Refined Hyers-Ulam approximation for Jensen and Euler-Lagrange Mappings

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ΠΕΡΙΛΗΨΗ

Το 1940 ο διάσημος Μαθηματικός S. M. Ulam πρότεινε για λύση το φημισμένο πρόβλημα ευστάθειας που φέρει το όνομά του. Στη συνέχεια το 1941 ο γνωστός Μαθηματικός D. H. Hyers έλυσε το παραπάνω πρόβλημα του Ulam για προσθετικές απεικονίσεις , υπό τον όρο ότι θα ισχύει η γνωστή συνθήκη του Hyers , για προσεγγιστικά προσθετικές απεικονίσεις. Σ’ αυτήν την εργασία γενικεύουμε το εν λόγω αποτέλεσμα του Hyers για εναλλακτικές απεικονίσεις τύπου Jensen , υπό τον όρο ότι θα ισχύει μία ασθενέστερη συνθήκη από την γνωστή συνθήκη του Hyers, ως προς γινόμενα δυνάμεων ορισμένων norm. Αυτή η διαδικασία (process) οδηγεί στη βελτίωση (refinement) της γνωστής προσέγγισης του Hyers . Επιπλέον εισάγουμε, στα Μαθηματικά για πρώτη φορά, τις εναλλακτικά προσθετικές απεικονίσεις πρώτης και δεύτερης μορφής και ερευνούμε αποτελέσματα ευστάθειας σχετικά με το πρόβλημα του Ulam. Παρόμοια ερευνούμε Euler - Lagrange τετραγωνικές απεικονίσεις και προσεγγιστικά προσθετικές απεικονίσεις που εκφυλίζονται σε γνήσια προσθετικές απεικονίσεις. Τα αποτελέσματα αυτά μπορούν να εφαρμοστούν στη στοχαστική ανάλυση, στα οικονομικά και ασφαλιστικά μαθηματικά, καθώς επίσης και στη ψυχολογία και κοινωνιολογία.

ABSTRACT

In 1940 S. M. Ulam proposed the famous Ulam stability problem. In 1941 D. H. Hyers solved this problem for Cauchy additive mappings subject to the Hyers condition on approximately additive mappings. In this paper we generalize the Hyers result for the Ulam stability problem for alternative Jensen type mappings, by considering approximately alternative Jensen type mappings satisfying conditions weaker than the Hyers condition, in terms of products of powers of norms. This process leads to a refinement of the well-known Hyers approximation for the Ulam stability problem. Besides we introduce alternative additive mappings of the first and second form and investigate pertinent stability results. Similarly we investigate Euler-Lagrange quadratic mappings and approximately additive mappings degenerating to genuine additive mappings. These stability results can be applied in stochastic analysis, financial and actuarial mathematics, as well as in psychology and sociology.

ABSTRAIT

En 1940 S.M. Ulam proposés le problème célèbre de stabilité d' Ulam. En 1941 D.H. Hyers a résolu ce problème pour les tracés additifs de Cauchy sujet à la condition de Hyers sur les tracés approximativement additifs. Dans cet article nous généralisons le résultat de Hyers pour le problème de stabilité d' Ulam pour le type alternatif tracés de Jensen, en considérant le type approximativement alternatif tracés de Jensen satisfaisant des conditions plus faible que la condition de Hyers, en termes des produits des puissances des normes. Ce processus mène à une amélioration de l'approximation bien connue de Hyers pour le problème de stabilité d' Ulam. Sans compter que nous présentons les tracés additifs alternatifs de la première et deuxième forme et étudions des résultats convenables de stabilité pour ces résultats de stabilité. De même nous étudions des tracés quadratique d' Euler-Lagrange et tracés approximativement additifs se dégénérant aux tracés additifs véritables. Ces résultats de stabilité peuvent être appliqués dans l'analyse stochastique, mathématiques financières et actuarielles, aussi bien qu'en la psychologie et la sociologie.

Key words and phrases: Ulam stability problem, Hyers condition, Cauchy sum, Jensen mapping, Euler-Lagrange mapping, genuine additive mapping.

AMS (MOS) Subject Classification: 39B.

1. Introduction

In 1940 and in 1964 S. M. Ulam [18] proposed the famous *Ulam stability problem*:

"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true ?"

In 1941 D. H. Hyers [4] solved this stability problem for Cauchy additive mappings subject to the following *Hyers condition*

$$\left\| f\left(x_1+x_2\right)-f\left(x_1\right)-f\left(x_2\right) \right\| \leq \delta \tag{HC}$$

on approximately Cauchy additive mappings $f: X \rightarrow Y$,for a fixed $\delta \geq 0$, and all x_1, $x_2 \in X$, where X is a real normed space and Y a real Banach space.

In 1951 D. G. Bourgin [1] was the second author to treat the Ulam problem for additive mappings. In 1978, according to P.M. Gruber [3], this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1982-2005 J. M. Rassias ([7-11],[14-17]) and in 2003 and 2005 M. J. Rassias and the first author ([13], [16]) solved the above Ulam problem for different mappings . In 1999 P. Gavruta [2] answered a question of ours [9] concerning the stability of the Cauchy equation. In 1998 S.- M. Jung [5] and in 2002-2003 M. J. Rassias and the first author ([12-13], [16]) investigated the Ulam stability for additive and quadratic mappings on restricted domains**.** In this paper we generalize the Hyers result for the Ulam stability problem for alternative Jensen type mappings, by considering approximately alternative Jensen and Jensen type mappings satisfying conditions weaker than the Hyers condition, in terms of products of powers of norms. Also we introduce alternative additive mappings of the first and the second form and investigate pertinent stability results. These results can be applied in stochastic analysis, financial and actuarial mathematics, as well as in psychology and sociology. In 1997, P. Malliavin [6] published an interesting reference book for stochastic analysis.

Throughout this paper, let X be a real normed space and Y be a real Banach space in the case of functional inequalities, as well as let X and Y be real linear spaces for functional equations. Besides let us denote with $\mathbb{N}=\{1,2,3,...\}$ the set of natural numbers, $\mathbb{R}$ the set of real numbers, and for some fixed

$$\alpha, \beta \in \mathbb{R}, \rho=\alpha+\beta \neq 1 .$$

If $P=(x_1, x_2) \in X^2$, then we introduce the following alternative sums:

$$S_0(P)=f(-(x_1+x_2))+f(x_1)+f(x_2)$$

the alternative Cauchy sum,

$$S_1(P)=f(x_1+x_2)+f(x_1-x_2)+2f(-x_1)$$

the alternative sum of first form ,

$$S_2(P) = f(x_1 + x_2) - f(x_1 - x_2) + 2f(-x_2)$$

the alternative sum of second form,

$$S_3(P) = 2f(-\frac{x_1 + x_2}{2}) + f(x_1) + f(x_2)$$

the alternative Jensen sum, and

$$S_4(P) = 2f(-\frac{x_1 - x_2}{2}) + f(x_1) - f(x_2)$$

the alternative Jensen type sum.

If

$$\mathrm{M}(\|x\|) = 2\delta \left|2 - 2^{\rho}\right|^{-1} \|x\|^{\rho}, \delta \geq 0, \rho \neq 1,$$

we consider the formula

$$A(x) = \lim_{n\to\infty}\begin{cases} 2^{-n} f(2^n x), & if\ -\infty < \rho < 1 \\ 2^n f(2^{-n} x), & if\ \rho > 1 \end{cases}. \tag{*}$$

Also we consider inequality

$$\|f(x) - A(x)\| \leq M(\|x\|) \tag{**}$$

for all $x \in X$.

If Q=$(\|x_1\|, \|x_2\|) \in \mathbb{R}^2$, then we denote

$$\Pi(Q) = \delta \|x_1\|^{\alpha} \|x_2\|^{\beta}, \delta \geq 0, \rho = \alpha + \beta \neq 1.$$

Definition 1.0. A mapping $A : X \to Y$ is called *alternative Cauchy* if A satisfies

$$A(-(x_1 + x_2)) = -[A(x_1) + A(x_2)] \tag{C}$$

for all $x_1, x_2 \in X$.

Definition 1.1. A mapping $A : X \to Y$ is called *alternative additive of the first form* if A satisfies the functional equation

$$A(x_1 + x_2) + A(x_1 - x_2) = -2A(-x_1) \tag{1}$$

for all $x_1, x_2 \in X$.

We note that (1) is equivalent to *the alternative Jensen equation*

$$A\left(-\frac{x+y}{2}\right) = -\frac{1}{2}\left[A(x) + A(y)\right], \tag{J}$$

for $x = x_1 + x_2$, $y = x_1 - x_2$. A mapping $A : X \to Y$ is called *alternative Jensen mapping* if A satisfies the functional equation (J).

Definition 1.2. A mapping $A : X \to Y$ is called *alternative additive of the second form* if A satisfies the functional equation

$$A(x_1 + x_2) - A(x_1 - x_2) = -2A(-x_2) \tag{2}$$

for all $x_1, x_2 \in X$.

We note that (2) is equivalent to *the alternative Jensen type equation*

$$A\left(-\frac{x-y}{2}\right) = -\frac{1}{2}\left[A(x) - A(y)\right], \tag{JT}$$

for $x = x_1 + x_2$, $y = x_1 - x_2$. A mapping $A : X \to Y$ is called *alternative Jensen type mapping* if A satisfies the functional equation (JT).

Our *main stability result* is the following:

Theorem 1.3. *If a mapping $f : X \to Y$ satisfies the approximately alternative Jensen type inequality*

$$\| S_4(P) \| \leq \Pi(Q) , \tag{3}$$

for all $P \in X^2$, $Q \in \mathbb{R}^2$, then there exists a unique alternative Jensen type mapping $A : X \to Y$, which satisfies the formula (*) and the inequality (**) for all $x \in X$.

If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$,

$$A(tx) = tA(x)$$

for all $x \in X$ and $t \in \mathbb{R}$.

2. Jensen Mappings and Outline of the Proof

(1) We note that *the Hyers condition (HC)* on approximately alternative Jensen type mappings is the corresponding inequality (3) ,when $\alpha = \beta = 0$;thus

$$\Pi(Q) = \delta .$$

(2) If we replace $S_4(P)$, *with* $S_i(P)$ $(i = 0,1,2,3)$ in the above inequality (3),then we easily establish stability results for the remaining four kinds of mappings corresponding to the alternative sums $S_i(P)$ $(i = 0,1,2,3)$.

In these four cases we prove that $M(\|x\|)$ of the inequality (**) has to be substituted by $M(\|x\|)/2$.

(3) To prove Theorem 1.3,we argue, as follows:

In fact, replacing $x_1 = x_2 = 0$ in (3),one gets $f(0) = 0$ .Setting $x_1 = 2x, x_2 = 0$ in (3), we get $f(2x) = -2f(-x)$.
Besides substituting $x_1 = x, x_2 = -x$ in (3), e obtain

$$\|f(-x) + f(x)\| \le \delta \|x\|^{\rho} .$$

Therefore,

$$\|f(2x) - 2f(x)\| = 2\|f(-x) + f(x)\| \le 2\delta \|x\|^{\rho} ,$$

or

$$\left\|f(x) - 2^{-1} f(2x)\right\| \le \delta \|x\|^{\rho} , \forall \rho \ne 1 .$$

Hence,

$$\left\|f(x) - 2^{-n} f(2^n x)\right\| \le \left\|f(x) - 2^{-1} f(2^1 x)\right\| + ... + 2^{-(n-1)} \left\|f(2^{n-1} x) - 2^{-1} f(2^n x\right\|$$

$$\le [1 - 2^{n(\rho-1)}](2 - 2^{\rho})^{-1} 2\delta \|x\|^{\rho} ,$$

for $\forall n \in \mathbb{N}$.

Assume $-\infty < \rho < 1$.
If one replaces x *with* $2^{-n} x$ in the above general inequality, then he finds

$$\left\|f(x) - 2^n f(2^{-n} x)\right\| \le [1 - 2^{n(1-\rho)}](2^{\rho} - \rho)^{-1} 2\delta \|x\|^{\rho} ,$$

for $\forall n \in \mathbb{N}$.

Assume $\rho > 1$
From the above two general inequalities, we prove that the sequence $\{f_n(x)\}$ with

$$f_n(x) = \{\begin{matrix} 2^{-n} f(2^n x), & \text{if } -\infty < \rho < 1 \\ 2^n f(2^{-n} x), & \text{if } \rho > 1 \end{matrix}$$

is a Cauchy sequence.
From (*),(3) and the completeness of Y, one proves that a well-defined mapping $A: X \to Y$ exists, such that the mapping $A : X \to Y$ satisfies the alternative Jensen type equation (JT).
It is easy to prove that

$$A(x) = \begin{cases} 2^{-n} A(2^n x), if & -\infty < \rho < 1 \\ 2^n A(2^{-n} x), if & \rho > 1 \end{cases}$$

The rest of the proof for the existence and uniqueness of $A: X \to Y$ is omitted as similar to our detailed proofs in references ([8]-[10]).
The proof of the last assertion in the above Theorem 1.3 is obvious according to our work [7],in 1982.

The singular case $\rho = 1$ *is open.*

We refer the reader to P. Gavruta [2] and ([7]-[10]) for analogous singular cases.

3. Euler-Lagrange Quadratic Mappings

The following theorem 3.1 is well-known for 2-dimensional Euler-Lagrange quadratic mappings.

Theorem 3.1 ([11], [17]). *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that f : X→Y is a mapping for which there exists a constant c (independent of* x_1, x_2*)* ≥ 0 *such that the Euler-Lagrange quadratic functional inequality*

$$\left\| f(x_1+x_2)+f(x_1-x_2)-2\left[f(x_1)+f(x_2)\right] \right\| \leq c$$

holds for all $(x_1, x_2) \in X^2$.

Then the limit

$$Q(x)= \lim_{n\to\infty} 2^{-2n} f(2^n x)$$

exists for all $x \in X$ *and all* $n \in N$ *and* $Q : X\to Y$ *is the unique Euler-Lagrange quadratic mapping satisfying the functional equation*

$$Q(x_1+x_2)+Q(x_1-x_2)=2\left[Q(x_1)+Q(x_2)\right]$$

for all $(x_1, x_2) \in X^2$, *such that inequality*

$$\left\| f(x) - Q(x) \right\| \leq \frac{1}{2}c ,$$

holds for all $x \in X$.

In this section we establish the Ulam stability for 3-dimensional quadratic mappings.

Definition 3.1. Let X be a normed linear space and let Y be a real complete normed linear space. Then a mapping $Q : X \to Y$, is called 3-dimensional *Euler-Lagrange quadratic* if the functional equation

$$Q(x_1+x_2+x_3) + Q(x_1-x_2+x_3) + Q(x_1+x_2-x_3) + Q(x_1-x_2-x_3) = 4\left[Q(x_1)+Q(x_2)+Q(x_3)\right] \quad (4)$$

holds for all $(x_1, x_2, x_3) \in X^3$ .Note that mapping Q is called *quadratic*, because the functional equation

$$Q(2^n x) = (2^n)^2 Q(x), \quad (5)$$

holds for all $x \in X$, and all $n \in N$.

In fact, substitution of $x_1 = x_2 = x_3 = 0$ in equation (4) yields that $Q(0) = 0$.

Substituting $x_1 = x_2 = x$, $x_3 = 0$ one gets that the functional equation $2Q(2x) + 2Q(0) = 4\left[2Q(x)+Q(0)\right]$,or $Q(2x) = (2)^2 Q(x)$,

holds for all $x \in X$.

Then induction on $n \in N$ with $x \to 2^{n-1}x$ yields equation (5).

Theorem 3.2. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that $f : X \to Y$ is a 3-dimensional mapping for which there exists a constant c (independent of x_1, x_2, x_3) ≥ 0 such that the Euler-Lagrange quadratic functional inequality*

$$\left\| f(x_1+x_2+x_3) + f(x_1-x_2+x_3) + f(x_1+x_2-x_3) + f(x_1-x_2-x_3) - 4\left[f(x_1)+f(x_2)+f(x_3)\right] \right\| \leq c \quad (6)$$

holds for all $(x_1, x_2, x_3) \in X^3$. Then the limit

$$Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x), \quad (7)$$

exists for all $x \in X$ and all $n \in N$ and $Q : X \to Y$ is the unique 3-dimensional Euler-Lagrange quadratic mapping satisfying the functional equation (4), such that inequality

$$\left\| f(x) - Q(x) \right\| \leq \frac{5}{24}\, c \ , \quad (8)$$

holds for all $x \in X$. Moreover, functional identity

$$Q(x) = 2^{-2n} Q(2^n x),$$

holds for all $x \in X$, and all $n \in N$.

Proof. Substitution of $x_1 = x_2 = x_3 = 0$ in inequality (6) yields that

$$\|f(0)\| \le \frac{c}{8} \ . \tag{9}$$

Moreover substituting $x_1 = x_2 = x$, $x_3 = 0$ in inequality (6) and employing (9) and the triangle inequality one concludes

$$\|2f(2x) + 2f(0) - 4[2f(x) + f(0)]\| \le c \ \text{, or } \ \|f(2x) - 4f(x) - f(0)\| \le \frac{c}{2} \text{ ,or}$$

$$\|f(2x) - 4f(x)\| \le \|f(2x) - 4f(x) - f(0)\| + \|f(0)\| \text{,or}$$

$$\|f(2x) - 4f(x)\| \le \frac{c}{2} + \frac{c}{8} = \frac{5}{8}c \text{, or}$$

thus one gets that *the basic functional inequality*

$$\|f(x) - 2^{-2} f(2x)\| \le \frac{5}{32}c = c_1 \ (1-2^{-2}) \ . \tag{10}$$

holds for all $x \in X$, where $c_1 = (5/24)c$. Replacing now x with $2x$ in (10) one concludes that $\|f(2x) - 2^{-2} f(2^2x)\| \le c_1 \ (1-2^{-2})$, or

$$\|2^{-2} f(2x) - 2^{-4} f(2^2x)\| \le c_1 \ (2^{-2} - 2^{-4}) \tag{10a}$$

holds for all $x \in X$ .Functional inequalities (10) - (10a) and the triangle inequality yield

$$\begin{aligned}\|f(x) - 2^{-4} f(2^2x)\| &\le \|f(x) - 2^{-2} f(2x)\| + \|2^{-2} f(2x) - 2^{-4} f(2^2x)\| \\ &\le c_1 \left[(1-2^{-2}) + (2^{-2} - 2^{-4})\right] \ ,\end{aligned}$$

or that the functional inequality

$$\|f(x) - 2^{-4} f(2^2x)\| \le c_1 \ (1\text{-}2^{-4}),$$

holds for all $x \in X$.

Similarly by induction on $n \in N$ with $x \to 2^{n\text{-}1}x$ in the basic inequality (10) claim that *the general functional inequality*

$$\|f(x) - 2^{-2n} f(2^n x)\| \le c_1 \ (1 - 2^{-2n}), \tag{11}$$

holds for all $x \in X$ and all $n \in N$ .In fact , the basic inequality (7) with $x \to 2^{n\text{-}1}x$ yield the functional inequality $\|f(2^{n-1}x) - 2^{-2} f(2^n x)\| \le c_1 \ (1-2^{-2})$,or the inequality

$$\|2^{-2(n-1)} f(2^{n-1}x) - 2^{-2n} f(2^n x)\| \le c_1 \ (2^{-2(n-1)} - 2^{-2n}), \tag{11a}$$

holds for all $x \in X$ .Moreover, by induction hypothesis with $n \to n\text{-}1$ in the general inequality (11) one gets that

$$\left\|f(x)-2^{-2(n-1)}f(2^{n-1}x)\right\| \leq c_1\ (1 - 2^{-2(n-1)})\ , \qquad (11b)$$

holds for all $x \in X$ .Thus functional inequalities (11a) - (11b) and the triangle inequality imply

$$\left\|f(x)-2^{-2n}f(2^n x)\right\| \leq \left\|f(x)-2^{-2(n-1)}f(2^{n-1}x)\right\| + \left\|2^{-2(n-1)}f(2^{n-1}x)-2^{-2n}f(2^n x)\right\|\ , \text{ or}$$

$$\left\|f(x)-2^{-2n}f(2^n x)\right\| \leq c_1\left[(1-2^{-2(n-1)})+(2^{-2(n-1)}-2^{-2n})\right] = c_1\ (1-2^{-2n})\ ,$$

completing the proof of the required general functional inequality (11).

Claim now that the sequence $\left\{2^{-2n}f(2^n \text{x})\right\}$ converges. Note that from the general inequality (11) and the completeness of Y , one proves that the above sequence is *a Cauchy sequence*. In fact , if $i > j > 0$, then

$$\left\|2^{-2i}f(2^i x)-2^{-2j}f(2^j x)\right\| = 2^{-2j}\left\|2^{-2(i-j)}f(2^i x)-f(2^j x)\right\|\ ,$$

holds for all $x \in X$, and all $i, j \in N$ .Setting $h = 2^j x$ in the above relation and employing the general inequality (11) one concludes that

$$\left\|2^{-2i}f(2^i x)-2^{-2j}f(2^j x)\right\| = 2^{-2j}\left\|2^{-2(i-j)}f(2^{i-j}h)-f(h)\right\| \leq 2^{-2j}c_1\ (1 - 2^{-2(i-j)})\ , \text{ or}$$

$$\left\|2^{-2i}f(2^i x)-2^{-2j}f(2^j x)\right\| \leq c_1\ (2^{-2j} - 2^{-2i}) < c_1 2^{-2j}\ , \text{or}$$

$$\lim_{j\to\infty}\left\|2^{-2i}f(2^i x)-2^{-2j}f(2^j x)\right\| = 0\ ,$$

completing the proof that the sequence $\left\{2^{-2n}f(2^n x)\right\}$ *converges*.

Hence $Q = Q(x)$ is *a well-defined mapping* via the formula (7).This means that the limit (7) exists for all $x \in X$.

In addition claim that mapping *Q satisfies the functional equation* (4) for all $(x_1 , x_2, x_3) \in X^3$ .In fact, it is clear from the functional inequality (6) and the limit (7) that the following inequality

$$\begin{aligned}2^{-2n}\Big\| & f(2^n x_1+2^n x_2+2^n x_3) + f(2^n x_1-2^n x_2+2^n x_3) + f(2^n x_1+2^n x_2-2^n x_3) \\ & + f(2^n x_1-2^n x_2-2^n x_3) - 4\left[f(2^n x_1)+f(2^n x_2)+f(2^n x_3)\right]\Big\| \leq 2^{-2n}\ c\ ,\end{aligned}$$

holds for all $(x_1 , x_2, x_3) \in X^3$, and all $n \in N$ .Therefore one gets

$$\begin{aligned}&\Big\|\lim_{n\to\infty}2^{-2n}f\left[2^n(x_1+x_2+x_3)\right] + \lim_{n\to\infty}2^{-2n}f\left[2^n(x_1-x_2+x_3)\right] + \lim_{n\to\infty}2^{-2n}f\left[2^n(x_1+x_2-x_3)\right] \\ &+ \lim_{n\to\infty}2^{-2n}f\left[2^n(x_1-x_2-x_3)\right] - 4\left[\lim_{n\to\infty}2^{-2n}f(2^n x_1) + \lim_{n\to\infty}2^{-2n}f(2^n x_2) + \lim_{n\to\infty}2^{-2n}f(2^n x_3)\right]\Big\| \\ &\leq \lim_{n\to\infty}(2^{-2n})c = 0\ ,\end{aligned}$$

or mapping Q satisfies the equation (4) for all $(x_1, x_2, x_3) \in X^3$. Thus Q is *a 3-dimensional quadratic mapping*. It is clear now from the general functional inequality (11), $n \to \infty$, and the formula (7) that inequality (8) holds in X, completing *the existence proof* of this Theorem 3.2.

The proof of *uniqueness* is omitted as obvious and thus the *stability* of this Theorem 3.2 is complete.

4. Genuine Additive Mappings

In this section we investigate approximate additive mappings degenerating to genuine additive mappings.

Definition 4.1. Let X and Y be real linear spaces.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in R^p - \{(0, 0, \ldots, 0)\}$. Then a mapping $A : X \to Y$ is called *additive* , if the additive functional equation

$$A\left(\sum_{i=1}^{p} a_i x_i\right) = \sum_{i=1}^{p} a_i A(x_i) \tag{12}$$

holds for every $x_i \in X$ $(i = 1, 2, \ldots, p)$,where

p is arbitrary but fixed and equals to 2, 3, . . . and any fixed a $(\neq 0)$: $0 < m = \sum_{i=1}^{p} a_i \neq 1$.

Definition 4.2. Let X and Y be real normed linear spaces.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in R^p - \{(0, 0, \ldots, 0)\}$.Then a mapping $f : X \to Y$ is called *approximately additive*, if the approximately additive functional inequality

$$\left\| f\left(\sum_{i=1}^{p} a_i x_i\right) - \sum_{i=1}^{p} a_i f(x_i) \right\| \leq c K_r\left(\|x_1\|, \|x_2\|, \ldots, \|x_p\|\right), \tag{13}$$

holds for every $(x_1, x_2, \ldots, x_p) \in X^p$,where p is arbitrary but fixed and equals to 2, 3, . . ., with a real constant $c \geq 0$ (independent of $x_1, x_2, \ldots, x_p \in X$), any fixed

a $(\neq 0)$: $0 < m = \sum_{i=1}^{p} a_i \neq 1$ and any fixed real $(1 \neq)$ $r \geq 0$:

$$K_r = K_r\left(\|x_1\|,\|x_2\|,\ldots,\|x_p\|\right)=\begin{cases} p^{r-1}\left(\sum_{i=1}^{p}\|x_i\|^r\right)-\left\|\sum_{i=1}^{p}x_i\right\|^r ,if\ \ r\rangle 1 \\ \left(\sum_{i=1}^{p}\|x_i\|\right)^r - p^{r-1}\left(\sum_{i=1}^{p}\|x_i\|^r\right),if\ \ 0\le r\langle 1\ \ , \end{cases} \tag{14}$$

holds for every $(x_1, x_2, \ldots, x_p) \in X^p$.

Lemma 4.1. *If K_r is given via* (14), *then $K_r \geq 0$ for any fixed real* $0 \leq r \neq 1$.

Proof. In fact, take a function $F = F(t) = t^r$ ($t \geq 0$ and $r \in R$).It is clear that $F''(t)=r(r-1)t^{r-2}\geq 0$ for $r \in R : r \geq 1$. Thus F is *convex* for $r \geq 1$.Therefore

$$F\left(\frac{1}{p}\sum_{i=1}^{p}t_i\right)\le\frac{1}{p}\left(\sum_{i=1}^{p}F(t_i)\right), \text{ or}$$

$$\left(\frac{1}{p}\sum_{i=1}^{p}t_i\right)^r\le\frac{1}{p}\left(\sum_{i=1}^{p}t_i^r\right)$$

for $r \in R : r \geq 1$, and $t_i \geq 0$ $(i = 1, 2, \ldots, p)$, where p is arbitrary but fixed and equals to 2, 3 , ... Taking $t_i=\|x_i\|\geq 0$ for $x_i \in X$ $(i = 1, 2, \ldots, p)$ and $r \in R : r \geq 1$, we get

$$\left(\frac{1}{p}\sum_{i=1}^{p}\|x_i\|\right)^r\le\frac{1}{p}\left(\sum_{i=1}^{p}\|x_i\|^r\right), \text{ or}$$

$$p^{r-1}\left(\sum_{i=1}^{p}\|x_i\|^r\right)\ge\left(\sum_{i=1}^{p}\|x_i\|\right)^r$$

for $r \geq 1$. But it is clear that

$$\left\|\sum_{i=1}^{p}x_i\right\|^r\le\left(\sum_{i=1}^{p}\|x_i\|\right)^r$$

for $r \geq 0$ .Therefore we have that

$$K_r = p^{r-1}\left(\sum_{i=1}^{p}\|x_i\|^r\right)-\left\|\sum_{i=1}^{p}x_i\right\|^r\ge 0$$

for $r > 1$ .Similarly $F''(t)=r(r-1)t^{r-2}\le 0$ for $0 \leq r< 1$. Thus F is *concave* for $r \in R : 0\leq r<1$. Therefore

$$\left(\frac{1}{p}\sum_{i=1}^{p}t_i\right)^r\ge\frac{1}{p}\left(\sum_{i=1}^{p}t_i^r\right).$$

Taking $t_i = \|x_i\| \geq 0$

($i = 1, 2, \ldots, p$), we get

$$K_r = \left(\sum_{i=1}^{p}\|x_i\|\right)^r - p^{r-1}\left(\sum_{i=1}^{p}\|x_i\|^r\right) \geq 0$$

for $0 \leq r < 1$, completing the proof of Lemma 4.1.

Let us denote

$$I_1 = \{ (r, m) \in R^2 : 0 \leq r < 1, m > 1 \text{ and } r > 1, 0 < m < 1 \},$$

and

$$I_2 = \{ (r, m) \in R^2 : 0 \leq r < 1, 0 < m < 1 \text{ and } r > 1, m > 1 \},$$

such that $m^{r-1} < 1$ for any $(r, m) \in I_1$, and $m^{1-r} < 1$ for any $(r, m) \in I_2$. Note that approximately additive mappings are not additive in case $K_r = 1$ and $m > 0$. In this case Y is assumed to be complete. Also $K_0 = 0$ and the singular case

$$K_1 = \sum_{i=1}^{p}\|x_i\| - \left\|\sum_{i=1}^{p} x_i\right\| \quad (\geq 0).$$

Theorem 4.1. *Let X and Y be normed linear spaces.*

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in R^p - \{(0, 0, \ldots, 0)\}: 0 < m = \sum_{i=1}^{p} a_i \neq 1$, *where p is arbitrary but fixed and equals to* 2,3, . . . *Assume in addition that* $f : X \to Y$ *is an approximately additive mapping satisfying* (13) *with* $1 \neq r \geq 0$. *Define*

$$f_n(x) = \begin{cases} m^{-n} f(m^n x) & , if\ (r,m) \in I_1 \\ m^n f(m^{-n} x) & , if\ (r,m) \in I_2 \end{cases}$$

for all $x \in X$ *and* $n \in N_o = \{0, 1, 2, \ldots\}$, *where*

$$I_1 = \{ (r, m) \in R^2 : 0 \leq r < 1, m > 1 \text{ and } r > 1, 0 < m < 1 \},$$

and

$$I_2 = \{ (r, m) \in R^2 : 0 \leq r < 1, 0 < m < 1 \text{ and } r > 1, m > 1 \}.$$

Then the formula

$$A(x) = f_n(x) \tag{15}$$

exists for all $x \in X$ *and* $n \in N_o$ *and* $A : X \to Y$ *is the unique additive mapping satisfying*

$$f(x) = A(x) \tag{15a}$$

for all $x \in X$.

Proof. It is useful for the following investigation to observe that, from (13) with $x_i = 0$ (i = 1, 2, . . . , p) and 0< $m \neq 1$, we get $|m-1|\,\|f(0)\| \leq 0$, or

$$f(0) = 0\ . \tag{16}$$

Now claim for $n \in N_0 = \{0, 1, 2, \ldots\}$ that

$$f(x) = f_n(x) \tag{16a}$$

holds for all $x \in X$. For $n = 0$ it is trivial. From (14) with $x_i = x$ ($i \in N_p = \{1, 2, \ldots, p\}$), we obtain

$$K_r = \|x\|^r \begin{cases} p^{r-1} \cdot p - p^r = 0, & if \quad r \rangle 1 \\ p^r - p^{r-1} \cdot p = 0, & if \quad 0 \leq r \langle 1 \end{cases} \qquad \text{or}$$

$$K_r = K_r(\|x\|, \|x\|, \ldots, \|x\|) = \|x\|^r \cdot 0 = 0\,, \tag{17}$$

for every $x \in X$ and any fixed real $r \in R : 0 \leq r \neq 1$ with $p = 2, 3, \ldots$ Similarly from (14) with $x_i = m^{-1}x$ ($i \in N_p$), we get

$$K_r = \|x\|^r m^{-r} \begin{cases} p^{r-1} \cdot p - p^r = 0, & if \quad r \rangle 1 \\ p^r - p^{r-1} \cdot p = 0, & if \quad 0 \leq r \langle 1 \end{cases}$$

or

$$K_r = K_r\left(m^{-1}\|x\|, m^{-1}\|x\|, \ldots, m^{-1}\|x\|\right) = \|x\|^r m^{-r} \cdot 0 = 0\ , \tag{18}$$

for every $x \in X$ and any fixed real $r \in R : 0 \leq r \neq 1$ with $p = 2, 3, \ldots$

From (13) and (17), with $x_i = x$ ($i \in N_p$), we get

$$\|f(mx) - mf(x)\| \leq cK_r(\|x\|, \|x\|, \ldots, \|x\|) = 0\,, \text{ or}$$

$$f(x) = m^{-1} f(mx)\,, \tag{19}$$

which is (16a) for $n = 1$, if I_1 holds. Similarly, from (13) and (18), with $x_i = m^{-1}x$ ($m \neq 0$) ($i \in N_p$), we obtain

$$\left\|f(x) - mf\left(m^{-1}x\right)\right\| \leq cK_r\left(m^{-1}\|x\|, m^{-1}\|x\|, \ldots, m^{-1}\|x\|\right) = 0 \quad \text{or}$$

$$f(x) = mf\left(m^{-1}x\right), \tag{20}$$

which is (16a) for $n = 1$, if I_2 holds.

Assume (16a) is true and from (19), with $m^n x$ on place of x, we get:

$$f\left(m^{n+1}x\right) = mf(m^n x) = mm^n f(x) = m^{n+1} f(x)\,. \tag{21}$$

Similarly, from (20) with $m^{-n}x$ on place of x, we obtain:

$$f\left(m^{-(n+1)}x\right)=m^{-1}f(m^{-n}x)=m^{-1}m^{-n}f(x)=m^{-(n+1)}f(x). \tag{22}$$

These formulas (21) and (22) by induction, prove formula (16a). It is obvious from (16a) that A *defines* a mapping $A : X \to Y$, given by (15).Finally, claim from (13) and (16a) we can get that $A : X \to Y$ is additive.

In fact, it is clear from the functional inequality (13), the Lemma 4.1 and the formula (16a) that the following functional inequality

$$m^{-n}\left\| f\left(\sum_{i=1}^{p} a_i m^n x_i\right) - \sum_{i=1}^{p} a_i f\left(m^n x_i\right)\right\| \le m^{-n} c K_r\left(\left\|m^n x_1\right\|, \left\|m^n x_2\right\|, \ldots, \left\|m^n x_p\right\|\right)$$

holds for all $(x_1, x_2, \ldots, x_p) \in X^p$, and all $n \in N_0$, with $f_n(x)=m^{-n}f\left(m^n x\right)$: I_1 holds. Therefore

$$\left\| f_n\left(\sum_{i=1}^{p} a_i x_i\right) - \sum_{i=1}^{p} a_i f_n(x_i)\right\| \le m^{n(r-1)} c K_r\left(\left\|x_1\right\|, \left\|x_2\right\|, \ldots, \left\|x_p\right\|\right) \text{ or}$$

$$\left\| A\left(\sum_{i=1}^{p} a_i x_i\right) - \sum_{i=1}^{p} a_i A(x_i)\right\| \le m^{n(r-1)} c K_r \xrightarrow[n\to\infty]{} 0 \ ,$$

because $m^{r-1} < 1$ for any $(r, m) \in I_1$, or

$$A\left(\sum_{i=1}^{p} a_i x_i\right) = \sum_{i=1}^{p} a_i A(x_i) \qquad ,$$

yielding that mapping $A : X \to Y$ satisfies the additive functional equation (12). Similarly, from (13), the Lemma 4.1 and (16a) we get that

$$m^{n}\left\| f\left(\sum_{i=1}^{p} a_i m^{-n} x_i\right) - \sum_{i=1}^{p} a_i f\left(m^{-n} x_i\right)\right\| \le m^{n} c K_r\left(\left\|m^{-n} x_1\right\|, \left\|m^{-n} x_2\right\|, \ldots, \left\|m^{-n} x_p\right\|\right)$$

holds for all $(x_1, x_2, \ldots, x_p) \in X^p$, and all $n \in N_0$, with $f_n(x)=m^{n}f\left(m^{-n} x\right)$: I_2 holds. Therefore

$$\left\| f_n\left(\sum_{i=1}^{p} a_i x_i\right) - \sum_{i=1}^{p} a_i f_n(x_i)\right\| \le m^{n(1-r)} c K_r\left(\left\|x_1\right\|, \left\|x_2\right\|, \ldots, \left\|x_p\right\|\right) \text{ or}$$

$$\left\| A\left(\sum_{i=1}^{p} a_i x_i\right) - \sum_{i=1}^{p} a_i A(x_i)\right\| \le m^{n(1-r)} c K_r \xrightarrow[n\to\infty]{} 0$$

because $m^{1-r} < 1$ for $(r, m) \in I_2$, implying that $A : X \to Y$ satisfies (12), completing the proof that A can be an additive mapping in X. This completes *the existence proof* of the above Theorem 4.1.

The Uniqueness proof of Theorem 4.1 is clear, because if $A : X \to Y$ and $A' : X \to Y$ are two additive mappings satisfying (15a) then A and A' satisfy

$$A(x) - A'(x) = f(x) - f(x) = 0 \text{ , or } \quad A(x) = A'(x) \text{ for all } x \in X.$$

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