DEFORMATION OF CURVES WITH AUTOMORPHISMS AND REPRESENTATIONS ON RIEMANN-ROCH SPACES.

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Abstract. We study the deformation theory of nonsigular projective curves defined over algebraic closed fields of positive characteristic. We show that under some assumptions the local deformation problem for automorphisms of powerseries can be reduced to a deformation problem for matrix representations. We study both equicharacteristic and mixed deformations in the case of two dimensional representations.

1. Introduction

Let X be a nonsingular projective curve of genus $g \geq 2$ defined over an algebraically closed field of characteristic $p > 0$. The automorphism group $G := \mathrm{Aut}(X)$ is known to be a finite group. The appearance of wild ramification in the cover $X \rightarrow X/\mathrm{Aut}(X)$ makes the theory of such covers more difficult than the corresponding theory in characteristic zero. For a point $P \in X$ the decomposition group $G(P) = \{\sigma \in G : \sigma(P) = P\}$ is known to be cyclic in characteristic zero and a non-abelian solvable group admitting a ramification filtration [35]. In [23] the author defined a faithful representation of the p-part of the decomposition group at a wild ramified point P:

$$\rho : G_1(P) \rightarrow GL(L(mP)), \tag{1}$$

where $L(mP) = \{f \in k(X) : \mathrm{div}(f) + mP \geq 0\} \cup \{0\}$. In this paper we would like to study the relation of two deformation theories, namely the deformation theory of representations of finite groups and the deformation theory of curves with automorphisms.

We will treat both mixed characteristic and equicharacteristic deformations. For the mixed characteristic case we consider Λ to be a complete Noetherian local ring with residue field k. Usually Λ is an algebraic extension of the ring of Witt vector $W(k)$. For the equicharacteristic case we take $\Lambda = k$.

Let $\mathcal{C}$ denote the category of local Artin Λ-algebras, which residue field k. Consider a subgroup G of the group $\mathrm{Aut}(X)$. A deformation of the couple (X, G) over the local Artin ring A is a proper, smooth family of curves

$$\mathcal{X} \rightarrow \mathrm{Spec}(A)$$

parametrized by the base scheme $\mathrm{Spec}(A)$, together with a group homomorphism $G \rightarrow \mathrm{Aut}_A(\mathcal{X})$, such that there is a G-equivariant isomorphism ϕ from the fibre over the closed point of A to the original curve X:

$$\phi : \mathcal{X} \otimes_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \rightarrow X.$$

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Two deformations $\mathcal{X}_1, \mathcal{X}_2$ are considered to be equivalent if there is a G-equivariant isomorphism ψ that reduces to the identity in the special fibre and making the following diagram commutative:

$$\begin{tikzcd} \mathcal{X}_1 \arrow[rr, "\psi"] \arrow[dr] & & \mathcal{X}_2 \arrow[dl] \\ & \mathrm{Spec} A & \end{tikzcd}$$

The global deformation functor is defined:

$$D_{\mathrm{gl}} : \mathcal{C} \rightarrow \text{Sets, A} \mapsto \left\{ \begin{array}{l} \text{Equivalence classes} \\ \text{of deformations of} \\ \text{couples } (X,G) \text{ over } A \end{array} \right\}$$

By the local-global theorems of J.Bertin and A. Mézard [2] and the formal patching theorems of D. Harbater, K. Stevenson [15], [16], the study of the functor D_{gl} can be reduced to the study of the following deformation functors attached to each wild ramification point P of the cover $X \rightarrow X/G$:

$$D_P : \mathcal{C} \rightarrow \text{Sets}, A \mapsto \left\{ \begin{array}{l} \text{lifts } G(P) \rightarrow \mathrm{Aut}(A[[t]]) \text{ of } \rho \text{ mod-} \\ \text{ulo conjugation with an element} \\ \text{of } \ker(\mathrm{Aut} A[[t]] \rightarrow k[[t]]) \end{array} \right\} \tag{2}$$

The theory of automorphisms of formal powerseries rings is not as well understood as is the theory of automorphisms of finite dimensional vector spaces, i.e the theory of general linear groups.

For a k-algebra A with maximal ideal m_A, consider the multiplicative group $L_n(A) < GL_n(A)$, of invertible lower triangular matrices with entries in A, and invertible elements λ in the diagonal, such that $\lambda - 1 \in m_A$. We consider the following functor from the category $\mathcal{C}$ of local Artin k-algebras to the category of sets

$$F : A \in Ob(\mathcal{C}) \mapsto \left\{ \begin{array}{l} \text{liftings of } \rho : G(P) \rightarrow L_n(k) \\ \text{to } \rho_A : G(P) \rightarrow L_n(A) \text{ modulo} \\ \text{conjugation by an element} \\ \text{of } \ker(L_n(A) \rightarrow L_n(k)) \end{array} \right\} \tag{3}$$

It is known that among the curves X with automorphism group $G = \mathrm{Aut}(X)$ divisible by the characteristic, the curves so that $G_2(P) = \{1\}$ for all ramified points are the most simple. We will call these curves *weakly ramified*. Many intractable problems for the theory of curves with general automorphism group are solved for weakly ramified curves. For example the computation of the G-module structure of spaces of holomorphic differentials [22] or the computation of the deformation rings of curves with automorphisms [4]. In our representation perspective it seems that the simplest curves are those with two dimensional representations at all wild ramified points. Notice that if a two dimensional representation is attached at the wild point P, then the group $G_1(P)$ is elementary abelian and has conductor $m > 1$ [23, example 3.].

In section 2 we show how to attach a deformation of a matrix representation to every deformation of the couple (X,G) over a complete local domain. Section 3 is devoted to the deformations of matrix representations. We focus on the two dimensional case and we construct a hull for these deformations. The deformation theory of such representations is closely related to deformations of products of $\mathbb{G}_a$ group schemes in the equal characteristic case or to $\mathcal{G}^{(\lambda)}$ group schemes in mixed characteristic. In section 4 we try to analyze further the relation between the functors $F(\cdot)$ and $D(\cdot)$. A matrix representation allows us to express a deformation $\tilde{\rho}_\sigma$ given as a formal series $\tilde{\rho}_\sigma(t) \in A[[t]]$ in the form of a root od rational function of t. For the case of two dimensional representations, where $V = G_1(P)$ is an elementary abelian group, we are able to compute the image of elements in $F(\cdot)$ in the tangent space $D(k[\epsilon]/\epsilon^2) = H^1(V, \mathcal{T}_\mathcal{O})$, see proposition 4.2. By combining these results to the computation of $H^1(V, \mathcal{T}_\mathcal{O})$ given by the author in [24, prop. 2.8] we are able to compute the Krull dimension of the hull's attached to every wild ramified point.

Finally, in section 5 we restrict ourselves to to the equicharacteristic case and we relate two dimensional matrix deformations to the deformation functor of R. Pries.

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2. Branch locus and liftings of matrix representations.

In this section we will show how the problem of deforming the representations attached at the wild ramified points give information on the problem of deformations of curves with automorphisms.

Select a wild ramified point P_i on every orbit of wild ramified points under the action of the group G. Define the functor $D_{loc} = \prod D_{P_i}$. J. Bertin and A. Mézard proved that there is a smooth morphism $\phi : D_{gl} \rightarrow D_{loc}$, and this morphism induces the following relation on the global deformation ring R_{gl} and of the deformation rings R_i of the deformation functors D_{P_i}.

$$R_{gl} = (R_1 \hat{\otimes} R_2 \hat{\otimes} \cdots \hat{\otimes} R_r)[[U_1, \ldots, U_N]],$$

where $N = \dim_k H^1(X/G, \pi_*^G(\mathcal{T}_X))$, and R_i is the deformation ring of D_{P_i}. For more information concerning this construction we refer to [2]. For an exact formula for N we refer to [24, sec. 3].

In the approach of Schlessinger [31] one wants to build deformations of (X, G) over Artin algebras, especially over the algebras $k[\epsilon]/\epsilon^n$, and study whether a deformation over $\mathrm{Spec} k[\epsilon]/\epsilon^n$ can be lifted to deformation over $\mathrm{Spec} k[\epsilon]/\epsilon^{n+1}$. More generaly a small extension A' of A is given by the the short exact sequence of local Artin algebras

$$0 \rightarrow \ker \pi \rightarrow A' \rightarrow A \rightarrow 0$$

such that $\ker \pi \cdot m_{A'} = 0$, where m_A is the maximal ideal of A' respectively. We would like to know if a deformation in $D(A)$ can be lifted to a deformation in $D(A')$. The obstructions of such liftings are elements in $H^2(G, \mathcal{T}_\mathcal{O})$. If there are no obstructions then we can construct a family over the formal scheme $\mathcal{X} \rightarrow \mathrm{Spf} R$ for some complete domain R. The scheme $\mathrm{Spf} R$ is a formal scheme and does not posses a generic fibre. J. Bertin and A. Mézard in [2] observed that an algebraization theorem of Grothendieck [12] gives that the formal scheme representing D_{gl} is algebraizable, and it corresponds to the formal completion of a proper smooth curve over Spec$R$. This means that every unobstructed deformation over a formal affine scheme can be extended to the generic fibre.

Assume that $\mathcal{X} \to \mathrm{Spec}R$ is a relative curve that is a solution to our deformation problem, where R is a complete local domain. Let $\sigma \in G_1(P)$, $\sigma \neq 1$, and let $\tilde{\sigma}$ be a lift of σ in $\mathcal{X}$. The scheme $\mathcal{X}$ is regular at P, and the completion of $\mathcal{O}_{\mathcal{X},P}$ is isomorphic to the ring $R[[T]]$. Weierstrass preparation theorem [3, prop. VII.6] implies that:

$$\tilde{\sigma}(T) - T = g_{\tilde{\sigma}}(T)u_{\tilde{\sigma}}(T),$$

where $g_{\tilde{\sigma}}(T)$ is a distinguished Weierstrass polynomial of degree $m+1$ and $u_{\tilde{\sigma}}(T)$ is a unit in $R[[T]]$.

The polynomial $g_{\tilde{\sigma}}(T)$ gives rise to a horizontal divisor that corresponds to the fixed points of $\tilde{\sigma}$. This horizontal divisor might not be reducible. The branch divisor corresponds to the union of the fixed points of any $\sigma \in G_1(P)$. Next lemma shows how to define a horizontal branch divisor for the relative curves $\mathcal{X} \to \mathcal{X}^G$ when G is not a cyclic group.

Lemma 2.1. *Let $\mathcal{X} \to \mathrm{Spec}A$ be an A-curve, admitting a fibrewise action of the finite group G, where A is a Noetherian local ring. Let $S = \mathrm{Spec}A$, and $\Omega_{\mathcal{X}/S}$, $\Omega_{\mathcal{Y}/S}$ be the sheaves of relative differentials of $\mathcal{X}$ over S and $\mathcal{Y}$ over S, respectively. Let $\pi : \mathcal{X} \to \mathcal{Y}$ be the quotient map. The sheaf*

$$\mathcal{L}(-D_{\mathcal{X}/\mathcal{Y}}) = \Omega^{-1}_{\mathcal{X}/S} \otimes_S \pi^*\Omega_{\mathcal{Y}/S}.$$

is the ideal sheaf the horizontal Cartier divisor $D_{\mathcal{X}/\mathcal{Y}}$. The intersection of $D_{\mathcal{X}/\mathcal{Y}}$ with the special and generic fibre of $\mathcal{X}$ gives the ordinary branch divisors for curves.

Proof. We will first prove that the above defined divisor $D_{\mathcal{X}/\mathcal{Y}}$ is indeed an effective Cartier divisor. According to [21, Cor. 1.1.5.2] it is enough to prove that

- $D_{\mathcal{X}/\mathcal{Y}}$ is a closed subscheme which is flat over S.
- for all geometric points $\mathrm{Spec}k \to S$ of S, the closed subscheme $D_{\mathcal{X}/\mathcal{Y}} \otimes_S k$ of $\mathcal{X} \otimes_S k$ is a Cartier divisor in $\mathcal{X} \otimes_S k/k$.

We are interested in deformations of nonsingular curves. Since the base is a local ring and the special fibre is nonsingular, the deformation $\mathcal{X} \to \mathrm{Spec}A$ is smooth. (See the remark after the definition 3.35 p.142 in [25]). The smoothness of the curves $\mathcal{X} \to S$, and $\mathcal{Y} \to S$, implies that the sheaves $\Omega_{\mathcal{X}/S}$ and $\Omega_{\mathcal{X}/S}$ are S-flat, [25, cor. 2.6 p.222].

On the other hand the sheaf $\Omega_{\mathcal{Y},\mathrm{Spec}A}$ is by [21, Prop. 1.1.5.1] $\O_{\mathcal{Y}}$-flat. Thus, $\pi^*(\Omega_{\mathcal{Y},\mathrm{Spec}A})$ is $\O_{\mathcal{X}}$-flat and therefore Spec$A$-flat [17, Prop. 9.2]. Finally, observe that the intersection with the special and generic fibre is the ordinary branch divisor for curves according to [17, IV p.301]. $\square$

Remark: Two horizontal branch divisors can collapse to the same point in the special fibre. For instance, this always happens if a deformation of curves from positive characteristic to characteristic zero with a wild ramification point is possible.

For a curve X and a branch point P of X we will denote by $i_{G,P}$ the order function of the filtration of G at P. The Artin representation of the group G is defined by $\mathrm{ar}_P(\sigma) = -f_P i_{G,P}(\sigma)$ for $\sigma \neq 1$ and $\mathrm{ar}_P(1) = f_P \sum_{\sigma\neq 1} i_{G,P}(\sigma)$ [35, VI.2]. We are going to use the Artin representation at both the special and generic fibre. In the special fibre we always have $f_P = 1$ since the field k is algebraically closed. The field of quotients of A should not be algebraically closed therefore a fixed point there might have $f_P \geq 1$. The integer $i_{G,P}(\sigma)$ is equal to the multiplicity of $P \times P$ in the intersection of $\Delta.\Gamma_\sigma$ in the relative A-surface $\mathcal{X} \times_{\mathrm{Spec}A} \mathcal{X}$, where Δ is the diagonal and Γ_σ is the graph of σ [35, p. 105].

Since the diagonals Δ_0, Δ_η and the graphs of σ in the special and generic fibres respectively of $\mathcal{X} \times_{\mathrm{Spec}A} \mathcal{X}$ are algebraically equivalent divisors we have:

Proposition 2.2. *Assume that A is an integral domain, and let $\mathcal{X} \rightarrow \mathrm{Spec}A$ be a deformation of X. Let $\bar{P}_i$, $i = 1, \cdots, s$ be the horizontal branch divisors that intersect at the special fibre, at point P, and let P_i be the corresponding points on the generic fibre. For the Artin representations attached to the points P, P_i we have:*

$$\mathrm{ar}_P(\sigma) = \sum_{i=1}^{s} \mathrm{ar}_{P_i}(\sigma).$$

This generalizes a result of J. Bertin [1]. Moreover if we set $\sigma = 1$ to the above formula we obtain a relation for the valuations of the differents in the special and the generic fibre, since the value of the Artin's representation at 1 is the valuation of the different [35, prop. 4.IV,prop. 4.VI]. This observetion is equivalent to claim 3.2 in [10] and is one direction of a local criterion for good reduction theorem proved in [10, 3.4], [20, sec. 5].

Corollary 2.3. *Assume that $V = G_1(P)$ is an elementary abelian group with more than one $\mathbb{Z}/p\mathbb{Z}$ components. If V can be lifted to characteristic zero, then $\frac{|V|}{p} \mid m+1$.*

Proof. The group V acts on the generic fibre, where the possible stabilizers of points are cyclic groups. Since V is not cyclic it can not fix any point P_i in the intersection of the branch locus with the generic fibre. Only a cyclic component of V can fix a point P_i. Since V act on the set of points P_i, each orbit has $|V|/p$ elements. For any element $\sigma \in V$ the Artin representation $\mathrm{ar}_{P_i}(\sigma) = 1$ (no wild ramification at the generic fibre). Therefore proposition 2.2 gives us that the number of $\{P_i\}$ is $m + 1$ and the desired result follows. □

Remark: Consider the case of equicharacteristic deformations of ordinary curves, together with a p-subgroup of the group of automorphisms. Then $|\mathrm{ar}_P(\sigma)| = 2$ for all $\sigma \in G(P) = G_1(P), \sigma \neq 1$ [28]. On the other hand the ramification at the points of the generic fibre is also wild and 2.2 implies that there is only one horizontal branch divisor extending every wild ramification point P.

Remark: The author finds amusing the following similarity to the theory of dynamical systems: It is known that autonomous (ordinary) differential equations on a manifold M induce an action of $\mathbb{R}$ on M. The fixed locus of this action, called *equilibrium locus* in the realm of differential equations, can split as the integrated vector fields depend on parameters. The study of this splitting is the object of *bifurcation theory* [13]. Notice also that $\mathbb{R}$ is not compact and the representation theory of $\mathbb{R}$ shares many difficulties with the corresponding representation theory of groups of order divided by the characteristic, because of the absence of a Haar measure on them.

Proposition 2.4. *Let R be a complete local regular integer domain. Let $\mathcal{X} \rightarrow \mathrm{Spec}R$ be a deformation of the couple (X, G), and let P be a wild ramified point of the special fibre X. Assume that there is a a 2-dimensional representation $\rho : G_1(P) \rightarrow \mathrm{GL}_k(H^0(X, \mathcal{L}(mP)))$ attached to P. Assume also that there is a G-invariant horizontal divisor that intersects the special fibre with multiplicity m.*

Then, there is a free R-module M of rank 2 generated by $1, \tilde{f}$ so that $M := \langle 1, \tilde{f}\rangle_R \subset H^0((\mathcal{X}, \mathcal{L}(\alpha D)))$, where $1 \leq \alpha \in \mathbb{N}$ and $M \otimes_R k = H^0(X, \mathcal{L}(mP))$. Moreover, the representation ρ can be lifted to a representation

$$\tilde{\rho} : G_1(P) \to \mathrm{GL}_R(\langle 1, \tilde{f}\rangle_R).$$

The elements $\tilde{\rho}_\sigma$ are lower triangular matrices.

Moreover the basis element $\tilde{f}$ is of the form

$$\tilde{f} = \frac{1}{(T^m + a_{m-1}T^{m-1} + \cdots + a_1T_1 + a_0)}u(T), \tag{4}$$

where $a_0, \ldots, a_{m-1} \in m_R$ and $u(T)$ is a unit in $R[[T]]$ reducing to $1 \bmod m_R$.

Proof. Let us consider the sheaf $\mathcal{L}(D)$. The space of global sections $H^0(\mathcal{X}, \mathcal{L}(D))$ has the structure of an R-module. For an arbitrary Cartier divisor D on $\mathcal{X}$ and for all $i \geq 0$ there is a natural map [17, prop. III 12.5]

$$\phi_i : H^i(\mathcal{X}, \mathcal{L}(D)) \otimes_R k \to H^i(X_s, \mathcal{L}(D \otimes k)).$$

We are interested in global sections *i.e.*, for the zero cohomology groups, but in general ϕ_0 can fail to be an isomorphism.

Instead of looking at D we will consider $a'D$, where a' is a sufficiently large natural number. We will employ the Riemann-Roch theorem in both the special and the generic fibre and we can choose a sufficiently big so that the index of speciality at both the generic and the special fibre is zero. P. Deligne - D. Mumford observed [6, 4. 78], [11, chap.3 sec.7] that since

$$H^1(\mathcal{X}_s, \mathcal{L}(a'D \otimes k)) = H^1(\mathcal{X}_\eta, \mathcal{L}(a'D \otimes K)) = 0$$

the R-module $H^0(\mathcal{X}, \mathcal{L}(a'D))$ is free. We can then select an element $\tilde{f} \in H^0(\mathcal{X}, \mathcal{L}(a'D))$ so that $\tilde{f} \equiv f \bmod m_R$. Consider the least a such that $\langle 1, \tilde{f}\rangle_R \subseteq H^0(\mathcal{X}, \mathcal{L}(aD))$ for some $1 \leq a \leq a'$. Since D is $G_1(P)$-invariant the R-module $H^0(\mathcal{X}, \mathcal{L}(aD))$ is equipped with a $G_1(P)$-action. The module M might not be the whole $H^0(\mathcal{X}, \mathcal{L}(aD))$ but it is the R-free part of it. Therefore $G_1(P)$ acts on M as well and the representation can be lifted:

$$\tilde{\rho} : G_1(P) \to \mathrm{GL}_R(M),$$

as required. Since $\sigma \mid_R = \mathrm{Id}_A$ this representation is given by lower triangular matrices.

The element $1/\tilde{f}$ is a holomorphic element in $R[[T]]$ reducing to $1/f = t^m$ modulo m_R. Thus, the reduced order of $1/\tilde{f}$ is m and eq. (4) follows by Weierstrass preparation theorem [3, prop. VII.6]. □

We will now try to give conditions for the existence of a $G_1(P)$-invariant divisor intersecting the special fibre at P with degree $m+1$. Let $T = \{\bar{P}_i\}_{i=1,\ldots,s}$ be the set of horizontal branch divisors that restricts to P in the special fibre of X. This space is acted on by $G_1(P)$, since $\bar{P}_i$ are all components of the branch divisor. Each of the $\bar{P}_i$ is fixed by some element of G but not necessarily by the whole group $G_1(P)$, unless of course $G_1(P)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Let $O(T)$ be the set of orbits of T under the action of the group $G_1(P)$, on T. A horizontal divisor D supported on T, is invariant under the action of $G_1(P)$ if and only if, the divisor D is of the form:

$$D = \sum_{C \in O(T)} n_C \sum_{P \in C} P, \tag{5}$$

i.e., horizontal Cartier divisors that are in the same orbit of the action of $G_1(P)$ must appear with the same weight in D. If the semigroup $\sum_{C\in O(T)} n_C \# C$, $n_C \in \mathbb{N}$, contains the Weierstrass semigroup of the branch point P of the special fibre, then we can select the desired $G_1(P)$-invariant divisor D supported on T.

If one orbit of $G_1(P)$ acting on T is a singleton, i.e., there is a $\bar{P}_i$ fixed by the whole group $G_1(P)$, then the semigroup

$$\sum_{C\in O(T)} n_C \# C, \quad n_C \in \mathbb{N},$$

is the semigroup of natural numbers, and we are done. This is the case when the group $G_1(P)$ is cyclic.

If $\#T \not\equiv 0 \mathrm{mod} p$ then there is at least one orbit that is a singleton. Indeed, if all orbits have more than one element then all orbits must have cardinality divisible by p, and since the set T is the disjoint union of orbits it must also have cardinality divisible by p.

Lemma 2.5. *If m is the first pole number that is not divisible by the characteristic, and $p \nmid m+1$ then there is an orbit that consists of only one element.*

Proof. By proposition 2.2 the Artin representation at the special fibre equals the sum of the Artin representations at the generic fibre. Let $\sigma \in G_1(P)$. The Artin representation of σ at the special fibre equals $m+1$. All $\bar{P}_i$ that are not fixed by σ do not contribute in the sum of the Artin representations at the generic fibre.

An element τ sends $\bar{P}_i$ which is fixed by $H \subset G_1(P)$ to $\tau\bar{P}_i$ which is fixed by $\tau H \tau^{-1}$. Since the representation attached to P is two dimensional the group $G_1(P)$ is abelian, and $\tau\bar{P}_i$ is fixed by $H = \tau H \tau^{-1}$.

If we now consider P_i that is fixed by $\langle \sigma \rangle$ then the above argument shows that the orbit of P_i under the action of the group $G_1(P)$ has p^a elements $0 \leq a$. If $a = 0$ then P_i is fixed by the whole group $G_1(P)$. If on the other hand for all P_i fixed by σ the coresponding orbit orders have more than one element then the set of $\bar{P}_i$ fixed by σ has order divisible by p. This implies that the sum of the Artin representations at the generic fibre is divisible by p, a contradiction. $\square$

We have thus obtained the following easy to apply

Corollary 2.6. *If $G_1(P)$ is cyclic or $p \nmid m+1$, then there is a horizontal branch divisor D, fixed under the action of $G_1(P)$, that intersects the special fibre at mP. In particular, the assumption of proposition 2.4 is satisfied and the two dimensional representation can be lifted.*

Lemma 2.7. *In the mixed characteristic case, if the elementary abelian group $G_1(P)$ has more than two cyclic components, then there is no horizontal $G_1(P)$-invariant divisor D contained in the branch locus and intersecting the special fibre at P with multiplicity m.*

Proof. Since the stabilizers of elements in the generic fibre are cyclic groups of order p, all orbits of elements are divisible by p. Therefore, a $G_1(P)$-invariant divisor should have degree divisible by p. This, can not happen since $(m,p) = 1$. $\square$

Remark 2.8. Lemma 2.7 shows that our method can not be used for lifting curves with elementary abelian action to characteristic zero. However, M. Matignon proved that such liftings exist [26].

We have seen how to relate a deformation of the couple (X, G) to a deformation of a matrix representation. Now we will see the effect of considering equivalent deformations of couples.

Lemma 2.9. *Let ϕ be a map $\mathcal{O}_{\mathcal{X},P} \to \mathcal{O}_{\mathcal{X},P}$ making the extensions $\tilde{\rho}_\sigma, \tilde{\rho}'_\sigma$ equivalent. The corresponding matrix representations are conjugate by a 2×2 matrix of the form $\begin{pmatrix} 1 & 0 \\ \mu & \lambda \end{pmatrix}$ where $\lambda \equiv 1 \bmod m_A$ and $\mu \equiv 0 \bmod m_A$.*

Conversely, every such matrix gives rise to a map $\phi : \mathcal{O}_{\mathcal{X},P} \to \mathcal{O}_{\mathcal{X},P}$ that reduces to the identity modulo m_A.

Proof. Assume that there is a map $\phi : \mathcal{O}_{\mathcal{X},P} \to \mathcal{O}_{\mathcal{X},P}$ making the extensions $\tilde{\rho}_\sigma, \tilde{\rho}'_\sigma$ equivalent. The local-global principle of J.Bertin-A.Mézard implies that this map can be extended to a map $\phi' : \mathcal{X} \to \mathcal{X}$ that makes the corresponding global deformations equivalent. Let $\tilde{f}$ be the generator given in proposition 2.4. Then $\phi'(\tilde{f}) \in H^0(\mathcal{X}, \mathcal{L}(aD))$, therefore $\phi'(\tilde{f}) = \lambda \tilde{f} + \mu$. This means that ϕ gives rise to a base change in $H^0(\mathcal{X}, \mathcal{L}(a'D))$, and two elements in $F(\cdot)$ are equivalent if they are conjugate by a 2×2 matrix of the desired form.

Conversely, assume that we have two equivalent matrix representations that are conjugate by a matrix Q of the form $\begin{pmatrix} 1 & 0 \\ \mu & \lambda \end{pmatrix}$ where $\lambda \equiv 1 \bmod m_A$ and $\mu \equiv 0 \bmod m_A$. Then Q sends $\tilde{f} \mapsto \lambda \tilde{f} + \mu$, i.e.

$$\frac{1}{\phi(T)^m + \sum_{\nu=0}^{m-1} a_\nu \phi(T)^n u} = \lambda \tilde{f}(T) + \mu.$$

A solution $\phi(T)$ of this polynomial equation exists by using Hensel's lemma. This solution gives rise to the desired map ϕ. $\square$

3. Deformations of Linear groups

We would like to represent the functor F defined in Eq. (3). We will employ the construction for universal deformation rings for matrix representations, explained by B. de Smit and H. W. Lenstra in [5]. Let H be a p-group with identity e and let $\rho : H \to L_n(k)$ be a faithful representation of H. Let $\Lambda[H, n]$ be the commutative Λ-algebra generated by X_{ij}^g for $g \in H, 1 \le j \le i \le n$, such that

$$X_{ij}^e = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$X_{ij}^{gh} = \sum_{l=1}^{n} X_{il}^g X_{lj}^h \text{ for } g, h \in H \text{ and } 1 \le i, j \le n. \tag{6}$$

and

$$X_{ij}^g = 0 \text{ for } i < j \text{ and for all } g \in H.$$

We will focus on representations on $L_n(A)$. For every Λ-algebra A we have a canonical bijection

$$\mathrm{Hom}_{\Lambda-\mathrm{Alg}}(\Lambda[H, n], A) \cong \mathrm{Hom}(H, L_n(A)),$$

where a Λ-algebra homomorphism $f : \Lambda[H, n] \to A$ corresponds to the group homomorphism ρ_f that sends $g \in H$ to the matrix $(f(X_{ij}^g))$. The representation

$\rho : H \to L_n(k)$ corresponds to a homomorphism $\Lambda[H, n] \to k$. Its kernel is a maximal ideal, which we denote by m_ρ. We take the completion $R(H)$ of $\Lambda[H, n]$ at m_ρ. The canonical map $\Lambda[H, n] \to R(H)$, gives rise to a map $\rho_{R(H)} : H \to L_n(R(H))$, such that the diagram:

$$\begin{tikzcd} H \arrow[r, "\rho_{R(H)}"] \arrow[d, "="'] & L_n(R(H)) \arrow[d] \\ H \arrow[r, "\rho"] & L_n(k) \end{tikzcd}$$

is commutative.

We have to distinguish two cases:

• The case of equicharacteristic deformations, i.e., R is a complete local domain so that $\mathrm{Quot}(R)$ is of characteristic p. Recall that in this case $\Lambda = k$. Since the generic fibre is of characteristic p we have $X_{22}^g = 1$ for all $1 \leq i \leq n$. Moreover, if we fix elements g_i generating H as an $\mathbb{Z}/p\mathbb{Z}$-vector space and monomials $x_i = X_{21}^{g_i} - c(g_i)$ for each g_i we easily see that $R(H) = k[[x_1, \ldots, x_n]]$.

• The case of liftings to characteristic zero, i.e. R is a complete local domain so that $\mathrm{Quot}(R)$ of characteristic 0. Let us again fix elements x_i, y_i for each generator g_i of H, so that $x_i = X_{21}^{g_i} - c(g_i)$, and $y_i = X_{22}^{g_i} - 1$.

In this case we have the conditions:

$$(X_{22}^g)^p = 1, \tag{7}$$

$$X_{21}^g \sum_{\nu=0}^{p-1} (X_{22}^g)^\nu = 0, \tag{8}$$

and the commuting relation: $(X_{21}^g - X_{21}^h + X_{22}^g X_{21}^h - X_{22}^h X_{21}^g) = 0$. Observe that $X_{22}^g \neq 1$. Indeed, if $X_{22}^g = 1$ then eq. (8) will give us that $X_{21}^g = 0$ and then the matrix is just the identity. Therefore, equations (7) and (8) reduce to the single equation $\sum_{\nu=0}^{p-1} (X_{22}^g)^\nu = 0$.

These conditions imply that:

$$R(H) = \Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_n]]/I,$$

where I is the ideal

$$I := \left\langle \sum_{\nu=0}^{p} (1 + y_i)^{\nu-1}, y_j(c(g_i) + x_i) - y_i(c(g_j) + x_j) \right\rangle.$$

The ring $R(H)$ defined above does not represent the deformation functor F, since A-equivalent deformations may correspond to different maps in $\mathrm{Hom}(R(H), A)$. If $n = 2$, i.e., in the case of a two dimensional representation, the conjugation action given by lemma 2.9 is easy to handle.

Considering the quotient of $R(H)$ in positive characteristic, for representations of dimension ≥ 3 is a difficult problem since the "trace" argument of characteristic zero does not work. (Characters do not distinguish equivalent representations in modular representation theory).

We focus now on the theory of two dimensional representations. This forces the group H to be elementary abelian. We compute that

$$\begin{pmatrix} 1 & 0 \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mu & \lambda \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \mu + \lambda x - y\mu & y \end{pmatrix}. \tag{9}$$

We will consider the effect of the conjugation action given in eq. (9). The elements y_i remain invariant while the elements $x_i \mapsto x_i + \lambda_a c(g_i) + \lambda_a x_i - \mu y_i$, where $\lambda = 1+\lambda_a$, $\lambda_a, \mu \in m_A$. If $A = k[\epsilon]/\epsilon^2$, then $x_i = x_i + c(g_i)\lambda_a$ since $\lambda_a x_i \in m_A^2 = 0$.

Let A be an object in $\mathcal{C}$. An element in the set $F(A)$ is determined by the conjugation equivalence class of a function $f : R(H) \to A$. Such a function should be defined on the generators x_i, y_j of the ring $R(H)$. Since $f(x_i)$ is equivalent to $f(x_i) + f(\lambda_a)c(g_i) \bmod m_A^2$, if there is a ring representing the functor $F(\cdot)$ then this should be a subring of $R(H)$ and $f(x_i) = 0$ for all generators x_i, as one sees by considering $\lambda_a = -x_i/c(g_i)$. Therefore the ring representing $F(\cdot)$ is the subring of $R(H)$ generated by $y_1, \ldots, y_n$ in the mixed characteristic case and is the zero ring in the equicharacteristic case. This is in accordance to remark 4.7.

According to remark 2.8 the case $n = 1$ is the only case we can handle using our approach in the mixed characteristic situation.

Remark 3.1. We consider the subring R of $R(H) = R(\mathbb{Z}/p\mathbb{Z})$ generated by y. The ring R is singular. Indeed, by the infinitesimal lifting property [17, II. exer. 8.6], [18, sec. 1.4] it is enough to provide a small extension $A' \to A \to 0$ and a homomorphism $h \in \mathrm{Hom}(R, A)$ that does not lift to a homomorphism to $\mathrm{Hom}(R, A')$. Let m_Λ be the maximal ideal of Λ. Consider the natural map $\pi : R \to R/m_\Lambda R = k[[y]]/\langle y^{p-1}\rangle =: A$. Consider also the ring A' given by $k[[y]]/\langle y^p\rangle$. Then $A' \to A$ is a small extension and there is no map $R \to A'$ lifting π. Indeed, every such homomorphism $R \to A'$ should factor through $\bmod m_\Lambda$. Therefore we obtain a nontrivial homomorphism $A \to A'$, a contradiction.

Remark: In [32] T. Sekiguchi, F. Oort, N. Suwa introduced the group schemes $\mathcal{G}^{(\lambda)}$ in order to deform the additive group schemes $\mathbb{G}_a$ to the multiplicative group schemes $\mathbb{G}_m$ and they were able to give a unified Artin-Schreier-Kummer theory [33],[34]. Many articles devoted to the deformations of automorphism groups from positive to zero characteristic are based on this theory, see for example [10].

Observe that if $H = \mathbb{Z}/p\mathbb{Z}$, i.e. we have an elementary abelian group with just one component, then the ring homomorphism

$$R(\mathbb{Z}/p\mathbb{Z}) \to A[[u, 1/(\epsilon u + 1)]]$$

sending y/ϵ to u gives rise to an injection of $\hat{\mathcal{G}}^{(\lambda)} \to \mathrm{Spec} R(\mathbb{Z}/p\mathbb{Z})$, where $\lambda = \epsilon$. Indeed, the diagonal elements $X_{22}^g = 1 + \epsilon y/\epsilon = 1 + \epsilon u$ are multiplied as elements in $\mathcal{G}^{(\epsilon)}$.

4. Relation to first order infinitesimal deformations

In this section we will relate the deformation functor of the two dimensional representations given in (3) to the deformation functor of actions in formal powerseries rings in (2). The advantage of this approach is that using the two dimensional representation we can contract the infinite powerseries representing the extended automorphism to a root of a rational function. Denote by V the elementary abelian group $G_1(P)$.

Assume that a two dimensional representation is attached on the wild ramification point P. By using the equation

$$\sigma\left(\frac{1}{t^m}\right) = \frac{1}{t^m} + c(\sigma),$$

we can define the following representation of V to automorphisms of formal powerseries rings:

$$\rho : V \to \mathrm{Aut}(k[[t]]),$$

$$\sigma \mapsto \rho_\sigma,$$

where

$$\rho_\sigma(t) = \frac{t}{(1+c(\sigma)t^m)^{1/m}} = t\left(1 + \sum_{\nu=1}^{\infty} \binom{-1/m}{\nu} c(\sigma)^\nu t^{\nu m}\right).$$

Let

$$0 \to \ker\pi \to A' \to A \to 0$$

be a small extension, i.e. $\ker\pi \cdot m_{A'} = 0$, where $m_{A'}, m_A$ are the maximal ideals of A, A' respectively. Assume that we have the following data: A deformation of the two dimensional representation given by $C(\sigma) = c(\sigma) + \delta(\sigma)$, $\lambda(\sigma) = 1 + \lambda_1(\sigma)$, where $\delta(\sigma), \lambda_1(\sigma) \in m_{A'}$ and the element $\tilde{f}$ given in proposition 2.4 extending f. Write $\tilde{f} = f + \Delta$, for some element $\Delta \in m_{A'}((t))$. Then we have:

$$\tilde{\rho}_\sigma\left(f + \Delta)\right) = \lambda(\sigma)(f + \Delta) + c(\sigma) + \delta(\sigma).$$

This implies that ($f = 1/t^m$):

$$\tilde{\rho}_\sigma\left(\frac{1}{t^m}\right) = \frac{\lambda(\sigma)}{t^m} + c(\sigma) + \big(\delta(\sigma) + \lambda(\sigma)\Delta - \tilde{\rho}_\sigma\Delta\big),$$

or equivalently:

$$\tilde{\rho}_\sigma(t) = \rho_\sigma(t) + t\left(\sum_{\nu=0}^{\infty} \binom{-1/m}{\nu} \sum_{k=1}^{\nu} \binom{\nu}{k} E^k c(\sigma)^{\nu-k} t^{m\nu}\right), \tag{10}$$

where

$$E = \delta(\sigma) + \lambda(\sigma)\Delta - \tilde{\rho}_\sigma\Delta + \frac{\lambda_1(\sigma)}{t^m} \in m_{A'}((t)).$$

Suppose that we can extend $\rho_\sigma(t)$ to a homomorphism $\tilde{\rho}_{\sigma,A} \in \mathrm{Aut}A[[t]]$. A further extension of ρ_σ over A' is then given by

$$\tilde{\rho}_{\sigma,A'}(t) = \tilde{\rho}_{\sigma,A}(t) + \rho'_\sigma(t),$$

where $\rho'_\sigma(t) \in \ker\pi[[t]]$. Since $\Delta \in m_{A'}((t))$ and since $\ker\pi \cdot m_{A'} = 0$

$$\tilde{\rho}_{\sigma,A'}(\Delta) = \tilde{\rho}_{\sigma,A}(\Delta).$$

Thus, equation (10) allows us to compute the value of $\tilde{\rho}_{\sigma,A'}(t)$ from the value of $\tilde{\rho}_{\sigma,A}(t)$.

Lemma 4.1. *Let $\tilde{\rho}_{\sigma,A} = \{\tilde{\rho}_{\sigma,A}(t)\}_{\sigma\in V}$ be a representation of $V \to \mathrm{Aut}A[[t]]$, and consider the corresponding element in $F(A)$. If this element in $F(A)$ can be lifted to an element in $F(A')$ then $\tilde{\rho}_{\sigma,A}$ can be lifted to a representation $V \to \mathrm{Aut}A'[[t]]$.*

Proof. According to [2, 3.2] every obstruction in lifting a representation in $D(A)$ to $D(A')$ is group theoretic. Consider extensions of the homomorphisms $\tilde{\rho}_{\sigma,A'} \in \mathrm{Aut}A'[[t]]$ for every $\sigma \in V$. The element $\tilde{\rho}_{\sigma,A'}\tilde{\rho}_{\tau,A'}\tilde{\rho}^{-1}_{\sigma\tau,A'}$ is a 2-cocycle and gives rise to a cohomology class in $H^2(V, \mathcal{T}_\mathcal{O})$.

In our case observe that if $\lambda_1(\sigma), \delta(\sigma)$ are functions $R(V) \to A'$ and therefore satisfy the 2×2 multiplication relations, then there is no group theoretic obstruction in lifting $\tilde{\rho}_{\sigma,A}$ to $\tilde{\rho}_{\sigma,A'}$ since a simple computation shows that the lifts defined by eq. (10) satisfy the relations

$$\tilde{\rho}_{\sigma,A'} \circ \tilde{\rho}_{\tau,A'} = \tilde{\rho}_{\sigma\tau,A'}.$$

Therefore any obstruction to lifting $\{\tilde{\rho}_\sigma\}$ reduces to the corresponding obstruction of lifting the matrix representation in $F(A)$ to $F(A')$. $\square$

Now we will focus on the small extension $k[\epsilon]/\epsilon^2 \to k$, and we will compute the image of matrix deformations in $H^1(V, \mathcal{T}_\mathcal{O})$. The general cocycle in $H^1(V, \mathcal{T}_\mathcal{O})$ is given by $d_1(t)\frac{d}{dt}$. In [24] the author proved that the map

$$\mathcal{T}_\mathcal{O} \to k[[t]]/t^{m+1} \tag{11}$$

$$f(t)\frac{d}{dt} \to f(t)/t^{m+1}$$

is a V-equivariant isomorphism.

Proposition 4.2. *Assume that P is a wild ramified point of X with a two dimensional representation attached to it. An extension $\tilde{\rho}_\sigma$ gives rise to the following cocyle in $H^1(V, \frac{1}{t^{m+1}}k[[t]])$:*

$$\alpha(\sigma) = \frac{1}{m}\left(\frac{\lambda_1(\sigma)}{t^m} + \lambda_1(\sigma)c(\sigma) - \delta(\sigma) + \sum_{\mu=0}^{m-1} \frac{2m-\mu}{m}\frac{a_{\mu,1}c(\sigma)}{t^{m-\mu}}\right),$$

modulo elements in $A[[t]]$.

Proof. We will compute the first order infinitesimal deformations of ρ_σ. We begin from

$$\tilde{\rho}_\sigma(f) = \lambda(\sigma)f + c(\sigma) + \delta(\sigma) + \lambda(\sigma)\Delta - \tilde{\rho}_\sigma\Delta.$$

Set $E_1 = \frac{\delta(\sigma)}{\lambda(\sigma)} + \Delta - \tilde{\rho}_\sigma\Delta\frac{1}{\lambda(\sigma)} - c(\sigma)\lambda_1(\sigma)$. Then

$$\tilde{\rho}_\sigma\left(\frac{1}{t^m}\right) = \lambda(\sigma)\frac{1 + t^m\frac{c(\sigma)}{\lambda(\sigma)} + t^m E'}{t^m}.$$

We compute

$$\begin{aligned}
\tilde{\rho}_\sigma(t) &= \frac{\lambda(\sigma)^{-\frac{1}{m}}t}{\left(1+t^m c(\sigma)\right)^{1/m}\left(1+\frac{E_1 t^m}{1+c(\sigma)t^m}\right)^{1/m}} \\
&= \frac{\lambda(\sigma)^{-\frac{1}{m}}\rho_\sigma(t)}{\left(1+\frac{E_1 t^m}{1+c(\sigma)t^m}\right)^{1/m}} \\
&= \lambda(\sigma)^{-\frac{1}{m}}(\rho_\sigma(t) - \frac{1}{m}E_1\rho_\sigma^{m+1}(t)) \bmod \epsilon^2 \\
&= \rho_\sigma(t) - \frac{1}{m}E_1\rho_\sigma^{m+1}(t) - \frac{1}{m}\lambda_1(\sigma)\rho_\sigma(t) \bmod \epsilon^2. \qquad (12)
\end{aligned}$$

We compute that

$$\tilde{\rho}_\sigma \circ \rho_\sigma^{-1}(t) = \frac{\tilde{\rho}_\sigma(t)}{(1 - c(\sigma)\tilde{\rho}_\sigma(t)^m)^{\frac{1}{m}}}.$$

Since the derivative of the function $x \mapsto \frac{x}{(1+Ax^m)^{\frac{1}{m}}}$ is the function $x \mapsto (1 + Ax^m)^{-\frac{m+1}{m}}$ we compute:

$$
\begin{aligned}
(13) \qquad \frac{d}{d\epsilon}\tilde{\rho}_\sigma \circ \rho_\sigma^{-1}\Big|_{\epsilon=0} &= \frac{t^{m+1}}{\rho_\sigma(t)^{m+1}} \frac{d}{d\epsilon}\tilde{\rho}_\sigma\Big|_{\epsilon=0} \\
&= -\frac{1}{m}t^{m+1} \, E_1|_{\epsilon=0} - \frac{1}{m}\lambda_1(\sigma)\frac{t^{m+1}}{\rho_\sigma(t)^m} \\
(14) \qquad &= -\frac{1}{m}t^{m+1} \, E_1|_{\epsilon=0} - \frac{\lambda_1(\sigma)}{m}\left(t + t^{m+1}c(\sigma)\right).
\end{aligned}
$$

We will now compute $(1 - \lambda(\sigma)^{-1}\tilde{\rho}_\sigma)\Delta$. Write $T = t + \epsilon g_1(t) \bmod \epsilon^2 A[[t]]$. Write $\tilde{f} = (T^m + \sum_{\mu=0}^{m-1} a_\mu T^\mu)^{-1}u$, where $a_\mu = \sum_{\nu \geq 1} a_{\mu,\nu}\epsilon^\nu$.

We compute:

$$
\begin{aligned}
\Delta = \tilde{f} - \frac{1}{t^m} &= \frac{1}{T^m(1 + \sum_{\mu=0}^{m-1} a_\mu T^{\mu-m})} - \frac{1}{t^m} \\
&= \frac{1}{T^m}\left(1 - \epsilon \sum_{\mu=0}^{m-1} a_{\mu,1}T^{\mu-m}\right) - \frac{1}{t^m} \bmod \epsilon^2 A[[T]] \\
&= \frac{1 - m\epsilon g_1(t)^{m-1}}{t^m}\left(1 - \epsilon \sum_{\mu=0}^{m-1} a_{\mu,1}T^{\mu-m}\right) - \frac{1}{t^m} \bmod \epsilon^2 A[[T]] \\
&= \epsilon m g_1(t)^{m-1}/t^m - \epsilon \sum_{\mu=0}^{m-1} a_{\mu,1}t^{\mu-2m} \bmod \epsilon^2 A[[T]].
\end{aligned}
$$

Consider the automorphism σ given by $\sigma(t) = t(1 + c(\sigma)t^m)^{-1/m}$. Observe that

$$
\sigma\left(\frac{1}{t^k}\right) = \frac{(1 + c(\sigma)t^m)^{\frac{k}{m}}}{t^k} = \frac{1}{t^k} + \sum_{\nu \geq 1}\binom{\frac{k}{m}}{\nu}c(\sigma)^\nu t^{m\nu-k},
$$

therefore

$$
(1 - \lambda_1(\sigma)\epsilon)\sigma\left(\frac{1}{t^k}\right) - \frac{1}{t^k} = \frac{k}{m}c(\sigma)t^{m-k} + \sum_{\nu \geq 2}\binom{\frac{k}{m}}{\nu}c(\sigma)^\nu t^{m\nu-k} - \frac{\epsilon\lambda_1(\sigma)}{t^k}.
$$

This means that for $k \leq m$ the quantity $(1 - \lambda(\sigma)^{-1}\tilde{\rho}_\sigma)(\epsilon t^{-k})$ is holomorphic in t modulo ϵ^2. Thus $(1 - \lambda(\sigma)^{-1}\tilde{\rho}_\sigma)\frac{g_1(t)^{m-1}}{t^m} \in \bmod A[[t]]$ and we arrive at:

$$
(\lambda(\sigma)^{-1}\sigma - 1)\epsilon\Delta = \sum_{\mu=0}^{m-1}\frac{2m-\mu}{m}\frac{a_{\mu,1}c(\sigma)}{t^{m-\mu}} \bmod \epsilon^2 + A[[t]].
$$

This result combined with eq. (14) gives us

(15)

$$
\frac{d}{d\epsilon}\tilde{\rho}_\sigma \circ \rho_\sigma^{-1}\Big|_{\epsilon=0} = \frac{t^{m+1}}{m}\left(\frac{\lambda_1(\sigma)}{t^m} + \lambda_1(\sigma)c(\sigma) - \delta(\sigma) + \sum_{\mu=0}^{m-1}\frac{2m-\mu}{m}\frac{a_{\mu,1}c(\sigma)}{t^{m-\mu}}\right)
$$

modulo elements in $A[[t]]$. The desired result follows by applying the map given in eq.(11). $\square$

Lemma 4.3. *Assume that $G_1(P) = \mathbb{Z}/p\mathbb{Z}$. The k-vector space $H^1(\mathbb{Z}/p\mathbb{Z}, k[[t]]/t^{m+1})$ is generated by the elements $\{b_i/t^i : b \leq i \leq m+1\}$ so that $\binom{i/m}{p-1} = 0$ and $b = 1$ if $p \mid m+1$ and $b = 2$ if $p \nmid m+1$, and $b_i \in \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, k)$.*

Proof. This is proposition 2.7 in [24] for $a = -m-1$. □

Consider the elementary abelian group $V = \oplus_{i=1}^s V_i$ where $V_i \cong \mathbb{Z}/p\mathbb{Z}$. The computation of the cohomology group $H^1(V, \mathcal{T}_{\mathcal{O}})$ seems complicated in the general case. However, under some mild assumptions we can prove the following:

Proposition 4.4. *Let $m+1 = \sum_{i \geq 0} b_i p^i$ be the p-adic expansion of m. If $\left\lfloor \frac{2b_0}{p} \right\rfloor = \left\lfloor \frac{b_0 + b_{\nu-1}}{p} \right\rfloor$ for all $2 \leq \nu \leq s$, then the map*

$$\Psi : H^1(V, \mathcal{T}_{\mathcal{O}}) \to \bigoplus_{\nu=1}^{s} H^1(V_\nu, \mathcal{T}_{\mathcal{O}}), \tag{16}$$

sending $v \mapsto \sum_{\nu=1}^s \mathrm{res}_{V \to V_i} v$ is an isomorphism. Moreover

$$H^1(V, \mathcal{T}_{\mathcal{O}}) \cong \bigoplus_{i=2, \binom{i/m}{p-1}=0}^{m+1} b_i \frac{1}{t^i}, \tag{17}$$

where $b_i \in \mathrm{Hom}(V, k)$.

Proof. Consider the maps $c_i \in \mathrm{Hom}(V_i, k)$ and extend them to maps $\bar{c}_i \in \mathrm{Hom}(V, k)$, by setting $\bar{c}_i(\sigma) = 0$ if $\sigma \notin V_i$. The image of $\sum_{\nu=1}^s \bar{c}_i$ under the map Ψ given in (16) is $(c_1, \ldots, c_s)$, therefore the map Ψ is onto and it is sufficient to prove that both spaces have the same dimension.

For the dimension $h_1(V, \mathcal{T}_{\mathcal{O}}) = \dim_k H^1(V, \mathcal{T}_{\mathcal{O}})$ the author has proved the following formula:

$$h_1(V, \mathcal{T}_{\mathcal{O}}) = \sum_{i=1}^{s} \left(\left\lfloor \frac{(m+1)(p-1) + a_i}{p} \right\rfloor - \left\lceil \frac{a_i}{p} \right\rceil \right), \tag{18}$$

where $a_1 = -(m+1)$, $a_i = \left\lceil \frac{a_{i-1}}{p} \right\rceil$ [24, prop. 2.9]. Observe that $a_i = -\left\lfloor \frac{m+1}{p^{i-1}} \right\rfloor$. We compute that

$$\frac{m+1}{p^k} = \sum_{\nu=0}^{k-1} \frac{b_i}{p^{k-\nu}} + \sum_{\nu \geq k} b_\nu p^{\nu-k}, \tag{19}$$

therefore

$$\left\lfloor \frac{m+1}{p^k} \right\rfloor = \sum_{\nu \geq k} b_\nu p^{\nu-k}. \tag{20}$$

Now we compute that

$$\left\lfloor \frac{m+1}{p} + \frac{1}{p} \left\lfloor \frac{m+1}{p^{i-1}} \right\rfloor \right\rfloor = \left\lfloor \frac{b_0 + b_{i-1}}{p} \right\rfloor + \sum_{\nu \geq 1} b_\nu p^{\nu-1} + \sum_{\nu \geq i} b_\nu p^{\nu-i}. \tag{21}$$

The desired result follows by plugging eq. (20),(21) into eq. (18).

Equation (17) folows by lemma 4.3. □

Remark: Consider the curves defined by

$$\sum_{\nu=0}^{s} a_n y^{p^n} = \sum_{\mu=0}^{m} b_\mu x^\mu,$$

so that $m \not\equiv 0 \bmod p$, $a_s, a_0, b_0 \neq 0$, $s \geq 1$, $mu \geq 2$ studied by H. Stichtenoth in [36]. The representation attached to the unique place P_∞ above the place p_∞ of the function field $k(x)$ is two dimensional if and only if $m < p^s$ [23]. In this case the assumptions of proposition 4.4 hold.

Corollary 4.5. *If $G_1(P) = \mathbb{Z}/p\mathbb{Z}$ or if the assumptions of proposition 4.4 hold then the tangent vector corresponding to $0 \neq \frac{d}{dt} \in H^1(V, \mathcal{T}_\mathcal{O})$ is an obstructed deformation.*

Proof. The element $\frac{d}{dt}$ corresponds to $\frac{1}{t^{m+1}} \in H^1(V, \frac{1}{t^{m+1}} k[[t]])$. Using proposition 4.4 we see that it is impossible to obtain a vector in the direction of $\frac{1}{t^{m+1}}$ using a matrix representation, i.e. an element in $F(\cdot)$.

Notice that since we have assumed that the representation attached to P is two dimensional we have that $m > 1$. □

Corollary 4.6. *Assume that $p \nmid m+1$ and the assumptions of proposition 4.4 hold. Assume also that V is an elementary abelian group with more than one component. Using the notation of eq. (17) unubstructed deformations should satisfy $b_i(\sigma) = \lambda_i c(\sigma)$ for some element $\lambda_i \in k$.*

Proof. Condition $p \nmid m+1$ implies that every deformation is coming from a matrix representation 2.6 and condition follows by using proposition 4.2. □

Remark 4.7. We see that the data $\delta(\sigma)$ of the matrix representation deformation do not affect the corresponding element in $H^1(V, \mathcal{T}_\mathcal{O})$ since they appear as coefficients of t^0 in the cocylce expression of proposition 4.2 and are cohomologous to zero. What seems to affect the tangent elements is the coefficients of the distinguished Weierstrass polynomial of the function $\tilde{f}$ defined in eq. (4).

On the other hand in the case of liftings from characteristic p to characteristic zero the diagonal element λ_1 appears as coefficient of the element $t\frac{d}{dt}$. This construction is similar to the one of J.Bertin and A. Mézard [2, lemme 4.2.2].

Following [2, th. 4.2.8] we can prove:

Proposition 4.8. *If R_P denotes the versal deformation ring at P, then there is a surjection*

$$R_P \to W(k)[[y]] \Big/ \left\langle \sum_{\nu=1}^{p} \binom{p}{\nu} y^{\nu-1} \right\rangle := R' \tag{22}$$

The ring R_P is not smooth.

Proof. We are in the mixed characteristic case so $V = \langle \sigma \rangle$. According to section 3, the ring R' gives rise to deformation of the two dimensional representation given by

$$\tilde{\rho}_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1+y \end{pmatrix}$$

which in turn gives rise to the deformation

$$\tilde{\rho}_\sigma(t) = \frac{(1+y)^{-\frac{1}{m}}}{\left(1+\frac{Et^m}{1+c(\sigma)t^m}\right)^{1/m}}\rho_\sigma(t),$$

for a suitable element E. The map $\mathrm{Hom}(R_P,\cdot) \rightarrow D(\cdot)$ is smooth (in the sence of Schlessinger [31, def. 2.2], [27, p. 278]), therefore there is a map $\phi : R_P \rightarrow R'$. In order to prove that R_P is not a smooth ring we proceed as follows: Consider the natural map $\pi : W(k) \rightarrow W(k)/p = k$. We obtain the following map

$$\phi \circ \pi : R_P \rightarrow k[[y]]/\langle y^{p-1}\rangle := A.$$

Consider the ring $A' = k[[y]]/\langle y^p\rangle$. Then $A' \rightarrow A$ is a small extension, and there is no map $R_P \rightarrow A'$ extending $R_P \rightarrow R' \stackrel{\mathrm{mod} p}{\longrightarrow} A$ by remark 3.1. In this way we obtain an obstruction to the infinitesimal affine lifting for the affine scheme $\mathrm{Spec} R_P$ therefore R_P is not smooth.

Alternatively one can compute the obstruction as an element in $H^2(V, \mathcal{T}_\mathcal{O})$ following [2, lemme 4.2.3]. □

Proposition 4.9. *Assume that the hypotheses of proposition 4.4 hold. Consider the ring R_1 defined by*

$$R_1 = \begin{cases} k & \textit{in the equicharacteristic case} \\ R' & \textit{in the mixed characteristic case (see eq. (22))} \end{cases}$$

Let $b = 1$ if $p \mid m+1$ and $b = 2$ if $p \nmid m+1$. Let Σ be the subset of numbers $b \leq i \leq m$ so that $\binom{\frac{i}{m}}{p-1} = 0$. Consider the ring $\bar{R} := R_1[[X_i : i \in \Sigma]]$ and the k-vector space $W \subset H^1(V, \mathcal{T}_\mathcal{O})/\langle d/dt\rangle$ generated by elements $\lambda_i c(\sigma)t^{m+1-i}\frac{d}{dt}$.

There is a surjection $R_P \rightarrow \bar{R}$ that induces an isomorphism $W \cong \mathrm{Hom}(\bar{R}, k[\epsilon]/\epsilon^2)$. The Krull dimension of R_P is equal to $\#\Sigma$.

Proof. We have observed in corollary 4.5 that deformations in the direction of d/dt are not coming from matrix representations. The elements $\frac{1}{t^i}$ for $i \in \Sigma$ are elements in $H^1\left(V, \frac{1}{t^{m+1}}k[[t]]\right)$ that give rise to elements $t^{m+1-i}\frac{d}{dt} \in H^1(V, \mathcal{T}_\mathcal{O})$. Every deformation on these directions is unobstructed by lemma 4.1. □

5. Relation to Deformations of Artin-Schreier curves

Let P be a wild ramified point of the cover $\pi : X \rightarrow Y = X/G$ so that the corresponding representation is two dimensional. In this section we will examine the dependence of the Artin-Schreier extension $X \rightarrow X/G_1(P)$ on the form of matrix representation $\rho : G_1(P) \rightarrow GL_2(k)$. Then we will restrict to the germs $\mathcal{O}_{X,P} \rightarrow \mathcal{O}_{Y,\pi(P)}$, and we will study the relation to the deformation functor introduced in [29] by R. Pries. The approach of R. Pries is to work with germs of curves and to deform the defining Artin-Schreier equation. Since the germs are living in local rings, that have only one maximal ideal, the effect of splitting the branch locus can not be studied. Therefore R. Pries considers only deformations that do not split the branch locus. According to proposition 2.2 it is impossible to lift a wild ramified action to characteristic zero, without splitting the branch locus. We will now restrict ourselves to the equicharacteristic deformation case.

Let X be a curve that has a 2-dimensional representation attached at a wild ramified point P. Denote by $\{1, f\}$ a basis of the 2-dimensional vector space $L(mP)$ where $m := v_P(f)$ is the highest jump in the upper ramification filtration.

We would like to write down an algebraic equation for the cover $X \to X/G_1(P)$. The representation $c = c_1 : G_1(P) \to k$ is a faithful homomorphism of additive groups. We consider the action of $G_1(P)$ on f: Let $\Phi(Y)$ be the additive polynomial with set of roots $\{c_1(\sigma) : \sigma \in G_1(P)\}$. The polynomial $\Phi(Y)$ can be computed as follows: The group $G_1(P)$ is by [23, sec. 3] elementary abelian so we express $G_1(P)$ as an $\mathbb{F}_p$ vector space with basis $\{\sigma_i\}$ such that $G_1(P) = \bigoplus_{i=1}^s \sigma_i \mathbb{F}_p$.

Let $\Delta(x_1, \ldots, x_n)$ denote the Moore determinant:

$$\Delta(x_1, \ldots, x_n) = \det \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^p & x_2^p & \cdots & x_n^p \\ \vdots & & & \vdots \\ x_1^{p^{n-1}} & x_2^{p^{n-1}} & \cdots & x_n^{p^{n-1}} \end{pmatrix}.$$

The additive polynomial Φ can be expressed in terms of the Moore determinant:

$$\Phi(Y) = \frac{\Delta(c(\sigma_1), \ldots, c(\sigma_s), Y)}{\Delta(c(\sigma_1), \ldots, c(\sigma_s))},$$

see [9, lemma 1.3.6],[7, eq. 3.6]. Thus, the cover $X \to X/G_1(P)$ is given in terms of the generalized Artin-Schreier equation

$$\Phi(Y) = \prod_{\sigma \in G_1(P)} \sigma f = N_{G_1(P)}(f).$$

We would like to represent the curve as a fibre product of Artin-Schreier curves and then using Garcia's-Stichtenoth's normalization [8] to write the curve in the form $y^{p^s} - y = u$, where u is an element in the function field of the curve $X/G_1(P)$.

There are elements $y_j \in k(X)$ so that $\sigma_i(y_j) = y_j + \delta_{ij}$. Using this notation we can see that the function field $k(X)$ can be recovered as the function field of the fibre product of the curves $y_i^p - y_i = u_i$. The constant elements u_i can be computed from the map $c : G_1(P) \to k$ as follows: Let $V_i = \bigoplus_{\nu=1, \nu \neq i}^s \sigma_\nu \mathbb{F}_p$. We compute an additive polynomial $\mathrm{ad}_i(Y)$ with roots the $\mathbb{F}_p$-vector space V_i using the Moore determinant:

$$\mathrm{ad}_i(Y) := \frac{\Delta(c(\sigma_1), \ldots, \widehat{c(\sigma_i)}, \ldots, c(\sigma_s), Y)}{\Delta(c(\sigma_1), \ldots, \widehat{c(\sigma_i)}, \ldots, c(\sigma_s))}.$$

These polynomials are invariants of the curve and the map c. Moreover we compute that $y_i' := \prod_{\sigma \in V_i} \sigma f = \prod_{v \in V_i} (f - v) = \mathrm{ad}_i(f)$. The element y_i' is invariant under the action of V_i and $\sigma_i(y_i') = y_i' + \mathrm{ad}(c(\sigma_i))$. We can normalize by setting $y_i = y_i'/\mathrm{ad}(c(\sigma_i))$. Then,

$$\sigma_j(y_i) = y_i + \delta_{ij}.$$

Following [8] we choose an $\mathbb{F}_p$ basis $\mu_1, \ldots, \mu_s$ of $\mathbb{F}_{p^s}$ and we set $y = \sum_{i=1}^s \mu_i y_i$. We observe that the function field can be recovered as the following extension of the field $k(X)^{G_1(P)}$:

$$y^{p^s} - y = N_{G_1(P)} \left(\sum_{i=1}^s \frac{\mu_i \mathrm{ad}_i(f)}{\mathrm{ad}_i(c(\sigma_i))} \right) =: u.$$

The element $u \in k(X)^{G_1(P)}$ is an invariant of the action of $G_1(P)$ on $k(X)$. Observe that

(23) $$\Delta\big(c(\sigma_1), \ldots, \widehat{c(\sigma_i)}, c(\sigma_s), c(\sigma_i)\big) = (-1)^{s-i}\Delta(c(\sigma_1), \ldots, c(\sigma_s)).$$

Let D be the operator sending $x \mapsto x^{p^s} - x$. Since $\mu_i \in \mathbb{F}_{p^s}$ we have $D(\mu_i x) = \mu_i D(x)$. The element u can thus also be expressed by

(24)
$$u = \sum_{i=1}^{s} \mu_i D\left(\frac{\mathrm{ad}_i(f)}{\mathrm{ad}_i(c(\sigma_i))}\right) = \sum_{i=1}^{s} \mu_i (-1)^{s-i} D\left(\frac{\Delta\big(c(\sigma_1), \ldots, \widehat{c(\sigma_i)}, \ldots, c(\sigma_s), f\big)}{\Delta\big(c(\sigma_1), \ldots, c(\sigma_s)\big)}\right)$$

Equation (24) allows us to express u in terms of the following determinant:

(25)
$$u_1 = \frac{1}{\Delta\big(c(\sigma_1), \ldots, c(\sigma_s)\big)} \det \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_s & 0 \\ c(\sigma_1) & c(\sigma_2) & \cdots & c(\sigma_s) & f \\ c(\sigma_1)^p & c(\sigma_2)^p & \cdots & c(\sigma_s)^p & f^p \\ \vdots & \vdots & & \vdots & \\ c(\sigma_1)^{p^{s-1}} & c(\sigma_2)^{p^{s-1}} & \cdots & c(\sigma_s)^{p^{s-1}} & f^{p^{s-1}} \end{pmatrix},$$

$$u = D(u_1).$$

Notice that u_1 is a polynomial of f of the form

$$u_1(f) = \sum_{\nu=1}^{s} o_\nu f^{p^{\nu-1}},$$

where o_ν can be computed, in terms of the function c, as minor determinants of the above matrix. Then $u(f)$ is a polynomial of f of the form

$$u(f) = \sum_{\nu=1}^{2s} a_i f^{p^{\nu-1}},$$

where for $1 \le \nu$, $a_{\nu+s} = -a_\nu^{p^s}$.

Now consider the relative situation: Consider the element $\tilde{f} \in A[[t]][t^{-1}]$ defined in proposition 2.4. Given such an element $\tilde{f}$ and a deformation of the representation $\rho : G_1(P) \to GL_2(L(mP))$ we will construct a deformation $\mathcal{O}_{X,P}$ of the germ $\mathcal{O}_{X,P}$ with Galois group $G_1(P)$.

We form again the additive polynomials:

$$\mathrm{Ad}_i(Y) =: \frac{\Delta(C(\sigma_1), \ldots, \widehat{C(\sigma_i)}, \ldots, C(\sigma_s), Y)}{\Delta(C(\sigma_1), \ldots, \widehat{C(\sigma_i)}, \ldots, C(\sigma_s))}.$$

Using the previous normalization procedure we arrive at the following deformed Artin-Schreier curve:

$$\begin{aligned} y^{p^s} - y &= \sum_{i=1}^{s} \mu_i D\left(\frac{\mathrm{Ad}_i(\tilde{f})}{\mathrm{Ad}_i(C(\sigma_i))}\right) = \\ &= \sum_{i=1}^{s} \mu_i (-1)^{s-i} D\left(\frac{\Delta\big(C(\sigma_1), \ldots, \widehat{C(\sigma_i)}, \ldots, C(\sigma_s), \tilde{f}\big)}{\Delta\big(C(\sigma_1), \ldots, C(\sigma_s)\big)}\right) := U. \end{aligned}$$

Notice that similar to equation (25) we have:

$$U_1 = \frac{1}{\Delta\big(C(\sigma_1), \ldots, C(\sigma_s)\big)} \det \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_s & 0 \\ C(\sigma_1) & C(\sigma_2) & \cdots & C(\sigma_s) & \tilde{f} \\ C(\sigma_1)^p & C(\sigma_2)^p & \cdots & C(\sigma_s)^p & \tilde{f}^p \\ \vdots & \vdots & & \vdots & \\ C(\sigma_1)^{p^{s-1}} & C(\sigma_2)^{p^{s-1}} & \cdots & C(\sigma_s)^{p^{s-1}} & \tilde{f}^{p^{s-1}} \end{pmatrix},$$

$$U = D(U_1).$$

The element $U \in A[[t]][t^{-1}]$ so that $U \equiv u \bmod m_A$.

5.1. **Relation with equivalence class of Artin-Schreier extensions.** In what follows we would like to consider isomorphism classes of Artin-Schreier curves. The following lemma identifies two Artin-Schreier extensions of the ring $A[[x]][x^{-1}]$, where A is a k-algebra that gives rise to an irreducible affine scheme, i.e. $A/\mathrm{rad}(A)$ is an integral domain.

Lemma 5.1. *Consider the extensions $y_1^{p^s} - y_1 = g_1$ and $y_2^{p^s} - y_2 = g_2$, where $g_1, g_2 \in A[[x]][x^{-1}]$. These extensions are isomorphic if and only if $g_1(x) = \zeta g_2(x) + d^{p^s} - d$, for some $d \in A[[x]][x^{-1}]$, and $\zeta \in \mathbb{F}_{p^s}^*$.*

Proof. If A is a field k then this is classical result due to Hasse [19]. For the general case we refer to [30, lemma 2.4]. □

Lemma 5.2. *The Artin-Schreier curve $y^{p^s} - y = f(x)$ where $f(x) \in A[[x]][x^{-1}]$ is isomorphic to $y^{p^s} - y = f(x) + g(x)$, where $g(x) \in A[[x]]$.*

Proof. Following [30, sec. 3] we observe that $g(x) = d^{p^s} - d$, where $d = \sum_{\nu=0}^{\infty} g(x)^{p^{s\nu}}$. The desired result follows by using lemma 5.1. □

Let m be the conductor, i.e. the highest jump in the upper ramification filtration. Since the group H is elementary abelian this is equal to the highest jump in the lower ramification filtration [24, lemma 1.8].

Lemma 5.3. *Consider an Artin-Schreier cover of $A[[x]][x^{-1}]$ given by:*

$$y^{p^s} - y = \sum_{\nu=0}^{\lambda} r_\nu (1/x)^{p^\nu},$$

where $r_\nu(T) \in A[T]$ are polynomials of degree d_ν, so that $\gcd(d_\nu, p) = 1$. The conductor of the cover equals to $\max_\nu d_\nu$.

Proof. R. Pries [30]. □

D. Harbater in [14] (see also [2, sec. 5.1]) gave a parametrization of the classes of cyclic $\mathbb{Z}_p$-covers of a local fields branched above the maximal ideal. For the more general case of $\mathbb{F}_{p^s}$-covers the space of classes of covers of $k((t'))$ is parametrized by the quotient:

$$C = \frac{k((t'))}{k[[t]] + D(k((t')))}, \tag{26}$$

where D denotes the map $x \mapsto x^{p^s} - x$. Indeed, by lemma 5.1 adding $D(a)$ does not alter the equivalence class of the Artin-Schreier curve and by lemma 5.3 the same holds for adding a holomorphic element.

R. Pries gave a moduli interpretation of p-group covers of the projective line and she proposed two approaches: either transform (by extending the base ring A) an arbitrary Artin-Schreier extension of $A[[x]][x^{-1}]$ to a class in (26) or define a fine moduli space by considering a category where all powers of the q-Frobenious maps are invertible elements. She introduced the following:

Definition 5.4. *Let A_1, A_2 be two k-algebras that give rise to irreducible affine schemes, i.e. $A_i/\mathrm{rad}(A_i)$, $i = 1, 2$ are integral domains. Consider the Artin-Schreier relative A-curves $C_i : y_i^{p^s} - y_i = f_i(x)$, where $f_i(x) \in A_i$. The two curves are considered to be equivalent if and only if there is an algebra extension A of both A_i, i.e. there are ring monomorphisms $A_i \hookrightarrow A$, so that the curves $C_i \times_{\mathrm{Spec} A_i} \mathrm{Spec} A$ are isomorphic covers of $A[[x]][x^{-1}]$.*

In general $U - u \in m_A[[t]][t^{-1}]$ and it is not an element in $m_A[[x]][x^{-1}]$. If the deformation $\tilde{\rho}_\sigma$ does not split the branch locus, then $U - u \in m_A[[x]][x^{-1}]$. After cutting the holomorphic part of $U - u$ and applying the transformation of lemma 5.1 we get an equivalence class of germs of Artin-Schreier curves given in eq. (26) to an element in the deformation functor of Pries.

Conversely for every Laurent polynomial $\Delta \in m_A((x))$ so that $n_0 = v_x(\Delta)$, satisfies $(n_0, p) < m$ we can consider the extension of $A((x))$ defined as

$$A((x))[y]/(y^{p^s} - y = f + \Delta).$$

This gives rise to an infinitesimal extension of the germ of X at P in the sense of Pries and according to the local-global theory developed by Harbater all this local deformations can be patched together to give a global deformation of the couple (X, G).

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