Research Article

Stability of General Newton Functional Equations for Logarithmic Spirals

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We investigate the generalized Hyers-Ulam stability of Newton functional equations for logarithmic spirals.

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1. Introduction

The starting point of studying the stability of functional equations seems to be the famous talk of Ulam [1] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

Let $G_1$ be a group and let $G_2$ be a metric group with a metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by Hyers [2] under the assumption that $G_1$ and $G_2$ are Banach spaces. Later, the result of Hyers was significantly generalized for additive mappings by Aoki [3] and for linear mappings by Rassias [4]. It should be remarked that we can find in the books [5–7] a lot of references concerning the stability of functional equations.

Recently, Jung and Sahoo [8] proved the generalized Hyers-Ulam stability of the functional equation $f(\sqrt{r^2 + 1}) = f(r) + \arctan \left(1/r\right)$ which is closely related to the square root spiral, for the case that $f(1) = 0$ and $f(r)$ is monotone increasing for $r > 0$ (see [9, 10]).
By $\mathcal{F}$ we denote the set of all functions $f : (0, \infty) \rightarrow \mathbb{R}$. Let $\Delta$ be the difference operator defined by
\[
(\Delta f)(r) = f(r + 1) - f(r) \quad (r > 0)
\] (1.1)
for any $f \in \mathcal{F}$. Throughout this paper, let $n$ be a fixed positive integer, and we define an operator $\Delta^n : \mathcal{F} \rightarrow \mathcal{F}$ by
\[
(\Delta^n f)(r) = \Delta (\Delta^{n-1} f)(r) \quad (r > 0)
\] (1.2)
for all $f \in \mathcal{F}$, where we set $\Delta^0 f = f$. For instance, we see that
\[
(\Delta^2 f)(r) = f(r + 2) - 2f(r + 1) + f(r),
\]
\[
(\Delta^3 f)(r) = f(r + 3) - 3f(r + 2) + 3f(r + 1) - f(r).
\] (1.3)

In this paper, we will investigate the generalized Hyers-Ulam stability of the Newton difference (operator) equations
\[
\Delta^n f(r) = A \ln R_n(r)
\] (1.4)
for all $r > 0$ and some fixed integer $n > 0$, where $A > 0$ is a constant and
\[
R_1(r) = \frac{r + 1}{r}, \quad R_k(r) = \frac{R_{k-1}(r + 1)}{R_{k-1}(r)}
\] (1.5)
for $k \in \{2, 3, \ldots, n\}$.
We will say that (1.4) has the generalized Hyers-Ulam stability whenever a (given) function $f : (0, \infty) \rightarrow \mathbb{R}$ satisfies the inequality
\[
|\Delta^n f(r) - A \ln R_n(r)| \leq \varphi_n(r)
\] (1.6)
for all $r > 0$, where $\varphi_n : (0, \infty) \rightarrow [0, \infty)$ is a given nonnegative function, there exists a solution of (1.4) which is not far from $f$.

2. Newton $n$-ary difference equation

The difference equation in (1.4) is called the Newton $n$-ary difference (operator) equation. In the following theorem, we give a partial solution to the generalized Hyers-Ulam stability problem of (1.4).

**Theorem 2.1.** If a function $f : (0, \infty) \rightarrow \mathbb{R}$ satisfies the inequality (1.6) for all $r > 0$ and some integer $n > 0$, where $\varphi_n : (0, \infty) \rightarrow [0, \infty)$ is a function which satisfies
\[
\Phi_n(r) = \sum_{k=0}^{\infty} \varphi_n(r + k) < \infty
\] (2.1)
for any $r > 0$, then there exists a unique function $F_n : (0, \infty) \rightarrow \mathbb{R}$ such that $\Delta F_n(r) = A \ln R_n(r)$ and
\[
|F_n(r) - \Delta^{n-1} f(r)| \leq \Phi_n(r)
\] (2.2)
for each $r > 0$. 

Proof. It follows from (1.6) that
\[
|\Delta^n f(r) - A \ln R_n(r)| \leq \varphi_n(r),
\]
\[
|\Delta^n f(r + 1) - A \ln R_n(r + 1)| \leq \varphi_n(r + 1),
\]
\[
\vdots
\]
\[
|\Delta^n f(r + m - 1) - A \ln R_n(r + m - 1)| \leq \varphi_n(r + m - 1)
\]
for any \( r > 0 \) and \( m \in \mathbb{N} \). In view of triangular inequality, the above inequalities yield
\[
\left| \sum_{k=0}^{m-1} \Delta^n f(r + k) - \sum_{k=0}^{m-1} A \ln R_n(r + k) \right| \leq \sum_{k=0}^{m-1} \varphi_n(r + k). \tag{2.4}
\]
Substitute \( r + \ell \) for \( r \) in (2.4) and then substitute \( k \) for \( k + \ell \) in the resulting inequality to obtain
\[
\left| \sum_{k=\ell}^{\ell+m-1} \Delta^n f(r + k) - \sum_{k=\ell}^{\ell+m-1} A \ln R_n(r + k) \right| \leq \sum_{k=\ell}^{\ell+m-1} \varphi_n(r + k). \tag{2.5}
\]
for all \( r > 0 \) and \( \ell, m \in \mathbb{N} \).

By some manipulation, we further have
\[
\left| \sum_{k=0}^{\ell+m-1} \Delta^n f(r + k) - \sum_{k=0}^{\ell+m-1} A \ln R_n(r + k) + \Delta^{n-1} f(r) \right.
\]
\[
- \sum_{k=0}^{\ell-1} \Delta^n f(r + k) + \sum_{k=0}^{\ell-1} A \ln R_n(r + k) - \Delta^{n-1} f(r) \right| \leq \sum_{k=0}^{\ell+m-1} \varphi_n(r + k) \tag{2.6}
\]
for every \( r > 0 \) and \( \ell, m \in \mathbb{N} \). Thus, considering (2.1), we see that the sequence
\[
\left\{ \sum_{k=0}^{m-1} \left[ \Delta^n f(r + k) - A \ln R_n(r + k) \right] + \Delta^{n-1} f(r) \right\}_{n=1,2,3,...} \tag{2.7}
\]
is a Cauchy sequence for any \( r > 0 \). Hence, we can define a function \( F_n : (0, \infty) \rightarrow \mathbb{R} \) by
\[
F_n(r) = \sum_{k=0}^{\infty} \left[ \Delta^n f(r + k) - A \ln R_n(r + k) \right] + \Delta^{n-1} f(r). \tag{2.8}
\]
By (2.8), we get
\[
\Delta F_n(r) = F_n(r + 1) - F_n(r)
\]
\[
= \sum_{k=1}^{\infty} \left[ \Delta^n f(r + k) - A \ln R_n(r + k) \right] + \Delta^{n-1} f(r + 1)
\]
\[
- \sum_{k=0}^{\infty} \left[ \Delta^n f(r + k) - A \ln R_n(r + k) \right] - \Delta^{n-1} f(r)
\]
\[
= A \ln R_n(r) \tag{2.9}
\]
for all \( r > 0 \). In view of (2.1) and (2.8), if we let \( m \) go to infinity in (2.4), then we obtain (2.2).
Let us define \( f \). If we set \( r > 0 \), we have
\[
H(r + m) - H(r) = \sum_{k=0}^{m-1} A \ln R_n(r + k)
\] (2.10)
for all \( r > 0 \) and \( m \in \mathbb{N} \). Now, assume that \( G_n : (0, \infty) \to \mathbb{R} \) satisfies \( \Delta G_n(r) = A \ln R_n(r) \) and the inequality (2.2) in place of \( F_n \). By (2.1), (2.2), and (2.10), we obtain
\[
|F_n(r) - G_n(r)| = |F_n(r + m) - G_n(r + m)| \leq 2\Phi_n(r + m) \to 0 \quad \text{as} \quad m \to \infty,
\] (2.11)
for any \( r > 0 \), which proves the uniqueness of \( F_n \).

3. Application to logarithmic spirals

For given \( a > 1 \) and \( c > 0 \), the equation
\[
r = ce^{\theta/\sqrt{a^2 - 1}}
\] (3.1)
represents a logarithmic spiral in the polar coordinates \((r, \theta)\). We know that this formula is equivalent to
\[
\theta = \sqrt{a^2 - 1}(\ln r - \ln c).
\] (3.2)
Let us define \( f(r) = \sqrt{a^2 - 1}(\ln r - \ln c) \) so that we can write the above expression in a simpler form, \( \theta = f(r) \). Then \( f \) is a solution of (3.4) for \( n = 1 \) and \( A = \sqrt{a^2 - 1} \), that is, \( f \) is a solution of the equation
\[
\Delta f(r) = \sqrt{a^2 - 1} \ln \frac{r + 1}{r},
\] (3.3)
which may be called the equation for logarithmic spirals.

We will now solve (3.3) by using [9, Theorem 1].

**Theorem 3.1.** If a function \( f : (0, \infty) \to \mathbb{R} \) satisfies (3.3), then there exists a periodic function \( \sigma : \mathbb{R} \to \mathbb{R} \) of period 1 such that
\[
f(r) = \sigma(r) + \sqrt{a^2 - 1} \ln r
\] (3.4)
for all \( r > 0 \).

**Proof.** If we set
\[
\varphi(r) = \sqrt{a^2 - 1} \ln \frac{r + 1}{r}
\] (3.5)
for all \( r > 0 \), then we have
\[
\varphi(r + s) - \varphi(r) = \sqrt{a^2 - 1} \ln \frac{r^2 + (s + 1)r}{r^2 + (s + 1)r + s} < 0
\] (3.6)
for any \( r, s > 0 \), which implies that \( \psi \) is monotonically decreasing. Moreover, we also see that
\[
\lim_{r \to \infty} \psi(r) = \sqrt{a^2 - 1} \lim_{r \to \infty} \ln \frac{r + 1}{r} = 0. \tag{3.7}
\]

According to [9, Theorem 1], the general solution of (3.3) is given by
\[
f(r) = \sigma(r) + \sum_{k=0}^{\infty} [\psi(k + 1) - \psi(r + k)] = \sigma(r) + \sqrt{a^2 - 1} \ln r, \tag{3.8}
\]
where \( \sigma \) is an arbitrary periodic function of period 1.

If we set \( n = 1 \) in Theorem 2.1 and apply Theorem 3.1, then we get the following corollary concerning the generalized Hyers-Ulam stability of (3.3).

**Corollary 3.2.** If a given function \( f : (0, \infty) \to \mathbb{R} \) satisfies the inequality
\[
\left| \Delta f(r) - \sqrt{a^2 - 1} \ln \frac{r + 1}{r} \right| \leq \psi(r) \tag{3.9}
\]
for all \( r > 0 \) and some \( \alpha > 1 \), where \( \psi : (0, \infty) \to [0, \infty) \) is a function which satisfies the condition
\[
\Phi(r) = \sum_{k=0}^{\infty} \psi(r + k) < \infty \tag{3.10}
\]
for any \( r > 0 \), then there exists a unique periodic function \( \sigma : \mathbb{R} \to \mathbb{R} \) of period 1 such that
\[
|f(r) - \sigma(r) - \sqrt{a^2 - 1} \ln r| \leq \Phi(r) \tag{3.11}
\]
for all \( r > 0 \).

**References**

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<thead>
<tr>
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<th>June 1, 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>September 1, 2008</td>
</tr>
<tr>
<td>Publication Date</td>
<td>December 1, 2008</td>
</tr>
</tbody>
</table>

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