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Special Issue on **Leonhard Paul Euler**'s:
Functional Equations and Inequalities (F. E. I.)

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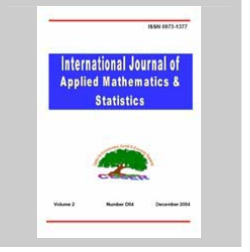

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PREFACE

This Euler's commemorating volume entitled :
Functional Equations , Integral Equations, Differential Equations and Applications (**F. I. D. A**), is a forum for exchanging ideas among eminent mathematicians and physicists, from many parts of the world, as a tribute to the tri-centennial birthday anniversary of Leonhard Paul Euler (April 15,1707 A.D., b. in Basel – September 18, 1783 A.D., d. in St. Petersburg).
This 998 pages long collection is composed of outstanding contributions in mathematical and physical equations and inequalities and other fields of mathematical, physical and life sciences.
In addition, this anniversary volume is unique in its target, as it strives to represent a broad and highly selected participation from across and beyond the scientific and technological country regions. It is intended to boost the cooperation among mathematicians and physicists working on a broad variety of pure and applied mathematical areas.
Moreover, this new volume will provide readers and especially researchers with a detailed overview of many significant insights through advanced developments on Euler's mathematics and physics. This transatlantic collection of mathematical ideas and methods comprises a wide area of applications in which equations, inequalities and computational techniques pertinent to their solutions play a core role.
Euler's influence has been tremendous on our everyday life, because new tools have been developed, and revolutionary research results have been achieved , bringing scientists of exact sciences even closer, by fostering the emergence of new approaches, techniques and perspectives.
The central scope of this commemorating 300birthday anniversary volume is broad, by deeper looking at the impact and the ultimate role of mathematical and physical challenges, both inside and outside research institutes, scientific foundations and organizations.
We have recently observed a more rapid development in the areas of research of Euler worldwide. Leonhard P. Euler (1707-1783) was actually the most influential mathematician and prolific writer of the eighteenth century, by having contributed to almost all the fundamental fields of mathematics and mathematical physics. In calculus of variations, according to C. Caratheodory, Euler's work: *Methodus inveniendi lineas curves…*(1740 A.D.) was one of the most beautiful works ever written. Euler was dubbed *Analysis Incarnate* by his peers for his incredible ability. He was especially great from his writings and that produced more academic work on mathematics than anyone. He could produce an entire new mathematical paper in about thirty minutes and had huge piles of his works lying on his desk. It was not uncommon to find Analysis Incarnate ruminating over a new subject with a child on his lap.
This volume is suitable for graduate students and researchers interested in functional equations, integral equations and differential equations and would make an ideal supplementary reading or independent study research text.
This item will also be of interest to those working in other areas of mathematics and physics. It is a work of great interest and enjoyable read as well as unique in market.

This Euler's volume (F. I. D. A.) consists of six (6) issues containing various parts of contemporary pure and applied mathematics with emphasis to Euler's mathematics and physics.
It contains sixty eight (68) fundamental research papers of one hundred one (101) outstanding research contributors from twenty seven (27) different countries.
In particular, these contributors come from:
Algerie (1 contributor); Belgique (2); Bosnia and Herzegovina (2); Brazil (2); Bulgaria (3); China (9); Egypt (1); France (3); Greece (2); India (8); Iran (3); Italy (1); Japan (7); Korea (7); Morocco (3); Oman (2); Poland (3); R. O. Belarus (8); Romania (2); Russia (3); Saudi Arabia (1); Serbia and Montenegro (5); The Netherlands (3); U. A. Emirates (1);U. K. (2); U. S. A. (15); Uzbekistan (2).

First Issue (F. E. I.) consisting of 14 research papers, 181 pages long, contains various parts of ***Functional Equations and Inequalities*****,**
namely:
Euler's Life and Work, Ulam stability, Hyers – Ulam stability and Ulam – Gavruta - Rassias stability of functional equations, Euler – Lagrange type and Euler – Lagrange – Rassias quadratic mappings in Banach and Hilbert spaces, Aleksandrov and isometry Ulam stability problems, stability of Pexider and Drygas functional equations, alternative of fixed point, and Hyers - Ulam stability of differential equations.

Second Issue (MT. PDE) consisting of 9 research papers, 117 pages long, contains various parts of ***Mixed Type Partial Differential Equations*,**
namely:
Tricomi - Protter problem of nD mixed type partial differential equations, solutions of generalized Rassias' equation, degenerated elliptic equations, mixed type oblique derivative problem, Cauchy problem for Euler – Poisson - Darboux equation, non - local boundary value problems, non-uniqueness of transonic flow past a flattened airfoil, multiplier methods for mixed type equations.
Third Issue (F . D . E.) consisting of 9 research papers, 146 pages long, contains various parts of ***Functional and Differential Equations*,**
namely:
Iterative method for singular Sturm - Liouville problems, Euler type boundary value problems in quantum mechanics, positive solutions of boundary value problems, controllability of impulsive functional semi-linear differential inclusions in Frechet spaces, asymptotic properties of solutions of the Emden-Fowler equation, comparison theorems for perturbed half-linear Euler differential equations, almost sure asymptotic estimations for solutions of stochastic differential delay equations, difference equations inspired by Euler's discretization method, extended oligopoly models.
Fourth Issue (D. E. I.) consisting of 9 research papers, 160 pages long, contains various parts of ***Differential Equations and Inequalities*,**
namely:
New spaces with wavelets and multi-fractal analysis, mathematical modeling of flow control and wind forces, free convection in conducting fluids, distributions in spaces, strong stability of operator–differential equations, slope – bounding procedure, sinc methods and PDE, Fourier type analysis and quantum mechanics.
Fifth Issue (DS. IDE.) consisting of 9 research papers, 159 pages long, contains various parts of ***Dynamical Systems and Integro - Differential Equations*,**
namely:
Semi-global analysis of dynamical systems, nonlinear functional-differential and integral equations, optimal control of dynamical systems, analytical and numerical solutions of singular integral equations, chaos control of classes of complex dynamical systems, second order integro-differential equation, integro-differential equations with variational derivatives generated by random partial integral equations, inequalities for positive operators, strong convergence for a family of non-expansive mappings.
Sixth Issue (M. T. A.) consisting of 18 research papers, 231 pages long, contains various parts of ***Mathematical Topics and Applications*,**
namely:
Maximal subgroups and theta pairs in a group, Euler constants on algebraic number fields, characterization of modulated Cox measures on topological spaces, hyper-surfaces with flat r-mean curvature and Ribaucour transformations, Leonhard Euler's methods and ideas live on in the thermodynamic hierarchical theory of biological evolution, zeroes of L-series in characteristic p,
Beck's graphs, best co-positive approximation function, Convexity in the theory of the Gamma function, analytical and differential – algebraic properties of Gamma function, Ramanujan's summation formula and related identities, ill – posed problems, zeros of the q-analogues of Euler polynomials, Eulerian and other integral representations for some families of hyper-geometric polynomials, group C*-algebras and their stable rank, complementaries of Greek means to Gini means, class of three- parameter weighted means, research for Bernoulli's inequality.

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Matina- Mathematics Ph. D. candidate of the Strathclyde University (Glasgow, United Kingdom), Katia- Senior student of Archaeology and History of Art of the National and Capodistrian University of Athens (Greece), and Vassiliki- M. B. A. of the University of La Verne, Marketing Manager in a FMCG company (Greece). Finally I express my special appreciation to:
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John Michael Rassias

Special Editor-in-Chief of Euler's volume F. I. D. A. – **IJAMAS**.
National and Capodistrian University of Athens, Greece
E-mail: jrassias@primedu.uoa.gr; jrassias@tellas.gr
URL: http://www.primedu.uoa.gr/~jrassias/

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LEONHARD PAUL EULER

HIS LIFE AND HIS WORK

John Michael Rassias[1] **and Matina John Rassias**[2]

[1] Pedagogical Department, E E
Section of Mathematics and Informatics
National and Capodistrian University of Athens
4, Agamemnonos Str., Aghia Paraskevi
Attikis 15342 Athens , GREECE
e-mail: jrassias@primedu.uoa.gr

and

[2] Statistics and Modelling Science
University of Strathclyde
Livingstone Tower, 26 Richmond Str.
Glasgow , Scotland UK G1 1XH
e-mail:matina@stams.strath.ac.uk

Abstract

Leonhard Paul Euler (April 15, 1707 - September 18, 1783) was born in Basel, Switzerland and died in St. Petersburg, Russia. He married and fathered thirteen children, one of which, Jean Albert, was a good applied mathematician. Leonhard was a student of John Bernoulli. In 1735 he lost sight in one eye, but even this did not stay his output. In 1741 he went to Berlin, at the call of Frederick the Great. In 1766 he returned from Berlin to St. Petersburg, at the request of Catherine the Great, where he soon went totally blind. Nevertheless, sustained by an uncommon memory and a remarkable facility in mental computation, he became the most prolific mathematician of all times. He averaged about 800 printed pages a year throughout his long life. Some of the books and 400 of his research papers were written after he became totally blind. He was the central figure in the mathematical activities of the 18^{th} *century. Although interested in all fields of mathematics, Euler was primarily an analyst. via his work analysis became more easily applicable to the fields of physics. He contributed much also to the progress of algebra and number theory, as well as to differential geometry and topology. Euler had, however, little concern for rigorous foundations. He was superbly inventive in methodology and a skilled technician. His collected works, although still incomplete, have been published by Teubner and O. Füssli in seventy-two big volumes.*

Key words and phrases: *Euler formula, Euler method, Euler equation, Euler numbers, Euler-Lagrange quadratic identity, Euler Monte Carlo Method for SDEs .*

AMS (MOS) Subject Classification: 00, 01.

1 LIFE OF LEONHARD PAUL EULER

The key figure in eighteenth-century mathematics and the dominant theoretical physicist of the century, the man who should be ranked with Archimedes, Cauchy, Gauss and Newton is Leonhard Paul Euler (1707-83).

Born near Basel to a preacher, who wanted him to study theology, Leonhard entered the university at Basel and completed his work at the age of fifteen.

While at Basel he learned mathematics from John Bernoulli. He decided to pursue the subject and began to publish papers at eighteen. At nineteen he won a prize from the French Académie des Sciences for a work on the masting of ships. Through the younger Bernoullis, Nicholas (1695-1726) and Daniel (1700-82), sons of John, Euler in 1733 secured an appointment at the St. Petersburg Academy in Russia. He started as an assistant to Daniel Bernoulli but soon succeeded him as a professor.

Though Euler passed some painful years (1733-41) under the autocratic government, he did an amazing amount of research, the results of which appeared in papers published by the St. Petersburg Academy. He also helped the Russian government on many physical problems.

In 1741, at the call of Frederick the Great, he went to Berlin, where he remained until 1766. At the request of Frederick the Great, Euler worked on *state problems of insurance* and the *design of canals and waterworks.* Even during his twenty-five years in Berlin he sent hundreds of papers to the St. Petersburg Academy and advised it on its affairs [3].

In 1766, at the request of Catherine the Great, Euler returned to Russia, although fearing the effect on his weakened sight (he had lost the sight of one eye in 1735) of the rigors of the climate there. He became, in effect, blind shortly after returning to Russia, and during the last seventeen years of his life was totally blind. Nevertheless, these years were no less fruitful than the preceding ones.

Euler's mathematical productivity is incredible. His major mathematical fields were the calculus, differential equations, analytic and differential geometry of curves and surfaces, the theory of numbers, series, and the calculus of variations. This mathematics he applied to the entire domain of physics. Chemistry, geography, and cartography also interested him, and he made *a map of Russia.* The applications were said to be an excuse for his mathematical investigations; but there can be no doubt that he liked both [3].

Euler wrote texts on mechanics, algebra, mathematical analysis, analytic and differential geometry, and the calculus of variations that were standard works for a hundred years and more afterward. The ones that will concern us in this paper are the two-volume *Introductio in Analysin Infinitorum* (1748), the first connected presentation of the calculus and elementary analysis; the more comprehensive *Institutiones Calculi Differentialis* (1755) ; and the three-volume *Institutiones Calculi Integralis* (1768-70) ; all are landmarks.

All of Euler's books contained some highly original features. His mechanics, as noted, was based on analytical rather than geometrical methods. He gave the first significant treatment of the calculus of variations.

Beyond texts he published original research papers of high quality at the rate of about eight hundred pages a year throughout his long life. The quality of these papers may be judged from the fact that he won so many prizes for them that these awards became an almost regular addition to his income.

Some of the books and four hundred of his research papers were written after he became totally blind.

A current edition of his collected works, when completed, will contain about eighty volumes. Euler did not open up new branches of mathematics. But no one was so prolific or could so cleverly handle mathematics; no one could muster and utilize the resources of algebra, geometry, and analysis to produce so many admirable results.

Euler was superbly inventive in methodology and a skilled technician. One finds his name in all branches of mathematics: there are formulas of Euler, polynomials of Euler, Euler constants, Eulerian integrals, Euler lines, and Euler circles [3].

One might suspect that such a volume of activity could be carried on only at the expense of all other interests. But Euler married and fathered thirteen children. One of his sons, named Jean Albert Euler, was born on November 16, 1734. This man was a good applied mathematician. Always attentive to his family and its welfare Leonhard Euler instructed his children and his grandchildren, constructed scientific games for them, and spent evenings reading the Holy Bible to them. He also loved to express himself on matters of philosophy. On September 18, 1783, after having discussed the topics of the day,

"He ceased to calculate and to live".

2 WORK OF LEONHARD PAUL EULER

Leonhard Euler was a student of John Bernoulli at the University, but he soon outstripped his teacher. His working life was spent as a member of the Academies of Science at Berlin and St. Petersburg, and most of his papers were published in the journals of these organizations. His business was mathematical research, and he knew his business [7]. He extended and perfected plane and solid analytic geometry, introduced the analytic approach to trigonometry, and was responsible for the modern treatment of the functions ln$x$ and e^x. He created a consistent theory of logarithms of negative and imaginary numbers, and discovered that ln$x$ has an infinite number of values. It was through his work that the symbols

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+\frac{1}{1!}+\frac{1}{2!}+\ldots \cong 2.71828\ldots,$$

$$\pi \cong 3.14159\ldots,$$

$$i = \sqrt{-1}\,,$$

became common currency for all mathematicians, and it was he who linked them together in the astonishing relation

$$e^{\pi i} = -1.$$

This is merely a special case of his famous formula

$$e^{i\theta} = \cos\theta + i\sin\theta\,,\ \theta = \pi,$$

which connects the exponential and trigonometric functions and is absolutely indispensable in higher analysis. Among his other contributions to standard mathematical notation were sin$x$, cos$x$, the use of $f(x)$ for an unspecified function, and the use of Σ for summation. Good notations are important, but the ideas behind them are what really count, and in this respect, Leonhard's fertility was almost beyond belief. He preferred concrete special problems to the general theories in vogue today, and his unique insight into the connections between apparently unrelated formulas blazed many trails into new areas of mathematics which he left for his successors to cultivate. He was the first and greatest master of infinite series, infinite products, and continued fractions, and his works are crammed with striking discoveries in these fields. James Bernoulli (John's older brother) found the sums of several infinite series, but he was not able to find the sum of the reciprocals of the squares,

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots\ .$$

He wrote, "If someone should succeed in finding this sum, and will tell me about it, I shall be much obliged to him." In 1736, long after James's death, Euler made the wonderful discovery that

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots = \frac{\pi^2}{6}\ .$$

When John heard about these feats, he wrote, "If only my brother were alive now."

His work in all departments of analysis strongly influenced the further development of this subject through the next three centuries.

He contributed many important ideas to *differential equations*, including substantial parts of the theory of second-order linear equations and the method of solution by power series. He gave the first systematic discussion of the *calculus of variations*, which he founded on his basic differential equation for a minimizing curve. He introduced the number now known as
Euler's constant,

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln n\right)=0.57721\ldots ,$$

which is *the most important special positive real number in mathematics after π and e.* He discovered the integral defining the gamma function

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt ,$$

which is often the first of the so-called higher transcendental functions that students meet beyond the level of calculus, and he developed many of its applications and special properties [9].
He also worked with *Fourier series*, encountered the Bessel functions in his study of the vibrations of a stretched circular membrane, and applied Laplace transforms to solve differential equations-all before Fourier, Bessel, and Laplace were born [7].
E.T. Bell, the well-known historian of mathematics, observed that
"One of the most remarkable features of Euler's universal genius was its equal strength in both of the main currents of mathematics, the continuous and the discrete." In the realm of the discrete, he was one of the originators of number theory and made many far- reaching contributions to this subject throughout his life.

In addition, the origins of *topology*-one of the dominant forces in modern mathematics-lie in his solution of the Königsberg bridge problem and his formula

$$V - E + F = 2$$

connecting the numbers of vertices, edges, and faces of a simple polyhedron. In the following paragraphs, we briefly describe his activities in these fields.

In *number theory,* Euler drew much of his inspiration from the challenging marginal notes left by Fermat in his copy of the works of Diophantus, and some of his achievements are mentioned in our account of Fermat.

He also initiated the *theory of partitions*, a little-known branch of number theory that turned out much later to have applications in statistical mechanics and the kinetic theory of gases. A typical problem of this subject is to determine the number $p(n)$ of ways in which a given positive integer

n can be expressed as a sum of positive integers, and if possible to discover some properties of this function.

For example, 4 can be partitioned into

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1,$$

so $p(4) = 5$ and similarly $p(5) = 7$ and $p(6) = 11$.

It is clear that $p(n)$ increases very rapidly wıth n. Euler began his investigations by noticing that $p(n)$ is the coefficient of x^n when the function

$$[(1 - x)\,(1 - x^2)\,(1 - x^3) \ldots\ ]^{-1}$$

is expanded in a power series:

$$1 + p(1)x + p(2)x^2 + p(3)x^3 + \ldots\ .$$

By building on this foundation, he derived many other remarkable identities related to a variety of problems about partitions.

The Königsberg bridge problem originated as a pastime of Sunday strollers in the town of Königsberg (now *Kaliningrad*) in what was formerly East Prussia. There were seven bridges across the river that flows through the town**.** The residents used to enjoy walking from one bank to the islands and then to the other bank and back again, and the conviction was widely held that it is impossible to do this by crossing αll seven bridges without crossing any bridge more than once. Euler analyzed the problem by examining the schematic diagram, in which the land areas are represented by points and the bridges by lines connecting these points. The points are called vertices, and a vertex is said to be odd or even according as the number of lines leading to it is odd or even. In modern terminology, the entire configuration is called *α graph,* and a path through the graph that traverses every line but no line more than once is called an *Euler path.* An Euler path need not end at the vertex where it began, but if it does, it is called an *Euler circuit.* By the use of combinatorial reasoning, Euler arrived at the following theorems about any such graph: (1) there are an even number of odd vertices; (2) if there are no odd vertices, there is an Euler circuit starting at any point; (3) if there are two odd vertices, there is no Euler circuit, but there is an Euler path starting at one odd vertex and ending at the other; (4) if there are more than two odd vertices, there are no Euler paths. The graph of the Königsberg bridges has four odd vertices, and therefore, by the last theorem, has no Euler paths. The branch of mathematics that has developed from these ideas is known as *graph theory*; it has applications to chemical bonding, economics, psychosociology, the properties of networks of roads and railroads, and other subjects [7].

In our note on Euclid we remarked that a polyhedron is a solid whose surface consists of a number of polygonal faces, and of that note displays the five regular polyhedra. Then it remained for Euler to discover the simplest of their common properties:

If V, E, and F denote the numbers of vertices, edges, and faces of any one of them, then in every case we have

$$V - E + F = 2 \quad .$$

This fact is known as *Euler's formula for polyhedra*. This formula is also valid for any irregular polyhedron as long as it is simple-which means that it has no "holes" in it, so that its surface can be deformed continuously into the surface of a sphere. However, Euler's formula must be extended to

$$V - E + F = 2 - 2p$$

in the case of a polyhedron with p holes (a simple polyhedron is one for which $p = 0$). The number $V - E + F$ has the same value for all maps on our surface, and is called the *Euler characteristic* of this surface.

The number p is called the *genus* of the surface. These two numbers, and the relation between them given by the equation

$$V - E + F = 2 - 2p,$$

are evidently unchanged when the surface is continuously deformed by stretching or bending. Intrinsic geometric properties of this kind-which have little connection with the type of geometry concerned with lengths, angles, and areas-are called *topological*. The serious study of such topological properties, has greatly increased during the past century, and has furnished valuable insights to many branches of mathematics and science.

Euler numbers $E_1, E_2, E_3, \ldots$:

$$\left.\begin{aligned} \sec x &= 1 + E_1 \frac{x^2}{2!} + E_2 \frac{x^4}{4!} + E_3 \frac{x^6}{6!} + \ldots \\ \sec h\, x &= 1 - E_1 \frac{x^2}{2!} + E_2 \frac{x^4}{4!} - E_3 \frac{x^6}{6!} + \ldots \end{aligned}\right. \quad , \; x < \frac{\pi}{2},$$

such that

$$E_1 = 1, E_2 = 5, E_3 = 61,$$

$$E_4 = 1385,\; E_5 = 50{,}521,\; E_6 = 2{,}702{,}765\,, \ldots$$

and

$$E_n = \frac{2^{2n+2}}{\pi^{2n+1}} (2n)! \{1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \ldots\}\,.$$

Euler differential equation (or Cauchy equation):

$$x^2 \frac{d^2 y}{dx^2} + P(x)\, x \frac{dy}{dx} + Q(x)\, y = R(x)\,.$$

3 TERMS CARRYING EULER'S NAME

Euler angles: Three angular parameters that specify the orientation of a body with respect to reference axes [5].

Euler characteristic of a topological space X:

$$\chi(X) = \sum (-1)^q \beta_q\ ,$$

where β_q is the *qth-Betti number* of *X*.

Euler diagram: A diagram consisting of closed curves, used to represent relations between logical propositions or sets; similar to a Venn diagram.

Euler equation: Expression for the energy removed from a gas stream by a rotating blade system (as a gas turbine), independent of the blade system (as a radial- or axial-flow system).

Euler equations of motion: A set of three differential equations expressing relations between the force moments, angular velocities, and angular accelerations of a rotating rigid body.

Euler force: The greatest load that a long, slender column can carry without buckling, according to the Euler formula for long columns.

Euler formula for long columns: A formula which gives the greatest axial load that a long, slender column can carry without buckling, in terms of its length, Young's modulus, and the moment of inertia about an axis along the center of the column.

Euler-Lagrange equation: A partial differential equation arising in the calculus of variations, which provides a necessary condition that y(x) minimize the integral over some finite interval of $f(x, y, y')dx$, where $y'=dy/dx$; the equation is

$$(\partial f(x, y, y')/\partial y) - (d/dx)(\partial f(x, y, y')/\partial y') = 0.$$

Also known as *Euler's equation.*

Euler method: A method of studying fluid motion and the mechanics of deformable bodies in which one considers volume elements at fixed locations in space, across which material flows; the Euler method is in contrast to the Lagrangian method.

Euler number 1: A dimensionless number used in the study of fluid friction in conduits, equal to the pressure drop due to friction divided by the product of the fluid density and the square of the fluid velocity.

Euler number 2: A dimensionless number equal to two times the Fanning friction factor. *Euler-Rodrigues parameter*: One of four numbers which may be used to specify the orientation of a rigid body; they are components of a quaternion.

Euler transformation: A method of obtaining from a given convergent series a new series which converges faster to the same limit, and for defining sums of certain divergent series; the transformation carries the series

$$a_0 - a_1 + a_2 - a_3 + \ldots$$

into a series whose *n*th term is ([4], [6], [8]):

$$\sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} a_r / 2^n ,$$

Eulerian coordinates: Any system of coordinates in which properties of a fluid are assigned to points in space at each given time, without attempt to identify individual fluid parcels from one time to the next; a sequence of synoptic charts is an Eulerian representation of the data ([1]-[2], [5]).

Eulerian correlation: The correlation between the properties of a flow at various points in space at a single instant of time. Also known as *synoptic correlation.*

Eulerian equation: A mathematical representation of the motions of a fluid in which the behavior and the properties of the fluid are described at fixed points in a coordinate system.

Eulerian path: A path that traverses each of the lines in a graph exactly once.

Eulerian wind: A wind motion only in response to the pressure force; the cyclostrophic wind is a special case of the Eulerian wind, which is limited in its meteorological applicability to those situations in which the Coriolis effect is negligible.

Euler's expansion: The transformation of a derivative (*d*/*dt*) describing the behavior of a moving particle with respect to time, into a local derivative (*δ*/*δt*) and three additional terms that describe the changing motion of a fluid as it passes through a fixed point. *Euler's formula* [3]:

$$e^{ix} = \cos x + i \sin x , \quad i = \sqrt{-1} .$$

Euler's theorem: For any polyhedron, $V - E + F = 2$, where V, E, F represent the number of vertices, edges, and faces ,respectively.

Euler-Lagrange quadratic identity ([6]-[12]):

$$|x+y|^2 + |x-y|^2 = 2[|x|^2 + |y|^2].$$

Euler scheme approximation and *Euler Monte Carlo Method*:

For numerical solution of ordinary and partial Stochastic Differential Equations (SDEs), [5].

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On the Double Quadratic Difference Property

Marcin Adam[1] **and Stefan Czerwik**[2]

[1]School of Labour Safety Management
Bankowa 8, PL-40-007 Katowice, Poland
e-mail: madam@wszop.edu.pl

[2]Institute of Mathematics
Silesian University of Technology
Kaszubska 23
PL-44-100 Gliwice, Poland
e-mail: Stefan.Czerwik@polsl.pl

ABSTRACT

Let X be a real normed space and Y a real Banach space. Denote by $C^n(X,Y)$ the class of n-times continuously differentiable functions $f\colon X \to Y$. We prove that the class C^n has so called the double quadratic difference property, that is if $Qf(x,y) := f(x+y) + f(x-y) - 2f(x) - 2f(y)$ and $Qf \in C^n(X \times X, Y)$, then there exists exactly one quadratic function $K\colon X \to Y$ such that $f - K \in C^n(X,Y)$.

Keywords: quadratic functional equation, double difference property, differentiable solutions of the quadratic functional equation.

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1 INTRODUCTION

Let X, Y be groups. A mapping $f\colon X \to Y$ is said to be quadratic iff it satisfies the following functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. We define Qf, the quadratic difference of f, by the formula

$$Qf(x,y) := f(x+y) + f(x-y) - 2f(x) - 2f(y) \tag{1.1}$$

for all $x, y \in X$.
Assume that X and Y are normed spaces. For a function $f\colon X \to Y$ we put

$$\|f\|_{\sup} := \sup_{x \in X} \|f(x)\|.$$

Let X, Y be a real normed space and a real Banach space, respectively. By $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{R}$ we denote the sets of all nonnegative integers, positive integers and real numbers, respectively. Let $f\colon X \to Y$ be an n-times differentiable function. By $D^n f$, $n \in \mathbb{N}$, we denote n-th derivative of f and $D^0 f$ stands for f. The space of all functions $f\colon X \to Y$ that are n-times differentiable on X will be denoted by $D^n(X,Y)$. By $C^n(X,Y)$ we denote the space of n-times continuously differentiable functions and $C^\infty(X,Y)$ stands for the space of infinitely many times continuously differentiable functions. By $\partial_k^n f$, $k = 1,2$, we denote, as usual n-th partial derivative of $f\colon X \times X \to Y$ with respect to the k-th variable.

Following an idea of J. Tabor and J. Tabor [9] we assume that we are given a norm in $X \times X$ such that $\|(x_1,x_2)\|$ is a function of $\|x_1\|$ and $\|x_2\|$ and the following condition is satisfied

$$\|(x,0)\| = \|(0,x)\| = \|x\|$$

for all $x \in X$.

Let $i_1\colon X \to X \times X$, $i_2\colon X \to X \times X$ be injections defined by

$$i_1(x) := (x,0),$$

$$i_2(y) := (0,y)$$

for all $x, y \in X$.

Let $L\colon X \times X \to X$ be a bounded linear mapping. It follows directly from the assumed conditions on the norm in $X \times X$ that

$$\|L \circ i_1\| \le \|L\| \; \|i_1\| = \|L\|,$$

$$\|L \circ i_2\| \le \|L\| \; \|i_2\| = \|L\|.$$

Therefore if $F\colon X \times X \to Y$ is n-times differentiable for $n \in \mathbb{N}$, then

$$\big\|\partial_1 F(x,y)\big\| = \big\|DF(x,y) \circ i_1\big\| \le \big\|DF(x,y)\big\|,$$

$$\big\|\partial_2 F(x,y)\big\| = \big\|DF(x,y) \circ i_2\big\| \le \big\|DF(x,y)\big\|$$

and

$$\left.\begin{aligned} \big\|\partial_1^{i-2}\partial_2^2 F(x,y)\big\| &\le \big\|D^i F(x,y)\big\|, \\ \big\|\partial_1^2\partial_2^{i-2} F(x,y)\big\| &\le \big\|D^i F(x,y)\big\| \end{aligned}\right\} \tag{1.2}$$

for all $x, y \in X$ and $i = 2,\ 3,\ \ldots,\ n.$

Let $n \in \mathbb{N}$ and let $f\colon X \to Y$ be n-times differentiable. Then Qf is n-times differentiable and by (1.2), we have

$$\big\|D^2 f(x+y) + D^2 f(x-y) - 2D^2 f(y)\big\| \le \big\|D^2(Qf)(x,y)\big\| \tag{1.3}$$

for all $x, y \in X$. Moreover, for $n \ge 3$, we obtain from (1.2)

$$\big\|D^i f(x+y) + D^i f(x-y)\big\| \le \big\|D^i(Qf)(x,y)\big\| \tag{1.4}$$

for all $x, y \in X$ and $i = 3,\ 4,\ \ldots,\ n.$

2 PRELIMINARIES

In the present section we will prove some lemmas that will play an essential role in further considerations.

Lemma 2.1. Let X, Y be abelian groups and $f\colon X \to Y$ be a function. Then Qf given by formula (1.1) satisfies the following functional equation

$$\begin{aligned} &Qf(x+s,y+t)+Qf(x-s,y-t)+2Qf(x,s)+2Qf(y,t) \\ &=Qf(x+y,s+t)+Qf(x-y,s-t)+2Qf(x,y)+2Qf(s,t) \end{aligned} \tag{2.1}$$

for all $x, y, s, t \in X$.

Proof. Using the definition (1.1) and calculating left and right sides of the equation (2.1), we get the desired conclusion. □

As a corollary from above lemma we obtain well know functional equations (cf. [2]).

Corollary 2.1. Let X, Y be abelian groups and $f\colon X \to Y$ be a function. Then Qf given by formula (1.1) satisfies the following functional equations

$$Qf(x+y,s)+Qf(x-y,s)+2Qf(x,y)=Qf(x+s,y)+Qf(x-s,y)+2Qf(x,s), \tag{2.2}$$

$$Qf(x+y,t)+Qf(x-y,t)+2Qf(x,y)=Qf(x,y+t)+Qf(x,y-t)+2Qf(y,t) \tag{2.3}$$

for all $x, y, s, t \in X$.

Proof. Setting $t = 0$ in (2.1) and again $s = 0$ in (2.1) and using the definition of Qf, we easily obtain (2.2) and (2.3), respectively. □

Lemma 2.2. Let X, Y be normed spaces and let $f\colon X \to Y$ be such a function that $Qf \in D^2(X \times X, Y)$. Then we have for all $x, y \in X$

$$\partial_2^2(Qf)(x+y,0)+\partial_2^2(Qf)(x-y,0)=2\partial_1^2(Qf)(x,y)+2\partial_2^2(Qf)(x,0), \tag{2.4}$$

$$\partial_2^2(Qf)(x+y,0)+\partial_2^2(Qf)(x-y,0)=2\partial_2^2(Qf)(x,y)+2\partial_2^2(Qf)(y,0). \tag{2.5}$$

Proof. Differentiating two times both sides of the equalities (2.2) and (2.3) with respect to s at the point $s = 0$ and with respect to t at the point $t = 0$, respectively, we obtain equations (2.4) and (2.5), which concludes the proof. □

From Lemmas 2.1 and 2.2 we easily obtain the following one.

Lemma 2.3. Let X, Y be normed spaces and let $f\colon X \to Y$ be such a function that $Qf \in D^2(X \times X, Y)$. Then we have

$$\partial_1(Qf)(0,0)=0, \tag{2.6}$$

$$\partial_{12}^2(Qf)(0,0)=0, \tag{2.7}$$

$$\partial_2^2(Qf)(0,0)=0. \tag{2.8}$$

Proof. Differentiating both sides of the equality (2.3) with respect to x at the point $x = 0$, we obtain (2.6) for $y = t = 0$. Differentiating two times both sides of the equality (2.3) with respect to x at the point $x = 0$ and next with respect to t at the point $t = 0$, we get (2.7) for $y = 0$. From (2.5), we easily obtain (2.8) for $x = y = 0$. □

S. Czerwik (see [2]) proved the following lemma for an even function $f\colon X \to Y$ such that $f \in D^2(X, Y)$. We prove this lemma for a weaker condition $Qf \in D^2(X \times X, Y)$, where f is any function.

Lemma 2.4. Let X, Y be normed spaces and let $f\colon X \to Y$ be such a function that $Qf \in D^2(X \times X, Y)$. Then we have for all $x, y \in X$

$$2\partial_{12}^2(Qf)(x,y) = \partial_2^2(Qf)(x+y,0) - \partial_2^2(Qf)(x-y,0). \tag{2.9}$$

Proof. Differentiating two times both sides of the equality (2.1) with respect to s at the point $s = 0$ and next with respect to t at the point $t = 0$, we obtain

$$2\partial_{12}^2(Qf)(x,y) = \partial_2^2(Qf)(x+y,0) - \partial_2^2(Qf)(x-y,0) + 2\partial_{12}^2(Qf)(0,0).$$

Since from (2.7) we have $\partial_{12}^2(Qf)(0,0) = 0$, so we get the desired conclusion. □

3 DOUBLE QUADRATIC DIFFERENCE PROPERTY

Let X and Y denote a real normed space and a real Banach space, respectively. We will prove that the class $C^n(X, Y)$ has so called the double quadratic difference property, i.e. if $f\colon X \to Y$ is such a function that $Qf \in C^n(X \times X, Y)$, then there exists exactly one quadratic function $K\colon X \to Y$ such that $f - K \in C^n(X, Y)$. The problem of the double difference property for a Cauchy difference $Cf \in C^n(X \times X, Y)$ has been investigated in the paper [9]. For more details about the double difference property the reader is referred to [6].

Lemma 3.1. Let $f\colon \mathbb{R} \to Y$ be a function and let $K\colon \mathbb{R} \to Y$ be a quadratic function such that $f - K \in C^2(X, Y)$ and $D^2(f - K)(0) = 0$. Then

$$K(x) = \frac{1}{2}\lim_{n\to\infty} n^2\left[f\left(2\frac{x}{n}\right) - 2f\left(\frac{x}{n}\right) + f(0)\right]$$

for all $x \in \mathbb{R}$.

Proof. Since $D^2(f - K)(0) = 0$, we have for all $x \in \mathbb{R}$, $x \neq 0$

$$\lim_{h\to 0}\frac{\left\|f(2h) - K(2h) - 2f(h) + 2K(h) + f(0) - K(0)\right\|}{h^2}$$
$$= \lim_{h\to 0}\frac{\left\|f(2h) - 2f(h) - 2K(h) + f(0)\right\|}{h^2}$$
$$= \lim_{n\to\infty}\frac{\left\|f\left(2\frac{x}{n}\right) - 2f\left(\frac{x}{n}\right) - 2K\left(\frac{x}{n}\right) + f(0)\right\|}{\left(\frac{x}{n}\right)^2} = 0,$$

i.e.

$$\lim_{n\to\infty}\left\|2K(x) - n^2\left[f\left(2\frac{x}{n}\right) - 2f\left(\frac{x}{n}\right) + f(0)\right]\right\| \cdot \frac{1}{x^2} = 0.$$

Then we have

$$K(x) = \frac{1}{2}\lim_{n\to\infty} n^2 \left[f\left(2\frac{x}{n}\right) - 2f\left(\frac{x}{n}\right) + f(0)\right]$$

for all $x \in \mathbb{R}$. It obviously also holds for $x = 0$. □

Theorem 3.1. Let $n \in \mathbb{N}\cup\{\infty\}$ and let $f\colon \mathbb{R} \to Y$ be such a function that $Qf \in C^n(\mathbb{R}\times\mathbb{R}, Y)$ for $n \geq 2$. Then there exists exactly one quadratic function $K\colon \mathbb{R} \to Y$ such that $f - K \in C^n(\mathbb{R}, Y)$ and $D^2(f-K)(0) = 0$. Moreover

$$\left\|D^k(f-K)(x)\right\| \leq \frac{1}{2}\left\|D^k(Qf)(x,0)\right\|, \qquad k \in \mathbb{N}\backslash\{1\},\ k \leq n$$

for all $x \in \mathbb{R}$.

Proof. Let $f_1(x) := f(x) - f(0)$ for all $x \in \mathbb{R}$. Then $Qf_1 = Qf + 2f(0) \in C^n(\mathbb{R}\times\mathbb{R}, Y)$ and $Qf_1(0,0) = 0$. Let us fix arbitrarily $x, y \in \mathbb{R}$ and consider a function

$$\varphi(t) := Qf_1(tx, ty)$$

for all $t \in \mathbb{R}$. Obviously $\varphi \in C^n(\mathbb{R}, Y)$. Then we have

$$D\varphi(t) = \partial_1(Qf_1)(tx,ty)(x) + \partial_2(Qf_1)(tx,ty)(y)$$

for all $t \in \mathbb{R}$. Hence and from (2.6), we get

$$D\varphi(0) = \partial_2(Qf_1)(0,0)(y).$$

Therefore we obtain

$$Qf_1(x,y) = \varphi(1) - \varphi(0) = \int_0^1 D\varphi(t)dt = \int_0^1\int_0^t D^2\varphi(u)dudt + D\varphi(0)$$

$$= \int_0^1\int_0^t D^2(Qf_1)(ux,uy)(x,y)dudt + \partial_2(Qf_1)(0,0)(y)$$

$$= \int_0^1\int_0^t \partial_1^2(Qf_1)(ux,uy)(x^2)dudt + 2\int_0^1\int_0^t \partial_{12}^2(Qf_1)(ux,uy)(xy)dudt$$

$$+ \int_0^1\int_0^t \partial_2^2(Qf_1)(ux,uy)(y^2)dudt + \partial_2(Qf_1)(0,0)(y).$$

Thus

$$Qf_1(x,y) = \int_0^1\int_0^t \partial_1^2(Qf_1)(ux,uy)(x^2)dudt + 2\int_0^1\int_0^t \partial_{12}^2(Qf_1)(ux,uy)(xy)dudt$$

$$+ \int_0^1\int_0^t \partial_2^2(Qf_1)(ux,uy)(y^2)dudt + \partial_2(Qf_1)(0,0)(y) \tag{3.1}$$

for all $x, y \in \mathbb{R}$. We define the function $K\colon \mathbb{R} \to Y$ by the formula

$$K(x) := f_1(x) + \frac{1}{2}\partial_2(Qf_1)(0,0)(x) - \frac{1}{2}\int_0^1\int_0^t \partial_2^2(Qf_1)(ux,0)(x^2)dudt$$

for all $x \in \mathbb{R}$. We show that K is a quadratic function. Making use of (2.4), (2.5), (2.9) and (3.1), we obtain for all $x, y \in \mathbb{R}$

$$K(x+y) + K(x-y) - 2K(x) - 2K(y) = Qf_1(x,y) - \partial_2(Qf_1)(0,0)(y)$$

$$-\frac{1}{2}\int_0^1\int_0^t \partial_2^2(Qf_1)(ux+uy,0)(x+y)^2dudt - \frac{1}{2}\int_0^1\int_0^t \partial_2^2(Qf_1)(ux-uy,0)(x-y)^2dudt$$

$$+\int_0^1\int_0^t \partial_2^2(Qf_1)(ux,0)(x^2)dudt + \int_0^1\int_0^t \partial_2^2(Qf_1)(uy,0)(y^2)dudt$$

$$= \int_0^1\int_0^t \partial_1^2(Qf_1)(ux,uy)(x^2)dudt + 2\int_0^1\int_0^t \partial_{12}^2(Qf_1)(ux,uy)(xy)dudt$$

$$+\int_0^1\int_0^t \partial_2^2(Qf_1)(ux,uy)(y^2)dudt - \frac{1}{2}\int_0^1\int_0^t \partial_2^2(Qf_1)(ux+uy,0)(x^2)dudt$$

$$-\int_0^1\int_0^t \partial_2^2(Qf_1)(ux+uy,0)(xy)dudt - \frac{1}{2}\int_0^1\int_0^t \partial_2^2(Qf_1)(ux+uy,0)(y^2)dudt$$

$$-\frac{1}{2}\int_0^1\int_0^t \partial_2^2(Qf_1)(ux-uy,0)(x^2)dudt + \int_0^1\int_0^t \partial_2^2(Qf_1)(ux-uy,0)(xy)dudt$$

$$-\frac{1}{2}\int_0^1\int_0^t \partial_2^2(Qf_1)(ux-uy,0)(y^2)dudt + \int_0^1\int_0^t \partial_2^2(Qf_1)(ux,0)(x^2)dudt$$

$$+\int_0^1\int_0^t \partial_2^2(Qf_1)(uy,0)(y^2)dudt$$

$$= \int_0^1\int_0^t \Big[\partial_1^2(Qf_1)(ux,uy) - \frac{1}{2}\partial_2^2(Qf_1)(ux+uy,0) - \frac{1}{2}\partial_2^2(Qf_1)(ux-uy,0)$$

$$+\partial_2^2(Qf_1)(ux,0)\Big](x^2)dudt$$

$$+\int_0^1\int_0^t \Big[2\partial_{12}^2(Qf_1)(ux,uy) - \partial_2^2(Qf_1)(ux+uy,0) + \partial_2^2(Qf_1)(ux-uy,0)\Big](xy)dudt$$

$$+\int_0^1\int_0^t \Big[\partial_2^2(Qf_1)(ux,uy) - \frac{1}{2}\partial_2^2(Qf_1)(ux+uy,0) - \frac{1}{2}\partial_2^2(Qf_1)(ux-uy,0)$$

$$+\partial_2^2(Qf_1)(uy,0)\Big](y^2)dudt = 0.$$

Therefore K is a quadratic function.
Now we prove that $f-K$ is differentiable. Applying the definition of the derivative of the function $f_1 - K$ and using the substitution method for the double integral, one can prove the formula

$$D\left[\int_0^1\int_0^t \partial_2^2(Qf_1)(ux,0)(x^2)dudt\right] = \int_0^x \partial_2^2(Qf_1)(s,0)ds \tag{3.2}$$

for all $x \in \mathbb{R}$.
Now applying (3.2) we see that $f - K = f_1 - K + f(0)$ is differentiable at every point $x \in \mathbb{R}$ and that

$$D(f-K)(x) = \frac{1}{2}\int_0^x \partial_2^2(Qf)(s,0)ds - \frac{1}{2}\partial_2(Qf)(0,0),$$

$$D^2(f-K)(x) = \frac{1}{2}\partial_2^2(Qf)(x,0).$$

Then $\partial_2^2(Qf)(x,0) \in C^{n-2}(\mathbb{R}\times\mathbb{R}, Y)$, and hence $\partial_2^2(Qf)(x,0) \in C^{n-2}(\mathbb{R}, Y)$. Therefore $D^2(f-K) \in C^{n-2}(\mathbb{R}, Y)$, i.e. $f - K \in C^n(\mathbb{R}, Y)$. Moreover, from (2.8) we have

$$D^2(f-K)(0) = \frac{1}{2}\partial_2^2(Qf)(0,0) = 0.$$

The proof of the uniqueness part of the theorem follows directly from the Lemma 3.1.
We claim that

$$\|D^k(f-K)(x)\| \le \frac{1}{2}\|D^k(Qf)(x,0)\|, \qquad k \in \mathbb{N}\backslash\{1\},\ k \le n \tag{3.3}$$

for all $x \in \mathbb{R}$. Let $f_1 := f-K$. Then $f_1 \in C^n(\mathbb{R}, Y)$ and consequently $Qf_1 = Qf \in C^n(\mathbb{R}\times\mathbb{R}, Y)$. Making use of (1.3) and the fact that $D^2(f-K)(0) = 0$, we obtain for all $x \in \mathbb{R}$

$$\|D^2 f_1(x)\| = \|D^2 f_1(x) - D^2 f_1(0)\| \le \frac{1}{2}\|D^2(Qf_1)(x,0)\|,$$

which proves (3.3) for $k = 2$. For $3 \le k \le n$, $k \in \mathbb{N}$, (3.3) follows directly from (1.4). The prove is complete. □

Corollary 3.1. Under the assumptions of Theorem 3.1 we have

$$\|D^k(f-K)(0)\| \le \frac{1}{2}\|D^k(Qf)(0,0)\|, \qquad k \in \mathbb{N}_0\backslash\{1\},\ k \le n,$$

$$\|D^k(f-K)\|_{\sup} \le \frac{1}{2}\|D^k(Qf)\|_{\sup}, \qquad k \in \mathbb{N}\backslash\{1\},\ k \le n.$$

Theorem 3.1 states, in particular, that the class of infinitely many times differentiable functions has the double quadratic difference property. We show that the class of analytic functions also has this property.

Corollary 3.2. Let $f\colon \mathbb{R} \to Y$ be a function such that Qf is analytic. Then there exists exactly one quadratic function $K\colon \mathbb{R} \to Y$ such that $f - K$ is analytic and $D^2(f-K)(0) = 0$.

Proof. By Theorem 3.1 there exists exactly one quadratic function $K\colon \mathbb{R} \to Y$ such that $F := f - K \in C^\infty(\mathbb{R}, Y)$ and $D^2F(0) = 0$. Then obviously $QF = Qf$ and hence QF is analytic. Making use of the equality

$$\frac{1}{2}\partial_2^2(QF)(x,0) = D^2F(x) - D^2F(0)$$

for all $x \in \mathbb{R}$, we obtain that D^2F is analytic, and consequently that F is analytic. □

Theorem 3.1 can be applied to the problem of Ulam-Hyers stability of the quadratic functional equation in some special class of differentiable functions with special norms (see [1]).

Remark. Theorem 3.1 can be also proved for a function defined on a normed space X.
For more information concerning similar problems and applications to the stability of functional equations the reader is referred to [3], [4], [5], [7], [8].

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Ulam-Găvruta-Rassias stability of the Pexider functional equation

Bouikhalene belaid[1] **and Elqorachi elhoucien**[2]

[1]University of Ibn Tofail, Faculty of Sciences,
Department of Mathematics, Kenitra,Morocco,
Email: bbouikhalene@yahoo.fr

[2]University of Ibn Zohr, Faculty of Sciences,
Department of Mathematics, Laboratory LAMA,
Harmonic Analysis and functional equations Team, Agadir, Morocco,
Email: elqorachi@hotmail.com

ABSTRACT

We will investigate the Ulam-Găvruta-Rassias stability of the Pexider functional equation

$$f_1(x+y) + f_2(x+\sigma(y)) = f_3(x) + f_4(y), \quad x, y \in E,$$

where E is a vector space and σ: $E \longrightarrow E$ is an involution.

Keywords: functional equation, Ulam-Găvruta-Rassias stability.

2000 Mathematics Subject Classification: 39B72.

1 Introduction

In 1940, S. M. Ulam [36] posed the following problem. Let E_1 be a group and let E_2 be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function f: $E_1 \longrightarrow E_2$ satisfies the functional inequality $d(f(xy), f(x)f(y)) \leq \delta$ for all x and y in E_1, then there exists a group homomorphism h: $E_1 \longrightarrow E_2$ with $d(f(x), h(x)) \leq \varepsilon$ for any $x \in E_1$.

The case of approximately additive mappings was solved by D. H. Hyers [8] under the assumption that G_1 and G_2 are Banach spaces.
In 1978, Th. M. Rassias [29] gave a generalization of the Hyers's result which allows the Cauchy difference to be unbounded, as follows:

Theorem 1.1. *[29] Let $f : V \longrightarrow X$ be a mapping between Banach spaces and let $p < 1$ be fixed. If f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|x\|^p) \tag{1.1}$$

for some $\theta \geq 0$ and for all $x, y \in V$ ($x, y \in V \setminus \{0\}$ if $p < 0$). Then there exists a unique additive mapping $T : V \longrightarrow X$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p \tag{1.2}$$

for all $x \in V$ ($x \in V \setminus \{0\}$ if $p < 0$).
If, in addition, $f(tx)$ is continuous in t for each fixed x, then T is linear.

In 1990, Th. M. Rassias during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for values of p greater or equal to one. Gajda Z. [6] following the same approach as in [29] provided an affirmative solution to Th. M. Rassias'question for p strictly greater than one. However, it was shown independently by J. Gajda [6] and Th. M. Rassias and P. Semrl [32] that a similar result for the case of value of p equal to one can not be obtained.

In 1982, J. M. Rassias [15] provided a generalization of Hyers's stability Theorem which allows the Cauchy difference to be unbounded, as follows:

Theorem 1.2. *[15] Let $f : E \longrightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the following Cauchy-Gavrută-Rassias inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon\|x\|^p\|y\|^p$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $0 \leq p < 1/2$. Then the limit

$$L(x) = \lim_{n \longrightarrow +\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and L: $E \longrightarrow E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\varepsilon}{2 - 2^{2p}}\|x\|^{2p} \qquad (*)$$

for all $x \in E$. If $p < 0$ then the above Cauchy-Gavrută-Rassias inequality holds for $x, y \neq 0$ and inequality $(*)$ for $x \neq 0$.
If $p > 1/2$ then the Cauchy-Gavrută-Rassias inequality holds for all $x, y \in E$ and the limit

$$A(x) = \lim_{n \longrightarrow +\infty} 2^n f(\frac{x}{2^n})$$

exists for all $x \in E$ and A: $E \longrightarrow E'$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{\varepsilon}{2^{2p} - 2}\|x\|^{2p}$$

for all $x \in E$. If in addition f: $E \longrightarrow E'$ is a mapping such that the transformation $t \longrightarrow f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then L is $\mathbb{R}$-linear mapping.

The case $p = 1/2$ in Cauchy-Gavrută-Rassias inequality is singular. A counter-example has been given by P. Gavrută ("An answer to a question of John M. Rassias concerning the stability of Cauchy equation", in: Advances in Equations and Inequalities, in: Hadronic Math. Ser. (1999), 67-71).

In 1994 P. Gavrută [7] provided a generalization of the above theorems by replacing the functions:
$(x, y) \longmapsto \theta(\| x \|^p + \| y \|^p)$; $(x, y) \longmapsto \theta(\| x \|^p \| y \|^p)$ by a mapping $\varphi(x, y)$ which satisfies the following condition

$$\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty \quad \text{or} \quad \sum_{n=0}^{\infty} 2^n \varphi(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}) < \infty \tag{1.3}$$

for every $x, y \in E_1$.
However in 1992; 1994, J. M. Rassias [18]-[19] provided stability results analogous to Gavrută's stability results [7] achieved in 1994.

Theorem 1.3. *[18] Let X be a real normed linear space and let Y be a complete normed linear space. Assume in addition that $f\colon X \longrightarrow Y$ is an approximately additive mapping for which there exists a constant $\theta \geq 0$ such that f satisfies generalized Cauchy-Gavrută-Rassias inequality*

$$\| f(\sum_{i=1}^{n} x_i) - \sum_{i=1}^{n} f(x_i) \| \leq \theta K(x_1, x_2, ..., x_n)$$

for all $(x_1, x_2, ..., x_n) \in X^n$ and $K\colon X^n \longrightarrow \mathbb{R}^+ \bigcup\{0\}$ is a non-negative real-valued function such that

$$R_n = R_n(x) = \sum_{j=0}^{\infty} n^{-j} K(n^j x, n^j x, ..., n^j x)(< \infty)$$

is a non-negative function of x, and the condition

$$\lim_{m \longrightarrow \infty} n^{-m} K(n^m x_1, n^m x_2, ..., n^m x_n) = 0$$

holds. Then there exists a unique additive mapping L_n: $X \longrightarrow Y$ satisfying

$$\| f(x) - L_n(x) \| \leq \frac{\theta}{n} R_n(x)$$

for all $x \in X$. If in addition $f\colon X \longrightarrow Y$ is a mapping such that the transformation $t \longrightarrow f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L_n is an $\mathbb{R}$-linear mapping.

Theorem 1.4. *[18] Let X be a real normed linear space and let Y be a complete normed linear space. Assume in addition that $f\colon X \longrightarrow Y$ is an approximately additive mapping such that f satisfies generalized Cauchy-Gavrută-Rassias inequality*

$$\| f(\sum_{i=1}^{n} x_i) - \sum_{i=1}^{n} f(x_i) \| \leq$$

N($x_1, x_2, ..., x_n) forall($x_1, x_2, ..., x_n) \in X^n$ and $N\colon X^n \longrightarrow \mathbb{R}^+ \bigcup\{0\}$ is a non-negative real-valued function such that $N(x, x, ..., x)$ is bounded on the unit ball of X, and

$$N(tx_1, tx_2, ..., tx_n) \leq k(t) N(x_1, x_2, ..., x_n)$$

for all $t \geq 0$, where $k(t) < \infty$ and

$$R_n^0 = R_n^0(x) = \sum_{j=0}^{\infty} n^{-j} k(n^j) < \infty.$$

If in addition f: $X \longrightarrow Y$ is a mapping such that the transformation $t \longrightarrow f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and f: $X \longrightarrow Y$ is bounded on some ball of X, then there exists a unique $\mathbb{R}$-linear mapping L_n: $X \longrightarrow Y$ satisfying

$$\| f(x) - L_n(x) \| \leq MN(x, x, ..., x),$$

for all $x \in X$, where $M = \sum_{m=0}^{\infty} n^{-(m+1)} k(n^m)$.

Stability problems of various functional equations have been extensively investigated by a number of authors. The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds. For more detailed definitions and further developments of stability concepts one is referred to [5], [9], [10], [12], [15], [16], [17], [19], [20], [21], [22]...
The generalized quadratic functional equation

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y) \quad x, y \in E_1, \tag{1.4}$$

where σ is an involution of E_1, i.e. $\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$ was solved by H. Stetkær in [35]. More precisely, a mapping f: $E_1 \longrightarrow E_2$ between vector space is a solution of the quadratic functional equation (1.4) if and only if there exists an additive function a: $E_1 \longrightarrow E_2$ and a biadditive, symmetric mapping b: $E_1 \times E_1 \longrightarrow E_2$ such that $a(\sigma(x)) = -a(x)$, $b(x, \sigma(y)) = -b(x, y)$ and $f(x) = b(x, x) + a(x)$ for any $x \in E_1$ and $y \in E_1$. This formulas is explained in [35] for the case where E_1 is an abelian group and E_2 is the set of the complex numbers, but the some proof holds for functions between vectors spaces.
A stability theorem for the functional equation (1.4) with $\sigma = -I$ was proved by F. Skof [34] for functions f: $E_1 \longrightarrow E_2$, where E_1 is a normed space and E_2 a Banach space. Her proof also works if E_1 is replaced by an abelian group G, the theorem was proved by P. W. Cholewa [3].
In a previous papers [1], [2] the authors and Th. M. Rassias deal with the Hyers-Ulam stability of the functional equation (1.4). It is convenient to state the stability theorem obtained in [1], the result will be used later.

Theorem 1.5. *[1] Let $(E_1, \|\|)$ be a real normed space and let $(E_2, \|\|)$ a Banach space. Let φ: $E_1 \times E_1 \longrightarrow [0, \infty)$ a mapping such that*

$$\psi(x,y) = \sum_{n=0}^{\infty} 2^{2(n+1)} [\varphi(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}) + (1 - \frac{1}{2^{n+1}})\varphi(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}, \frac{y}{2^{n+2}} + \frac{\sigma(y)}{2^{n+2}})] < \infty \tag{1.5}$$

for all $x, y \in E_1$.
Assume that the function f: $E_1 \longrightarrow E_2$ satisfies the inequality

$$\| f(x+y) + f(x+\sigma(y)) - 2f(x) - 2f(y) \| \leq \varphi(x, y) \tag{1.6}$$

for all $x, y \in E_1$. Then, there exists a unique solution q: $E_1 \longrightarrow E_2$ of equation (1.4) satisfying

$$\| f(x) - q(x) \| \leq \frac{1}{4}\psi(x, x) \tag{1.7}$$

for all $x \in E_1$.

In the present paper the stability problem in the spirit of Găvruta of the Pexider functional equation

$$f_1(x+y) + f_2(x+\sigma(y)) = f_3(x) + f_4(y) \tag{1.8}$$

shall be proved. The results are a natural extension of the ones obtained by the authors and Th. M. Rassias in [1] and [2] and by S.-M. Jung [13].

2 Stability of the Pexider functional equation

Throughout this section $(E_1, \|\|)$ be a real normed space and $(E_2, \|\|)$ a Banach space, φ: $E_1 \times E_1 \longrightarrow [0, \infty)$ is a given mapping which satisfies the following conditions: $\varphi(x, y) = \varphi(y, x)$; $\varphi(x, \sigma(y)) = \varphi(x, y)$ and

$$\psi(x,y) = \sum_{n=0}^{\infty} 2^{2(n+1)}[\varphi(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}) + (1 - \frac{1}{2^{n+1}})\varphi(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}, \frac{y}{2^{n+2}} + \frac{\sigma(y)}{2^{n+2}})] < \infty \quad (2.1)$$

for all $x, y \in E_1$.
It is easy to verify that $\psi(0,0) = \varphi(0,0) = 0$.

For any function φ which satisfies the above conditions, we introduce the new functions

$$\phi(x,y) = 2\varphi(x,y) + \varphi(x + \sigma(y), 0) + \varphi(x + y, 0)$$

$$\omega(x,y) = \sum_{n=0}^{\infty} 2^n \phi(\frac{x}{2^n}, \frac{y}{2^n}).$$

In this section, we will prove the Ulam-Gavrută-Rassias stability of the Pexider functional equation

$$f_1(x+y) + f_2(x + \sigma(y)) = f_3(x) + f_4(y) \;\; x, y \in E_1 \quad (2.2)$$

by following some ideas from the papers [1] and [13].

Theorem 2.1. *If the functions f_1, f_2, f_3, f_4: $E_1 \longrightarrow E_2$ satisfy the inequality*

$$\| f_1(x+y) + f_2(x+\sigma(y)) - f_3(x) - f_4(y) \| \leq \varphi(x,y), \quad (2.3)$$

for all $x, y \in E_1$, then there exists a unique function q: $E_1 \longrightarrow E_2$ solution of equation (1.4), there exists a function ν: $E_1 \longrightarrow E_2$ solution of equation

$$\nu(x+y) = \nu(x + \sigma(y)), \;\; x, y \in E_1, \quad (2.4)$$

there exist exactly two additive functions a_1, a_2: $E_1 \longrightarrow E_2$ such that $a_i \circ \sigma = -a_i$ ($i = 1, 2$),

$$\| f_1(x) - \frac{1}{2}a_1(x) - \frac{1}{2}a_2(x) - \frac{1}{2}\nu(x) - \frac{1}{2}q(x) - f_1(0) \| \quad (2.5)$$

$$\leq \omega(x,x) + \frac{1}{4}\psi(x,x) + \frac{1}{2}\psi(x,0) + \varphi(\frac{x}{2}, \frac{x}{2}) + 2\varphi(\frac{x}{2}, 0) + \varphi(x, 0),$$

$$\| f_2(x) - \frac{1}{2}a_1(x) + \frac{1}{2}a_2(x) + \frac{1}{2}\nu(x) - \frac{1}{2}q(x) - f_2(0) \| \quad (2.6)$$

$$\leq \omega(x,x) + \frac{1}{4}\psi(x,x) + \frac{1}{2}\psi(x,0) + \varphi(\frac{x}{2}, \frac{x}{2}) + 2\varphi(\frac{x}{2}, 0) + \varphi(x, 0),$$

$$\| f_3(x) - a_2(x) - q(x) - f_3(0) \| \leq \varphi(x,0) + \frac{1}{2}\psi(x,x) + \psi(x,0) + \omega(x,x) \quad (2.7)$$

and

$$\| f_4(x) - a_1(x) - q(x) - f_4(0) \| \leq \varphi(x,0) + \frac{1}{2}\psi(x,x) + \psi(x,0) + \omega(x,x). \quad (2.8)$$

for all $x \in E_1$.

Proof. In the proof, we use ideas and methods that are analogous to those used in [1] for the case where $\varphi(x,y)=\delta$, for some $\delta>0$. For any function f_i: $E_1 \longrightarrow E_2$, we put $F_i = f_i - f_i(0)$ and we denote by F_i^e (resp. F_i^o) the even part $F_i^e(x)=\frac{F_i(x)+F_i(\sigma(x))}{2}$ and the odd part $F_i^o(x)=\frac{F_i(x)-F_i(\sigma(x))}{2}$ of F_i.
By replacing x and y in (2.3) by $\sigma(x)$ and $\sigma(y)$, respectively, we obtain

$$\| F_1(\sigma(x)+\sigma(y)) + F_2(\sigma(x)+y) - F_3(\sigma(x)) - F_4(\sigma(y)) \| \leq \varphi(\sigma(x),\sigma(y)) = \varphi(x,y). \quad (2.9)$$

Consequently, from inequalities (2.3)-(2.9) and the triangle inequality, we get

$$\| F_1^o(x+y) + F_2^o(x+\sigma(y)) - F_3^o(x) - F_4^o(y) \| \leq \varphi(x,y) \quad (2.10)$$

and

$$\| F_1^e(x+y) + F_2^e(x+\sigma(y)) - F_3^e(x) - F_4^e(y) \| \leq \varphi(x,y), \quad (2.11)$$

for all $x, y \in E_1$.
Thus, substituting $y=0$ in inequality (2.11), one gets

$$\| F_1^e(x) + F_2^e(x) - F_3^e(x) \| \leq \varphi(x,0), \quad (2.12)$$

for all $x \in E_1$
Taking $x=0$ in inequality (2.11)

$$\| F_1^e(y) + F_2^e(y) - F_4^e(y) \| \leq \varphi(0,y), \quad (2.13)$$

for all $y \in E_1$.
Hence the inequalities (2.3)-(2.12)-(2.13) imply

$$\begin{aligned} &\| F_1^e(x+y) + F_2^e(x+\sigma(y)) - (F_1^e+F_2^e)(x) - (F_1^e+F_2^e)(y) \| \quad (2.14)\\ &\leq \| F_1^e(x+y) + F_2^e(x+\sigma(y)) - F_3^e(x) - F_4^e(y) \| \\ &+ \| F_1^e(x) + F_2^e(x) - F_3^e(x) \| + \| F_1^e(y) + F_2^e(y) - F_4^e(y) \| \\ &\leq \varphi(x,y) + \varphi(x,0) + \varphi(y,0). \end{aligned}$$

If we add (subtract) the argument of the norm of the inequality (2.14) to the like inequality by replacing y by $\sigma(y)$, we obtain

$$\begin{aligned} &\| (F_1^e+F_2^e)(x+y) + (F_1^e+F_2^e)(x+\sigma(y)) - 2(F_1^e+F_2^e)(x) - 2(F_1^e+F_2^e)(y) \| \quad (2.15)\\ &\leq 2\varphi(x,y) + 2\varphi(x,0) + 2\varphi(y,0) \end{aligned}$$

and

$$\| (F_1^e-F_2^e)(x+y) - (F_1^e-F_2^e)(x+\sigma(y)) \| \leq 2\varphi(x,y) + 2\varphi(x,0) + 2\varphi(y,0). \quad (2.16)$$

Hence, there exists ν: $E_1 \longrightarrow E_2$ defined by $\nu(x) = (F_1^e - F_2^e)(\frac{x+\sigma(x)}{2})$ solution of equation

$$\nu(x+y) = \nu(x+\sigma(y)), \quad x, y \in E_1 \quad (2.17)$$

that satisfies the inequality

$$\| (F_1^e - F_2^e)(x) - \nu(x) \| \leq 2\varphi(\frac{x}{2}, \frac{x}{2}) + 4\varphi(\frac{x}{2}, 0) \tag{2.18}$$

for all $x \in E_1$.
Applying Theorem 1.5 and the inequality (2.15), one gets that there exists a unique mapping q solution of equation (1.4) such that

$$\| (F_1^e + F_2^e)(x) - q(x) \| \leq \frac{1}{2}\psi(x,x) + \psi(x,0) \text{ for every } x \in E_1. \tag{2.19}$$

Therefore, from (2.18)-(2.19) and the triangle inequality, we obtain

$$\| 2F_2^e(x) - q(x) + \nu(x) \| \tag{2.20}$$
$$\leq \frac{1}{2}\psi(x,x) + \psi(x,0) + 2\varphi(\frac{x}{2}, \frac{x}{2}) + 4\varphi(\frac{x}{2}, 0),$$

$$\| 2F_1^e(x) - q(x) - \nu(x) \| \tag{2.21}$$
$$\leq \frac{1}{2}\psi(x,x) + \psi(x,0) + 2\varphi(\frac{x}{2}, \frac{x}{2}) + 4\varphi(\frac{x}{2}, 0).$$

Thus, from inequalities (2.12)-(2.13) and the triangle inequality, we establish

$$\| F_3^e(x) - q(x) \| \leq \varphi(x,0) + \frac{1}{2}\psi(x,x) + \psi(x,0) \tag{2.22}$$

$$\| F_4^e(x) - q(x) \| \leq \varphi(x,0) + \frac{1}{2}\psi(x,x) + \psi(x,0). \tag{2.23}$$

Now, if we put $y = 0$ resp. $x = 0$ in (2.10), one finds

$$\| F_3^o(x) - F_1^o(x) - F_2^o(x) \| \leq \varphi(x,0) \tag{2.24}$$

resp.

$$\| F_4^o(y) - F_1^o(y) + F_2^o(y) \| \leq \varphi(0,y), \tag{2.25}$$

for all $x, y \in E_1$.
Thus, from inequalities (2.3)-(3.25) and the triangle inequality, we get

$$\| F_4^o(x+y) + F_4^o(x+\sigma(y)) - 2F_4^o(x) \| \tag{2.26}$$
$$\leq \| F_4^o(x+y) - F_1^o(x+y) + F_2^o(x+y) \|$$
$$+ \| F_4^o(x+\sigma(y)) - F_1^o(x+\sigma(y)) + F_2^o(x+\sigma(y)) \|$$
$$+ \| F_1^o(y+x) + F_2^o(y+\sigma(x)) - F_3^o(y) - F_4^o(x) \|$$
$$+ \| F_1^o(\sigma(y)+x) + F_2^o(\sigma(y)+\sigma(x)) - F_3^o(\sigma(y)) - F_4^o(x) \|$$
$$\leq \phi(x,y) = 2\varphi(x,y) + \varphi(x+\sigma(y),0) + \varphi(x+y,0)$$

for all $x, y \in E_1$.
Functional inequalities (2.3)-(2.24) and the triangle inequality imply

$$\| F_3^o(x+y) + F_3^o(x+\sigma(y)) - 2F_3^o(x) \| \tag{2.27}$$

$$\leq \| F_3^o(x+y) - F_1^o(x+y) - F_2^o(x+y) \|$$
$$+ \| F_3^o(x+\sigma(y)) - F_1^o(x+\sigma(y)) - F_2^o(x+\sigma(y)) \|$$
$$+ \| F_1^o(x+y) + F_2^o(x+\sigma(y)) - F_3^o(x) - F_4^o(y) \|$$
$$+ \| F_1^o(x+\sigma(y)) + F_2^o(x+y) - F_3^o(x) - F_4^o(\sigma(y)) \|$$
$$\leq \phi(x,y) = 2\varphi(x,y) + \varphi(x+\sigma(y),0) + \varphi(x+y,0)$$

for all $x, y \in E_1$.

Next, we will prove the stability of the functional inequalities (2.26) and (2.27). The inequality (2.27) imply

$$\| F_3^o(y+x) + F_3^o(y+\sigma(x)) - 2F_3^o(y) \| \leq \phi(x,y) \tag{2.28}$$

then, from (2.27)-(2.28) and the triangle inequality, we obtain

$$\| F_3^o(x+y) - F_3^o(x) - F_3^o(y) \| \leq \phi(x,y) \tag{2.29}$$

for all $x, y \in E_1$.

By using now [7], there exists an unique additive function

a_1: $E_1 \longrightarrow E_2$ such that

$$\| F_3^o(x) - a_1(x) \| \leq \omega(x,x) \tag{2.30}$$

for all $x \in E_1$. Furthermore, $a_1(\sigma(x)) = -a_1(x)$, because $\varphi(x,\sigma(y)) = \varphi(x,y)$, $F_3^o(\sigma(x)) = -F_3^o(x)$ for all $x, y \in E_1$ and a_1 is the unique additive function which satisfies (2.30).

By similar proof, we deduce that there exists another additive function a_2: $E_1 \longrightarrow E_2$ such that $a_2(\sigma(x)) = -a_2(x)$ and

$$\| F_4^o(x) - a_2(x) \| \leq \omega(x,x) \tag{2.31}$$

for all $x \in E_1$. Thus, from inequalities (2.24)-(2.25) and the triangle inequality, we obtain

$$\| 2F_1^o(x) - F_3^o(x) - F_4^o(x) \| \leq 2\varphi(x,0) \tag{2.32}$$

$$\| 2F_2^o(x) - F_3^o(x) + F_4^o(x) \| \leq 2\varphi(x,0), \tag{2.33}$$

for all $x \in E_1$. Consequently, from inequalities (2.30)-(2.31), we obtain

$$\| 2F_2^o(x) - a_1(x) + a_2(x) \| \leq 2\varphi(x,0) + 2\omega(x,x) \tag{2.34}$$

and

$$\| 2F_1^o(x) - a_1(x) - a_2(x) \| \leq 2\varphi(x,0) + 2\omega(x,x). \tag{2.35}$$

Now, According to the inequalities (2.20)-(2.21)-(2.22)-(2.23)-(2.30)-(2.31)-(2.34)-(2.35), one gets the inequalities (2.5)-(2.6)-(2.7)-(2.8).

The uniqueness of q, a_i, ($i = 1, 2$) follows by applying some argument used in [13]. This completes the proof of the theorem. □

From Theorem 2.1, we can deduce the well known results obtained in [4], [13], [15], [16], [17], [37],...

Corollary 2.2. *If the functions* f_1, f_2, f_3, f_4*:* $E_1 \longrightarrow E_2$ *satisfy the inequality*

$$\| f_1(x+y) + f_2(x+\sigma(y)) - f_3(x) - f_4(y) \| \leq \theta(\| x \|^p + \| y \|^p) \tag{2.36}$$

for all $x, y \in E_1$*, for some* $\theta \geq 0$ *and* $p > 2$*, then there exists a unique function* q*:* $E_1 \longrightarrow E_2$ *solution of equation (1.4), there exists a function* ν*:* $E_1 \longrightarrow E_2$ *solution of equation*

$$\nu(x+y) = \nu(x+\sigma(y)), \quad x, y \in E_1, \tag{2.37}$$

there exist exactly two additive functions a_1, a_2*:* $E_1 \longrightarrow E_2$ *such that* $a_i \circ \sigma = -a_i$ *(*$i = 1, 2$*),*

$$\| f_1(x) - \frac{1}{2}a_1(x) - \frac{1}{2}a_2(x) - \frac{1}{2}q(x) - \frac{1}{2}\nu(x) - f_1(0) \| \tag{2.38}$$

$$\leq 2\theta[\frac{4+2^p}{2(2^p-2)} + \frac{2}{2^p-4} + \frac{2}{2^p} + \frac{1}{2}] \| x \|^p + 2\theta[\frac{1}{2(2^p-2)} + \frac{2}{2^p(2^p-4)}] \| x + \sigma(x) \|^p$$

$$\| f_2(x) - \frac{1}{2}a_1(x) + \frac{1}{2}a_2(x) - \frac{1}{2}q(x) + \frac{1}{2}\nu(x) - f_2(0) \| \tag{2.39}$$

$$\leq 2\theta[\frac{4+2^p}{2(2^p-2)} + \frac{2}{2^p-4} + \frac{2}{2^p} + \frac{1}{2}] \| x \|^p + 2\theta[\frac{1}{2(2^p-2)} + \frac{2}{2^p(2^p-4)}] \| x + \sigma(x) \|^p,$$

$$\| f_3(x) - a_2(x) - q(x) - f_3(0) \| \tag{2.40}$$

$$\leq \theta[1 + \frac{8}{2^p-4} + \frac{(4+2^p)}{2^p-2}] \| x \|^p + \theta[\frac{1}{2^p-2} + \frac{8}{2^p(2^p-4})] \| x + \sigma(x) \|^p$$

and

$$\| f_4(x) - a_1(x) - q(x) - f_4(0) \| \tag{2.41}$$

$$\leq \theta[1 + \frac{8}{2^p-4} + \frac{(4+2^p)}{2^p-2}] \| x \|^p + \theta[\frac{1}{2^p-2} + \frac{8}{2^p(2^p-4})] \| x + \sigma(x) \|^p$$

for all $x \in E_1$*.*

Corollary 2.3. *If the functions* f_1, f_2, f_3, f_4*:* $E_1 \longrightarrow E_2$ *satisfy the inequality*

$$\| f_1(x+y) + f_2(x+\sigma(y)) - f_3(x) - f_4(y) \| \leq \theta(\| x \|^p \| y \|^p) \tag{2.42}$$

for all $x, y \in E_1$*, for some* $\theta \geq 0$*,* $p > 1$*, then there exists a unique function* q*:* $E_1 \longrightarrow E_2$ *solution of equation (1.4), there exists a function* ν*:* $E_1 \longrightarrow E_2$ *solution of equation*

$$\nu(x+y) = \nu(x+\sigma(y)), \quad x, y \in E_1, \tag{2.43}$$

there exist exactly two additive functions a_1, a_2*:* $E_1 \longrightarrow E_2$ *such that* $a_i \circ \sigma = -a_i$ *(*$i = 1, 2$*),*

$$\| f_1(x) - \frac{1}{2}a_1(x) - \frac{1}{2}a_2(x) - \frac{1}{2}q(x) - \frac{1}{2}\nu(x) - f_1(0) \| \tag{2.44}$$

$$\leq \theta[\frac{2}{2^{2p}-2} + \frac{1}{2^{2p}} + \frac{1}{2^{2p}-4}] \| x \|^{2p} + \frac{\theta}{2^{2p}(2^{2p}-4)} \| x + \sigma(x) \|^{2p}$$

$$\| f_2(x) - \frac{1}{2}a_1(x) + \frac{1}{2}a_2(x) - \frac{1}{2}q(x) + \frac{1}{2}\nu(x) - f_2(0) \| \tag{2.45}$$

$$\leq \theta[\frac{2}{2^{2p}-2}+\frac{1}{2^{2p}}+\frac{1}{2^{2p}-4}] \parallel x \parallel^{2p} + \frac{\theta}{2^{2p}(2^{2p}-4)} \parallel x+\sigma(x) \parallel^{2p},$$

$$\parallel f_3(x)-a_2(x)-q(x)-f_3(0) \parallel \tag{2.46}$$

$$\leq 2\theta[\frac{1}{2^{2p}-4}+\frac{1}{2^{2p}-2}] \parallel x \parallel^{2p} + \frac{2\theta}{2^{2p}(2^{2p}-4)} \parallel x+\sigma(x) \parallel^{2p}$$

and

$$\parallel f_4(x)-a_1(x)-q(x)-f_4(0) \parallel \tag{2.47}$$

$$\leq 2\theta[\frac{1}{2^{2p}-4}+\frac{1}{2^{2p}-2}] \parallel x \parallel^{2p} + \frac{2\theta}{2^{2p}(2^{2p}-4)} \parallel x+\sigma(x) \parallel^{2p}$$

for all $x \in E_1$.

Corollary 2.4. *Let* $\sigma = -I$. *If the functions* f_1, f_2, f_3, f_4*:* $E_1 \longrightarrow E_2$ *satisfy the inequality*

$$\parallel f_1(x+y)+f_2(x-y)-f_3(x)-f_4(y) \parallel \leq \varphi(x,y) \tag{2.48}$$

for all $x, y \in E_1$, *then there exists a unique function* q*:* $E_1 \longrightarrow E_2$ *solution of equation (1.4), there exists* $\alpha \in E_2$, *there exist exactly two additive functions* a_1, a_2*:* $E_1 \longrightarrow E_2$ *such that*

$$\parallel f_1(x)-\frac{1}{2}a_1(x)-\frac{1}{2}a_2(x)-\frac{1}{2}q(x)-f_1(0)-\alpha \parallel \tag{2.49}$$

$$\leq \omega(x,x)+\frac{1}{4}\psi(x,x)+\frac{1}{2}\psi(x,0)+\varphi(\frac{x}{2},\frac{x}{2})+2\varphi(\frac{x}{2},0)+\varphi(x,0),$$

$$\parallel f_2(x)-\frac{1}{2}a_1(x)+\frac{1}{2}a_2(x)-\frac{1}{2}q(x)-f_2(0)+\alpha \parallel \tag{2.50}$$

$$\leq \omega(x,x)+\frac{1}{4}\psi(x,x)+\frac{1}{2}\psi(x,0)+\varphi(\frac{x}{2},\frac{x}{2})+2\varphi(\frac{x}{2},0)+\varphi(x,0),$$

$$\parallel f_3(x)-\frac{1}{2}a_2(x)-\frac{1}{2}q(x)-f_3(0) \parallel \leq \frac{1}{2}\varphi(x,0)+\frac{1}{4}\psi(x,x)+\frac{1}{2}\psi(x,0)+\frac{1}{2}\omega(x,x) \tag{2.51}$$

and

$$\parallel f_4(x)-\frac{1}{2}a_1(x)-\frac{1}{2}q(x)-f_4(0) \parallel \leq \frac{1}{2}\varphi(x,0)+\frac{1}{4}\psi(x,x)+\frac{1}{2}\psi(x,0)+\frac{1}{2}\omega(x,x). \tag{2.52}$$

for all $x \in E_1$.

Corollary 2.5. *Let* $\sigma = I$. *If the functions* f_1, f_2, f_3*:* $E_1 \longrightarrow E_2$ *satisfy the inequality*

$$\parallel f_1(x+y)-f_2(x)-f_3(y) \parallel \leq \varphi(x,y), \tag{2.53}$$

for all $x, y \in E_1$, *then there exist exactly three additive functions* a_1, a_2, a_3*:* $E_1 \longrightarrow E_2$ *such that*

$$\parallel f_1(x)-a_2(x)-a_3(x)-f_1(0) \parallel \tag{2.54}$$

$$\leq 2\omega(x,x)+\frac{1}{2}\psi(x,x)+\psi(x,0)+2\varphi(\frac{x}{2},\frac{x}{2})+4\varphi(\frac{x}{2},0)+2\varphi(x,0)$$

$$\| f_2(x) - a_2(x) - a_3(x) - f_2(0) \| \leq \varphi(x,0) + \frac{1}{2}\psi(x,x) + \psi(x,0) + \omega(x,x) \tag{2.55}$$

and

$$\| f_3(x) - a_1(x) - a_3(x) - f_3(0) \| \leq \varphi(x,0) + \frac{1}{2}\psi(x,x) + \psi(x,0) + \omega(x,x) \tag{2.56}$$

for all $x \in E_1$*.*

In the following corollary, we generalize the stability result obtained in [14] for Drygas functional equation.

Corollary 2.6. *If the function* $f\colon E_1 \longrightarrow E_2$ *satisfies the inequality*

$$\| f(x+y) + f(x+\sigma(y)) - 2f(x) - f(y) - f(\sigma(y)) \| \leq \varphi(x,y), \tag{2.57}$$

for all $x, y \in E_1$*, then there exists a unique additive function* $a\colon E_1 \longrightarrow E_2$ *and a unique quadratic function* $q\colon E_1 \longrightarrow E_2$ *such that* $a \circ \sigma = -a$ *and*

$$\| f(x) - q(x) - a(x) - f(0) \| \leq \varphi(x,0) + \frac{1}{2}\psi(x,x) + \psi(x,0) + \omega(x,x) \tag{2.58}$$

for all $x \in E_1$*.*

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The Alternative Of Fixed Point And Stability Results For Functional Equations

Liviu Cădariu[1] **and Viorel Radu**[2]

[1]University "Politehnica" of Timişoara,
Department of Mathematics,
Piaţa Victoriei No.2, 300006 Timişoara, Romania
email: lcadariu@yahoo.com, liviu.cadariu@mat.upt.ro

[2]West University of Timişoara,
Faculty of Mathematics and Computer Science,
Bv. Vasile Pârvan 4, 300223 Timişoara, Romania
Email: radu@math.uvt.ro

ABSTRACT

We present Ulam-Hyers-Bourgin stability theorems for functional equations of quadratic-type, slightly extending some results in [12], [17], [18], [22], [26], [27], [29], [30] and [31]. Both the direct method and the fixed point method are used.

Keywords: Functional equations, Fixed points, Ulam-Hyers-Bourgin stability.

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1 INTRODUCTION

Different methods to obtain stability properties for functional equations are known. The *direct method* revealed by Hyers in [19], where the Ulam's problem concerning the stability of group homomorphisms was affirmatively answered for Banach spaces, arrived at a very large extent and successful use (see, e.g., [3], [5], [34], [15], [12], [22] and [32]). The interested reader may consult [16], [20], [13], [14] and [21] for details.
On the other hand, in [28], [8] and [7] a *fixed point method* was proposed, by showing that many theorems concerning the stability of Cauchy and Jensen equations are consequences of the fixed point alternative. Subsequently, the method has been successfully used, e.g., in [9], [10], [33] or [24]. It is worth noting that the fixed point method introduces a metrical context and better clarifies the ideas of stability, which is seen to be unambiguously related to fixed points of concrete contractive-type operators on suitable (function) spaces.
We present some *stability results of the Hyers-Aoki-Bourgin type* for functional equations of additive-quadratic type and bi-quadratic type. By using both the direct method and the fixed point method we slightly extend the results in [26], [27], [12], [18], [22], [29], [30] and [31].

2 FUNCTIONAL EQUATIONS OF AddQ–TYPE

Let X_1, X_2 and Y be real linear spaces and consider the cartesian product $Z := X_1 \times X_2$ together with the linear selfmappings P_{X_1}, P_{X_2} and S, where $P_{X_1}(u) = (u_1, 0),\ P_{X_2}(u) = (0, u_2),\ \forall u = (u_1, u_2) \in Z$, and $S = S_{X_1} := P_{X_1} - P_{X_2}$. A function $F : Z \to Y$ is called an $AddQ-$*type mapping* iff it satisfies, for all $u, v \in Z$, the following equation:

$$D_F(u,v) := F(u+v) + F(u+S(v)) - \\ 2\left(F(u) + F(v) + F\left(\frac{u+S(u)+v-S(v)}{2}\right) + F\left(\frac{u-S(u)+v+S(v)}{2}\right)\right) = 0. \tag{2.1}$$

Notice that, whenever Z is an inner product space, $F(u) = a \cdot ||P_{X_1}u|| \cdot ||P_{X_2}u||^2$ defines a solution of (2.1) for each $a \in \mathbb{R}$. If F is a solution of (2.1) for $X_1 = X_2 = X$, then $X \times X \ni u = (x,z) \to f(x,z) := F(u) \in Y$ is an *additive-quadratic mapping on* X, i.e. it verifies the following equation [27]:

$$f(x+y, z+w) + f(x+y, z-w) = \\ = 2\left(f(x,z) + f(y,w) + f(x,w) + f(y,z)\right),\ \forall x,y,z,w \in X. \tag{2.2}$$

Recall that a mapping $g : X \to Y$, between linear spaces, is called *additive* if it satisfies the Cauchy equation

$$g(x+y) = g(x) + g(y), \forall x, y \in X, \tag{2.3}$$

and $h : X \to Y$ is called *quadratic* if it satisfies the following equation:

$$h(x+y) + h(x-y) = 2h(x) + 2h(y), \forall x, y \in X. \tag{2.4}$$

Remark 2.1. Any solution F of (2.1) has the following properties:
(i) $F(0) = 0$ and F is an odd mapping;
(ii) $F(2^n \cdot u) = 2^{3n} \cdot F(u),\ \forall u \in Z,\ \forall n \in \mathbb{N}$;
(iii) $F \circ S = F$;
(iv) $F \circ P_{X_1} = F \circ P_{X_2} = 0.$
Moreover, if $f(x,z) = F(u)$, where $u = (x,z)$, then f is additive in the first variable and quadratic in the second variable. We also have:

Lemma 2.1. *Suppose* $F : Z \to Y$ *is of the form*

$$F(u) = f_2(z) f_1(x),\ \forall u = (x,z) \in Z = X_1 \times X_2,$$

with arbitrary nonzero mappings $f_1 : X_1 \to Y$ *and* $f_2 : X_2 \to \mathbb{R}$. *Then:*
(i) F *is of* $AddQ-$*type if* f_1 *is additive and* f_2 *is quadratic;*
(ii) f_1 *is additive if* F *is of* $AddQ-$*type and* f_2 *is quadratic;*
(iii) f_2 *is quadratic if* F *is of* $AddQ-$*type and* f_1 *is additive.*

2.1. The Ulam-Hyers-Bourgin Stability for AddQ–Type Equations

Consider a control mapping $\Phi : Z \times Z \to [0, \infty)$ such that

$$\Psi(u,v) := \sum_{i=0}^{\infty} \frac{\Phi(2^i u, 2^i v)}{8^{i+1}} < \infty, \forall u, v \in Z, \tag{2.5}$$

and suppose Y is a Banach space.

Theorem 2.2. *Let $F : Z \to Y$ be such a mapping that $F \circ P_{X_1} = 0$ and suppose that*

$$||D_F(u,v)|| \le \Phi(u,v), \forall u,v \in Z. \tag{2.6}$$

Then there exists a unique $AddQ-$type mapping $B : Z \to Y$ for which

$$||F(u) - B(u)|| \le \Psi(u,u),\ \forall u \in Z. \tag{2.7}$$

Proof. We shall use the Hyers' (direct) method. Letting $u = v$ in (2.6), we obtain

$$\left|\left|\frac{F(2u)}{8} - F(u)\right|\right| \le \frac{\Phi(u,u)}{8},\ \forall u \in Z.$$

In the next step, one shows that

$$\left|\left|\frac{F(2^p u)}{8^p} - \frac{F(2^m u)}{8^m}\right|\right| \le \sum_{i=p}^{m-1} \frac{\Phi(2^i u, 2^i u)}{8^{i+1}},\ \forall u \in Z, \tag{2.8}$$

for given integers p, m, with $0 \le p < m$. Using (2.5) and (2.8), $\{\frac{F(2^n u)}{8^n}\}_{n\ge 0}$ is a Cauchy sequence for any $u \in Z$. Since Y is complete, we can define the mapping $B : Z \to Y$,

$$B(u) = \lim_{n\to\infty} \frac{F(2^n u)}{8^n},$$

for all $u \in Z$. By using (2.8) for $p = 0$ and $m \to \infty$ we obtain the estimation (2.7).
By (2.6), we have

$$\left|\left|\frac{F(2^n(u+v))}{8^n} + \frac{F(2^n(u+S(v)))}{8^n} - \right.\right.$$
$$-2\left(\frac{F(2^n(u))}{8^n} + \frac{F(2^n(v))}{8^n} + \frac{1}{8^n}F\left(2^n\left(\frac{u+S(u)+v-S(v)}{2}\right)\right) + \right.$$
$$\left.\left.\left.\frac{1}{8^n}F\left(2^n\left(\frac{u-S(u)+v+S(v)}{2}\right)\right)\right)\right|\right| \le \frac{\Phi(2^n u, 2^n v)}{8^n},$$

for all $u, v \in Z$. Using (2.5) and letting $n \to \infty$, we immediately see that B is an $AddQ-$type mapping.
Let B_1 be an $AddQ-$type mapping which satisfies (2.7). Then

$$||B(u) - B_1(u)|| \le$$
$$\le \left|\left|\frac{B(2^n u)}{8^n} - \frac{F(2^n u)}{8^n}\right|\right| + \left|\left|\frac{F(2^n u)}{8^n} - \frac{B_1(2^n u)}{8^n}\right|\right| \le$$
$$\le 2 \cdot \sum_{k=n}^{\infty} \frac{\Phi(2^k u, 2^k u)}{8^{k+1}} \longrightarrow 0, \text{for } n \to \infty.$$

Hence the uniqueness claim for B holds true. □
We show that the stability result in ([27], Theorem 7) is a direct consequence of our theorem.
Let us consider a mapping $\varphi : X \times X \times X \times X \to [0,\infty)$ such that

$$\psi(x,z,y,w) := \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i z, 2^i y, 2^i w)}{8^{i+1}} < \infty, \forall x,y,z,w \in X.$$

Corollary 2.3. *Suppose that X is a real linear space and Y is a real Banach space. Let $f : X \times X \to Y$ be a mapping such that*

$$||f(x+y, z+w) + f(x+y, z-w)-$$

$$-2\,(f(x,z) + f(y,w) + f(x,w) + f(y,z))|| \leq \varphi(x,z,y,w)$$

and $f(x,0) = 0$, for all $x, y, z, w \in X$. Then there exists a unique additive-quadratic mapping $b : X \times X \to Y$, such that

$$||f(x,z) - b(x,z)|| \leq \psi(x,z,x,z),\ \forall x, z \in X. \tag{2.9}$$

Proof. Let us define, for $X_1 = X_2 = X$, $u, v \in X \times X, u = (x,z)$ and $v = (y,w)$,

$$F(u) = f(x,z) \text{ and } \Phi(u,v) = \varphi(x,z,y,w).$$

Since $\Psi(u,v) = \psi(x,z,y,w) < \infty$, then we can apply Theorem 2.2. Clearly, the mapping b, defined by $b(x,z) = B(u)$ is additive-quadratic and verifies (2.9). □

2.2. Stability Results of Aoki-Rassias Type

For particular forms of the mapping Φ in (2.5), we can obtain interesting consequences. We identify stability properties with unbounded control conditions invoking sums (T. Aoki, [3]) and products (J. M. Rassias, [29, 30, 31]) of powers of norms.
Let X_1, X_2 and Y be real linear spaces. Suppose that $Z := X_1 \times X_2$ is endowed with a norm $||u||_Z$ and that Y is a real Banach space.

Corollary 2.4. *Let $F : Z \to Y$ be a mapping such that*

$$||D_F(u,v)||_Y \leq \varepsilon(||u||_Z^p + ||v||_Z^q),\ \forall u, v \in Z,$$

where $p, q \in [0,3)$ and $\varepsilon \geq 0$ are fixed. If $F \circ (I+S) = 0$, then there exists a unique $AddQ-$type mapping $B : Z \to Y$, such that

$$||F(u) - B(u)||_Y \leq \frac{\varepsilon}{2^3 - 2^p} \cdot ||u||_Z^p + \frac{\varepsilon}{2^3 - 2^q} \cdot ||u||_Z^q,\ \forall u \in Z.$$

Proof. Consider the mapping $\Phi : Z \times Z \to [0,\infty)$, $\Phi(u,v) = \varepsilon(||u||_Z^p + ||v||_Z^q)$. Then

$$\Psi(u,v) := \sum_{i=0}^{\infty} \frac{\Phi(2^i u, 2^i v)}{8^{i+1}} = \varepsilon \cdot \frac{||u||_Z^p}{2^3 - 2^p} + \varepsilon \cdot \frac{||v||_Z^q}{2^3 - 2^q} < \infty,\ \forall u, v \in Z,$$

and the conclusion follows directly from Theorem 2.2. □

Corollary 2.5. *Let $F : Z \to Y$ be a mapping such that*

$$||D_F(u,v)||_Y \leq \varepsilon \cdot ||u||_Z^p \cdot ||v||_Z^q,\ \forall u, v \in Z,$$

where $\varepsilon, p, q \geq 0$ are fixed and $p + q < 3$. If $F \circ (I+S) = 0$, then there exists a unique $AddQ-$type mapping $B : Z \to Y$, such that

$$||F(u) - B(u)||_Y \leq \frac{\varepsilon}{2^3 - 2^{p+q}} \cdot ||u||_Z^{p+q},\ \forall u \in Z.$$

Proof. Consider the mapping $\Phi : Z \times Z \to [0, \infty)$, $\Phi(u, v) = \varepsilon \cdot ||u||_Z^p \cdot ||v||_Z^q$. Then

$$\Psi(u, v) := \sum_{i=0}^{\infty} \frac{\Phi(2^i u, 2^i v)}{8^{i+1}} = \varepsilon \cdot \frac{||u||_Z^p \cdot ||v||_Z^q}{2^3 - 2^{p+q}} < \infty, \ \forall u, v \in Z,$$

so that we can apply Theorem 2.2. □

Now, suppose that $X_1 = X_2 = X$, where X is a real normed space, and consider the function $X \times X \ni u = (x, z) \to F(u) = f(x, z)$, where f is mapping $X \times X$ into the real Banach space Y. Although the functions of the form $u \to ||u|| := (||x||^r + ||z||^s)^{\frac{1}{t}}$ may not be norms, the above proofs work as well, and we obtain the following stability properties for additive-quadratic equations:

Corollary 2.6. *Let $f : X \times X \to Y$ be a mapping such that $f(x, 0) = 0$ and*

$$||f(x + y, z + w) + f(x + y, z - w) + f(x - y, z + w) + f(x - y, z - w) -$$

$$-4\left(f(x, z) + f(y, w) + f(x, w) + f(y, z)\right)||_Y \le \varepsilon\left(||x||^p + ||y||^p + ||z||^q + ||w||^q\right),$$

for all $x, y, z, w \in X$ and for some fixed $\varepsilon, p, q \ge 0$ with $p, q < 3$. Then there exists a unique additive-quadratic mapping $b : X \times X \to Y$, such that

$$||f(x, z) - b(x, z)||_Y \le \frac{2\varepsilon}{2^3 - 2^p} \cdot ||x||^p + \frac{2\varepsilon}{2^3 - 2^q} \cdot ||z||^q, \ \forall x, z \in X.$$

Corollary 2.7. *Let $f : X \times X \to Y$ be a mapping such that*

$$||f(x + y, z + w) + f(x + y, z - w) + f(x - y, z + w) + f(x - y, z - w) -$$

$$-4\left(f(x, z) + f(y, w) + f(x, w) + f(y, z)\right)||_Y \le \varepsilon \cdot (||x||^p + ||z||^p) \cdot (||y||^q + ||w||^q),$$

for all $x, y, z, w \in X$ and for some fixed $\varepsilon, p, q \ge 0$, with $p + q < 3$. If $f(x, 0) = 0$, for all $x \in X$, then there exists a unique additive-quadratic mapping $b : X \times X \to Y$, such that

$$||f(x, z) - b(x, z)||_Y \le \frac{\varepsilon}{2^3 - 2^{p+q}} \cdot (||x||^p + ||z||^p) \cdot (||x||^q + ||z||^q), \ \forall x, z \in X.$$

2.3. Applications to Additive Equations and to Quadratic Equations

For the sake of convenience, we recall the following Ulam-Hyers-Bourgin stability properties of the additive and quadratic functional equations:

Proposition 2.8. ([18]) *Let X be a real normed vector space and Y a real Banach space. If $\bar{\varphi}$ and $\bar{\phi} : X \times X \to [0, \infty)$ verify the condition*

$$\bar{\phi}(x, y) := \sum_{i=0}^{\infty} \frac{\bar{\varphi}(2^i x, 2^i y)}{2^{i+1}} < \infty, \text{ for all } x, y \in X \tag{2.10}$$

and the mapping $\bar{f} : X \to Y$ satisfies the relation

$$\left\|\bar{f}(x + y) - \bar{f}(x) - \bar{f}(y)\right\| \le \bar{\varphi}(x, y), \text{ for all } x, y \in X, \tag{2.11}$$

then there exists a unique additive mapping $\bar{a} : X \to Y$ which satisfies the inequality

$$\left\|\bar{f}(x) - \bar{a}(x)\right\| \le \bar{\phi}(x, x), \text{ for all } x \in X. \tag{2.12}$$

Proposition 2.9. ([22], Theorem 2.2) *Let X be a real normed vector space and Y a real Banach space. If $\tilde{\varphi}$ and $\tilde{\phi} : X \times X \to [0, \infty)$ verify the condition*

$$\tilde{\phi}(z, w) := \sum_{i=0}^{\infty} \frac{\tilde{\varphi}(2^i z, 2^i w)}{4^{i+1}} < \infty, \text{ for all } z, w \in X \tag{2.13}$$

and the mapping $\tilde{f} : X \to Y$, with $\tilde{f}(0) = 0$, satisfies the relation

$$\left\| \tilde{f}(z + w) + \tilde{f}(z - w) - 2\tilde{f}(z) - 2\tilde{f}(w) \right\| \leq \tilde{\varphi}(z, w), \text{ for all } z, w \in X, \tag{2.14}$$

then there exists a unique quadratic mapping $\tilde{q} : X \to Y$ which satisfies the inequality

$$\left\| \tilde{f}(z) - \tilde{q}(z) \right\| \leq \tilde{\phi}(z, z), \text{ for all } z \in X. \tag{2.15}$$

As a matter of fact, we can show that Proposition 2.8 is a consequence of our Theorem 2.2. Namely, we have

Theorem 2.10. *The stability of the equation* (2.1) *implies the Ulam-Hyers-Bourgin stability of the additive equation* (2.3)*.*

Proof. Let $X, Y, \bar{\varphi} : X \times X \to [0, \infty)$ and $\bar{f} : X \to Y$ as in Proposition 2.8. We take $X_1 = X$ and consider a linear space X_2 such that there exist a quadratic function $\bar{h} : X_2 \to \mathbb{R}$, with $\bar{h}(0) = 0$ and an element $z_0 \in X_2$, such that $\bar{h}(z_0) \neq 0$ (In inner product spaces such a function is, e.g., $z \to ||z||^2$). If we set, for $u = (x, z), v = (y, w) \in X \times X_2$,

$$\Phi(u, v) = \Phi(x, z, y, w) = 2|\bar{h}(z) + \bar{h}(w)| \cdot \bar{\varphi}(x, y)$$

and

$$F(u) = F(x, z) = \bar{h}(z) \cdot \bar{f}(x),$$

then by using (2.10) and the properties of the quadratic mapping we easily get:

$$\Psi(u, v) = \frac{1}{2}|\bar{h}(z) + \bar{h}(w)| \sum_{i=0}^{\infty} \frac{\bar{\varphi}(2^i x, 2^i y)}{2^{i+1}} < \infty,$$

for all $u, v \in X \times X_2$. At the same time, by (2.11),

$$||D_F(u, v)|| = 2|\bar{h}(z) + \bar{h}(w)| \cdot ||\bar{f}(x + y) + \bar{f}(x) - \bar{f}(y)|| \leq$$

$$\leq 2|\bar{h}(z) + \bar{h}(w)| \cdot \bar{\varphi}(x, y) = \Phi(u, v), \ \forall u, v \in X \times X_2.$$

Therefore, by Theorem 2.2, there exists a unique mapping of $AddQ-$type, $B : X \times X_2 \to Y$, such that $||F(u) - B(u)|| \leq \Psi(u, u)$ and

$$B(u) = \lim_{n \to \infty} \frac{F(2^n u)}{8^n} = \lim_{n \to \infty} \bar{h}(z) \cdot \frac{\bar{f}(2^n x)}{2^n}, \forall u = (x, z) \in X \times X_2.$$

We know that $\bar{h}(z_0) \neq 0$. Therefore the limit

$$\bar{a}(x) = \lim_{n \to \infty} \frac{\bar{f}(2^n x)}{2^n}$$

exists for every $x \in X$ and, moreover, $B(u) = \bar{h}(z) \cdot \bar{a}(x), \forall u = (x, z) \in Z$. Since $||\bar{h}(z)\bar{f}(x) - B(u)|| \leq \bar{h}(z) \cdot \bar{\phi}(x, x),\ \forall u = (x, z) \in X \times X_2$, then the estimation (2.12) is easily seen to hold. By Lemma 2.1, $\bar{a}$ is additive. If an additive mapping $\bar{a}_1$ satisfies (2.12), then $(x, z) \to \bar{h}(z)\bar{a}_1(x)$ is of $AddQ-$type (again by Lemma 2.1) and has to coincide with B, that is $\bar{h}(z)\bar{a}_1(x) = \bar{h}(z)\bar{a}(x)$, for all $u = (x, z) \in X \times X_2$. Since $\bar{h}$ is nonzero, then $\bar{a}_1(x) = \bar{a}(x)$, for all $x \in X$. Hence $\bar{a}$ is unique. □

As very particular cases, we obtain the results in (T. Aoki, [3]) and (J. M. Rassias, [29]) for additive equations:

Corollary 2.11. *Let $\bar{f} : X \to Y$ be a mapping such that*

$$\left\|\bar{f}(x+y) - \bar{f}(x) - \bar{f}(y)\right\| \leq \varepsilon\left(||x||^p + ||y||^p\right), \text{ for all } x, y \in X,$$

and for any fixed $\varepsilon, p \geq 0$, with $p < 1$. If $\bar{f}(0) = 0$, then there exists a unique additive mapping $\bar{a} : X \to Y$ which satisfies the estimation

$$\left\|\bar{f}(x) - \bar{a}(x)\right\| \leq \frac{2\varepsilon}{2 - 2^p} \cdot ||x||^p, \text{ for all } x \in X.$$

Proof. Let $\bar{h} : \mathbb{R} \to \mathbb{R}, \bar{h}(z) = z^2$ and $\bar{f} : X \to Y$, where X is a normed space and Y a Banach space. We apply Theorem 2.2 for $X_1 = X, X_2 = \mathbb{R}$, $u, v \in X \times \mathbb{R}$, with $u = (x, z), v = (y, w)$ and the mappings

$$F(u) = F(x, z) = z^2 \cdot \bar{f}(x),$$

$$\Phi(u, v) = \Phi(x, z, y, w) = 2(z^2 + w^2) \cdot \varepsilon(||x||^p + ||y||^p),$$

to obtain the existence of a unique additive mapping $\bar{a}$ and the required estimation.

Corollary 2.12. *Let $\bar{f} : X \to Y$ be a mapping such that*

$$\left\|\bar{f}(x+y) - \bar{f}(x) - \bar{f}(y)\right\| \leq \theta\left(||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}}\right), \text{ for all } x, y \in X,$$

and for any fixed $\theta, p \geq 0$, with $p < 1$. If $\bar{f}(0) = 0$, then there exists a unique additive mapping $\bar{a} : X \to Y$ which satisfies the estimation

$$\left\|\bar{f}(x) - \bar{a}(x)\right\| \leq \frac{\theta}{2 - 2^p} \cdot ||x||^p, \text{ for all } x \in X.$$

Indeed, one can use *either* the mappings $F(u) = F(x, z) = z^2 \cdot \bar{f}(x)$, and $\Phi(u, v) = \Phi(x, z, y, w) = 2(z^2 + w^2) \cdot \theta \cdot ||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}}$, *or* the means inequality: $\theta\left(||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}}\right) \leq \frac{\theta}{2}\left(||x||^p + ||y||^p\right)$ in the preceding corollary.

Remark 2.2. As it is wellknown (see [17, 20]), Gajda showed that the additive equation (2.3) is **not stable** for $\bar{\varphi}(x, y)$ of the form $\varepsilon(||x|| + ||y||)$, ε being a given positive constant. In fact, he proved that there exists a mapping $\bar{f}_0 : \mathbb{R} \to \mathbb{R}$ such that (2.11) holds with the above $\bar{\varphi}$, and there exists **no** additive mapping $\bar{a}$ to verify

$$\left\|\bar{f}_0(x) - \bar{a}(x)\right\| \leq c(\varepsilon)||x||, \text{ for all } x \in \mathbb{R}.$$

This suggests the following:

Example 2.1: Let X_1 be a real normed space, Y a Banach space, and $\bar{h} : X_2 = \mathbb{R} \to \mathbb{R}$ a quadratic function with $\bar{h}(0) = 0, \bar{h}(1) = 1$. Then **the equation** (2.1) **is not stable** for

$$\Phi(u, v) = \Phi(x, z, y, w) = 2\varepsilon \cdot (||x|| + ||y||)(h(z) + h(w)). \tag{2.16}$$

In fact, we can show that there exists an F for which the relation (2.6) holds and there exists **no** $AddQ-$type mapping $B : X_1 \times X_2 \to Y$ to verify

$$||F(u) - B(u)|| \leq c(\varepsilon)\bar{h}(z)||x||, \ \forall u = (x, z) \in X_1 \times X_2. \tag{2.17}$$

Indeed, for $F(u) = F(x, z) = \bar{h}(z) \cdot \bar{f}_0(x)$, and Φ as in (2.16), (2.6) holds. Therefore

$$\left\|\bar{f}_0(x+y) - \bar{f}_0(x) - \bar{f}_0(y)\right\| \leq \varepsilon\left(||x|| + ||y||\right), \text{ for all } x, y \in X_1.$$

Let us suppose, for a contradiction, that there exists an $AddQ-$type mapping B which verifies (2.17). By Remark 2.1, the mapping $\bar{a} : X_1 \to Y, \bar{a}(x) := B(x, 1)$ is a solution for (2.3). The estimation (2.17) gives us

$$||\bar{f}_0(x) - \bar{a}(x)|| \leq c(\varepsilon)||x||, \ \forall x \in X_1,$$

in contradiction with the assertions in Remark 2.2.
We can also show that Theorem 2.2 does imply Proposition 2.9:

Theorem 2.13. *The stability of the equation* (2.1) *implies the Ulam-Hyers-Bourgin stability of the quadratic equation* (2.4)*.*

Proof. Let $X, Y, \tilde{\varphi} : X \times X \to [0, \infty)$ and $\tilde{f} : X \to Y$ as in Proposition 2.9. We take $X_2 = X$ and consider a real linear space X_1 together with an additive function $\tilde{h} : X_1 \to \mathbb{R}$ and an element $x_0 \in X_1$, such that $\tilde{h}(x_0) \neq 0$. If we set, for $u = (x, z), v = (y, w) \in X \times X_2$,

$$\Phi(u, v) = \Phi(x, z, y, w) = |\tilde{h}(x) + \tilde{h}(y)| \cdot \tilde{\varphi}(z, w) \text{ and } F(u) = F(x, z) = \tilde{h}(x) \cdot \tilde{f}(z),$$

then we easily get, by using (2.13) and the properties of the additive mappings:

$$\Psi(u, v) = \frac{1}{2}|\tilde{h}(x) + \tilde{h}(y)| \sum_{i=0}^{\infty} \frac{\tilde{\varphi}(2^i z, 2^i w)}{4^{i+1}} < \infty, \ \forall u, v \in X_1 \times X.$$

At the same time, by (2.14),

$$||D_F(u, v)|| = |\tilde{h}(x) + \tilde{h}(y)| \cdot ||\tilde{f}(z + w) + \tilde{f}(z - w) - 2\tilde{f}(z) - 2\tilde{f}(w)|| \leq$$

$$\leq |\tilde{h}(x) + \tilde{h}(y)| \cdot \tilde{\varphi}(z, w) = \Phi(u, v), \ \forall u, v \in X_1 \times X.$$

Now, by our Theorem 2.2, there exists a unique mapping $B : X_1 \times X \to Y$, of $AddQ-$type, such that

$$||F(u) - B(u)|| \leq \Psi(u, u),$$

and

$$B(u) = \lim_{n \to \infty} \frac{F(2^n u)}{8^n} = \lim_{n \to \infty} \tilde{h}(x) \cdot \frac{\tilde{f}(2^n z)}{4^n}, \forall u = (x, z) \in X_1 \times X.$$

Since $\tilde{h}(x_0) \neq 0$, then the limit $\tilde{q}(z) = \lim_{n\to\infty} \frac{\tilde{f}(2^n z)}{4^n}$ exists for every $z \in X$ and $B(u) = \tilde{h}(x) \cdot \tilde{q}(z), \forall u = (x,z) \in X_1 \times X$. Since $||\tilde{h}(z)\tilde{f}(x) - \tilde{h}(z)\tilde{q}(x)|| \leq \tilde{h}(z) \cdot \tilde{\phi}(x,x),\ \forall (x,z) \in X_1 \times X$, then the estimation (2.15) holds. By Lemma 2.1, $\tilde{q}$ is quadratic. If a quadratic mapping $\tilde{q}_1$ satisfies (2.15), then $(x,z) \to \tilde{h}(x)\tilde{q}_1(z)$ is of $AddQ-$type (again by Lemma 2.1) and has to coincide with B, that is $\tilde{h}(x)\tilde{q}_1(z) = \tilde{h}(x)\tilde{q}(z)$, for all $u = (x,z) \in X_1 \times X$. Since $\tilde{h}$ is nonzero, then $\tilde{q}_1(z) = \tilde{q}(z) \forall z \in X$, hence $\tilde{q}$ is unique. □

In particular, we obtain a stability property of type Aoki for quadratic equations ([12]):

Corollary 2.14. *Let $\tilde{f}$ be a mapping, from a real linear space X into a real Banach space Y, such that*

$$\left\|\tilde{f}(z+w) + \tilde{f}(z-w) - 2\tilde{f}(z) - 2\tilde{f}(w)\right\| \leq \varepsilon\left(||z||^p + ||w||^p\right), \text{ for all } z, w \in X,$$

and for some fixed $\varepsilon, p \geq 0$, with $p < 2$. If $\tilde{f}(0) = 0$, then there exists a unique quadratic mapping $\tilde{q} : X \to Y$ which satisfies the estimation

$$\left\|\tilde{f}(z) - \tilde{q}(z)\right\| \leq \frac{2\varepsilon}{2^2 - 2^p} \cdot ||z||^p, \text{ for all } z \in X.$$

For the *proof*, let $\tilde{h} : \mathbb{R} \to \mathbb{R}, \tilde{h}(x) = x$. We apply Theorem 2.2 for $X_2 = X, X_1 = \mathbb{R}, u, v \in \mathbb{R} \times X$, with $u = (x,z), v = (y,w)$ and the mappings $F(u) = F(x,z) = x \cdot \tilde{f}(z),\ \Phi(u,v) = \Phi(x,z,y,w) = |x+y| \cdot \varepsilon(||z||^p + ||w||^p)$, to obtain the existence of a unique quadratic mapping $\tilde{q}$ and the required estimation.
Similarly, by choosing $F(u) = F(x,z) = x \cdot \tilde{f}(z)$ and $\Phi(u,v) = \Phi(x,z,y,w) = |x+y| \cdot \varepsilon \cdot ||z||^p \cdot ||w||^q$, we obtain a stability of type J. M. Rassias [31]:

Corollary 2.15. *Let $\tilde{f}$ mapping a real linear space X into a real Banach space Y such that $\tilde{f}(0) = 0$ and*

$$\left\|\tilde{f}(z+w) + \tilde{f}(z-w) - 2\tilde{f}(z) - 2\tilde{f}(w)\right\| \leq \varepsilon \cdot ||z||^p \cdot ||w||^q, \forall z, w \in X,$$

for some fixed $\varepsilon, p, q \geq 0$, with $p + q < 2$. Then there exists a unique quadratic mapping $\tilde{q} : X \to Y$ which satisfies the estimation

$$\left\|\tilde{f}(z) - \tilde{q}(z)\right\| \leq \frac{\varepsilon}{2^2 - 2^{p+q}} \cdot ||z||^{p+q},\ \forall z \in X.$$

Remark 2.3. As shown in [12, 20], the quadratic equation (2.4) is **not stable** for $\tilde{\varphi}(z,w)$ of the form $\varepsilon(z^2 + w^2),\ \varepsilon$ being a given positive constant. In fact, Czervick proved that there exists a mapping $\tilde{f}_0 : \mathbb{R} \to \mathbb{R}$ such that (2.14) holds with the above $\tilde{\varphi}$, and there exists **no** quadratic mapping $\tilde{q}$ to verify

$$\left\|\tilde{f}_0(z) - \tilde{q}(z)\right\| \leq c(\varepsilon)||z||^2, \text{ for all } z \in \mathbb{R}.$$

This suggests the following:

Example 2.2: Consider a real normed space X_2, a real Banach space Y and an additive function $\tilde{h} : X_1 = \mathbb{R} \to \mathbb{R}$, with $\tilde{h}(1) = 1$. Then the equation (2.1) is **not stable** for

$$\Phi(u,v) = \Phi(x,z,y,w) = \varepsilon \cdot (||z||^2 + ||w||^2) \cdot |\tilde{h}(x) + \tilde{h}(y)|. \tag{2.18}$$

In fact, we shall construct an F for which the relation (2.6) does hold, and there exists **no** $AddQ-$type mapping $B : X_1 \times X_2 \to Y$ which verifies

$$||F(u) - B(u)|| \leq c(\varepsilon) \cdot |\tilde{h}(x)| \cdot ||z||^2, \; \forall u = (x, z) \in X_1 \times X_2. \tag{2.19}$$

Indeed, if we set $F(u) = F(x, z) = \tilde{h}(x) \cdot \tilde{f}_0(z)$ and Φ as in (2.18), then (2.6) holds. Suppose, for a contradiction, that there exists an $AddQ-$type mapping B which verifies (2.19). Then, by Remark 2.1, the mapping $\tilde{q} : X_2 \to Y, \tilde{q}(z) = B(1, z)$, is a solution for the quadratic equation (2.4). Moreover, the estimation (2.19) gives: $||\tilde{f}_0(z) - \tilde{q}(z)|| \leq c(\varepsilon)||z||^2, \; \forall z \in X_2$, in contradiction with the assertions on f_0 in Remark 2.3.

3 A SECOND STABILITY RESULT BY THE FIXED POINT METHOD

We will show that Corollary 2.4 and Corollary 2.5 can be essentially extended by using *a fixed point method*. As we already remarked, the method is seen to be a meaningful tool, and is plainly related to some fixed point of a concrete operator. Specifically, each of our control conditions is perceived to be responsible for three fundamental facts: Actually, they ensure
1) the *contraction property* of a Schröder type operator J and
2) the first two successive approximations, f and Jf, to be at a *finite distance*.
And, moreover, they force
3) the fixed point function of J to be a *solution of the initial equation*.
For the sake of convenience, we recall a celebrated result in fixed point theory, namely *the fixed point alternative* (compare with [23]):

Lemma 3.1. *Suppose we are given a complete generalized metric space* (X, d) (*i.e.* d *may assume infinite values*) *and a strictly contractive mapping* $J : X \to X$, *that is,*

$$d(Jx, Jy) \leq Ld\,(x, y)\,, \forall x, y \in X, \tag{$\mathbf{B_1}$}$$

for some $L < 1$*. Then, for each fixed element* $x \in X$, ***either*** $(\mathbf{A_1})$ $d(J^n x, J^{n+1} x) = +\infty$, $\forall n \geq 0$, ***or*** $(\mathbf{A_2})$ $d(J^n x, J^{n+1} x) < +\infty, \forall n \geq n_0$, *for some natural number* n_0*. In fact, if* $(\mathbf{A_2})$ *holds, then:*

$(\mathbf{A_{21}})$ *The sequence* $(J^n x)$ *is convergent to a fixed point* y^* *of* J;

$(\mathbf{A_{22}})$ y^* *is the unique fixed point of* J *in* $Y = \{y \in X, d\,(J^{n_0} x, y) < +\infty\}\,;$

$(\mathbf{A_{23}})$ $d\,(y, y^*) \leq \dfrac{1}{1 - L} d\,(y, Jy)\,, \forall y \in Y.$

Let X_1, X_2 be linear spaces, $Z := X_1 \times X_2, Y$ a Banach space, and consider an arbitrary mapping $\Phi : Z \times Z \to [0, \infty)$.

Theorem 3.2. *Let* $F : Z \to Y$ *be a mapping such that* $F \circ (I + S) = 0$ *and suppose that*

$$||D_F(u, v)|| \leq \Phi(u, v), \; \forall u, v \in Z. \tag{3.1}$$

If there exists $L < 1$ *such that the mapping*

$$u \to \Omega(u) = \Phi\left(\frac{u}{2}, \frac{u}{2}\right)$$

verifies the condition

$$\Omega(u) \leq L \cdot 2^3 \cdot \Omega\left(\frac{u}{2}\right), \forall u \in Z, \tag{$\mathbf{H}$}$$

and the mapping Φ *has the property*

$$\lim_{n\to\infty} \frac{\Phi\left(2^n u, 2^n v\right)}{2^{3n}} = 0, \forall u, v \in Z, \tag{$\mathbf{H}^*$}$$

then there exists a unique $AddQ-$*type mapping* $B : Z \to Y$*, such that*

$$||F(u) - B(u)|| \leq \frac{L}{1-L}\Omega(u),\ \forall u \in Z. \tag{$\mathbf{Est}$}$$

Proof. We introduce a *complete generalized metric* on the set $\mathcal{F} := \{G : Z \to Y\}$:

$$d\left(G, H\right) = d_{\Omega}\left(G, H\right) = \inf\left\{K \in \mathbb{R}_{+}, \left\|G\left(u\right) - H\left(u\right)\right\| \leq K\Omega(u), \forall u \in Z\right\}.$$

Now, we consider the (linear) *operator of cubic type*

$$J : \mathcal{F} \to \mathcal{F},\ JG\left(u\right) := \frac{G\left(2u\right)}{2^3}.$$

We have, for any $G, H \in \mathcal{F}$:

$$d(G,H) < K \Longrightarrow \left\|G\left(u\right) - H\left(u\right)\right\| \leq K\Omega(u), \forall u \in Z \Longrightarrow$$
$$\left\|\frac{1}{2^3}G\left(2u\right) - \frac{1}{2^3}H\left(2u\right)\right\| \leq LK\Omega(u), \forall u \in Z \ \Longrightarrow$$
$$d\left(JG, JH\right) \leq LK.$$

Therefore

$$d\left(JG, JH\right) \leq L \cdot d\left(G, H\right), \forall G, H \in \mathcal{F},$$

that is J is a *strictly contractive* self-mapping of $\mathcal{F}$. If we set $u = v = t$ in the relation (3.1), then we see that

$$\left\|F\left(2t\right) - 8F(t)\right\| \leq \Omega(2t), \forall t \in Z.$$

Using the hypothesis $(\mathbf{H})$ we obtain

$$\left\|\frac{F\left(2t\right)}{8} - F(t)\right\| \leq \frac{\Omega(2t)}{8} \leq L\Omega(t), \forall t \in Z.$$

which shows that $d\left(F, JF\right) \leq L < \infty$. Thus, we can apply the fixed point alternative and we obtain the existence of a mapping $B : \mathcal{F} \to \mathcal{F}$ such that:

- B is a fixed point of the operator J, that is

$$B\left(2u\right) = 8B\left(u\right), \forall u \in Z. \tag{3.2}$$

The mapping B is the unique fixed point of J in the set $\{G \in \mathcal{F},\ d\left(F, G\right) < \infty\}$. This says that B is the unique mapping with *both* the properties (3.2) and (3.3), where

$$\exists K \in (0, \infty) \text{ such that} \left\|B\left(u\right) - F\left(u\right)\right\| \leq K\Omega(u), \forall u \in Z. \tag{3.3}$$

Moreover,

- $d(J^n F, B) \xrightarrow[n\to\infty]{} 0$, which implies the equality

$$\lim_{n\to\infty} \frac{F(2^n u)}{2^{3n}} = B(u), \forall u \in Z. \tag{3.4}$$

- $d(F, B) \le \frac{1}{1-L} d(F, JF)$, which implies the inequality

$$d(F,B) \le \frac{L}{1-L},$$

that is $(\mathbf{Est})$ is seen to be true. The statement that B is an $AddQ-$type mapping is easily seen: If we replace u by $2^n u$ and v by $2^n v$ in (3.1), then we obtain

$$\left\| \frac{F(2^n(u+v))}{8^n} + \frac{F(2^n(u+S(v)))}{8^n} - \right.$$
$$-2\left(\frac{F(2^n(u))}{8^n} + \frac{F(2^n(v))}{8^n} + \frac{1}{8^n}F\left(2^n\left(\frac{u+S(u)+v-S(v)}{2}\right)\right) + \right.$$
$$\left.\left.\frac{1}{8^n}F\left(2^n\left(\frac{u-S(u)+v+S(v)}{2}\right)\right)\right)\right\| \le \frac{\Phi(2^n u, 2^n v)}{8^n},$$

for all $u, v \in Z$. By using (3.4) and $(\mathbf{H}^*)$ and letting $n \to \infty$, we see that B satisfies (2.1). $\square$

Example 3.1: If we apply Theorem 3.2 with the mappings $\Phi : Z \times Z \to [0, \infty)$ given by $(u, v) \to \varepsilon(||u||_Z^p + ||v||_Z^q)$ and $(u, v) \to \varepsilon||u||_Z^p \cdot ||v||_Z^q$, then we obtain the stability results in Corollary 2.4 and Corollary 2.5, respectively.

Remark 3.1. In [10], we used the cubic type operator and essentially the same method as above in order to prove a generalized stability theorem for the following equation, considered by J. M. Rassias in [32]:

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) = 0$$

We only note the following Aoki type extension of the result of J. M. Rassias:

Corollary 3.3. *Let E be a (real or complex) normed space, F a Banach space and the numbers $\varepsilon, p \ge 0$ with $p \ne 3$. Suppose that the mapping $f : E \to F$ verifies, for all $x, y \in E$, the following condition:*

$$\|f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p). \tag{$\mathbf{C}_p$}$$

Then there exists ***a unique cubic mapping*** *$C : E \to F$ which satisfies the estimation*

$$\|f(x) - C(x)\| \le \frac{7\varepsilon}{6} \cdot \frac{1}{|2^3 - 2^p|} \cdot \|x\|^p, \forall x \in E. \tag{$\mathbf{Est}_p$}$$

4 FUNCTIONAL EQUATIONS OF BiQ-TYPE

A function $F : Z \to Y$ is called a $BiQ-$*type mapping* iff it satisfies the following equation

$$F(u+v)+F(u-v)+F(u+S(v))+F(u-S(v))=$$
$$=4\left(F(u)+F(v)+F\left(\frac{u+S(u)+v-S(v)}{2}\right)+F\left(\frac{u-S(u)+v+S(v)}{2}\right)\right) \tag{4.1}$$

for all $u, v \in Z$. Notice that, in inner product spaces, the function

$$Z \ni u \to F(u) = a \cdot ||P_{X_1}u||^2 \cdot ||P_{X_2}u||^2 \tag{4.2}$$

is a solution of (4.1) for any $a \in \mathbb{R}$. If F verifies (4.1) for $X_1 = X_2 = X$, then $u = (x, z) \to f(x, z) := F(u)$ is *a bi-quadratic mapping*, verifying the following equation [26]:

$$f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)=$$
$$=4\left(f(x,z)+f(y,w)+f(x,w)+f(y,z)\right),\ \forall x,y,z,w \in X. \tag{4.3}$$

Remark 4.1. Every solution F of (4.1) has the following properties:
(i) $F(0) = 0$ and F is an even mapping;
(ii) $F(2^n \cdot u) = 2^{4n} \cdot F(u),\ \forall u \in Z$ and $\forall n \in \mathbb{N}$;
(iii) $F \circ S = F$;
(iv) $F \circ P_{X_1} = F \circ P_{X_2} = 0$;
(v) If $f(x, z) = F(u)$, then f is quadratic in each variable.

We also note the following easily verified result:

Lemma 4.1. *Suppose that $F : Z \to Y$ is of the form*

$$F(u) = f_2(z)f_1(x),\ \forall u = (x, z) \in Z = X_1 \times X_2,$$

with arbitrary nonzero mappings $f_1 : X_1 \to Y$ and $f_2 : X_2 \to \mathbb{R}$. Then F is of $BiQ-$type if and only if f_1 and f_2 are quadratic.

Let us fix the following notation, related to (4.1):

$$Q_F(u,v) := F(u+v)+F(u-v)+F(u+S(v))+F(u-S(v))-$$
$$-4\left(F(u)+F(v)+F\left(\frac{u+S(u)+v-S(v)}{2}\right)+F\left(\frac{u-S(u)+v+S(v)}{2}\right)\right). \tag{4.4}$$

4.1 The Ulam-Hyers-Bourgin Stability For Biq–Type Equations

Let us consider a control mapping $\Phi : Z \times Z \to [0, \infty)$ such that

$$\Psi(u,v) := \sum_{i=0}^{\infty} \frac{\Phi(2^i u, 2^i v)}{16^{i+1}} < \infty, \forall u, v \in Z, \tag{4.5}$$

and suppose Y is a Banach space.

Theorem 4.2. *Let $F : Z \to Y$ be a mapping such that $F \circ P_{X_1} = F \circ P_{X_2} = 0$ and suppose that*

$$||Q_F(u,v)|| \le \Phi(u,v), \forall u,v \in Z. \tag{4.6}$$

Then there exists a unique $BiQ-$type mapping $B : Z \to Y$, such that

$$||F(u) - B(u)|| \le \Psi(u,u),\ \forall u \in Z. \tag{4.7}$$

Proof (sketch). We shall use the Hyers method. Letting $u = v$ in (4.6), we obtain

$$\left|\left|\frac{F(2u)}{16} - F(u)\right|\right| \le \frac{\Phi(u,u)}{16},\ \forall u \in Z.$$

In the next step, as usual, one shows that

$$\left|\left|\frac{F(2^p u)}{16^p} - \frac{F(2^m u)}{16^m}\right|\right| \le \sum_{i=p}^{m-1} \frac{\Phi(2^i u, 2^i u)}{16^{i+1}},\ \forall u \in Z,\ 0 \le p < m, \tag{4.8}$$

so that we can define the mapping $B : Z \to Y$,

$$B(u) = \lim_{n\to\infty} \frac{F(2^n u)}{16^n},\ \forall u \in Z.$$

By using (4.8) for $p = 0$ and $m \to \infty$ we obtain the estimation (4.7). By (4.6) and (4.5), we immediately see that B is a $BiQ-$type mapping. Let B_1 be a $BiQ-$type mapping, which satisfies (4.7). Then

$$||B(u) - B_1(u)|| \le$$
$$\le \left|\left|\frac{B(2^n u)}{16^n} - \frac{F(2^n u)}{16^n}\right|\right| + \left|\left|\frac{F(2^n u)}{16^n} - \frac{B_1(2^n u)}{16^n}\right|\right| \le$$
$$\le 2 \cdot \sum_{k=n}^{\infty} \frac{\Phi(2^k u, 2^k u)}{16^{k+1}} \longrightarrow 0, \text{for } n \to \infty.$$

Hence the uniqueness claim for B holds true. □

Remark 4.2. In the above proof we actually used the following fact only:

$$F \circ P_{X_1} + F \circ P_{X_2} = 0.$$

Let us consider a mapping $\varphi : X \times X \times X \times X \to [0,\infty)$ such that

$$\psi(x,z,y,w) := \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i z, 2^i y, 2^i w)}{16^{i+1}} < \infty, \forall x,y,z,w \in X.$$

As a direct consequence of Theorem 4.2, we obtain the result in ([26], Theorem 7):

Corollary 4.3. *Suppose that X is a real linear space, Y is a real Banach space and let $f : X \times X \to Y$ be a mapping such that*

$$||f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w)-$$
$$-4\left(f(x,z) + f(y,w) + f(x,w) + f(y,z)\right)|| \le \varphi(x,z,y,w),$$

and let $f(x,0) + f(0,z) = 0$, for all $x,y,z,w \in X$. Then there exists a unique bi-quadratic mapping $b : X \times X \to Y$, such that

$$||f(x,z) - b(x,z)|| \le \psi(x,z,x,z),\ \forall x,z \in X. \tag{4.9}$$

Proof. Let consider $X_1 = X_2 = X$, $u, v \in X \times X, u = (x, z), v = (y, w)$, $F(u) = f(x, z)$, and $\Phi(u, v) = \varphi(x, z, y, w)$. Since $\Psi(u, v) = \psi(x, z, y, w) < \infty$, then we can apply Theorem 4.2. Clearly, the mapping b, defined by $b(x, z) = B(u)$ is bi-quadratic and verifies (4.9). □

In the next two corollaries we give stability results of Aoki type for the equation (4.1). Let us consider a Banach space Y, the linear spaces X_1 and X_2, and suppose that $Z := X_1 \times X_2$ is endowed with a norm $||u||_Z$.

Corollary 4.4. *Let $F : Z \to Y$ be a mapping such that*

$$||Q_F(u, v)||_Y \leq \varepsilon(||u||_Z^p + ||v||_Z^p),\ \forall u, v \in Z,$$

where $p \in [0, 4)$ and $\varepsilon \geq 0$ are fixed. If $F \circ (I - S) = 0$ and $F \circ (I + S) = 0$, then there exists a unique $BiQ-$type mapping $B : Z \to Y$, such that

$$||F(u) - B(u)||_Y \leq \frac{2\varepsilon}{2^4 - 2^p} \cdot ||u||_Z^p,\ \forall u \in Z.$$

Proof. Consider the mapping $\Phi : Z \times Z \to [0, \infty)$, $\Phi(u, v) = \varepsilon(||u||_Z^p + ||v||_Z^p)$. Then

$$\Psi(u, v) := \sum_{i=0}^{\infty} \frac{\Phi(2^i u, 2^i v)}{16^{i+1}} = \varepsilon \cdot \frac{||u||_Z^p + ||v||_Z^p}{2^4 - 2^p},\ \forall u, v \in Z,$$

and we can apply Theorem 4.2. □

Now, let us consider, as in Corollarry 4.4: $X_1 = X_2 = X$, where X is a normed space, $u, v \in X \times X, u = (x, z), v = (y, w)$, $F(u) = f(x, z)$, with $f : X \times X \to Y$ and $||u|| = ||u||_p := \sqrt[p]{||x||^p + ||z||^p}, p \geq 0$. Although the functions $||\cdot||_p$ is not a norm, the above proof works as well. Actually, we obtain the following

Corollary 4.5. *Let $f : X \times X \to Y$ be a mapping such that*

$$||f(x + y, z + w) + f(x + y, z - w) + f(x - y, z + w) + f(x - y, z - w) -$$

$$-4\left(f(x, z) + f(y, w) + f(x, w) + f(y, z)\right)|| \leq \varepsilon\left(||x||^p + ||y||^p + ||z||^p + ||w||^p\right),$$

for all $x, y, z, w \in X$ and for some fixed ε, p, $0 \leq p < 4$, $\varepsilon \geq 0$. If $f(x, 0) = 0$ and $f(0, y) = 0$, for all $x, y \in X$, then there exists a unique bi-quadratic mapping $b : X \times X \to Y$, such that

$$||f(x, z) - b(x, z)|| \leq \frac{2\varepsilon}{2^4 - 2^p} \cdot \left(||x||^p + ||z||^p\right),$$

for all $x, z \in X$.

Furthermore, by using the means inequality or directly, two interesting results of J. M. Rassias type can be obtained for products:

Corollary 4.6. *Let $F : Z \to Y$ be a mapping such that*

$$||Q_F(u, v)||_Y \leq \varepsilon \cdot ||u||_Z^p \cdot ||v||_Z^p),\ \forall u, v \in Z,$$

where $p \in [0, 2)$ and $\varepsilon \geq 0$ are fixed. If $F \circ (I - S) = 0$ and $F \circ (I + S) = 0$, then there exists a unique $BiQ-$type mapping $B : Z \to Y$, such that

$$||F(u) - B(u)||_Y \leq \frac{\varepsilon}{2^4 - 2^{2p}} \cdot ||u||_Z^{2p},\ \forall u \in Z.$$

Corollary 4.7. *Let* $f : X \times X \to Y$ *be a mapping such that*

$$||f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) + f(x-y, z-w)-$$

$$-4\left(f(x,z) + f(y,w) + f(x,w) + f(y,z)\right)|| \le \varepsilon\left(||x||^p \cdot ||y||^p \cdot ||z||^p \cdot ||w||^p\right),$$

for all $x, y, z, w \in X$ *and for some fixed* $\varepsilon \ge 0,\ p \in [0,1)$. *If* $f(x,0) = 0$ *and* $f(0,y) = 0$*, for all* $x, y \in X$*, then there exists a unique bi-quadratic mapping* $b : X \times X \to Y$*, such that*

$$||f(x,z) - b(x,z)|| \le \frac{\varepsilon}{2^4 - 2^{4p}} \cdot ||x||^{2p} \cdot ||z||^{2p},\ \forall x, z \in X.$$

Remark 4.3. Let $\tilde{h} : \mathbb{R} \to \mathbb{R}, \tilde{h}(x) = x^2$ and $\tilde{f} : X \to Y,$ where X is a real normed space and Y a real Banach space. By using Lemma 4.1, we could apply Theorem 4.2 for $X_2 = X, X_1 = \mathbb{R}$, $u, v \in \mathbb{R} \times X$, with $u = (x,z), v = (y,w)$ and the mappings given by $F(u) = F(x,z) = x^2 \cdot \tilde{f}(z),\ \Phi(u,v) = \Phi(x,z,y,w) = (x^2 + y^2) \cdot \tilde{\varphi}(z,w)$, to obtain a new proof of Proposition 2.9.

5 THE FIXED POINT METHOD FOR BiQ–TYPE EQUATIONS

Let Y be a real Banach space and $\Phi : Z \times Z \to [0,\infty)$ be an arbitrary given function.

Theorem 5.1. *Let* $F : Z \to Y$ *be such that* $F \circ (I - S) = 0,\ F \circ (I + S) = 0$ *and*

$$||Q_F(u,v)|| \le \Phi(u,v),\ \forall u, v \in Z. \tag{5.1}$$

If, moreover, there exists $L < 1$ *such that the mapping* $u \to \Omega(u) = \Phi\left(\frac{u}{2}, \frac{u}{2}\right)$ *verifies the condition*

$$\Omega(u) \le L \cdot 2^4 \cdot \Omega\left(\frac{u}{2}\right), \forall u \in Z \tag{$\mathbf{H}$}$$

and the mapping Φ *has the property*

$$\lim_{n\to\infty} \frac{\Phi\left(2^n u, 2^n v\right)}{2^{4n}} = 0, \forall u, v \in Z, \tag{$\mathbf{H}^*$}$$

then there exists a unique $BiQ-$*type mapping* $B : Z \to Y$*, such that*

$$||F(u) - B(u)|| \le \frac{L}{1-L}\Omega(u),\ \forall u \in Z. \tag{$\mathbf{Est}$}$$

Proof. We define a *complete generalized metric* on the set $\mathcal{F} := \{G : Z \to Y\}$:

$$d(G,H) = d_\Omega(G,H) = \inf\left\{K \in \mathbb{R}_+, \|G(u) - H(u)\| \le K\Omega(u), \forall u \in Z\right\}$$

and the (linear) *operator of quartic type*

$$J : \mathcal{F} \to \mathcal{F}, JG(u) := \frac{G(2u)}{2^4}.$$

It is easy to see that $d(JG, JH) \le L \cdot d(G,H), \forall G, H \in \mathcal{F}$, so that J is a *strictly contractive* self-mapping of $\mathcal{F}$. If we set $u = v = t$ in the relation (5.1), then we see that $\|F(2t) - 16F(t)\| \le \Omega(2t), \forall t \in Z$. Using the hypothesis $(\mathbf{H})$ we have

$$\left\|\frac{F(2t)}{16} - F(t)\right\| \le \frac{\Omega(2t)}{16} \le L\Omega(t), \forall t \in Z,$$

hence $d(F, JF) \leq L < \infty$. We can apply the fixed point alternative, and we obtain the existence of a mapping $B : \mathcal{F} \to \mathcal{F}$ such that:

- B is a fixed point of the operator J, that is

$$B(2u) = 16B(u), \forall u \in Z. \tag{5.2}$$

The mapping B is the unique fixed point of J in the set $\{G \in \mathcal{F},\ d(F, G) < \infty\}$. Thus B is the unique mapping with *both* the properties (5.2) and (5.3), where

$$\exists K \in (0, \infty) \text{ such that } \|B(u) - F(u)\| \leq K\Omega(u), \forall u \in Z. \tag{5.3}$$

Moreover,

- $d(J^n F, B) \xrightarrow[n\to\infty]{} 0$, which implies the equality

$$\lim_{n\to\infty} \frac{F(2^n u)}{2^{4n}} = B(u), \forall u \in Z. \tag{5.4}$$

- $d(F, B) \leq \frac{1}{1-L} d(F, JF)$, which implies $(\mathbf{Est})$.

The statement that B is a $BiQ-$type mapping follows from (5.1).□

Example 5.1: Consider a real Banach space Y and the real linear spaces X_1, X_2 such that $Z := X_1 \times X_2$ is endowed with a norm $||u||_Z$. If we apply Theorem 5.1 with the mappings $\Phi : Z \times Z \to [0, \infty)$ given by $(u, v) \to \varepsilon(||u||_Z^p + ||v||_Z^p)$ and $(u, v) \to \varepsilon||u||_Z^p \cdot ||v||_Z^p$, then we obtain the stability results in Corollary 4.4 and Corollary 4.6, respectively.

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On Drygas functional equation on groups

Valeriĭ A. Faĭziev[1] **and Prasanna K. Sahoo**[2]

[1]Tver State Agricultural Academy
Tver Sakharovo, Russia
vfaiz@tvcom.ru

[2]Department of Mathematics
University of Louisville
Louisville, Kentucky 40292 USA
sahoo@louisville.edu

ABSTRACT

Drygas (1987) introduced the functional equation $f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1})$ in connection with the characterization of quasi-inner-product spaces. In this paper, we study the system of functional equations $f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1})$ and $f(yx) + f(y^{-1}x) = 2f(x) + f(y) + f(y^{-1})$ on groups. Here f is a real-valued function that takes values on a group. On groups, this system generalizes the functional equation introduced by Drygas.

Keywords: Additive character of a group, bihomomorphism, Drygas functional equation, free group, homomorphism, Jensen functional equation, n-Abelian group, and quadratic functional equation.

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1 Introduction

Drygas (1987) obtained a Jordan and von Neumann type characterization theorem for quasi-inner product spaces. In Drygas's characterization of quasi-inner product spaces the functional equation

$$f(x) + f(y) = f(x-y) + 2\left\{f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right)\right\}$$

played an important role. If we replace y by $-y$ in the above functional equation and add the resulting equation to the above equation, then we obtain

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) \tag{1.1}$$

for all $x, y \in \mathbb{R}$ (the set of real numbers). This functional equation was studied by Szabó (1983) and Skof (2002). The Drygas functional equation (1.1) on an arbitrary group G takes the form

$$f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) = 0 \tag{1.2}$$

for all $x, y \in G$. The stability of the functional equation (1.1) was studied by Jung and Sahoo (2002). For stability of other functional equations the interested reader referred to Rassias (1982, 1984, 1989, 2001, 2002), and Rasiass and Rassias (2003). In Ebanks, Kannappan and Sahoo (1992), the system of equations

$$f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) = 0, \qquad f(zyx) = f(zxy) \tag{1.3}$$

for all $x, y, z \in G$ was solved without any regularity assumption on f. It was shown there that the general solution $f : G \to K$ (a commutative field with characteristic different from two) of the functional equation (1.3) is given by

$$f(x) = H(x, x) + A(x)$$

where $H : G \times G \to K$ is a symmetric bihomomorphism and $A : G \to K$ is a homomorphism.
In this paper, we consider the following system of equations:

$$\begin{cases} f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) = 0, \\ f(yx) + f(y^{-1}x) - 2f(x) - f(y) - f(y^{-1}) = 0 \end{cases} \tag{1.4}$$

for all $x, y \in G$. Here $f : G \to \mathbb{R}$ is an unknown function to be determined. The main goal of this paper is to study the system of functional equations (1.4) on groups.
In this sequel, we will write the arbitrary group G in multiplicative notation so that 1 will denote the identity element of G.

2 Jensen, Quadratic and Drygas equations on groups

The functional equation

$$f(xy) + f(xy^{-1}) - 2f(x) = 0 \tag{2.1}$$

for all $x, y \in G$ is called the Jensen functional equation on a group G. A function $f : G \to \mathbb{R}$ is said to be a Jensen function if f satisfies (2.1).
It is evident that a constant function satisfies the equation (2.1). The set of Jensen functions will be denoted by $J(G)$, and the subset of $J(G)$ consisting of functions f satisfying condition $f(1) = 0$ will be denoted by $J_0(G)$. It is clear that the linear space $J(G)$ is direct sum of its subspaces $C(G)$ and $J_0(G)$. Here $C(G)$ denotes the set of constant functions on G. The subspace of $J_0(G)$ consisting of real additive characters will be denoted by $X(G)$.

Lemma 2.1. *If G is an arbitrary group and $f \in J_0(G)$, then $f(x^n) = nf(x)$ for any $x \in G$ and any $n \in \mathbb{Z}$ (the set of integers).*

Proof. If we put $y = x$, then from (2.1) we get

$$f(x^2) = 2f(x).$$

From (2.1) it follows

$$f(x^k y) + f(x^k y^{-1}) - 2f(x^k) = 0.$$

Now if we put $y = x$, then we get

$$f(x^{k+1}) + f(x^{k-1}) - 2f(x^k) = 0.$$

Now by induction on k we get $f(x^n) = nf(x)$. □

Lemma 2.2. *If G is an arbitrary group, $f \in J_0(G)$, and f is constant on every class of conjugate elements, then $f \in X(G)$.*

Proof. From (2.1), for any $x, y \in G$, we get

$$f(xy) + f(xy^{-1}) - 2f(x) = 0$$

and

$$f(yx) + f(yx^{-1}) - 2f(y) = 0,$$

respectively. Taking into account relations $f(xy) = f(yx)$ and $f(yx^{-1}) = -f(xy^{-1})$, we get

$$2f(xy) - 2f(x) - 2f(y) = 0$$

which is

$$f(xy) = f(x) + f(y).$$

This completes the proof of the lemma. □

Corollary 2.3. *For any abelian group G we have the relation $J_0(G) = X(G)$.*

Corollary 2.4. *Let G be an arbitrary group and x, y be elements of G such that $xy = yx$. Then for any $f \in J_0(G)$ the following relation holds $f(xy) = f(x) + f(y)$.*

A functional equation of the form

$$f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) = 0 \tag{2.2}$$

for all $x, y \in G$ is called the quadratic equation. If a function $f : G \to \mathbb{R}$ satisfies the quadratic equation, then f is called a quadratic function. The set of functions satisfying (2.2) will be denoted by $Q(G)$.

Lemma 2.5. *If G is an arbitrary group and $f \in Q(G)$, then $f(1) = 0$ and $f(x^{-1}) = f(x)$ for any $x \in G$.*

Proof. If $x = y = 1$, then (2.2) implies $f(1) = 0$. Now if $x = 1$, then (2.2) implies $f(y) + f(y^{-1}) - 2f(1) - 2f(y) = 0$. Hence it follows that $f(y) = f(y^{-1})$ for any $y \in G$. □

Lemma 2.6. *If G is an arbitrary group and $f \in Q(G)$, then*

1. *f is constant on any class of conjugate elements of G.*
2. $f(x^n) = n^2 f(x), \quad \forall x \in G$ *and* $\forall n \in \mathbb{Z}$.

Proof. 1. Since $f \in Q(G)$, we have

$$f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) = 0$$

and

$$f(yx) + f(yx^{-1}) - 2f(x) - 2f(y) = 0$$

for all $x, y \in G$. Thus, we see that

$$f(xy) + f(xy^{-1}) = f(yx) + f(yx^{-1}). \tag{2.3}$$

Since $f(yx^{-1}) = f((xy^{-1})^{-1})$, from Lemma 2.5 and (2.3), we get

$$f(xy) = f(yx), \tag{2.4}$$

for any $x, y \in G$.

2. If $y = x$, then from (2.2) and Lemma 2.5 it follows that

$$f(x^2) = 4f(x).$$

Now suppose that for $i = 1, 2, \ldots, k$ the formula

$$f(x^i) = i^2 f(x)$$

has been already established. Let us check it for $i = k + 1$. We have

$$f(x^k x) + f(x^k x^{-1}) = 2f(x^k) + 2f(x),$$
$$f(x^{k+1}) + f(x^{k-1}) = 2f(x^k) + 2f(x).$$

Therefore

$$\begin{aligned} f(x^{k+1}) &= -f(x^{k-1}) + 2f(x^k) + 2f(x) \\ &= [-(k-1)^2 + 2k^2 + 2]f(x) \\ &= (k+1)^2 f(x), \end{aligned}$$

and the proof of the lemma is now complete. □

Lemma 2.7. *If f satisfies (1.4), then $\varphi(x) := f(x) + f(x^{-1})$ belongs to $Q(G)$ and $\psi(x) := f(x) - f(x^{-1})$ be belongs to $J_0(G)$.*

Proof. Since

$$\begin{aligned} &\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x) - 2\varphi(x) \\ &\quad = f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \\ &\qquad + f(y^{-1}x^{-1}) + f(yx^{-1}) - 2f(x^{-1}) - 2f(y^{-1}) \\ &\quad = f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) \\ &\qquad + f(y^{-1}x^{-1}) + f(yx^{-1}) - 2f(x^{-1}) - f(y) - f(y^{-1}) \\ &\quad = 0, \end{aligned}$$

$\varphi \in Q(G)$. It is clear that $\psi(1) = 0$, and since

$$\begin{aligned}
&\psi(xy) + \psi(xy^{-1}) - 2\psi(x) \\
&\quad = f(xy) + f(xy^{-1}) - 2f(x) \\
&\qquad\qquad -[f(y^{-1}x^{-1}) + f(yx^{-1}) - 2f(x^{-1})] \\
&\quad = f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) \\
&\qquad\qquad -[f(y^{-1}x^{-1}) + f(yx^{-1}) - 2f(x^{-1}) - f(y) - f(y^{-1})] \\
&\quad = 0,
\end{aligned}$$

$\phi \in J_0(G)$. The proof of the lemma is now complete. □

Let $D(G)$ denote the set of all functions satisfying the system (1.4). It is evident that $D(G)$ is a linear space.

Theorem 2.8. *The linear space $D(G)$ is a direct sum of $J_0(G)$ and $Q(G)$, that is $D(G) = J_0(G) \oplus Q(G)$.*

Proof. Let us show that $J_0(G)$ belongs to $D(G)$. Let $f \in J_0(G)$. Then we have $f(x^{-1}) = -f(x)$ for all $x \in G$. In order to show that f satisfies the system (1.4) it is necessary and sufficient to show that f satisfies the system

$$\begin{cases} f(xy) + f(xy^{-1}) - 2f(x) = 0 \\ f(yx) + f(y^{-1}x) - 2f(x) = 0 \end{cases} \tag{2.5}$$

for all $x, y \in G$. The first equation in (2.5) is the Jensen equation. Let us show that the second equation of this system is equivalent to the first one.
If we interchange x by x^{-1} in the second equation in (2.5), we get

$$f(yx^{-1}) + f(y^{-1}x^{-1}) - 2f(x^{-1}) = 0.$$

The last equation can be rewritten as

$$f((xy^{-1})^{-1}) + f((xy)^{-1}) - 2f(x^{-1}) = 0$$

or

$$-[f((xy^{-1})) + f(xy) - 2f(x)] = 0$$

or

$$f((xy^{-1})) + f(xy) - 2f(x) = 0$$

or

$$f(xy) + f(xy^{-1}) - 2f(x) = 0.$$

So, $J_0(G) \subseteq D(G)$. It is clear that every quadratic function on G satisfies the system (1.4), so $Q(G) \subseteq D(G)$. Let us show that $J_0(G) \cap Q(G) = \{0\}$. Indeed, if $f \in J_0(G)$ then for any $x \in G$ we have $f(x^{-1}) = -f(x)$, and if $f \in Q(G)$ then for any $x \in G$ we have $f(x^{-1}) = f(x)$, therefore for any $f \in J_0(G) \cap Q(G)$ we have $f(x) = -f(x)$. Hence $f \equiv 0$. Let $f \in D(G)$. Then by Lemma 2.7, $f = \frac{1}{2}\varphi + \frac{1}{2}\psi$, where φ and ψ are functions as defined in Lemma 2.7. So $D(G) = J_0(G) \oplus Q(G)$ and the proof of the theorem is now complete. □

Let n be an integer. A group G is said to be an n-Abelian group if $(xy)^n = x^n y^n$ for every x and y in G. For more on n-Abelian groups the interested reader should refer to Levi (1944, 1945), Baer (1953), Gallian and Reid (1993), and Li (1998). For any $n \in \mathbb{N}$ (the set of natural numbers), let $\mathcal{K}_n$ denote the class of groups G satisfying the relation

$$(xy)^n = x^n y^n = y^n x^n \tag{2.6}$$

for any $x, y \in G$. Obviously, the class $\mathcal{K}_n$ is a subclass of the class of n-Abelian groups. For $n \in \mathbb{N}$, let G^n be the subgroup of G generated by the set $\{\, x^n \mid x \in G\}$.

Theorem 2.9. *Let G belong to the class $\mathcal{K}_n$. Then*

1. $J_0(G) = X(G)$.
2. *Any element f from $Q(G)$ is representable in the form $f(x) = \frac{1}{n^2}\varphi(x^n)$, where $\varphi \in Q(G^n)$. Further $Q(G) = Q(G^n)$.*
3. $D(G) = X(G) \oplus Q(G^n)$.

Proof. 1. The subgroup G^n is abelian. Hence $J_0(G^n) = X(G^n)$. Let $f \in J_0(G)$. Then for any $x \in G$, the element x^n belongs to G^n. Thus by Lemma 2.1, we have

$$\begin{aligned} f(xy) &= \tfrac{1}{n}\; n\; f(xy) \\ &= \tfrac{1}{n}\, f((xy)^n) \\ &= \tfrac{1}{n}\, f(x^n y^n) \\ &= \tfrac{1}{n}\, f(x^n) + \tfrac{1}{n}\, f(y^n) \\ &= f(x) + f(y). \end{aligned}$$

2. Let $f \in Q(G)$, and $\varphi = f\big|_{G^n}$. Then $\varphi \in Q(G^n)$. Now for any $\varphi \in Q(G^n)$ define a function $f : G \to \mathbb{R}$ by $f(x) := \frac{1}{n^2}\,\varphi(x^n)$. Let us verify that $f \in Q(G)$. Taking into account Lemma 2.6 item 2, we get

$$\begin{aligned} &f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \\ &\quad = \tfrac{n^2}{n^2} f(xy) + \tfrac{n^2}{n^2} f(xy^{-1}) - 2\tfrac{n^2}{n^2} f(x) - 2\tfrac{n^2}{n^2} f(y) \\ &\quad = \tfrac{1}{n^2}\left[\,\varphi((xy)^n) + \varphi((xy^{-1})^n) - 2\varphi(x^n) - 2\varphi(y^n)\,\right] \\ &\quad = \tfrac{1}{n^2}\left[\,\varphi(x^n y^n) + \varphi(x^n (y^{-1})^n) - 2\varphi(x^n) - 2\varphi(y^n)\,\right] = 0, \end{aligned}$$

and thus $f \in Q(G)$. It is easy to check that the mapping $f \mapsto \varphi$ is an isomorphism of $Q(G)$ onto $Q(G^n)$. So we have $Q(G) = Q(G^n)$.

3. Follows from Theorem 2.8, and items 1 and 2 of this theorem.

The proof of the theorem is now complete. □

3 Quadratic equation on groups with two generators

Lemma 3.1. *Let G be a group with two generators a, b. If $f \in Q(G)$ and $f(a) = f(b) = f(ab) = 0$, then $f \equiv 0$.*

Proof. We may assume that

$$f(a) = f(b) = f(ab) = f(ab^{-1}) = 0. \tag{3.1}$$

Since $f \in Q(G)$, we have

$$f(a^n b^{m+1}) + f(a^n b^{m-1}) - 2f(a^n b^m) - 2f(b) = 0.$$

Thus using (3.1), we obtain

$$f(a^n b^{m+1}) + f(a^n b^{m-1}) = 2f(a^n b^m). \tag{3.2}$$

Similarly we get

$$f(a^{n+1} b^m) + f(a^{n-1} b^m) = 2f(a^n b^m). \tag{3.3}$$

Using (3.1), (3.2), and (3.3) and by induction on n and m we see that $f(a^n b^k) = 0$ for all $n, m \in \mathbb{Z}$.
Let A and B be subgroups of G generated by elements a and b, respectively. Every nonunit element of G is conjugate to an element of the form

$$a_1 b_1 \cdots a_k b_k \quad \text{where} \quad a_i \in A \quad \text{and} \quad b_i \in B. \tag{3.4}$$

Now by induction on k we show that if an element g has the form (3.4), then $f(g) = 0$.
We have

$$f(a_1 b_1 \cdots a_k b_k a_{k+1} b_{k+1}) + f(a_1 b_1 \cdots a_k b_k (a_{k+1} b_{k+1})^{-1}) \\ -2f(a_1 b_1 \cdots a_k b_k) - 2f(a_{k+1} b_{k+1}) = 0.$$

That is

$$f(a_1 b_1 \cdots a_k b_k a_{k+1} b_{k+1}) + f(a_1 b_1 \cdots a_k b_k b_{k+1}^{-1} a_{k+1}^{-1}) \\ -2f(a_1 b_1 \cdots a_k b_k) - 2f(a_{k+1} b_{k+1}) = 0. \tag{3.5}$$

From induction hypothesis we have $f(a_1 b_1 \cdots a_k b_k) = f(a_{k+1} b_{k+1}) = 0$. It is clear that element $a_1 b_1 \cdots a_k b_k b_{k+1}^{-1} a_{k+1}^{-1}$ is conjugate to $(a_{k+1}^{-1} a_1) b_1 \cdots a_k (b_k b_{k+1}^{-1})$, hence by induction hypothesis we get $f((a_{k+1}^{-1} a_1) b_1 \cdots a_k (b_k b_{k+1}^{-1})) = 0$. Now from (3.5) it follows that $f(a_1 b_1 \cdots a_k b_k a_{k+1} b_{k+1}) = 0$. Therefore $f \equiv 0$ on G and the proof is now complete. □

Theorem 3.2. *Let F be a free group with two generators a and b. Then every element f in $Q(F)$ is representable in the form $f(x) = \psi(\tau(x))$, where $\psi \in Q(A)$, $A = \mathbb{Z} \times \mathbb{Z}$ is a free abelian group of rank two, and $\tau : F \to A$ is an epimorphism such that $\tau(a) = v$ and $\tau(b) = w$. Here v and w denote the free generators of A.*

Proof. Let $f(a) = \alpha, f(b) = \beta, f(ab) = \gamma$, and let v and w be the free generators of A. Let ψ be an element of $Q(\mathbb{Z} \times \mathbb{Z})$ such that $\psi(v) = \alpha, \psi(w) = \beta, \psi(vw) = \gamma$. Then we have $\varphi = f - \psi \circ \tau \in Q(F)$ and $\varphi(a) = \varphi(b) = \varphi(ab) = 0$. Now from Lemma 3.1 we get $\varphi \equiv 0$. Hence $f = \psi \circ \tau$. □

Theorem 3.3. *Let G be a group with two generators, and G' be the commutator subgroup of G and $\tau : G \to G/G'$ be the natural epimorphism from G onto G/G'. Then every element f in $Q(G)$ is representable in the form $f(x) = \varphi(\tau(x))$, where $\varphi \in Q(G/G')$.*

Proof. Let us verify that f is constant on every coset G' of group G. Suppose that there are $x \in G$ and $u \in G'$ such that $f(xu) \neq f(x)$. Let $\pi : F \to G$ be the natural epimorphism of F onto G and let $\varphi = f \circ \pi$. Then for some $t \in F$ and $v \in F'$ (the commutator subgroup of F) we have $\pi(t) = x$, $\pi(v) = u$. Therefore $\pi(tv) = xu$ and $\varphi(tv) = f(\pi(tv)) = f(xu) \neq f(x) = f(\pi(t)) = \varphi(t)$.
Thus it follows that f is constant on every coset G' of G and we can define $\varphi \in Q(G/G')$ such that $f = \varphi \circ \tau$. □

Theorem 3.4. *Suppose $\{a, b\}$ is a basis of the free abelian group $\mathbb{Z} \times \mathbb{Z}$ of rank two. Then any element $f \in Q(\mathbb{Z} \times \mathbb{Z})$ is uniquely representable in the form $f(a^n b^m) = n^2\alpha + m^2\beta + nm\lambda$ for some $\alpha, \beta, \lambda \in \mathbb{R}$.*

Proof. Let $\{a, b\}$ be a basis of $G = \mathbb{Z} \times \mathbb{Z}$. Denote by A and B subgroups of $G = \mathbb{Z} \times \mathbb{Z}$ generated by elements a and b respectively. Let π_1 and π_2 be canonical projections of G onto A and B respectively. Denote by ξ_1 and ξ_2 additive characters of A and B respectively such that $\xi_1(a^n) = n$ and $\xi_2(b^n) = n$. It is easy to see that the functions $e_i(x) = \xi_i^2(\pi_i(x))$ are quadratic functions on G. Denote by $e_3(x)$ a function defined by the rule: $e_3(x) = e_1(x)e_2(x)$.
It is easy to verify that e_3 is also a quadratic function on G. Let us verify that functions e_1, e_2, e_3 form a basis of the space $Q(G)$. First let us verify that these functions are linearly independent. Suppose that there are $\alpha_1, \alpha_2, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \equiv 0$. It follows that

$$\alpha_1 e_1(a) + \alpha_2 e_2(a) + \alpha_3 e_3(a) = 0. \tag{3.6}$$

But $e_2(a) = e_3(a) = 0$. So we see that $\alpha_1 = 0$. Similarly $e_1(b) = e_3(b) = 0$ and we get $\alpha_2 = 0$. Now from relations (3.6) and $e_3(ab) = 1$ it follows that $\alpha_3 = 0$.
Now let us verify that every element $\varphi \in Q(G)$ is a linear combination of the functions e_i, $i = 1, 2, 3$. Let $\varphi(a) = \alpha, \varphi(b) = \beta, \varphi(ab) = \gamma$, consider the function $\psi(x) = \alpha e_1(x) + \beta e_2(x) + (\gamma - \alpha - \beta)e_3(x)$. Then the function $f(x) = \varphi(x) - \psi(x)$ is an element of $Q(G)$, such that $f(a) = 0, f(b) = 0, f(ab) = 0$. Lemma 3.1 implies that $f \equiv 0$. So, $\varphi \equiv \psi$, or $\varphi(x) = \alpha e_1(x) + \beta e_2(x) + (\gamma - \alpha - \beta)e_3(x)$, and the proof of the theorem is complete. □

4 Drygas equation on a certain group with two generators

Let G be an arbitrary group. For $a, b, c \in G$, we define $[a, b] = a^{-1}b^{-1}ab$. Consider the group H over two generators a, b and the following defining relations:

$$[b, a]a = a[b, a], \quad b[b, a] = [b, a]b.$$

If we set $c = [b, a]$ we get the following presentation of H in terms of generators and defining relations:

$$H = \langle\, a, b \mid c = [b, a], \quad [c, a] = [c, b] = 1 \,\rangle\,. \tag{4.1}$$

It is well known that each element of H can be uniquely represented as $g = a^m b^n c^k$, where $m, n, k \in \mathbb{Z}$. The mapping

$$g = a^m b^n c^k \to \begin{bmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix}$$

is an isomorphism between H and $UT(3, \mathbb{Z})$.

Lemma 4.1. *A function $\phi : H \to \mathbb{R}$ defined by the formula $\phi(a^m b^n c^k) = mn - 2k$ is an element of $J_0(H)$.*

Proof. It is clear that $\phi(1) = 0$. Now let $x = a^m b^n c^k$, $y = a^{m_1} b^{n_1} c^{k_1}$, then $xy^{-1} = a^m b^n c^k c^{-k_1} b^{-n_1} a^{-m_1}$ $= a^{m-m_1} b^{n-n_1} c^{m_1 n_1 - m_1 n + k - k_1}$. Hence

$$\begin{aligned} &\phi(xy) + \phi(xy^{-1}) - 2\phi(x) \\ &\quad = (m + m_1)(n + n_1) - 2(m_1 n + k + k_1) \\ &\quad\quad +(m - m_1)(n - n_1) - 2(m_1 n_1 - m_1 n + k - k_1) - 2(mn - 2k) \\ &\quad = 0 \end{aligned}$$

and the proof of the lemma is now complete. □

Lemma 4.2. *Let $f \in J_0(H)$ and $f(a) = f(b) = f(c) = 0$, then $f \equiv 0$.*

Proof. Since H has the presentation

$$H = \langle\, a, b \mid c = [b, a], \quad [c, a] = [c, b] = 1 \,\rangle,$$

we have $ba = abc$. Further it follows that $b^p a^q = a^q b^p c^{pq}$.
Let $x = a^m b^n c^k$, $y = a^{m_1} b^{n_1} c^{k_1}$. Hence we see that

$$xy = a^{m+m_1} b^{n+n_1} c^{m_1 n + k + k_1} \tag{4.2}$$

and

$$yx = a^{m+m_1} b^{n+n_1} c^{m n_1 + k + k_1}. \tag{4.3}$$

Element c belongs to center of H. Therefore from the Corollary 2.4 and relations (4.2) and (4.3) we get

$$f(xy) = f(a^{m+m_1} b^{n+n_1}) + f(c^{m_1 n + k + k_1}) \tag{4.4}$$

and

$$f(yx) = f(a^{m+m_1} b^{n+n_1}) + f(c^{m n_1 + k + k_1}). \tag{4.5}$$

Now taking into account equality $f(c) = 0$ we obtain

$$f(xy) = f(a^{m+m_1} b^{n+n_1}) \tag{4.6}$$

and

$$f(yx) = f(a^{m+m_1} b^{n+n_1}). \tag{4.7}$$

From the two last relations we see that $f(xy) = f(yx)$ for any $x, y \in H$. Then by Lemma 2.2, $f \in X(H)$. But $f(a) = f(b) = 0$. Hence $f \equiv 0$. □

Theorem 4.3. *Every element $f \in J_0(H)$ is uniquely representable in the form $f(x) = \xi(x) + \mu\phi(x)$, where $\xi \in X(H),\ \mu \in \mathbb{R}$ and ϕ is the function defined in the Lemma 4.1.*

Proof. Let ξ be an element of $X(H)$ such that $\xi(a) = f(a)$ and $\xi(b) = f(b)$. Further let $f(c) = \lambda$. Note that $\xi(c) = 0$ since c belongs to the commutator subgroup of H. Define $\psi(x) = \xi(x) - \frac{1}{2}\lambda\phi(x)$ then $\psi \in J_0(H)$ and the function $f(x) - \psi(x)$ is equal to zero on elements a, b, c. By Lemma 4.2 we get $f(x) \equiv \psi(x)$ and hence $f(x) = \xi(x) - \frac{1}{2}\lambda\phi(x)$. This completes the proof of the theorem. □

Now from Theorems 2.8, 4.3, 3.2 and 3.4 we have the following theorem.

Theorem 4.4. *Let the group H has the presentation*

$$H = \langle\, a, b \mid c = [b, a], \quad [c, a] = [c, b] = 1 \,\rangle \,.$$

Then every element f in $D(H)$ is uniquely representable in the form

$$f(a^n b^m c^k) = \xi(a^n b^m c^k) + \mu\phi(a^n b^m c^k) + \alpha n^2 + \beta m^2 + \gamma nm \tag{4.8}$$

for some $\xi \in X(H)$ and $\alpha, \beta, \gamma, \mu \in \mathbb{R}$. Here $\phi : H \to \mathbb{R}$ is the function as defined in the Lemma 4.1, and $m, n \in \mathbb{Z}$.

Remark 4.1*.* The function ϕ is a solution of the system (1.4). It is clear that ϕ is not a constant function on conjugacy classes in H, so ϕ is not a solution of (1.3).

Remark 4.2*.* If a function f satisfies system (1.3), then it satisfies system (1.4).

Proof. From the relation $f(zxy) = f(zyx)$ it follows that $f(xy) = f(yx)$. Now from this relation and the first equation of (1.4) we get $f(yx) + f(y^{-1}x) - 2f(x) - f(y) - f(y^{-1}) = 0$. □

So, we see that the system (1.4) is a generalization of the system (1.3).

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Stability of Drygas functional equation on $T(3, \mathbb{R})$

Valeriĭ A. Faĭziev[1] **and Prasanna K. Sahoo**[2]

[1]Tver State Agricultural Academy
Tver Sakharovo, Russia
vfaiz@tvcom.ru

[2]Department of Mathematics
University of Louisville
Louisville, Kentucky 40292 USA
sahoo@louisville.edu

ABSTRACT

In this paper, we study the stability of the system of functional equations $f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1})$ and $f(yx) + f(y^{-1}x) = 2f(x) + f(y) + f(y^{-1})$ on non-abelian group $T(3, \mathbb{R})$. Here f is a real-valued function that takes values on a group. We show that this system is stable on the group $T(3, \mathbb{R})$.

Keywords: bihomomorphism, Drygas function, homomorphism, Jensen functional equation, pseudojensen function, pseudoquadratic function, quadratic function, quasidrygas function, quasijensen function, quasiquadratic function, semidirect product of groups, and $T(3, \mathbb{R})$ group.

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1 Introduction

Given an operator T and a solution class $\{u\}$ with the property that $T(u) = 0$, when does $\|T(v)\| \leq \delta$ for a $\delta > 0$ imply that $\|u-v\| \leq \varepsilon(\delta)$ for some u and for some $\varepsilon > 0$? This problem is called the stability of the functional transformation (see Ulam (1964)). A great deal of work has been done in connection with the ordinary and partial differential equations. If f is a function from a normed vector space into a Banach space, and $\|f(x+y)-f(x)-f(y)\| \leq \delta$, Hyers (1941) proved that there exists an additive function A such that $\|f(x)-A(x)\| \leq \delta$ (cf. Rassias (1978)). If $f(x)$ is a real continuous function of x over $\mathbb{R}$, and $|f(x+y) - f(x) - f(y)| \leq \delta$, it was shown by Hyers and Ulam (1952) that there exists a constant k such that $|f(x) - kx| \leq 2\delta$. Taking these results into account, we say that the additive Cauchy equation $f(x+y) = f(x) + f(y)$ is stable in the sense of Hyers and Ulam. The interested reader should refer to Rassias (1982, 1988, 1989, 1992, 1994, 2001, 2002), Rassias and Rassias (2003, 2005), and the book by Hyers, Isac and Rassias (1998) for an in depth account on the subject of stability of functional equations.

Drygas (1987) obtained a Jordan and von Neumann type characterization theorem for quasi-inner product spaces. In Drygas's characterization of quasi-inner product spaces the functional equation

$$f(x) + f(y) = f(x-y) + 2\left\{ f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right)\right\}$$

played an important role. If we replace y by $-y$ in the above functional equation and add the resulting equation to the above equation, then we obtain

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y). \tag{1.1}$$

The Drygas functional equation (1.1) on an arbitrary group G takes the form

$$f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) = 0 \tag{1.2}$$

for all $x, y \in G$. The stability of the equation (1.2) was studied in Jung and Sahoo (2002) when the domain of f was a real vector space and the range was a Banach space. In Yang (2004), the stability of (1.2) was treated when the domain of f was an Abelian group and the range of f was a Banach space.

In Ebanks, Kannappan and Sahoo (1992) the system of equations

$$f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) = 0, \qquad f(zyx) = f(zxy) \tag{1.3}$$

for all $x, y, z \in G$ was solved without any regularity assumption on f. It was shown there that the general solution $f : G \to K$ (a commutative field with characteristic different from two) of the system of functional equations (1.3) is given by

$$f(x) = H(x,x) + A(x)$$

where $H : G \times G \to K$ is a symmetric bihomomorphism and $A : G \to K$ is a homomorphism.

In this sequel, we will write the arbitrary group G in multiplicative notation so that 1 will denote the identity element of G. In Faiziev and Sahoo (2005c), it was shown that on an arbitrary group G, the system of equations

$$\begin{cases} f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) = 0, \\ f(yx) + f(y^{-1}x) - 2f(x) - f(y) - f(y^{-1}) = 0 \end{cases} \tag{1.4}$$

for all $x, y \in G$, is a generalization of the system (1.3). Here $f : G \to \mathbb{R}$ (the set of real numbers) is the unknown function to be determined.

The system (1.4) is said to be stable if for any f satisfying the system of inequalities

$$\begin{cases} |f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1})| \le \delta, \\ |f(yx) + f(y^{-1}x) - 2f(x) - f(y) - f(y^{-1})| \le \delta \end{cases} \tag{1.5}$$

for some positive number δ there is a φ, a solution of (1.4), and a positive number ε such that

$$|f(x) - \varphi(x)| \le \varepsilon, \quad \forall x \in G.$$

In Faiziev and Sahoo (2005d), the stability of the system (1.4) was studied on an arbitrary group G. It was shown that the system is stable on Abelian groups. It was also shown that the system (1.4) is stable on Heisenberg groups and n-Abelian groups. In this paper, we examine the stability of the system (1.4) on the noncommutative group $T(3,\mathbb{R})$. To establish the stability of the system (1.4), we first show the stability of quadratic functional equation as well as Jensen functional equation on $T(3,\mathbb{R})$.

2 Definitions and preliminaries

Let G be a group. A function $f : G \to \mathbb{R}$ is said to be a quadratic function if f satisfies the quadratic functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y) \qquad \forall\, x, y \in G.$$

A function $f : G \to \mathbb{R}$ is said to be a Jensen function if f satisfies the Jensen functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) \qquad \forall\, x, y \in G.$$

Definition 2.1. The function $f : G \to \mathbb{R}$ is said to be a quasijensen function if the set

$$\Delta J = \big\{ f(xy) + f(xy^{-1}) - 2f(x) \;\big|\; \forall\, x, y \in G \big\}$$

is bounded. A quasijensen function $f : G \to \mathbb{R}$ is said to be a pseudojensen function if it satisfies the condition $f(x^n) = nf(x)$ for all $n \in \mathbb{Z}$ (the set of integers). The set of quasijensen functions will be denoted by $KJ(G)$. The set of pseudojensen functions will be denoted by $PJ(G)$.

Definition 2.2. The function $f : G \to \mathbb{R}$ is said to be a quasiquadratic function if the set

$$\Delta Q = \big\{ f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \;\big|\; \forall\, x, y \in G \big\},$$

is bounded. The quasiquadratic function $f : G \to \mathbb{R}$ that satisfies the condition $f(x^n) = n^2 f(x)$ for all $n \in \mathbb{Z}$ will be called a pseudoquaratic function. The set of quasiquadratic functions will be denoted by $KQ(G)$. The set of pseudoquadratic functions will be denoted by $PQ(G)$.

Definition 2.3. The function $f : G \to \mathbb{R}$ is said to be a quasidrygas function if the sets

$$\Delta D_1 = \big\{ f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) \;\big|\; \forall\, x, y \in G \big\}$$
$$\Delta D_2 = \big\{ f(yx) + f(y^{-1}x) - 2f(x) - f(y) - f(y^{-1}) \;\big|\; \forall\, x, y \in G \big\}$$

are bounded. A quasidrygas function $f : G \to \mathbb{R}$ is said to be a pseudodrygas function if there is a decomposition $f(x) = \varphi(x) + \psi(x)$, such that $\varphi \in PQ(G)$ and $\psi \in PJ(G)$. The set of quasidrygas functions will be denoted by $KD(G)$, where as the set of pseudodrygas functions on G will be denoted by $PD(G)$.

The following lemma was established by authors in Faiziev and Sahoo (2005d), and it will serve as one of the main tools for proving the stability of the system of Drygas equations.

Lemma 2.1. *The system* (1.4) *is stable over group* G *if and only if* $PD(G) = D(G)$*. In other words the system* (1.4) *is stable if and only if* $PQ(G) = Q(G)$ *and* $PJ(G) = J_0(G)$*.*

Let $\mathbb{R}^*$ be the set of nonzero real numbers. Let G denote the group $T(3, \mathbb{R})$ consisting of triangular 3-by-3 matrices. That is,

$$G = T(3, \mathbb{R}) = \left\{ \begin{bmatrix} \alpha & y & t \\ 0 & \beta & x \\ 0 & 0 & \gamma \end{bmatrix} \;\middle|\; \alpha, \beta, \gamma \in \mathbb{R}^*; \;\; x, y, t \in \mathbb{R}. \right\}.$$

We denote by H, D the subgroups of $T(3, \mathbb{R})$ consisting of the matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix},$$

respectively, where $a, b, c \in \mathbb{R}^*$ and $x, y, z \in \mathbb{R}$. It is clear that $H \triangleleft G$. We have the following semidirect product: $G = D \cdot H$.

3 Stability of Jensen equation on $T(3, \mathbb{R})$

Let

$$E = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

It is easy to see that the following relations holds:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{3.1}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}; \tag{3.2}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.3}$$

Let $f \in PJ(G)$. It is easy to see that every element of E has the order two. Hence by Lemma 3.12 in Faiziev and Sahoo (2005a), we have $f^e = f$ for any $e \in E$. Here f^e denotes $f(x^e)$ for $x \in G$, and x^e denotes $e^{-1}xe$. Denote by $A,\ B,\ C$ the subgroups of G consisting of matrices of the form

$$\begin{bmatrix} 1 & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}; \quad \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

respectively. Letting $v = \begin{bmatrix} 1 & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $e = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and using the relation (3.1) and the fact that $f \in PJ(G)$, we get

$$f(v) = f^e(v) = f(e^{-1}ve) = f(v^{-1}) = -f(v).$$

Hence $f(v) = 0$ for all $v \in A$, and $f\big|_A = 0$. Similarly, from (3.2) and (3.3) one can show that $f\big|_B = 0$ and $f\big|_C = 0$, respectively. Hence, we obtain $f\big|_{A\cup B\cup C} \equiv 0$. Let φ be the restriction of f to H. Then $\varphi \in PJ(H)$.

Let M be a subgroup of G generated by B and C. It is clear that $M = B \times C = \mathbb{R} \times \mathbb{R}$. The group M is an abelian group so, by Theorem 3.11 in Faiziev and Sahoo (2005a), we have $PJ(M) = X(M)$. Now from (3.1), (3.2), (3.3) it follows that $\varphi \equiv 0$ on M. We have semidirect product $H = A \cdot M$. Let $\pi : H \to A$ an epimorphism defined by the rule: $\pi(au) = a$, where $a \in A, u \in M$. Then the mapping $\pi^* : PJ(A) \to PJ(H)$ defined by the rule $\pi^*(\psi)(x) = \psi(\pi(x))$ is an embedding of $PJ(A)$ into $PJ(H)$. Let $\varphi \in PJ(H)$ and $\psi = \varphi\big|_A$. Then $\pi^*(\psi)(x) = \psi(\pi(x)) = \varphi(x)$ for any $x \in A$. So, $\omega = \varphi - \pi^*(\psi)$ is an element of $PJ(H)$ such that $\omega\big|_A \equiv 0$. From the relations (3.1), (3.2), (3.3) it follows that $\omega\big|_M \equiv 0$, so, $\omega\big|_{A\cup M} \equiv 0$. Let δ be a positive number such that

$$|\omega(xy) + \omega(xy^{-1}) - 2\omega(x)| \leq \delta \quad \text{for any} \quad x, y \in G. \tag{3.4}$$

Now let $x = au$ and $y = bv$, where $a, b \in A$ and $u, v \in M$. Then

$$xy = abu^b v \quad \text{and} \quad xy^{-1} = ab^{-1}(uv^{-1})^{b^{-1}}.$$

Now from (3.4) it follows that

$$|\omega(abu^b v) + \omega(ab^{-1}(uv^{-1})^{b^{-1}}) - 2\omega(au)| \leq \delta$$

If we put $a = b$ and $u = 1$, then from the last inequality we get

$$|\omega(a^2 v) + \omega((v^{-1})^{a^{-1}}) - 2\omega(a)| \leq \delta.$$

Since $\omega|_A = 0$ and $\omega|_{A\cup M} = 0$, the last inequality yields

$$|\omega(a^2 v)| \leq \delta.$$

From the last inequality it follows that for any $a \in A$ and $u \in M$

$$|\omega(au)| = \frac{1}{2}|\omega((au)^2)| = \frac{1}{2}|\omega(a^2 u^a u)| \leq \frac{1}{2}\delta.$$

Therefore ω is a bounded function on H. Since $\omega \in PJ(H)$, it follows that $\omega \equiv 0$ and $\varphi = \pi^*(\psi) \in X(H)$. Taking into account (3.1), (3.2), (3.3) we get $\varphi \equiv 0$ and $PJ(H) = X(H)$.

We have $G = D \cdot H$, $f \in PJ(G)$ and $\varphi = f\big|_H \equiv 0$. Let $\tau : G \to D$ an epimorphism defined by the rule: $\tau(au) = a$, where $a \in D, u \in H$. Then the mapping $\tau^* : PJ(D) \to PJ(G)$ defined by the rule $\tau^*(\psi)(x) = \psi(\tau(x))$ is an embedding of $PJ(D)$ into $PJ(G)$. Let $\varphi \in J(G)$ and $\psi = \varphi\big|_D$. Then $\tau^*(\psi)(x) = \psi(\tau(x)) = \varphi(x)$ for any $x \in D$. So, $\omega = \varphi - \tau^*(\psi)$ is an element of $PJ(G)$ such that $\omega\big|_D \equiv 0$.

Arguing as above we get $f \in X(G)$. So $PJ(G) = J(G) = X(G)$. Hence we have the following theorem:

Theorem 3.1. *If* $G = T(3, \mathbb{R})$*, then* $PJ(G) = J(G) = X(G)$*.*

Remark 3.1*.* By Corollary 3.3 in Faiziev and Sahoo (2005a) we see that Jensen functional equation is stable on $T(3, \mathbb{R})$.

4 Stability of quadratic equation on $T(3, \mathbb{R})$

In this section we establish that $PQ(G) = Q(G)$.

Let $f \in PQ(G)$. We will assume that $f\big|_D \equiv 0$. One can check that the following holds:

$$\begin{bmatrix} \alpha & x & z \\ 0 & \beta & y \\ 0 & 0 & \gamma \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha & x & z \\ 0 & \beta & y \\ 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} 1 & \frac{a\beta}{\alpha} & \frac{ay}{\alpha} + \frac{c\gamma}{\alpha} - \frac{bx\gamma}{\alpha\beta} \\ 0 & 1 & \frac{b\gamma}{\beta} \\ 0 & 0 & 1 \end{bmatrix}.$$

From the last equality we see that if an element g is conjugate of $\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$, then g has the form

$$\begin{bmatrix} 1 & \frac{a\beta}{\alpha} & \frac{ay}{\alpha} + \frac{c\gamma}{\alpha} - \frac{bx\gamma}{\alpha\beta} \\ 0 & 1 & \frac{b\gamma}{\beta} \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.1}$$

Lemma 4.1. *If* $a \neq 0$ *and* $b \neq 0$*, then* $\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ *is a conjugate of* $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Proof. We must show that there are $\alpha, \beta, \gamma \in \mathbb{R}^*$ and $x, y, z \in \mathbb{R}$ such that

$$\begin{cases} \frac{a\beta}{\alpha} = 1, \\ \frac{b\gamma}{\beta} = 1, \\ \frac{ay}{\alpha} + \frac{c\gamma}{\alpha} - \frac{bx\gamma}{\alpha\beta} = 0. \end{cases} \tag{4.2}$$

If $\beta = 1$ we get

$$\begin{cases} \frac{a}{\alpha} = 1, \\ b\gamma = 1, \\ ay + c\gamma - bx\gamma = 0. \end{cases}$$

So, $\alpha = a$, $\gamma = \frac{1}{b}$, and $ay + c\frac{1}{b} - bx\frac{1}{b} = 0$. If we put $y = 0$ we get $c\frac{1}{b} = bx\frac{1}{b}$, or $x = \frac{c}{b}$. Hence the system (4.2) has a solution, and the proof of the lemma is now complete. □

Lemma 4.2. *If $a \neq 0$ and $b = 0$, then* $\begin{bmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *is a conjugate of* $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Proof. We must show that there are $\alpha, \beta, \gamma \in \mathbb{R}^*$ and $x, y, z \in \mathbb{R}$ such that

$$\begin{cases} \frac{a\beta}{\alpha} = 1, \\ \frac{ay}{\alpha} + \frac{c\gamma}{\alpha} = 0. \end{cases}$$

Let $\beta = 1$, then $\alpha = a$ and $ay + c\gamma = 0$. Let $\gamma = 10$, then we get $y = -\frac{c}{a}$. This completes the proof of the lemma. □

Lemma 4.3. *If $a = 0$ and $b \neq 0$, then* $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ *is a conjugate of* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Proof. We must show that there are $\alpha, \beta, \gamma \in \mathbb{R}^*$ and $x, y, z \in \mathbb{R}$ such that

$$\begin{cases} \frac{b\gamma}{\beta} = 1, \\ \frac{c\gamma}{\alpha} - \frac{bx\gamma}{\alpha\beta} = 0. \end{cases}$$

Let $\beta = 1$, then $\gamma = \frac{1}{b}$ and $c = bx$, $x = \frac{c}{b}$. The proof of the lemma is now complete. □

Lemma 4.4. *If $a = 0$, $b = 0$ and $c \neq 0$, then* $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *is a conjugate of* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Proof. We must show that there are $\alpha, \beta, \gamma \in \mathbb{R}^*$ and $x, y, z \in \mathbb{R}$ such that

$$\begin{cases} \frac{a\beta}{\alpha} = 0, \\ \frac{b\gamma}{\beta} = 0, \\ \frac{ay}{\alpha} + \frac{c\gamma}{\alpha} - \frac{bx\gamma}{\alpha\beta} = 1. \end{cases}$$

That is, $\frac{c\gamma}{\alpha} = 1$, and we see that we can put $\alpha = 1$, $\gamma = \frac{1}{c}$ is a solution. The proof of the lemma is now complete. □

Lemma 4.5. *Let matrix A belongs to the set*

$$P = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

then A^2 conjugate to A.

Proof. From the relations

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and from Lemmas $4.1 - 4.4$ it follows that A^2 is conjugate to A. The proof of the lemma is now complete. □

Lemma 4.6. *Let matrix A belongs to the set*

$$P = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

then for any $f \in PQ(G)$ we have $f(A) = 0$.

Proof. The function f is constant on every class of conjugate elements in G. From Lemma 4.5 it follows $f(A^2) = f(A)$. On the other hand f is a pseudoquadratic function. Therefore $f(A^2) = 2f(A)$. Hence we have $2f(A) = f(A)$, and $f(A) = 0$. This completes the proof. □

Lemma 4.7. *For any $f \in PQ(G)$ we have $f\big|_H \equiv 0$.*

Proof. Any element g of H is conjugate of an element of the set P. Therefore by Lemma 4.6 we have $f(g) = 0$. Hence $f\big|_H \equiv 0$ and proof of the lemma is now finished. □

We have $G = D \cdot H$. Suppose that $f \in PQ(G)$ and $f\big|_{D\cup H} \equiv 0$. Then for some positive number ε the following relation holds:

$$|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)| \leq \varepsilon, \quad \forall x, y \in G. \tag{4.3}$$

If we put $x = aua$, $y = u$, $a \in D,\ u \in H$, then we obtain

$$|f(auau) + f(auau^{-1}) - 2f(aua) - 2f(u)| \leq \varepsilon.$$

Since $f\big|_{D\cup H} \equiv 0$, hence from the last inequality we have

$$|4f(au) + f(a^2u^au^{-1}) - 2f(a^2u^a)| \leq \varepsilon. \tag{4.4}$$

Similarly, letting $x = u$ and $y = a$ for $a \in D$ and $u \in H$ in (4.3), we get

$$|f(ua) + f(ua^{-1}) - 2f(u) - 2f(a)| \leq \varepsilon.$$

Therefore

$$|f(ua) + f(ua^{-1})| \leq \varepsilon$$

for all $u \in H$ and all $a \in D$. Replacing u by a, and a by u^{-1} in the last inequality we have

$$|f(au^{-1}) + f(au)| \leq \varepsilon. \tag{4.5}$$

Again letting $x = au$, $y = a$ for $a \in D$ and $u \in H$, we obtain

$$|f(aua) + f(aua^{-1}) - 2f(au) - 2f(a)| \leq \varepsilon.$$

Since $f\big|_{D\cup H} \equiv 0$, we obtain

$$|f(a^2u^a) - 2f(au)| \leq \varepsilon. \tag{4.6}$$

From (4.4) and (4.6) it follows that

$$|4f(au) + f(a^2u^au^{-1}) - 4f(au)| \leq \varepsilon + 2\varepsilon.$$

Therefore we have

$$|f(a^2u^au^{-1})| \leq 3\varepsilon. \tag{4.7}$$

Since f is invariant relative to inner automorphisms (see Proposition 2.12 in Faiziev and Sahoo (2005d)), we get from (4.6) that

$$|f(a^2u) - 2f(au)| \leq \varepsilon. \tag{4.8}$$

Substituting u^au^{-1} for u we get

$$|f(a^2u^au^{-1}) - 2f(au^au^{-1})| \leq \varepsilon.$$

Now using (4.7) in the last inequality, we get

$$|2f(au^au^{-1})| \leq 4\varepsilon$$

which is

$$|f(au^au^{-1})| \leq 2\varepsilon. \tag{4.9}$$

Let $u = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$ and $a = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$, then we have

$$u^a = \begin{bmatrix} \alpha^{-1} & 0 & 0 \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & \gamma^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} 1 & \alpha^{-1}\beta x & \alpha^{-1}\gamma z \\ 0 & 1 & \beta^{-1}\gamma y \\ 0 & 0 & 1 \end{bmatrix}.$$

So

$$\begin{aligned} au^au^{-1} &= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \cdot \begin{bmatrix} 1 & \alpha^{-1}\beta x & \alpha^{-1}\gamma z \\ 0 & 1 & \beta^{-1}\gamma y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha & (\beta - \alpha)x & (\alpha - \beta)xy + (\gamma - \alpha)z \\ 0 & \beta & (\gamma - \beta)y \\ 0 & 0 & \gamma \end{bmatrix}. \end{aligned}$$

From (4.9) it follow that the function f is bounded on the set

$$M_1 = \left\{ \begin{bmatrix} \alpha & (\beta - \alpha)x & (\alpha - \beta)xy + (\gamma - \alpha)z \\ 0 & \beta & (\gamma - \beta)y \\ 0 & 0 & \gamma \end{bmatrix} \,\middle|\, \alpha, \beta, \gamma \in \mathbb{R}^*,\ x, y, z \in \mathbb{R} \right\},$$

and

$$|f(g)| \leq 2\varepsilon, \quad \text{for any} \quad g \in M_1. \tag{4.10}$$

Lemma 4.8. *If α, β, γ are pair wise distinct elements, then the matrix* $\begin{bmatrix} \alpha & u & w \\ 0 & \beta & v \\ 0 & 0 & \gamma \end{bmatrix}$ *can be represented in the form* $\begin{bmatrix} \alpha & (\beta-\alpha)x & (\alpha-\beta)xy+(\gamma-\alpha)z \\ 0 & \beta & (\gamma-\beta)y \\ 0 & 0 & \gamma \end{bmatrix} \in M_1$.

Proof. We must show that the system

$$\begin{cases} (\beta-\alpha)x = u \\ (\gamma-\beta)y = v \\ (\alpha-\beta)xy+(\gamma-\alpha)z = w \end{cases}$$

has a solution. It is clear that $x = \frac{u}{(\beta-\alpha)}$, $y = \frac{v}{(\gamma-\beta)}$, and

$$\begin{aligned} z &= \frac{w+(\beta-\alpha)xy}{(\gamma-\alpha)} \\ &= \frac{w+(\beta-\alpha)\frac{u}{(\beta-\alpha)}\frac{v}{(\gamma-\beta)}}{(\gamma-\alpha)} \\ &= \frac{w+\frac{uv}{(\gamma-\beta)}}{(\gamma-\alpha)} \\ &= \frac{w}{(\gamma-\alpha)} + \frac{uv}{(\gamma-\beta)(\gamma-\alpha)}. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 4.9. *If $f \in PQ(G)$ and $f\big|_D \equiv 0$, then $f \equiv 0$ on G.*

Proof. For any $a, b \in D$ and any $u, v \in H$ we have

$$|f(aubv) + f(auv^{-1}b^{-1}) - 2f(au) - 2f(bv)| \le \varepsilon,$$

that is

$$|f(abu^b v) + f(ab^{-1}u^{b^{-1}}(v^{-1})^{b^{-1}}) - 2f(au) - 2f(bv)| \le \varepsilon.$$

Let (α, β, γ) be the diagonal elements of the matrix a and $(\alpha_1, \beta_1, \gamma_1)$ be the diagonal elements of the matrix b. Let us choose the matrix b such that the numbers $(\alpha_1, \beta_1, \gamma_1)$ be pair wise distinct. So let $\alpha\alpha_1, \beta\beta_1, \gamma\gamma_1$ be pair wise distinct. Similarly, let the numbers $\alpha\alpha_1^{-1}, \beta\beta_1^{-1}, \gamma\gamma_1^{-1}$ be pair wise distinct too. Then by Lemma 4.8 and inequality (4.10), we obtain

$$\begin{aligned} |2f(au)| &= |f(abu^b v) + f(ab^{-1}u^{b^{-1}}(v^{-1})^{b^{-1}}) - 2f(au) - 2f(bv) \\ &\quad -f(abu^b v) - f(ab^{-1}u^{b^{-1}}(v^{-1})^{b^{-1}}) + 2f(bv)| \\ &\le |f(abu^b v) + f(ab^{-1}u^{b^{-1}}(v^{-1})^{b^{-1}}) - 2f(au) - 2f(bv)| \\ &\quad +|f(abu^b v) + f(ab^{-1}u^{b^{-1}}(v^{-1})^{b^{-1}}) - 2f(bv)| \\ &\le |f(abu^b v) + f(ab^{-1}u^{b^{-1}}(v^{-1})^{b^{-1}}) - 2f(au) - 2f(bv)| \\ &\quad +|f(abu^b v)| + |f(ab^{-1}u^{b^{-1}}(v^{-1})^{b^{-1}})| + |2f(bv)| \\ &\le \varepsilon + 3\cdot 2\varepsilon = 7\varepsilon. \end{aligned}$$

Therefore f is bounded function on G. Hence it follows that $f \equiv 0$ on G. The proof of the lemma is now complete. □

Theorem 4.10. *If $G = T(3,\mathbb{R})$, then $PQ(G) = Q(G)$.*

Proof. We have semidirect product $G = D \cdot H$. Let $\pi : G \to D$ an epimorphism defined by the rule: $\pi(au) = a$, where $a \in D, u \in H$. Then the mapping $\pi^* : PQ(D) \to PQ(G)$ defined by the rule $\pi^*(\psi)(x) = \psi(\pi(x))$ is an embedding of $PQ(D)$ into $PQ(G)$. Let $\varphi \in PQ(G)$ and $\psi = \varphi\big|_D$. Then $\pi^*(\psi)(x) = \psi(\pi(x)) = \varphi(x)$ for any $x \in D$. So, $\omega = \varphi - \pi^*(\psi)$ is an element of $PQ(G)$ such that $\omega\big|_D \equiv 0$. From Lemma 4.9 it follows that $\omega \equiv 0$ on G. So, $\varphi = \pi^*(\psi) \in PQ(D) \subset PQ(G)$. The group D is abelian therefore $\varphi \in Q(G)$ and $PQ(G) = Q(G)$. The proof of the theorem is now complete. □

Remark 4.1. By Theorem 3.2 in Faiziev and Sahoo (2005b) we see that quadratic functional equation is stable on $T(3,\mathbb{R})$.

5 Main result

Now we are ready to establish our out main result.

Theorem 5.1. *Let $G = T(3,\mathbb{R})$. The system* (1.4) *is stable on G.*

Proof. The proof of the theorem follows from Lemma 2.1, Theorem 3.1, and Theorem 4.10. □

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The spaces $\mathcal{S}^\nu$: New spaces defined with wavelet coefficients and related to multifractal analysis

Jean-Marie Aubry, Françoise Bastin, Sophie Dispa, Stéphane Jaffard

J.-M. Aubry (Jean-Marie.Aubry@math.cnrs.fr), S. Jaffard (jaffard@univ-paris12.fr)
Université de Paris 12
Laboratoire d'analyse et de mathématiques appliquées
Faculté des Sciences et Technologies
Avenue du Général de Gaulle, 61, 94010 Créteil Cédex, France

F. Bastin (F.Bastin@ulg.ac.be), S. Dispa (sdispa@ulg.ac.be)
Université de Liège
Département de Mathématique (B37), B-4000 Liège, Belgique

ABSTRACT

In the context of multifractal analysis, more precisely in the context of the study of Hölder regularity, Stphane Jaffard introduced new spaces of functions related to the distribution of wavelet coefficients, the $\mathcal{S}^\nu$ spaces. From a functional analysis point of view, one can define the corresponding sequence spaces, endow them with natural topologies and study their properties. The results lead to construct probability Borel measures with applications in the context of multifractal analysis.

Keywords: Sequence spaces, wavelet coefficients, topological vector spaces, prevalence

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1 Introduction

The representation of a signal by means of its wavelet coefficients is a widely used tool; in this context, we say that a property of signals is robust if it has an expression in terms of wavelet coefficients independent of the chosen wavelet basis. In that case, such a property can be intrinsically studied using sequence spaces with the functional analysis point of view. The pointwise Hölder regularity is an example of such a property; this has applications in multifractal analysis.

In various domains, the natural and classical functional setting to study signals consists is considering Besov spaces. Nevertheless, it appeared that these spaces are not sufficient to handle all the accurate information contained in the distribution of the wavelet coefficients or histograms ([9]), which is usually the data which is concretely accessible. In this context, spaces of type $\mathcal{S}^\nu$ have been introduced by S. Jaffard ([9]).

The goal of this paper is to study these spaces as sequence spaces. These new spaces constitute a suitable setting for our purpose since they do not depend on the chosen wavelet

basis. In what follows, we will present the main results of [4],[3], focusing on the functional analysis point of view. We present topological results about these spaces, as well as results involving probability theory. Consequences of these results for typical multifractal analysis considerations can be found in [3], [6]. In the present paper, we just introduce a link between the two points of view.
Nevertheless, in order to give more motivation to the reader, let us briefly introduce “multifractal analysis”.
Multifractal analysis is concerned with the computation of the spectrum of singularities d_f of a locally bounded function f, that is to say

$$d_f(h) = dim\{x \in \mathbb{R} \; : \; h_f(x) = h\}$$

where dim means “Hausdorff dimension” and where $h_f(x)$ is the Hölder exponent of f at x. In this context, multifractal formalisms are formulas which give estimations for d_f ([5]). Multifractal analysis was certainly motivated by problems occurring in turbulence; in particular, an important problem is to study the possible multifractal properties of solutions of some PDEs ([10]). The importance of multifractal analysis and formalisms is also especially underlined by recent results in the context of the study of DNA sequences ([1],[13] and references therein).

2 The $\mathcal{S}^\nu$ spaces: definition and topological properties

For convenience and simplicity of notations, we will only consider functions of one real variable; natural generalizations concerning definition, topology and measure results on $\mathcal{S}^\nu$ can be directly obtained for functions defined on $\mathbb{R}^n$.

2.1 The context

If ψ is a mother wavelet of $L^2(\mathbb{R})$ which is also in the Schwartz class, we consider the periodic functions

$$\psi_{j,k}(x) = 2^{j/2} \sum_{l \in \mathbb{Z}} \psi\left(2^j(x-l) - k\right), \; j \geq 0, \; k \in \{0, \dots 2^j - 1\};$$

it is well known that with the constant function 1, they form an orthonormal basis of $L^2([0,1])$. In what follows, to simplify formulas and notations, we use a L^∞ normalization for wavelets and, for a signal f, we define its wavelet coefficents by

$$c_{j,k} = 2^j \int_0^1 f(x)\psi_{j,k}(x)\, dx, \quad j \geq 0, \; k \in \{0, \dots 2^j - 1\}.$$

Somehow, since the characterization of the Hölder regularity amounts to examine the asymptotic behaviour of the sequences $2^{j\alpha}|c_{j,k}|$ for all $\alpha \in \mathbb{R}$, and since practical information about wavelet coefficients is often given by histograms of coefficients, it is then rather natural to define the sets

$$E_j(C,\alpha)(f) = \{k : |c_{j,k}| \geq C 2^{-\alpha j}\}$$

and the *wavelet profile* of f

$$\nu_f(\alpha) = \lim_{\varepsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\ln(\# E_j(1, \alpha + \varepsilon)(f))}{\ln 2^j} \right) \right), \quad \alpha \in \mathbb{R}.$$

This definition formalizes the idea that there are about $2^{\nu_f(\alpha)j}$ coefficients larger than $2^{-j\alpha}$. The function ν_f takes its values in $\{-\infty\} \cup [0,1]$, is nondecreasing and right-continuous. Moreover, it contains strictly more information than the Besov spaces whenever it is not concave (see [9]). Note that since we deal with periodic signals, one gets the existence of α_0 such that $\sup_{j,k} 2^{\alpha_0 j}|c_{j,k}| < +\infty$ hence

$$\nu_f(\alpha) = -\infty, \quad \forall \alpha < \alpha_0.$$

In order to provide a new context in which more accurate information about signals and multifractal properties could be derived, S. Jaffard proposed the following definition ([9]). Given a nondecreasing and right-continuous function $\nu \ : \ \mathbb{R} \to \{-\infty\} \cup [0,1]$, a function f belongs to the space $\mathcal{S}^\nu$ if its wavelet coefficients satisfy the following properties

$$\forall \alpha \in \mathbb{R}, \ \forall \varepsilon > 0, \ \forall C > 0, \exists J \geq 0 \ : \ \#E_j(C,\alpha)(f) \leq 2^{(\nu(\alpha)+\varepsilon)j}, \ \forall j \geq J.$$

S. Jaffard also proved that these spaces are robust, which allows a specific study as sequence spaces (i.e. only considering wavelet coefficients). Multifractal analysis is concerned with Hölder exponent of signals; this exponent can be characterized by means of the wavelet coefficients. It follows that the results obtained in the specific functional analysis context (sequence spaces) can thus be interpreted in terms of wavelet series and lead to applications in multifractal analysis ([3],[6]).

2.2 Sequence spaces $\mathcal{S}^\nu$ and related spaces

Let $\nu \ : \ \mathbb{R} \to \{-\infty\} \cup [0,1]$ be a nondecreasing and right-continuous function and $\alpha_0 \in \mathbb{R}$ be such that $\nu(\alpha) = -\infty$ iff $\alpha < \alpha_0$.

The space of all sequences of complex numbers $\vec{c} = (c_{j,k})_{j\geq 0, k=0,\ldots 2^j-1}$ will be denoted by Ω. The natural product topology on this space will be referred as "pointwise topology".

Definition 2.1. Given such a function ν, we define the sequence space $\mathcal{S}^\nu$ (just in the spirit of [9]) as the set of all sequences $\vec{c}$ such that

$$\forall \alpha \in \mathbb{R}, \ \forall \varepsilon > 0, \ \forall C > 0, \exists J \geq 0 \ : \ \#E_j(C,\alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\varepsilon)j}, \ \forall j \geq J$$

where

$$E_j(C,\alpha)(\vec{c}) = \{k : |c_{j,k}| \geq C2^{-\alpha j}\}$$

Just as before, we also define the profile $\nu_{\vec{c}}$ of any sequence $\vec{c} \in \Omega$.

Definition 2.2. If $\vec{c} \in \Omega$, we define

$$\nu_{\vec{c}}(\alpha) = \lim_{\varepsilon \to 0^+} \nu^*_{\vec{c}}(\alpha + \varepsilon)$$

where

$$\nu^*_{\vec{c}}(\alpha) = \limsup_{j \to +\infty} \left(\frac{\ln(\#E_j(1,\alpha)(\vec{c}))}{\ln 2^j} \right), \quad \alpha \in \mathbb{R}.$$

The function $\nu_{\vec{c}}$ is nondecreasing, right-continuous and such that

$$\nu^*_{\vec{c}}(\alpha) \leq \nu_{\vec{c}}(\alpha) \leq \nu^*_{\vec{c}}(\alpha'), \quad \forall \alpha, \alpha' \in \mathbb{R}, \alpha < \alpha'.$$

The following characterization of the space $\mathcal{S}^\nu$ holds.

Proposition 2.3. ([4]) The space $\mathcal{S}^\nu$ is a vector space and satisfies

$$\mathcal{S}^\nu = \{\vec{c} \in \Omega \ : \nu_{\vec{c}}(\alpha) \leq \nu(\alpha) \ \forall \alpha \in \mathbb{R}\}.$$

Let us also give the following remarks, which illustrate some other properties of the function $\nu_{\vec{c}}$ and then of the set $\mathcal{S}^\nu$.

1. For any sequence $\vec{c}$, one has

   $$\sup_{j,k} 2^{\alpha j}|c_{j,k}| < +\infty \ \Rightarrow \ \nu_{\vec{c}}(\alpha') = -\infty \ \forall \alpha' < \alpha \quad \text{and} \quad \nu_{\vec{c}}(\alpha) = -\infty \ \Rightarrow \sup_{j,k} 2^{\alpha j}|c_{j,k}| < +\infty.$$

   In particular, every sequence $\vec{c}$ with only a finite number of non zero coefficients satisfies $\nu_{\vec{c}}(\alpha) = -\infty \ \forall \alpha$, hence belongs to $\mathcal{S}^\nu$. Indeed, if $\sup_{j,k} 2^{\alpha j}|c_{j,k}| \leq R$, then for every $\varepsilon > 0$ there is a J such that $\sup_k 2^{j\alpha}|c_{j,k}| < 2^{j\varepsilon}, \ \forall j \geq J$, hence $E_j(1, \alpha - \varepsilon) = \emptyset$ and finally $\nu_{\vec{c}}(\alpha') = -\infty, \ \forall \alpha' < \alpha$. Conversely, if $\nu_{\vec{c}}(\alpha) = -\infty$, then there is an $\varepsilon > 0$ and there is a J such that $\ln(\# E_j(1, \alpha + \varepsilon)) < 0, \ \forall j \geq J$; it follows that $E_j(1, \alpha + \varepsilon) = \emptyset, \ \forall j \geq J$, hence $\sup_k |c_{j,k}| < 2^{-(\alpha+\varepsilon)j} \leq 2^{-\alpha j} \ \forall j \geq J$, which concludes the proof.

2. For every $\alpha \in \mathbb{R}$, the sequence $\vec{c} = (2^{-\alpha j})_{j,k}$ is such that $\nu_{\vec{c}}(\alpha) = 1$. In particular, if $\alpha < \alpha_0$ it does not belong to $\mathcal{S}^\nu$.

3. The set $\mathcal{S}^\nu$ is not closed under the product topology (i.e. the pointwise topology).

   Indeed, it is clear using the two particular cases of the previous remarks.

In order to endow the vector space $\mathcal{S}^\nu$ with a natural (metric, complete) topology and examine its properties, let us focus on steps leading to the general definition of $\mathcal{S}^\nu$, i.e. to spaces denoted by $E(\alpha, \beta)$ which are defined in what follows. At this point, let us mention that since $\mathcal{S}^\nu$ is not a closed set of Ω endowed with the product topology, the classical distance derived from this (i.e. Ω is a Frechet space, hence metrizable) does not define a complete metric distance on $\mathcal{S}^\nu$ and then is not a good candidate for our purposes.

Definition 2.4. Let $\alpha \in \mathbb{R}, \ \beta \in \{-\infty\} \cup [0, +\infty[$. The set $E(\alpha, \beta)$ is defined as follows

$$\vec{c} \in E(\alpha, \beta) \ \Leftrightarrow \ \exists C, C' \geq 0 : \# E_j(C, \alpha)(\vec{c}) \leq C' 2^{\beta j}, \ \forall j \geq 0.$$

Let us remark that

1. in case $\beta = -\infty$, then $E(\alpha, \beta)$ is the set of all sequences such that $\sup_{j,k} 2^{j\alpha}|c_{j,k}| < +\infty$ (i.e. $E(\alpha, \beta)$ is the Hölder sequence space C^α);
2. in case $\beta \in [1, +\infty[$, we have $E(\alpha, \beta) = \Omega$ (i.e. $E(\alpha, \beta)$ is the set of all sequences).

The following property clearly shows the natural link between these related spaces and $\mathcal{S}^\nu$.

Proposition 2.5. (Theorem 5.1, [4]) We have

$$\mathcal{S}^\nu = \bigcap_{\varepsilon > 0} \bigcap_{\alpha \in \mathbb{R}} E(\alpha, \nu(\alpha) + \varepsilon) = \bigcap_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} E(\alpha_n, \nu(\alpha_n) + \varepsilon_m)$$

for every sequence α_n $(n \in \mathbb{N})$ dense in $\mathbb{R}$ and every sequence ε_m $(m \in \mathbb{N})$ in $]0, +\infty[$ which converges to 0.

This result is also true if one considers a sequence α_n dense in $[\alpha_0', +\infty[$ with $\alpha_0' < \alpha_0$. Indeed, it suffices to improve the proof of Theorem 5.1. remarking that, in case $\nu(\alpha) = -\infty$, one has $\alpha < \alpha_0$ and

$$E(\alpha, \nu(\alpha) + \varepsilon) = C^{\alpha} \supset C^{\alpha_n} = E(\alpha_n, \nu(\alpha_n) + \varepsilon)$$

for some $\alpha_n \in [\alpha, \alpha_0[$.

In order to go on with the definition of a natural topology on $\mathcal{S}^{\nu}$, let us give the following proposition, defining a natural distance on each "step" $E(\alpha, \beta)$ and presenting some of its properties, very useful to get the desired results on $\mathcal{S}^{\nu}$.

Definition 2.6. The function $d_{\alpha,\beta}$ is defined as

$$d_{\alpha,\beta}(\vec{c}, \vec{d}) = \inf\{C + C' \ : \ C, C' \geq 0 \text{ and } \#E_j(C, \alpha)(\vec{c} - \vec{d}) \leq C' 2^{\beta j}, \forall j \geq 0\}$$

for $\vec{c}, \vec{d} \in E(\alpha, \beta)$.

Proposition 2.7. ([4]) If $\alpha \in \mathbb{R}, \beta \in \{-\infty\} \cup [0, +\infty[$, the set $E(\alpha, \beta)$ is a vector space and the function $d_{\alpha,\beta}$ is a distance on this space, which is translation invariant and satisfies

$$d_{\alpha,\beta}(\lambda \vec{c}, \vec{0}) \leq \sup\{1, |\lambda|\} d_{\alpha,\beta}(\vec{c}, \vec{0}).$$

Moreover,

1. The sum is a continuous operation on $(E(\alpha, \beta), d_{\alpha,\beta})$; however, if $\beta \in [0, 1]$, then the product
$$\mathbb{C} \times (E(\alpha, \beta), d_{\alpha,\beta}) \ \to (E(\alpha, \beta), d_{\alpha,\beta}) \quad (\lambda, \vec{c}) \mapsto \lambda \vec{c}$$
is not continuous.

2. If $\beta = -\infty$, then the distance $d_{\alpha,\beta}$ is defined by the sup norm $\|\vec{c}\| = \sup_{j,k} 2^{j\alpha} |c_{j,k}|$ (i.e. the natural norm of the Hölder sequence space C^{α}).

3. If $\beta > 1$, the topology defined by the distance $d_{\alpha,\beta}$ is equivalent to the pointwise topology.

4. The space $(E(\alpha, \beta), d_{\alpha,\beta})$ has a stronger topology than the pointwise topology and every Cauchy sequence in $(E(\alpha, \beta), d_{\alpha,\beta})$ is also a pointwise Cauchy sequence; however, the bounded sets are different,

5. The metric space $(E(\alpha, \beta), d_{\alpha,\beta})$ is complete.

In fact, in the terminology of [12], the function $\vec{c} \mapsto d_{\alpha,\beta}(\vec{c}, \vec{0})$ fulfils each required property to be an F-norm[1] on $E(\alpha, \beta)$, but one: in case $\beta \in [0, 1]$, the property ($\lambda_m \to 0 \ \Rightarrow d_{\alpha,\beta}(\lambda_m \vec{c}, \vec{0}) \to 0$) is not satisfied (see [4]). To illustrate this, let us simply give the following example: if $\alpha > 0$ and $\beta = 1$, then for every $\lambda \neq 0, \ d_{\alpha,1}(\lambda \vec{1}, \vec{0}) = 1$, where $\vec{1}$ denotes the sequence with each coefficient equal to 1.

The properties of each space $(E(\alpha, \beta), d_{\alpha,\beta})$ can then be regarded as a little bit strange and useless. Nevertheless, because of natural links between all these topologies (see [4]) and the

[1] A function $\|.\|$ on a vector space is called an F-norm if the following properties are satisfied: (F1) $\|x\| \geq 0$; (F2) $x = 0$ if $\|x\| = 0$; (F3) $\|\lambda x\| \leq \|x\|$ if $|\lambda| \leq 1$; (F4) $\|x + y\| \leq \|x\| + \|y\|$; (F5) $\|\lambda x_n\| \to 0$ if $\|x_n\| \to 0$; (F6) $\|\lambda_n x\| \to 0$ if $\lambda_n \to 0$.

natural way they were defined, they can be used to define a very useful (and natural) metric topology on the space $\mathcal{S}^\nu$, which in fact is our goal (for prevalent results in the context of multifractal analysis, see [3], [6]). A key point to obtain this is certainly the following property, which brings some converse to the following fact (straightforward to prove)

$$\text{if } \alpha' \leq \alpha,\ \beta' \geq \beta \text{ then } E(\alpha,\beta) \subset E(\alpha',\beta') \text{ and } d_{\alpha',\beta'} \leq d_{\alpha,\beta}.$$

Property 2.8. ([4]) If $\alpha' < \alpha,\ \beta' > \beta$ and B is a bounded set in $(E(\alpha,\beta), d_{\alpha,\beta})$, then every sequence in B which converges for the pointwise topology also converges in $(E(\alpha',\beta'), d_{\alpha',\beta'})$. In particular, if $\vec{c} \in E(\alpha,\beta)$ and $\lambda_m \to 0$, then $d_{\alpha',\beta'}(\lambda_m \vec{c}, \vec{0}) \to 0$.

Now, let us give the general topological results obtained on $\mathcal{S}^\nu$ thanks to these auxiliary spaces $E(\alpha,\beta)$.

Proposition 2.9. Let α_n $(n \in \mathbb{N})$ be a dense sequence in $\mathbb{R}$ and let ε_m $(m \in \mathbb{N})$ be a sequence of strictly positive numbers which converges to 0. Let

$$d_{m,n} = d_{\alpha_n, \nu(\alpha_n)+\varepsilon_m}, \quad E_{m,n} = E(\alpha_n, \nu(\alpha_n)+\varepsilon_m), \quad m,n \in \mathbb{N}$$

then

$$d = \sum_{m,n} 2^{-(m+n)} \frac{d_{m,n}}{1+d_{m,n}}$$

is a distance on $\mathcal{S}^\nu$ with the following properties

1. The function d is a distance on $\mathcal{S}^\nu$, defining the weakest topology τ on $\mathcal{S}^\nu$ such that the identity maps $(\mathcal{S}^\nu, \tau) \mapsto (E_{m,n}, d_{m,n})$ are continuous.
2. The topology defined by the distance d is stronger than the pointwise topology.
3. The metric space $(\mathcal{S}^\nu, d)$ is a topological vector vector space, complete and separable.

In fact, it can be showed directly that the topology defined by d is independent from such sequences α_n, ε_m. Moreover, this result can be generalized using the closed graph theorem for metrizable topological vector spaces: all distances on $\mathcal{S}^\nu$, defining a complete topological vector topology stronger than the pointwise topology, define equivalent topologies.
Let us also mention that the compact sets of $(\mathcal{S}^\nu, d)$ have been completely characterized (see [4]).
In the same reference, the following (density) result has also been proved: for every $\vec{c} \in \mathcal{S}^\nu$, one has $\lim_{N\to+\infty} \vec{c}^N = \vec{c}$ in the metric space $\mathcal{S}^\nu$, where $\vec{c}^N$ is defined as $c^N_{j,k} = c_{j,k}$ for all k if $j \leq N$ and $c^N_{j,k} = 0$ for all k if $j > N$. This leads to the following remark: if $\alpha \geq \alpha_0$, the application

$$(\mathcal{S}^\nu, d) \to \{-\infty\} \cup [0,1] \quad \vec{c} \to \nu_{\vec{c}}(\alpha)$$

is not continuous. Indeed, let $\vec{c}$ be the sequence defined as follows: for every j, $c_{j,k} = 2^{-\alpha_0 j}$ for a set of index k with cardinality equal to the entire part of $2^{\nu(\alpha_0)j}$ and 0 for the others. Then $\nu_{\vec{c}}(\alpha) = -\infty$ if $\alpha < \alpha_0$, $\nu_{\vec{c}}(\alpha) = \nu(\alpha_0) \in [0,1]$ if $\alpha \geq \alpha_0$; in particular, it belongs to $\mathcal{S}^\nu$. We can then conclude since $\nu_{\vec{c}^N}(\alpha) = -\infty$ for all N and all α.

3 An appropriate measure on the sequence space $\mathcal{S}^\nu$

3.1 Notion of prevalence

The concept of *prevalence* allows to define a notion of "almost surely" or "almost everywhere" in infinite dimensional spaces. For more informations about this, we refer to [8], [3] and references therein. Nevertheless, let us briefly recall some definitions and properties related to this concept.

In $\mathbb{R}^n$, the Tonelli-Fubini theorem shows that a Borel set A has Lebesgue measure zero iff there is a compactly supported probability Borel measure $\mathbb{P}$ such that $\mathbb{P}(x+A)=0$ for all $x\in\mathbb{R}^n$. This result suggests the following infinite-dimensional extension.

Definition 3.1. Let X be a complete metric vector space. A Borel set $A\subset X$ is called *shy* if there exists a Borel measure $\mathbb{P}$, strictly positive on some compact set $K\subset X$, such that

$$\mathbb{P}(x+A)=0,\quad \forall x\in X.$$

A subset of X is *prevalent* if its complement is included in a shy Borel set.

With this terminology, to show that a Borel set A of X is prevalent, it suffices to construct a Borel probability measure $\mathbb{P}$ on the space X such that $\mathbb{P}(x+A)=1$ for all $x\in X$.

In case X is a metric and complete topological vector space which is separable, then the requirement concerning the existence of K in the previous definition is automatically fulfilled (see [8], [14]).

3.2 Measure on the sequence space $\mathcal{S}^\nu$

In order to get prevalent properties in the context of multifractal analysis, we are now going to consider the set of the sequences $\vec{c}$ such that $\nu_{\vec{c}}=\nu$ (see [5], [3]). The key point is to construct a Borel probability measure on the (metric, complete and separable) space $\mathcal{S}^\nu$ for which

$$\mathbb{P}\left(\left\{\vec{c}\in\mathcal{S}^\nu\ :\nu_{\vec{c}-\vec{d}}=\nu\right\}\right)=1,\quad \forall\vec{d}\in\mathcal{S}^\nu,$$

which means that the set of elements of $\mathcal{S}^\nu$ whose profile is equal to ν is prevalent. For applications in the context of multifractal analysis, we refer to [5], [3]. Let us also remark that in this context, i.e. to apply our results to multifractal analysis which involves Hölder exponents of functions, we assume that $\alpha_0>0$.

3.2.1 On the sequence space Ω

The first step is to construct a suitable probability Borel measure on the space Ω of all sequences.

For each $j\geq 0$, let

$$F_j(\alpha):=\begin{cases}2^{-j}\sup\{j^2,2^{j\nu(\alpha)}\} & \text{if } \alpha\geq\alpha_0\\ 0 & \text{if } \alpha<\alpha_0\end{cases}$$

and

$$\rho_j(]\alpha,\beta]) := F_j(\beta) - F_j(\alpha),\ \alpha,\beta \in \mathbb{R},\ \alpha < \beta, \qquad \rho_j(\{+\infty\}) := 1 - \lim_{\alpha\to+\infty} F_j(\alpha).$$

With this definition and because of the conditions imposed on ν, the function ρ_j defines a Borel probability measure on $]-\infty,+\infty]$; in case $\lim_{\alpha\to+\infty}\nu(\alpha) = 1$, which gives $\rho_j(\{+\infty\}) = 0$, let us remark that it is also a Borel probability measure on $\mathbb{R}$. The measure satisfies

$$\limsup_{j\to+\infty} \frac{\log_2(2^j\rho_j(]-\infty,\alpha]))}{j} = \nu(\alpha), \quad \forall\alpha\in\mathbb{R}$$

which will be crucial in what follows. For each $j \geq 0$, let also g_j be the function

$$g_j\ :]-\infty,+\infty]\times\mathcal{C} \to \mathbb{C} \qquad (\alpha,u)\mapsto \begin{cases} 2^{-j\alpha}u & \text{if } \alpha\in\mathbb{R} \\ 0 & \text{if } \alpha=+\infty. \end{cases}$$

If μ is the standard uniformly distributed measure on the unit cercle $\mathcal{C}$, then for each $j \geq 0$, the function Pr_j defined as

$$\mathbb{P}_j(B) = (\rho_j\times\mu)\left(g_j^{-1}(B)\right)$$

is a Borel probability measure on $\mathbb{C}$ such that

$$\mathbb{P}_j\left(\left\{c\in\mathbb{C}\ :\ |c|\geq 2^{-\alpha j}\right\}\right) = \rho_j(]-\infty,\alpha]) = F_j(\alpha)$$

and

$$\mathbb{P}_j(\{0\}) = \rho_j(\{+\infty\}).$$

The uniformity of μ is a key point in the proof of the result concerning prevalence in this context ([3]).

As illustrations, let us give some examples.

The $\mathbb{P}_j$ measure of a point different from the origin is 0, as well as the $\mathbb{P}_j$ measure of a line which does not contain the origin.

If H denotes the closed half-plane of all complex numbers with positive imaginary part (resp. with negative imaginary part), then $g_j^{-1}(H) = (]-\infty,+\infty[\times\{e^{i\theta}:\theta\in[0,\pi]\})\ \cup\ (\{+\infty\}\times\{e^{i\theta}:\theta\in[0,2\pi]\})$. It follows that $\mathbb{P}_j(H) = \frac{1+\rho_j(\{+\infty\})}{2}$.

The $\mathbb{P}_j$ measure of a closed half-plane determined by a line containing 0 is greater or equal to $\frac{1}{2}$ (obtained using the rotation invariance and the previous example). It follows that $\mathbb{P}_j(B)\geq\frac{1}{2}$, where $B = \{c\in\mathbb{C}\ :\ |c-z_0|\geq|c|\}$; this example is important in the proof of the prevalent result below (see also [3]) and is simply due to the fact that, for instance, we have the following description (with $x_0 = \Re z_0, y_0 = \Im z_0$)

$$B = \{(x,y)\in\mathbb{R}^2\ :\ (x-x_0)^2+(y-y_0)^2\geq x^2+y^2\} = \{(x,y)\in\mathbb{R}^2\ :\ x_0^2+y_0^2\geq 2(x_0x+y_0y)\},$$

which shows that B is the closed half-plane containing 0 and determined by the line containing $z_0/2$ and orthogonal to the one containing $0, z_0$.

Now, using a classical result of measure theory and using the $\mathbb{P}_j$ constructed above, one gets a probability measure $\mathbb{P}$ on the sequence space $(\Omega,\mathcal{F})$ where $\mathcal{F}$ is the Borel $\sigma-$algebra for the topology of pointwise convergence (i.e. the product topology). For each α, let us now consider the function

$$(\Omega, \mathcal{F}) \to \mathbb{R} \quad \vec{c} \mapsto \#E_j(1,\alpha)(\vec{c}).$$

This function is $\mathbb{P}-$measurable (it is a sum of Borel functions), thus defines a random variable with expected value equal to $2^j \rho_j(]-\infty, \alpha])$ for every $j \geq 0$ and $\alpha \in \mathbb{R}$ (which is equal to $2^{j\nu(\alpha)}$ if j is large enough and $\nu(\alpha) > 0$).

3.2.2 Some questions of measurability

In order to consider $\mathbb{P}$ directly on $\mathcal{S}^\nu$ and to show that it defines in fact a Borel measure on this metric space, we have first to examine the measurability of some "key sets".

Let us first remark that because of the nondecreasing property of $\nu_{\vec{c}}$ one has

$$\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) \leq \nu(\alpha) \ \forall \alpha \geq \alpha_0'\} = \{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) \leq \nu(\alpha) \ \forall \alpha\},$$

where α_0' is any real number strictly smaller than α_0.The same equalities between sets hold with "$\geq, =$".

Property 3.2. a) For every $\alpha \in \mathbb{R}$, the following sets are $\mathbb{P}-$measurable

$$\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) \ \leq \ \nu(\alpha)\}, \quad \{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) \ \geq \ \nu(\alpha)\}, \quad \{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) \ = \nu(\alpha)\}.$$

b) It follows that

$$\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) \ \leq \text{(resp. } \ \geq, =) \ \nu(\alpha), \forall \alpha \in \mathbb{R}\}$$

are $\mathbb{P}-$measurable. In particular, $\mathcal{S}^\nu$ is $\mathbb{P}-$measurable.

Proof. Let us first prove item a) with $\nu^*_{\vec{c}}$ instead of $\nu_{\vec{c}}$. Let us consider the case "$=$"; the other ones are analogous.

For any $\alpha \in \mathbb{R}, r \geq 0$, we have

$$\{\vec{c} \in \Omega : \nu^*_{\vec{c}}(\alpha) = -\infty\} = \bigcup_{J \in \mathbb{N}} \bigcap_{j \geq J} \{\vec{c} \in \Omega : \#E_j(1,\alpha)(\vec{c}) = 0\}$$

and

$$\{\vec{c} \in \Omega : \nu^*_{\vec{c}}(\alpha) = r\} = \bigcap_{m \in \mathbb{N}} \bigcup_{J \in \mathbb{N}} \bigcap_{j \geq J} \left\{\vec{c} \in \Omega : 2^{j(r-1/m)} \leq \#E_j(1,\alpha)(\vec{c}) \leq 2^{j(r+1/m)}\right\}.$$

Using the measurability of the function $\vec{c} \to \#E_j(1,\alpha)(\vec{c})$ we conclude.

Using the definition of $\nu_{\vec{c}}$ by means of $\nu^*_{\vec{c}}$, we then obtain item a).

b) If α_n $(n \in \mathbb{N})$ is a dense sequence in $\mathbb{R}$, then

$$\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) \ \leq \ \nu(\alpha), \forall \alpha \in \mathbb{R}\} = \bigcap_{n \in \mathbb{N}} \{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha_n) \ \leq \ \nu(\alpha_n)\}$$

since ν and $\nu_{\vec{c}}$ are right-continuous. Hence the conclusion using item a).□

3.2.3 Some computations of measures

Let us also recall the following lemma (see [5], [15]).

Lemma 3.3. There is $C > 0$ such that for all $0 \leq a < b$ and $j > 0$ such that $2^j \rho_j(]a,b]) \geq j^2$, we have

$$\mathbb{P}\left(\left\{\vec{c} \in \Omega : 2^{j-1}\rho_j(]a,b]) \leq \#\{k : 2^{-bj} \leq |c_{j,k}| < 2^{-aj}\} \leq 2^{j+1}\rho_j(]a,b])\right\}\right) \geq 1 - C\frac{2^j}{j^2}e^{-Cj^2}.$$

Let us remark that with our hypothesis, we have $\rho_j(]-\infty,b]) = \rho_j(]0,b])$ for all $b > 0$ and all j.

Property 3.4. For every $\alpha < \alpha_0$, one has

$$\mathbb{P}\left(\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) = -\infty\}\right) = 1.$$

It follows that

$$\mathbb{P}\left(\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) = -\infty \ \forall \alpha < \alpha_0\}\right) = \mathbb{P}\left(\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) = -\infty \ \forall \alpha \in [s_0, \alpha_0[\}\right) = 1$$

for any $s_0 < \alpha_0$.

Proof. Let $\alpha < \alpha' < \alpha_0$. Using the definition of $\mathbb{P}$, one gets

$$\mathbb{P}\left(\left\{\vec{c} \in \Omega \ : \ \#E_j(1,\alpha')(\vec{c}) = 0\right\}\right) = 1, \ \forall j$$

hence

$$\mathbb{P}\left(\bigcap_{j \geq 0}\{\vec{c} \in \Omega \ : \ \#E_j(1,\alpha')(\vec{c}) = 0\}\right) = 1.$$

Since

$$\bigcap_{j \geq 0}\left\{\vec{c} \in \Omega \ : \ \#E_j(1,\alpha')(\vec{c}) = 0\right\} \subset \{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) = -\infty\}$$

we are done.

For the second part, it suffices to remark that

$$\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) = -\infty \ \forall s_0 \leq \alpha < \alpha_0\} = \bigcap_{n \in \mathbb{N}}\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha_n) = -\infty\}$$

for any sequence α_n, dense in $[s_0, \alpha_0[$.□

Property 3.5. For every $\alpha \geq \alpha_0$, one has $\mathbb{P}\left(\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) = \nu(\alpha)\}\right) = 1$. It follows that

$$\mathbb{P}\left(\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) = \nu(\alpha) \ \forall \alpha \geq \alpha_0\}\right) = 1.$$

Proof. If α_n $(n \in \mathbb{N})$ is a dense sequence in $[\alpha_0, +\infty[$, we have

$$\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) = \nu(\alpha) \ \forall \alpha \geq \alpha_0\} = \bigcap_{n \in \mathbb{N}}\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha_n) = \nu(\alpha_n)\}$$

because of the right-continuity of the functions $\nu_{\vec{c}}, \nu$. Hence it suffices to show the first equality.

Let us fix $\alpha \geq \alpha_0$. Using the definition of the ρ_j, we have $2^j \rho_j(]-\infty, \alpha'] \geq j^2$ for all $j \geq 1$ and $\alpha' \geq \alpha$. From Lemma 3.3, it follows that

$$\mathbb{P}\left(\left\{\vec{c} \in \Omega \ : \ \frac{1}{2}2^j \rho_j(]-\infty,\alpha']) \leq \#E_j(1,\alpha')(\vec{c}) \leq 22^j \rho_j(]-\infty,\alpha'])\right\}\right) \geq 1 - C\frac{2^j}{j^2}e^{-Cj^2}, \ \forall \alpha' \geq \alpha, j \geq 1.$$

By the Borel-Cantelli lemma, it follows that

$$1 = \mathbb{P}\left(\bigcup_{J\geq 1}\bigcap_{j\geq J}\left\{\vec{c} \in \Omega : \frac{1}{2}2^j \rho_j(]-\infty,\alpha']) \leq \#E_j(1,\alpha')(\vec{c}) \leq 22^j \rho_j(]-\infty,\alpha'])\right\}\right), \ \forall \alpha' \geq \alpha.$$

Since

$$\begin{aligned} &\bigcup_{J\geq 1}\bigcap_{j\geq J}\left\{\vec{c} \in \Omega : \frac{1}{2}2^j \rho_j(]-\infty,\alpha']) \leq \#E_j(1,\alpha')(\vec{c}) \leq 22^j \rho_j(]-\infty,\alpha'])\right\} \\ &\subset \left\{\vec{c} \in \Omega : \nu(\alpha') \leq \nu^*_{\vec{c}}(\alpha') = \limsup_{j\to+\infty} \frac{\log_2(\#E_j(1,\alpha')(\vec{c}))}{j} \leq \nu(\alpha')\right\}\end{aligned}$$

we get

$$1 = \mathbb{P}\left(\left\{\vec{c} \in \Omega : \nu(\alpha') = \nu^*_{\vec{c}}(\alpha')\right\}\right), \ \forall \alpha' \geq \alpha.$$

Using a dense sequence and the relation between $\nu_{\vec{c}}, \nu^*_{\vec{c}}$, we finally get

$$1 = \mathbb{P}\left(\{\vec{c} \in \Omega : \nu(\alpha) = \nu_{\vec{c}}(\alpha)\}\right)$$

as announced.□
As a consequence of the previous properties, we get

$$\mathbb{P}\left(\{\vec{c} \in \Omega : \nu_{\vec{c}}(\alpha) = \nu(\alpha), \ \forall \alpha\}\right) = 1.$$

and also

$$\mathbb{P}\left(\mathcal{S}^\nu\right) = 1.$$

In [3], the following result is proved.

Theorem 3.6. ([3]) For every $\vec{d} \in \Omega$, we have

$$\mathbb{P}\left(\vec{d} + \{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) \geq \nu(\alpha), \ \forall \alpha\}\right) = \mathbb{P}\left(\left\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}-\vec{d}}(\alpha) \geq \nu(\alpha), \ \forall \alpha\right\}\right) = 1.$$

□
It follows that

$$\begin{aligned}\mathbb{P}\left(\vec{d} + \{\vec{c} \in \Omega \ : \ \nu_{\vec{c}}(\alpha) = \nu(\alpha), \ \forall \alpha\}\right) &= \mathbb{P}\left(\left\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}-\vec{d}}(\alpha) = \nu(\alpha), \ \forall \alpha\right\}\right) \\ &\geq \mathbb{P}\left(\left\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}-\vec{d}}(\alpha) \leq \nu(\alpha), \ \forall \alpha\right\}\right).\end{aligned}$$

If $\vec{d} \in \mathcal{S}^\nu$, we get

$$\begin{aligned}\mathbb{P}\left(\left\{\vec{c} \in \Omega \ : \ \nu_{\vec{c}-\vec{d}}(\alpha) = \nu(\alpha), \ \forall \alpha\right\}\right) &\geq \mathbb{P}\left(\left\{\vec{c} \in \mathcal{S}^\nu \ : \ \nu_{\vec{c}-\vec{d}}(\alpha) \leq \nu(\alpha), \ \forall \alpha\right\}\right) \\ &= \mathbb{P}\left(\vec{d} + \{\vec{c} \in \mathcal{S}^\nu \ : \ \nu_{\vec{c}}(\alpha) \leq \nu(\alpha), \ \forall \alpha\}\right) \\ &= \mathbb{P}(\mathcal{S}^\nu) = 1\end{aligned}$$

hence also

$$\mathbb{P}\left(\left\{\vec{c} \in \mathcal{S}^\nu \ : \ \nu_{\vec{c}-\vec{d}}(\alpha) = \nu(\alpha), \ \forall\alpha\right\}\right) = 1.$$

At that point, since $\mathcal{S}^\nu$ is not a metric complete space under the product topology, we can not conclude immediately and say that the set of sequences in $\mathcal{S}^\nu$ for which the profile is equal to ν is prevalent. In fact, as will be showed in the next section, $\mathbb{P}$ defines a probability Borel measure on the metric (complete and separable) space $\mathcal{S}^\nu$; hence the previous result indeed means that the set of all sequences $\vec{c}$ in $\mathcal{S}^\nu$ such that $\nu_{\vec{c}} = \nu$ is prevalent.

3.2.4 Probability measure on the metric sequence space $\mathcal{S}^\nu$

In the paper [3], the following result is proved.

Proposition 3.7. All the open subsets of the metric space $\mathcal{S}^\nu$ belongs to $\mathcal{F}$. □

Since $\mathbb{P}(\mathcal{S}^\nu) = 1$, this implies that $\mathbb{P}$ defines a probability Borel measure on the (complete and separable) metric space $\mathcal{S}^\nu$. As mentioned in the introduction about prevalence, this implies the following result.

Theorem 3.8. The set of all sequences $\vec{c}$ in $\mathcal{S}^\nu$ such that $\nu_{\vec{c}} = \nu$ is prevalent.

For related results in the context of random wavelet series and multifractal analysis (involving Hölder exponent), we refer to [5],[3],[6]. Nevertheless, let us briefly mention some properties concerning fundamental links between signals expressed in terms of their wavelet coefficients and sequences spaces introduced here above.

4 A link with random wavelet series and multifractal analysis

In this section, we use the notations introduced in section 2.1 (untitled "The context") of this paper.

We recall that a $1-$ periodic function f on $\mathbb{R}$ belongs to the Hölder space C^α (where $\alpha > 0$) if and only if $\sup_{j,k} 2^{\alpha j}|c_{j,k}| < +\infty$. In that case, the Hölder exponent of f at x is given by $\limsup_{j\to+\infty} \inf_k \frac{\ln(|c_{j,k}|)}{\ln(2^{-j}+|k2^{-j}-x|)}$ provided that the wavelet has enough vanishing moments (see for example [9], [5]).

Assume now that we have a sequence ρ_j of Borel probabilities on $]-\infty,+\infty]$. Then, using a standard procedure of measure theory, one can construct a Borel probability $\mathbb{P}$ on the sequence space Ω such that

$$\mathbb{P}\left(\left\{\vec{c} \ : \ |c_{jk}| \geq 2^{-\alpha j}\right\}\right) = \rho_j(]-\infty,\alpha]), \quad \forall k \in \{0,\ldots,2^j-1\}, \ \forall j \geq 0 \ (*).$$

In this context, the events $|c_{jk}| \geq 2^{-\alpha j}, \ k = 0,\ldots,2^j-1, \ j \geq 0$ are independant; using Borel-Cantelli lemmas, it follows that

$$\sum_{j=0}^{+\infty} 2^j \rho_j(]-\infty,\alpha]) \left\{ \begin{array}{l} = +\infty \\ < +\infty \end{array} \right. \quad \Rightarrow \quad \mathbb{P}\left(\bigcap_{J\geq 0}\bigcup_{j\geq J}\{\vec{c} : \exists k : |c_{jk}| \geq 2^{-\alpha j}\}\right) = \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right.$$

which can be interpreted as the fact that with probability 1, we have $|c_{jk}| \geq 2^{-\alpha j}$ for infinitely many coefficients (resp. $|c_{jk}| \leq 2^{-\alpha j}$ but for a finite number of coefficients) if the series $\sum_{j=0}^{+\infty} 2^j \rho_j(]-\infty,\alpha])$ diverges (resp. converges).

Considering multifractal analysis and Hölder regularity in the context of probability theory, it is then natural to say that

$$f = \sum_{j,k} c_{jk}\psi_{jk} \textit{ is a random wavelet series}$$

in case the coefficients are random variables such that (*) and

$$\exists \alpha_0 > 0 : \sum_{j=0}^{+\infty} 2^j \rho_j(]-\infty, \alpha]) < +\infty \ \forall \alpha < \alpha_0$$

(see [5]). In this context, the study of the profile ν_f also leads to the following definition

$$\nu(\alpha) = \limsup_{j\to+\infty} \frac{\log_2(2^j \rho_j(]-\infty,\alpha])}{j},\ \alpha \in \mathbb{R}.$$

This function is nondecreasing, right-continuous and $\nu(\alpha) < 0$ if $\alpha < \alpha_0$.
This can be considered as a starting point for the study of $\mathcal{S}^\nu$ (nothing is changed if one considers $\nu = -\infty$ on $]-\infty, \alpha_0[$).
Finally, in the context of multifractal analysis (see the introduction for the definition of d_f) and random wavelet series, let us mention the following result, obtained using Theorem 3.8 and results of [5].

Proposition 4.1. ([3],[6]) If $h_0 = \inf\{\frac{h'}{\nu(h')} : h' \geq \alpha_0\}$ and

$$d_\nu(h) = \begin{cases} h \sup\left\{\frac{\nu(h')}{h'} : h' \in]0,h]\right\} & \text{if } h \leq h_0 \\ 1 & \text{if } h > h_0 \end{cases}$$

then the following set is prevalent

$$\{f \in \mathcal{S}^\nu \ : \ d_f(h) = d_\nu(h), \forall h \leq h_0,\ d_f(h) = -\infty,\ \forall h > h_0\}.$$

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Hyers-Ulam Stability of Linear Differential Equations of First Order, I

Soon-Mo Jung

Mathematics Section
College of Science and Technology
Hong-Ik University, 339-701 Chochiwon, Korea
smjung@hongik.ac.kr

ABSTRACT

Let X be a complex Banach space and let $I = (a, b)$ be an open interval with $-\infty \le a < b \le \infty$. Suppose $\varphi : I \to [0, \infty)$ is a function and $e^{-\mu t}\varphi(t)$ is integrable on each subinterval of I, where μ is a fixed complex number. In this paper, we will prove that if a differentiable function $f : I \to X$ satisfies $\|f'(t) - \mu f(t)\| \le \varphi(t)$ for all $t \in I$, then there exists a unique $x \in X$ such that $\|f(t) - e^{\mu t}x\| \le e^{\Re(\mu)t}\left|\int_t^b e^{-\mu v}\varphi(v)dv\right|$ for all $t \in I$.

Keywords: Hyers-Ulam stability, differential equation.

2000 Mathematics Subject Classification: 26D10, 34A40.

1 Introduction

In 1940, S. M. Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [18]). Among those was the question concerning the stability of homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In the following year, D. H. Hyers affirmatively answered in his paper [6] the question of Ulam for the case where G_1 and G_2 are Banach spaces. Furthermore, the result of Hyers has been generalized by Th. M. Rassias (ref. [16]; see also [2]). Since then, the stability problems of various functional equations have been investigated by many authors (see [3, 4, 5, 7, 8, 9]).

Let X be a normed space and let I be an open interval. Assume that for any n times differentiable function $f : I \to X$ satisfying the inequality

$$\|a_n(t)f^{(n)}(t) + a_{n-1}(t)f^{(n-1)}(t) + \cdots + a_1(t)f'(t) + a_0(t)f(t) + h(t)\| \le \varepsilon$$

for all $t \in I$ and for some $\varepsilon \ge 0$, there exists a solution $f_0 : I \to X$ of the differential equation

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + h(t) = 0$$

such that $\|f(t) - f_0(t)\| \leq K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression of ε only. Then, we say that the above differential equation has the Hyers-Ulam stability.
If the above statement is also true when we replace ε and $K(\varepsilon)$ by $\varphi(t)$ and $\Phi(t)$, where $\varphi, \Phi : I \to [0, \infty)$ are functions not dependent on f and f_0 explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability).
We may apply these terminologies for other (linear or nonlinear) differential equations. For more detailed definitions of the Hyers-Ulam stability and the generalized Hyers-Ulam stability, we refer the reader to [3, 7, 8, 9].

C. Alsina and R. Ger were the first authors who investigated the Hyers-Ulam stability of differential equations: They proved in [1] that if a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(t) - y(t)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \leq 3\varepsilon$ for any $t \in I$.
This result of Alsina and Ger has been generalized by S.-E. Takahasi, T. Miura and S. Miyajima. They proved in [17] that the Hyers-Ulam stability holds true for the Banach space valued differential equation

$$y'(t) = \mu y(t) \tag{1.1}$$

(see also [11, 12]):

> *Let X be a Banach space, let μ be a complex number with $\Re(\mu) \neq 0$, and let I be an open interval. If a differentiable function $f : I \to X$ satisfies $\|f'(t) - \mu f(t)\| \leq \varepsilon$ for all $t \in I$ and for some $\varepsilon > 0$, then there exists a solution $f_0 : I \to X$ of the differential equation (1.1) such that*
>
> $$\|f(t) - f_0(t)\| \leq \frac{\varepsilon}{|\Re(\mu)|}\left(1 - \frac{m}{M}\right) \quad (\textit{for any } t \in I),$$
>
> *where $m = \inf\{e^{-\Re(\mu)t} : t \in I\}$ and $M = \sup\{e^{-\Re(\mu)t} : t \in I\}$.*

Recently, T. Miura, S. Miyajima and S.-E. Takahasi [13] investigated the Hyers-Ulam stability of linear differential equations of n-th order, $a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 = 0$, with complex coefficients.
In [14], T. Miura, S. Miyajima and S.-E. Takahasi also proved the Hyers-Ulam stability of linear differential equations of first order, $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function. Indeed, they dealt with the differential inequality $\|y'(t) + g(t)y(t)\| \leq \varepsilon$ for some $\varepsilon > 0$.
Recently, the author proved the Hyers-Ulam stability of differential equations of the form $c(t)y'(t) = y(t)$ (see [10].)

The aim of this paper is to investigate the generalized Hyers-Ulam stability of the first order linear differential equation (1.1). That is, this paper deals with a direct generalization of the paper [17].
We assume that X is a complex Banach space, μ is a fixed complex number, and that $I = (a, b)$ is an arbitrary interval with $-\infty \leq a < b \leq \infty$. We moreover assume that $\varphi : I \to [0, \infty)$ is

a function such that $e^{-\mu t}\varphi(t)$ is integrable on every subinterval of I. We will prove that if a differentiable function $f : I \to X$ satisfies the inequality

$$\|f'(t) - \mu f(t)\| \leq \varphi(t) \tag{1.2}$$

for all $t \in I$, then there exists a unique $x \in X$ such that

$$\|f(t) - e^{\mu t}x\| \leq e^{\Re(\mu)t}\left|\int_t^b e^{-\mu v}\varphi(v)dv\right| \tag{1.3}$$

for all $t \in I$, where $\Re(\mu)$ denotes the real part of μ.

2 Main Results

Throughout this section, let $I = (a, b)$ be an open interval, where $a, b \in \mathbb{R}\cup\{\pm\infty\}$ are arbitrarily given with $a < b$.

Following ideas of the papers [11, 12, 15, 17] (see also [1]), we will prove the generalized Hyers-Ulam stability of the linear differential equation (1.1).

Theorem 2.1. *Let X be a complex Banach space, let μ be a complex number, and let $I = (a, b)$ be an open interval as above. Suppose $\varphi : I \to [0, \infty)$ is a function and $e^{-\mu t}\varphi(t)$ is integrable on each subinterval of I. If a differentiable function $f : I \to X$ satisfies the inequality (1.2) for all $t \in I$, then there exists a unique $x \in X$ such that the equality (1.3) holds for every $t \in I$.*

Proof. We denote by X^* the dual space of X. For any given $\lambda \in X^*$, we define a function $f_\lambda : I \to \mathbb{C}$ by $f_\lambda(t) = \lambda(f(t))$ for all $t \in I$. Since λ is continuous, we can easily verify that $(f_\lambda)'(t) = \lambda(f'(t))$ for every $t \in I$.

Let us define a function $z : I \to X$ by $z(t) = e^{-\mu t}f(t)$ for any $t \in I$. It then follows from (1.2) that

$$\begin{aligned}
|\lambda(z(t) - z(s))| &= \left|e^{-\mu t}f_\lambda(t) - e^{-\mu s}f_\lambda(s)\right| \\
&= \left|\int_s^t \frac{d}{dv}\left[e^{-\mu v}f_\lambda(v)\right]dv\right| \\
&= \left|\int_s^t e^{-\mu v}\{(f_\lambda)'(v) - \mu f_\lambda(v)\}dv\right| \\
&\leq \left|\int_s^t e^{-\mu v}\left|\lambda(f'(v) - \mu f(v))\right|dv\right| \\
&\leq \left|\int_s^t e^{-\mu v}\|\lambda\|\|f'(v) - \mu f(v)\|dv\right| \\
&\leq \|\lambda\|\left|\int_s^t e^{-\mu v}\varphi(v)dv\right|
\end{aligned}$$

for any $s, t \in I$. Since $\lambda \in X^*$ was selected arbitrarily, we may conclude from the above inequality that

$$\|z(t) - z(s)\| \leq \left|\int_s^t e^{-\mu v}\varphi(v)dv\right| \tag{2.1}$$

for all $s, t \in I$.

Due to the integrability hypothesis, for any given $\varepsilon > 0$, there exists a $t_0 \in I$ such that $\|z(t) - z(s)\| < \varepsilon$ for any $s, t \in I$ with $s, t \geq t_0$. Thus, $\{z(s)\}_{s \in I}$ is a Cauchy net and hence there exists an $x \in X$ such that $z(s)$ converges to x as $s \to b$, since X is complete.
Therefore, it follows from (2.1) that

$$\begin{aligned} \|f(t) - e^{\mu t}x\| &= \|e^{\mu t}(z(t) - x)\| \\ &\leq e^{\Re(\mu)t}\|z(t) - z(s)\| + e^{\Re(\mu)t}\|z(s) - x\| \\ &\leq e^{\Re(\mu)t}\left|\int_s^t e^{-\mu v}\varphi(v)dv\right| + e^{\Re(\mu)t}\|z(s) - x\| \end{aligned}$$

for all $s, t \in I$. If we let $s \to b$ in the above inequality, then we obtain

$$\|f(t) - e^{\mu t}x\| \leq e^{\Re(\mu)t}\left|\int_b^t e^{-\mu v}\varphi(v)dv\right|$$

for each $t \in I$, which proves the validity of inequality (1.3).
Now, it remains to prove the uniqueness of x. Assume that inequality (1.3) holds for another $x_1 \in X$ in place of x. It then follows from (1.3) that

$$\begin{aligned} e^{\Re(\mu)t}\|x_1 - x\| &\leq \|(f(t) - e^{\mu t}x) - (f(t) - e^{\mu t}x_1)\| \\ &\leq \|f(t) - e^{\mu t}x\| + \|f(t) - e^{\mu t}x_1\| \\ &\leq 2e^{\Re(\mu)t}\left|\int_t^b e^{-\mu v}\varphi(v)dv\right| \end{aligned}$$

for all $t \in I$. If we divide by $e^{\Re(\mu)t}$ both the sides of the above inequality and if we let $t \to b$ in the last inequality, then the integrability hypothesis implies that $x = x_1$, which means the uniqueness of x. □

Remark 2.1. We may now remark that $y(t) = e^{\mu t}x$ is the general solution of the differential equation (1.1), where x is an arbitrary element of X.

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Stability of an Euler–Lagrange–Rassias type additive mapping

Chun-Gil Park[1]

Dedicated to the Tricentennial Birthday Anniversary of Leonhard Euler

[1]Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
cgpark@cnu.ac.kr

ABSTRACT

Let X, Y be Banach modules over a C^*-algebra and let r, s be positive real numbers. We prove the stability of the following functional equation in Banach modules over a unital C^*-algebra:

$$rf\left(s(x-y)\right)+sf\left(r(y-x)\right)+(r+s)f\left(rx+sy\right)=(r+s)\left(rf(x)+sf(y)\right). \tag{0.1}$$

We show that if $r = s$ and an odd mapping $f : X \to Y$ satisfies the functional equation (0.1) then the odd mapping $f : X \to Y$ is Cauchy additive. As an application, we show that every almost linear bijection $h : A \to B$ of a unital C^*-algebra A onto a unital C^*-algebra B is a C^*-algebra isomorphism when $h((2r)^d uy) = h((2r)^d u)h(y)$ for all unitaries $u \in A$, all $y \in A$, and all $d \in \mathbf{Z}$.

Keywords: Euler–Lagrange–Rassias type additive mapping, stability, isomorphism between C^*-algebras.

2000 Mathematics Subject Classification: 39B52, 46L05, 47B48.

1 Introduction

J.M. Rassias [6] introduced and solved the stability problem of Ulam for the Euler–Lagrange type quadratic functional equation

$$f(rx+sy)+f(sx-ry)=(r^2+s^2)[f(x)+f(y)], \tag{1.1}$$

motivated from the following pertinent algebraic equation

$$|ax+by|^2+|bx-ay|^2=(a^2+b^2)(|x|^2+|y|^2). \tag{1.2}$$

The solution of the functional equation (1.1) is called an *Euler–Lagrange type quadratic mapping*. J.M. Rassias [7, 8] introduced and investigated the relative functional equations. In addition, J.M. Rassias [9] generalized the above algebraic equation (1.2) to the following equation

$$mn|ax+by|^2+|nbx-may|^2=(ma^2+nb^2)(n|x|^2+m|y|^2),$$

and introduced and investigated the general pertinent Euler–Lagrange quadratic mappings. Analogous quadratic mappings were introduced and investigated in [10, 11].
These Euler–Lagrange mappings could be named *Euler–Lagrange–Rassias mappings* and the corresponding Euler–Lagrange equations might be called *Euler–Lagrange–Rassias equations*. Before 1992, these mappings and equations were not known at all in functional equations and inequalities. However, a completely different kind of Euler–Lagrange partial differential equations are known in calculus of variations. Therefore, we think that our introduction of Euler–Lagrange mappings and equations in functional equations and inequalities provides an interesting cornerstone in analysis. Already some mathematicians have employed these Euler–Lagrange mappings.
Recently, Jun and Kim [1] solved the stability problem of Ulam for another Euler–Lagrange–Rassias type quadratic functional equation. Jun and Kim [2] introduced and investigated the following quadratic functional equation of Euler–Lagrange–Rassias type

$$\begin{aligned}\sum_{i=1}^{n} r_i Q\left(\sum_{j=1}^{n} r_j(x_i-x_j)\right)+\left(\sum_{i=1}^{n} r_i\right) Q\left(\sum_{i=1}^{n} r_i x_i\right)\\ =\left(\sum_{i=1}^{n} r_i\right)^2 \sum_{i=1}^{n} r_i Q(x_i),\end{aligned}$$

whose solution is said to be a generalized quadratic mapping of Euler–Lagrange–Rassias type.
In this paper, we introduce the following functional equation

$$\begin{aligned}rL\left(s(x-y)\right)+sL\left(r(y-x)\right)+(r+s)L\left(rx+sy\right)\\ =(r+s)\left(rL(x)+sL(y)\right), \qquad r,s\in(0,\infty)\end{aligned} \tag{1.3}$$

whose solution is called an *Euler–Lagrange–Rassias type additive mapping*. We investigate the stability of an Euler–Lagrange–Rassias type additive mapping in Banach modules over a C^*-algebra. These results are applied to investigate C^*-algebra isomorphisms between unital C^*-algebras.

2 Stability of an Euler–Lagrange–Rassias type additive mapping in Banach modules over a C^*-algebra

Throughout this section, assume that A is a unital C^*-algebra with norm $|\cdot|$ and unitary group $U(A)$, and that X and Y are left Banach modules over a unital C^*-algebra A with norms $||\cdot||$ and $\|\cdot\|$, respectively.

For a given mapping $f : X \to Y$ and a given $u \in U(A)$, we define $D_u f : X^2 \to Y$ by

$$D_u f(x,y) := rf\left(s(ux-uy)\right) + sf\left(r(uy-ux)\right) + (r+s)f\left(rux+suy\right) \\ -(r+s)\left(ruf(x)+suf(y)\right)$$

for all $x, y \in X$.

Lemma 2.1. Assume that a mapping $L : X \to Y$ satisfies the functional equation (1.3) and that $L(0) = 0$. Then we have

$$L((r+s)^k x) = (r+s)^k L(x) \tag{2.1}$$

for all $x \in X$ and all $k \in \mathbf{Z}$.

Proof. Putting $x = y$ in (1.3), we get $(r+s)L((r+s)x) = (r+s)^2 L(x)$ for all $x \in X$. So we get

$$L((r+s)^k x) = (r+s)^k L(x) \tag{2.2}$$

for all $x \in X$ by induction on $k \in \mathbf{N}$.
It follows from (2.2) that

$$L\left(\frac{x}{(r+s)^k}\right) = \frac{1}{(r+s)^k} L(x)$$

for all $x \in X$ and all $k \in \mathbf{N}$. So we get the equality (2.1). $\square$

We investigate the stability of an Euler–Lagrange–Rassias type additive mapping in Banach spaces.

Theorem 2.2. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^2 \to [0, \infty)$ such that

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} \frac{1}{(r+s)^j} \varphi((r+s)^j x, (r+s)^k y) < \infty, \tag{2.3}$$

$$\|D_1 f(x,y)\| \le \varphi(x,y) \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique Euler–Lagrange–Rassias type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{(r+s)^2} \widetilde{\varphi}(x,x) \tag{2.5}$$

for all $x \in X$.

Note that if $r + s = 1$ in (2.3), then φ is identically zero. So $f = L$ is itself an Euler–Lagrange–Rassias type additive mapping. Thus we assume that $r + s \neq 1$.

Proof. Letting $x = y$ in (2.4), we get the following inequality

$$\left\|(r+s)f((r+s)x) - (r+s)^2 f(x)\right\| \le \varphi(x,x) \tag{2.6}$$

for all $x \in X$. It follows from (2.6) that

$$\left\| f(x) - \frac{f((r+s)x)}{r+s} \right\| \leq \frac{1}{(r+s)^2}\varphi(x,x) \tag{2.7}$$

for all $x \in X$. Now applying a standard procedure of direct method [4, 5] to the inequality (2.7), we obtain that for all nonnegative integers k, l with $k > l$

$$\left\| \frac{f((r+s)^l x)}{(r+s)^l} - \frac{f((r+s)^k x)}{(r+s)^k} \right\| \leq \frac{1}{(r+s)^2} \sum_{j=l}^{k-1} \frac{1}{(r+s)^j}\varphi((r+s)^j x, (r+s)^j x) \tag{2.8}$$

for all $x \in X$. Since the right hand side of (2.8) tends to zero as $l \to \infty$, the sequence $\{\frac{f((r+s)^k x)}{(r+s)^k}\}$ is a Cauchy sequence for all $x \in X$, and thus converges by the completeness of Y. Thus we can define a mapping $L : X \to Y$ by

$$L(x) = \lim_{k\to\infty} \frac{f((r+s)^k x)}{(r+s)^k}$$

for all $x \in X$. Letting $l = 0$ in (2.8), we obtain

$$\left\| f(x) - \frac{f((r+s)^k x)}{(r+s)^k} \right\| \leq \frac{1}{(r+s)^2} \sum_{j=0}^{k-1} \frac{1}{(r+s)^j}\varphi((r+s)^j x, (r+s)^j x) \tag{2.9}$$

for all $x \in X$ and all $k \in \mathbf{N}$. Taking the limit as $k \to \infty$ in (2.9), we obtain the desired inequality (2.5).

It follows from (2.3) and (2.4) that

$$\begin{aligned} \|D_1 L(x,y)\| &= \lim_{k\to\infty} \frac{1}{(r+s)^k} \|D_1 f((r+s)^k x, (r+s)^k y)\| \\ &\leq \lim_{k\to\infty} \frac{1}{(r+s)^k} \varphi((r+s)^k x, (r+s)^k y) = 0. \end{aligned} \tag{2.10}$$

Therefore, the mapping $L : X \to Y$ satisfies the equation (1.3) and hence L is an Euler–Lagrange–Rassias type additive mapping.

To prove the uniqueness, let L' be another Euler–Lagrange–Rassias type additive mapping satisfying (2.5). By Lemma 2.1, we get $L'((r+s)^k x) = (r+s)^k L'(x)$ for all $x \in X$ and all $k \in \mathbf{N}$. Thus we have, for any positive integer k,

$$\begin{aligned} \|L(x) - L'(x)\| &\leq \frac{1}{(r+s)^k} \Big\{ \Big\| L((r+s)^k x) - f((r+s)^k x) \Big\| \\ &\quad + \Big\| f((r+s)^k x) - L'((r+s)^k x) \Big\| \Big\} \\ &\leq \frac{2}{(r+s)^2} \sum_{j=0}^{\infty} \frac{1}{(r+s)^{k+j}} \varphi((r+s)^{k+j} x, (r+s)^{k+j} x). \end{aligned}$$

Taking the limit as $k \to \infty$, we conclude that $L(x) = L'(x)$ for all $x \in X$. □

Corollary 2.3. Let $\epsilon \geq 0$ and let p be a real number with $0 < p < 1$ if $r + s > 1$ and with $p > 1$ if $r + s < 1$. Assume that a mapping $f : X \to Y$ satisfies the inequality

$$\|D_1 f(x,y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Then there exists a unique Euler–Lagrange–Rassias type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{(r+s)^2 - (r+s)^{p+1}}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y) := \epsilon(||x||^p + ||y||^p)$, and apply Theorem 2.2. □

Theorem 2.4. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^2 \to [0, \infty)$ such that

$$\widetilde{\varphi}(x, y) := \sum_{j=1}^{\infty}(r+s)^j \varphi(\frac{x}{(r+s)^j}, \frac{y}{(r+s)^j}) < \infty, \tag{2.11}$$

$$\|D_1 f(x, y)\| \leq \varphi(x, y) \tag{2.12}$$

for all $x, y \in X$. Then there exists an Euler–Lagrange–Rassias type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{1}{(r+s)^2}\widetilde{\varphi}(x, x) \tag{2.13}$$

for all $x \in X$.

Proof. It follows from (2.6) that

$$\left\|f(x) - (r+s)f(\frac{1}{r+s}x)\right\| \leq \frac{1}{r+s}\varphi(\frac{x}{r+s}, \frac{x}{r+s}) \tag{2.14}$$

for all $x \in X$. Now applying a standard procedure of direct method [4, 5] to the inequality (2.14), we obtain that for all nonnegative integers k, l with $k > l$

$$\begin{aligned} &\left\|(r+s)^l f(\frac{x}{(r+s)^l}) - (r+s)^k f(\frac{x}{(r+s)^k})\right\| \\ &\leq \frac{1}{(r+s)^2} \sum_{j=l+1}^{k} (r+s)^j \varphi(\frac{x}{(r+s)^j}, \frac{x}{(r+s)^j}) \end{aligned} \tag{2.15}$$

for all $x \in X$. Since the right hand side of (2.15) tends to zero as $l \to \infty$, the sequence $\{(r+s)^k f(\frac{x}{(r+s)^k})\}$ is a Cauchy sequence for all $x \in X$, and thus converges by the completeness of Y. Thus we can define a mapping $L : X \to Y$ by

$$L(x) = \lim_{k\to\infty}(r+s)^k f(\frac{x}{(r+s)^k})$$

for all $x \in X$. Letting $l = 0$ in (2.15), we obtain

$$\left\|f(x) - (r+s)^k f(\frac{x}{(r+s)^k})\right\| \leq \frac{1}{(r+s)^2}\sum_{j=1}^{k}(r+s)^j \varphi(\frac{x}{(r+s)^j}, \frac{x}{(r+s)^j}) \tag{2.16}$$

for all $x \in X$ and all $k \in \mathbf{N}$. Taking the limit as $k \to \infty$ in (2.16), we obtain the desired inequality (2.13).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. Let $\epsilon \geq 0$ and let p be a real number with $0 < p < 1$ if $r + s < 1$ and with $p > 1$ if $r + s > 1$. Assume that a mapping $f : X \to Y$ satisfies the inequality

$$\|D_1 f(x, y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Then there exists a unique Euler–Lagrange–Rassias type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{(r+s)^{p+1} - (r+s)^2}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y) := \epsilon(||x||^p + ||y||^p)$, and apply Theorem 2.4. □

3 Stability of linear mappings in Banach modules over a C^*-algebra

Throughout this section, assume that $r = s \in (0, \infty)$. Let A be a unital C^*-algebra with norm $|\cdot|$ and unitary group $U(A)$, and let X and Y be left Banach modules over a unital C^*-algebra A with norms $||\cdot||$ and $\|\cdot\|$, respectively.

Lemma 3.1. If an odd mapping $L : X \to Y$ satisfies (1.3) for all $x, y \in X$, then L is Cauchy additive.

Proof. Assume that $L : X \to Y$ satisfies (1.3) for all $x, y \in X$.
Note that $L(0) = 0$ and $L(-x) = -L(x)$ for all $x \in X$ since L is an odd mapping. It follows from (1.3) that

$$2rL(rx + ry) = 2r^2\big(L(x) + L(y)\big) \tag{3.1}$$

for all $x, y \in X$. Letting $y = 0$ in (3.1), $2rL(rx) = 2r^2L(x)$ for all $x \in X$. So

$$2r^2L(x + y) = 2r^2\big(L(x) + L(y)\big)$$

for all $x, y \in X$. Thus L is Cauchy additive. □

Theorem 3.2. Let $f : X \to Y$ be an odd mapping for which there is a function $\varphi : X^2 \to [0, \infty)$ such that

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{(2r)^j}\varphi((2r)^j x, (2r)^j y) < \infty, \tag{3.2}$$

$$\|D_u f(x, y)\| \leq \varphi(x, y) \tag{3.3}$$

for all $u \in U(A)$ and all $x, y \in X$. Then there exists a unique A-linear Euler–Lagrange–Rassias type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{1}{(2r)^2}\widetilde{\varphi}(x, x) \tag{3.4}$$

for all $x \in X$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Let $u = 1 \in U(A)$. By Theorem 2.2, there exists a unique Euler–Lagrange–Rassias type additive mapping $L : X \to Y$ satisfying (3.4).

By the assumption, for each $u \in U(A)$, we get

$$\|D_u L(x,0)\| = \lim_{d\to\infty} \frac{1}{(2r)^d}\|D_u f((2r)^d x, 0)\| \le \lim_{d\to\infty} \frac{1}{(2r)^d}\varphi((2r)^d x, 0) = 0$$

for all $x \in X$. So

$$2rL(rux) = 2r \cdot ruL(x)$$

for all $u \in U(A)$ and all $x \in X$. By Lemma 3.1,

$$L(ux) = \frac{1}{r}L(rux) = uL(x) \tag{3.5}$$

for all $u \in U(A)$ and all $x \in X$.

By the same reasoning as in the proofs of [4] and [5],

$$L(ax + by) = L(ax) + L(by) = aL(x) + bL(y)$$

for all $a, b \in A(a, b \neq 0)$ and all $x, y \in X$. And $L(0x) = 0 = 0L(x)$ for all $x \in X$. So the unique Euler–Lagrange–Rassias type additive mapping $L : X \to Y$ is an A-linear mapping. □

Corollary 3.3. Let $\epsilon \ge 0$ and let p be a real number with $0 < p < 1$ if $2r > 1$ and with $p > 1$ if $2r < 1$. Let $f : X \to Y$ be an odd mapping such that

$$\|D_u f(x,y)\| \le \epsilon(||x||^p + ||y||^p)$$

for all $u \in U(A)$ and all $x, y \in X$. Then there exists a unique A-linear Euler–Lagrange–Rassias type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{(2r)^2 - (2r)^{p+1}}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x,y) = \epsilon(||x||^p + ||y||^p)$, and apply Theorem 3.2. □

Theorem 3.4. Let $f : X \to Y$ be an odd mapping for which there is a function $\varphi : X^2 \to [0,\infty)$ satisfying (3.3) such that

$$\widetilde{\varphi}(x,y) := \sum_{j=1}^{\infty}(2r)^j\varphi(\frac{1}{(2r)^j}x, \frac{1}{(2r)^j}y) < \infty$$

for all $x, y \in X$. Then there exists a unique A-linear Euler–Lagrange–Rassias type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{(2r)^2}\widetilde{\varphi}(x,x) \tag{3.6}$$

for all $x \in X$.

Proof. The proof is similar to the proofs of Theorems 2.4 and 3.2. □

Corollary 3.5. Let $\epsilon \geq 0$ and let p be a real number with $0 < p < 1$ if $2r < 1$ and with $p > 1$ if $2r > 1$. Let $f : X \to Y$ be an odd mapping such that

$$\|D_u f(x,y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $u \in U(A)$ and all $x, y \in X$. Then there exists a unique A-linear Euler–Lagrange–Rassias type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{(2r)^{p+1} - (2r)^2}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x,y) = \epsilon(||x||^p + ||y||^p)$, and apply Theorem 3.2. □

4 Isomorphisms between unital C^*-algebras

Throughout this section, assume that $r = s \in \mathbf{Q} \cap (0, \infty)$. Assume that A is a unital C^*-algebra with norm $||\cdot||$ and unit e, and that B is a unital C^*-algebra with norm $\|\cdot\|$. Let $U(A)$ be the set of unitary elements in A.

We investigate C^*-algebra isomorphisms between unital C^*-algebras.

Theorem 4.1. Let $h : A \to B$ be an odd bijective mapping satisfying $h((2r)^d uy) = h((2r)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $d \in \mathbf{Z}$, for which there exists a function $\varphi : A^2 \to [0, \infty)$ such that

$$\sum_{j=0}^{\infty} \frac{1}{(2r)^j}\varphi((2r)^j x, (2r)^j y) < \infty, \tag{4.1}$$

$$\|D_\mu h(x,y)\| \leq \varphi(x,y),$$

$$\|h((2r)^d u^*) - h((2r)^d u)^*\| \leq \varphi((2r)^d u, (2r)^d u) \tag{4.2}$$

for all $\mu \in S^1 := \{\lambda \in \mathbf{C} \mid \ |\lambda| = 1\}$, all $u \in U(A)$, all $d \in \mathbf{Z}$ and all $x, y \in A$. Assume that

$$\text{(4.i)} \qquad \lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d e) \text{ is invertible.}$$

Then the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Consider the C^*-algebras A and B as left Banach modules over the unital C^*-algebra $\mathbf{C}$. By Theorem 3.2, there exists a unique $\mathbf{C}$-linear Euler–Lagrange–Rassias type additive mapping $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{1}{(2r)^2}\widetilde{\varphi}(x,x) \tag{4.3}$$

for all $x \in A$. The Euler–Lagrange–Rassias type additive mapping $H : A \to B$ is given by

$$H(x) = \lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d x)$$

for all $x \in A$.

By (4.1) and (4.2), we get

$$\begin{aligned} H(u^*) &= \lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d u^*) = \lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d u)^* \\ &= (\lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d u))^* = H(u)^* \end{aligned}$$

for all $u \in U(A)$. Since H is $\mathbf{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [3]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbf{C}, u_j \in U(A))$,

$$\begin{aligned} H(x^*) &= H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* = (\sum_{j=1}^m \lambda_j H(u_j))^* \\ &= H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^* \end{aligned}$$

for all $x \in A$.

Since $h((2r)^d uy) = h((2r)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $d \in \mathbf{Z}$,

$$H(uy) = \lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d uy) = \lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d u)h(y) = H(u)h(y) \tag{4.4}$$

for all $u \in U(A)$ and all $y \in A$. By the additivity of H and (4.4),

$$(2r)^d H(uy) = H((2r)^d uy) = H(u((2r)^d y)) = H(u)h((2r)^d y)$$

for all $u \in U(A)$ and all $y \in A$. Hence

$$H(uy) = \frac{1}{(2r)^d} H(u)h((2r)^d y) = H(u)\frac{1}{(2r)^d} h((2r)^d y) \tag{4.5}$$

for all $u \in U(A)$ and all $y \in A$. Taking the limit in (4.5) as $d \to \infty$, we obtain

$$H(uy) = H(u)H(y) \tag{4.6}$$

for all $u \in U(A)$ and all $y \in A$. Since H is $\mathbf{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbf{C}, u_j \in U(A))$, it follows from (4.6) that

$$\begin{aligned} H(xy) &= H(\sum_{j=1}^m \lambda_j u_j y) = \sum_{j=1}^m \lambda_j H(u_j y) = \sum_{j=1}^m \lambda_j H(u_j)H(y) = H(\sum_{j=1}^m \lambda_j u_j)H(y) \\ &= H(x)H(y) \end{aligned}$$

for all $x, y \in A$.

By (4.4) and (4.6),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in A$. Since $\lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d e) = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in A$.

Therefore, the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism. □

Corollary 4.2. Let $\epsilon \geq 0$ and let p be a real number with $0 < p < 1$ if $2r > 1$ and with $p > 1$ if $2r < 1$. Let $h : A \to B$ be an odd bijective mapping satisfying $h((2r)^d uy) = h((2r)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $d \in \mathbf{Z}$, such that

$$\|D_\mu h(x, y)\| \leq \epsilon(||x||^p + ||y||^p),$$
$$\|h((2r)^d u^*) - h((2r)^d u)^*\| \leq 2 \cdot (2r)^{dp}\epsilon$$

for all $\mu \in S^1$, all $u \in U(A)$, all $d \in \mathbf{Z}$, and all $x, y \in A$. Assume that $\lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d e)$ is invertible. Then the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Define $\varphi(x, y) = \epsilon(||x||^p + ||y||^p)$, and apply Theorem 4.1. □

Theorem 4.3. Let $h : A \to B$ be an odd bijective mapping satisfying $h((2r)^d uy) = h((2r)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $d \in \mathbf{Z}$, for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying (4.1), (4.2), and (4.i) such that

$$\|D_\mu h(x, y)\| \leq \varphi(x, y) \tag{4.7}$$

for $\mu = 1, i$, and all $x, y \in A$. If $h(tx)$ is continuous in $t \in \mathbf{R}$ for each fixed $x \in A$, then the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Put $\mu = 1$ in (4.7). By the same reasoning as in the proof of Theorem 2.2, there exists a unique Euler–Lagrange–Rassias type additive mapping $H : A \to B$ satisfying (4.3). By the same reasoning as in the proofs of [4, 5], the Euler–Lagrange–Rassias type additive mapping $H : A \to B$ is $\mathbf{R}$-linear.
Put $\mu = i$ in (4.7). By the same method as in the proof of Theorem 3.2, one can obtain that

$$H(ix) = \lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d ix) = \lim_{d\to\infty} \frac{i}{(2r)^d} h((2r)^d x) = iH(x)$$

for all $x \in A$.
For each element $\lambda \in \mathbf{C}$, $\lambda = s + it$, where $s, t \in \mathbf{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) = (s + it)H(x) \\ &= \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbf{C}$ and all $x \in A$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbf{C}$, and all $x, y \in A$. Hence the Euler–Lagrange–Rassias type additive mapping $H : A \to B$ is $\mathbf{C}$-linear.
The rest of the proof is the same as in the proof of Theorem 4.1. □

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Hyers–Ulam stability of an Euler–Lagrange type additive mapping

Chun-Gil Park[1] **and John Michael Rassias**[2]

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[1]Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
cgpark@cnu.ac.kr

[2]Pedagogical Department, E. E.
National and Capodistrian University of Athens
Section of Mathematics and Informatics
4, Agamemnonos Str., Aghia Paraskevi
Athens 15342, Greece
jrassias@primedu.uoa.gr

ABSTRACT

Let X, Y be Banach modules over a C^*-algebra and let r, s be positive real numbers. We prove the stability of the following functional equation in Banach modules over a unital C^*-algebra:

$$rf(s(x-y)) + sf(r(y-x)) + (r+s)f(rx+sy) = (r+s)(rf(x)+sf(y)). \qquad (0.1)$$

We show that if $r = s$ and an odd mapping $f : X \to Y$ satisfies the functional equation (0.1) *then the odd mapping $f : X \to Y$ is Cauchy additive. As an application, we show that every almost linear bijection $h : A \to B$ of a unital C^*-algebra A onto a unital C^*-algebra B is a C^*-algebra isomorphism when $h((2r)^d uy) = h((2r)^d u)h(y)$ for all unitaries $u \in A$, all $y \in A$, and all $d \in \mathbb{Z}$.*

Keywords: Euler–Lagrange type additive mapping, Hyers–Ulam stability, isomorphism between C^*-algebras.

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1 Introduction

In 1940, S.M. Ulam [19] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an $f : G \to G'$ an *approximate homomorphism*.

In 1941, D.H. Hyers [1] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

No continuity conditions are required for this result, but if $f(tx)$ is continuous in the real variable t for each fixed $x \in E$, then L is linear, and if f is continuous at a single point of E then $L : E \to E'$ is also continuous. A generalization of this result was proved by J.M. Rassias [7, 8, 9, 13]. J.M. Rassias assumed *the following weaker inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \cdot ||x||^p \cdot ||y||^q, \qquad \forall x, y \in E,$$

involving a product of different powers of norms, where $\theta > 0$ and real p, q such that $r = p + q \neq 1$, and retained the condition of continuity $f(tx)$ in t for fixed x. J.M. Rassias [11] investigated that it is possible to replace ϵ in the above Hyers inequality by a non-negative real-valued function such that the pertinent series converges and other conditions hold and still obtain stability results. In all the cases investigated in this article, the approach to the existence question was to prove *asymptotic type formulas* of the form $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$, or $L(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$. However, in 2002, J.M. Rassias and M.J. Rassias [17] considered and investigated quadratic equations involving a product of powers of norms in which an approximate quadratic mapping *degenerates* to a *genuine* quadratic mapping. Analogous results could be investigated with additive type equations involving a product of powers of norms.

Theorem 1.1. ([7, 8, 13]) Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p \in \mathbb{R} - \{1\}$ such that f satisfies inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \cdot ||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}}$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^p - 2|} ||x||^p$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Theorem 1.2. ([9]) Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \cdot ||x||^p \cdot ||y||^q$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|}||x||^r$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Theorem 1.3. ([11]) Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exists a constant $\theta \geq 0$ such that f satisfies inequality

$$\|f(\sum_{i=1}^{n} x_i) - \sum_{i=1}^{n} f(x_i)\| \leq \theta K(x_1, \cdots, x_n)$$

for all $(x_1, \cdots, x_n) \in X^n$ and $K : X^n \to \mathbb{R}^+ \cup \{0\}$ is a non-negative real-valued function such that

$$R_n(x) = \sum_{j=0}^{\infty} \frac{1}{n^j} K(n^j x, \cdots, n^j x) < \infty$$

is a non-negative function of x, and the condition

$$\lim_{m \to \infty} \frac{1}{n^m} K(n^m x_1, \cdots, n^m x_n) = 0$$

holds. Then there exists a unique additive mapping $L_n : X \to Y$ satisfying

$$\|f(x) - L_n(x)\| \leq \frac{\theta}{n} R_n(x)$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L_n is an $\mathbb{R}$-linear mapping.

Example 1.4. *If we take* $n = 2, x_1 = x, x_2 = y$ *and thus* $K(x_1, x_2) = K(x, y) = ||x||^p \cdot ||y||^q$ *such that* $-\infty < r = p + q < 1$*; and respectively,* $r > 1$*, then we obtain*

$$\begin{aligned} R_2(x) &= \frac{2}{2 - 2^r}||x||^r \qquad (r < 1); \\ R_2(x) &= \frac{2}{2^r - 2}||x||^r \qquad (r > 1). \end{aligned}$$

Therefore, we get the following inequality

$$\|f(x) - L_2(x)\| \leq \frac{\theta}{|2^r - 2|}||x||^r, \qquad r \neq 1$$

with $L_2 = L$ *and hence the stability result of Theorem 1.2 is achieved.*

Theorem 1.5. ([11]) Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping such that f satisfies inequality

$$\|f(\sum_{i=1}^{n} x_i) - \sum_{i=1}^{n} f(x_i)\| \leq N(x_1, \cdots, x_n)$$

for all $(x_1, \cdots, x_n) \in X^n$ and $N : X^n \to \mathbb{R}^+ \cup \{0\}$ is a non-negative real-valued function such that $N(x, \cdots, x)$ is bounded on the unit ball of X, and

$$N(tx_1, \cdots, tx_n) \leq k(t) N(x_1, \cdots, x_n)$$

for all $t \geq 0$, where $k(t) < \infty$ and

$$R_n^0 = \sum_{j=0}^{\infty} \frac{1}{n^j} k(n^j) < \infty.$$

If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $f : X \to Y$ is bounded on some ball of X, then there exists a unique $\mathbb{R}$-linear mapping $L_n : X \to Y$ satisfying

$$\|f(x) - L_n(x)\| \leq MN(x, \cdots, x)$$

for all $x \in X$, where $M = \sum_{m=0}^{\infty} \frac{1}{n^{m+1}} k(n^m)$.

Example 1.6. *If we take* $n = 2, x_1 = x, x_2 = y$ *and thus* $N(x_1, x_2) = N(x, y) = \theta \cdot ||x||^p \cdot ||y||^q$*, then we obtain* $N(tx_1, tx_2) = |t|^r N(x_1, x_2)$ *with* $k(t) = |t|^r < \infty$ *and* $N(x_1, x_2) = \theta$ *on the unit ball* $U = \{x \in X : ||x|| = 1\}$ *and, in general,* $N(x_1, x_2) = \theta ||x||^r$ *for all* $x \in X$*, where* $-\infty < r = p + q < 1$*; and respectively,* $r > 1$*, such that*

$$\begin{aligned} R_2^0 &= \frac{2}{2 - 2^r} \qquad (r < 1), \\ M &= \frac{1}{2 - 2^r} \qquad (r < 1), \\ MN(x, x) &= \frac{\theta}{2 - 2^r} ||x||^r \qquad (r < 1); \end{aligned}$$

$$\begin{aligned} R_2^0 &= \frac{2}{2^r - 2} \qquad (r > 1), \\ M &= \frac{1}{2^r - 2} \qquad (r > 1), \\ MN(x, x) &= \frac{\theta}{2^r - 2} ||x||^r \qquad (r > 1). \end{aligned}$$

Therefore, we get the following inequality

$$\|f(x) - L_2(x)\| \leq \frac{\theta}{|2^r - 2|} ||x||^r, \qquad r \neq 1$$

with $L_2 = L$ *and hence the stability result of Theorem 1.2 is achieved.*

J.M. Rassias [10] introduced and solved the stability problem of Ulam for the Euler–Lagrange type quadratic functional equation

$$f(rx+sy)+f(sx-ry)=(r^2+s^2)[f(x)+f(y)], \tag{1.1}$$

motivated from the following pertinent algebraic equation

$$|ax+by|^2+|bx-ay|^2=(a^2+b^2)(|x|^2+|y|^2). \tag{1.2}$$

The solution of the functional equation (1.1) is called an *Euler–Lagrange type quadratic mapping*. J.M. Rassias [12, 14] introduced and investigated the relative functional equations. In addition, J.M. Rassias [15] generalized the above algebraic equation (1.2) to the following equation

$$mn|ax+by|^2+|nbx-may|^2=(ma^2+nb^2)(n|x|^2+m|y|^2),$$

and introduced and investigated the general pertinent Euler–Lagrange quadratic mappings. Analogous quadratic mappings were introduced and investigated in [16, 18].

These mappings could be named *Euler–Lagrange mappings* and the corresponding equations might be called *Euler–Lagrange equations*. Before 1992, these mappings and equations were not known at all in functional equations and inequalities. However, a completely different kind of Euler–Lagrange partial differential equations are known in calculus of variations. Therefore, we think that J.M. Rassias' introduction of Euler–Lagrange mappings and equations in functional equations and inequalities provides an interesting cornerstone in analysis. Already some mathematicians have employed these Euler–Lagrange mappings.

Recently, Jun and Kim [2] solved the stability problem of Ulam for another Euler–Lagrange type quadratic functional equation. Jun and Kim [3] introduced and investigated the following quadratic functional equation of Euler–Lagrange type

$$\begin{aligned}\sum_{i=1}^{n} r_i Q\left(\sum_{j=1}^{n} r_j(x_i-x_j)\right)+\left(\sum_{i=1}^{n} r_i\right) Q\left(\sum_{i=1}^{n} r_i x_i\right) \\ =\left(\sum_{i=1}^{n} r_i\right)^2 \sum_{i=1}^{n} r_i Q(x_i),\end{aligned}$$

whose solution is said to be a generalized quadratic mapping of Euler–Lagrange type.

In this paper, we introduce the following functional equation

$$\begin{aligned} rL\left(s(x-y)\right)+sL\left(r(y-x)\right)+(r+s)L\left(rx+sy\right) \\ =(r+s)\left(rL(x)+sL(y)\right), \qquad r,s\in(0,\infty)\end{aligned} \tag{1.3}$$

whose solution is called an *Euler–Lagrange type additive mapping*. We investigate the Hyers–Ulam stability of an Euler–Lagrange type additive mapping in Banach modules over a C^*-algebra. These results are applied to investigate C^*-algebra isomorphisms between unital C^*-algebras.

2 Hyers–Ulam stability of an Euler–Lagrange type additive mapping in Banach modules over a C^*-algebra

Throughout this section, assume that A is a unital C^*-algebra with norm $|\cdot|$ and unitary group $U(A)$, and that X and Y are left Banach modules over a unital C^*-algebra A with norms $||\cdot||$ and $\|\cdot\|$, respectively.
For a given mapping $f : X \to Y$ and a given $u \in U(A)$, we define $D_u f : X^2 \to Y$ by

$$D_u f(x,y) := rf\left(s(ux-uy)\right) + sf\left(r(uy-ux)\right) + (r+s)f\left(rux+suy\right) \\ -(r+s)\left(ruf(x)+suf(y)\right)$$

for all $x, y \in X$.

Lemma 2.1. Assume that a mapping $L : X \to Y$ satisfies the functional equation (1.3) and that $L(0) = 0$. Then we have

$$L((r+s)^k x) = (r+s)^k L(x) \tag{2.1}$$

for all $x \in X$ and all $k \in \mathbb{Z}$.

Proof. Putting $x = y$ in (1.3), we get $(r+s)L((r+s)x) = (r+s)^2 L(x)$ for all $x \in X$. So we get

$$L((r+s)^k x) = (r+s)^k L(x) \tag{2.2}$$

for all $x \in X$ by induction on $k \in \mathbb{N}$.
It follows from (2.2) that

$$L\left(\frac{x}{(r+s)^k}\right) = \frac{1}{(r+s)^k} L(x)$$

for all $x \in X$ and all $k \in \mathbb{N}$. So we get the equality (2.1). □

We investigate the Hyers–Ulam stability of an Euler–Lagrange type additive mapping in Banach spaces.

Theorem 2.2. Let $\epsilon \geq 0$ and let p be a real number with $0 < p < \frac{1}{2}$ if $r+s > 1$ and with $p > \frac{1}{2}$ if $r+s < 1$. Assume that a mapping $f : X \to Y$ satisfies inequality

$$\|D_1 f(x,y)\| \leq \epsilon \cdot ||x||^p \cdot ||y||^p \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique Euler–Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{(r+s)^2 - (r+s)^{2p+1}} ||x||^{2p} \tag{2.4}$$

for all $x \in X$.

Proof. Letting $x = y$ in (2.3), we get the following inequality

$$\left\|(r+s)f((r+s)x) - (r+s)^2 f(x)\right\| \le \epsilon ||x||^{2p} \tag{2.5}$$

for all $x \in X$. It follows from (2.5) that

$$\left\|f(x) - \frac{f((r+s)x)}{r+s}\right\| \le \frac{\epsilon ||x||^{2p}}{(r+s)^2} \tag{2.6}$$

for all $x \in X$. Now applying a standard procedure of direct method [1, 5] to the inequality (2.6), we obtain that for all nonnegative integers k, l with $k > l$

$$\left\|\frac{f((r+s)^l x)}{(r+s)^l} - \frac{f((r+s)^k x)}{(r+s)^k}\right\| \le \frac{\epsilon}{(r+s)^2} \sum_{j=l}^{k-1} \frac{(r+s)^{2pj}}{(r+s)^j} ||x||^{2p} \tag{2.7}$$

for all $x \in X$. Since the right hand side of (2.7) tends to zero as $l \to \infty$, the sequence $\{\frac{f((r+s)^k x)}{(r+s)^k}\}$ is a Cauchy sequence for all $x \in X$, and thus converges by the completeness of Y. Thus we can define a mapping $L : X \to Y$ by

$$L(x) = \lim_{k\to\infty} \frac{f((r+s)^k x)}{(r+s)^k}$$

for all $x \in X$. Letting $l = 0$ in (2.7), we obtain

$$\left\|f(x) - \frac{f((r+s)^k x)}{(r+s)^k}\right\| \le \frac{\epsilon}{(r+s)^2} \sum_{j=0}^{k-1} \frac{(r+s)^{2pj}}{(r+s)^j} ||x||^{2p} \tag{2.8}$$

for all $x \in X$ and all $k \in \mathbb{N}$. Taking the limit as $k \to \infty$ in (2.8), we obtain the desired inequality (2.4).

It follows from (2.3) that

$$\begin{aligned} \|D_1 L(x,y)\| &= \lim_{k\to\infty} \frac{1}{(r+s)^k} \|D_1 f((r+s)^k x, (r+s)^k y)\| \\ &\le \lim_{k\to\infty} \frac{\epsilon (r+s)^{2pk}}{(r+s)^k} |||x||^{2p} = 0. \end{aligned} \tag{2.9}$$

Therefore, the mapping $L : X \to Y$ satisfies the equation (1.3) and hence L is an Euler–Lagrange type additive mapping.

To prove the uniqueness, let L' be another Euler–Lagrange type additive mapping satisfying (2.4). By Lemma 2.1, we get $L'((r+s)^k x) = (r+s)^k L'(x)$ for all $x \in X$ and all $k \in \mathbb{N}$. Thus we have, for any positive integer k,

$$\begin{aligned} \|L(x) - L'(x)\| &\le \frac{1}{(r+s)^k} \Big\{ \left\|L((r+s)^k x) - f((r+s)^k x)\right\| \\ &\quad + \left\|f((r+s)^k x) - L'((r+s)^k x)\right\| \Big\} \\ &\le \frac{2\epsilon}{(r+s)^2 - (r+s)^{2p+1}} \sum_{j=0}^{\infty} \frac{(r+s)^{2pk+2pj}}{(r+s)^{k+j}} ||x||^{2p}. \end{aligned}$$

Taking the limit as $k \to \infty$, we conclude that $L(x) = L'(x)$ for all $x \in X$. □

Theorem 2.3. Let $\epsilon \geq 0$ and let p be a real number with $0 < p < \frac{1}{2}$ if $r+s<1$ and with $p > \frac{1}{2}$ if $r+s>1$. Assume that a mapping $f : X \to Y$ satisfies inequality

$$\|D_1 f(x,y)\| \leq \epsilon \cdot ||x||^p \cdot ||y||^p \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique Euler–Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{(r+s)^{2p+1} - (r+s)^2}||x||^{2p} \tag{2.11}$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$\left\|f(x) - (r+s)f(\frac{1}{r+s}x)\right\| \leq \frac{\epsilon}{(r+s)^{2p+1}}||x||^{2p} \tag{2.12}$$

for all $x \in X$. Now applying a standard procedure of direct method [1, 5] to the inequality (2.12), we obtain that for all nonnegative integers k, l with $k > l$

$$\left\|(r+s)^l f(\frac{x}{(r+s)^l}) - (r+s)^k f(\frac{x}{(r+s)^k})\right\| \tag{2.13}$$
$$\leq \frac{\epsilon}{(r+s)^{2p+1}} \sum_{j=l}^{k-1} \frac{(r+s)^j}{(r+s)^{2pj}}||x||^{2p}$$

for all $x \in X$. Since the right hand side of (2.13) tends to zero as $l \to \infty$, the sequence $\{(r+s)^k f(\frac{x}{(r+s)^k})\}$ is a Cauchy sequence for all $x \in X$, and thus converges by the completeness of Y. Thus we can define a mapping $L : X \to Y$ by

$$L(x) = \lim_{k\to\infty}(r+s)^k f(\frac{x}{(r+s)^k})$$

for all $x \in X$. Letting $l = 0$ in (2.13), we obtain

$$\left\|f(x) - (r+s)^k f(\frac{x}{(r+s)^k})\right\| \leq \frac{\epsilon}{(r+s)^{2p+1}} \sum_{j=0}^{k-1} \frac{(r+s)^j}{(r+s)^{2pj}}||x||^{2p} \tag{2.14}$$

for all $x \in X$ and all $k \in \mathbb{N}$. Taking the limit as $k \to \infty$ in (2.14), we obtain the desired inequality (2.11).

The rest of the proof is similar to the proof of Theorem 2.2. □

3 Stability of linear mappings in Banach modules over a C^*-algebra

Throughout this section, assume that $r = s \in (0, \infty)$. Let A be a unital C^*-algebra with norm $|\cdot|$ and unitary group $U(A)$, and let X and Y be left Banach modules over a unital C^*-algebra A with norms $||\cdot||$ and $\|\cdot\|$, respectively.

Lemma 3.1. If an odd mapping $L : X \to Y$ satisfies (1.4) for all $x, y \in X$, then L is Cauchy additive.

Proof. Assume that $L : X \to Y$ satisfies (1.4) for all $x, y \in X$.
Note that $L(0) = 0$ and $L(-x) = -L(x)$ for all $x \in X$ since L is an odd mapping. It follows from (1.4) that

$$2rL(rx + ry) = 2r^2\big(L(x) + L(y)\big) \tag{3.1}$$

for all $x, y \in X$. Letting $y = 0$ in (3.1), $2rL(rx) = 2r^2L(x)$ for all $x \in X$. So

$$2r^2L(x + y) = 2r^2\big(L(x) + L(y)\big)$$

for all $x, y \in X$. Thus L is Cauchy additive. □

Theorem 3.2. Let $\epsilon \geq 0$ and let p be a real number with $0 < p < \frac{1}{2}$ if $2r > 1$ and with $p > \frac{1}{2}$ if $2r < 1$. Let $f : X \to Y$ be an odd mapping such that

$$\|D_u f(x, y)\| \leq \epsilon \cdot ||x||^p \cdot ||y||^p \tag{3.2}$$

for all $u \in U(A)$ and all $x, y \in X$. Then there exists a unique A-linear Euler–Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{(2r)^2 - (2r)^{2p+1}}||x||^{2p} \tag{3.3}$$

for all $x \in X$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Let $u = 1 \in U(A)$. By Theorem 2.2, there exists a unique Euler–Lagrange type additive mapping $L : X \to Y$ satisfying (3.3).
By the assumption, for each $u \in U(A)$, we get

$$\|D_u L(x, x)\| = \lim_{d\to\infty} \frac{1}{(2r)^d}\|D_u f((2r)^d x, (2r)^d x)\| \leq \lim_{d\to\infty} \frac{\epsilon(2r)^{2pd}}{(2r)^d}||x||^{2p} = 0$$

for all $x \in X$. So

$$2rL(2rux) = 2r \cdot 2ruL(x)$$

for all $u \in U(A)$ and all $x \in X$. By Lemma 3.1,

$$L(ux) = \frac{1}{2r}L(2rux) = uL(x) \tag{3.4}$$

for all $u \in U(A)$ and all $x \in X$.
By the same reasoning as in the proofs of [5] and [6],

$$L(ax + by) = L(ax) + L(by) = aL(x) + bL(y)$$

for all $a, b \in A (a, b \neq 0)$ and all $x, y \in X$. And $L(0x) = 0 = 0L(x)$ for all $x \in X$. So the unique Euler–Lagrange type additive mapping $L : X \to Y$ is an A-linear mapping. □

Theorem 3.3. Let $\epsilon \geq 0$ and let p be a real number with $0 < p < \frac{1}{2}$ if $2r < 1$ and with $p > \frac{1}{2}$ if $2r > 1$. Let $f : X \to Y$ be an odd mapping such that

$$\|D_u f(x,y)\| \leq \epsilon \cdot ||x||^p \cdot ||y||^p \tag{3.5}$$

for all $u \in U(A)$ and all $x, y \in X$. Then there exists a unique A-linear Euler–Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{(2r)^{2p+1} - (2r)^2} ||x||^{2p} \tag{3.6}$$

for all $x \in X$.

Proof. The proof is similar to the proofs of Theorems 2.2 and 3.2. □

4 Isomorphisms between unital C^*-algebras

Throughout this section, assume that $r = s \in \mathbb{Q} \cap (0, \infty)$. Assume that A is a unital C^*-algebra with norm $||\cdot||$ and unit e, and that B is a unital C^*-algebra with norm $\|\cdot\|$. Let $U(A)$ be the set of unitary elements in A.

We investigate C^*-algebra isomorphisms between unital C^*-algebras.

Theorem 4.1. Let $\epsilon \geq 0$ and let p be a real number with $0 < p < \frac{1}{2}$ if $2r > 1$ and with $p > \frac{1}{2}$ if $2r < 1$. Let $h : A \to B$ be an odd bijective mapping satisfying $h((2r)^d uy) = h((2r)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $d \in \mathbb{Z}$, such that

$$\|D_\mu h(x,y)\| \leq \epsilon \cdot ||x||^p \cdot ||y||^p, \tag{4.1}$$

$$\|h((2r)^d u^*) - h((2r)^d u)^*\| \leq 2 \cdot (2r)^{dp} \epsilon \tag{4.2}$$

for all $\mu \in S^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, all $u \in U(A)$, all $d \in \mathbb{Z}$, and all $x, y \in A$. Assume that

$$\lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d e) \text{ is invertible.} \tag{4.3}$$

Then the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Consider the C^*-algebras A and B as left Banach modules over the unital C^*-algebra $\mathbb{C}$. By Theorem 3.2, there exists a unique $\mathbb{C}$-linear Euler–Lagrange type additive mapping $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{\epsilon}{(2r)^2 - (2r)^{2p+1}} ||x||^{2p} \tag{4.4}$$

for all $x \in A$. The Euler–Lagrange type additive mapping $H : A \to B$ is given by

$$H(x) = \lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d x)$$

for all $x \in A$.

By (4.2), we get

$$\begin{aligned} H(u^*) &= \lim_{d\to\infty}\frac{1}{(2r)^d}h((2r)^d u^*) = \lim_{d\to\infty}\frac{1}{(2r)^d}h((2r)^d u)^* \\ &= (\lim_{d\to\infty}\frac{1}{(2r)^d}h((2r)^d u))^* = H(u)^* \end{aligned}$$

for all $u \in U(A)$. Since H is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [4]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(A)$),

$$\begin{aligned} H(x^*) &= H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* = (\sum_{j=1}^m \lambda_j H(u_j))^* \\ &= H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^* \end{aligned}$$

for all $x \in A$.
Since $h((2r)^d uy) = h((2r)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $d \in \mathbb{Z}$,

$$H(uy) = \lim_{d\to\infty}\frac{1}{(2r)^d}h((2r)^d uy) = \lim_{d\to\infty}\frac{1}{(2r)^d}h((2r)^d u)h(y) = H(u)h(y) \tag{4.5}$$

for all $u \in U(A)$ and all $y \in A$. By the additivity of H and (4.5),

$$(2r)^d H(uy) = H((2r)^d uy) = H(u((2r)^d y)) = H(u)h((2r)^d y)$$

for all $u \in U(A)$ and all $y \in A$. Hence

$$H(uy) = \frac{1}{(2r)^d}H(u)h((2r)^d y) = H(u)\frac{1}{(2r)^d}h((2r)^d y) \tag{4.6}$$

for all $u \in U(A)$ and all $y \in A$. Taking the limit in (4.6) as $d \to \infty$, we obtain

$$H(uy) = H(u)H(y) \tag{4.7}$$

for all $u \in U(A)$ and all $y \in A$. Since H is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(A)$), it follows from (4.7) that

$$\begin{aligned} H(xy) &= H(\sum_{j=1}^m \lambda_j u_j y) = \sum_{j=1}^m \lambda_j H(u_j y) = \sum_{j=1}^m \lambda_j H(u_j)H(y) = H(\sum_{j=1}^m \lambda_j u_j)H(y) \\ &= H(x)H(y) \end{aligned}$$

for all $x, y \in A$.
By (4.5) and (4.7),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in A$. Since $\lim_{d\to\infty}\frac{1}{(2r)^d}h((2r)^d e) = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in A$.
Therefore, the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism. □

Theorem 4.2. Let $\epsilon \geq 0$ and let p be a real number with $0 < p < \frac{1}{2}$ if $2r < 1$ and with $p > \frac{1}{2}$ if $2r > 1$. Let $h : A \to B$ be an odd bijective mapping satisfying $h((2r)^d uy) = h((2r)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $d \in \mathbb{Z}$, (4.1) and (4.2). Assume that

$$\lim_{d\to\infty} (2r)^d h(\frac{e}{(2r)^d}) \text{ is invertible.} \tag{4.8}$$

Then the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. The proof is similar to the proof of Theorem 4.1. □

Theorem 4.3. Let $\epsilon \geq 0$ and let p be a real number with $0 < p < \frac{1}{2}$ if $2r > 1$ and with $p > \frac{1}{2}$ if $2r < 1$. Let $h : A \to B$ be an odd bijective mapping satisfying $h((2r)^d uy) = h((2r)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$ and all $d \in \mathbb{Z}$, (4.2) and (4.3) such that

$$\|D_\mu h(x, y)\| \leq \epsilon \cdot ||x||^p \cdot ||y||^p \tag{4.9}$$

for $\mu = 1, i$, and all $x, y \in A$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Put $\mu = 1$ in (4.9). By the same reasoning as in the proof of Theorem 2.2, there exists a unique Euler–Lagrange type additive mapping $H : A \to B$ satisfying (4.4). By the same reasoning as in the proofs of [5, 6], the Euler–Lagrange type additive mapping $H : A \to B$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (4.9). By the same method as in the proof of Theorem 3.2, one can obtain that

$$H(ix) = \lim_{d\to\infty} \frac{1}{(2r)^d} h((2r)^d ix) = \lim_{d\to\infty} \frac{i}{(2r)^d} h((2r)^d x) = iH(x)$$

for all $x \in A$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) = (s + it)H(x) \\ &= \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in A$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in A$. Hence the Euler–Lagrange type additive mapping $H : A \to B$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 4.1. □

Theorem 4.4. Let $\epsilon \geq 0$ and let p be a real number with $0 < p < \frac{1}{2}$ if $2r < 1$ and with $p > \frac{1}{2}$ if $2r > 1$. Let $h : A \to B$ be an odd bijective mapping satisfying $h((2r)^d uy) = h((2r)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$ and all $d \in \mathbb{Z}$, (4.2), (4.8) and (4.9). If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. The proof is similar to the proofs of Theorems 4.1 and 4.3. □

By the same method as above, one can obtain similar results to Theorem 1.2 for the case $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$.

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Refined ULAM Stability for Euler–Lagrange Type Mappings in Hilbert Spaces

John Michael Rassias[1] and **Matina John Rassias**[2]

[1] Pedagogical Department, E E
Section of Mathematics and Informatics,
National and Capodistrian University of Athens
Athens , GREECE
e-mail: jrassias@primedu.uoa.gr

and

[2] Statistics and Modelling Science, University of Strathclyde
Livingstone Tower, 26 Richmond Str. Glasgow , Scotland UK G1 1XH
e-mail:matina@stams.strath.ac.uk

Abstract

We have introduced the Euler-Lagrange quadratic mappings, motivated from the pertinent algebraic equation $|a_1x_1+a_2x_2|^2+|a_2x_1-a_1x_2|^2=(a_1^2+a_2^2)[|x_1|^2+|x_2|^2]$, *for the special case* $a_i=1\ (i=1,2)$. *Thus we introduced and investigated the relative functional equation in our publications. In addition we generalized the above algebraic equation to the following equation*

$$m_1m_2|a_1x_1+a_2x_2|^2+|m_2a_2x_1-m_1a_1x_2|^2=(m_1|a_1|^2+m_2|a_2|^2)[m_2|x_1|^2+m_1|x_2|^2]$$

Therefore we introduced and investigated the general pertinent Euler-Lagrange quadratic mappings via our paper. Analogous quadratic mappings were introduced and investigated through our following publications. Before 1992 these mappings and equations were not known at all in functional equations and inequalities. However, a completely different kind of Euler-Lagrange partial differential equations is known in calculus of variations. We are now introducing the general Euler-Lagrange type equation

$\|ax+by\|^2+\|bx+ay\|^2+\|ax-by\|^2+\|bx-ay\|^2=2(a^2+b^2)[\|x\|^2+\|y\|^2]$ *which holds in real Hilbert spaces and establish the refined Ulam stability for these mappings.*

Key words and phrases: Euler-Lagrange mapping; Ulam stability; Hilbert space.

AMS (MOS) Subject Classification: 39B.

1. Introduction

We note that we have introduced the *Euler-Lagrange quadratic mappings*, motivated from the following pertinent algebraic equation

$$|a_1x_1+a_2x_2|^2+|a_2x_1-a_1x_2|^2=(a_1^2+a_2^2)[|x_1|^2+|x_2|^2], \tag{1.1}$$

for the special case $a_i = 1$ $(i = 1, 2)$. Thus we introduced and investigated the relative functional equation in our publications below ([1]-[3]). In addition we generalized the above algebraic equation (1.1) to the following equation

$$m_1 m_2 \left| a_1 x_1 + a_2 x_2 \right|^2 + \left| m_2 a_2 x_1 - m_1 a_1 x_2 \right|^2 = (m_1 \left| a_1 \right|^2 + m_2 \left| a_2 \right|^2)[m_2 \left| x_1 \right|^2 + m_1 \left| x_2 \right|^2] \qquad (1.2)$$

Therefore we introduced and investigated the general pertinent Euler-Lagrange quadratic mappings via our paper [4]. Analogous quadratic mappings were introduced and investigated through our following publications ([5]-[6]).

Before 1992 these mappings and equations were not known at all in functional equations and inequalities. However, a completely different kind of Euler-Lagrange partial differential equations is known in calculus of variations. Therefore we think that our introduction of Euler-Lagrange mappings and equations in functional equations and inequalities provides an interesting cornerstone in analysis. Already some mathematicians have employed these Euler-Lagrange mappings.

We are now introducing the general equation

$$\left\| ax + by \right\|^2 + \left\| bx + ay \right\|^2 + \left\| ax - by \right\|^2 + \left\| bx - ay \right\|^2 = 2(a^2 + b^2)[\left\| x \right\|^2 + \left\| y \right\|^2] \qquad (1.3)$$

which holds in real Hilbert spaces. Motivated from this equation we introduce the general Euler-Lagrange type quadratic difference operator

$$D_f(x, y) = f(ax + by) + f(bx + ay) + f(ax - by) + f(bx - ay) - 2\lambda[f(x) + f(y)] \qquad (1.4)$$

with $0 < \lambda = a^2 + b^2 \neq 1; a, b \in \mathbb{R}/\{0\}$ *and* $\mathbb{R} = (-\infty, \infty)$ the set of all real numbers.

Definition 1.1. A mapping $f : X \to Y$ is called *quadratic* if *f* satisfies equation

$$f(x + y) + f(x - y) - 2[f(x) + f(y)] = 0 \qquad (1.5)$$

for all $x, y \in X$.

Let us denote by $\mathbb{N}$ the set of all natural numbers. Substituting $(0,0); (x,x)$ on (x, y) in functional equation (1.5) ,one gets $f(0) = 0$ and $f(2x) = 2^2 f(x)$ and by induction on $n \in \mathbb{N}$, we obtain $f(2^n x) = 2^{2n} f(x)$ for all $x \in X$.

Furthermore substituting $(0, x)$ *on* (x, y) in (1.5), one obtains $f(-x) = f(x)$ for all $x \in X$.

Definition 1.2. A mapping $f : X \to Y$ is called *Euler-Lagrange* if *f* satisfies functional equation

$$f(ax + by) + f(bx - ay) - \lambda[f(x) + f(y)] = 0 \qquad (1.6)$$

for all $x, y \in X$,where $0 < \lambda = a^2 + b^2 \neq 1$; $a \in \mathbb{R}/\{0\}$. ([3]-[6])

Substituting $(0,0); (x,0); (ax,bx)$ on (x, y) in functional equation (1.6) ,one gets $f(0) = 0$ and $f(ax) + f(bx) = \lambda f(x)$ as well as $f(\lambda x) = \lambda[f(ax) + f(bx)]$. Thus

$f(\lambda x) = \lambda^2 f(x)$ for all $x \in X$ and by induction on $n \in \mathbb{N}$, we obtain $f(\lambda^n x) = \lambda^{2n} f(x)$ for all $x \in X$.

Furthermore substituting $(0, x); (x, 0)$ *on* (x, y) in (1.6), one obtains $f(-ax) + f(bx) = \lambda f(x)$ and $f(ax) + f(bx) = \lambda f(x)$ .Therefore $f(-x) = f(x)$ for all $x \in X$.

Equation (1.5) is called also *Euler-Lagrange equation* as a special case of (1.6). ([1]-[2]).

Definition 1.3. A mapping $f: X \to Y$ is called *Euler-Lagrange type* if *f* satisfies functional equation

$$D_f(x, y) = 0 \tag{1.7}$$

for all $x, y \in X$.

Let us denote $\mu = a^2 - b^2 \neq 0, \nu = 2ab$, such that $\mu^2 + \nu^2 = \lambda^2$ and introduce the *quadratic weighted means of first and second form*, respectively:

$$\overline{f(x)} = \frac{f(ax) + f(bx)}{\lambda} \quad and \quad \overline{\overline{f(x)}} = \frac{f(\mu x) + f(\nu x)}{\lambda^2} \tag{1.8}$$

Let us assume *quadratic mean condition*

$$\overline{\overline{f(x)}} = f(x) \tag{*}$$

for all $x \in X$.

Substituting $(0,0); (x,0); (ax, bx)$ on (x, y) in functional equation (1.7) ,one gets $f(0) = 0$ and $f(ax) + f(bx) = \lambda f(x)$ as well as $f(\lambda x) = 2\lambda[f(ax) + f(bx)] - \lambda^2 \overline{\overline{f(x)}}$ $= \lambda^2[2f(x) - \overline{\overline{f(x)}}]$. From the mean condition (*), one now gets $f(\lambda x) = \lambda^2 f(x)$ for all $x \in X$ and by induction on $n \in \mathbb{N}$, we obtain $f(\lambda^n x) = \lambda^{2n} f(x)$ for all $x \in X$. Furthermore substituting $(ax, bx); (bx, ax)$ *on* (x, y) in (1.7), one obtains

$f(\lambda x) + f(\mu x) + f(\nu x) + f(0) = 2\lambda[f(ax) + f(bx)]$ and

$f(\lambda x) + f(-\mu x) + f(\nu x) + f(0) = 2\lambda[f(ax) + f(bx)]$ for all $x \in X$.

If we assume $a \neq \pm b$ (or $\mu \neq 0$) then we get $f(-x) = f(x)$ for all $x \in X$.

Let us denote $\delta > 0, \delta_1 > 0$ *and* $\varepsilon = (\lambda + 1)\delta + \delta_1 + (2\lambda^2 + 1)\|f(0)\|$ and

$r = p + q$, $r \in \mathbb{R}/\{2\}$; $\vartheta = \delta|a|^p|b|^q$; $\varepsilon_1 = \vartheta + \delta_1$.

Throughout this paper, let X be a real normed space and Y a real Banach space in the case of functional inequalities, as well as let X and Y be real linear spaces for functional equations.

2. Hyers - Ulam stability

We state and prove our following theorem on the Hyers - Ulam stability.

Theorem 2.1. *Assume that a mapping* $f: X \to Y$ *satisfies the Euler-Lagrange type quadratic inequality*

$$\|D_f(x, y)\| \le \delta \tag{2.1}$$

for all $x, y \in X$. *If the condition*

$$\left\|\overline{\overline{f(x)}} - f(x)\right\| \le \frac{\delta_1}{\lambda^2} \tag{2.2}$$

holds for $\lambda > 1$ *and all* $x \in X$, *then there exists a unique Euler-Lagrange type quadratic mapping* $Q: X \to Y$ *which satisfies the following inequality*

$$\|f(x)-Q(x)\| \le \frac{\varepsilon}{\lambda^2-1} \tag{2.3}$$

for $\lambda > 1$ *and all* $x \in X$ *,such that the general inequality*

$$\left\|f(x)-\lambda^{-2n}f(\lambda^n x)\right\| \le \frac{\varepsilon}{\lambda^2-1}(1-\lambda^{-2n}) \tag{2.4}$$

holds for $\lambda > 1$ *and all* $n \in \mathbb{N}$ *and* $x \in X$ *as well as Q satisfies the formula*

$$Q(x) = \lim_{n\to\infty} \lambda^{-2n} f(\lambda^n x) \tag{2.5}$$

for $\lambda > 1$ *and all* $x \in X$.

Proof. Substituting $(ax, bx)\ on\ (x, y)$ in the functional inequality (2.1), we obtain

$$\|f(\lambda x)+f(\mu x)+f(\nu x)+f(0)-2\lambda[f(ax)+f(bx)]\| \le \delta \tag{2.6}$$

or

$$\left\|\lambda^{-2}f(\lambda x)+\overline{\overline{f(x)}}-2\overline{f(x)}\right\| \le \frac{\delta+\|f(0)\|}{\lambda^2}, \quad \lambda>1 \tag{2.7}$$

Substituting $(x,0)\ on\ (x,y)$ in the functional inequality (2.1), we obtain

$$\left\|\overline{f(x)}-f(x)\right\| \le \frac{\delta}{2\lambda}+\|f(0)\|, \quad \lambda>1 \tag{2.8}$$

Therefore from the condition (2.2) and (2.7) and (2.8), one gets

$$\begin{aligned} &\left\|f(x)-\lambda^{-2}f(\lambda x)\right\| (= \left\|\lambda^{-2}f(\lambda x)-f(x)\right\|) \\ &\le \left\|\lambda^{-2}f(\lambda x)+\overline{\overline{f(x)}}-2\overline{f(x)}\right\| + 2\left\|\overline{f(x)}-f(x)\right\| + \left\|-[\overline{\overline{f(x)}}-f(x)]\right\| \\ &\le \frac{\varepsilon}{\lambda^2}\left(=\frac{\varepsilon}{\lambda^2-1}(1-\lambda^{-2})\right), \quad \lambda>1 \end{aligned} \tag{2.9}$$

By induction on $n \in \mathbb{N}$ we prove from (2.9) that (2.4) holds. The rest of the proof is standard. We note that substituting $(0,0)\ on\ (x,y)$ in the functional inequality (2.1), one obtains $\|f(0)\| \le \frac{\delta}{4(\lambda-1)}, \quad \lambda>1$.

Furthermore substituting $(ax,bx);(bx,ax)\ on\ (x,y)$ in inequality (2.1), one obtains

$\|f(\lambda x)+f(\mu x)+f(\nu x)+f(0)-2\lambda[f(ax)+f(bx)]\| \le \delta$ and

$\|f(\lambda x)+f(-\mu x)+f(\nu x)+f(0)-2\lambda[f(ax)+f(bx)]\| \le \delta$ for all $x \in X.$

If we assume $|a| \ne |b|$ (or $\mu \ne 0$) then we get $\|f(-\mu x)-f(\mu x)\| \le 2\delta$, or

$\|f(-x)-f(x)\| \le 2\delta$ for all $x \in X.$

3. Refined Ulam stability

We state and prove our following two theorems on a refined Ulam stability.

Theorem 3.1. *Assume that a mapping $f: X \to Y$ satisfies the Euler-Lagrange type quadratic inequality*

$$\|D_f(x,y)\| \le \delta \|x\|^p \|y\|^q \quad (3.1)$$

for $p, q \in \mathbb{R}$ *and all* $x, y \in X$. *If the condition*

$$\left\| \overline{\overline{f(x)}} - f(x) \right\| \le \frac{\delta_1}{\lambda^2} \|x\|^r \quad (3.2)$$

holds for $\lambda > 1$, $r = p+q < 2, or\ 0 < \lambda < 1, r > 2, \quad or\ \lambda^{r-2} < 1$ *and all* $x \in X$ *, then there exists a unique Euler-Lagrange type quadratic mapping* $Q: X \to Y$ *which satisfies the following inequality*

$$\|f(x) - Q(x)\| \le \frac{\varepsilon_1}{\lambda^2 - \lambda^r} \|x\|^r \quad (3.3)$$

for $\lambda > 1$, $r = p+q < 2, or\ 0 < \lambda < 1, r > 2, \quad or\ \lambda^{r-2} < 1$*, and* $x \in X$, *such that the general inequality*

$$\left\| f(x) - \lambda^{-2n} f(\lambda^n x) \right\| \le \frac{\varepsilon_1}{\lambda^2 - \lambda^r} [1 - \lambda^{(r-2)n}] \|x\|^r \quad (3.4)$$

holds for $\lambda > 1$, $r = p+q < 2, or\ 0 < \lambda < 1, r > 2, \quad or\ \lambda^{r-2} < 1$ *and all* $n \in \mathbb{N}$ *and* $x \in X$ *as well as Q satisfies the formula*

$$Q(x) = \lim_{n\to\infty} \lambda^{-2n} f(\lambda^n x) \quad (3.5)$$

for $\lambda > 1$, $r = p+q < 2, or\ 0 < \lambda < 1, r > 2, \quad or\ \lambda^{r-2} < 1$ *and all* $x \in X$.

Proof. Substituting $(0,0)\ on\ (x,y)$ in the functional inequality (3.1), one gets $f(0) = 0$.

Substituting $(ax, bx)\ on\ (x,y)$ in the functional inequality (3.1), we obtain

$$\|D_f(ax,bx)\| = \|f(\lambda x) + f(\mu x) + f(\nu x) + f(0) - 2\lambda[f(ax) + f(bx)]\| \le \vartheta \|x\|^r \quad (3.6)$$

or

$$\left\| \lambda^{-2} f(\lambda x) + \overline{\overline{f(x)}} - 2\overline{f(x)} \right\| \le \frac{\vartheta}{\lambda^2} \|x\|^r \quad (3.7)$$

Substituting $(x,0)\ on\ (x,y)$ in the functional inequality (3.1), we obtain

$$\overline{f(x)} = f(x) \quad (3.8)$$

Therefore from the condition (3.2) and (3.7) and (3.8), one gets

$$\begin{aligned} &\left\| f(x) - \lambda^{-2} f(\lambda x) \right\| (= \left\| \lambda^{-2} f(\lambda x) - f(x) \right\|) \\ &\le \left\| \lambda^{-2} f(\lambda x) + \overline{\overline{f(x)}} - 2\overline{f(x)} \right\| + 2\left\| \overline{f(x)} - f(x) \right\| + \left\| -[\overline{\overline{f(x)}} - f(x)] \right\| \quad (3.9) \\ &\le \frac{\varepsilon_1}{\lambda^2} \|x\|^r \end{aligned}$$

By induction on $n \in \mathbb{N}$ we prove from (3.9) that (3.4) holds. The rest of the proof is standard.

Theorem 3.2. *Assume that a mapping* $f : X \to Y$ *satisfies the Euler-Lagrange type quadratic inequality*

$$\left\| D_f(x, y) \right\| \le \delta \left\| x \right\|^p \left\| y \right\|^q \tag{3.10}$$

for $p, q \in \mathbb{R}$ *and all* $x, y \in X$. *If the condition*

$$\left\| \overline{\overline{f(\frac{x}{\lambda})}} - f(\frac{x}{\lambda}) \right\| \le \frac{\delta_1}{\lambda^{r+2}} \left\| x \right\|^r \tag{3.11}$$

holds for $0 < \lambda < 1$, $r = p + q < 2, or\ \lambda > 1, r > 2, \quad or\ \lambda^{2-r} < 1$ *and all* $x \in X$, *then there exists a unique Euler-Lagrange type quadratic mapping* $Q : X \to Y$ *which satisfies the following inequality*

$$\left\| f(x) - Q(x) \right\| \le \frac{\varepsilon_1}{\lambda^r - \lambda^2} \left\| x \right\|^r \tag{3.12}$$

for $0 < \lambda < 1$, $r = p + q < 2, or\ \lambda > 1, r > 2, \quad or\ \lambda^{2-r} < 1$, *and* $x \in X$, *such that the general inequality*

$$\left\| f(x) - \lambda^{2n} f(\lambda^{-n} x) \right\| \le \frac{\varepsilon_1}{\lambda^r - \lambda^2} [1 - \lambda^{(2-r)n}] \left\| x \right\|^r \tag{3.13}$$

holds for $0 < \lambda < 1$, $r = p + q < 2, or\ \lambda > 1, r > 2, \quad or\ \lambda^{2-r} < 1$ *and all* $n \in \mathbb{N}$ *and* $x \in X$ *as well as Q satisfies the formula*

$$Q(x) = \lim_{n \to \infty} \lambda^{2n} f(\lambda^{-n} x) \tag{3.14}$$

for $0 < \lambda < 1$, $r = p + q < 2, or\ \lambda > 1, r > 2, \quad or\ \lambda^{2-r} < 1$ *and all* $x \in X$.

Proof. We replace $\frac{x}{\lambda}\ on\ x$ through the above proof of Theorem 3.1 to obtain the corresponding proof of Theorem 3.2.

Corollary 3.1. *Assume that a mapping* $f : X \to Y$ *satisfies the Euler-Lagrange type quadratic inequality*

$$\left\| D_f(x, y) \right\| \le \delta$$

for all $x, y \in X$. *If the condition*

$$\left\| \overline{\overline{f(\frac{x}{\lambda})}} - f(\frac{x}{\lambda}) \right\| \le \frac{\delta_1}{\lambda^2}$$

holds for $0 < \lambda < 1$ *and all* $x \in X$, *then there exists a unique Euler-Lagrange type quadratic mapping* $Q : X \to Y$ *which satisfies the following inequality*

$$\left\| f(x) - Q(x) \right\| \le \frac{\varepsilon}{1 - \lambda^2}$$

for $0 < \lambda < 1$ *and all* $x \in X$ *,such that the general inequality*

$$\left\| f(x) - \lambda^{2n} f(\lambda^{-n} x) \right\| \leq \frac{\varepsilon}{1-\lambda^2}(1-\lambda^{2n})$$

holds for $0 < \lambda < 1$ *and all* $n \in \mathbb{N}$ *and* $x \in X$ *as well as Q satisfies the formula*

$$Q(x) = \lim_{n\to\infty} \lambda^{2n} f(\lambda^{-n} x)$$

for $0 < \lambda < 1$ *and all* $x \in X$.

OPEN QUESTIONS.

1. To investigate pertinent Theorems 4.1 and 4.2 with inequalities (3.1) and (3.10) controlled by a general nonnegative phi mapping $\varphi : X^2 \to [0,\infty)$.

   Suggestion: One might employ our above substitutions.
2. To establish stability in restricted domains.
3. To achieve stability in Banach modules.

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On the ALEKSANDROV and Triangle Isometry ULAM Stability Problems

John Michael Rassias[1] **Shuhuang Xiang**[2] **and Matina John Rassias**[3]

[1]Department of Mathematics, National and Capodistrian University of Athens,
Athens, Greece,
Email: jrassias@tellas.gr

[2]Department of Applied Mathematics, Central South University,
Changsha, Hunan 410083
Email: xiangsh@mail.csu.edu.cn

[3]Statistic and Modelling Science, University of Strathclyde,
Glasgow,UK
Email: matina@stams.strath.ac.uk

ABSTRACT

it This paper provides an overall account of the Aleksandrov problem and Rassias problems for mappings which preserve distances, the area of any planar convex hull of any points in $[0,1]^3$*, or preserve the volume of any convex hull of any points in* $[0,1]^3$*, and considers the triangle isometry Ulam stability problem on bounded domains and Ulam-Gavruta-Rassias stability for additive mappings.*

Keywords: isometry, preserving distance, stability.

2000 Mathematics Subject Classification: 39B42, 46B04, 51k05

1 Aleksandrov and Rassias Problems

The theory of isometry had its beginning in the important paper by S. Mazur and S. Ulam (cf. [1]) in 1932. Let X, Y be two metric spaces, d_1, d_2 the distances on X, Y. A mapping $f : X \to Y$ is defined to be an isometry if $d_2(f(x), f(y)) = d_1(x, y)$ for all elements x, y of X. Mazur and Ulam [1] proved that every isometry of a normed real vector space onto a normed real vector space is a linear mapping up to translation.

Given two metric spaces X and Y and a mapping $f : X \to Y$, what do we really need to know about f in order to be sure that f is an isometry?

A mapping $f : X \to Y$ is called preserving the distance r if for all x, y of X with $d_1(x, y) = r$ then $d_2(f(x), f(y)) = r$. A. D. Aleksandrov (cf. [2]) has posed the following problem:

Aleksandrov Problem. *Whether the existence of a single conservative distance for some mapping* f *implies that* f *is an isometry* .

Let X, Y be two normed vector spaces. Consider the following conditions for $f : X \to Y$ introduced for the first time by Rassias and Šemrl [3]: *distance one preserving property* **(DOPP)** and *strongly distance one preserving property* **(SDOPP)**.

(DOPP) *Let* $x, y \in X$ *with* $\|x - y\|_X = 1$*. Then* $\|f(x) - f(y)\|_Y = 1$*.*

(SDOPP) *Let* $x, y \in X$*. Then* $\|f(x) - f(y)\|_Y = 1$ *if and only if* $\|x - y\|_X = 1$*.*

A number of authors have discussed Aleksandrov problem under certain additional conditions for a given mapping satisfying DOPP in order to be an isometry and have posed several interesting and new open problems (cf. [2-12,15-24]). Even if X, Y are normed vector spaces the above problem is not easy to answer. For example, the following question posed by Rassias (cf. [15]) has not been solved yet: *Is a mapping* f *from* $\mathbf{R}^2$ *to* $\mathbf{R}^3$ *preserving unit distance (simply DOPP) necessarily an isometry*?

Aleksandrov problem has been solved for Euclidean spaces $X = Y = \mathbf{R}^n$. If $n = 1$, Beckman and Quarles [4] pointed out that such a mapping f does not need to be an isometry. For example ([4]), let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x + 1, & \text{if } x \text{ is an integer,} \\ x, & \text{otherwise.} \end{cases}$$

f satisfies (DOPP) but f is not an isometry.

If $2 \leq n < \infty$, f must be an isometry due to the theorem of Beckman and Quarles [4], Bishop [5] and in a special case Zvengrovski [8, Appendix to Chapter II] independently:

Theorem 1.1. *Let* f *be a transformation (possibly many-valued) of* $\mathbf{R}^n (2 \leq n < \infty)$ *into itself. Let* $d(p, q)$ *be the distance between points* p *and* q *of* $\mathbf{R}^n$*, and let* $f(p), f(q)$ *be any images of* p *and* q*, respectively. If there is a length* $a > 0$ *such that* $d(f(p), f(q)) = a$ *whenever* $d(p, q) = a$*, then* f *is a Euclidean transformation of* $\mathbf{R}^n$ *onto itself. That is,* f *is a linear isometry up to translation.*

For $X = \mathbf{R}^\infty$ the Hilbert space of infinite sequences of real numbers $x = (x_1, x_2, \cdots)$ satisfying $\sum_{j=1}^{\infty} x_j^2 < \infty$, an example of a unit distance preserving mapping that is not an isometry has been given by Beckman and Quarles [4].

For $f : \mathbf{R}^n \to \mathbf{R}^m$ in case $n \neq m$, up to now, the Aleksandrov problem is only partially solved. Rassias in [16] gave some interesting counterexamples for $f : \mathbf{R}^2 \to \mathbf{R}^8$ and $f' : \mathbf{R}^2 \to \mathbf{R}^6$ that f and f' satisfy (DOPP) but f and f' are not isometries. Generally, Rassias [16] proved the following result:

Theorem 1.2. *For each positive integer* $n \geq 1$*, there exists an integer* $k(n)$ *such that there exists a non-isometric mapping* $f : \mathbf{R}^n \to \mathbf{R}^{k(n)}$ *satisfying (DOPP).*

A geometric interpretation is that for a mapping $f : \mathbf{R}^n \to \mathbf{R}^m$, and for arbitrary three points p_1, p_2, p_3 in $\mathbf{R}^n$ forming an equilateral triangle with unit length, if $f(p_1)$, $f(p_2)$ and $f(p_3)$ also form an equilateral triangle with unit length and $1 < m = n < \infty$,

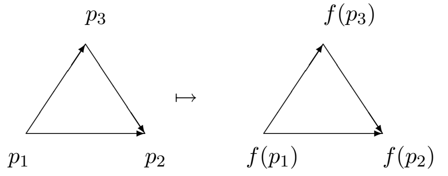

Fig. 1.1

Then f must be a linear isometry up to translation. If $m = n = 1$ or $m = n = \infty$ or $m \neq n$, for arbitrary three points p_1, p_2, p_3 in $\mathbf{R}^n$ forming an equilateral triangle with unit length, if $f(p_1)$, $f(p_2)$ and $f(p_3)$ also form an equilateral triangle with unit length, f may be not an isometry.

What happens if we require, instead of one conservative distance for a mapping between normed vector spaces, two conservative distances? An answer to this question was given by Benz and Berens [6] who proved the following theorem and pointed out that the condition that Y is strictly convex can not be relaxed.

Theorem 1.3. *Let X and Y be real normed vector spaces. Assume that $dimX \geq 2$ and Y is strictly convex. Suppose $f : X \to Y$ satisfies that: f preserves the two distances ρ and $\lambda\rho$ for some integer $\lambda \geq 2$. That is, for all $x, y \in X$ with $\|x - y\| = \rho$, $\|f(x) - f(y)\| \leq \rho$; and for all $x, y \in X$ with $\|x - y\| = \lambda\rho$, $\|f(x) - f(y)\| \geq \lambda\rho$. Then f is a linear isometry up to translation.*

Rassias Problem. *If f preserves two distances with a noninteger ratio, and X and Y are real normed vector spaces such that Y is strictly convex and $dimX \geq 2$, whether or not f must be an isometry (cf. [15]).*

Let X and Y be real Hilbert spaces.

Definition 1.4. Suppose $f : X \to Y$ is a mapping. The distance r is called contractive by f if and only if for all $x, y \in X$ with $\|x - y\| = r$, it follows that $\|f(x) - f(y)\| \leq r$; The distance r is called extensive by f if and only if for all $x, y \in X$ with $\|x - y\| = r$, it follows that $\|f(x) - f(y)\| \geq r$.

It is obvious by the triangle inequality that if the distance r is contractive by f, then the distance nr is contractive by f, $n = 1, 2, \ldots$; and that if the distance r is contractive by f, and the distance nr is extensive by f for some positive integer n, then the distances r and nr are both preserved by f.

Theorem 1.5. *Suppose that $f : X \to Y$ satisfies (DOPP) and the dimension of X is greater than or equal to 3. If one of the distances*

$$m\sqrt{n^2(2 \cdot 4^k - \frac{4^k - 1}{3})4^l - \frac{4^l - 1}{3}}$$

or

$$m\sqrt{n^2(3 \cdot 4^k - \frac{4^k - 1}{3})4^l - \frac{4^l - 1}{3}}$$

is preserved by f, $m, n = 1, 2, \ldots$ and $k, l = 0, 1, 2, \ldots$. Then f must be a linear isometry up to translation.

A geometric interpretation is that for $f : X \to Y$, X and Y being Hilbert spaces, and for arbitrary three points p_1, p_2, p_3 in X constructing an isosceles triangle with $\|p_1 - p_2\| = \sqrt{2}$ (or $\|p_1 - p_2\| = \sqrt{3}$), $\|p_1 - p_3\| = 1 = \|p_2 - p_3\| = 1$, if $f(p_1)$, $f(p_2)$ and $f(p_3)$ also constructs an isosceles triangle with the same size $\|f(p_1) - f(p_2)\| = \sqrt{2}$ (or $\|f(p_1) - f(p_2)\| = \sqrt{3}$), $\|f(p_1) - f(p_3)\| = \|f(p_2) - f(p_3)\| = 1$,

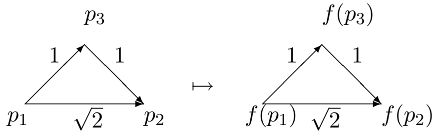

Fig. 1.2

then f must be an isometry.

Corollary 1.6. *Suppose that $f : X \to Y$ satisfies (DOPP) and the dimension of X is greater than or equal to 3. If the distance r is contractive by f and one of the distances*

$$m\sqrt{n^2(2\cdot 4^k - \frac{4^k-1}{3})4^l - \frac{4^l-1}{3}} + jr$$

or

$$m\sqrt{n^2(3\cdot 4^k - \frac{4^k-1}{3})4^l - \frac{4^l-1}{3}} + jr$$

is extensive by f, $m, n = 1, 2, \ldots$ and $k, l, j = 0, 1, 2, \ldots$. Then f must be a linear isometry up to translation.

Remarks 1. By Corollary 1.6, if $f : X \to Y$ satisfies (DOPP) and the dimension of X is infinite and one of the distances

$$m\sqrt{n^2(2\cdot 4^k - \frac{4^k-1}{3})4^l - \frac{4^l-1}{3}} + m_1\sqrt{n_1^2(2\cdot 4^{k_1} - \frac{4^{k_1}-1}{3})4^{l_1} - \frac{4^{l_1}-1}{3}} + j$$

or

$$m\sqrt{n^2(3\cdot 4^k - \frac{4^k-1}{3})4^l - \frac{4^l-1}{3}} + m_1\sqrt{n_1^2(3\cdot 4^{k_1} - \frac{4^{k_1}-1}{3})4^{l_1} - \frac{4^{l_1}-1}{3}} + j$$

is extensive by f, $m, n, = 1, 2, \ldots$, and $k, l, k_1, l_1, m_1, n_1, j = 0, 1, 2, \ldots$. Then f must be a linear isometry up to translation.

If f preserves two distances with a non-integer ratio, and X and Y are real Hilbert spaces with $dimX \geq 2$, the Rassias Problem is also true for the following special case.

Theorem 1.7. *Let X and Y be real Hilbert spaces with the dimension of X greater than one. Suppose that $f : X \to Y$ is a mapping and the distance a is contractive by f, the distance $\frac{\sqrt{7}+1}{2}a$ is extensive by f, where a is a positive number. Then f must be an isometry.*

The above result can be extended to

Corollary 1.8. *Let X and Y be real Hilbert spaces with the dimension of X greater than one. Suppose that $f : X \to Y$ is a mapping and the distance a is contractive by f, the distance*

$\dfrac{1+k\sqrt{4k^2+3}}{k^2+1}a$ *is extensive by* f *for some positive integer* k*, where* a *is a positive number. Then* f *must be an isometry.*

Corollary 1.9. *Let* X *and* Y *be real Hilbert spaces with the dimension of* X *greater than one. Suppose that* $f : X \to Y$ *is a mapping and the distances* a *and* b *are contractive by* f*, the distance* $\dfrac{1+k\sqrt{4k^2+3}}{k^2+1}a+b$ *is extensive by* f*, where* a *and* b *are positive numbers. Then* f *must be an isometry.*

Remark 2. From Corollary 1.9, suppose that $f : X \to Y$ is a mapping and the distance a is contractive by f, and one of the distances $\dfrac{1+k\sqrt{4k^2+3}}{k^2+1}a + \dfrac{1+k_1\sqrt{4k_1^2+3}}{k_1^2+1}a + la$ is extensive by f, where a is a positive number, $k = 1, 2, \ldots$ and $k_1, l = 0, 1, 2, \ldots$. Then f must be an isometry.

For mappings which preserve the area of any planar convex hull of any points in $[0,1]^3$, or preserve the volume of any convex hull of any points in $[0,1]^3$. Mike Th.Rassias [13] posed the following problems:

(PROBLEM 5053 (Mike Th.Rassias [13])) Let $[0,1]^3$ be the unit cube in $\mathbf{R}^3$. Find all transformations

$$T : [0,1]^3 \to [0,1]^3$$

which preserve the area of any planar convex hull of any points in $[0,1]^3$.

(PROBLEM 5054 (Mike Th.Rassias [13])) Let $[0,1]^3$ be the unit cube in $\mathbf{R}^3$. Find all transformations

$$T : [0,1]^3 \to [0,1]^3$$

which preserve the volume of any convex hull of any points in $[0,1]^3$.

Define the mappings from $[0,1]^3$ to $[0,1]^3$ as follows:

$$I(x,y,z) = (x,y,z), I_x(x,y,z) = (1-x,y,z), I_y(x,y,z) = (x,1-y,z),$$

$$I_z(x,y,z) = (x,y,1-z), I_{xy}(x,y,z) = (1-x,1-y,z), I_{yz} = (x,1-y,1-z),$$

and

$$I_{zx} = (1-x,y,1-z), I_{xyz} = (1-x,1-y,1-z).$$

It is easy to verify that each one of the above mapping preserves the area of any convex planar subset in $[0,1]^3$ as well as the volume of any closed subset in $[0,1]^3$, and that its inverse is the mapping itself.

Theorem 1.10. *Let* $[0,1]^3$ *be the unit cube in* $\mathbf{R}^3$ *and* $T : [0,1]^3 \to [0,1]^3$ *preserve the area of any planar convex hull of any points in* $[0,1]^3$*. Then* T *must be a composition of the rotation transformation with some of the following mappings:* I*,* I_x*,* I_y*,* I_z*,* I_{xy}*,* I_{yz}*,* I_{zx}*,* I_{xyz}*, where the rotation transformation is rotation about the central point* $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ *and the rotation angle is* $k\pi/2$ *for some integer* k*.*

Theorem 1.11. *Let* $[0,1]^3$ *be the unit cube in* $\mathbf{R}^3$ *and* $T : [0,1]^3 \to [0,1]^3$ *which preserves the volume of any convex hull of any points in* $[0,1]^3$*. Then* T *must be a composition of the rotation* *transformation with some of the following mappings:* I, I_x, I_y, I_z, I_{xy}, I_{yz}, I_{zx}, I_{xyz}, *where the rotation transformation is rotation about the central point* $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ *and the rotation angle is* $k\pi/2$ *for some integer* k.

2 Triangle perimeter isometry stability on bounded domains

In 1940 S. M. Ulam gave a talk before the mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms:

Ulam Problem. *We are given a group* G *and a metric group* G' *with metric* $\rho(\cdot,\cdot)$. *Given* $\varepsilon > 0$, *does there exist a* $\delta > 0$ *such that if* $f : G \to G'$ *satisfies* $\rho(f(xy), f(x)f(y)) < \delta$ *for all* x, y *in* G, *then a homomorphism* $h : G \to G'$ *exists with* $\rho(f(x), h(x)) < \varepsilon$ *for all* x *in* G?

Let X and Y be real Banach spaces. A mapping $I : X \to Y$ introduced by John M. Rassias, is called a triangle perimeter isometry if I satisfies the triangle perimeter identity

$$\|I(x) - I(y)\| + \|I(x)\| + \|I(y)\| = \|x - y\| + \|x\| + \|y\| \qquad (*)$$

for all $x, y \in X$.

In this section, we establish isometry stability results pertinent to the famous Ulam stability problem and the triangle perimeter mapping $T : X \to Y$,

$$T_i(x, y) = T(x, y) = \|x - y\| + \|x\| + \|y\|$$

with respect to a triangle ABC of vertices $A(0)$, $B(x)$, $C(y)$, and the corresponding mapping

$$T_f(x, y) = T(f(x), f(y)) = \|f(x) - f(y)\| + \|f(x)\| + \|f(y)\|$$

as well as the difference operator D_f such that

$$D_f(x, y) = T_f(x, y) - T_i(x, y) = \|f(x) - f(y)\| + \|f(x)\| + \|f(y)\| - [\|x - y\| + \|x\| + \|y\|]$$

in the ball $B = \{x \in X : \|x\| \le r\}(0 < r \le 1)$ of a real Hilbert space X associated with an inner product $< .,. >$, where the norm $\|x\|$ is given by the formula $\|x\|^2 =< x, x >$.

Theorem 2.1. *If a mapping* $f : X \to X$ *satisfies the following triangle perimeter inequality*

$$|D_f(x, y)| \le \vartheta \|x - y\|^p \qquad (2.1)$$

for all $x, y \in B(\subseteq X)$ *and some* $0 \le \vartheta$, *and* $p > 1$, *then there exists a unique linear triangle perimeter isometry* $I : X \to X$ *such that the following inequality*

$$\|f(x) - I(x)\| \le \theta \frac{2^{(p-1)/2}}{2^{(p-1)/2} - 1} \|x\|^{(1+p)/2} \qquad (2.2)$$

holds for all $x \in B$, *where*

$$\theta = \sqrt{\vartheta}\sqrt{1 + \frac{1}{2}\vartheta r^{p-1}} (\le \frac{\sqrt{6}}{2}\sqrt{\vartheta} < 1.5\sqrt{\vartheta} \text{ if } 0 \le \vartheta \le 1), \qquad (2.3)$$

and $I(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$ *for all* $x \in B$.

Proof. Let $x, y \in B$. Substituting $x = y = 0$ in (2.1), one obtains $f(0) = 0$. Putting $y = x$ in (2.1), we get $\|f(x)\| = \|x\|$ for all $x \in B$. Therefore replacing $y = \frac{1}{2}x$ in (2.1) and then applying the triangle inequality, yields the following inequality

$$\|f(x) - f(\frac{1}{2}x)\| \leq \frac{1}{2}\|x\| + 2^{-p}\vartheta\|x\|^p \tag{2.4}$$

for all $x \in B$. Thus

$$(\frac{1}{2}\|x\| + 2^{-p}\vartheta\|x\|^p)^2 \geq \|f(x) - f(\frac{1}{2}x)\|^2 = \|f(x)\|^2 + \|f(\frac{1}{2}x)\|^2 - 2 < f(x), f(x) > \tag{2.5}$$

for all $x \in B$. Employing $\|f(x)\| = \|x\|$ and (2.5) and the following quadratic identity

$$\frac{1}{2}\|f(x) - 2f(\frac{1}{2}x)\|^2 = -\frac{1}{2}\|f(x)\|^2 + \|f(\frac{1}{2}x)\|^2 + \|f(x) - f(\frac{1}{2}x)\|^2 \tag{2.6}$$

we get

$$\begin{aligned} & \tfrac{1}{2}\|f(x) - 2f(\tfrac{1}{2}x)\|^2 = -\tfrac{1}{4}\|f(x)\|^2 + \|f(\tfrac{1}{2}x)\|^2 \\ \leq \quad & -\tfrac{1}{4}\|f(x)\|^2 + (\tfrac{1}{2}\|x\| + 2^{-p}\vartheta\|x\|^p)^2 = 2^{-p}\vartheta\|x\|^{1+p} + 2^{-2p}\vartheta^2\|x\|^{2p} \end{aligned}$$

or

$$\begin{aligned} & \|f(x) - 2f(\tfrac{1}{2}x)\|^2 \\ \leq \quad & 2^{1-p}\vartheta\|x\|^{1+p} + 2^{1-2p}\vartheta^2\|x\|^{2p} \leq \vartheta(1 + \tfrac{1}{2}\vartheta r^{p-1})\|x\|^{1+p} \end{aligned}$$

or the fundamental inequality

$$\|f(x) - 2f(\frac{1}{2}x)\| \leq \theta\|x\|^{\frac{1+p}{2}} \tag{2.7}$$

for all $x \in B$, where θ is given by the afore-mentioned relation (2.3). Therefore, by (or without) induction on n, we obtain the general inequality

$$\|f(x) - 2^n f(2^{-n}x)\| \leq \theta \left[\sum_{j=0}^{n-1} 2^{\frac{j(1-p)}{2}}\right] \|x\|^{(1+p)/2} = \theta \frac{1 - 2^{\frac{n(1-p)}{2}}}{1 - 2^{\frac{1-p}{2}}} \tag{2.8}$$

on every natural number n and all $x \in B$ and some $0 \leq \vartheta$, and $p > 1$.
From (2.8) it is obvious that the sequence $\{I_n(x)\}$, with $I_n(x) = 2^n f(2^{-n}x)$, is a Cauchy sequence, because X is a complete space and $p > 1$. Therefore the limit

$$I(x) = \lim_{n\to\infty} I_n(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$$

exists and satisfies $(*)$ for all $x \in B$, yielding the existence of a triangle perimeter isometry $I : X \to X$. The proof for the linearity and uniqueness of this mapping $I : X \to X$ follows standard techniques.

3 Ulam-Gavruta-Rassias stability

In 1941 D. H. Hyers considered the case of approximately additive mappings $f : E \to E'$ where E and E' are Banach spaces and f satisfies Hyers inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$$

for all x, y in $E.$ It was shown that the limit

$$L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \varepsilon.$$

Two generalizations, Hyers-Ulam- Rassias stability and Ulam-Gavruta-Rassias stability, of this result were proved. Hyers-Ulam- Rassias stability [14] refers to the case that the initially considered inequalities are controlled by

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \quad \forall x, y \in E$$

And Ulam-Gavruta-Rassias [9,10,11,12] stability refers to the case that the initially considered inequalities are controlled by

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p \|y\|^q), \quad \forall x, y \in E.$$

In 1982, J. M. Rassias [9,10,11,12] provided a generalization of Hyers' stability Theorem which allows the Cauchy difference to be unbounded, as follows:

Theorem 3.1 ([9,10,12]). *Let $f : E \to E'$ be a mapping from normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \|x\|^p \|y\|^p \tag{3.1}$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $0 \leq p < 1/2$. Then the limit

$$L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\varepsilon}{2 - 2^{2p}} \|x\|^{2p} \tag{3.2}$$

for all $x \in E.$ If $p < 0$ then inequality (3.1) holds for $x, y \neq 0$ and (3.2) for $x \neq 0.$ If $p > 1/2$ then inequality (3.1) holds for all $x, y \in E$ and the limit

$$A(x) = \lim_{n\to\infty} 2^n f(2^{-n} x)$$

exists for all $x \in E$ and $A : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{\varepsilon}{2^{2p} - 2} \|x\|^{2p} \tag{3.3}$$

for all $x \in E.$ If in addition $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbf{R}$ for each fixed $x \in X$, then L and A are $\mathbf{R}$-linear mappings.

Theorem 3.2 ([11]). *Let X be a real normed linear space and let Y be a real complete normed linear space. Assume in addition that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbf{R}$ and f satisfies inequality such that $r = p+q \neq 1$ and f satisfies inequality*

$$\|f(x+y) - [f(x) + f(y)]\| \leq \theta \|x\|^p \|y\|^q$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2 - 2^r|}\|x\|^r \tag{3.4}$$

for all $x \in E$. If in addition $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbf{R}$ for each fixed $x \in X$, then L is $\mathbf{R}$-linear mapping.

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On The ULAM–Gavruta–Rassias Stability Of The Orthogonally Euler-Lagrange Type Functional Equation

K. Ravi[1] and **M. Arunkumar**[2]

[1] Department of Mathematics, Sacred Heart College,
Tirupattur-635 601, TamilNadu, India.
[1] shckravi@yahoo.co.in

[2] Department of Mathematics, Sacred Heart College,
Tirupattur-635 601, TamilNadu, India.
[2] annarun2002@yahoo.co.in

ABSTRACT

In 1940 S. M. Ulam proposed at the Mathematics club of the University of Wisconsin the problem: "Given conditions in order for a linear mappings near an approximately linear mappings to exists." In 1941 D. H. Hyers solved the Ulam problem for linear mappings. In 1951 D. G. Bourgin solved the Ulam problem for additive mappings. In 1982-2004 J. M. Rassias, M. J. Rassias established the Hyers-Ulam stability for the Ulam problem for different mappings. In 1992-2000, J. M. Rassias investigated the Ulam stability for Euler-Lagrange mappings. In 2005 J. M. Rassias solved the Ulam problem for Euler-Lagrange type quadratic functional equations.On the other hand, the orthogonal Cauchy functional equation with an abstract orthogonality relation, was first investigated by S. Gudder and D. Strawther. J. Rätz introduced a new definition of orthogonality by using more restrictive axioms than of S. Gudder and D. Strawther. Moreover, he investigated the structure of orthogonally additive mappings. J. Rätz and Gy. Szabò investigated the problem in a rather more general framework. The orthogonally standard quadratic equation was first investigated by F. Vajzovic when X in a Hilbert space, Y is the scalar field. Many more mathematicians like H. Drljevic, M. Fochi, M. S. Moslehian, Gy. Szabò generalized this result. In this paper the authors wish is to prove the Ulam-Gavruta-Rassias stability for the orthogonally Euler-Lagrange type functional equation.

Key Words: *Hyers – Ulam stability, Ulam- Gavruta- Rassias stability, Orthogonally Euler-Lagrange functional equation, Orthogonality space, Quadratic mapping.*

Mathematics Subject Classification: 39B55, 39B52, 39B82, 46H25

1.INTRODUCTION

In 1940 S. M. Ulam [49] gave a wide ranging talk before the Mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality

$d(h(xy),\, h(x)\, h(y)) < \delta$ *for all* $x, y \in G_1$, *then there exists a homomorphism* $H : G_1 \to G_2$ *with* $d(h(x),\, H(x)) < \varepsilon$ *for all* $x \in G_1$?

The case of approximately additive functions was solved by D. H. Hyers [18] under the assumption that G_1 and G_2 are Banach spaces. In 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [39]. In 1982 and in 1987, L. Fenyö [9,10] established the stability of the Ulam problem for quadratic and other mappings. In 1987 Gadja and Ger [13] showed that one can get analogous stability results for sub additive multi functions.

In 1982-2004, J. M. Rassias [26-39] solved the Ulam problem for different mappings. In 1992, J. M. Rassias introduced the Euler-Lagrange quadratic mappings using the algebraic identities the of form

$$\left|a_1x_1 + a_2x_2\right|^2 + \left|a_1x_1 - a_2x_2\right|^2 = \left(a_1^2 + a_2^2\right)\left[\left|x_1\right|^2 + \left|x_2\right|\right]. \quad \textbf{(1.1)}$$

and

$$m_1m_2\left|a_1x_1 + a_2x_2\right|^2 + \left|m_2a_1x_1 - m_1a_2x_2\right|^2 = \left(m_1\left|a_1\right|^2 + m_2\left|a_2\right|^2\right)\left(m_1\left|x_1\right|^2 + m_2\left|x_2\right|^2\right) \quad \textbf{(1.2)}$$

and investigated the relative functional equations, one can refer to [34,38]. J. M. Rassias [38] solved Euler- Lagrange type quadratic functional equation of the form

$$Q\left(m_1a_1x_1 + m_2a_2x_2\right) + m_1m_2Q\left(a_1x_1 - a_2x_2\right) = \left(m_1a_1^2 + m_2a_2^2\right)\left(m_1 Q\left(x_1\right) + m_2 Q\left(x_2\right)\right) \quad \textbf{(1.3)}$$

and discussed its Ulam stability problem.

In 1992-2000 [29-36] J. M. Rassias introduced many Euler- Lagrange type quadratic mappings analogous to quadratic mappings and solved the Ulam stability problem. The stability of different functional equations were obtained by numerous authors (see, for instance, [4,5,14,26-40,45,46]).

F. Skof [46] was the first author to solve the Ulam problem for additive mappings on a restricted domain and he investigated the stability of the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad \textbf{(1.4)}$$

by considering f from a normed space X into a Banach space Y. Quadratic functional equations were used to characterize inner product spaces [3,20]. A square norm on an inner product space satisfies the important parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

The orthogonal Cauchy functional equation

$$f\left(x+y\right) = f\left(x\right) + f\left(y\right),\ x \perp y \quad \textbf{(1.5)}$$

in which $\perp$ is an abstract orthogonally was first investigated by S. Gudder and D. Strawther [17]. R. Ger and J. Sikorska discussed the orthogonal stability of the equation (1.5) in [16]. The orthogonally quadratic functional equation (1.4) was first investigated by F. Vajzovic [50] when X in a Hilbert space, Y is the scalar field, f in continuous and $\perp$ means the Hilbert space orthogonality. This result was then generalized by H. Drljevic [8], M. Fochi [11], Gy. Szabo [48].

We now introduce the concepts of orthogonality, orthogonality space and then proceed to prove results for the orthogonally Euler – Lagrange functional equation.

DEFINITION 1.1. A vector space X is called an *orthogonally vector space* if there is a relation $x \perp y$ on X such that

(i) $x \perp 0, 0 \perp x$ for all $x \in X$;

(ii) if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;

(iii) $x \perp y, \text{then } ax \perp by$ for all $a, b \in \mathbb{R}$;

(iv) if P is an two-dimensional subspace of X ; then

(a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;

(b) there exists vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.

Any vector space can be made into an orthogonally vector space if we define $x \perp 0, 0 \perp x$ for all x and for non zero vector x, y define $x \perp y$ iff x, y are linearly independent. The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. The pair $(X, \perp)$ is called an *orthogonality space*. It becomes *orthogonality normed space* when the orthogonality space equipped with a norm.

In this paper, we introduce a new type of Euler- Lagrange functional equation of the form

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+4f(x)-2f(y) \qquad \textbf{(1.6)}$$

and study the Hyers-Ulam Stability in the concept of orthogonality and Ulam -Gavruta – Rassias stabilty. Note that the functional equation (1.6) is equivalent to the standard Euler – Lagrange equation (1.4).

A mapping $f : X \to Y$ is called *orthogonal quadratic* if it satisfies the quadratic functional equation (1.6) for all $x, y \in X$ with $x \perp y$ where X be an orthogonality space and Y be a real Banach space.

2. STABILITY OF THE ORTHOGONALLY EULER- LAGRANGE TYPE FUNCTIONAL EQUATION (1.6)

In this section, let $(E,\perp)$ denotes an orthogonality normed space with norm $\|\cdot\|_E$ and $(F,\|\cdot\|_F)$ is a Banach space.

THEOREM 2.1. Let θ and $p(p<2)$ be nonnegative real numbers. Let $f:E\to F$ be a mapping fulfilling

$$\begin{aligned}&\left\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-4f(x)+2f(y)\right\|_F\\&\qquad\le\theta\left\{\|x\|_E^p+\|y\|_E^p\right\}\end{aligned}\tag{2.1}$$

for all $x,y\in E$ with $x\perp y$. Then there exists a unique orthogonally quadratic mapping $Q:E\to F$ such that

$$\left\|f(x)-Q(x)\right\|_F\le\frac{\theta}{8-2^{p+1}}\|x\|_E^p\tag{2.2}$$

for all $x\in E$.

Proof. Letting $x=y=0$ in (2.1), we get $f(0)=0$. Setting $y=0$ in (2.1), we obtain

$$\left\|2f(2x)-8f(x)\right\|_F\le\theta\left\{\|x\|_E^p\right\}\tag{2.3}$$

for all $x\in E$. Since $x\perp 0$, we have

$$\left\|\frac{f(2x)}{4}-f(x)\right\|_F\le\frac{\theta}{8}\|x\|_E^p\tag{2.4}$$

for all $x\in E$. Now replacing x by $2x$ and divided by 4 in (2.4) and summing the resulting inequality with (2.4), we arrive

$$\left\|\frac{f(2^2x)}{4^2}-f(x)\right\|_F\le\frac{\theta}{8}\left\{1+\frac{2^p}{4}\right\}\|x\|_E^p\tag{2.5}$$

for all $x\in E$. Using the induction on n, we obtain that

$$\left\|\frac{f(2^nx)}{2^{2n}}-f(x)\right\|_F\le\frac{\theta}{8}\sum_{k=0}^{n-1}\frac{2^{pk}}{2^{2k}}\|x\|_E^p\le\frac{\theta}{8}\sum_{k=0}^{\infty}\frac{2^{pk}}{2^{2k}}\|x\|_E^p\tag{2.6}$$

for all $x\in E$. In order to prove the convergence of the sequence $\left\{f(2^nx)/2^{2n}\right\}$ replace x by 2^mx and divide by 2^{2m} in (2.6), for any $n,m>0$ we obtain,

$$\left\|\frac{f(2^n 2^m x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}\right\|_F = \frac{1}{2^{2m}}\left\|\frac{f(2^n 2^m x)}{2^{2n}} - f(2^m x)\right\|_F$$
$$\leq \frac{1}{2^{2m}}\frac{\theta}{8}\sum_{k=0}^{n-1}\frac{2^{pk}}{2^{2k}}\left\|2^m x\right\|_E^p$$
$$\leq \frac{\theta}{8}\sum_{k=0}^{\infty}\frac{2^{p(k+m)}}{2^{2(k+m)}}\left\|x\right\|_E^p$$
$$\leq \frac{\theta}{8}\sum_{k=0}^{\infty}\frac{1}{2^{(2-p)(k+m)}}\left\|x\right\|_E^p. \quad \textbf{(2.7)}$$

As $p<2$, the R.H.S of (2.7) tends to 0 as $m\to\infty$ for all $x\in E$. Thus $\left\{f(2^n x)/2^{2n}\right\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q:E\to F$ such that

$$Q(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^{2n}},\ \forall x\in E.$$

Letting $n\to\infty$ in (2.6), we arrive at the formula (2.2) for all $x\in E$. To prove Q satisfies (1.6), replace (x,y) by $(2^n x, 2^n y)$ in (2.1) and divide by 2^{2n} then it follows that

$$\frac{1}{2^{2n}}\left\|f(2^n(2x+y))+f(2^n(2x-y))-2f(2^n(x+y))-2f(2^n(x-y))-4f(2^n x)-2f(2^n y)\right\|_F$$
$$\leq \frac{\theta}{2^{2n}}\left\{\left\|2^n x\right\|_E^p+\left\|2^n y\right\|_E^p\right\}.$$

Taking limit as $n\to\infty$ in the above inequality, we get

$$\left\|Q(2x+y)+Q(2x-y)-2Q(x+y)-2Q(x-y)-4Q(x)-2Q(y)\right\|_F\leq 0$$

which gives

$$Q(2x+y)+Q(2x-y)=2Q(x+y)+2Q(x-y)+4Q(x)-2Q(y)$$

for all $x,y\in E$ with $x\perp y$. Therefore $Q:E\to F$ is an orthogonally quadratic mapping which satisfies (1.6).

To prove the uniqueness: Let Q' be another quadratic function satisfying (1.6) and the inequality (2.2). Then

$$\left\|Q(x)-Q'(x)\right\|_F = \frac{1}{2^{2n}}\left\|Q(2^n x)-Q'(2^n x)\right\|_F$$
$$\leq \frac{1}{2^{2n}}\left(\left\|Q(2^n x)-f(2^n x)\right\|_F+\left\|f(2^n x)-Q'(2^n x)\right\|_F\right)$$
$$\leq \frac{1}{2^{2n}}\frac{2\theta}{8-2^{p+1}}\left\|2^n x\right\|_E^p$$
$$\leq \frac{\theta}{4-2^p}\frac{2^{pn}}{2^{2n}}\left\|x\right\|_E^p$$
$$\to 0 \text{ as } n\to\infty$$

for all $x \in E$. Therefore Q is unique. This completes the proof of the theorem.

THEOREM 2.2. Let θ and $p\left(p > 2\right)$ be nonnegative real numbers. Let $f: E \to F$ be a mapping fulfilling

$$\left\| f\left(2x+y\right)+f\left(2x-y\right)-2f\left(x+y\right)-2f\left(x-y\right)-4f\left(x\right)+2f\left(y\right) \right\|_F \leq \theta\left\{\|x\|_E^p+\|y\|_E^p\right\} \quad \textbf{(2.8)}$$

for all $x, y \in E$ with $x \perp y$. Then there exists a unique orthogonally quadratic mapping $Q: E \to F$ such that

$$\left\| f\left(x\right)-Q\left(x\right) \right\|_F \leq \frac{\theta}{2^{p+1}-8}\|x\|_E^p \quad \textbf{(2.9)}$$

for all $x \in E$.

Proof. Replacing x by $\frac{x}{2}$ in (2.3), we get

$$\left\| f\left(x\right)-4f\left(\frac{x}{2}\right) \right\|_F \leq \frac{\theta}{2^{p+1}}\|x\|_E^p \quad \textbf{(2.10)}$$

for all $x \in E$. Now replacing x by $\frac{x}{2}$ and multiply by 4 in (2.10), we arrive

$$\left\| 4f\left(\frac{x}{2}\right)-4^2 f\left(\frac{x}{2^2}\right) \right\|_F \leq \frac{4\theta}{2^{p+1}}\left\|\frac{x}{2}\right\|_E^p. \quad \textbf{(2.11)}$$

From (2.10) and (2.11), we obtain

$$\left\| f\left(x\right)-4^2 f\left(\frac{x}{2^2}\right) \right\|_F \leq \left\| f\left(x\right)-4f\left(\frac{x}{2}\right) \right\|_F+\left\| 4f\left(x\right)-4^2 f\left(\frac{x}{2^2}\right) \right\|_F$$
$$\leq \frac{\theta}{2^{p+1}}\|x\|_E^p+\frac{4\theta}{2^{p+1}}\left\|\frac{x}{2}\right\|_E^p.$$

Hence , we get

$$\left\| f\left(x\right)-\left(2^2\right)^2 f\left(\frac{x}{2^2}\right) \right\|_F \leq \frac{\theta}{2^{p+1}}\left\{1+\frac{4}{2^p}\right\}\|x\|_E^p \quad \textbf{(2.12)}$$

for all $x \in E$. Using the induction on n, we obtain that

$$\left\| f\left(x\right)-2^{2n} f\left(\frac{x}{2^n}\right) \right\|_F \leq \frac{\theta}{2^{p+1}}\sum_{k=0}^{n-1}\frac{2^{2k}}{2^{pk}}\|x\|_E^p \leq \frac{\theta}{2^{p+1}}\sum_{k=0}^{\infty}\frac{2^{2k}}{2^{pk}}\|x\|_E^p \quad \textbf{(2.13)}$$

for all $x \in E$. In order to prove the convergence of the sequence $\left\{2^{2n} f\left(\frac{x}{2^n}\right)\right\}$, replace x by $\frac{x}{2^m}$ and multiply by 2^{2m} in (2.13), for any $n, m > 0$ we obtain,

$$\left\| 2^{2(n+m)} f\left(\frac{x}{2^{2m+2n}}\right) - 2^{2m} f\left(\frac{x}{2^{2m}}\right)\right\|_F = 2^{2m} \left\| 2^{2n} f\left(\frac{x}{2^{2m+2n}}\right) - f\left(\frac{x}{2^{2m}}\right)\right\|_F$$

$$\leq 2^{2m} \frac{\theta}{2^{p+1}} \sum_{k=0}^{n-1} \frac{2^{2k}}{2^{pk}} \left\|\frac{x}{2^{2m}}\right\|_E^p$$

$$\leq \frac{\theta}{2^{p+1}} \sum_{k=n}^{\infty} \frac{2^{2k}}{2^{(p-2)m}} \|x\|_E^p \ . \qquad \textbf{(2.14)}$$

As $p > 2$, the R.H.S of (2.14) tends to 0 as $m \to \infty$ for all $x \in E$. Thus $\left\{2^{2n} f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q : E \to F$ such that

$$Q(x) = \lim_{n\to\infty} 2^{2n} f\left(\frac{x}{2^n}\right), \ \forall x \in E.$$

Letting $n \to \infty$ in (2.13), we arrive (2.9). To show that Q is unique and it satisfies equation (1.6), the proof is similar to that of Theorem 2.1.

3. STABILITY OF THE ORTHOGONALLY EULER- LAGRANGE TYPE FUNCTIONAL EQUATION (1.6) INVOLVING PRODUCT OF POWERS OF NORMS

We state the following Theorem 3.1, which is due to J. M. Rassias [26], without proof.

THEOREM 3.1. [26] Let $f : E \to E'$ be a mapping from a normed vector space E in to a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \|x\|^p \|y\|^p \qquad \textbf{(*)}$$

for all $x, y \in E$ where ε and p are constants with $\varepsilon > 0$ and $0 \leq p < 1/2$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x, y \in E$, and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\varepsilon}{2 - 2^{2p}} \|x\|^{2p} . \qquad \textbf{(**)}$$

for all $x \in E$ .If $p < 0$ then inequality (*) holds for $x, y \neq 0$ and (**) for $x \neq 0$.

If $p > 1/2$ then inequality (*) holds for all $x, y \in E$ and the limit

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x \in E$ and $A : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{\varepsilon}{2^{2p} - 2} \|x\|^{2p}$$

for all $x \in E$ .If in addition $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $f \in \mathbb{R}$ for each fixed $x \in X$, then L is $\mathbb{R}-$ linear mapping.

In 1982, J. M. Rassias [26] for the first time introduced this Theorem 3.1 involving the product of powers of norms. He proved this theorem has a generalization of Hyers stability theorem which allows the Cauchy difference to be bounded. We use the above theorem 3.1 to prove our following theorems.

THEOREM 3.2. Let $f : E \to F$ be a mapping from a normed vector space E in to a Banach space F subject to the inequality

$$\left\| f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-4f(x)+2f(y)\right\|_F \le \varepsilon \|x\|_E^p \|y\|_E^p \quad \textbf{(3.1)}$$

for all $x, y \in E$, where ε and p are constants with, $\varepsilon > 0$ and $0 \le p < 1$. Then the limit

$$Q(x)=\lim_{n\to\infty} \frac{f(3^n x)}{3^{2n}}$$

exists for all $x \in E$ and $Q : E \to F$ is the unique quadratic mapping such that

$$\left\| f(x)- Q(x)\right\|_F \le \frac{\varepsilon}{9-3^{2p}} \|x\|_E^{2p} \quad \textbf{(3.2)}$$

for all $x \in E$.

Proof. Letting $y = x$ in (3.1), we get

$$\left\| f(3x)-9f(x)\right\|_F \le \varepsilon \left\{\|x\|_E^{2p}\right\} \quad \textbf{(3.3)}$$

for all $x \in E$ which gives

$$\left\| \frac{f(3x)}{9} - f(x)\right\|_F \le \frac{\varepsilon}{9} \|x\|_E^{2p} \quad \textbf{(3.4)}$$

for all $x \in E$. Now replacing x by $3x$ and divided by 9 in (3.4) and summing the resulting inequality with (3.4), we arrive

$$\left\| \frac{f(3^2 x)}{9^2} - f(x)\right\|_F \le \frac{\varepsilon}{9} \left\{1+\frac{3^{2p}}{9}\right\} \|x\|_E^{2p} \quad \textbf{(3.5)}$$

for all $x \in E$. Using the induction on $n,$ we obtain that

$$\left\| \frac{f(3^n x)}{3^{2n}} - f(x)\right\|_F \le \frac{\varepsilon}{9} \sum_{k=0}^{n-1} \frac{3^{2pk}}{3^{2k}} \|x\|_E^{2p} \le \frac{\varepsilon}{9} \sum_{k=0}^{\infty} \frac{3^{2pk}}{3^{2k}} \|x\|_E^{2p} \quad \textbf{(3.6)}$$

for all $x \in E$. In order to prove the convergence of the sequence $\left\{ f\left(3^n x\right)/3^{2n} \right\}$ replace x by $3^m x$ and divide by 3^{2m} in (3.6), for any $n, m > 0$ we obtain,

$$\begin{aligned} \left\| \frac{f\left(3^n 3^m x\right)}{3^{2(n+m)}} - \frac{f\left(3^m x\right)}{3^{2m}} \right\|_F &= \frac{1}{3^{2m}} \left\| \frac{f\left(3^n 3^m x\right)}{3^{2n}} - f\left(3^m x\right) \right\|_F \\ &\leq \frac{1}{3^{2m}} \frac{\varepsilon}{9} \sum_{k=0}^{n-1} \frac{3^{2pk}}{3^{2k}} \left\| 3^m x \right\|_E^{2p} \\ &\leq \frac{\varepsilon}{9} \sum_{k=0}^{\infty} \frac{3^{2p(k+m)}}{3^{2(k+m)}} \left\| x \right\|_E^{2p} \\ &\leq \frac{\varepsilon}{9} \sum_{k=0}^{\infty} \frac{1}{3^{2(1-p)(k+m)}} \left\| x \right\|_E^{2p}. \end{aligned} \tag{3.7}$$

As $p < 1$, the R.H.S of (3.7) tends to 0 as $m \to \infty$ for all $x \in E$. Thus $\left\{ f\left(3^n x\right)/3^{2n} \right\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q : E \to F$ such that

$$Q(x) = \lim_{n \to \infty} \frac{f\left(3^n x\right)}{3^{2n}}, \ \forall x \in E.$$

Letting $n \to \infty$ in (3.6), we arrive at the formula (3.2) for all $x \in E$. To show that Q is unique and it satisfies equation (1.6), the proof is similar to that of Theorem 2.1.

THEOREM 3.3. Let $f : E \to F$ be a mapping from a normed vector space E in to a Banach space F subject to the inequality

$$\left\| f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y) \right\|_F \leq \varepsilon \left\| x \right\|_E^p \left\| y \right\|_E^p \tag{3.8}$$

for all $x, y \in E$, where ε and p are constants with, $\varepsilon > 0$ and $p > 1$. Then the limit

$$Q(x) = \lim_{n \to \infty} 3^{2n} f\left(\frac{x}{3^n}\right)$$

exists for all $x \in E$ and $Q : E \to F$ is the unique quadratic mapping such that

$$\left\| f(x) - Q(x) \right\|_F \leq \frac{\varepsilon}{3^{2p} - 9} \left\| x \right\|_E^{2p} \tag{3.9}$$

for all $x \in E$.

Proof. Replacing x by $\frac{x}{3}$ in (3.3), we get

$$\left\| f(x) - 9f\left(\frac{x}{3}\right) \right\|_F \leq \frac{\varepsilon}{3^{2p}} \left\| x \right\|_E^{2p} \tag{3.10}$$

for all $x \in E$. Now replacing x by $\frac{x}{3}$ and multiply by 9 in (3.10) and summing the resultant inequality with (3.10) we arrive

$$\left\| f(x)-\left(3^2\right)^2 f\left(\frac{x}{3^2}\right)\right\|_F \leq \frac{\varepsilon}{3^{2p}}\left\{1+\frac{9}{3^{2p}}\right\}\|x\|_E^{2p} \tag{3.11}$$

for all $x \in E$. Using the induction on $n,$ we obtain that

$$\left\| f(x)-3^{2n} f\left(\frac{x}{3^n}\right)\right\|_F \leq \frac{\varepsilon}{3^{2p}}\sum_{k=0}^{n-1}\frac{3^{2k}}{3^{2pk}}\|x\|_E^{2p} \leq \frac{\varepsilon}{3^{2p}}\sum_{k=0}^{\infty}\frac{3^{2k}}{3^{2pk}}\|x\|_E^{2p} \tag{3.12}$$

for all $x \in E$. In order to prove the convergence of the sequence $\left\{3^{2n} f\left(\frac{x}{3^n}\right)\right\}$, replace x by $\frac{x}{3^m}$ and multiply by 3^{2m} in (3.12), for any $n,m>0$ we obtain,

$$\begin{aligned}\left\|3^{2(n+m)} f\left(\frac{x}{3^{m+n}}\right)-3^{2m} f\left(\frac{x}{3^m}\right)\right\|_F &= 3^{2m}\left\|3^{2n} f\left(\frac{x}{3^{m+n}}\right)-f\left(\frac{x}{3^m}\right)\right\|_F \\ &\leq 3^{2m}\frac{\varepsilon}{3^{2p}}\sum_{k=0}^{n-1}\frac{3^{2k}}{3^{2pk}}\left\|\frac{x}{3^m}\right\|_E^{2p} \\ &\leq \frac{\varepsilon}{3^{2m(p-1)}3^{2p}}\sum_{k=0}^{\infty}\frac{1}{3^{2k(p-1)}}\|x\|_E^{2p}\end{aligned} \tag{3.13}$$

As $p>1$, the R.H.S of (3.13) tends to 0 as $m\to\infty$ for all $x \in E$. Thus $\left\{3^{2n} f\left(\frac{x}{3^n}\right)\right\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q:E\to F$ such that

$$Q(x)=\lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right), \ \forall x \in E.$$

Letting $n\to\infty$ in (3.12), we arrive (3.9). To show that Q is unique and it satisfies equation (1.6), the proof is similar to that of Theorem 2.1.

THEOREM 3.4. Let E be a real normed linear space and F be a real complete normed linear space. Assume in addition $f:E\to F$ is an approximately quadratic mappings for which there exists a constant $\theta>0$ and $p,q\in\mathbb{R}$ such that $r=p+q\neq 1$ and f satisfies

$$\left\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-4f(x)+2f(y)\right\|_F \leq \theta\|x\|_E^p\|y\|_E^q \tag{3.14}$$

for all $x,y\in E$. Then there exists a unique quadratic mappings $Q:E\to F$ such that

$$\left\|f(x)-Q(x)\right\|_F \leq \frac{\theta}{3^r-9}\|x\|_E^r \tag{3.15}$$

for all $x\in E$. In addition $f:E\to F$ is a mapping such that the transformation $t\to f(tx)$ is continuous.

THEOREM 3.5. Let E be a real normed linear space and F be a real complete normed linear space. Assume in addition $f: E \to F$ is an approximately quadratic mappings for which there exists a constant $\theta > 0$ such that f satisfies

$$\left\| f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y) \right\|_F \le \theta K(x,y) \quad \textbf{(3.16)}$$

for all $(x,y) \in E^2$ and $K: E^2 \to \mathbb{R}^+ \cup \{0\}$ is a non negative real valued function such that

$$R(x) = \sum_{j=0}^{\infty} 3^{-2j} K\left(3^j x, 3^j x\right)(<\infty) \quad \textbf{(3.17)}$$

is a non negative function on x , and the condition

$$\lim_{m\to\infty} 3^{-2m} K\left(3^m x, 3^m x\right) = 0 \quad \textbf{(3.18)}$$

holds . Then there exists a unique quadratic mappings $Q: E \to F$ such that

$$\left\| f(x) - Q(x) \right\|_F \le \frac{\theta}{9} R(x) \quad \textbf{(3.19)}$$

for all $x \in E$. In addition $f: E \to F$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $f \in \mathbb{R}$ for each fixed $x \in E$, then Q is $\mathbb{R}-$ linear mapping.

Proof. Letting $y = x$ in (3.16), we get

$$\left\| f(3x) - 9f(x) \right\|_F \le \theta K(x,x) \quad \textbf{(3.20)}$$

for all $x \in E$ which gives

$$\left\| 3^{-2} f(3x) - f(x) \right\|_F \le 3^{-2} \theta K(x,x) \quad \textbf{(3.21)}$$

for all $x \in E$. Now replacing x by $3x$ and multiply by 3^{-2} in (3.21) and summing the resulting inequality with (3.21), we arrive

$$\left\| 3^{-4} f\left(3^2 x\right) - f(x) \right\|_F \le 3^{-2} \theta \left\{ K(x,x) + 3^{-2} k(3x,3x) \right\} \quad \textbf{(3.22)}$$

for all $x \in E$. Using the induction on $n,$ we obtain that

$$\left\| 3^{-2n} f\left(3^n x\right) - f(x) \right\|_F \le 3^{-2} \theta \sum_{j=0}^{n-1} 3^{-2j} K\left(3^j x, 3^j x\right) \quad \textbf{(3.23)}$$

for all $x \in E$. In order to prove the convergence of the sequence $\left\{ 3^{-2n} f\left(3^n x\right) \right\}$ replace x by $3^m x$ and multiply by 3^{-2m} in (3.23), for any $n,m > 0$ we obtain,

$$\begin{aligned} \left\| 3^{-2(n+m)} f\left(3^n 3^m x\right) - 3^{-2m} f\left(3^m x\right) \right\|_F &= 3^{-2m} \left\| 3^{-2n} f\left(3^n 3^m x\right) - f\left(3^m x\right) \right\|_F \\ &\le 3^{-2m} \theta \sum_{j=0}^{\infty} 3^{-2(j+1)} K\left(3^{j+m} x, 3^{j+m} x\right) \quad \textbf{(3.24)} \\ &\to 0 \quad \text{as} \quad m \to \infty \end{aligned}$$

for all $x \in E$. Thus $\left\{3^{-2n} f\left(3^n x\right)\right\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q: E \to F$ such that

$$Q(x) = \lim_{n \to \infty} 3^{-2n} f\left(3^n x\right), \ \forall x \in E.$$

Letting $n \to \infty$ in (3.23) and using (3.17) we arrive at the formula (3.19) for all $x \in E$.

REMARK: 3.6.

If we take $K(x, y) = K(x, x) = \|x\|_E^p \|x\|_E^q$ such that $-\infty < r = p + q \neq 1$ and respectively, $r > 1$ then we obtain obtain that

$$R(x) = \sum_{j=0}^{\infty} 3^{-2j} K\left(3^j x, 3^j x\right) = \sum_{j=0}^{\infty} 3^{-2j} \left\|3^j x\right\|^r = \frac{3^2 \|x\|^r}{3^2 - 3^r} \quad \text{for } -\infty < r = p + q \neq 1 \quad \textbf{(3.25)}$$

and

$$R(x) = \sum_{j=0}^{\infty} 3^{2j} K\left(3^j x, 3^j x\right) = \sum_{j=0}^{\infty} 3^{2j} \left\|3^j x\right\|^r = \frac{3^2 \|x\|^r}{3^r - 3^2} \quad \text{for r>1.} \quad \textbf{(3.26)}$$

Using (3.25) and (3.26) in (3.19), we get

$$\|f(x) - Q(x)\|_F \leq \frac{\theta \|x\|^r}{\left|3^r - 3^2\right|}.$$

Hence the stability result for Theorem 3.4 is proved.

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Ulam-Găvruţă-Rassias Stability of a Linear Functional Equation

M. Ait SiBaha[1], B. Bouikhalene[2] and E. Elqorachi[3]

[1]University of Ibn Tofail, Faculty of Sciences,
Department of Mathematics, Kenitra, Morocco,
E-mail: mohaait@yahoo.fr

[2]University of Ibn Tofail, Faculty of Sciences,
Department of Mathematics, Kenitra, Morocco,
E-mail: bbouikhalene@yahoo.fr

[3]Laboratory LAMA, Harmonic Analysis and functional equations Team
Department of Mathematics, Faculty of Sciences,
University Ibn Zohr, Agadir, Morocco
E-mail: elqorachi@hotamail.com

ABSTRACT

In this paper, we give the Hyers-Ulam-Rassias stability of the linear functional equation

$$f(x+y+a)+f(x+y+b)=2f(x)+2f(y),\ \ x,y\in E_1,$$

where $f:E_1\longrightarrow E_2$ *and* a,b *are two arbitraries elements of a vector space* E_1 *and* E_2 *is a Banach space.*

Keywords: functional equation, Hyers-Ulam stability.

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1 Introduction

In [15] S. M. Ulam proposed the following stability problem: Under what conditions does there exist an additive mapping near an approximately additive mapping ?
In 1941 D. H. Hyers (see [6]), gave an answer to the Ulam's stability problem . More precisely, he proved the following theorem.

Theorem 1.1*.* Let E_1 be a vector space, let E_2 a Banach space and let δ be a positive number and $f:E_1\longrightarrow E_2$ a function that satisfies the inequality

$$\|f(x+y)-f(x)-f(y)\|\leq\delta$$

for every $x,y\in E_1$. Then there exists a unique additive function $g:E_1\longrightarrow E_2$ such that

$$\|f(x)-g(x)\|\leq\delta$$

for every $x\in E_1$.

Stability problems of various functional equations have been extensively investigated by a number of authors including D. H. Hyers [6,7,8], Th. M. Rassias [14], Z. Gajda [3], P. Găvruta [4,5] and J. M. Rassias [9,10,11].

The modified Hyers-Ulam stability problem with the generalization control function was proved by Găvruta in the following way.

Theorem 1.2. Let E_1 be a vector space, let E_2 a Banach space and let $\varphi : E_1 \times E_1 \longrightarrow [0, +\infty[$ be a function satisfying

$$\psi(x,y) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \varphi(2^k x, 2^k y) < +\infty$$

for all $x, y \in E_1$. If a function $f : E_1 \longrightarrow E_2$ satisfies the functional inequality $\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y), \;\; x, y \in E_1,$
then there exists a unique additive function $T : E_1 \longrightarrow E_2$ which satisfies $\|f(x) - T(x)\| \leq \psi(x,x)$ for all $x \in E_1$.

In [9,10,11,12,13] J. M. Rassias gave generalizations of the Hyers'result in the following way

Theorem 1.3. Let X be a real normed linear space and let Y be a real complete normed linear space. Assume in addition that if $f : X \longrightarrow Y$ is a mapping for which there exist constant $\theta > 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \longrightarrow Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 1|} \|x\|^r$$

for all $x \in X$.

The main purpose of this paper is to generalize the results obtained in [3], [4] and [9] for the linear functional equation

$$f(x+y+a) + f(x+y+b) = 2f(x) + 2f(y), \; x, y \in E_1. \tag{1.1}$$

A particular case of this functional equation is

$$f(x+y+a) = f(x) + f(y), \; x, y \in E_1. \tag{1.2}$$

In a previous papers [1] and [2] we dealt with the stability problem of the functional equation (1.2) and

$$f(x+y+a) + f(x+y+b) = 2f(x)f(y), \; x, y \in E_1. \tag{1.3}$$

When $a = b = 0$, (1.1) reduces to the Cauchy equation

$$f(x+y) = f(x) + f(y), x, y \in E_1. \tag{1.4}$$

If f is a solution of (1.4) it is said to be additive or satisfies the Cauchy equation.

This paper is organized as follows: In the first section after this introduction we establish the general solution of (1.1). In the others sections we prove the stability problem in the spirit of Hyers-Ulam, P. Găvruta and J. M. Rassias for the Cauchy linear functional equation (1.1).

2 Solution of (1.1)

In this section we present the general solution of (1.1)

Theorem 2.1. A function $f : E_1 \longrightarrow E_2$ between vectors spaces satisfies the functional equation (1.1) if and only if $f(x) = \frac{A(x+a)+A(x+b)}{2}$ for all $x \in E_1$, where A is an additive function of E_1.

Proof. If $A : E_1 \longrightarrow E_2$ is an additive function, then it's elementary to verify that $f(x) = \frac{A(x+a)+A(x+b)}{2}$ is a solution of (1.1). Conversely, let f be a solution of (1.1). By setting $x = -a$, $y = -b$ in (1.1) we get that $f(-a) + f(-b) = 0$. Now let $B : E_1 \longrightarrow E_2$ be the new function given by $B(x) =: \frac{f(x-a)+f(x-b)}{2}$. Firstly, we show that B is an addtive function. For all $x, y \in E_1$, we have

$$\begin{aligned} 2(B(x) + B(y)) &= f(x-a) + f(x-b) + f(y-a) + f(y-b) \\ &= [f(x-a) + f(y-b)] + [f(x-b) + f(y-a)] \\ &= \frac{f(x+y-b) + f(x+y-a)}{2} + \frac{f(x+y-b) + f(x+y-a)}{2} \\ &= f(x+y-a) + f(x+y-b) \\ &= 2B(x+y). \end{aligned}$$

Since $f(-a) + f(-b) = 0$ it follows that

$$\begin{aligned} 2(B(x+a) + B(x+b)) &= f(x) + f(x+a-b) + f(x+b-a) + f(x) \\ &= 2f(x) + f(x+a-b) + f(x+b-a) \\ &= 2f(x) + f(x) + 2f(-a) + f(x) + 2f(-b). \end{aligned}$$

Hence $f(x) = \frac{B(x+a)+B(x+b)}{2}$ for all $x \in E_1$. □

3 Hyers-Ulam stability of (1.1)

In this section, we give the Hyers-Ulam stability for the linear Cauchy functional equation (1.1). The results obtained here extend the ones obtained by D. H. Hyers in [6].

Theorem 3.1. Let E_1 be a vector space and E_2 a Banach space. If a function $f : E_1 \longrightarrow E_2$ satisfies the functional inequality

$$\|f(x+y+a) + f(x+y+b) - 2f(x) - 2f(y)\| \leq 2\delta, \tag{3.1}$$

for all $x, y \in E_1$, for some $\delta > 0$, then the limit

$$T(x) = \lim_{n \longrightarrow \infty} \{\frac{1}{2^{2n}} \sum_{m=1}^{2^n} f(2^n x + (2^n - m)a + (m-1)b)\} \tag{3.2}$$

exists for all $x \in E_1$ and $T : E_1 \longrightarrow E_2$ is the unique function such that

$$T(x+y+a) + T(x+y+b) = 2T(x) + 2T(y) \tag{3.3}$$

and

$$\|f(x) - T(x)\| \leq \delta$$

for any $x, y \in E_1$.

Proof. By letting $x = y$ in (3.1), one obtain the inequality

$$\|f(2x+a) + f(2x+b) - 4f(x)\| \leq 2\delta. \tag{3.4}$$

Now, make the inductive assumption

$$H_n(x) = \|f(x) - \frac{1}{2^{2n}} \sum_{m=1}^{2^n} f(2^n x + (2^n - m)a + (m-1)b)\| \leq \delta(1 - \frac{1}{2^n}) \tag{3.5}$$

for some positive integer n. Clearly the inductive assumption is true for the case $n = 1$, since replacing n by 1 in (3.5) would give (3.4). for $n+1$ we get by using the following identities

$$\sum_{m=1}^{2^{n+1}} F(m) = \sum_{p=1}^{2^n} F(2p) + \sum_{p=0}^{2^n - 1} F(2p+1), \tag{3.6}$$

$$\sum_{m=0}^{2^n - 1} F(m) = \sum_{m=1}^{2^n} F(m-1)$$

for any mapping $F : \mathbb{N} \longrightarrow E_2$,

$$\sum_{m=1}^{2^{n+1}} f(2^{n+1}x + (2^{n+1} - m)a + (m-1)b)$$

$$= \sum_{m=1}^{2^n} f(2^{n+1}x + (2^{n+1} - 2m)a + (2m-1)b) + \sum_{m=0}^{2^n - 1} f(2^{n+1}x + (2^{n+1} - 2m - 1)a + 2mb)$$

$$= \sum_{m=1}^{2^n} [f(2^{n+1}x + (2^{n+1} - 2m)a + (2m-1)b) + f(2^{n+1}x + (2^{n+1} - 2m + 1)a + (2m-2)b)].$$

Hence

$$H_{n+1}(x) = \|f(x) - \frac{1}{2^{2n+2}} [\sum_{m=1}^{2^{n+1}} f(2^{n+1}x + (2^{n+1} - m)a + (m-1)b)]\|$$

$$\leq \frac{1}{2^{2n+2}} \|2^{2n+2} f(x) - 4 \sum_{m=1}^{2^n} f(2^n x + (2^n - m)a + (m-1)b)\|$$

$$+ \frac{1}{2^{2n+2}} \sum_{m=1}^{2^n} \|f(2^{n+1}x + (2^{n+1} - 2m)a + (2m-1)b) + f(2^{n+1}x + (2^{n+1} - 2m + 1)a + (2m-2)b)$$

$$-4f(2^n x + (2^n - m)a + (m-1)b)\|.$$

By setting $X_{n,m} = 2^n x + (2^n - m)a + (m-1)b$, we get

$$2X_{n,m} + a = 2^{n+1}x + (2^{n+1} - 2m + 1)a + (2m-2)b$$

and

$$2X_{n,m} + b = 2^{n+1}x + (2^{n+1} - 2m)a + (2m-1)b.$$

Therefore

$$\begin{aligned} H_{n+1}(x) &\leq H_n(x) + \frac{1}{2^{2n+2}}\sum_{m=1}^{2^n}\|f(2X_{n,m}+a)+f(2X_{n,m}+b)-4f(X_{n,m})\| \\ &\leq H_n(x) + \frac{1}{2^{2n+2}}\sum_{m=1}^{2^n} 2\delta \\ &\leq \delta(1-\frac{1}{2^n}) + \frac{1}{2^{2n+2}}2^{n+1}\delta = \delta(1-\frac{1}{2^{n+1}}). \end{aligned}$$

We claim that the sequence

$$T_n(x) = \{\frac{1}{2^{2n}}\sum_{m=1}^{2^n} f(2^n x + (2^n - m)a + (m-1)b)\}_{n\in\mathbb{N}}$$

is a Cauchy sequence. Indeed for all positive integer n we have

$$\|T_{n+1}(x) - T_n(x)\| \tag{3.7}$$

$$\leq \frac{1}{2^{2n+2}}\sum_{m=1}^{2^n}\|f(2^{n+1}x + (2^{n+1}-2m)a + (2m-1)b)$$

$$+f(2^{n+1}x + (2^{n+1}-2m+1)a + (2m-2)b) - 4f(2^n x + (2^n - m)a + (m-1)b)\|$$

$$= \frac{1}{2^{2n+2}}\sum_{m=1}^{2^n}\|f(2X_{n,m}+a)+f(2X_{n,m}+b)-4f(X_{n,m})\|$$

$$\leq \frac{1}{2^{2n+2}}\sum_{m=1}^{2^n} 2\delta = \frac{\delta}{2^{n+1}},$$

where $X_{n,m} = 2^n x + (2^n - m)a + (m-1)b$. As $n \longrightarrow +\infty$ the right-hand side converges to zero. The limit of this sequence exists and is in E_2. Define $T : E_1 \longrightarrow E_2$ by

$$T(x) = \lim_{n\longrightarrow\infty}\frac{1}{2^{2n}}\sum_{m=1}^{2^n} f(2^n x + (2^n - m)a + (m-1)b).$$

According to (3.5) it follows that $\|f(x) - T(x)\| \leq \delta$ for every $x \in E_1$. For all $x, y \in E_1$ and for all positive integer n we get by using (3.6)

$$T_n(x+y+a) + T_n(x+y+b)$$

$$= \frac{1}{2^{2n}}\sum_{m=1}^{2^n}[f(2^n x + 2^n y + (2^{n+1}-m)a + (m-1)b) + f(2^n x + 2^n y + (2^n - m)a + (2^n + m - 1)b)]$$

$$= \frac{1}{2^{2n}}\sum_{m=1}^{2^{n-1}}[f(2^n x + 2^n y + (2^{n+1}-2m)a + (2m-1)b) + f(2^n x + 2^n y + (2^n - 2m)a + (2^n + 2m - 1)b)$$

$$+f(2^n x + 2^n y + (2^{n+1}-2m+1)a + (2m-2)b) + f(2^n x + 2^n y + (2^n - 2m + 1)a + (2^n + 2m - 2)b)].$$

On the other hand

$$T_n(x) = \frac{1}{2^{2n}}\sum_{m=1}^{2^{n-1}} f(2^n x + (2^n - m)a + (m-1)b) + \frac{1}{2^{2n}}\sum_{m=2^{n-1}+1}^{2^n} f(2^n x + (2^n - m)a + (m-1)b)$$

$$= \frac{1}{2^{2n}} \sum_{m=1}^{2^{n-1}} [f(2^n x + (2^n - m)a + (m-1)b) + f(2^n x + (2^{n-1} - m)a + (m + 2^{n-1} - 1)b)].$$

By setting

$$X_{n,m} = 2^n x + (2^n - m)a + (m-1)b,$$

$$Y_{n,m} = 2^n y + (2^n - m)a + (m-1)b,$$

$$X'_{n,m} = 2^n x + (2^{n-1} - m)a + (m + 2^{n-1} - 1)b$$

and

$$Y'_{n,m} = 2^n y + (2^{n-1} - m)a + (m + 2^{n-1} - 1)b,$$

then we have

$$T_n(x+y+a) + T_n(x+y+b) - 2T_n(x) - 2T_n(y)$$

$$= \frac{1}{2^{2n}} \sum_{m=1}^{2^{n-1}} [f(2^n x + 2^n y + (2^{n+1} - 2m)a + (2m-1)b) + f(2^n x + 2^n y + (2^{n+1} - 2m+1)a + (2m-2)b)$$

$$-2f(2^n x + (2^n - m)a + (m-1)b) - 2f(2^n y + (2^n - m)a + (m-1)b)]$$

$$+ \frac{1}{2^{2n}} \sum_{m=1}^{2^{n-1}} [f(2^n x + 2^n y + (2^n - 2m)a + (2^n + 2m - 1)b) + f(2^n x + 2^n y + (2^n - 2m+1)a + (2^n + 2m - 2)b)$$

$$-2f(2^n x + (2^{n-1} - m)a + (m + 2^{n-1} - 1)b) - 2f(2^n y + (2^{n-1} - m)a + (m + 2^{n-1} - 1)b)].$$

Then

$$(3.8) \qquad T_n(x+y+a) + T_n(x+y+b) - 2T_n(x) - 2T_n(y)$$

$$= \frac{1}{2^{2n}} \sum_{m=1}^{2^{n-1}} \{f(X_{n,m} + Y_{n,m} + a) + f(X_{n,m} + Y_{n,m} + b) - 2f(X_{n,m}) - 2f(Y_{n,m})\}$$

$$+ \frac{1}{2^{2n}} \sum_{m=1}^{2^{n-1}} \{f(X'_{n,m} + Y'_{n,m} + a) + f(X'_{n,m} + Y'_{n,m} + b) - 2f(X'_{n,m}) - 2f(Y'_{n,m})\},$$

so

$$\|T_n(x+y+a) + T_n(x+y+b) - 2T_n(x) - 2T_n(y)\| \le \frac{1}{2^{2n}} \sum_{m=1}^{2^{n-1}} (2\delta + 2\delta) = \frac{\delta}{2^{n-1}}.$$

As $n \longrightarrow +\infty$ if follows that

$$(3.9) \qquad T(x+y+a) + T(x+y+b) = 2T(x) + 2T(y), \qquad x, y \in E_1$$

which establishes (3.3). In the next we will show the uniqueness of T. Suppose there exists an other mapping $T' : E_1 \longrightarrow E_2$ solution of (3.3) with the property $\|f(x) - T'(x)\| \le \delta$. By setting $x = y$ in (3.9) it follows that

$$(3.10) \qquad 4T(x) = T(2x+a) + T(2x+b).$$

By induction we will show that

$$2^{2n} T(x) = \sum_{m=1}^{2^n} T(2^n x + (2^n - m)a + (m-1)b), \text{ for } x \in E_1.$$

The case $n = 1$ is clearly obtained from (3.10). For $n + 1$ we have

$$2^{2n+2}T(x) = 4(2^{2n}T(x)) = \sum_{m=1}^{2^n} 4T(2^n x + (2^n - m)a + (m-1)b)$$

$$= \sum_{m=1}^{2^n} [T(2^{n+1}x + (2^{n+1} - 2m + 1)a + (2m-2)b) + T(2^{n+1}x + (2^{n+1} - 2m)a + (2m-1)b)]$$

$$= \sum_{m=1}^{2^{n+1}} T(2^{n+1}x + (2^{n+1} - m)a + (m-1)b).$$

Consequently, it follows that

$$\|T(x) - T'(x)\|$$

$$= \|\frac{1}{2^{2n}} \sum_{m=1}^{2^n} T(2^n x + (2^n - m)a + (m-1)b) - \frac{1}{2^{2n}} \sum_{m=1}^{2^n} T'(2^n x + (2^n - m)a + (m-1)b)\|$$

$$\leq \|\frac{1}{2^{2n}} \sum_{m=1}^{2^n} T(2^n x + (2^n - m)a + (m-1)b) - \frac{1}{2^{2n}} \sum_{m=1}^{2^n} f(2^n x + (2^n - m)a + (m-1)b)\|$$

$$+\|\frac{1}{2^{2n}} \sum_{m=1}^{2^n} f(2^n x + (2^n - m)a + (m-1)b) - \frac{1}{2^{2n}} \sum_{m=1}^{2^n} T'(2^n x + (2^n - m)a + (m-1)b)\|$$

$$\leq \frac{1}{2^{2n}} \sum_{m=1}^{2^n} \delta + \frac{1}{2^{2n}} \sum_{m=1}^{2^n} \delta = \frac{\delta}{2^{n-1}}.$$

Thus $\lim_n \|T(x) - T'(x)\| = 0$ for any $x \in E_1$. This completes the proof of Theorem. □

4 Găvruta-Rassias stability of (1.1)

In this section, we obtain stability in the spirit of Găvruta (see [4,5]) and J. M. Rassias (see [9,10]) for the linear Cauchy functional equation (1.1). This result generalizes and modifies the Hyers-Ulam stability.

Theorem 4.1. Let E_1 be a vector space, let E_2 a Banach spaces and let $\varphi : E_1 \times E_1 \longrightarrow [0, +\infty[$ be a function satisfying

$$\sum_{k=0}^{\infty} \frac{1}{2^{2k+1}} \sum_{m=1}^{2^k} \varphi(2^k x + (2^k - m)a + (m-1)b, 2^k y + (2^k - m)a + (m-1)b) < +\infty \tag{4.1}$$

for all $x, y \in E_1$. If a function $f : E_1 \longrightarrow E_2$ satisfies the functional inequality

$$\|f(x+y+a) + f(x+y+b) - 2f(x) - 2f(y)\| \leq 2\varphi(x, y), \quad x, y \in E_1, \tag{4.2}$$

then there exists a unique function $T : E_1 \longrightarrow E_2$ solution of the functional equation

$$T(x+y+a) + T(x+y+b) = 2T(x) + 2T(y), \; x, y \in E_1 \tag{4.3}$$

and

$$\|f(x) - T(x)\| \leq \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}} \sum_{m=1}^{2^k} \varphi(2^k x + (2^k - m)a + (m-1)b, 2^k x + (2^k - m)a + (m-1)b) \tag{4.4}$$

for any $x, y \in E_1$.

Proof. By letting $x = y$ in (4.2) yields

$$\|f(2x+a) + f(2x+b) - 4f(x)\| \leq 2\varphi(x,x). \tag{4.5}$$

Now, make the induction assumption

$$\begin{aligned}(4.6)\quad K_n(x) &= \|f(x) - \frac{1}{2^{2n}}\sum_{m=1}^{2^n} f(2^n x + (2^n - m)a + (m-1)b)\| \\ &\leq \sum_{k=0}^{n-1}\frac{1}{2^{2k+1}}\sum_{m=1}^{2^k}\varphi(2^k x + (2^k - m)a + (m-1)b, 2^k x + (2^k - m)a + (m-1)b)\end{aligned}$$

for any positive integer n. By considering (4.5) it follows that the inductive assumption is true for the case $n = 1$. For $n+1$ we have by using the same way as in previous sections

$$\begin{aligned}K_{n+1}(x) &\leq K_n(x) + \frac{1}{2^{2n+2}}\sum_{m=1}^{2^k}\|f(2X_{n,m}+a) + f(2X_{n,m}+b) - 4f(X_{n,m})\| \\ &\leq K_n(x) + \frac{1}{2^{2n+2}}\sum_{m=1}^{2^n} 2\varphi(X_{n,m}, X_{n,m}).\end{aligned}$$

According to inductive assumption (4.6) we have

$$\begin{aligned}K_{n+1}(x) &\leq \sum_{k=0}^{n-1}\frac{1}{2^{2k+1}}\sum_{m=1}^{2^k}\varphi(2^k x + (2^k - m)a + (m-1)b, 2^k x + (2^k - m)a + (m-1)b) \\ &+ \frac{1}{2^{2n+1}}\sum_{m=1}^{2^n}\varphi(2^n x + (2^n - m)a + (m-1)b, 2^n x + (2^n - m)a + (m-1)b) \\ &= \sum_{k=0}^{n}\frac{1}{2^{2k+1}}\sum_{m=1}^{2^k}\varphi(2^k x + (2^k - m)a + (m-1)b, 2^k x + (2^k - m)a + (m-1)b)\end{aligned}$$

which ends the proof of (4.6).

For any positive integer n and $x \in E_1$, we get

$$\begin{aligned}\|T_{n+1}(x) - T_n(x)\| &\leq \frac{1}{2^{2n+2}}\sum_{m=1}^{2^n}\|f(2X_{n,m}+a) + f(2X_{n,m}+b) - 4f(X_{n,m})\| \\ &\leq \frac{1}{2^{2n+1}}\sum_{m=1}^{2^n} 2\varphi(X_{n,m}, X_{n,m}).\end{aligned}$$

By using the triangular inequality we have for all $n > p$

$$\begin{aligned}\|T_n(x) - T_p(x)\| &\leq \sum_{k=p}^{n-1}\|T_{k+1}(x) - T_k(x)\| \\ &\leq \sum_{k=p}^{n-1}\frac{1}{2^{2k+1}}\sum_{m=1}^{2^k}\varphi(X_{k,m}, X_{k,m}).\end{aligned}$$

From (4.1) it follows that the right-hand side converges to zero. Thus the sequence $\{T_n(x) = \frac{1}{2^{2n}}\sum_{m=1}^{2^n} f(2^n x + (2^n - m)a + (m-1)b)\}$ is a Cauchy sequence for each $x \in E_1$. As E_2 is complete we can define $T : E_1 \longrightarrow E_2$ by

$$T(x) = \lim_{n\longrightarrow\infty} \frac{1}{2^{2n}} \sum_{m=1}^{2^n} f(2^n x + (2^n - m)a + (m-1)b).$$

According to (4.2) we get for all $x, y \in E_1$ and for all positive integer n

$$\begin{aligned}
& \|T_n(x+y+a) + T_n(x+y+b) - 2T_n(x) - 2T_n(y)\| \\
= \;& \|\frac{1}{2^{2n}} \sum_{m=1}^{2^{n-1}} [f(X_{n,m} + Y_{n,m} + a) + f(X_{n,m} + Y_{n,m} + b) - 2f(X_{n,m}) - 2f(Y_{n,m})] \\
+ \;& \frac{1}{2^{2n}} \sum_{m=1}^{2^{n-1}} [f(X'_{n,m} + Y'_{n,m} + a) + f(X'_{n,m} + Y'_{n,m} + b) - 2f(X'_{n,m}) - 2f(Y'_{n,m})] \\
\leq \;& \frac{1}{2^{2n-1}} \sum_{m=1}^{2^{n-1}} \varphi(X_{n,m}, Y_{n,m}) + \frac{1}{2^{2n-1}} \sum_{m=1}^{2^{n-1}} \varphi(X'_{n,m}, Y'_{n,m}).
\end{aligned}$$

According to (4.1) it follows that $\frac{1}{2^{2n-1}} \sum_{m=1}^{2^{n-1}} \varphi(X_{n,m}, Y_{n,m})$ converges to 0 when $n \longrightarrow +\infty$. So that it follows that

$$T(x+y+a) + T(x+y+b) = 2T(x) + 2T(y)$$

for all $x, y \in E_1$. Taking the limit in (4.6) we obtain the inequality (4.4).
The uniqueness of T follows by applying some argument used in page 8. This ends the proof of theorem. □

The next corollary extends the results obtained by J. M. Rassias in [9,10,11,12,13].

Corollary 4.2. If a function $f : E_1 \longrightarrow E_2$ satisfies the functional inequality

$$\|f(x+y+a) + f(x+y+b) - 2f(x) - 2f(y)\| \leq 2\theta\|x\|^p\|y\|^q, \;\; x, y \in E_1, \qquad (4.6)$$

for some $\theta > 0$ and $p, q \in \mathbb{R}$ such that $p + q < 2$. Then there exists a unique function $T : E_1 \longrightarrow E_2$ solution of the functional equation

$$T(x+y+a) + T(x+y+b) = 2T(x) + 2T(y), \; x, y \in E_1 \qquad (4.7)$$

and

$$\|f(x) - T(x)\| \leq \theta \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}} \sum_{m=1}^{2^k} \|2^k x + (2^k - m)a + (m-1)b\|^{p+q} \qquad (4.8)$$

for any $x \in E_1$.

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