International Journal of Pure & Applied Mathematical Sciences
ISSN 0972-9828 Vol.2 No.1(2005), pp. 92-101

http://www.gbspublisher.com/ijpams.htm

On the Cauchy-Ulam Stability of the Jensen Equation in C*-Algebras

John Michael Rassias

Pedagogical Department E.E.
National and Capodistrian University of Athens
4, Agamemnonos Str., Aghia Paraskevi
Attikis 15342, Greece.
E-mail: jrassias@primedu.uoa.gr

Abstract

In 1964, Ulam raised the general problem: “When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?” In this article, we consider almost unital approximately linear mappings $h : A \to B$, in unital C*-algebras. Besides, we give conditions in order for h to be a $*$-homomorphism and establish results for $*$-derivations. Furthermore, we investigate the Cauchy-Ulam stability of the Jensen equation in unital C*-algebras. Finally we establish the Cauchy-Ulam stability and $*$-homomorphisms, as well as $*$-derivations.

2000 Mathematics Subject Classification: 39B, 46.

Key words and phrases: Ulam problem, Cauchy-Ulam stability, Jensen equation, C*-algebra, almost unital mapping, unital C*-algebra, $*$-homomorphism, $*$-derivation.

1. Introduction

In 1940 S. M. Ulam [12] proposed at the University of Wisconsin the problem: *"Give conditions in order for a linear mapping near an approximately linear mapping to exist."* In 1964 he proposed the more general problem: " When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?” In 1978 P. M. Gruber [2] proposed the analogous problem: "Suppose a mathematical object satisfies a certain property approximately. Is it then possible to approximate this object by objects, satisfying the property exactly?" According to P.M. Gruber this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types.

Theorem 1. *Let X be a real normed linear space and let Y be a real complete normed linear space. Assume in addition that* $f: X \to Y$ *is a mapping for which there exist constants* $\theta \geq 0$ *and* $p \in \mathbb{R} - \{1\}$ *such that*

$$\|f(x+y) - [f(x)+f(y)]\| \leq \theta \|x\|^{p/2} \|y\|^{p/2}$$

for all $x, y \in X$. *Then there exists a unique additive mapping* $L: X \to Y$ *satisfying*

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^p - 2|} \|x\|^p$$

for all $x \in X$. *If in addition* $f: X \to Y$ *is a mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then L is* $\mathbb{R}$*-linear mapping.*

Remark. The above Ulam stability theorem was obtained by D.H. Hyers [3] for the case $p = 0$, and by the author ([6]-[10]) for the case $p \in (-\infty, 1) \cup (1, \infty)$.In particular, P. Gavruta [1] gave a counter-example for the case $p = 1$. Besides, T. Trif [11] established an analogous stability of the Jensen type functional equation deriving from an inequality of T.Popoviciu[5] for convex functions. However, T. Trif [11] generalized the Popoviciu equation

$$3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) = 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right].$$

Let us consider two real linear spaces *X* and *Y* and the Jensen functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y) \tag{1}$$

for $f: X \to Y$ and all $x, y \in X$, and the additive Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{2}$$

for all $x, y \in X$ .Therefore we establish below an interesting theorem and a proposition connecting the Jensen equation (1) with the Cauchy equation (2).

Theorem 2. *Let X be a real normed linear space and let Y be a real complete normed linear space. Assume in addition that* $f: X \to Y$ *with f(0) = 0 is a mapping for which there exists a constant* $\theta \geq 0$ *such that*

$$\left\| 2f\left(\frac{x+y}{2}\right) - [f(x)+f(y)] \right\| \leq \theta \tag{$*$}$$

for all $x, y \in X$. *Then there exists a unique Jensen mapping* $J: X \to Y$ *satisfying the functional equation* (1) *and the functional inequality*

$$\|f(x)-J(x)\| \le \theta \qquad (**)$$

for all $x \in X$.

Proof. Substituting $x = 0$ and $y = 2x$ in the above inequality (∗) and employing condition $f(0) = 0$, we get

$$\|2f(x)-[f(0)+f(2x)]\| \le \theta \text{ ,or } \|f(x)-2^{-1}f(2x)\| \le \frac{\theta}{2} = \theta\left(1-2^{-1}\right),$$

for all $x \in X$. Thus, in general, one establishes $\left\|f(x)-2^{-n}f\left(2^n x\right)\right\| \le \theta\left(1-2^{-n}\right) \to \theta$ as $n \to \infty$,for all $x \in X$. Thus taking the limiting form $J(x) = \lim_{n\to\infty} 2^{-n} f\left(2^n x\right)$, and employing classical techniques on these concepts, we prove the inequality (∗∗) for a unique mapping $J : X \to Y$ of the above limiting form. The rest of the proof is omitted as analogous to the proofs of our old theorems ([6]-[10]).

Note . If we had replaced $x = 0$ and $y = x$ in (∗), then

$$\left\|f(x)-2f\left(2^{-1}x\right)\right\| \le \theta$$

and, in general, $\left\|f(x)-2^n f\left(2^{-n}x\right)\right\| \le \theta\left(1+2+...+2^{n-1}\right) = \theta\frac{1-2^n}{1-2} \to \infty$, as $n \to \infty$.

Thus the inequality (∗∗) does not hold.

Proposition 1. *A function* $f : X \to Y$ *between two real linear spaces X and Y satisfies the Jensen equation* **(1)** *for all* $x, y \in X$ *if and only if there exists an additive Cauchy mapping* $C : X \to Y$ *satisfying* **(2)** *and such that*

$$f(x) = C(x) + f(0) \qquad (3)$$

for all $x \in X$

Proof. *Necessity(⇒)*. Let us assume that a mapping $f : X \to Y$ satisfies the Jensen equation (1). We consider $C : X \to Y$ and $g : X \to Y$ two functions given by the formulas

$$C(x) = \frac{1}{2}[f(x)-f(-x)] \quad \text{and} \quad g(x) = \frac{1}{2}[f(x)+f(-x)]-f(0), \qquad (4)$$

respectively, for all $x \in X$. Therefore

$$f(x) = C(x)+g(x)+f(0), \qquad (5)$$

for all $x \in X$. Claim that C satisfies the additive Cauchy equation (2) and that $g(x) = 0$ in X. In fact,

$$2C\left(\frac{x+y}{2}\right) = C(x)+C(y) \qquad (6)$$

and

$$2g\left(\frac{x+y}{2}\right)=g(x)+g(y) \qquad (7)$$

for all $x, y \in$ X, because f satisfies (1) for all $x, y \in$ X and (4)-(5) hold. By virtue of (4) we find the relations

$$C(-x)=-C(x) \quad \text{and} \quad g(-x)=g(x) \qquad (8)$$

for all $x \in X$. Setting $x = 0$ in the former form of (4) and $y = 0$ in (6), we get $C(0) = 0$ and

$$2C\left(2^{-1}x\right)=C(x) \qquad (9)$$

for all $x \in X$. From (6) and (9), one obtains

$$C(x+y)-C(x)-C(y)=2C\left(2^{-1}(x+y)\right)-C(x)-C(y)$$
$$=[C(x)+C(y)]-C(x)-C(y)=0 \text{ ,or}$$
$$C(x+y)=C(x)+C(y) \qquad (10)$$

for all $x, y \in X$. Therefore, C is an additive Cauchy mapping.

On the other hand, substituting $x = 0$ in the latter form of (4) and $y = 0$ in (7), we find $g(0) = 0$ and

$$2g\left(2^{-1}x\right)=g(x) \qquad (11)$$

for all $x \in X$. Taking into account (7) and (11), we get

$$g(x+y)=g(x)+g(y) \qquad (12)$$

for all $x, y \in X$. Placing $y = -x$ in (12) yields $g(0)=g(x)+g(-x)$.From the eveness of g, by the latter form of (8) and $g(0) = 0$, one establishes $g(x) = 0$ for all $x \in X$.
Sufficiency ($\Leftarrow$). The converse is omitted as clear, completing the proof of the Proposition 1.

Corollary 1. *Let X be a real normed linear space and let Y be a real normed linear space. A mapping $f : X \to Y$ with $f(0) = 0$ satisfies the Jensen functional equation* **(1)** *if and only if the mapping $f : X \to Y$ satisfies the additive Cauchy functional equation* **(2).**

Corollary 2. *A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies Jensen equation* **(1) (or(7))** *for all $x, y \in \mathbb{R}$ if and only if it has the form $f(x) = \alpha x + \beta$,with α and β arbitrary real constants.*

Let us introduce below some basic terminology, from *functional analysis.*

An *algebra R* is a linear space over $\mathbb{C}$ together with a multiplication such that $x(yz)=(xy)z$, $x(y+z)=xy+xz$, $(x+y)z=xz+yz$, and $\lambda(xy)=(\lambda x)y=x(\lambda y)$ for $x, y, z \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. A Banach space R is called a *Banach algebra* (or *normed ring*) if $\|xy\| \le \|x\|\|y\|$ is satisfied for all $x, y \in \mathbb{R}$ .When a Banach algebra contains a unity element e with respect to the multiplication we call it *unital* and we can suppose $\| e \| = 1$.An *involution* in a Banach algebra R is an operation $x \to x^*$ from $\mathbb{R}$ into itself that satisfies the properties: $(x+y)^* = x^* + y^*$; $(\lambda x)^* = \bar{\lambda} x^*$; $(xy)^* = y^* x^*$; $(x^*)^* = x$ for $x, y, \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. A Banach algebra with an involution such that $\|x^*\| = \|x\|$ is called a *Banach* $*$*-algebra*. A mapping $h : A \to B$ between two Banach $*$-algebras A and B is a homomorphism if $h(xy) = h(x)h(y)$ for all $x, y \in A$. A $*$-*homomorphism* $h : A \to B$ between two Banach $*$-algebras A and B is a homomorphism which preserves involutions, i.e. $h(x^*) = h(x)^*$. A Banach $*$-algebra R satisfying $\|x^* x\| = \|x\|^2$ for all $x \in \mathbb{R}$ is called a *C*-algebra*. A $\mathbb{C}$-linear mapping $\delta: A \to A$ in a C^*-algebra A is called a *derivation* in A if its domain $D(\delta)$ is a dense subalgebra of A and $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in D(\delta)$. If, moreover, $x \in D(\delta)$ implies $x^* \in D(\delta)$ and $\delta(x^*) = \delta(x)^*$, then δ is a $*$-*derivation*. Finally a mapping $h : A \to B$ is *an almost unital mapping* if an element $\lim_{n\to\infty} 2^n h(2^{-n} e)$ in B is invertible.

J.-R. Lee and D.-Y. Shin [4] achieved the Cauchy-Ulam stability of the Trif functional equation in C^*-algebras.In this paper we apply our Theorem 1 to almost unital mappings $h : A \to B$ between unital C^*-algebras A and B and give conditions in order for h to be a $*$-homomorphism. We also investigate the Cauchy-Ulam stability of the Jensen equation $2h\left(\frac{x+y}{2}\right) = h(x) + h(y)$ in C^*-algebras. Finally we establish the Cauchy-Ulam stability and $*$-homomorphism, as well as, $*$-derivations.

We assume throughout this paper, that A and B are unital C^*-algebras with unit e. Besides we denote with $U(A)$ the set of all unitary elements. We note that $U(A) = \{u \in A \mid u^* u = uu^* = 1\}$ is the unitary group in A.

2. The Jensen Equation and Almost Unital Mappings

Let us denote $A_\mu h(x,y) = 2h\left(\frac{\mu}{2}(x+y)\right) - \mu[h(x) + h(y)]$ for given mapping h: $A \to B$, any $\mu \in L^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and for all all $x, y \in A$.

Theorem 3.*Let h: $A \to B$ be an almost unital mapping such that $h(0)=0$, and $h(2^n xu) = h(x)h(2^n u)$ for all $x \in A$, all $u \in U(A)$, and all sufficiently large integers n.If the condition*

$$\| A_\mu h(x,y) \| \leq \theta, \ \theta \geq 0, \tag{13}$$

holds for all $\mu \in L^1$ and all $x, y \in A$ then h is a homomorphism. If, in addition, the condition

$$\left\| h\left(2^n u^*\right) - h\left(2^n u\right)^* \right\| \le \theta \tag{14}$$

holds for all u ∈ U(A),and all sufficiently large integers n, then h is a ∗-homomorphism.

Proof. Setting $\mu = 1$ in (13) and employing Theorem 1 with $p = 0$ and Proposition 1 with $f(0)=0$ one proves that there exists a unique additive mapping $L : A \to B$ defined by

$$L(x) = \lim_{n\to\infty} 2^{-n} h\left(2^n x\right)$$

for all $x \in A$ ([6] – [10]).Placing $y = x$ in (13), we find the inequality

$$\left\| A_\mu(x,x) \right\| = 2\left\| h(\mu x) - \mu h(x) \right\| \le \theta$$

for all $\mu \in L^1$ and all $x \in A$. Substituting x in this inequality with $2^n x$, one gets

$$2^{-n}\left\| h\left(\mu 2^n x\right) - \mu h\left(2^n x\right) \right\| \le 2^{-(n+1)}\theta \quad \text{,or} \quad \lim_{n\to\infty} 2^{-n}\left\| h\left(\mu 2^n x\right) - \mu h\left(2^n x\right) \right\| = 0 \text{ ,or}$$

$$\lim_{n\to\infty} 2^{-n} h\left(\mu 2^n x\right) = \mu \lim_{n\to\infty} 2^{-n} h\left(2^n x\right)$$

for all $\mu \in L^1$ and all $x \in A$.

Therefore $L(\mu x) = \lim_{n\to\infty} 2^{-n} h\left(2^n \mu x\right) = \mu \lim_{n\to\infty} 2^{-n} h\left(2^n x\right) = \mu L(x)$, for all $\mu \in L^1$ and all $x \in A$. But it is well-known that *if an additive mapping $L : A \to B$ satisfies $L(\mu x) = \mu L(x)$ for all $\mu \in L^1$, then L is a $\mathbb{C}$-linear mapping* ([5], [6], [8]). Thus L is $\mathbb{C}$-linear. We now claim that L is a homomorphism. In fact, from the hypothesis that $h\left(2^n xu\right) = h(x) h\left(2^n u\right)$ we get

$$L(xu) = \lim_{n\to\infty} 2^{-n} h\left(2^n xu\right) = h(x) \lim_{n\to\infty} 2^{-n} h\left(2^n u\right) = h(x) L(u).$$

Thus from $\mathbb{C}$-linearity of L, one finds $L(xu) = 2^{-n} L\left(2^n xu\right) = 2^{-n} h\left(2^n u\right) L(u)$.

Therefore

$$\lim_{n\to\infty} L(xu) = \left[\lim_{n\to\infty} 2^{-n} h\left(2^n x\right)\right] L(u), \text{ or } \quad L(xu) = L(x)L(u)$$

for all $x \in A$ and all $u \in U(A)$.

But any element in a C^*-algebra is a finite linear combination of unitary elements in A and so any $y \in A$ is of the form $y = \sum_{j=1}^{m} \eta_j v_j$ for $\eta_j \in C$ and $v_j \in U(A)$.

Hence $L(xy) = \sum_{j=1}^{m} \eta_j L\left(x v_j\right) = L(x) L\left(\sum_{j=1}^{m} \eta_j v_j\right) = L(x) L(y)$

for all $x,y \in A$, yielding that L is *a homomorphism*.Besides $h(x)L(e) = L(xe) = L(x)$ $L(e)$ for all $x \in A$, because e is a unitary element of A.

Therefore the identity

$$L(x) = h(x) \tag{15}$$

holds for all $x \in A$, because $L(e) = \lim_{n\to\infty} 2^{-n} h(2^n e)$ is invertible as h is almost unital. Hence h is *a homomorphism*. From (14) we find

$$\lim_{n\to\infty} 2^{-n} \| h(2^n u^*) - h(2^n u)^* \| \le \theta\left(\lim_{n\to\infty} 2^{-n}\right) = 0, \text{ or}$$
$$\lim_{n\to\infty} 2^{-n} h(2^n u^*) = \lim_{n\to\infty} 2^{-n} h(2^n u)^*, \quad \text{or}$$
$$L(u^*) = L(u)^*$$

for all $u \in U(A)$. But, in addition, the $\mathbb{C}$-linear map L is a homomorphism. We now claim that L is a $*$-homomorphism. In fact, any $x \in A$ is of the form $x = \sum_{i=1}^{k} \xi_i u_i$ for $\xi_i \in \mathbb{C}$ and $u_i \in U(A)$. Thus $L(x^*) = L\left(\left(\sum_{i=1}^{k} \xi_i u_i\right)^*\right) = \sum_{i=1}^{k} \overline{\xi_i} L(u_i^*) = \sum_{i=1}^{k} \overline{\xi_i} L(u_i)^*$

$$= \left(\sum_{i=1}^{k} \overline{\xi_i} L(u_i)\right)^* = L(x)^* \tag{16}$$

for all $x \in A$, $\xi_i \in \mathbb{C}$ and $u_i \in U(A)$, yielding L is a $*$-homomorphism. Therefore from (15) – (16), one gets $h(x^*) = h(x)^*$ for all $x \in A$ and thus h is also *a $*$-homomorphism*, completing the proof of the Theorem 3.

3. The Cauchy-Ulam Stability and $*$-Homomorphisms

Let us denote $H_\mu h(x,y,z,w) = 2h\left(\frac{\mu}{2}(x+y) + \frac{1}{2}zw\right) - \mu[h(x) + h(y)] - h(z)h(w)$ for any $\mu \in L^1$ and for all $x, y, z, w \in A$, and for a given mapping $h : A \to B$ from a unital C^*-algebra A to a unital C^*-algebra B.

Theorem 4. *Let $h : A \to B$ be a mapping such that $h(0) = 0$. If two conditions*

$$\| H_\mu h(x,y,z,w) \| \le \theta, \quad \theta \ge 0, \tag{17}$$

$$\| h(2^n u^*) - h(2^n u)^* \| \le \theta \tag{18}$$

hold for all $\mu \in L^1$, all $u \in U(A)$ and all $x, y, z, w \in A$, all sufficiently large integers n, then there exists a unique $*$-homomorphism $L : A \to B$ satisfying

$$\| h(x) - L(x) \| \le \theta, \tag{19}$$

for all $x \in A$.

Proof. Setting $\mu = 1$ and $z = w = 0$ in (17) and employing ideas from the proof of the above Theorems 1-3, one gets that there exists a unique $\mathbb{C}$-linear mapping $L : A \to B$

satisfying (19) and is given by $L(x) = \lim_{n\to\infty} 2^{-n} h(2^n x)$ for all $x \in A$. Therefore from (18) one finds $L(x^*) = L(x)^*$ for all $x \in A$. We claim that $L : A \to B$ is a homomorphism. In fact, putting $x = y = 0$ and $h(0) = 0$ in (17), we obtain

$$\|H_\mu h(0,0,z,w)\| \le \theta, \text{ or } \left\| 2h\left(\frac{zw}{2}\right) - h(z)h(w) \right\| \le \theta$$

for all $z, w \in A$. Thus placing $2^n z$ and $2^n w$ on z and w, respectively, and multiplying by 2^{-2n} we get

$$\left\| 2^{-2n} \cdot 2h\left(2^{2n} \frac{zw}{2}\right) - 2^{-n} h(2^n z) 2^{-n} h(2^n w) \right\| \le 2^{-2n}\theta \text{ for all } z, w \in A.$$

Therefore

$$\lim_{n\to\infty} 2^{-2n} \cdot 2h\left(2^{2n} \frac{zw}{2}\right) = \lim_{n\to\infty} 2^{-n} h(2^n z) \lim_{n\to\infty} 2^{-n} h(2^n w) \text{ for all } z, w \in A.$$

But $L(x) = \lim_{n\to\infty} 2^{-n} h(2^n x) = \lim_{n\to\infty} 2^{-2n} h(2^{2n} x)$ for all $x \in A$. Thus

$$L(zw) = 2L\left(\frac{zw}{2}\right) = 2\lim_{n\to\infty} 2^{-2n} h\left(2^{2n} \frac{zw}{2}\right) = \lim_{n\to\infty} 2^{-2n} \cdot 2h\left(2^{2n} \frac{zw}{2}\right)$$

$$= \left[\lim_{n\to\infty} 2^{-n} h(2^n z)\right]\left[\lim_{n\to\infty} 2^{-n} h(2^n w)\right] = L(z)L(w) \text{ for all } z, w \in A.$$

Hence, L is a unique $*$-homomorphism satisfying the identity (15),completing the proof of the Theorem 4.

4. The Cauchy-Ulam Stability and $*$-Derivations

Let us denote

$$D_\mu h(x,y,z,w) = 2h\left(\frac{\mu}{2}(x+y) + \frac{zw}{2}\right) - \mu[h(x) + h(y)] - zh(w) - h(z)w$$

for any $\mu \in L^1$ and for all $x, y, z, w \in A$, and for a given mapping $h : A \to A$ from a unital C^*-algebra A to itself.

Theorem 5. *Let* $h : A \to A$ *be a mapping such that* $h(0) = 0$. *If conditions*

$$\|D_\mu h(x,y,z,w)\| \le \theta, \quad \theta \ge 0, \tag{20}$$

and **(18)** *) hold for all* $\mu \in L^1$ *and all* $x, y, z, w \in A$, *then there exists a unique*$*$*-derivation* $\delta : A \to A$ *satisfying*

$$\| h(x) - \delta(x) \| \le \theta, \tag{21}$$

for all $x \in A$.

Proof. Setting $\mu = 1$ and $z = w = 0$ in (21) and employing ideas from the proof of the above Theorems 1-3, one gets that there exists a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (21) and is given by $\delta(x) = \lim_{n\to\infty} 2^{-n} h(2^n x)$, for all $x \in A$. Therefore $\delta(x^*) = \delta(x)^*$ for all $x \in A$. We claim that $\delta : A \to A$ is a derivation. In fact, putting $x = y = 0$ and $h(0) = 0$ in (20), we obtain

$$\left\| D_\mu h(0,0,z,w) \right\| \le \theta, \text{ or } \left\| h(zw) - [zh(w) + h(z)w] \right\| \le \theta$$

for all $z, w \in A$. Thus placing $2^n z$ and $2^n w$ on z and w, respectively and multiplying by 2^{-2n}, we get

$$\left\| 2^{-2n} h(2^{2n} zw) - \left[2^{-n} zh(2^n w) + h(2^n z) 2^{-n} w\right] \right\|$$
$$= \left\| 2^{-2n} \cdot 2h\left(2^{2n} \frac{zw}{2}\right) - \left[z\left(2^{-n} h(2^n w)\right) + \left(2^{-n} h(2^n z)\right) w\right] \right\| \le 2^{-2n} \theta$$

for all $z, w \in A$. Therefore

$$\lim_{n\to\infty} 2^{-2n} \cdot 2h\left(2^{2n} \frac{zw}{2}\right) = \lim_{n\to\infty} \left[z\left(2^{-n} h(2^n w)\right) + \left(2^{-n} h(2^n z) w\right)\right]$$
$$= z\left[\lim_{n\to\infty} 2^{-n} h(2^n w)\right] + \left[\lim_{n\to\infty} 2^{-n} h(2^n z)\right] w,$$

for all $z, w \in A$. But $\delta(x) = \lim_{n\to\infty} 2^{-n} h(2^n x) = \lim_{n\to\infty} 2^{-2n} h(2^{2n} x)$ for all $x \in A$. Thus

$$\delta(zw) = 2\delta\left(\frac{zw}{2}\right) = 2\lim_{n\to\infty} 2^{-2n} \cdot 2h\left(2^{2n} \frac{zw}{2}\right) = \lim_{n\to\infty} 2^{-2n} \cdot 2h\left(2^{2n} \frac{zw}{2}\right)$$
$$= z\left[\lim_{n\to\infty} 2^{-n} h(2^n w)\right] + \left[\lim_{n\to\infty} 2^{-n} h(2^n z)\right] w = z\delta(w) + \delta(z) w$$

for all $z, w \in A$. Hence, δ is a unique $*$-derivation satisfying the identity (21), completing the proof of the Theorem 5.

References

[1] Gavruta, P., An answer to a question of John M. Rassias concerning the stability of Cauchy equation, *"Advances in Equations and Inequalities"*, The Hadronic Press Math. Series, 1999, 67-71.

[2] Gruber, P.M., Stability of Isometries, *Trans.Amer. Math. Soc.*, **245** (1978), 263- 277.

[3] Hyers, D.H., On the stability of the linear functional equation, *Proc. Natl. Acad. Sci.*, U.S.A., **27**(1941), 222-224.

[4] Lee, J.-R. and Shin, D.-Y., On the Cauchy-Rassias stability of the Trif functional equation in C*-algebras, *J. Math. Anal. & Appl.*, **296**(2004), 351-363.

[5] Popoviciu, T., Sur certaines inéqualités qui caractérisent les fonctions convexes, *An. Stiint. Univ. "Al. I. Cuza" Iasi Sect. I a Mat*. **11**(1965), 155-164.

[6] Rassias, J.M., On Approximation of Approximately Linear Mappings by Linear Mappings, *J. Funct. Anal.* **46**(1982), 126-130.

[7] Rassias, J.M., On Approximation of Approximately Linear Mappings by Linear Mappings, *Bull. Sc. Math.* **108**(1984), 445-446.

[8] Rassias, J.M., Solution of a Problem of Ulam, *J. Approx. Th.* **57**(1989), 268-273.

[9] Rassias, J.M., Solution of a Stability Problem of Ulam, *Discuss. Math.* **12** (1992), 95-103.

[10] Rassias,J.M., Complete Solution of the Multi-dimensional Problem of Ulam, *Discuss. Math.* **14** (1994), 101-107.

[11] Trif, T., On the stability of a functional equation deriving from an inequality of Popoviciu for convex functions, *J. Math. Anal. & Appl.*, **272**(2002), 604-616.

[12] Ulam, S.M., *Problems in Modern Mathematics,* John Wiley & Sons, Inc., New York, 1964, Chapter VI.