Tatra Mt. Math. Publ. **32** (2006), 1–5

TATRA MOUNTAINS
Mathematical Publications

THE ULAM PROBLEM FOR 3-DIMENSIONAL QUADRATIC MAPPINGS

John Michael Rassias — Matina John Rassias

ABSTRACT. In 1940 and in 1964 S. M. Ulam proposed the general problem: When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true? In this paper we investigate the 3-dimensional quadratic mappings $Q\colon X \to Y$, satisfying the functional equation

$$Q(x_1 + x_2 + x_3) + Q(x_1 - x_2 + x_3) + Q(x_1 + x_2 - x_3) + Q(x_1 - x_2 - x_3) = 4[Q(x_1) + Q(x_2) + Q(x_3)]$$

and then solve the corresponding Ulam stability problem.

1. Two-dimensional mappings

S. M. Ulam ([10]) proposed the general problem: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" In this paper we investigate the Ulam stability problem for 3-dimensional mappings. In this section we consider the Ulam stability problem for 2-dimensional mappings.

Theorem 1.1. ([4]–[6]) *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that $f\colon X \to Y$ is a mapping for which there exists a constant $c \geqslant 0$ (independent of x_1, x_2) such that the quadratic functional inequality*

$$\left\| f(x_1 + x_2) + f(x_1 - x_2) - 2\left[f(x_1) + f(x_2)\right] \right\| \leqslant c$$

holds for all $(x_1, x_2) \in X^2$. Then the limit $Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x)$ exists for all $x \in X$ and all $n \in \mathbb{N} = \{1, 2, \dots\}$ and $Q\colon X \to Y$ is the unique quadratic mapping satisfying the functional equation

$$Q(x_1 + x_2) + Q(x_1 - x_2) = 2\left[Q(x_1) + Q(x_2)\right]$$

2000 Mathematics Subject Classification: Primary 39B; Secondary 26D.
Keywords: Ulam stability, quadratic mapping, functional equation.

for all $(x_1, x_2) \in X^2$, *such that*

$$\|f(x) - Q(x)\| \leq \frac{1}{2}c,$$

holds for all $x \in X$.

2. Three-dimensional mappings

In this section we establish the Ulam stability for 3-dimensional quadratic mappings.

Definition 2.1. Let X be a normed linear space and let Y be a real complete normed linear space. Then a mapping $Q\colon X \to Y$, is called *3-dimensional quadratic* if the functional equation

$$Q(x_1 + x_2 + x_3) + Q(x_1 - x_2 + x_3) + Q(x_1 + x_2 - x_3) + Q(x_1 - x_2 - x_3) = 4\big[Q(x_1) + Q(x_2) + Q(x_3)\big] \quad (2.1)$$

holds for all $(x_1, x_2, x_3) \in X^3$. Note that mapping Q is called *quadratic*, because the functional equation

$$Q(2^n x) = (2^n)^2 Q(x), \quad (2.2)$$

holds for all $x \in X$, and all $n \in \mathbb{N}$ ([4]–[6], [8], [9]).

In fact, substitution of $x_1 = x_2 = x_3 = 0$ in equation (2.1) yields that $Q(0) = 0$.

Substituting $x_1 = x_2 = x,\ x_3 = 0$ one gets that the functional equation

$$2Q(2x) + 2Q(0) = 4\big[2Q(x) + Q(0)\big], \quad \text{or} \quad Q(2x) = (2)^2 Q(x),$$

holds for all $x \in X$.

Then induction on $n \in \mathbb{N}$ with $x \to 2^{n-1}x$ yields equation (2.2).

Theorem 2.1. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that $f\colon X \to Y$ is a 3-dimensional mapping for which there exists a constant c (independent of x_1, x_2, x_3) $\geqslant 0$ such that the quadratic functional inequality*

$$\big\|f(x_1 + x_2 + x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 - x_2 - x_3) - 4\big[f(x_1) + f(x_2) + f(x_3)\big]\big\| \leqslant c \quad (2.3)$$

holds for all $(x_1, x_2, x_3) \in X^3$. Then the limit

$$Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x), \quad (2.4)$$

exists for all $x \in X$ *and all* $n \in \mathbb{N}$ *and* $Q\colon X \to Y$ *is the unique 3-dimensional quadratic mapping satisfying the functional equation* (2.1), *such that*

$$\|f(x) - Q(x)\| \leqslant \frac{5}{24}\,c\,. \tag{2.5}$$

holds for all $x \in X$. *Moreover, functional identity*

$$Q(x) = 2^{-2n} Q(2^n x)\,,$$

holds for all $x \in X$, *and all* $n \in \mathbb{N}$.

P r o o f. Substitution of $x_1 = x_2 = x_3 = 0$ in inequality (2.3) yields that

$$\|f(0)\| \leqslant \frac{c}{8}\,. \tag{2.6}$$

Moreover substituting $x_1 = x_2 = x,\ x_3 = 0$ in inequality (2.3) and employing (2.6) and the triangle inequality one concludes $\|2f(2x) + 2f(0) - 4[2f(x) + f(0)]\| \leqslant c$, or $\|f(2x) - 4f(x)f(0)\| \leqslant \frac{c}{2}$, or $\|f(2x) - 4f(x)\| \leqslant \|f(2x) - 4f(x) - f(0)\| + \|f(0)\|$, or $\|f(2x) - 4f(x)\| \leqslant \frac{c}{2} + \frac{c}{8} = \frac{5}{8}c$, or thus one gets that *the basic functional inequality*

$$\|f(x) - 2^{-2} f(2x)\| \leqslant \frac{5}{32}\,c = c_1(1 - 2^{-2})\,. \tag{2.7}$$

holds for all $x \in X$, where $c_1 = (5/24)c$. Replacing now x with $2x$ in (2.7) one concludes that $\|f(2x) - 2^{-2} f(2^2 x)\| \leqslant c_1(1 - 2^{-2})$, or

$$\|2^{-2} f(2x) - 2^{-4} f(2^2 x)\| \leqslant c_1(2^{-2} - 2^{-4}) \tag{2.7a}$$

holds for all $x \in X$. Functional inequalities (2.7)–(2.7a) and the triangle inequality yield

$$\begin{aligned}\|f(x) - 2^{-4} f(2^2 x)\| &\leqslant \|f(x) - 2^{-2} f(2x)\| + \|2^{-2} f(2x) - 2^{-4} f(2^2 x)\| \\ &\leqslant c_1\left[(1 - 2^{-2}) + (2^{-2} - 2^{-4})\right],\end{aligned}$$

or that the functional inequality

$$\|f(x) - 2^{-4} f(2^2 x)\| \leqslant c_1(1 - 2^{-4})\,,$$

holds for all $x \in X$.

Similarly by induction on $n \in \mathbb{N}$ with $x \to 2^{n-1}x$ in the basic inequality (2.7) claim that the general functional inequality

$$\|f(x) - 2^{-2n} f(2^n x)\| \leqslant c_1(1 - 2^{-2n})\,, \tag{2.8}$$

holds for all $x \in X$ and all $n \in \mathbb{N}$. In fact, the basic inequality (2.7) with $x \to 2^{n-1}x$ yield the functional inequality $\|f(2^{n-1}x) - 2^{-2} f(2^n x)\| \leqslant c_1(1 - 2^{-2})$, or the inequality

$$\|2^{-2(n-1)} f(2^{n-1}x) - 2^{-2n} f(2^n x)\| \leqslant c_1(2^{-2(n-1)} - 2^{-2n})\,, \tag{2.8a}$$

holds for all $x \in X$. Moreover, by induction hypothesis with $n \to n-1$ in the general inequality (2.8) one gets that

$$\left\| f(x) - 2^{-2(n-1)} f(2^{n-1}x) \right\| \leqslant c_1 \left(1 - 2^{-2(n-1)}\right), \tag{2.8b}$$

holds for all $x \in X$. Thus functional inequalities (2.8a)–(2.8b) and the triangle inequality imply

$$\left\| f(x) - 2^{-2n} f(2^n x) \right\| \leqslant \left\| f(x) - 2^{-2(n-1)} f(2^{n-1}x) \right\| + \left\| 2^{-2(n-1)} f(2^{n-1}x) - 2^{-2n} f(2^n x) \right\|$$

or

$$\left\| f(x) - 2^{-2n} f(2^n x) \right\| \leqslant c_1 \left[\left(1 - 2^{-2(n-1)}\right) + \left(2^{-2(n-1)} - 2^{-2n}\right) \right] = c_1 \left(1 - 2^{-2n}\right),$$

completing the proof of the required general functional inequality (2.8).

Claim now that the sequence $\left\{ 2^{-2n} f(2^n x) \right\}$ converges. Note that from the general inequality (2.8) and the completeness of Y, one proves that the above sequence is a *Cauchy sequence*. In fact, if $i > j > 0$, then

$$\left\| 2^{-2i} f(2^i x) - 2^{-2j} f(2^j x) \right\| = 2^{-2j} \left\| 2^{-2(i-j)} f(2^i x) - f(2^j x) \right\|,$$

holds for all $x \in X$, and all $i, j \in \mathbb{N}$. Setting $h = 2^j x$ in the above relation and employing the general inequality (2.8) one concludes that

$$\left\| 2^{-2i} f(2^i x) - 2^{-2j} f(2^j x) \right\| = 2^{-2j} \left\| 2^{-2(i-j)} f(2^{i-j} h) - f(h) \right\| \leqslant 2^{-2j} c_1 \left(1 - 2^{-2(i-j)}\right),$$

or

$$\left\| 2^{-2i} f(2^i x) - 2^{-2j} f(2^j x) \right\| \leqslant c_1 \left(2^{-2j} - 2^{-2i}\right) < c_1 2^{-2j}$$

or

$$\lim_{j \to \infty} \left\| 2^{-2i} f(2^i x) - 2^{-2j} f(2^j x) \right\| = 0,$$

completing the proof that the sequence $\left\{ 2^{-2n} f(2^n x) \right\}$ converges. Hence $Q = Q(x)$ is a *well-defined mapping* via the formula (2.4). This means that the limit (2.4) exists for all $x \in X$. In addition claim that mapping Q *satisfies the functional equation* (2.1) for all $(x_1, x_2, x_3) \in X^3$. In fact, it is clear from the functional inequality (2.3) and the limit (2.4) that the following inequality

$$2^{-2n} \left\| f(2^n x_1 + 2^n x_2 + 2^n x_3) + f(2^n x_1 - 2^n x_2 + 2^n x_3) + f(2^n x_1 + 2^n x_2 - 2^n x_3) + f(2^n x_1 - 2^n x_2 - 2^n x_3) - 4\left[f(2^n x_1) + f(2^n x_2) + f(2^n x_3) \right] \right\| \leqslant 2^{-2n} c,$$

holds for all $(x_1, x_2, x_3) \in X^3$, and all $n \in \mathbb{N}$. Therefore one gets

$$\Big\| \lim_{n \to \infty} 2^{-2n} f\left[2^n (x_1 + x_2 + x_3)\right] + \lim_{n \to \infty} 2^{-2n} f\left[2^n (x_1 - x_2 + x_3)\right] + \lim_{n \to \infty} 2^{-2n} f\left[2^n (x_1 + x_2 - x_3)\right] + \lim_{n \to \infty} 2^{-2n} f\left[2^n (x_1 - x_2 - x_3)\right] - 4\Big[\lim_{n \to \infty} 2^{-2n} f(2^n x_1) + \lim_{n \to \infty} 2^{-2n} f(2^n x_2) + \lim_{n \to \infty} 2^{-2n} f(2^n x_3) \Big] \Big\| \leqslant \lim_{n \to \infty} (2^{-2n}) c = 0,$$

or mapping Q satisfies the equation (2.1) for all $(x_1, x_2, x_3) \in X^3$.

Thus Q is a *3-dimensional quadratic mapping*. It is clear now from the general functional inequality (2.8), $n \to \infty$, and the formula (2.4) that inequality (2.5) holds in X, completing *the existence proof* of this Theorem 2.1.

The proof of *uniqueness* is omitted as obvious (see: [1–9]) and thus the *stability* of this Theorem 2.1 is complete. □

REFERENCES

[1] RASSIAS, J. M.: *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal. **46** (1982), 126–130.

[2] RASSIAS, J. M.: *On approximation of approximately linear mappings by linear mappings*, Bull. Soc. Math. **108** (1984), 445–446.

[3] RASSIAS, J. M.: *Solution of a problem of Ulam*, J. Approx. Theory **57** (1989), 268–273.

[4] RASSIAS, J. M.: *On the stability of the general Euler-Lagrange functional equation*, Demonstratio Math. **29** (1996), 755–766.

[5] RASSIAS, J. M.: *Solution of the Ulam stability problem for Euler-Lagrange quadratic mappings*, J. Math. Anal. Appl. **220** (1998), 613–639.

[6] RASSIAS, J. M.: *On the Ulam stability of mixed type mappings on restricted domains*, J. Math. Anal. Appl. **276** (2002), 747–762.

[7] RASSIAS, J. M.—RASSIAS, M. J.: *On the Ulam stability of Jensen and Jensen type mappings on restricted domains*, J. Math. Anal. Appl. **281** (2003), 516–524.

[8] RASSIAS, J. M.: *Asymptotic behavior of mixed type functional equations*, Austr. J. Math. Anal. Appl. **1** (2004), 1–21.

[9] RASSIAS, J. M.: *The Ulam stability problem in approximation of approximately quadratic mappings by quadratic mappings*, J. Inequalities of Pure Appl. Math. **5** (2004), 1–9.

[10] ULAM, S. M.: *Problems in Modern Mathematics*, Wiley-Interscience, New York, 1964, Chapter VI.

Received November 12, 2004

John Michael Rassias
Pedagogical department EE
Section of Mathematics and Informatics
National and Capodistrian University of Athens
4, Agamemnosos str., Aghia Paraskevi
Attikis, Athens 15342
GREECE
E-mail: jrassias@primedu.uoa.gr

Matina John Rassias
Statistics and Modelling Science
University of Strathclyde
Livingstone Tower, 26 Richmond str.
Glasgow, Scotland
UK, G1 1XH
E-mail: matina@stams. strath.ac.uk