x_2^{i_2} \cdots x_{\nu-1}^{i_{\nu-1}} x_{\nu+1}^{i_{\nu+1}} \cdots x_{s-1}^{i_{s-1}} & \text{if } i_\nu = k-1 \end{cases}$$

Denote by $\bar{x}^{\bar{i}} = x_1^{i_1} x_2^{i_2} \cdots x_{s-1}^{i_{s-1}}$. We now compute

Case I For $1 \leq \nu \leq s-1$:

$$\bar{x}^{\bar{i}} \cdot x_\nu \cdot \left(\overline{\bar{x}^{\bar{i}} \cdot x_\nu} \right)^{-1} = \begin{cases} \bar{x}^{\bar{i}} \cdot x_\nu \cdot x_1^{-i_1} x_2^{-i_2} \cdots x_\nu^{-i_\nu - 1} \cdots x_{s-1}^{-i_{s-1}} & \text{if } i_\nu < k-1 \\ \bar{x}^{\bar{i}} \cdot x_\nu \cdot x_1^{-i_1} x_2^{-i_2} \cdots x_{\nu-1}^{-i_{\nu-1}} x_{\nu+1}^{-i_{\nu+1}} \cdots x_{s-1}^{-i_{s-1}} & \text{if } i_\nu = k-1. \end{cases}$$

Case II For $\nu = s-1$:

$$\bar{x}^{\bar{i}} \cdot x_{s-1} \cdot \left(\overline{\bar{x}^{\bar{i}} \cdot x_{s-1}} \right)^{-1} =$$

$$= \begin{cases} 1 & \text{if } i_{s-1} < k-1 \\ \left(x_1^{i_1} x_2^{i_2} \cdots x_{s-2}^{i_{s-2}} \right) \cdot x_{s-1}^k \cdot \left(x_1^{i_1} x_2^{i_2} \cdots x_{s-2}^{i_{s-2}} \right)^{-1} & \text{if } i_{s-1} = k-1. \end{cases}$$

The generators of the free group $R_{k,s-1}$ are falling in the following categories:

$$\begin{aligned} A_{s-1} &= \left\{ (x_{s-1}^k)^{x_1^{i_1} \cdots x_{s-2}^{i_{s-2}}} \right\} \\ A_\nu &= \left\{ x_{1,\nu-1} \cdot x_\nu^{i_\nu} x_{\nu+1,s-1} \cdot x_\nu \cdot x_{\nu+1,s-1}^{-1} \cdot x_\nu^{-i_\nu - 1} \cdot x_{1,\nu-1}^{-1} \right\} \\ &= \left\{ [x_{\nu+1,s-1}, x_\nu]^{x_{1,\nu-1} \cdot x_\nu^{i_\nu}} \right\} \\ A'_\nu &= \left\{ x_1^{i_1} \cdots x_{\nu-1}^{i_{\nu-1}} \cdot x_\nu^{k-1} \cdot x_{\nu+1}^{i_{\nu+1}} \cdots x_{s-1}^{i_{s-1}} \cdot x_\nu \cdot x_{\nu+1,s-1}^{-1} \cdot x_{1,\nu-1}^{-1} \right\} \\ &= \left\{ \left(x_\nu^k [x_\nu^{-1}, x_{\nu+1,s-1}] \right)^{x_{1,\nu-1}} \right\}, \end{aligned} \tag{40}$$

where $x_{\ell_1,\ell_2} = x_{\ell_1}^{i_{\ell_1}} x_{\ell_1+1}^{i_{\ell_1+1}} \cdots x_{\ell_2}^{i_{\ell_2}}$. We now count the sizes of the above sets.

$$\begin{aligned} \# A_{s-1} &= k^{s-2} \\ \# A_\nu &= (k-1) \cdot k^{\nu-1} \cdot (k^{s-1-\nu} - 1), \text{ for } 1 \leq \nu \leq s-2 \\ \# A'_\nu &= k^{\nu-1} \cdot k^{s-1-\nu} = k^{s-2}, \text{ for } 1 \leq \nu \leq s-2 \end{aligned}$$

which gives in total

$$\#A_{s-1} + \sum_{\nu=1}^{s-2} \#A_\nu + \sum_{\nu=1}^{s-2} \#A'_\nu = (s-2)\cdot k^{s-1} + 1. \tag{41}$$

5.5.2. *Elements stabilized.* Observe first that the group generated by x_{s-1} stabilizes $(x_{s-1}^k)^{x_{1,s-2}}$. In this way we see that all k^{s-2} elements of A_{s-1} have non trivial stabilizer. Now we observe that

$$\begin{aligned} x_{1,\nu-1}\cdot x_\nu^{k-1}\cdot x_{\nu+1,s-1}\cdot x_\nu \cdot x_{\nu+1,s-1}^{-1}\cdot x_{1,\nu-1}^{-1} &= \\ = [x_\nu^{k-1}, x_{\nu+1,s-1}]^{x_{1,\nu-1}} \cdot (x_\nu^k)^{x_{1,\nu-1}\cdot x_{\nu+1,s-1}}. \end{aligned}$$

Observe that $\langle x_\nu\rangle$ stabilizes the k^{s-2} elements of A'_ν of the form $(x_\nu^k)^{x_{1,\nu-1}\cdot x_{\nu+1,s-1}}$ and the element $\langle x_1\cdots x_{s-1}\rangle$ stabilizes all elements $((x_1\cdots x_{s-1})^k)^{x_{1,s-2}}$, which are k^{s-2}.

Invariant element γ_i	Cardinal	Fixed by
$(x_{s-1}^k)^{x_{1,s-2}}$	k^{s-2}	$\langle x_{s-1}\rangle$
$(x_\nu^k)^{x_{1,\nu-1}\cdot x_{\nu+1,s-1}}$	$(s-2)k^{s-2}$	$\langle x_\nu\rangle$, where $1\le \nu\le s-2$
$((x_1\cdots x_{s-1})^k)^{x_{1,s-2}}$	k^{s-2}	$\langle x_1\cdots x_{s-1}\rangle$

In total we have sk^{s-2} fixed elements γ_i. Because of the above relations and the following computation

$$\begin{aligned} &[x_{\nu+1,s-1}, x_\nu]^{x_{1,\nu-1}\cdot x_\nu^{i_\nu}} = \\ &\qquad ([x_{\nu+1}^{i_{\nu+1}}, x_\nu]^{x_\nu{}^{i_\nu}} + [x_{\nu+2}^{i_{\nu+2}}, x_\nu]^{x_\nu{}^{i_\nu}\cdot x_{\nu+1}{}^{i_{\nu+1}}} + \ldots + [x_{s-1}^{i_{s-1}}, x_\nu]^{x_{\nu,s-2}})^{x_{1,\nu-1}}. \end{aligned}$$

We can also take a new basis insteed of the old one. This is the following:

$$\begin{aligned} A_{s-1} &= \{(x_{s-1}^k)^{x_{1,s-2}}\} \\ A_\nu &= \{(x_\nu^k)^{x_{1,\nu-1}\cdot x_{\nu+1,s-1}}\}, \text{ for } 1\le\nu\le s-2 \\ A'_\nu &= \{[x_j, x_\nu]^{x_{1,\nu-1}\cdot x_\nu^{i_\nu}\cdot x_{\nu+1,s-1}}\}, \text{ for } 1\le\nu\le s-2. \end{aligned}$$

Remark 39. For the homology of the closed curve we have:

$$H_1(F_{k,s-1},\mathbb{Z}) = \frac{R_{k,s-1}/R'_{k,s-1}}{\langle\gamma_1,\ldots,\gamma_{sk^{s-1}}\rangle}.$$

Using eq. (38) and the fact that $\mathrm{rank} H_1(F_{k,s-1},\mathbb{Z}) = 2g_{F_{k,s-1}}$ it is easy to verify that

$$(s-2)k^{s-1} + 1 - (s\cdot k^{s-2} - 1) = 2g_{F_{k,s-1}}. \tag{42}$$

In the above formula we have subtracted one from the number of invariant elements γ_i since $\gamma_1\cdots\gamma_{sk^{s-2}} = 1$.

5.6. The group $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ as an $\mathscr{A}$-module.

5.6.1. *Alexander modules for generalised Fermat curves.* It is clear that the group $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ is generated as an $\mathscr{A}$-module by the elements $[x_i, x_j]$ for $1\le i<j\le s-1$.

The structure of $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ as an $\mathscr{A}$-module is expressed in terms of the Crowell exact sequence, see section 4.1, related to the short exact sequence:

$$1 \to \mathfrak{F}'_{s-1} \to \mathfrak{F}_{s-1} \xrightarrow{\psi} \mathfrak{F}^{\mathrm{ab}}_{s-1} \to 1,$$

$$0 \to (\mathfrak{F}'_{s-1})^{\mathrm{ab}} = \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} \to A_\psi \to \mathbb{Z}_\ell[[u_1,\ldots,u_{s-1}]] \to \mathbb{Z}_\ell \to 0,$$

where $A_\psi = A_\psi^{\mathfrak{F}'_{s-1},\{1\}}$ is the Alexander module and

$$\mathscr{A} = \mathscr{A}^{\mathfrak{F}_{s-1},\{1\}} = \mathbb{Z}_\ell[[u_1,\ldots,u_{s-1}]].$$

Remark 40. The submodule of a free module is not necessarily a free module and $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ is not necessarily free. It contains the free module (see [37],[27, Th. 5.39])

$$\Lambda_{s-1} = (\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1})^{\mathrm{prim}} := \left\{ (\lambda_j u_1 \cdots \hat{u}_j \cdots u_{s-1})_{j=1,\ldots,s-1} : \lambda_j \in \mathscr{A}, \sum_{j=1}^{s-1} \lambda_j = 0 \right\}.$$

Set $w = u_1 \cdots u_{s-1}$ a basis of Λ_{s-1} is given by

$$v_1 = \left(-\frac{w}{u_1}, \frac{w}{u_2}, 0, \ldots, 0 \right)^t, \ldots, v_{s-2} = \left(0, \ldots, 0, -\frac{w}{u_{s-2}}, \frac{w}{u_{s-2}} \right)^t.$$

In the case of Fermat curves, i.e. $s = 2$ we have that $\Lambda_1 = (\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1})^{\mathrm{prim}} = \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ and $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ is a free $\mathscr{A}$-module.

The injective map $d : \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} \xrightarrow{d} \mathscr{A}^{s-1}$ is given by sending a representative

$$\begin{aligned} [x_i, x_j] \to d([x_i, x_j]) &= (1 - x_i)dx_j - (1 - x_j)dx_i \\ &= u_i \cdot dx_j - u_j \cdot dx_i. \end{aligned}$$

Set $w_{ij} = w/(u_i u_j)$. The elements

$$w_{ij} \cdot d([x_i, x_j]) = w/u_j \cdot dx_j - w/u_i \cdot dx_i \in \Lambda_{s-1}.$$

This means that the image of the commutators $d([x_i, x_j])$ there is some contribution to the torsion part of $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$.

5.6.2. *Application to Generalized Fermat curves.* Consider the the smallest closed normal subgroup $\mathfrak{R}_k$ of $\mathfrak{F}_{s-1}$ containing all $x_i^{\ell^k}$ for $1 \le i \le s-1$. Define also

$$\mathfrak{F}_{s-1,k} = \mathfrak{F}_{s-1}/\mathfrak{R}_k.$$

Set $\bar{\lambda} = \{0, 1, \infty, \lambda_1, \ldots, \lambda_{s-3}\}$ and let $\mathscr{M}$ be the maximum pro-ℓ extension of $K = \bar{k}(t)$ unramified outside the set of points $\bar{\lambda}$. Consider the function field of the generalized Fermat curves

$$K_k := K\left(t^{\frac{1}{\ell^k}}, (t-1)^{1/\ell^k}, (t-\lambda_1)^{1/\ell^k}, \ldots, (t-\lambda_{s-3})^{1/\ell^k} \right).$$

Let K_k^{ur} and K_k^{urab} be the maximal unramified and maximal abelian unramified extensions of K_k respectively. Also let K' be the maximum unramified extension of K and K'' be the maximum unramified extension of K'. By covering space theory the fields K', K'' correspond to the groups $\mathfrak{F}'_{s-1}$ and $\mathfrak{F}''_{s-1}$, respectively. The function field K_k corresponds to the group $\mathfrak{F}'_{s-1}\mathfrak{R}_k$ and is equal to the function field of the generalized Fermat curve.

We have

$$K' = \bigcup_k K_k, \qquad\qquad K' \cap K_k^{\mathrm{ur}} = K_k$$

$$K'' = \bigcup_k K_k^{\mathrm{urab}}, \qquad\qquad K'' \cap K_k^{\mathrm{ur}} = K_k^{\mathrm{urab}}$$

The Galois correspondence is given as follows:

$$\begin{tikzcd}
 & \mathscr{M} \arrow[dl, dash] \arrow[dr, dash] & \\
K'' \arrow[d, dash] \arrow[drr, dash] & & K_k^{\mathrm{ur}} \arrow[d, dash] \\
K' \arrow[dr, dash] & & K_k^{\mathrm{urab}} \arrow[dl, dash] \\
 & K_k \arrow[d, dash] & \\
 & K &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{1\} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathfrak{F}''_{s-1} \arrow[d, dash] \arrow[drr, dash] & & \mathfrak{R}_k \arrow[d, dash] \\
\mathfrak{F}'_{s-1} \arrow[dr, dash] & & \mathfrak{F}''_{s-1}\mathfrak{R}_k \arrow[dl, dash] \\
 & \mathfrak{F}'_{s-1}\mathfrak{R}_k \arrow[d, dash] & \\
 & \mathfrak{F}_{s-1} &
\end{tikzcd}$$

Using standard isomorphism theorems in group theory and the definitions we see

$$\mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} \cong \mathfrak{F}'_{s-1}/\left(\mathfrak{F}'_{s-1} \cap \mathfrak{F}''_{s-1}\mathfrak{R}_k\right) \cong \mathfrak{F}'_{s-1}\mathfrak{R}_k/\mathfrak{F}''_{s-1}\mathfrak{R}_k \cong \mathrm{Gal}(K_k^{\mathrm{urab}}/K_k)$$

is an abelian group, a free $\mathbb{Z}_\ell$-module of rank $2g$, where g is the genus of the generalized Fermat curve, $F_{\ell^k,s-1}$ so that

$$2g_{(\ell^k,s-1)} = 2 + \ell^{k(s-2)}((s-2)(\ell^k-1)-2). \tag{43}$$

Observe that according to eq. (39) we have

$$\mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} \cong H_1(F_{k,s-1},\mathbb{Z}_\ell).$$

The last genus computation also follow from the following proposition which identifies unramified $\mathbb{Z}/\ell^k\mathbb{Z}$ extensions of a curve X with the group of ℓ^k-torsion points of the Jacobian $J(X)$.

Proposition 41. *Let Y be a complete nonsingular algebraic curve defined over a field of characteristic prime to ℓ. The étale Galois covers of Y with Galois group $\mathbb{Z}/\ell^k\mathbb{Z}$ are classified by the étale cohomology group $H^1_{\mathrm{et}}(Y,\mathbb{Z}/\ell^k\mathbb{Z})$ which is equal to the group of ℓ^k-torsion points of* $\mathrm{Pic}(Y)$.

Proof. See [19, Ex. 2.7], [40, sec. 19]. □

5.6.3. *Crowell sequence for generalized Fermat curves.* Here we use the presentation $\mathfrak{F}_{s-1} = \mathfrak{F}_s/\langle x_1\cdots x_s\rangle$. Let $H_k = (\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}$. We have the short exact sequence

$$1 \to \mathfrak{F}'_{s-1,k} = (\mathfrak{F}_{s-1}/\mathfrak{R}_k)' \to \mathfrak{F}_{s-1}/\mathfrak{R}_k \xrightarrow{\psi} H_k \to 1$$

We will use the Crowell Exact sequence [35, chap. 9]

$$0 \to (\mathfrak{F}'_{s-1,k})^{\mathrm{ab}} = \mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} \xrightarrow{\theta_1} \mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k} \xrightarrow{\theta_2} \mathscr{A}^{\mathfrak{F}_{s-1},\mathfrak{R}_k} \xrightarrow{\varepsilon_{\mathscr{A}_k}} \mathbb{Z}_\ell \to 0, \tag{44}$$

where

$$\mathscr{A}^{\mathfrak{F}_{s-1},\mathfrak{R}_k} = \mathbb{Z}_\ell[[H_k]] = \mathbb{Z}_\ell[[(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}]],$$

and

$$\mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k} = \mathrm{coker}Q, \qquad \mathbb{Z}_\ell[H_k]^{s+1} \xrightarrow{Q} \mathbb{Z}_\ell[H_k]^s \tag{45}$$

The Alexander module for $\mathfrak{F}_{s-1}/\mathfrak{R}_k$ was computed on example 16. Notice that $\mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$ and the Crowell sequence know the genus of the generalised Fermat curve, see eq. (25).

5.6.4. *Representation theory on Generalized Fermat Curves.* These are representations on the free $\mathbb{Z}_\ell$-modules

$$\rho_k : G \to \mathrm{GL}(H_1(F_{k,s-1}, \mathbb{Z}_\ell))$$

Where G is either the absolute Galois group or the braid group B_{s-1} or B_s.

Let us now consider eq. (27) for $\Gamma = \mathfrak{R}_k$:

$$\begin{tikzcd} 0 \arrow[r] & (\mathfrak{F}'_{s-1})^{\mathrm{ab}} = \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} \arrow[r] \arrow[d] & \mathscr{A}^{s-1} \arrow[r, "d_1"] \arrow[d] & \mathscr{A} \arrow[r] \arrow[d, "\omega"] & \mathbb{Z}_\ell \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (\mathfrak{F}'_{s-1,k})^{\mathrm{ab}} = \mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} \arrow[r, "\theta"] & \left(\mathscr{A}^{\mathfrak{F}_{s-1},\mathfrak{R}_k}\right)^{s-1} \arrow[r, "d_{1,k}"] & \mathscr{A}^{\mathfrak{F}_{s-1},\mathfrak{R}_k} \arrow[r] & \mathbb{Z}_\ell \arrow[r] & 0 \end{tikzcd} \tag{46}$$

It is clear that $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ can be considered through the vertical map ω as an $\mathscr{A}$-module.

In order to understand $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ we will use its θ embedding in $\left(\mathscr{A}^{\mathfrak{F}_{s-1},\mathfrak{R}_k}\right)^{s-1}$. The map θ sends $[x_i, x_j]$ to $d[x_i, x_j] = (1 - x_i)dx_j - (1 - x_j)dx_i$. Observe that the elements of the form

$$\sum_{\nu=0}^{n} \bar{x}_i^{\nu}, \qquad 1 \leq i \leq s$$

annihilate $[x_i, x_j]$. We can see this by direct computations or by observing that

$$\Sigma_i \cdot \beta_i = \beta_{s+1} x_1 \cdots x_{i-1}.$$

and the image $\theta[x_i, x_j]$ has $\beta_{s+1} = 0$. The above observation generalises the definition of ideal $\mathfrak{a}_n$ in eq. (8) in the article of Ihara, [23].

5.6.5. *On Jacobian variety of Generalized Fermat curves.* Consider the ℓ-adic Tate module $T(\mathrm{Jac}(F_{k,s-1}))$ of the Jacobian of the generalized Fermat curves $F_{k,s-1}$:

$$T(\mathrm{Jac}(F_{k,s-1})) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathrm{Jac}(F_{k,s-1}))(\bar{\mathbb{Q}}) = H_1(F_{k,s-1}, \mathbb{Z}) \otimes \mathbb{Z}_\ell = \frac{\mathfrak{F}'_{s-1,k}}{\mathfrak{F}''_{s-1,k}}.$$

Following Ihara we consider

$$\mathbb{T} := \varprojlim_k T(\mathrm{Jac}(F_{k,s-1})) = \varprojlim_k \frac{\mathfrak{F}'_{s-1,k}}{\mathfrak{F}''_{s-1,k}},$$

where the inverse limit is considered with respect to the maps $T(\mathrm{Jac}(F_{k+1,s-1})) \to T(\mathrm{Jac}(F_{k,s-1}))$, which is induced by the map

$$(x_0, \ldots, x_{s-1}) \mapsto (x_0^\ell, \ldots, x_{s-1}^\ell).$$

Let $\bar{F}_{k,s-1} = F_{k,s-1} \otimes_{\mathrm{Spec}\mathbb{Q}} \mathrm{Spec}\bar{\mathbb{Q}}$. Consider also the inverse limit

$$\varprojlim_k \mathrm{Gal}(\bar{F}_{k,s-1}/\mathbb{P}^1_{\bar{\mathbb{Q}}}) = \varprojlim_k (\mathbb{Z}/\ell^k\mathbb{Z})^{s-1} = \mathbb{Z}_\ell^{s-1}.$$

Therefore

$$\varprojlim_k \mathbb{Z}_\ell[\mathrm{Gal}(\bar{F}_{k/s}/\mathbb{P}^1_{\bar{\mathbb{Q}}})] \cong \mathscr{A}$$

and $\mathbb{T}$ can be considered as an $\mathscr{A}$-module. Using eq. (46) we obtain

$$\frac{\mathfrak{F}'_{s-1}}{\mathfrak{F}''_{s-1}} \cong \mathbb{T}.$$

See [2, sec. 13] for the explicit isomorphism in the case of Fermat curves.

This construction leads to the definition of a subspace $\mathbb{T}^{\text{prim}} \subset \mathbb{T}$ which is a free $\mathscr{A}$-module of rank $s-1$.

An element in $g \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ induces an action on both $\mathbb{T}$ and $\mathbb{T}^{\text{prim}}$. In particular the subspace $\mathbb{T}^{\text{prim}}$ is a free $\mathscr{A}$-module. Thus we have a representation

$$\begin{aligned} \rho : G &\to \mathrm{GL}_{s-1}(\mathscr{A}) \\ \sigma &\longmapsto (a_{ij}(\sigma)) \end{aligned}$$

This matrix representation is defined by writing

$$\sigma(w_{ij} d[x_i, x_j]) = \sum_{\nu<\mu} a_{\nu,\mu}(\sigma) w_{\nu\mu} d[x_\nu, x_\mu].$$

5.6.6. *From generalized Fermat curves to cyclic covers of* $\mathbb{P}^1$. We will now relate the Crowell sequences for the generalized Fermat curves and cyclic covers $\bar{Y}_{\ell^k}$ of the projective line as where defined in section 5 using the results of section 4.1.2. Set

$$\bar{R}_{\ell^k} = R_{\ell^k}/\Gamma = \left\langle (x_2 x_1^{-1})^{x_1^\nu}, \ldots, (x_{s-1} x_1^{-1})^{x_1^\nu} : 0 \leq \nu < \ell^k \right\rangle$$

and recall that

$$R = \left\langle \left(x_j x_1^{-1}\right)^{x_1^\nu} : \nu \in \mathbb{Z} \right\rangle$$

The fixed field of R/Γ is the function field K_{ℓ^k} of the curve $\bar{Y}_{\ell^k}$, $K(C_s)$ is the function field of the curve C_s. The group R' corresponds to the maximal unramified abelian extension $K(C_s)^{\text{ur}}$ of $K(C_s)$ while Γ_{ℓ^k} corresponds to the maximal unramified extension $K(C_s)^{\text{unrab}}$. The group $R' \cdot \Gamma_{\ell^k}$ corresponds to the maximal abelian unramified $K_{\ell^k}^{\text{unrab}}$ extension of K_{ℓ^k}. The groups $F'_{s-1} \cdot \Gamma_{\ell^k}$ and $F''_{s-1} \cdot \Gamma_{\ell^k}$ correspond to the generalized Fermat curve $F_{k,s-1}$ and the maximal unramified extension $F_{k,s-1}^{\text{unrab}}$. The groups F'_{s-1}, F''_{s-1} correspond to the maximal abelian unramified extension of K_0 and the maximal abelian unramified extension of K' respectively.

$$\begin{tikzcd}
 & \mathscr{M} \arrow[dl, dash] \arrow[dr, dash] & & \\
K'' \arrow[d, dash] \arrow[dr, dash] & & K(C_s)^{\text{ab}} \arrow[dl, dash] \arrow[d, dash] & K(C_s)^{\text{unrab}} \arrow[dl, dash] \arrow[d, dash] \\
K' \arrow[dr, dash] & F_{k,s-1}^{\text{unrab}} \arrow[r] \arrow[d, dash] & K_{\ell^k}^{\text{unrab}} \arrow[d, dash, "H_1(C_{\ell^k},\mathbb{Z})"] & K(C_s) \arrow[dl, dash] \\
 & F_{k,s-1} \arrow[r] \arrow[dr, dash] & K_{\ell^k} \arrow[d, dash] & \\
 & & K_0 = k(t) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{1\} \arrow[dl, dash] \arrow[dr, dash] & & \\
F''_{s-1} \arrow[d, dash] \arrow[dr, dash] & & \Gamma_{\ell^k} \arrow[dl, dash] \arrow[d, dash] & R' \arrow[dl, dash] \arrow[d, dash] \\
F'_{s-1} \arrow[dr, dash] & F''_{s-1} \cdot \Gamma_{\ell^k} \arrow[r, hook] \arrow[d, dash] & R' \cdot \Gamma_{\ell^k} \arrow[d, dash, "H_1(C_{\ell^k},\mathbb{Z})"] & R \arrow[dl, dash, bend left] \\
 & F'_{s-1} \cdot \Gamma_{\ell^k} \arrow[r, hook] \arrow[dr, dash] & R \cdot \Gamma_{\ell^k} \arrow[d, dash] & \\
 & & F_{s-1} &
\end{tikzcd}$$

As in the case of generalized Fermat curves we can form the Tate module

$$\mathbb{T}_A := \lim_{\substack{\leftarrow \\ k}} (\mathrm{Jac}(Y_{\ell^k})) = \lim_{\substack{\leftarrow \\ k}} (\bar{R}_{\ell^k}/\mathfrak{R}_k)^{\mathrm{ab}}$$

We now compare the Crowell sequences for the cyclic covers and the Fermat covers:

$$\begin{tikzcd}
 & & 0 & 0 & & \\
0 \arrow[r] & R^{\mathrm{ab}} = \mathbb{T}_R \arrow[r] & \mathscr{A}_{\psi}^{R,\mathfrak{R}_k} \arrow[r] \arrow[u] & \mathscr{A}^{R,\mathfrak{R}_k} = \mathbb{Z}_\ell[[\mathbb{Z}/\ell^k\mathbb{Z}]] \arrow[r] \arrow[u] & \mathbb{Z}_\ell \arrow[r] & 0 \\
0 \arrow[r] & (\mathfrak{F}'_{s-1})^{\mathrm{ab}} = \mathbb{T}\phi_1 \arrow[r] \arrow[u] & \mathscr{A}_{\psi}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} \arrow[r] \arrow[u, "\phi_2"] & \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} = \mathbb{Z}_\ell[[(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}]] \arrow[r] \arrow[u, "\phi_3"] & \mathbb{Z}_\ell \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The map $\mathbb{T} \to \mathbb{T}_R$ on Tate modules is given by the first vertical map. The action module structure is given by the commutating diagram

$$\begin{tikzcd}
\mathscr{A} \times \mathbb{T} \arrow[r] \arrow[d] & \mathbb{T} \arrow[d] \\
\mathscr{A}^{R,\mathfrak{R}_k} \times \mathbb{T}_R \arrow[r] & \mathbb{T}_R
\end{tikzcd}$$

where the horizontal maps are the module actions and the first vertical map sends $(a, t) \mapsto (\phi_3(a), \phi_1(t))$. The map ϕ_3 is the reduction identifying the variables $x_1, x_2, \ldots, x_{s-1}$. In particular from the reduction $\mathbb{T} \to \mathbb{T}_R$ we obtain the diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[dr] & \mathrm{GL}_{s-1}(\mathscr{A}) \arrow[d] \\
 & \mathrm{GL}_{s-1}(\mathscr{A}^{R,\mathfrak{R}_k})
\end{tikzcd}$$

References

[1] Greg W. Anderson. Torsion points on Fermat Jacobians, roots of circular units and relative singular homology. *Duke Math. J.*, 54(2):501–561, 1987.

[2] Greg W. Anderson. The hyperadelic gamma function. *Invent. Math.*, 95(1):63–131, 1989.

[3] Greg W. Anderson. Torsion points on Jacobians of quotients of Fermat curves and p-adic soliton theory. *Invent. Math.*, 118(3):475–492, 1994.

[4] Jannis A. Antoniadis and Aristides Kontogeorgis. On cyclic covers of the projective line. *Manuscripta Math.*, 121(1):105–130, 2006.

[5] G. V. Belyĭ. Galois extensions of a maximal cyclotomic field. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(2):267–276, 479, 1979.

[6] Joan S. Birman. *Braids, links, and mapping class groups*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 82.

[7] Carl-Friedrich Bödigheimer and Ulrike Tillmann. Embeddings of braid groups into mapping class groups and their homology. In *Configuration Spaces*, CRM Series, pages 173–191. Scuola Normale Superiore, 2012.

[8] Oleg Bogopolski. *Introduction to group theory*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. Translated, revised and expanded from the 2002 Russian original.

[9] Mariela Carvacho, Rubén A. Hidalgo, and Saúl Quispe. Jacobian variety of generalized Fermat curves. *Q. J. Math.*, 67(2):261–284, 2016.

[10] Weiyan Chen. Homology of braid groups, the Burau representation, and points on superelliptic curves over finite fields. *Israel J. Math.*, 220(2):739–762, 2017.
[11] R. H. Crowell. ;l. *Advances in Math.*, 6:210–238 (1971), 1971.
[12] Rachel Davis, Rachel Pries, Vesna Stojanoska, and Kirsten Wickelgren. Galois action on the homology of Fermat curves. In *Directions in number theory*, volume 3 of *Assoc. Women Math. Ser.*, pages 57–86. Springer, [Cham], 2016.
[13] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. *Analytic pro-p groups*, volume 61 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1999.
[14] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.
[15] Hershel M. Farkas and Irwin Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1980.
[16] Kazuhiro Fujiwara and Kato Fumiharu. *Foundations of Rigid Geometry I*. European Mathematical Society, 2018.
[17] Gabino González-Diez, Rubén A. Hidalgo, and Maximiliano Leyton. Generalized Fermat curves. *J. Algebra*, 321(6):1643–1660, 15 March 2009.
[18] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.
[19] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
[20] Ruben A Hidalgo. Holomorphic differentials of generalized fermat curves. *arXiv [math.AG]*, October 2017.
[21] Rubén A. Hidalgo, Aristides Kontogeorgis, Maximiliano Leyton-Álvarez, and Panagiotis Paramantzoglou. Automorphisms of generalized Fermat curves. *J. Pure Appl. Algebra*, 221(9):2312–2337, 2017.
[22] Yasutaka Ihara. Some remarks on the number of rational points of algebraic curves over finite fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 28(3):721–724 (1982), 1981.
[23] Yasutaka Ihara. Profinite braid groups, galois representations and complex multiplications. *Ann. Math.*, 121(2):351–376, March 1985.
[24] Yasutaka Ihara. Arithmetic analogues of braid groups and Galois representations. In *Braids (Santa Cruz, CA, 1986)*, volume 78 of *Contemp. Math.*, pages 245–257. Amer. Math. Soc., Providence, RI, 1988.
[25] MM Kapranov and A Smirnov. Cohomology determinants and reciprocity laws. 1995.
[26] M. Kneser. *Lectures on Galois cohomology of classical groups*. Tata Institute of Fundamental Research, Bombay, 1969. With an appendix by T. A. Springer, Notes by P. Jothilingam, Tata Institute of Fundamental Research Lectures on Mathematics, No. 47.
[27] Hisatoshi Kodani, Masanori Morishita, and Yuji Terashima. Arithmetic Topology in Ihara Theory. *Publ. Res. Inst. Math. Sci.*, 53(4):629–688, 2017.
[28] Aristides Kontogeorgis. The group of automorphisms of cyclic extensions of rational function fields. *J. Algebra*, 216(2):665–706, 1999.
[29] Aristides Kontogeorgis. Field of moduli versus field of definition for cyclic covers of the projective line. *J. Théor. Nombres Bordeaux*, 21(3):679–692, 2009.
[30] Serge Lang. *Introduction to algebraic and abelian functions*, volume 89 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second edition, 1982.
[31] Michel Lazard. Sur les groupes nilpotents et les anneaux de Lie. *Ann. Sci. Ecole Norm. Sup. (3)*, 71:101–190, 1954.
[32] C. Maclachlan. Abelian groups of automorphisms of compact Riemann surfaces. *Proc. London Math. Soc. (3)*, 15:699–712, 1965.
[33] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory*. Dover Publications, Inc., Mineola, NY, second edition, 2004. Presentations of groups in terms of generators and relations.
[34] Curtis T. McMullen. Braid groups and Hodge theory. *Math. Ann.*, 355(3):893–946, 2013.
[35] M Morishita. *Knots and Primes: An Introduction to Arithmetic Topology*. SpringerLink : Bücher. Springer-Verlag London Limited, 2011.
[36] Jürgen Neukirch. *Class field theory*. Springer, Heidelberg, 2013. The Bonn lectures, edited and with a foreword by Alexander Schmidt, Translated from the 1967 German original by F. Lemmermeyer and W. Snyder, Language editor: A. Rosenschon.

[37] T Oda. Note on meta-abelian quotients of pro-l free groups. *preprint*, 1985.

[38] V. V. Prasolov and A. B. Sossinsky. *Knots, links, braids and 3-manifolds*, volume 154 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1997. An introduction to the new invariants in low-dimensional topology, Translated from the Russian manuscript by Sossinsky [Sosinskiĭ].

[39] Luis Ribes and Pavel Zalesskii. *Profinite groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2010.

[40] Jean-Pierre Serre. Sur la topologie des variétés algébriques en caractéristique p. In *Symposium internacional de topología algebraica International symposium on algebraic topology*, pages 24–53. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.

[41] Pavlos Tzermias. The group of automorphisms of the Fermat curve. *J. Number Theory*, 53(1):173–178, 1995.

[42] A. B. Sossinsky V. V. Prasolov. *Knots, Links, Braids and 3-Manifolds: An Introduction to the New Invariants in Low-Dimensional Topology (Translations of Mathematical Monographs)*. Translations of Mathematical Monographs. Amer Mathematical Society, 1996.

Department of Mathematics, National and Kapodistrian University of Athens Panepistimioupolis, 15784 Athens, Greece
E-mail address: `kontogar@math.uoa.gr`

Department of Mathematics, National and Kapodistrian University of Athens Panepistimioupolis, 15784 Athens, Greece
E-mail address: `pan_par@math.uoa.gr`