Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 336 (2007) 277–296
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Generalization of Ulam stability problem for Euler–Lagrange quadratic mappings

Hark-Mahn Kim [a,*,1], John Michael Rassias [b]

[a] Department of Mathematics, Chungnam National University, 220 Yuseong-Gu, Daejeon 305-764, Republic of Korea
[b] National and Capodistrian University of Athens, Pedagogical Department E.E., Section of Mathematics and Informatics, 4, Agamemnonos street, Aghia Paraskevi, Athens 15342, Greece

Received 25 June 2006
Available online 3 March 2007
Submitted by Steven G. Krantz

Abstract

In 1968 S.M. Ulam proposed the problem: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" In 1978 P.M. Gruber proposed the Ulam type problem: "Suppose a mathematical object satisfies a certain property approximately. Is it then possible to approximate this object by objects, satisfying the property exactly?" In this paper we solve the generalized Ulam stability problem for non-linear Euler–Lagrange quadratic mappings satisfying approximately a mean equation and an Euler–Lagrange type functional equations in quasi-Banach spaces and p-Banach spaces.

Keywords: Generalized Euler–Lagrange mapping; Ulam stability problem; Quadratic mapping; Quasi-Banach space; p-Banach space

1. Introduction

A definition of stability in the case of homomorphisms between groups was suggested by a problem posed by S.M. Ulam [21] in 1940: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric

* Corresponding author.
E-mail addresses: hmkim@math.cnu.ac.kr (H.-M. Kim), jrassias@tellas.gr (J.M. Rassias).
1 This work was supported by the second Brain Korea 21 Project in 2006.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.02.075

group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta(\varepsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d\big(h(x * y), h(x) \diamond h(y)\big) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d\big(h(x), H(x)\big) < \varepsilon$$

for all $x \in G_1$? In other words, if a mapping is almost homomorphism then there is a true homomorphism near it with small error as much as possible. If the answer is affirmative, we would call that the equation $H(x * y) = H(x) \diamond H(y)$ of homomorphism is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? In 1941 D.H. Hyers [5] solved this problem for linear mappings. In 1951 D.G. Bourgin [2] was the second author to treat the Ulam problem for additive mappings. T.M. Rassias [18] succeeded in extending the result of Hyers by weakening the condition for the Cauchy difference. In 1982 J.M. Rassias [9] extended Hyers result [5] by weakening the pertinent inequality controlled by a product of powers of norms. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found, for example, in the papers [4,9,10,19] and references therein. In 1978 according to P.M. Gruber [4] this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. Furthermore we quote that these stability results can be applied to mathematical statistics, stochastic analysis, algebra, geometry as well as psychology and sociology. We wish to note that stability properties of different functional equations can have applications to unrelated fields. For instance, Zhou [22] used a stability property of the functional equation $f(x-y) + f(x+y) = 2f(x)$ to prove a conjecture of Z. Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the associated Bernstein polynomials.

It is well known that a mapping f between real vector spaces satisfies the following quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

for all x, y if and only if there is a unique symmetric biadditive mapping B such that $f(x) = B(x,x)$ for all x, where B is given by $B(x,y) = \frac{1}{4}(f(x+y) - f(x-y))$ (see [6]). A stability problem for the quadratic functional equation (1.1) was solved by a lot of authors [3,8,19]. In particular, we note that J.M. Rassias introduced the *Euler–Lagrange quadratic mappings,* motivated from the following pertinent algebraic equation

$$|a_1x_1 + a_2x_2|^2 + |a_2x_1 - a_1x_2|^2 = \big(a_1^2 + a_2^2\big)\big[|x_1|^2 + |x_2|^2\big]. \tag{1.2}$$

Thus the second author of this paper introduced and investigated the stability problem of Ulam for the relative *Euler–Lagrange functional equation*

$$f(a_1x_1 + a_2x_2) + f(a_2x_1 - a_1x_2) = \big(a_1^2 + a_2^2\big)\big[f(x_1) + f(x_2)\big] \tag{1.3}$$

in the publications [11–13]. In addition J.M. Rassias [13] generalized the above algebraic equation (1.2) to the following equation

$$m_1m_2|a_1x_1+a_2x_2|^2+|m_2a_2x_1-m_1a_1x_2|^2=\left(m_1a_1^2+m_2a_2^2\right)\left[m_2|x_1|^2+m_1|x_2|^2\right].$$

Therefore the coauthor of this paper introduced the general pertinent Euler–Lagrange quadratic mappings via his paper [14] and investigated the stability problem of Ulam for the following generalized functional equation of (1.3)

$$\begin{aligned} &m_1m_2Q(a_1x_1+a_2x_2)+Q(m_2a_2x_1-m_1a_1x_2)\\ &\quad=\left(m_1a_1^2+m_2a_2^2\right)\left[m_2Q(x_1)+m_1Q(x_2)\right] \end{aligned}\tag{1.4}$$

for all vectors $x_1, x_2 \in X$ with X a normed linear space and Y a Banach space, and any fixed pair (a_1, a_2) of nonzero reals and any fixed pair (m_1, m_2) of positive reals. Analogous quadratic mappings were introduced and investigated through J.M. Rassias' publications [15–17]. Therefore these Euler–Lagrange mappings could be named *generalized Euler–Lagrange mappings* and the corresponding Euler–Lagrange equations might be called *generalized Euler–Lagrange equations*. Before 1992 these mappings and equations were not known at all in functional equations and inequalities. However, a completely different kind of Euler–Lagrange partial differential equations is known in calculus of variations. Therefore we think that our introduction of Euler–Lagrange mappings and equations in functional equations and inequalities provides an interesting cornerstone in analysis. Already some mathematicians have employed these Euler–Lagrange mappings.

If $\triangle ABC$ is a triangle in an inner product space and I is the interior point of the side $\overline{BC}$ with $a|\overline{BI}|=b|\overline{CI}|$ for some $a, b \in \mathbb{N}$, then we see that the following geometric identity

$$a^2\|\overrightarrow{AB}\|^2+ab\|\overrightarrow{AC}\|^2=(a+b)\left(a\|\overrightarrow{AI}\|^2+b\|\overrightarrow{CI}\|^2\right).$$

Employing the above identity, we obtain a functional equation,

$$(a+b)aQ(x)+(a+b)bQ(y)=Q(ax+by)+abQ(x-y),\tag{1.5}$$

of which geometric interpretation leads to (1.5) on the triangle $\triangle ABC$ with $a|\overline{BI}|=b|\overline{CI}|$ for the point I in the side $\overline{BC}$ [7]. In particular, if $a=b$ in (1.5) then Eq. (1.5) reduces to the quadratic functional equation (1.1). On the other hand, if either E is the exterior point of the half-line $\overrightarrow{BC}$ with $a|\overline{CE}|=b|\overline{BE}|$ or E is the exterior point of the half-line $\overrightarrow{CB}$ with $a|\overline{EB}|=b|\overline{EC}|$, then the corresponding geometric equation gives similarly rise to a functional equation

$$(a-b)bQ(y)+Q(ax-by)=a(a-b)Q(x)+abQ(x-y)\tag{1.6}$$

for all $x, y \in X$, and for a mapping $Q : X \to Y$ and given positive integers $a, b \in \mathbb{N}$ $(a > b)$ [7].

We note that functional equations (1.3) and (1.5) are special cases of (1.4). In this paper, using the direct method and ideas inspired by [14], we are going to solve the generalized Ulam stability problem for non-linear Euler–Lagrange quadratic mappings $f : X \to Y$, satisfying approximately an Euler–Lagrange mean equation and an Euler–Lagrange quadratic functional equation (1.4) controlled by a nonnegative function and a constant

$$m=\frac{(m_1+m_2)(m_1a_1^2+m_2a_2^2)}{m_1m_2+1}>0.$$

As results, we obtain the generalized theorems of the results in the papers [7,11,12,15,17].

2. Stability of (1.4)

Let X be a normed linear space and Y a Banach space throughout this paper unless we give some specific reference. Then consider a non-linear mapping $Q : X \to Y$ satisfying the *fundamental Euler–Lagrange functional equation*

$$m_1^2 m_2 Q(a_1 x) + m_1 Q(m_2 a_2 x) = m_0^2 m_2 Q\left(\frac{m_1}{m_0} a_1 x\right) + m_0^2 m_1 Q\left(\frac{m_2}{m_0} a_2 x\right) \tag{2.1}$$

with

$$m_0 := \frac{m_1 m_2 + 1}{m_1 + m_2}, \qquad m := \frac{(m_1 + m_2)(m_1 a_1^2 + m_2 a_2^2)}{m_1 m_2 + 1}$$

for all $x \in X$, and any fixed nonzero reals a_i and any fixed positive reals m_i $(i = 1, 2)$. A non-linear mapping $Q : X \to Y$ is called *generalized Euler–Lagrange quadratic* if the mapping Q satisfies Eqs. (2.1) and (1.4). They say that the non-linear mappings $\overline{Q} : X \to Y$, and $\overline{\overline{Q}} : X \to Y$ are 2-*dimensional Euler–Lagrange quadratic weighted means of first, and second form* if

$$\overline{Q}(x) = \frac{m_0^2 m_2 Q(\frac{m_1}{m_0} a_1 x) + m_0^2 m_1 Q(\frac{m_2}{m_0} a_2 x)}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)},$$

and

$$\overline{\overline{Q}}(x) = \frac{m_1 m_2 Q(a_1 x) + Q(m_2 a_2 x)}{m_2 (m_1 a_1^2 + m_2 a_2^2)}$$

hold for all $x \in X$, respectively.

Note that the fundamental functional equation (2.1) is equivalent to the *Euler–Lagrange quadratic mean functional equation*

$$\overline{\overline{Q}}(x) = \overline{Q}(x) \tag{2.2}$$

for all $x \in X$. Moreover, note that in the case of Eqs. (1.4) and (2.1) are of the form

$$\overline{\overline{Q}}(x) = \overline{Q}(x) = Q(x) \tag{2.3}$$

for all $x \in X$.

Lemma 2.1. *(See [14].) Let $Q : X \to Y$ be a generalized Euler–Lagrange quadratic mapping satisfying Eq.* (1.4). *If $m \neq 1$, then Q satisfies the equation*

$$Q(0) = 0, \qquad Q\big(m^n x\big) = m^{2n} Q(x) \tag{2.4}$$

for all $x \in X$ and all integers $n \in \mathbb{Z}$.

For notational convenience, given a mapping $f : X \to Y$ we define a generalized Euler–Lagrange difference operator $D_{m_1,m_2}^{a_1,a_2}$ of Eq. (1.4) as

$$\begin{aligned} D_{m_1,m_2}^{a_1,a_2} f(x_1, x_2) := {} & m_1 m_2 f(a_1 x_1 + a_2 x_2) + f(m_2 a_2 x_1 - m_1 a_1 x_2) \\ & - \big(m_1 a_1^2 + m_2 a_2^2\big)\big[m_2 f(x_1) + m_1 f(x_2)\big], \end{aligned}$$

which is called the approximate remainder of the functional equation (1.4) and acts as a perturbation of the equation for all vectors $x_1, x_2 \in X$, and any fixed pair (a_1, a_2) of nonzero reals and any fixed pair (m_1, m_2) of positive reals.

Now we will investigate under what conditions it is then possible to find a true generalized Euler–Lagrange quadratic mapping Q near an approximate generalized Euler–Lagrange quadratic mapping f.

Theorem 2.2. *Assume that $f : X \to Y$ is a mapping for which there exist mappings $\varphi_1 : X^2 \to [0,\infty)$ and $\psi_1 : X \to [0,\infty)$ such that the Euler–Lagrange functional inequality*

$$\left\| D^{a_1,a_2}_{m_1,m_2} f(x,y)\right\| \leqslant \varphi_1(x,y) \tag{2.5}$$

holds for all $x, y \in X$ and

$$\left\| m_1^2 m_2 f(a_1x) + m_1 f(m_2a_2x) - m_0^2 m_2 f\left(\frac{m_1}{m_0}a_1x\right) - m_0^2 m_1 f\left(\frac{m_2}{m_0}a_2x\right)\right\| \leqslant \psi_1(x) \tag{2.6}$$

for all $x \in X$. Suppose that

$$m := \frac{(m_1+m_2)(m_1a_1^2+m_2a_2^2)}{m_1m_2+1} > 1$$

and the series

$$\Phi_1(x,y) := \sum_{i=0}^{\infty} \frac{\varphi_1(m^ix, m^iy)}{m^{2i}}, \qquad \Psi_1(x) := \sum_{i=0}^{\infty} \frac{\psi_1(m^ix)}{m^{2i}} \tag{2.7}$$

converge for all $x, y \in X$.

Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(m^nx)}{m^{2n}}$$

exists for all $x \in X$, and $Q : X \to Y$ is the unique generalized Euler–Lagrange quadratic mapping satisfying Eq. (1.4)*, that is, $D^{a_1,a_2}_{m_1,m_2} Q(x,y) = 0$, the fundamental functional equation* (2.1) *and mean functional equation* (2.2) *such that*

$$\left\| f(x) - Q(x)\right\| \leqslant \frac{\Phi_1(x,0)}{m_0mm_2} + \frac{1}{m^2m_1m_2}\Phi_1\left(\frac{m_1a_1}{m_0}x, \frac{m_2a_2}{m_0}x\right) + \frac{\Psi_1(x)}{m_0mm_1m_2} + \frac{m^2m_1^2+1}{m_1m_2(m^2-1)}\left\| f(0)\right\| \tag{2.8}$$

holds for all $x \in X$, where $\|f(0)\| \leqslant \frac{\varphi_1(0,0)}{(m_1m_2+1)(m-1)}$.

Proof. Observe that the functional inequality (2.6) can be written by

$$\left\| \overline{\overline{f}}(x) - \overline{f}(x)\right\| \leqslant \frac{\psi_1(x)}{m_1m_2(m_1a_1^2+m_2a_2^2)} = \frac{\psi_1(x)}{m_0mm_1m_2} \tag{2.9}$$

for all $x \in X$. Substitution of $x = y = 0$ in inequality (2.5) yields that

$$\left\| m_1m_2f(0) + f(0) - m_0m(m_1+m_2)f(0)\right\| \leqslant \varphi_1(0,0),$$

or

$$\|f(0)\| \leqslant \frac{\varphi_1(0,0)}{(m_1m_2+1)(m-1)}.$$

Moreover substituting $y=0$ in inequality (2.5), one concludes the functional inequality

$$\big\|m_1m_2f(a_1x)+f(m_2a_2x)-m_0m\big[m_2f(x)+m_1f(0)\big]\big\| \leqslant \varphi_1(x,0),$$

or

$$\|\overline{\overline{f}}(x)-f(x)\| \leqslant \frac{\varphi_1(x,0)}{m_0mm_2}+\frac{m_1\|f(0)\|}{m_2}. \tag{2.10}$$

In addition replacing x, y in inequality (2.5) by $\frac{m_1a_1}{m_0}x$, $\frac{m_2a_2}{m_0}x$, respectively, one gets the functional inequality

$$\left\|m_1m_2f(mx)+f(0)-m_0m\left[m_2f\left(\frac{m_1a_1}{m_0}x\right)+m_1f\left(\frac{m_2a_2}{m_0}x\right)\right]\right\| \\ \leqslant \varphi_1\left(\frac{m_1a_1}{m_0}x, \frac{m_2a_2}{m_0}x\right),$$

or

$$\left\|\frac{f(mx)}{m^2}-\overline{f}(x)\right\| \leqslant \frac{1}{m^2m_1m_2}\left[\varphi_1\left(\frac{m_1a_1}{m_0}x, \frac{m_2a_2}{m_0}x\right)+\|f(0)\|\right]. \tag{2.11}$$

Using the functional inequalities (2.9)–(2.11) and the triangle inequality, we have the basic inequality

$$\begin{aligned} &\left\|f(x)-\frac{f(mx)}{m^2}\right\| \\ &\quad\leqslant \|f(x)-\overline{\overline{f}}(x)\|+\|\overline{\overline{f}}(x)-\overline{f}(x)\|+\left\|\overline{f}(x)-\frac{f(mx)}{m^2}\right\| \\ &\quad\leqslant \frac{\varphi_1(x,0)}{m_0mm_2}+\frac{1}{m^2m_1m_2}\varphi_1\left(\frac{m_1a_1}{m_0}x, \frac{m_2a_2}{m_0}x\right)+\frac{\psi_1(x)}{m_0mm_1m_2}+\frac{(m^2m_1^2+1)\|f(0)\|}{m^2m_1m_2} \\ &\quad:= \varepsilon(x), \quad \forall x\in X. \end{aligned} \tag{2.12}$$

Now substituting m^jx for x in (2.12) one gets the inequality

$$\left\|\frac{f(m^jx)}{m^{2j}}-\frac{f(m^{j+1}x)}{m^{2(j+1)}}\right\| \leqslant \frac{\varepsilon(m^jx)}{m^{2j}},$$

which yields the following general functional inequality

$$\begin{aligned} \left\|f(x)-\frac{f(m^nx)}{m^{2n}}\right\| &\leqslant \sum_{j=0}^{n-1}\frac{\varepsilon(m^jx)}{m^{2j}} \\ &\leqslant \sum_{j=0}^{n-1}\left[\frac{\varphi_1(m^jx,0)}{m_0mm_2m^{2j}}+\frac{1}{m^2m_1m_2m^{2j}}\varphi_1\left(\frac{m_1a_1}{m_0}m^jx, \frac{m_2a_2}{m_0}m^jx\right)\right. \\ &\quad\left.+\frac{\psi_1(m^jx)}{m_0mm_1m_2m^{2j}}+\frac{(m^2m_1^2+1)\|f(0)\|}{m^2m_1m_2m^{2j}}\right] \end{aligned} \tag{2.13}$$

for all $x \in X$ and all nonnegative integer n. We claim that a sequence $\{g_n(x) \mid n \in \mathbb{N}\}$ of mappings $g_n(x) := \frac{f(m^n x)}{m^{2n}}$ converges for all $x \in X$. In fact, for any nonnegative integers n, l with $n > l$, we figure out by (2.13) and triangle inequality

$$\begin{aligned}\|g_l(x) - g_n(x)\| &\leqslant \frac{1}{m^{2l}}\left\| f\left(m^l x\right) - \frac{f(m^{n-l} m^l x)}{m^{2(n-l)}}\right\| \leqslant \frac{1}{m^{2l}} \sum_{j=0}^{n-l-1} \frac{\varepsilon(m^j m^l x)}{m^{2j}} \\ &\leqslant \sum_{j=l}^{n-1}\left[\frac{\varphi_1(m^j x, 0)}{m_0 m m_2 m^{2j}} + \frac{1}{m^2 m_1 m_2 m^{2j}} \varphi_1\left(\frac{m_1 a_1}{m_0} m^j x, \frac{m_2 a_2}{m_0} m^j x\right)\right. \\ &\quad \left. + \frac{\psi_1(m^j x)}{m_0 m m_1 m_2 m^{2j}} + \frac{(m^2 m_1^2 + 1)\|f(0)\|}{m^2 m_1 m_2 m^{2j}}\right] \to 0 \quad \text{as } l \to \infty,\end{aligned}$$

which shows that $\{g_n(x)\}$ is a Cauchy sequence in Y. Therefore we see that a mapping $Q : X \to Y$ defined by

$$Q(x) := \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}}$$

exists for all $x \in X$. Taking the limit as $n \to \infty$ in (2.13), we find that the mapping Q near the approximate mapping $f : X \to Y$ of Eq. (1.4) satisfies the inequality (2.8).

In addition, we claim that the mapping Q satisfies Eq. (1.4) for all $x, y \in X$. In fact, it is clear from (2.5) that the following inequality

$$\frac{1}{m^{2n}}\left\| D_{m_1,m_2}^{a_1,a_2} f\left(m^n x, m^n y\right)\right\| \leqslant \frac{1}{m^{2n}} \varphi_1\left(m^n x, m^n y\right)$$

holds for all $x, y \in X$ and all $n \in \mathbb{N}$. Taking the limit $n \to \infty$, we see from (2.7) and the definition of Q that Q satisfies the equation

$$D_{m_1,m_2}^{a_1,a_2} Q(x, y) = 0,$$

that is, Q is a generalized Euler–Lagrange quadratic mapping satisfying Eq. (1.4). Moreover, from (2.6), (2.7) and (2.9), one proves that

$$\begin{aligned}\|\overline{\overline{Q}}(x) - \overline{Q}(x)\| &= \lim_{n\to\infty} \frac{1}{m^{2n}} \left\|\overline{\overline{f}}\left(m^n x\right) - \overline{f}\left(m^n x\right)\right\| \\ &\leqslant \lim_{n\to\infty} \frac{\psi_1(m^n x)}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2) m^{2n}} = 0,\end{aligned}$$

completing the proof that Q satisfies the fundamental functional equation (2.1) and so the mean functional equation (2.2).

Let $\check{Q} : X \to Y$ be another generalized Euler–Lagrange quadratic mapping satisfying the equation

$$D_{m_1,m_2}^{a_1,a_2} \check{Q}(x, y) = 0$$

and the approximate error bound

$$\begin{aligned}\|f(x) - \check{Q}(x)\| &\leqslant \frac{\Phi_1(x, 0)}{m_0 m m_2} + \frac{1}{m^2 m_1 m_2} \Phi_1\left(\frac{m_1 a_1}{m_0} x, \frac{m_2 a_2}{m_0} x\right) + \frac{\Psi_1(x, 0)}{m_0 m m_1 m_2} \\ &\quad + \frac{m^2 m_1^2 + 1}{m_1 m_2 (m^2 - 1)} \|f(0)\| \qquad (2.14)\end{aligned}$$

for all $x \in X$. To prove the above-mentioned uniqueness we employ Lemma 2.1, so that

$$Q(x) = m^{-2n} Q\big(m^n x\big), \qquad \check{Q}(x) = m^{-2n} \check{Q}\big(m^n x\big)$$

hold for all $x \in X$ and all $n \in \mathbb{N}$. Thus the triangle inequality and inequalities (2.8), (2.14) yield the inequality

$$\begin{aligned}
\big\| Q(x) - \check{Q}(x) \big\| &= \frac{1}{m^{2n}} \big\| Q\big(m^n x\big) - \check{Q}\big(m^n x\big) \big\| \\
&\leqslant \frac{1}{m^{2n}} \big(\big\| Q\big(m^n x\big) - f\big(m^n x\big) \big\| + \big\| f\big(m^n x\big) - \check{Q}\big(m^n x\big) \big\| \big) \\
&\leqslant 2 \bigg[\frac{\Phi_1(m^n x, 0)}{m_0 m m_2 m^{2n}} + \frac{1}{m^2 m_1 m_2 m^{2n}} \Phi_1\bigg(\frac{m_1 a_1}{m_0} m^n x, \frac{m_2 a_2}{m_0} m^n x \bigg) \\
&\quad + \frac{\Psi_1(m^n x, 0)}{m_0 m m_1 m_2 m^{2n}} + \frac{m^2 m_1^2 + 1}{m_1 m_2 (m^2 - 1) m^{2n}} \big\| f(0) \big\| \bigg]
\end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Therefore from $n \to \infty$, one establishes

$$Q(x) - \check{Q}(x) = 0$$

for all $x \in X$, completing the proof of uniqueness. The proof of Theorem 2.2 is now complete. □

Theorem 2.3. *Assume that $f : X \to Y$ is a mapping for which there exist mappings $\varphi_2 : X^2 \to [0, \infty)$ and $\psi_2 : X \to [0, \infty)$ such that the Euler–Lagrange functional inequality*

$$\big\| D^{a_1, a_2}_{m_1, m_2} f(x, y) \big\| \leqslant \varphi_2(x, y) \tag{2.15}$$

holds for all $x, y \in X$ and

$$\begin{aligned}
&\bigg\| m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_2 f\bigg(\frac{m_1}{m_0} a_1 x \bigg) - m_0^2 m_1 f\bigg(\frac{m_2}{m_0} a_2 x \bigg) \bigg\| \\
&\quad \leqslant \psi_2(x)
\end{aligned} \tag{2.16}$$

for all $x \in X$. Suppose that

$$0 < m := \frac{(m_1 + m_2)(m_1 a_1^2 + m_2 a_2^2)}{m_1 m_2 + 1} < 1$$

and the series

$$\Phi_2(x, y) := \sum_{i=1}^{\infty} m^{2i} \varphi_2\bigg(\frac{x}{m^i}, \frac{y}{m^i} \bigg), \qquad \Psi_2(x) := \sum_{i=1}^{\infty} m^{2i} \psi_2\bigg(\frac{x}{m^i} \bigg) \tag{2.17}$$

converge for all $x, y \in X$.

Then the limit

$$Q(x) = \lim_{n \to \infty} m^{2n} f\bigg(\frac{x}{m^n} \bigg)$$

exists for all $x \in X$, and $Q : X \to Y$ is the unique generalized Euler–Lagrange quadratic mapping satisfying Eq. (1.4)*, that is, $D^{a_1, a_2}_{m_1, m_2} Q(x, y) = 0$, the fundamental functional equation* (2.1) *and mean functional equation* (2.2) *such that*

$$\begin{aligned}\|f(x)-Q(x)\| &\leqslant \frac{\Phi_2(x,0)}{m_0mm_2}+\frac{1}{m^2m_1m_2}\Phi_2\left(\frac{m_1a_1}{m_0}x,\frac{m_2a_2}{m_0}x\right)+\frac{\Psi_2(x)}{m_0mm_1m_2}\\ &\quad+\frac{m^2m_1^2+1}{m_1m_2(1-m^2)}\|f(0)\| \end{aligned}\tag{2.18}$$

holds for all $x\in X$, where $\|f(0)\|\leqslant\frac{\varphi_2(0,0)}{(m_1m_2+1)(1-m)}$.

Proof. Using the same argument as those of (2.9)–(2.13), we obtain the crucial inequality

$$\left\|f(x)-m^2f\left(\frac{x}{m}\right)\right\|\leqslant m^2\varepsilon_2\left(\frac{x}{m}\right),\tag{2.19}$$

where

$$\begin{aligned}\varepsilon_2(x) &:= \frac{\varphi_2(\frac{x}{m},0)}{m_0mm_2}+\frac{1}{m^2m_1m_2}\varphi_2\left(\frac{m_1a_1}{m_0}\frac{x}{m},\frac{m_2a_2}{m_0}\frac{x}{m}\right)+\frac{\psi_2(\frac{x}{m})}{m_0mm_1m_2}\\ &\quad+\frac{(m^2m_1^2+1)\|f(0)\|}{m^2m_1m_2}\quad \forall x\in X,\end{aligned}$$

which induces similarly

$$\begin{aligned}\left\|f(x)-m^{2n}f\left(\frac{x}{m^n}\right)\right\| &\leqslant \sum_{j=1}^{n}\left[\frac{m^{2j}\varphi_2(\frac{x}{m^j},0)}{m_0mm_2}+\frac{m^{2j}}{m^2m_1m_2}\varphi_2\left(\frac{m_1a_1}{m_0}\frac{x}{m^j},\frac{m_2a_2}{m_0}\frac{x}{m^j}\right)\right.\\ &\quad\left.+\frac{m^{2j}\psi_2(\frac{x}{m^j})}{m_0mm_1m_2}+\frac{m^{2j}(m^2m_1^2+1)\|f(0)\|}{m^2m_1m_2}\right]\end{aligned}\tag{2.20}$$

for all $x\in X$ and all $n\in\mathbb{N}$.

Utilizing the last functional inequality (2.20) and the similar argument to the corresponding proof of Theorem 2.2, we obtain the conclusion of this theorem. □

We observe that if $m_1=m_2=1$, then $m_0=1$ and

$$\overline{\overline{f}}(x)=\frac{f(a_1x)+f(a_2x)}{a_1^2+a_2^2}=\overline{f}(x).$$

Thus one has the generalized Ulam stability problem for Eq. (1.3) if $m=a_1^2+a_2^2\neq 1$ and the corresponding series (2.7) or (2.17) converges. In turn, note that if $m_1=1, m_2>0$, then $m_0=1$ and

$$\overline{\overline{f}}(x)=\frac{m_2f(a_1x)+f(m_2a_2x)}{m_2(a_1^2+m_2a_2^2)}=\overline{f}(x).$$

Thus they have the generalized Ulam stability problem for the equation

$$m_2f(a_1x_1+a_2x_2)+f(m_2a_2x_1-a_1x_2)=\left(a_1^2+m_2a_2^2\right)\left[m_2f(x_1)+f(x_2)\right]$$

if $m=a_1^2+m_2a_2^2\neq 1$ and the corresponding series (2.7) or (2.17) converges.

In particular, given $\varphi_i(x,y):=c_1$ and $\psi_i(x):=c_2$ for some nonnegative constants c_1,c_2 in the main theorems, one gets the result of J.M. Rassias [14]. As a special case, if one takes $m_1:=a$, $m_2:=b$, $a_1:=-1$, $a_2:=1$ and switches x with y, and then considers $\varphi_1(y,x):=\varphi(x,y)$ in Theorem 2.2, then one has the following corollary.

Corollary 2.4. *Assume that $f : X \to Y$ is a mapping for which there exist mappings $\varphi : X^2 \to [0,\infty)$ and $\psi : X \to [0,\infty)$ such that the Euler–Lagrange functional inequality*

$$\left\| f(ax+by) + abf(x-y) - (a+b)af(x) - (a+b)bf(y) \right\| \leqslant \varphi(x,y)$$

holds for all $x, y \in X$ and any fixed positive integers a, b and

$$\left\| a^2bf(-x) + af(bx) - \left(\frac{ab+1}{a+b}\right)^2 \left[bf\left(-\left(\frac{a+b}{ab+1}\right)ax\right) + af\left(\left(\frac{a+b}{ab+1}\right)bx\right)\right]\right\| \leqslant \psi(x)$$

for all $x \in X$. Suppose that the series

$$\Phi(x,y) := \sum_{i=0}^{\infty} \frac{\varphi(m^i x, m^i y)}{m^{2i}}, \qquad \Psi(x) := \sum_{i=0}^{\infty} \frac{\psi(m^i x)}{m^{2i}}$$

converge for all $x, y \in X$, where $m := \frac{(a+b)^2}{ab+1}$.

Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}}$$

exists for all $x \in X$, and $Q : X \to Y$ is the unique generalized Euler–Lagrange quadratic mapping satisfying Eq. (1.5)*, that is, $D^{a_1,a_2}_{m_1,m_2} Q(x,y) = 0$, the fundamental functional equation* (2.1) *and mean functional equation* (2.2) *such that*

$$\left\| f(x) - Q(x) \right\| \leqslant \frac{\Phi(0,x)}{(a+b)b} + \frac{(ab+1)^2}{ab(a+b)^4} \Phi\left(\frac{(a+b)bx}{ab+1}, -\frac{(a+b)ax}{ab+1}\right) + \frac{\Psi(x)}{ab(a+b)} + \frac{[(a+b)^2a^2 + (ab+1)^2]}{ab[(a+b)^4 - (ab+1)^2]} \left\| f(0) \right\|$$

holds for all $x \in X$, where $\|f(0)\| \leqslant \frac{\varphi(0,0)}{(a+b)^2-(ab+1)}$.

Corollary 2.5. *Assume that $f : X \to Y$ is a mapping for which there exist nonnegative reals c_1, c_2 and positive reals p_1, p_2, r such that the Euler–Lagrange functional inequality*

$$\left\| D^{a_1,a_2}_{m_1,m_2} f(x,y) \right\| \leqslant c_1 \|x\|^{p_1} \|y\|^{p_2}$$

holds for all $x, y \in X$ and

$$\left\| m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_2 f\left(\frac{m_1}{m_0} a_1 x\right) - m_0^2 m_1 f\left(\frac{m_2}{m_0} a_2 x\right) \right\| \leqslant c_2 \|x\|^r$$

for all $x \in X$.

Then there exists a unique generalized Euler–Lagrange quadratic mapping $Q : X \to Y$ satisfying Eq. (1.4)*, that is, $D^{a_1,a_2}_{m_1,m_2} Q(x,y) = 0$, the fundamental functional equation* (2.1) *and mean functional equation* (2.2) *such that*

$$\|f(x)-Q(x)\| \leqslant \begin{cases} \frac{c_1|m_1a_1|^{p_1}|m_2a_2|^{p_2}\|x\|^{p_1+p_2}}{m_1m_2m_0^{p_1+p_2}(m^2-m^{p_1+p_2})} + \frac{c_2m\|x\|^r}{m_0m_1m_2(m^2-m^r)}, \\ \quad \textit{if } m>1,\ r<2 \textit{ and } p_1+p_2<2 \\ \quad (\textit{or } m<1,\ r>2 \textit{ and } p_1+p_2>2), \\ \frac{c_1|m_1a_1|^{p_1}|m_2a_2|^{p_2}\|x\|^{p_1+p_2}}{m_1m_2m_0^{p_1+p_2}(m^{p_1+p_2}-m^2)} + \frac{c_2m\|x\|^r}{m_0m_1m_2(m^r-m^2)}, \\ \quad \textit{if } m<1,\ r<2 \textit{ and } p_1+p_2<2 \\ \quad (\textit{or } m>1,\ r>2 \textit{ and } p_1+p_2>2) \end{cases}$$

holds for all $x \in X$.

Proof. Take account of $\varphi_i(x,y) := c_1\|x\|^{p_1}\|y\|^{p_2}$ and $\psi_i(x) := c_2\|x\|^r$ and then apply Theorems 2.2 and 2.3 for each cases. □

Corollary 2.6. *Assume that* $f : X \to Y$ *is a mapping for which there exist nonnegative reals* c_1, c_2 *and positive reals* p_1, p_2, r *such that the Euler–Lagrange functional inequality*

$$\left\|D^{a_1,a_2}_{m_1,m_2} f(x,y)\right\| \leqslant c_1\left(\|x\|^{p_1}+\|y\|^{p_2}\right)$$

holds for all $x, y \in X$ *and*

$$\left\|m_1^2m_2 f(a_1x)+m_1 f(m_2a_2x)-m_0^2m_2 f\left(\frac{m_1}{m_0}a_1x\right)-m_0^2m_1 f\left(\frac{m_2}{m_0}a_2x\right)\right\| \leqslant c_2\|x\|^r$$

for all $x \in X$.

Then there exists a unique generalized Euler–Lagrange quadratic mapping $Q : X \to Y$ *satisfying Eq.* (1.4)*, that is,* $D^{a_1,a_2}_{m_1,m_2} Q(x,y) = 0$*, the fundamental functional equation* (2.1) *and mean functional equation* (2.2) *such that*

$$\begin{aligned} &\|f(x)-Q(x)\| \\ &\leqslant \begin{cases} \frac{c_1m\|x\|^{p_1}}{m_0m_2(m^2-m^{p_1})} + \frac{c_1|m_1a_1|^{p_1}\|x\|^{p_1}}{m_1m_2m_0^{p_1}(m^2-m^{p_1})} + \frac{c_1|m_2a_2|^{p_2}\|x\|^{p_2}}{m_1m_2m_0^{p_2}(m^2-m^{p_2})} + \frac{c_2m\|x\|^r}{m_0m_1m_2(m^2-m^r)}, \\ \quad \textit{if } m>1,\ \textit{and } r, p_1, p_2<2\ (\textit{or } m<1,\ \textit{and } r, p_1, p_2>2), \\ \frac{c_1m\|x\|^{p_1}}{m_0m_2(m^{p_1}-m^2)} + \frac{c_1|m_1a_1|^{p_1}\|x\|^{p_1}}{m_1m_2m_0^{p_1}(m^{p_1}-m^2)} + \frac{c_1|m_2a_2|^{p_2}\|x\|^{p_2}}{m_1m_2m_0^{p_2}(m^{p_2}-m^2)} + \frac{c_2m\|x\|^r}{m_0m_1m_2(m^r-m^2)}, \\ \quad \textit{if } m<1,\ \textit{and } r, p_1, p_2<2\ (\textit{or } m>1,\ \textit{and } r, p_1, p_2>2) \end{cases} \end{aligned}$$

holds for all $x \in X$.

Proof. Take account of $\varphi_i(x,y) := c_1(\|x\|^{p_1}+\|y\|^{p_2})$ and $\psi_i(x) := c_2\|x\|^r$ and then apply Theorems 2.2 and 2.3 for each cases. □

Now, in the next theorem we consider a singular case $m := \frac{(m_1+m_2)(m_1a_1^2+m_2a_2^2)}{m_1m_2+1} = 1$ of Theorems 2.2 and 2.3.

Theorem 2.7. *Assume that* $f : X \to Y$ *is a mapping for which there exists a mapping* $\varphi_3 : X^2 \to [0,\infty)$ *such that the Euler–Lagrange functional inequality*

$$\left\|D^{a_1,a_2}_{m_1,m_2} f(x,y)\right\| \leqslant \varphi_3(x,y) \tag{2.21}$$

holds for all $x, y \in X$. *Suppose that*

$$m := \frac{(m_1+m_2)(m_1a_1^2+m_2a_2^2)}{m_1m_2+1} = 1,$$

2-dimensional vectors (m_1, m_2), (a_2, a_1) *are linearly dependent, and that the series*

$$\Phi_3(x, y) := \sum_{i=0}^{\infty} \frac{\varphi_3(l^i x, l^i y)}{l^{2i}}, \quad \text{if } l^2 > 1 \quad \left(\Phi_3(x, y) := \sum_{i=1}^{\infty} l^{2i} \varphi_3\left(\frac{x}{l^i}, \frac{y}{l^i}\right), \quad \text{if } l^2 < 1\right)$$

converges for all $x, y \in X$, *where* $l := a_1 + a_2$ *is given with* $l^2 \neq 0, 1$.

Then there exists a unique generalized Euler–Lagrange quadratic mapping $Q : X \to Y$ *satisfying Eq.* (1.4)*, that is,* $D^{a_1,a_2}_{m_1,m_2} Q(x, y) = 0$, *such that*

$$\|f(x) - Q(x)\| \leqslant \begin{cases} \frac{\Phi_3(x,x)}{m_1 m_2 l^2} + \frac{\|f(0)\|}{m_1 m_2 (l^2-1)}, & \text{if } l^2 > 1, \\ \frac{\Phi_3(x,x)}{m_1 m_2 l^2} + \frac{\|f(0)\|}{m_1 m_2 (1-l^2)}, & \text{if } l^2 < 1, \end{cases}$$

holds for all $x \in X$. *The mapping* $Q : X \to Y$ *is given by*

$$Q(x) = \lim_{n\to\infty} \frac{f(l^n x)}{l^{2n}}, \quad \text{if } l^2 > 1 \quad \left(Q(x) = \lim_{n\to\infty} l^{2n} f\left(\frac{x}{l^n}\right), \quad \text{if } l^2 < 1\right).$$

Moreover, if there exists a mapping $\psi_3 : X \to [0, \infty)$ *for which the mapping* f *satisfies approximately the following fundamental functional equation as follows*

$$\left\| m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_2 f\left(\frac{m_1}{m_0} a_1 x\right) - m_0^2 m_1 f\left(\frac{m_2}{m_0} a_2 x\right)\right\| \leqslant \psi_3(x)$$

and

$$\Psi_3(x) := \sum_{i=0}^{\infty} \frac{\psi_3(l^i x)}{l^{2i}}, \quad \text{if } l^2 > 1 \quad \left(\Psi_3(x) := \sum_{i=1}^{\infty} l^{2i} \psi_3\left(\frac{x}{l^i}\right), \quad \text{if } l^2 < 1\right)$$

converges for all $x \in X$, *then the mapping* $Q : X \to Y$ *satisfies further the fundamental functional equation* (2.1) *and mean functional equation* (2.2).

Proof. Note that

$$\frac{m_1 m_2 + 1}{m_1 m_2} = \frac{m_1^2 a_1 + a_2}{m_1^2 a_1} = (a_1 + a_2)^2$$

according to $m_2 a_2 = m_1 a_1$ and

$$m := \frac{(m_1 + m_2)(m_1 a_1^2 + m_2 a_2^2)}{m_1 m_2 + 1} = 1.$$

Replacing y by x in (2.21), we obtain

$$\|f(lx) - l^2 f(x)\| \leqslant \frac{\varphi_3(x, x)}{m_1 m_2} + \frac{\|f(0)\|}{m_1 m_2},$$

which yields the following crucial functional inequality

$$\left\|\frac{f(l^n x)}{l^{2n}} - f(x)\right\| \leqslant \frac{1}{m_1 m_2 l^2} \sum_{i=0}^{n-1} \frac{\varphi_3(l^i x, l^i x) + \|f(0)\|}{l^{2i}}, \quad \text{if } l^2 > 1,$$

$$\left\|f(x) - l^{2n} f\left(\frac{x}{l^n}\right)\right\| \leqslant \frac{1}{m_1 m_2 l^2} \sum_{i=1}^{n} l^{2i} \left[\varphi_3\left(\frac{x}{l^i}, \frac{x}{l^i}\right) + \|f(0)\|\right], \quad \text{if } l^2 < 1$$

for all $x \in X$ and all nonnegative integer n. Applying the similar argument to the proof of Theorems 2.2 and 2.3 to the last functional inequality for each two cases, one has indeed the desired results. □

3. Stability of (1.4) in quasi-Banach spaces

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 3.1. (See [1,20].) Let X be a linear space. A *quasi-norm* $\|\cdot\|$ is a real-valued function on X satisfying the following:

(1) $\|x\| \geqslant 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(3) There is a constant K such that $\|x + y\| \leqslant K(\|x\| + \|y\|)$ for all $x, y \in X$.

The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X. A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leqslant 1)$ if

$$\|x + y\|^p \leqslant \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space*.

Clearly, p-norms are continuous, and in fact, if $\|\cdot\|$ is a p-norm on X, then the formula $d(x, y) := \|x - y\|^p$ defines an translation invariant metric for X and $\|\cdot\|^p$ is a p-homogeneous F-norm. The Aoki–Rolewicz theorem [1,20] guarantees that each quasi-norm is equivalent to some p-norm for some $0 < p \leqslant 1$. In this section, we are going to prove the generalized Ulam stability of mappings satisfying approximately Eq. (1.4) in quasi-Banach spaces, and in p-Banach spaces, respectively. Let X be a quasi-normed space and Y a quasi-Banach space. Let $K \geqslant 1$ be the modulus of concavity of $\|\cdot\|$ throughout this section.

Theorem 3.2. *Assume that $f : X \to Y$ is a mapping for which there exist mappings $\varphi_1 : X^2 \to [0, \infty)$ and $\psi_1 : X \to [0, \infty)$ such that the Euler–Lagrange functional inequality*

$$\left\| D^{a_1,a_2}_{m_1,m_2} f(x, y)\right\| \leqslant \varphi_1(x, y)$$

holds for all $x, y \in X$ and

$$\left\| m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_2 f\left(\frac{m_1}{m_0} a_1 x\right) - m_0^2 m_1 f\left(\frac{m_2}{m_0} a_2 x\right)\right\| \leqslant \psi_1(x) \tag{3.1}$$

for all $x \in X$. Suppose that

$$m := \frac{(m_1 + m_2)(m_1 a_1^2 + m_2 a_2^2)}{m_1 m_2 + 1} > \sqrt{K}$$

and the series

$$\Phi_1(x,y) := \sum_{i=0}^{\infty} \frac{K^i \varphi_1(m^i x, m^i y)}{m^{2i}}, \qquad \Psi_1(x) := \sum_{i=0}^{\infty} \frac{K^i \psi_1(m^i x)}{m^{2i}} \tag{3.2}$$

converge for all $x, y \in X$.

Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}}$$

exists for all $x \in X$, and $Q : X \to Y$ is the unique generalized Euler–Lagrange quadratic mapping satisfying Eq. (1.4)*, that is, $D^{a_1,a_2}_{m_1,m_2} Q(x,y) = 0$, the fundamental functional equation* (2.1) *and mean functional equation* (2.2) *such that*

$$\begin{aligned}\|f(x) - Q(x)\| &\leqslant \frac{K^3 \Phi_1(x,0)}{m_0 m m_2} + \frac{K^3}{m^2 m_1 m_2}\Phi_1\left(\frac{m_1 a_1}{m_0}x, \frac{m_2 a_2}{m_0}x\right) + \frac{K^3 \Psi_1(x)}{m_0 m m_1 m_2} \\ &\quad + \frac{K^3(m^2 m_1^2 + 1)\|f(0)\|}{m_1 m_2 (m^2 - K)}\end{aligned} \tag{3.3}$$

holds for all $x \in X$, where $\|f(0)\| \leqslant \frac{\varphi_1(0,0)}{(m_1 m_2 + 1)(m-1)}$.

Proof. Using the functional inequalities (2.9)–(2.11) and the property of quasi-norm $\|\cdot\|$, we have the basic inequality

$$\begin{aligned}\left\| f(x) - \frac{f(mx)}{m^2} \right\| &\leqslant K\|f(x) - \overline{\overline{f}}(x)\| + K\left\|\overline{\overline{f}}(x) - \frac{f(mx)}{m^2}\right\| \\ &\leqslant K\|f(x) - \overline{\overline{f}}(x)\| + K^2\|\overline{\overline{f}}(x) - \overline{f}(x)\| + K^2\left\|\overline{f}(x) - \frac{f(mx)}{m^2}\right\| \\ &\leqslant K^2 \varepsilon(x), \quad \forall x \in X,\end{aligned} \tag{3.4}$$

where $\varepsilon(x)$ is defined as in (2.12). From the functional inequality (3.4), we show by applying a standard procedure of the induction argument on n that

$$\left\| f(x) - \frac{f(m^n x)}{m^{2n}} \right\| \leqslant K^3 \sum_{i=0}^{n-2} \left(\frac{K}{m^2}\right)^i \varepsilon(m^i x) + K^2 \left(\frac{K}{m^2}\right)^{n-1} \varepsilon(m^{n-1} x) \tag{3.5}$$

for all $x \in X$ and all $n \geqslant 1$, which is considered to be (2.12) for $n = 1$. In fact, we figure out by the inequality (3.5),

$$\begin{aligned}&\left\| f(x) - \frac{f(m^{n+1}x)}{m^{2(n+1)}} \right\| \\ &\quad \leqslant K\left\| f(x) - \frac{f(mx)}{m^2}\right\| + K\left\|\frac{f(mx)}{m^2} - \frac{f(m^{n+1}x)}{m^{2(n+1)}}\right\| \\ &\quad \leqslant K^3\varepsilon(x) + \frac{K}{m^2}\left[K^3 \sum_{i=0}^{n-2}\left(\frac{K}{m^2}\right)^i \varepsilon(m^{i+1}x) + K^2\left(\frac{K}{m^2}\right)^{n-1}\varepsilon(m^n x)\right] \\ &\quad = K^3 \sum_{j=0}^{n-1}\left(\frac{K}{m^2}\right)^j \varepsilon(m^j x) + K^2\left(\frac{K}{m^2}\right)^n \varepsilon(m^n x)\end{aligned} \tag{3.6}$$

which yields (3.5) for $n + 1$. Thus one obtains that for all nonnegative integers n, l with $n > l$

$$\left\|\frac{f(m^l x)}{m^{2l}} - \frac{f(m^n x)}{m^{2n}}\right\| = \frac{1}{m^{2l}}\left\|f\left(m^l x\right) - \frac{f(m^{n-l}\cdot m^l x)}{m^{2(n-l)}}\right\|$$
$$\leqslant \frac{K^3}{m^{2l}}\sum_{i=0}^{n-l-2}\frac{K^i\varepsilon(m^{l+i}x)}{m^{2i}} + \frac{K^2}{m^{2l}}\frac{K^{n-l-1}\varepsilon(m^{n-1}x)}{m^{2(n-l-1)}}$$
$$= \frac{K^3}{K^l}\sum_{j=l}^{n-2}\frac{K^j\varepsilon(m^j x)}{m^{2j}} + \frac{K^2}{K^l}\frac{K^{n-1}\varepsilon(m^{n-1}x)}{m^{2(n-1)}}, \tag{3.7}$$

which tends to zero by (3.2) as $l \to \infty$.

Therefore a mapping $Q : X \to Y$ given by

$$Q(x) = \lim_{n\to\infty}\frac{f(m^n x)}{m^{2n}}, \quad x \in X,$$

is well defined. Thus passing the limit $n \to \infty$ in (3.5), we have the inequality (3.3). To prove the uniqueness, let Q' be another mapping satisfying (3.3). Then we get by Lemma 2.1 that $Q'(m^n x) = m^{2n}Q'(x)$ for all $x \in X$ and all $n \in \mathbb{N}$. Thus we have

$$\left\|Q(x) - Q'(x)\right\| \leqslant \frac{1}{m^{2n}}\left\{K\left\|Q\left(m^n x\right) - f\left(m^n x\right)\right\| + K\left\|f\left(m^n x\right) - Q'\left(m^n x\right)\right\|\right\}$$
$$\leqslant \frac{2K^4\Phi_1(m^n x, 0)}{m_0 m m_2 m^{2n}} + \frac{2K^4}{m^2 m_1 m_2 m^{2n}}\Phi_1\left(\frac{m_1 a_1}{m_0}m^n x, \frac{m_2 a_2}{m_0}m^n x\right)$$
$$+ \frac{2K^4\Psi_1(m^n x)}{m_0 m m_1 m_2 m^{2n}} + \frac{2K^4(m^2 m_1^2 + 1)\|f(0)\|}{m_1 m_2(m^2 - K)m^{2n}} \tag{3.8}$$

for all $x \in X$. Taking the limit as $n \to \infty$, then we conclude that $Q(x) = Q'(x)$ for all $x \in X$.

The rest of the proof of this theorem is omitted as similar to the corresponding that of Theorem 2.2. □

Theorem 3.3. *Assume that $f : X \to Y$ is a mapping for which there exist mappings $\varphi_2 : X^2 \to [0, \infty)$ and $\psi_2 : X \to [0, \infty)$ such that the Euler–Lagrange functional inequality*

$$\left\|D^{a_1,a_2}_{m_1,m_2} f(x, y)\right\| \leqslant \varphi_2(x, y) \tag{3.9}$$

holds for all $x, y \in X$ and

$$\left\|m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_2 f\left(\frac{m_1}{m_0}a_1 x\right) - m_0^2 m_1 f\left(\frac{m_2}{m_0}a_2 x\right)\right\|$$
$$\leqslant \psi_2(x) \tag{3.10}$$

for all $x \in X$. Suppose that

$$0 < m := \frac{(m_1 + m_2)(m_1 a_1^2 + m_2 a_2^2)}{m_1 m_2 + 1} < \frac{1}{\sqrt{K}}$$

and the series

$$\Phi_2(x, y) := \sum_{i=1}^{\infty} K^i m^{2i}\varphi_2\left(\frac{x}{m^i}, \frac{y}{m^i}\right), \qquad \Psi_2(x) := \sum_{i=1}^{\infty} K^i m^{2i}\psi_2\left(\frac{x}{m^i}\right) \tag{3.11}$$

converge for all $x, y \in X$.

Then the limit

$$Q(x) = \lim_{n\to\infty} m^{2n} f\left(\frac{x}{m^n}\right)$$

exists for all $x \in X$, and $Q : X \to Y$ is the unique generalized Euler–Lagrange quadratic mapping satisfying Eq. (1.4)*, that is, $D^{a_1,a_2}_{m_1,m_2} Q(x,y) = 0$, the fundamental functional equation* (2.1) *and mean functional equation* (2.2) *such that*

$$\begin{aligned}\|f(x) - Q(x)\| \leqslant{}& \frac{K^2 \Phi_2(x,0)}{m_0 m m_2} + \frac{K^2}{m^2 m_1 m_2}\Phi_2\left(\frac{m_1 a_1}{m_0}x, \frac{m_2 a_2}{m_0}x\right) + \frac{K^2 \Psi_2(x)}{m_0 m m_1 m_2} \\ &+ \frac{K^3(m^2 m_1^2 + 1)\|f(0)\|}{m_1 m_2 (1 - K m^2)}\end{aligned} \tag{3.12}$$

holds for all $x \in X$, where

$$\|f(0)\| \leqslant \frac{\varphi_2(0,0)}{(m_1 m_2 + 1)(1 - m)}.$$

Proof. Applying the same argument as that of (3.5)–(3.6) to (3.4), we obtain the crucial inequality

$$\left\|f(x) - m^{2n} f\left(\frac{x}{m^n}\right)\right\| \leqslant K^2 \sum_{i=1}^{n-1} K^i m^{2i} \varepsilon_2\left(\frac{x}{m^i}\right) + K^{n+1} m^{2n} \varepsilon_2\left(\frac{x}{m^n}\right) \tag{3.13}$$

for all $x \in X$ and all $n \in \mathbb{N}$, where $\varepsilon_2(x)$ is defined as in (2.19).

Utilizing the last functional inequality (3.13) and the similar argument to the corresponding process (3.7)–(3.8) of Theorem 3.2, we obtain the conclusion of this theorem. □

Remark 3.4. It will be interesting to investigate the stability problem of Ulam for the case of m with $1/\sqrt{K} \leqslant m \leqslant \sqrt{K}$ in view of Theorems 3.2 and 3.3.

We now investigate the general Ulam stability problem for the functional equation (1.4) in p-Banach spaces.

Theorem 3.5. *Let X be a quasi-normed space and Y a p-Banach space. Assume that $f : X \to Y$ is a mapping for which there exist mappings $\varphi_1 : X^2 \to [0,\infty)$ and $\psi_1 : X \to [0,\infty)$ such that the Euler–Lagrange functional inequality*

$$\left\|D^{a_1,a_2}_{m_1,m_2} f(x,y)\right\| \leqslant \varphi_1(x,y)$$

holds for all $x, y \in X$ and

$$\left\|m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_2 f\left(\frac{m_1}{m_0}a_1 x\right) - m_0^2 m_1 f\left(\frac{m_2}{m_0}a_2 x\right)\right\| \leqslant \psi_1(x)$$

for all $x \in X$. Suppose that $m := \frac{(m_1+m_2)(m_1 a_1^2 + m_2 a_2^2)}{m_1 m_2 + 1} > 1$ and the series

$$\Phi_1(x,y) := \sum_{i=0}^{\infty} \frac{\varphi_1(m^i x, m^i y)^p}{m^{2ip}}, \qquad \Psi_1(x) := \sum_{i=0}^{\infty} \frac{\psi_1(m^i x)^p}{m^{2ip}} \tag{3.14}$$

converge for all $x, y \in X$.

Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}}$$

exists for all $x \in X$, *and* $Q : X \to Y$ *is the unique generalized Euler–Lagrange quadratic mapping satisfying Eq.* (1.4)*, that is,* $D^{a_1,a_2}_{m_1,m_2} Q(x,y) = 0$, *the fundamental functional equation* (2.1) *and mean functional equation* (2.2) *such that*

$$\|f(x) - Q(x)\| \leqslant \left[\frac{\Phi_1(x,0)}{(m_0 m m_2)^p} + \frac{1}{(m^2 m_1 m_2)^p}\Phi_1\left(\frac{m_1 a_1}{m_0}x, \frac{m_2 a_2}{m_0}x\right) + \frac{\Psi_1(x)}{(m_0 m m_1 m_2)^p} + \frac{(m^{2p} m_1^{2p} + 1)\|f(0)\|^p}{m_1^p m_2^p (m^{2p} - 1)}\right]^{1/p} \tag{3.15}$$

holds for all $x \in X$, *where*

$$\|f(0)\| \leqslant \frac{\varphi_1(0,0)}{(m_1 m_2 + 1)(m-1)}.$$

Proof. Using the functional inequalities (2.9)–(2.11) and the property of quasi-norm $\|\cdot\|$, we have the basic inequality

$$\begin{aligned}
\left\|f(x) - \frac{f(mx)}{m^2}\right\|^p &\leqslant \|f(x) - \overline{\overline{f}}(x)\|^p + \|\overline{\overline{f}}(x) - \overline{f}(x)\|^p + \left\|\overline{f}(x) - \frac{f(mx)}{m^2}\right\|^p \\
&\leqslant \left(\frac{\varphi_1(x,0)}{(m_0 m m_2)} + \frac{m_1\|f(0)\|}{m_2}\right)^p + \frac{\psi_1(x)^p}{(m_0 m m_1 m_2)^p} \\
&\quad + \frac{1}{(m^2 m_1 m_2)^p}\left(\varphi_1\left(\frac{m_1 a_1}{m_0}x, \frac{m_2 a_2}{m_0}x\right) + \|f(0)\|\right)^p, \\
&\leqslant \frac{\varphi_1(x,0)^p}{(m_0 m m_2)^p} + \frac{1}{(m^2 m_1 m_2)^p}\varphi_1\left(\frac{m_1 a_1}{m_0}x, \frac{m_2 a_2}{m_0}x\right)^p \\
&\quad + \frac{\psi_1(x)^p}{(m_0 m m_1 m_2)^p} + \frac{(m^{2p} {m_1}^{2p} + 1)\|f(0)\|^p}{m^{2p} m_1^p m_2^p} \quad \forall x \in X.
\end{aligned} \tag{3.16}$$

Thus one obtains from the last inequality that for all nonnegative integers n, l with $n > l$

$$\begin{aligned}
&\left\|\frac{f(m^l x)}{m^{2l}} - \frac{f(m^n x)}{m^{2n}}\right\|^p \\
&\leqslant \sum_{j=l}^{n-1}\left\|\frac{f(m^j x)}{m^{2j}} - \frac{f(m^{j+1} x)}{m^{2(j+1)}}\right\|^p \\
&\leqslant \sum_{j=l}^{n-1}\left[\frac{\varphi_1(m^j x, 0)^p}{(m_0 m m_2)^p m^{2jp}} + \frac{1}{(m^2 m_1 m_2)^p m^{2jp}}\varphi_1\left(\frac{m_1 a_1}{m_0}m^j x, \frac{m_2 a_2}{m_0}m^j x\right)^p \right.\\
&\quad \left. + \frac{\psi_1(m^j x)^p}{(m_0 m m_1 m_2)^p m^{2jp}} + \frac{(m^{2p} {m_1}^{2p} + 1)\|f(0)\|^p}{m^{2p} m_1^p m_2^p m^{2jp}}\right],
\end{aligned} \tag{3.17}$$

which tends to zero by (3.14) as $l \to \infty$. Therefore a mapping $Q : X \to Y$ given by

$$Q(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}}, \quad x \in X,$$

is well defined. Letting $l := 0$ and then passing the limit $n \to \infty$ in (3.17), we have the inequality (3.15). To prove the uniqueness, let Q' be another mapping satisfying (3.15). Then we get by Lemma 2.1 that $Q'(m^n x) = m^{2n} Q'(x)$ for all $x \in X$ and all $n \in \mathbb{N}$. Thus we have

$$\begin{aligned}
&\|Q(x) - Q'(x)\|^p \\
&\leqslant \frac{1}{m^{2np}}\left\{\left\|Q\left(m^n x\right) - f\left(m^n x\right)\right\|^p + \left\|f\left(m^n x\right) - Q'\left(m^n x\right)\right\|^p\right\} \\
&\leqslant 2\sum_{j=n}^{\infty}\left[\frac{\varphi_1(m^j x, 0)^p}{(m_0 m m_2)^p m^{2jp}} + \frac{1}{(m^2 m_1 m_2)^p m^{2jp}}\varphi_1\left(\frac{m_1 a_1}{m_0} m^j x, \frac{m_2 a_2}{m_0} m^j x\right)^p \right. \\
&\quad \left. + \frac{\psi_1(m^j x)^p}{(m_0 m m_1 m_2)^p m^{2jp}} + \frac{(m^{2p} {m_1}^{2p} + 1)\|f(0)\|^p}{m^{2p} m_1^p m_2^p m^{2jp}}\right]
\end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, then we conclude that $Q(x) = Q'(x)$ for all $x \in X$.

The rest of the proof of this theorem is omitted as similar to the corresponding that of Theorem 2.2. □

Theorem 3.6. *Let X be a quasi-normed space and Y a p-Banach space. Assume that $f : X \to Y$ is a mapping for which there exist mappings $\varphi_2 : X^2 \to [0, \infty)$ and $\psi_2 : X \to [0, \infty)$ such that the Euler–Lagrange functional inequality*

$$\left\|D_{m_1,m_2}^{a_1,a_2} f(x, y)\right\| \leqslant \varphi_2(x, y)$$

holds for all $x, y \in X$ and

$$\left\|m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_2 f\left(\frac{m_1}{m_0} a_1 x\right) - m_0^2 m_1 f\left(\frac{m_2}{m_0} a_2 x\right)\right\| \leqslant \psi_2(x)$$

for all $x \in X$. Suppose that

$$0 < m := \frac{(m_1 + m_2)(m_1 a_1^2 + m_2 a_2^2)}{m_1 m_2 + 1} < 1$$

and the series

$$\Phi_2(x, y) := \sum_{i=1}^{\infty} m^{2pi} \varphi_2\left(\frac{x}{m^i}, \frac{y}{m^i}\right)^p, \qquad \Psi_2(x) := \sum_{i=1}^{\infty} m^{2pi} \psi_2\left(\frac{x}{m^i}\right)^p \tag{3.18}$$

converge for all $x, y \in X$.

Then the limit

$$Q(x) = \lim_{n\to\infty} m^{2n} f\left(\frac{x}{m^n}\right)$$

exists for all $x \in X$, and $Q : X \to Y$ is the unique generalized Euler–Lagrange quadratic mapping satisfying Eq. (1.4)*, that is, $D_{m_1,m_2}^{a_1,a_2} Q(x, y) = 0$, the fundamental functional equation* (2.1) *and mean functional equation* (2.2) *such that*

$$\begin{aligned}
\|f(x) - Q(x)\| \leqslant \Bigg[&\frac{\Phi_2(x, 0)}{(m_0 m m_2)^p} + \frac{1}{(m^2 m_1 m_2)^p}\Phi_2\left(\frac{m_1 a_1}{m_0} x, \frac{m_2 a_2}{m_0} x\right) + \frac{\Psi_2(x)}{(m_0 m m_1 m_2)^p} \\
&+ \frac{(m^{2p} m_1^{2p} + 1)\|f(0)\|^p}{m_1^p m_2^p (1 - m^{2p})}\Bigg]^{1/p}
\end{aligned} \tag{3.19}$$

holds for all $x \in X$, where

$$\|f(0)\| \leqslant \frac{\varphi_2(0,0)}{(m_1m_2+1)(1-m)}.$$

Proof. Now by the similar argument to the functional inequalities (3.16) and (3.17), we get

$$\begin{aligned}
&\left\|m^{2l}f\left(\frac{x}{m^l}\right)-m^{2n}f\left(\frac{x}{m^n}\right)\right\|^p \\
&\leqslant \sum_{j=l+1}^{n}\left\|m^{2j}f\left(\frac{x}{m^j}\right)-m^{2(j+1)}f\left(\frac{x}{m^{j+1}}\right)\right\|^p \\
&\leqslant \sum_{j=l+1}^{n}\left[\frac{m^{2pj}}{(m_0mm_2)^p}\varphi_2\left(\frac{x}{m^j},0\right)^p+\frac{m^{2pj}}{(m^2m_1m_2)^p}\varphi_2\left(\frac{m_1a_1}{m_0}\frac{x}{m^j},\frac{m_2a_2}{m_0}\frac{x}{m^j}\right)^p\right. \\
&\quad\left.+\frac{m^{2pj}}{(m_0mm_1m_2)^p}\psi_2\left(\frac{x}{m^j}\right)^p+\frac{m^{2pj}(m^{2p}{m_1}^{2p}+1)\|f(0)\|^p}{m^{2p}m_1^pm_2^p}\right]
\end{aligned}\tag{3.20}$$

for all $x \in X$ and all integers l, n with $n > l \geqslant 0$.

It follows from (3.18) and (3.20) that a sequence $\{m^{2n}f(\frac{x}{m^n})\}$ is Cauchy sequence for all $x \in X$. Since Y is complete, we may define a mapping $Q : X \to Y$ by

$$Q(x)=\lim_{m\to\infty}m^{2n}f\left(\frac{x}{m^n}\right)$$

for all $x \in X$. Letting $l = 0$ and taking the limit as $n \to \infty$ in (3.20), one has the inequality (3.19).

The rest of the proof goes through by the same way as that of Theorem 3.5. This completes the proof. □

Remark 3.7. The result for the case $K = 1$ in Theorem 3.2 (Theorem 3.3) is the same as the result for the case $p = 1$ in Theorem 3.5 (Theorem 3.6).

References

[1] Y. Benyamini, J. Lindenstrauss, Geometric Nonlinear Functional Analysis, vol. 1, Colloq. Publ., vol. 48, Amer. Math. Soc., Providence, RI, 2000.

[2] D.G. Bourgin, Classes of transformations and bordering transformations, Bull. Amer. Math. Soc. 57 (1951) 223–237.

[3] S. Czerwik, The stability of the quadratic functional equation, in: T.M. Rassias, J. Tabor (Eds.), Stability of Mappings of Hyers–Ulam Type, Hadronic Press, Florida, 1994, pp. 81–91.

[4] P.M. Gruber, Stability of isometries, Trans. Amer. Math. Soc. 245 (1978) 263–277.

[5] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA 27 (1941) 222–224.

[6] D.H. Hyers, G. Isac, T.M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Basel, 1998.

[7] K.W. Jun, H.M. Kim, Ulam stability problem for quadratic mappings of Euler–Lagrange, Nonlinear Anal. 61 (2005) 1093–1104.

[8] S.M. Jung, On the Hyers–Ulam stability of the functional equations that have the quadratic property, J. Math. Anal. Appl. 222 (1998) 126–137.

[9] J.M. Rassias, On approximation of approximately linear mappings by linear mappings, J. Funct. Anal. 46 (1982) 126–130.

[10] J.M. Rassias, Solution of a problem of Ulam, J. Approx. Theory 57 (1989) 268–273.

[11] J.M. Rassias, On the stability of the Euler–Lagrange functional equation, Chinese J. Math. 20 (1992) 185–190.

[12] J.M. Rassias, On the stability of the non-linear Euler–Lagrange functional equation in real normed linear spaces, J. Math. Phys. Sci. 28 (1994) 231–235.
[13] J.M. Rassias, On the stability of the general Euler–Lagrange functional equation, Demonstratio Math. 29 (1996) 755–766.
[14] J.M. Rassias, Solution of the Ulam stability problem for Euler–Lagrange quadratic mappings, J. Math. Anal. Appl. 220 (1998) 613–639.
[15] J.M. Rassias, On the stability of the multi-dimensional Euler–Lagrange functional equation, J. Indian Math. Soc. (N.S.) 66 (1999) 1–9.
[16] J.M. Rassias, Solution of the Ulam stability problem for an Euler type quadratic functional equation, Southeast Asian Bull. Math. 26 (2002) 101–112.
[17] M.J. Rassias, J.M. Rassias, On the Ulam stability for Euler–Lagrange type quadratic functional equations, Austral. J. Math. Anal. Appl. 2 (2005) 1–10.
[18] T.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300.
[19] T.M. Rassias, On the stability of functional equations in Banach spaces, J. Math. Anal. Appl. 251 (2000) 264–284.
[20] S. Rolewicz, Metric Linear Spaces, Reidel, Dordrecht, 1984.
[21] S.M. Ulam, A Collection of Mathematical Problems, Interscience, New York, 1960.
[22] Ding-Xuan Zhou, On a conjecture of Z. Ditzian, J. Approx. Theory 69 (1992) 167–172.