The Australian Journal of Mathematical Analysis and Applications

AJMAA

**Volume 4**, Issue 1, Article 17, pp. 1-20, 2007

# POWER AND EULER-LAGRANGE NORMS

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*Received 3 May, 2006; accepted 15 February, 2007; published 28 May, 2007.*

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ABSTRACT. We introduce the notions of power and Euler-Lagrange norms by replacing the triangle inequality, in the definition of norm, by appropriate inequalities. We prove that every usual norm is a power norm and vice versa. We also show that every norm is an Euler-Lagrange norm and that the converse is true under certain condition.

*Key words and phrases:* Norm, power norm, Euler-Lagrange norm, convexity.

2000 *Mathematics Subject Classification.* Primary 44B20; Secondary 46C05.

ISSN (electronic): 1449-5910



This work was written whilst the first author was visiting the University of Leeds during his sabbatical leave in 2006. He would like to sincerely thank the members of School of Mathematics for their warm hospitality.

## 1. INTRODUCTION AND PRELIMINARIES

We introduce the notions of power norm and Euler-Lagrange by replacing the triangle inequality, in the definition of norm, by interesting inequalities. The reader is referred to [2] for undefined terms and notations.

We shall need the following lemma [1]. For the sake of completeness we state its proof.

**Lemma 1.1.** *Let $\mathcal{X}$ be a real or complex linear space. Let $\|\cdot\| : \mathcal{X} \to [0, \infty)$ be a mapping satisfying (1) and (2) in the definition of a $q$-norm. Then $\|\cdot\|$ is a norm if and only if the set $B = \{x \mid \|x\| \leq 1\}$ is convex.*

*Proof.* If $\|\cdot\|$ is a norm, then $B$ is clearly a convex set. Conversely, let $B$ be convex and $x, y \in \mathcal{X}$. We can assume that $x \neq 0, y \neq 0$. Putting $x' = \frac{x}{\|x\|}$ and $y' = \frac{y}{\|y\|}$ we have $x', y' \in B$.

Now $\lambda x' + (1-\lambda)\, y' \in B$ for all $0 \leq \lambda \leq 1$. In particular, for $\lambda = \frac{\|x\|}{\|x\|+\|y\|}$ we obtain

$$\| \frac{x}{\|x\|+\|y\|} + \frac{y}{\|x\|+\|y\|} \| = \|\lambda x' + (1-\lambda)\, y'\| \leq 1.$$

So that $\|x+y\| \leq \|x\| + \|y\|$. ∎

## 2. POWER NORM

We start this section with the definition of power norm by using a more general inequality than the triangle inequality.

**Definition 2.1.** Let $\mathcal{X}$ be a real or complex linear space, $q, p, r$ be non-negative fixed numbers such that $q \geq 2$ and $\frac{p}{r} = \alpha + \sqrt{\alpha^2 - 1}$ with $\alpha = 2^{q-1} - 1$. A mapping $\|\cdot\| : \mathcal{X} \to [0, \infty)$ is called a power norm on $\mathcal{X}$ if it satisfies the following conditions:

(1) $\|x\| = 0 \Leftrightarrow x = 0$,
(2) $\|\lambda x\| = \|\lambda\|\|x\|$ for all $x \in \mathcal{X}$ and all scalar $\lambda$,
(3) $\frac{\|x_1+x_2\|^q}{p+r} \leq \frac{\|x_1\|^q}{p} + \frac{\|x_2\|^q}{r}$, for all $x, y \in \mathcal{X}$.

**Remark 2.1.** Let $q \geq 2$ be given. The condition $\frac{p}{r} = \alpha + \sqrt{\alpha^2-1}$, where $\alpha = 2^{q-1} - 1$ implies that $\frac{p}{r}$ satisfies the equation $x^2 + (2 - 2^q)x + 1 = 0$, which is converted, in turn, to $(p+r)^2 = 2^q pr$.

Our first result reads as follows.

**Proposition 2.1.** *Every usual norm is a power norm.*

*Proof.* The function $f(t) = \frac{1}{p} + \frac{t^q}{r} - \frac{(1+t)^q}{p+r}$ has the nonnegative derivative $f'(t) = \frac{q}{r}t^{q-1} - \frac{q}{p+r}(1+t)^{q-1}$ on the interval $[1, \infty)$ and thus it is monotonically increasing. In fact, the condition $\frac{p}{r} \geq \alpha$ implies that for $t \geq 1$, we have $\frac{1}{\sqrt[q-1]{1+\frac{p}{r}}-1} \leq 1 \leq t$ and so $(1+\frac{1}{t})^{q-1} \leq \frac{p}{r} + 1$ or $\frac{1}{r}t^{q-1} \geq \frac{1}{p+r}(1+t)^{q-1}$.

Therefore $f(t) \geq f(1) = \frac{1}{p} + \frac{1}{r} - \frac{1}{p+r}2^q \geq 0$ for all $t \geq 1$. Note that $\frac{1}{p} + \frac{1}{r} - \frac{1}{p+r}2^q \geq 0$ holds whenever $pr2^q \leq (p+r)^2$.

Thus $\frac{1}{p} + \frac{(\frac{\|y\|}{\|x\|})^q}{r} - \frac{(1+\frac{\|y\|}{\|x\|})^q}{p+r} \geq 0$ whenever $\|x\| \leq \|y\|$. Therefore $\frac{\|x+y\|^q}{p+r} \leq \frac{(\|x\|+\|y\|)^q}{p+r} \leq \frac{\|x\|^q}{p} + \frac{\|y\|^q}{r}$ for all $x, y \in \mathcal{X}$. It follows that $\|.\|$ is a power norm. ∎

Using some ideas of [1], we prove our second result.

**Theorem 2.2.** *Every power norm is a usual norm.*

*Proof.* We shall show that $B = \{x : \|x\| \leq 1\}$ is convex. Let $x, y \in B$. Then we have

$$\|x+y\|^q \leq (p+r)\left(\frac{\|x\|^q}{p} + \frac{\|y\|^q}{r}\right) \leq (p+r)\left(\frac{1}{p} + \frac{1}{r}\right) = 2^q,$$

whence $\|\frac{x+y}{2}\|^q \leq 1$, so $\frac{1}{2}x + \left(1 - \frac{1}{2}\right) y \in B$. Thus if

$$A := \left\{\frac{k}{2^n} \mid n = 1, 2, \ldots; k = 0, 1, \ldots, n\right\},$$

then for each $\lambda \in A$ we have $\lambda x + (1-\lambda) y \in B$.

Let $0 \leq \lambda \leq 1$ and $z = \lambda x + (1-\lambda) y$. Since $A$ is dense in $[0,1]$, there exists a decreasing sequence $\{r_n\}$ in $A$ such that $\lim\limits_n r_n = \lambda$. Put $\beta_n = \frac{1-r_n}{1-\lambda}$. Obviously $0 \leq \beta_n \leq 1$, $\lim\limits_n \beta_n = 1$ and $\frac{r_n+\beta_n-1}{r_n} \leq 1$. Since $\frac{r_n+\beta_n-1}{r_n} x \in B$ and $r_n \in A$ we conclude that

$$\beta_n z = \lambda \beta_n x + (1-\lambda)\beta_n y = r_n \frac{r_n + \beta_n - 1}{r_n} x + (1 - r_n) y \in B.$$

Thus $\beta_n\|z\| = \|\beta_n z\| \leq 1$ for all $n$. Tending $n$ to infinity we get $\|z\| \leq 1$, i.e. $z \in B$. ∎

## 3. Euler-Lagrange norm

We introduce the concept of Euler-Lagrange norm by replacing the triangle inequality by an Euler-Lagrange type inequality; cf. [3].

**Definition 3.1.** Let $\mathcal{X}$ be a real or complex linear space, $m, m_1, m_2, a_1, a_2$ be non-negative fixed numbers such that $m = m_1a_1^2 + m_2a_2^2$. A mapping $\|\cdot\| : \mathcal{X} \to [0,\infty)$ is called an Euler-Lagrange norm on $\mathcal{X}$ if it satisfies the following conditions:

(1) $\|x\| = 0 \Leftrightarrow x = 0$,
(2) $\|\lambda x\| = \|\lambda\|\|x\|$ for all $x \in \mathcal{X}$ and all scalar $\lambda$,
(3) $\frac{\|a_1x_1+a_2x_2\|^2}{m} \leq \frac{\|x_1\|^2}{m_1} + \frac{\|x_2\|^2}{m_2}$, for all $x, y \in \mathcal{X}$.

We are ready to prove our next result.

**Proposition 3.1.** *Every usual norm is an Euler-Lagrange norm.*

*Proof.* Assume that $m_2a_2 \leq m_1a_1$. Consider the function $f(t) = \frac{t^2}{m_2} + \frac{1}{m_1} - \frac{(a_1+ta_2)^2}{m}$ having the derivative $f'(t) = \frac{2t}{m_2} - \frac{2a_2(a_1+a_2t)}{m}$. Evidently, $f'(t) \geq 0$ if and only if $\frac{m_2a_2}{m_1a_1} \leq t$. Hence $f$ is monotonically increasing on $[\frac{m_2a_2}{m_1a_1}, \infty)$. In particular, for all $t \geq 1$, we have $f(t) \geq f(1) \geq f(\frac{m_2a_2}{m_1a_1}) = 0$.

Thus $f(\frac{\|y\|}{\|x\|}) = \frac{(\frac{\|y\|}{\|x\|})^2}{m_2} + \frac{1}{m_1} - \frac{(a_1+\frac{\|y\|}{\|x\|}a_2)^2}{m} \geq 0$ whenever $\|x\| \leq \|y\|$. Therefore $\frac{\|a_1x+a_2y\|^2}{m} \leq \frac{(a_1\|x\|+a_2\|y\|)^2}{m} \leq \frac{\|x\|^2}{m_1} + \frac{\|y\|^2}{m_2}$ for all $x, y \in \mathcal{X}$. It follows that $\|.\|$ is an Euler-Lagrange norm.

In the case that $m_1a_1 \leq m_2a_2$ we can apply the same method by using the function $f(t) = \frac{t^2}{m_1} + \frac{1}{m_2} - \frac{(ta_1+a_2)^2}{m}$. ∎

Our last result is the following.

**Theorem 3.2.** *Every Euler-Lagrange norm is a usual norm if* $m_1a_1^2 = m_2a_2^2$.

*Proof.* Let $B = \{x : \|x\| \leq 1\}$ and let $x, y \in B$. We have

$$\begin{aligned}
\|\frac{x+y}{2}\|^2 &\leq \frac{\|a_1\frac{x}{a_1} + a_2\frac{y}{a_2}\|^2}{4} \\
&\leq \frac{(a_1\frac{\|x\|}{a_1} + a_2\frac{\|y\|}{a_2})^2}{4} \\
&\leq \frac{m}{4}\left(\frac{1}{m_1}\frac{\|x\|^2}{a_1^2} + \frac{1}{m_2}\frac{\|y\|^2}{a_2^2}\right) \\
&\leq \frac{m^2}{4m_1m_2a_1^2a_2^2} \\
&= 1,
\end{aligned}$$

whence $\frac{1}{2}x + \left(1 - \frac{1}{2}\right)y \in B$.

The rest of the proof is similar to the last part of the proof of Theorem 2.2. ∎

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