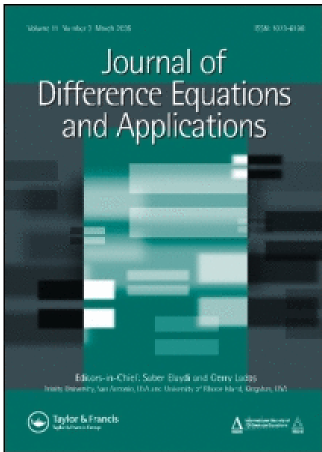


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# Extended Hyers–Ulam stability for Cauchy–Jensen mappings†

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In 1940, Ulam proposed the famous Ulam stability problem. In 1941, Hyers solved the well-known Ulam stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. In 2003–2006, the last author of this paper investigated the Hyers–Ulam stability of additive and Jensen type mappings. In this paper, we improve results obtained in 2003 and 2005 for Jensen type mappings and establish new theorems about the Ulam stability of additive and alternative additive mappings. These stability results can be applied in stochastic analysis, financial and actuarial mathematics, as well as in psychology and sociology.

*Keywords*: Ulam stability problem; Jensen equation; Euler–Lagrange mappings; Jensen type mappings

*2000 Mathematics Subject Classification*: 39B82; 46L05; 39B52

## 1. Introduction

In 1940 and in 1964, Ulam [31] proposed the famous Ulam stability problem:

"When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

For very general functional equations, the concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

In 1941, Hyers [11] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. In 1951, Bourgin [3] was the second author to treat the Ulam stability problem for additive mappings. In 1978, Gruber [10] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types. We wish to note that stability properties of different functional equations can have applications to unrelated fields. For instance, Zhou [32] used a stability property of the functional equation $f(x-y)+f(x+y)=2f(x)$ to prove

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a conjecture of Z. Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the associated Bernstein polynomials. Rassias [28] and then Găvruta [8] obtained generalized results of Hyers' Theorem which allow the Cauchy difference to be unbounded. In 1987, Gajda and Ger [7] showed that one can get analogous stability results for subadditive multifunctions. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found, for example, in the papers [1,4–6,12,13,29] and references therein. In 1982–2005, Rassias [17,18,20–24] established the Hyers–Ulam stability of linear and nonlinear mappings. In 2003–2006, J.M. Rassias and M.J. Rassias [25,26] and Rassias[27] solved the above Ulam problem for Jensen and Jensen type mappings. In 1999, Găvruta [9] answered a question of Rassias [19] concerning the stability of the Cauchy equation.

We note that J.M. Rassias [20, 22–23] introduced the Euler–Lagrange quadratic mappings, motivated from the following pertinent algebraic equation

$$|a_1x_1 + a_2x_2|^2 + |a_2x_1 - a_1x_2|^2 = \left(a_1^2 + a_2^2\right)\left[|x_1|^2 + |x_2|^2\right]$$

Thus, the third author of this paper introduced and investigated the stability problem of Ulam for the relative Euler–Lagrange functional equation

$$f(a_1x_1 + a_2x_2) + f(a_2x_1 - a_1x_2) = \left(a_1^2 + a_2^2\right)[f(x_1) + f(x_2)]. \tag{1.1}$$

in the publications [20–22]. Analogous quadratic mappings were introduced and investigated through J.M. Rassias' publications [23,26]. Before 1992, these mappings and equations were not known at all in functional equations and inequalities. However, a completely different kind of Euler–Lagrange partial differential equations is known in calculus of variations. In this paper, we introduce Cauchy and Cauchy–Jensen mappings of Euler–Lagrange and thus generalize Ulam stability results controlled by more general mappings, by considering approximately Cauchy and Cauchy–Jensen mappings of Euler–Lagrange satisfying conditions much weaker than Hyers and J.M. Rassias conditions on approximately Cauchy and Cauchy–Jensen mappings of Euler–Lagrange. These stability results can be applied in stochastic analysis [16], financial and actuarial mathematics, as well as in psychology and sociology.

Throughout this paper, let $X$ be a real normed space and $Y$ a real Banach space in the case of functional inequalities, as well as let $X$ and $Y$ be real linear spaces for functional equations. Besides let us denote by $\mathbb{N}$ the set of all natural numbers and $\mathbb{R}$ the set of all real numbers.

DEFINITION 1.1. *A mapping $f: X \to Y$ is called additive iff $f$ satisfies the functional equation*

$$A(x + y) = A(x) + A(y) \tag{1.2}$$

*for all $x, y \in X$. We note that the equation* (1.2) *is equivalent to the Jensen equation*

$$2A\left(\frac{x+y}{2}\right) = A(x) + A(y)$$

*for all $x, y \in X$ and $A(0) = 0$.*

Now, we consider a mapping $A : X \longrightarrow Y$, which may be analogously called Euler–Lagrange additive, satisfying the functional equation

$$A(ax + by) + A(bx + ay) = (a + b)[A(x) + A(y)] \tag{1.3}$$

for all $x$, y $\in X$, where $a, b \in \mathbb{R}$ are nonzero fixed reals with $a + b \neq 0$. Next, we consider a mapping $A$: $X \longrightarrow Y$ of Euler–Lagrange satisfying the functional equation

$$A(ax + by) + A(ax - by) = 2aA(x) \tag{1.4}$$

which is equivalent to the equation of Jensen type

$$A(x) + A(y) = 2aA\left(\frac{x + y}{2a}\right)$$

for all $x$, $y \in X$, where $a, b \in \mathbb{R}$ are nonzero fixed reals. It is easy to see that the equation (1.4) together with $A(0) = 0$ is equivalent to the equation (1.2). Alternatively, we investigate the functional equation of Euler–Lagrange

$$A(ax + by) - A(ax - by) = 2bA(y) \tag{1.5}$$

for all $x$, $y \in X$. We note that the equation (1.5) is equivalent to the equation

$$A(x) - A(y) = 2bA\left(\frac{x - y}{2b}\right) \tag{1.6}$$

for all $x$, $y \in X$, where $a$, $b \in \mathbb{R}$ are nonzero fixed reals. It follows that the equation (1.5) is equivalent to the equation (1.2).

## 2. Stability of Euler–Lagrange additive mappings

We will investigate under what conditions it is then possible to find a true Euler–Lagrange additive mapping near an approximate Euler–Lagrange additive mapping with small error. We note that if $\lambda = 1$ in the next two theorems, then the mapping $\varphi$ is identically zero by the convergence of series and thus $f$ is itself the solution of the equation (1.3). Thus, we may assume without loss of generality that $\lambda \neq 1$ in these theorems.

THEOREM 2.1. *Assume that there exists a mapping $\varphi : X^2 \longrightarrow [0, \infty)$ for which a mapping $f$: $X \longrightarrow Y$ satisfies the inequality*

$$\|f(ax + by) + f(bx + ay) - (a + b)[f(x) + f(y)]\| \leq \varphi(x, y) \tag{2.1}$$

*and the series*

$$\sum_{i=0}^{\infty} \frac{\varphi(\lambda^i x, \lambda^i y)}{|\lambda|^i} < \infty \tag{2.2}$$

*for all $x$, $y \in X$, where $\lambda := a + b$. Then, there exists a unique Euler–Lagrange additive mapping $A$: $X \longrightarrow Y$ which satisfies the equation* (1.3) *and the inequality*

$$\|f(x) - A(x)\| \leq \frac{1}{2|\lambda|} \sum_{i=0}^{\infty} \frac{\varphi(\lambda^i x, \lambda^i x)}{|\lambda|^i} \tag{2.3}$$

*for all $x \in X$.*

*Proof.* Substituting $x$, $y := x$ in the functional inequality (2.1), we obtain

$$2\|f(\lambda x) - \lambda f(x)\| \le \varphi(x, x), \quad \left\| f(x) - \frac{f(\lambda x)}{\lambda} \right\| \le \frac{1}{2|\lambda|}\varphi(x, x) \tag{2.4}$$

for all $x \in X$. Therefore from (2.4) with $\lambda^{i}x$ in place of $x$ ($i = 1, \ldots, n-1$) and iterative method, one gets

$$\left\| f(x) - \frac{f(\lambda^n x)}{\lambda^n} \right\| \le \frac{1}{2|\lambda|}\sum_{i=0}^{n-1} \frac{\varphi(\lambda^i x, \lambda^i x)}{|\lambda|^i} \tag{2.5}$$

for all $x \in X$ and all $n \in \mathbb{N}$. By (2.5), for any $n > m \ge 0$ we have

$$\left\| \frac{f(\lambda^m x)}{\lambda^m} - \frac{f(\lambda^n x)}{\lambda^n} \right\| \le \frac{1}{|\lambda|^m} \left\| f(\lambda^m x) - \frac{f(\lambda^{n-m}\lambda^m x)}{\lambda^{n-m}} \right\| \quad \le \frac{1}{2|\lambda|}\sum_{i=0}^{n-m-1} \frac{\varphi(\lambda^{i+m}x, \lambda^{i+m}x)}{|\lambda|^{i+m}},$$

which tends to zero as $m$ tends to infinity. Thus, it follows that a sequence $\{(f(\lambda^n x))/(\lambda^n)\}$ is Cauchy in $Y$ and it thus converges. Therefore, we see that a mapping $A$: $X \to Y$ defined by

$$A(x) := \lim_{n\to\infty} \frac{f(\lambda^n x)}{\lambda^n} = \lim_{n\to\infty} \frac{f((a+b)^n x)}{(a+b)^n}$$

exists for all $x \in X$. In addition, it is clear from (2.1) that the following inequality

$$\begin{aligned}
&\|A(ax+by) + A(bx+ay) - (a+b)[A(x) + A(y)]\| \\
&\quad = \lim_{n\to\infty} |\lambda|^{-n} \|f(\lambda^n(ax+by)) + f(\lambda^n(bx+ay)) - (a+b)[f(\lambda^n x) + f(\lambda^n y)]\| \\
&\quad \le \lim_{n\to\infty} |\lambda|^{-n} \varphi(\lambda^n x, \lambda^n y) = 0
\end{aligned}$$

holds for all $x$, $y \in X$. Thus, taking the limit $n \to \infty$ in (2.5), we find that the mapping $A$ is Euler–Lagrange additive mapping satisfying the inequality (2.3) near the approximate mapping $f: X \to Y$ of the equation (1.3).

To prove the afore-mentioned uniqueness, we assume now that there is another Euler–Lagrange additive mapping $\check{A} : X \to Y$ which satisfies the equation (1.3) and the inequality (2.3). Then, it follows easily that by setting $y := x$ in (1.3) we get

$$A(x) = \lambda^{-n}A(\lambda^n x), \;\; \check{A}(x) = \lambda^{-n}\check{A}(\lambda^n x)$$

for all $x \in X$ and all $n \in \mathbb{N}$. Thus from the last equality and (2.3) one proves that

$$\begin{aligned}
\|A(x) - \check{A}(x)\| &= \frac{1}{|\lambda|^n} \|A(\lambda^n x) - \check{A}(\lambda^n x)\| \\
&\le \frac{1}{|\lambda|^n} (\|A(\lambda^n x) - f(\lambda^n x)\| + \|f(\lambda^n x) - \check{A}(\lambda^n x)\|) \\
&\le \sum_{i=0}^{\infty} \frac{\varphi(\lambda^{i+n}x, \lambda^{i+n}x)}{|\lambda|^{i+n+1}}
\end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Therefore from $n \to \infty$, one establishes

$$A(x) - \check{A}(x) = 0$$

for all $x \in X$, completing the proof of uniqueness. □

THEOREM 2.2. *Assume that there exists a mapping $\varphi : X^2 \to [0,\infty)$ for which a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(ax+by)+f(bx+ay)-(a+b)[f(x)+f(y)]\| \le \varphi(x,y)$$

*and the series*

$$\sum_{i=1}^{\infty} |\lambda|^i \varphi\Big(\frac{x}{\lambda^i},\frac{y}{\lambda^i}\Big) < \infty$$

*for all $x, y \in X$, where $\lambda := a+b$. Then there exists a unique Euler–Lagrange additive mapping $A : X \to Y$ which satisfies the equation* (1.3) *and the inequality*

$$\|f(x)-A(x)\| \le \frac{1}{2|\lambda|}\sum_{i=1}^{\infty} |\lambda|^i \varphi\Big(\frac{x}{\lambda^i},\frac{x}{\lambda^i}\Big)$$

*for all $x \in X$.*

We obtain the following corollary concerning the stability for approximately Euler–Lagrange additive mappings in terms of a product of powers of norms.

COROLLARY 2.3. *If a mapping $f : X \to Y$ satisfies the functional inequality*

$$\|f(ax+by)+f(bx+ay)-(a+b)[f(x)+f(y)]\| \le \delta\|x\|^{\alpha}\|y\|^{\beta},$$

*for all $x, y \in X$ ($X\backslash\{0\}$ if $\alpha, \beta \le 0$) and for some fixed $\alpha, \beta \in \mathbb{R}$, such that $\rho$: $\alpha+\beta \in \mathbb{R}$, $\rho \ne 1$, $\lambda := a+b \ne 1$ and $\delta \ge 0$, then there exists a unique Euler–Lagrange additive mapping $A : X \to Y$ which satisfies the equation* (1.3) *and the inequality*

$$\|f(x)-A(x)\| \le \begin{cases} \frac{\delta\|x\|^{\rho}}{2(|\lambda|-|\lambda|^{\rho})}, & \text{if } |\lambda|>1, \rho<1, (|\lambda|<1, \rho>1,) \\ \frac{\delta\|x\|^{\rho}}{2(|\lambda|^{\rho}-|\lambda|)}, & \text{if } |\lambda|>1, \rho>1, (|\lambda|<1, \rho<1) \end{cases}$$

*for all $x \in X$ ($X\backslash\{0\}$ if $\rho \le 0$). The mapping $A$ is defined by the formula*

$$A(x) = \begin{cases} \lim_{n\to\infty} \frac{f(\lambda^n x)}{\lambda^n}, & \text{if } |\lambda|>1, \rho<1, (|\lambda|<1, \rho>1,) \\ \lim_{n\to\infty} \lambda^n f\big(\frac{x}{\lambda^n}\big), & \text{if } |\lambda|>1, \rho>1, (|\lambda|<1, \rho<1) \end{cases}$$

Now we are going to investigate the stability problem of the Euler–Lagrange type equation (1.4). Similarly, we remark that if $\lambda = 1$ in the next two theorems, then the mapping $\varphi$ is identically zero by the convergence of series and thus $f$ is itself the solution of the equation (1.4). Thus, we may assume without loss of generality that $\lambda \ne 1$ in these theorems.

THEOREM 2.4. *Assume that there exists a mapping $\varphi : X^2 \to [0,\infty)$ for which a mapping $f$: $X \to Y$ satisfies the inequality*

$$\|f(ax+by)+f(ax-by)-2af(x)\| \le \varphi(x,y) \tag{2.6}$$

*and the series*

$$\sum_{i=0}^{\infty} \frac{\varphi(\lambda^i x, \lambda^i y)}{|\lambda|^i} < \infty \tag{2.7}$$

*for all x, y* $\in X$*, where* $\lambda := 2a \neq 1$. *Then, there exists a unique additive mapping* $A : X \to Y$ *of Euler–Lagrange which satisfies the equation* (1.4) *and the inequality*

$$\left\| f(x) - \frac{f(0)}{\lambda - 1} - A(x) \right\| \leq \frac{1}{|\lambda|} \sum_{i=0}^{\infty} \frac{\varphi\left(\lambda^i x, \frac{a}{b}\lambda^i x\right)}{|\lambda|^i}$$

*for all* $x \in X$. *If, moreover, f is measurable or f(tx) is continuous in t for each fixed* $x \in X$ *then* $A(tx) = tA(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

*Proof.* Substituting $(x, y)$ for $(x, (a/b)x)$ in the functional inequality (2.6), we obtain

$$\|f(2ax) + f(0) - 2af(x)\| \leq \varphi\left(x, \frac{a}{b}x\right), \quad \left\| g(x) - \frac{g(2ax)}{2a} \right\| \leq \frac{1}{2|a|}\varphi\left(x, \frac{a}{b}x\right), \tag{2.8}$$

for all $x \in X$, where $g(x) := f(x) - f(0)/(2a - 1)$. From (2.8), one gets

$$\left\| g(x) - \frac{g(\lambda^n x)}{\lambda^n} \right\| \leq \frac{1}{|\lambda|} \sum_{i=0}^{n-1} \frac{\varphi\left(\lambda^i x, \frac{a}{b}\lambda^i x\right)}{|\lambda|^i} \tag{2.9}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Thus, it follows from (2.7) that a sequence $\{\lambda^{-n} g(\lambda^n x)\}$ is Cauchy in $Y$ and it thus converges. Therefore, we see that a mapping $A$: $X \to Y$ defined by

$$A(x) := \lim_{n\to\infty} \frac{g(\lambda^n)}{\lambda^n} = \lim_{n\to\infty} \frac{f(\lambda^n)}{\lambda^n}$$

exists for all $x \in X$. We observe that $A(0) = 0$ since $f(0) = 0 = \varphi(0, 0)$ by the convergence of (2.7) if $\lambda = -1$ or $|\lambda| < 1$, and that $A(0) = 0$ by the definition of $A$ if $|\lambda| > 1$. Utilizing the last functional inequality (2.9) and the similar argument to Theorem 2.1, we can obtain the conclusion of this theorem.

The proof of the last assertion in our Theorem 2.4 is obvious according to J.M. Rassias work [17]. □

THEOREM 2.5. *Assume that there exists a mapping* $\varphi : X^2 \to [0, \infty)$ *for which a mapping* $f: X \to Y$ *satisfies the inequality*

$$\|f(ax + by) + f(ax - by) - 2af(x)\| \leq \varphi(x, y)$$

*and the series*

$$\sum_{i=1}^{\infty} |\lambda|^i \varphi\left(\frac{x}{\lambda^i}, \frac{y}{\lambda^i}\right) < \infty \tag{2.10}$$

*for all x, y* $\in X$*, where* $\lambda := 2a \neq 1$. *Then, there exists a unique additive mapping* $A{:}X \to Y$ *of Euler–Lagrange which satisfies the equation* (1.4) *and the inequality*

$$\left\| f(x) - \frac{f(0)}{\lambda - 1} - A(x) \right\| \leq \frac{1}{|\lambda|} \sum_{i=1}^{\infty} |\lambda|^i \varphi\left(\frac{x}{\lambda^i}, \frac{a}{b}\frac{x}{\lambda^i}\right)$$

*for all* $x \in X$. *If, moreover, f is measurable or f(tx) is continuous in t for each fixed* $x \in X$ *then* $A(tx) = tA(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

COROLLARY 2.6. *If a mapping* $f$*:* $X \to Y$ *satisfies the functional inequality*

$$\|f(ax + by) + f(ax - by) - 2af(x)\| \leq \delta\|x\|^{\alpha}\|y\|^{\beta},$$

*for all* $x,y \in X$ $(X \setminus \{0\}$ *if* $\alpha, \beta \leq 0)$ *and for some fixed* $\alpha, \beta \in \mathbb{R}$*, such that* $\rho := \alpha + \beta \in \mathbb{R}$*,* $\rho \neq 1$*,* $\lambda := 2a \neq 1$ *and* $\delta \geq 0$*, then there exists a unique additive mapping* $A: X \to Y$ *of Euler–Lagrange which satisfies the equation* (1.4) *and the inequality*

$$\left\| f(x) - \frac{f(0)}{2a-1} - A(x) \right\| \leq \begin{cases} \left|\frac{a}{b}\right|^{\beta} \frac{\delta \|x\|^{\rho}}{|\lambda| - |\lambda|^{\rho}}, & \text{if } |\lambda| > 1, \rho < 1, (|\lambda| < 1, \rho > 1,) \\ \left|\frac{a}{b}\right|^{\beta} \frac{\delta \|x\|^{\rho}}{|\lambda|^{\rho} - |\lambda|}, & \text{if } |\lambda| > 1, \rho > 1, (|\lambda| < 1, \rho < 1) \end{cases}$$

*for all* $x \in X(X \setminus \{0\}$ *if* $\rho \leq 0)$.

We will investigate under what conditions it is then possible to find a true additive Euler–Lagrange mapping of equation (1.5) near an approximate additive Euler–Lagrange mapping of equation (1.5) with small error.

THEOREM 2.7. *Assume that there exists a mapping* $\varphi : X^2 \to [0, \infty)$ *for which a mapping f:* $X \to Y$ *satisfies the inequality*

$$\| f(ax + by) - f(ax - by) - 2bf(y) \| \leq \varphi(x, y)$$

*and the series*

$$\sum_{i=0}^{\infty} \frac{\varphi(\lambda^i x, \lambda^i y)}{|\lambda|^i} < \infty \quad \left( \sum_{i=1}^{\infty} |\lambda|^i \varphi(\lambda^{-i} x, \lambda^{-i} y) < \infty, \quad \text{respectively} \right)$$

*for all* $x, y \in X$*, where* $\lambda := 2b \neq 1$*. Then, there exists a unique additive mapping* $A: X \to Y$ *of Euler–Lagrange which satisfies the equation* (1.5) *and the inequality*

$$\left\| f(x) + \frac{f(0)}{\lambda - 1} - A(x) \right\| \leq \frac{1}{|\lambda|} \sum_{i=0}^{\infty} \frac{\varphi\left(\frac{b}{a} \lambda^i x, \lambda^i x\right)}{|\lambda|^i}$$

$$\left( \left\| f(x) + \frac{f(0)}{\lambda - 1} - A(x) \right\| \leq \frac{1}{|\lambda|} \sum_{i=1}^{\infty} |\lambda|^i \varphi\left( \frac{b}{a} \lambda^{-i} x, \lambda^{-i} x \right), \quad \text{respectively} \right)$$

*for all* $x \in X$*. If, moreover, f is measurable or f(tx) is continuous in t for each fixed* $x \in X$ *then* $A(tx) = tA(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

*Proof.* The proof of this theorem is similar to that of Theorem 2.1 and Theorem 2.4. □

COROLLARY 2.8. *If a mapping f:* $X \to Y$ *satisfies the functional inequality*

$$\| f(ax + by) - f(ax - by) - 2bf(y) \| \leq \delta \|x\|^{\alpha} \|y\|^{\beta},$$

*for all* $x, y \in X$ $(X \setminus \{0\}$ *if* $\alpha, \beta \leq 0)$ *and for some fixed* $\alpha, \beta \in \mathbb{R}$*, such that* $\rho := \alpha + \beta \in \mathbb{R}$*,* $\rho \neq 1$*,* $\lambda := 2b \neq 1$ *and* $\delta \geq 0$*, then there exists a unique additive mapping* $A: X \to Y$ *of Euler–Lagrange which satisfies the equation* (1.5) *and the inequality*

$$\left\| f(x) + \frac{f(0)}{\lambda - 1} - A(x) \right\| \leq \begin{cases} \left|\frac{b}{a}\right|^{\alpha} \frac{\delta \|x\|^{\rho}}{|\lambda| - |\lambda|^{\rho}}, & \text{if } |\lambda| > 1, \rho < 1, (|\lambda| < 1, \rho > 1,) \\ \left|\frac{b}{a}\right|^{\alpha} \frac{\delta \|x\|^{\rho}}{|\lambda|^{\rho} - |\lambda|}, & \text{if } |\lambda| > 1, \rho > 1, (|\lambda| < 1, \rho < 1) \end{cases}$$

*for all* $x \in X(X \setminus \{0\}$ *if* $\rho \leq 0)$.

## 3. $C^*$-algebra isomorphisms between unital $C^*$-algebras

Throughout this section, assume that $\mathcal{A}$ and $\mathcal{B}$ are unital $C*$-algebras. Let $U(\mathcal{A})$ the unitary group of $\mathcal{A}$, $\mathcal{A}_{\text{in}}$ the set of invertible elements in $\mathcal{A}$, $\mathcal{A}_{\text{sa}}$ the set of self-adjoint elements in $\mathcal{A}$, $\mathcal{A}_1 := \{a \in \mathcal{A} : |a| = 1\}$, $\mathcal{A}^+$ the set of positive elements in $\mathcal{A}$. As an application, we are going to investigate $C^*$-algebra isomorphisms between unital $C*$-algebras. We denote $\mathbb{N}_0$ by the set of nonnegative integers.

THEOREM 3.1. *Let $h:\mathcal{A} \to \mathcal{B}$ be a bijective mapping with $h(0) = 0$ for which there exist mappings $\varphi : \mathcal{A}^2 \to R^+ := [0, \infty)$ satisfying* (2.7), *$\psi_1 : \mathcal{A} \times \mathcal{A} \to R^+$, and $\psi : \mathcal{A} \to R^+$ such that*

$$\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) - 2a\mu h(x)\| \leq \varphi(x, y), \tag{3.1}$$

$$\|h(\lambda^n ux) - h(\lambda^n u)h(x)\| \leq \psi_1(\lambda^n u, x), \tag{3.2}$$

$$\|h(\lambda^n u^*) - h(\lambda^n u)^*\| \leq \psi(\lambda^n u) \tag{3.3}$$

*for all $\mu \in S^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$, all $u \in U(\mathcal{A})$, all $x,y \in \mathcal{A}$ and all $n \in \mathbb{N}_0$, where $\lambda := 2a \neq 1$. Assume that*

$$\lim_{n\to\infty} \lambda^{-n}\psi_1(\lambda^n u, x) = 0, \text{ for all } u \in U(\mathcal{A}), \ x \in \mathcal{A}, \tag{3.4}$$

$$\lim_{n\to\infty} \lambda^{-n}\psi(\lambda^n u) = 0, \text{ for all } u \in U(\mathcal{A}), \tag{3.5}$$

$$\lim_{n\to\infty} \lambda^{-n}h(\lambda^n u_0) \in \mathcal{B}_{\text{in}}, \text{ for some } u_0 \in \mathcal{A}. \tag{3.6}$$

*Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is in fact a $C^*$-algebra isomorphism.*

*Proof.* Consider the $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ as Banach left modules over the unital $C^*$-algebra $\mathbb{C}$. From (2.7), (3.1) with $\mu = 1$ and by Theorem 2.4, there exists a unique mapping $H : \mathcal{A} \to \mathcal{B}$, defined by $H(x) := \lim_{n\to\infty}\lambda^{-n}h(\lambda^n x)$, satisfying the equation (1.4) and the inequality

$$\|h(x) - H(x)\| \leq \frac{1}{|\lambda|}\sum_{i=0}^{\infty} \frac{\varphi\left(\lambda^i x, \frac{a}{b}\lambda^i x\right)}{|\lambda|^i} \tag{3.7}$$

for all $x \in \mathcal{A}$. We claim that the mapping $H$ is $\mathbb{C}$-linear. For this, putting $y := 0$ in (1.4) one has $H(ax) = aH(x)$ for all $x \in \mathcal{A}$. Now replacing y by $(ay/b)$ in (1.4) we get $H(ax + ay) + H(ax - ay) = 2aH(x)$ and so $H(x + y) + H(x - y) = 2H(x)$, which means that $H$ is additive. On the other hand, we obtain by (2.7) and (3.1) that $H(a\mu x + b\mu y) + H(a\mu x - b\mu y) - 2a\mu H(x) = 0$ for all $x, y \in \mathcal{A}$ and so

$$H(\mu x) - \mu H(x) = 0 \tag{3.8}$$

for all $x \in \mathcal{A}$ and all $\mu \in S^1 = U(\mathbb{C})$. Now, let $\eta$ be a nonzero element in $\mathbb{C}$ and $K$ a positive integer greater than $4|\eta|$. Then, we have $|(\eta/K)| < (1/4) < 1 - (2/3)$. By Ref. [14, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in S^1$ such that

$3(\eta/K) = \mu_1 + \mu_2 + \mu_3$. Thus, we calculate by (3.8)

$$\begin{aligned} H(\eta x) &= H\left(\frac{K}{3}\cdot 3\frac{\eta}{K}x\right) = \left(\frac{K}{3}\right)H(\mu_1 x + \mu_2 x + \mu_3 x) \\ &= \left(\frac{K}{3}\right)(H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x)) \\ &= \left(\frac{K}{3}\right)(\mu_1 + \mu_2 + \mu_3)H(x) = \left(\frac{K}{3}\right)\cdot 3\frac{\eta}{K}g(x) = \eta H(x) \end{aligned}$$

for all $\eta \in \mathbb{C}(\eta \neq 0)$ and all $x \in \mathcal{A}$. So the unique mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear mapping, as desired.

By (3.3) and (3.5), we have

$$H(u^*) = \lim_{n\to\infty} \lambda^{-n} h(\lambda^n u^*) = \lim_{n\to\infty} \lambda^{-n} h(\lambda^n u)^* = \left(\lim_{n\to\infty} \lambda^{-n} h(\lambda^n u)\right)^* = H(u)^* \tag{3.9}$$

for all $u \in U(\mathcal{A})$. Since each $x \in \mathcal{A}$ is a finite linear combination of unitary elements ([15, Theorem 4.1.7]), i.e. $x = \sum_{j=1}^m c_j u_j$ ($c_j \in \mathbb{C}$, $u_j \in U(\mathcal{A})$), we get by (3.9)

$$\begin{aligned} H(x^*) &= H\left(\sum_{j=1}^m \bar{c}_j u_j^*\right) = \sum_{j=1}^m \bar{c}_j H(u_j^*) = \sum_{j=1}^m \bar{c}_j H(u_j)^* = \left(\sum_{j=1}^m c_j H(u_j)\right) \\ &= H\left(\sum_{j=1}^m c_j u_j\right) = H(x)^* \end{aligned}$$

for all $x \in \mathcal{A}$. So the mapping $H$ is preserved by involution.

Using the relations (3.2) and (3.4), we get

$$H(ux) = \lim_{n\to\infty} \lambda^{-n} h(\lambda^n ux) = \lim_{n\to\infty} \lambda^{-n} h(\lambda^n u)h(x) = H(u)h(x) \tag{3.10}$$

for all $u \in U(\mathcal{A})$ and all $x \in \mathcal{A}$. On the other hand, it follows from (3.10) and the additivity of $H$ that the equation

$$H(ux) = \lambda^{-n} H(\lambda^n ux) = \lambda^{-n} H(u\lambda^n x) = \lambda^{-n} H(u)h(\lambda^n x) = H(u)\lambda^{-n} h(\lambda^n x)$$

holds for all $u \in U(\mathcal{A})$ and all $x \in \mathcal{A}$. Taking the limit as $n \to \infty$ in the last equation, we obtain

$$H(ux) = H(u)H(x) \tag{3.11}$$

for all $u \in U(\mathcal{A})$ and all $x \in \mathcal{A}$. Now, let $z \in \mathcal{A}$ be an arbitrary element. Then $z = \sum_{j=1}^m c_j u_j$ ($c_j \in \mathbb{C}$, $u_j \in U(\mathcal{A})$), and it follows from (3.10) that

$$\begin{aligned} H(zx) &= H\left(\sum_{j=1}^m c_j u_j x\right) = \sum_{j=1}^m c_j H(u_j x) = \sum_{j=1}^m c_j H(u_j)h(x) \\ &= H\left(\sum_{j=1}^m c_j u_j\right)h(x) = H(z)h(x) \end{aligned} \tag{3.12}$$

for all $z, x \in \mathcal{A}$. Using the similar argument to (3.11), we see that

$$H(zx) = H(z)H(x) \tag{3.13}$$

for all $z, x \in \mathcal{A}$. Hence, the mapping $H$ is multiplicative.

Finally, it follows from (3.12) and (3.13) that

$$H(u_0)H(x) = H(u_0x) = H(u_0)h(x)$$

for all $x \in \mathcal{A}$. Since $H(u_0) = \lim_{n\to\infty}\lambda^{-n}h(\lambda^n u_0)$ is invertible by (3.6), we see that $H(x) = h(x)$ for all $x \in \mathcal{A}$. Hence, the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is in fact a $C$*-algebra isomorphism, as desired. □

THEOREM 3.2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and (3.6) for which there exist a mapping $\varphi : \mathcal{A}^2 \to R^+$ satisfying (2.7), and mappings $\psi_1, \psi$ such that*

$$\begin{aligned} \|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) - 2a\mu h(x)\| &\le \varphi(x, y), \\ \|h(\lambda^n ux) - h(\lambda^n u)h(x)\| &\le \psi_1(\lambda^n u, x), \end{aligned} \tag{3.14}$$

$$\|h(\lambda^n u^*) - h(\lambda^n u)^*\| \le \psi(\lambda^n u) \tag{3.15}$$

*for all $\mu \in S^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$, all $u \in \mathcal{A}_1^+ \cup \{i\}$ and all $x, y \in \mathcal{A}$ and all $n \in \mathbb{N}_0$. Assume that*

$$\lim_{n\to\infty}\lambda^{-n}\psi_1(\lambda^n u, x) = 0, \ \textit{for all} \ u \in \mathcal{A}_1^+ \cup \{i\}, \ \textit{all} \ x \in \mathcal{A}, \tag{3.16}$$

$$\lim_{n\to\infty}\lambda^{-n}\psi(\lambda^n u) = 0, \ \textit{for all} \ \ u \in \mathcal{A}_1^+ \cup \{i\}. \tag{3.17}$$

*Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is in fact a $C$*-algebra isomorphism.*

*Proof.* By the same reasoning as in the proof of Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$, defined by $H(x) := \lim_{n\to\infty}\lambda^{-n}h(\lambda^n x)$, satisfying the equation (1.4) and the functional inequality (3.7).

By (3.15) and (3.17), we have $H(u^*) = H(u)^*$ for all $u \in \mathcal{A}_1^+ \cup \{i\}$, and so

$$H(v^*) = H\left(|v|\cdot\frac{v^*}{|v|}\right) = |v|H\left(\frac{v^*}{|v|}\right) = \left[|v|H\left(\frac{v}{|v|}\right)\right]^* = H(v)^* \tag{3.18}$$

for all nonzero $v \in \mathcal{A}^+ \cup \{i\}$. Now, for any element $v \in A$, $v = v_1 + iv_2$, where $v_1, v_2 \in \mathcal{A}_{sa}$; furthermore, $v = v_1^+ - v_1^- + iv_2^+ - iv_2^-$, where $v_1^+, v_1^-, v_2^+$ and $v_2^-$ are all positive elements (see [2, Lemma 38.8]). Since $H$ is $\mathbb{C}$-linear, we figure out by (3.18)

$$\begin{aligned} H(v^*) &= H\left(\left(v_1^+ - v_1^- + iv_2^+ - iv_2^-\right)^*\right) \\ &= H\left(v_1^+\right)^* - H\left(v_1^-\right)^* + H\left(\left(iv_2^+\right)^*\right) - H\left(\left(iv_2^-\right)^*\right) \\ &= H\left(v_1^+\right)^* - H\left(v_1^-\right)^* - iH\left(v_2^+\right)^* + iH\left(v_2^-\right)^* \\ &= \left[H\left(v_1^+ - v_1^- + iv_2^+ - iv_2^-\right)\right]^* = H(v)^* \end{aligned}$$

for all $v \in \mathcal{A}$.

Using (3.14) and (3.16), we get $H(ux) = H(u)h(x)$ for all $u \in \mathcal{A}_1^+ \cup \{i\}$ and all $x \in \mathcal{A}$, and so $H(vx) = H(v)h(x)$ for all $v \in \mathcal{A}^+ \cup \{i\}$ and all $x \in \mathcal{A}$ because

$$\begin{aligned} H(vx) &= H\left(|v|\frac{v}{|v|}\cdot x\right) = |v|H\left(\frac{v}{|v|}\cdot x\right) \\ &= |v|H\left(\frac{v}{|v|}\right)\cdot h(x) = H(v)h(x), \quad \forall v \in \mathcal{A}^+. \end{aligned} \tag{3.19}$$

Now, for any element $v \in A$, $v = v_1^+ - v_1^- + iv_2^+ - iv_2^-$, where $v_1^+, v_1^-, v_2^+$ and $v_2^-$ are positive elements (see [2, Lemma 38.8]). Thus, we calculate by (3.19) and the additivity of $H$

$$\begin{aligned} H(vx) &= H\left(v_1^+x - v_1^-x + iv_2^+x - iv_2^-x\right) \\ &= H\left(v_1^+x\right) - H\left(v_1^-x\right) + iH\left(v_2^+x\right) - iH\left(v_2^-x\right) \\ &= \left(H\left(v_1^+\right) - H\left(v_1^-\right) + iH\left(v_2^+\right) - iH\left(v_2^-\right)\right)h(x) \\ &= H(v)h(x) \end{aligned} \tag{3.20}$$

for all $v, x \in \mathcal{A}$. By (3.20) and the additivity of $H$, one has

$$H(vx) = \lambda^{-n}H(\lambda^n vx) = \lambda^{-n}H(v\lambda^n x) \quad = \lambda^{-n}H(v)h(\lambda^n x) = H(v)\lambda^{-n}h(\lambda^n x),$$

which yields by taking the limit as $n \to \infty$

$$H(vx) = H(v)H(x) \tag{3.21}$$

for all $v, x \in \mathcal{A}$.

It follows from (3.20) and (3.21) that for a given $u_0$ subject to (3.6)

$$H(u_0)H(x) = H(u_0x) = H(u_0)h(x)$$

for all $x \in \mathcal{A}$. Since $H(u_0) = \lim_{n\to\infty}\lambda^{-n}h(\lambda^n u_0) \in \mathcal{B}_{\text{in}}$, we see that $H(x) = h(x)$ for all $x \in \mathcal{A}$. Hence, the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a $C$*-algebra isomorphism, as desired. □

THEOREM 3.3. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping with $h(0) = 0$ satisfying* (2.7)*,* (3.2) *and* (3.3) *such that*

$$\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) - 2a\mu h(x)\| \le \varphi(x, y) \tag{3.22}$$

*holds for $\mu = 1, i$. Assume that the conditions* (3.4)–(3.6) *are satisfied, and that $h$ is measurable or $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$. Then, the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C*-algebra isomorphism.*

*Proof.* Fix $\mu = 1$ in (3.22). By the same reasoning as in the proof of Theorem 3.1, there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying the equation (1.4) and the inequality (3.7).

By the assumption that $h$ is measurable or $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, the mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear, that is, $H(tx) = tH(x)$ for all $t \in \mathbb{R}$ and all $x \in \mathcal{A}$ [12,17,28]. Put $\lambda = i$ in (3.22). Then applying the same argument to (3.8) as in the proof of Theorem 3.1, we obtain that

$$H(ix) = iH(x),$$

and so for any $\mu = s + it \in \mathbb{C}$

$$H(\mu x) = H(sx + itx) = H(sx) + H(itx) = sH(x) + itH(x) \quad = (s + it)H(x) = \mu H(x)$$

for all $x \in \mathcal{A}$. Hence, the mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as the proof of Theorem 3.1. □

THEOREM 3.4. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping with $h(0) = 0$ satisfying* (2.7), (3.6), (3.14) *and* (3.15) *such that*

$$\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) - 2a\mu h(x)\| \leq \varphi(x, y) \tag{3.23}$$

*holds for $\mu = 1, i$. Assume that the equations* (3.16) *and* (3.17) *are satisfied, and that $h$ is measurable or $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$. Then, the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C*-algebra isomorphism.*

*Proof.* The proof is the similar to that of Theorem 3.3. □

## 4. Derivations mapping into the radicals of Banach algebras

Throughout this section, assume that $\mathcal{A}$ is a complex Banach algebra with norm $\|\cdot\|$. As an application, we are going to investigate the stability of derivations on Banach algebras and to consider the range of derivations on Banach algebras.

LEMMA 4.1. *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $h(0) = 0$ for which there exist a mapping $\varphi : \mathcal{A}^2 \to R^+$ satisfying* (2.7) *and a mapping $\psi : \mathcal{A}^2 \to R^+$ satisfying*

$$\lim_{n\to\infty} \frac{\psi(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}} = 0 \tag{4.1}$$

*for all $x, y \in X$, where $\lambda := 2a \neq 1$, such that*

$$\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) - 2a\mu h(x)\| \leq \varphi(x, y), \tag{4.2}$$

$$\|h(xy) - h(x)y - xh(y)\| \leq \psi(x, y) \tag{4.3}$$

*for all $\mu \in S^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$ and all $x, y \in \mathcal{A}$. Then, there exists a unique $\mathbb{C}$-linear derivation $H : \mathcal{A} \to \mathcal{A}$ which satisfies the inequality*

$$\|h(x) - H(x)\| \leq \frac{1}{|\lambda|} \sum_{i=0}^{\infty} \frac{\varphi\left(\lambda^i x, \frac{a}{b}\lambda^i x\right)}{|\lambda|^i} \tag{4.4}$$

*for all $x \in \mathcal{A}$.*

*Proof.* By the same reasoning as in the proof of Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{A}$, defined by $H(x) := \lim_{n\to\infty} \lambda^{-n} h(\lambda^n x)$, satisfying the equation (1.4) and the functional inequality (4.4).

Replacing $x$ and $y$ in (4.3) by $\lambda^n x$ and $\lambda^n y$, respectively, and dividing the result by $|\lambda|^{2n}$, we obtain

$$\left\| \frac{h(\lambda^{2n}xy)}{\lambda^{2n}} - \frac{h(\lambda^n x)}{\lambda^n} y - x \frac{h(\lambda^n y)}{\lambda^n} \right\| \le \frac{\psi(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}}$$

for all $x, y \in \mathcal{A}$. Taking the limit in the last inequality, one obtains that

$$H(xy) - H(x)y - xH(y) = 0$$

for all $x, y \in \mathcal{A}$ because $\lim_{n\to\infty}(\psi(\lambda^n x, \lambda^n y)/|\lambda|^{2n}) = 0$ and $\lim_{n\to\infty}(h(\lambda^{2n}xy)/\lambda^{2n}) = H(xy)$. Thus, the mapping $H : \mathcal{A} \to \mathcal{A}$ is a unique $\mathbb{C}$-linear derivation satisfying the functional inequality (4.4). □

LEMMA 4.2. *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $h(0) = 0$ for which there exist a mapping $\varphi : \mathcal{A}^2 \to R^+$ satisfying* (2.10) *and a mapping $\psi : \mathcal{A}^2 \to R^+$ satisfying*

$$\lim_{n\to\infty} |\lambda|^{2n} \psi\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right) = 0$$

*for all $x, y \in X$, where $\lambda := 2a \neq 1$, such that*

$$\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) - 2a\mu h(x)\| \le \varphi(x, y),$$

$$\|h(xy) - h(x)y - xh(y)\| \le \psi(x, y)$$

*for all $\mu \in S^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$ and all $x, y \in \mathcal{A}$. Then, there exists a unique $\mathbb{C}$-linear derivation $H : \mathcal{A} \to \mathcal{A}$ which satisfies the inequality*

$$\|h(x) - H(x)\| \le \frac{1}{|\lambda|} \sum_{i=1}^{\infty} |\lambda|^i \varphi\left(\frac{x}{\lambda^i}, \frac{a}{b}\frac{x}{\lambda^i}\right)$$

*for all $x \in \mathcal{A}$.*

COROLLARY 4.3. *Let $p, \alpha, \beta$ be reals such that either $p < 1$, $\alpha + \beta < 2$, $1 < |\lambda := 2a|$ or $p > 1$, $\alpha + \beta > 2$, $1 > |\lambda := 2a| > 0$. Assume that $h : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying $h(0) = 0$ for which there exist nonnegative constants $\varepsilon_1$, $\varepsilon_2$, such that*

$$\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) - 2a\mu h(x)\| \le \varepsilon_1(\|x\|^p + \|y\|^p),$$

$$\|h(xy) - h(x)y - xh(y)\| \le \varepsilon_2 \|x\|^\alpha \|y\|^\beta$$

*for all $\mu \in S^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$ and all $x, y \in \mathcal{A}$. Then, there exists a unique $\mathbb{C}$-linear derivation $H : \mathcal{A} \to \mathcal{A}$ which satisfies the inequality*

$$\|h(x) - H(x)\| \le \frac{\varepsilon_1 \|x\|^p \left(1 + \left|\frac{a}{b}\right|^p\right)}{\left|\, |\lambda| - |\lambda|^p \,\right|}$$

*for all $x \in \mathcal{A}$.*

LEMMA 4.4. *Let $h : \mathcal{A} \to \mathcal{A}$ be a linear mapping for which there exists a mapping $\psi : \mathcal{A}^2 \to R^+$ satisfying either*

$$\lim_{n\to\infty} \frac{\psi(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}} = 0 \quad or, \qquad \lim_{n\to\infty} |\lambda|^{2n} \psi\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right) = 0$$

*for all $x, y \in X$, where $\lambda := 2a \neq 0, 1$, such that*

$$\|h(xy) - h(x)y - xh(y)\| \leq \psi(x, y)$$

*for all $x, y \in \mathcal{A}$. Then, the mapping $h$ is in fact a derivation on $\mathcal{A}$.*

*Proof.* Taking $\varphi(x, y) := 0$ in the previous two lemmas, then we have the desired result. □

THEOREM 4.5. *Let $\mathcal{A}$ be a commutative Banach algebra. Let $h : \mathcal{A} \to \mathcal{A}$ be a given linear mapping and an approximate derivation with difference $Dh$ bounded by $\psi$, that is, there exists a mapping $\psi : \mathcal{A} \times \mathcal{A} \to R^+$ such that*

$$\|Dh(x, y) := h(xy) - h(x)y - xh(y)\| \leq \psi(x, y) \tag{4.5}$$

*for all $x, y \in \mathcal{A}$. Assume that there exists a nonzero real number $\lambda$ with $\lambda \neq 1$ such that the limit*

$$\lim_{n\to\infty} \frac{\psi(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}} = 0 \quad \left(\lim_{n\to\infty} |\lambda|^{2n} \psi\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right) = 0, respectively\right) \tag{4.6}$$

*for all $x, y \in \mathcal{A}$. Then, the mapping $h$ is in fact a linear derivation and maps the algebra into its radical.*

*Proof.* By Lemma 4.4, the mapping $h$ is in fact a linear derivation and maps the algebra into its radical by Thomas' result [30]. □

It is well-known that all linear derivations on commutative semi-simple Banach algebras are zero [30]. We remark that every linear mapping $h$ on a commutative semi-simple Banach algebra, which is an approximate derivation satisfying (4.5) and (4.6), is also zero.

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