SOLUTION OF THE HYERS-ULAM STABILITY PROBLEM FOR QUADRATIC TYPE FUNCTIONAL EQUATIONS IN SEVERAL VARIABLES

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Received 30 March 2005; accepted 9 June 2005; published 19 December 2005.

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ABSTRACT. In 1940 (and 1968) S. M. Ulam proposed the well-known Ulam stability problem. In 1941 D. H. Hyers solved the Hyers-Ulam problem for linear mappings. In 1951 D. G. Bourgin has been the second author treating the Ulam problem for additive mappings. In 1978 according to P. M. Gruber this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1982-2004 we established the Hyers-Ulam stability for the Ulam problem for different mappings. In this article we solve the Hyers-Ulam problem for quadratic type functional equations in several variables. These stability results can be applied in stochastic analysis, financial and actuarial mathematics, as well as in psychology and sociology.

Key words and phrases: Hyers-Ulam stability, Quadratic functional equation, Several variables.

2000 Mathematics Subject Classification. Primary 39B. Secondary 26D.
1. Introduction

In 1940 (and 1968) S. M. Ulam [24] proposed the Ulam stability problem: “When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"


In this paper we introduce the following quadratic type functional equation in several variables

\[ (1.1) \sum_{\varepsilon_j \in \{-1,1\}} Q \left( x_1 + \sum_{j=2}^{p} \varepsilon_j x_j \right) = 2^{p-1} \sum_{i=1}^{p} Q(x_i) \]

for \( p \) arbitrary but fixed and equal to 2, 3, 4, . . . , with mappings \( Q : X \to Y \).

**Definition 1.1.** A mapping \( Q : X \to Y \) is called quadratic type, if the above-mentioned quadratic type functional equation (1.1) holds for every \( p \)-dimensional vector \((x_1, x_2, \ldots, x_p) \in X^p\) with an arbitrary but fixed \( p = 2, 3, 4, \ldots \).

We consider the approximately quadratic type functional inequality

\[ (1.2) \left\| \sum_{\varepsilon_j \in \{-1,1\}} f \left( x_1 + \sum_{j=2}^{p} \varepsilon_j x_j \right) - 2^{p-1} \sum_{i=1}^{p} f(x_i) \right\| \leq cK_r (x_1, x_2, \ldots, x_p) \]

with approximately quadratic type mappings \( f : X \to Y \), where

\[ K_r (x_1, x_2, \ldots, x_p) = \sum_{\varepsilon_j \in \{-1,1\}} \left\| x_1 + \sum_{j=2}^{p} \varepsilon_j x_j \right\|^r \]

and a constant \( c \geq 0 \) (independent of \( x_1, x_2, \ldots, x_p \in X \), \( r \in \mathbb{R} - \{2\} \). If we denote \( Q_k = Q_k (x_1, x_2, \ldots, x_p) \) for \( k = 0, 1, 2, 3, \ldots, p-2, p-1 \) with an arbitrary but fixed \( p = 2, 3, 4, \ldots, E_p = x_1 + \sum_{j=2}^{p} \varepsilon_j x_j \), where for \( j = 2, 3, \ldots, p \), such that

\[ Q_0 = Q(x_1 + x_2 + \cdots + x_p) : \text{with all of the } \varepsilon_j = 1 \text{ in } E_p, \]

\[ Q_1 = Q(x_1 - x_2 + x_3 + \cdots + x_p) + Q(x_1 + x_2 - x_3 + \cdots + x_p) + \cdots + Q(x_1 + x_2 + x_3 + \cdots + x_p) \]
\( \cdots - x_{p-1} + x_p \) + \( Q(x_1 + x_2 + x_3 + \cdots + x_{p-1} - x_p) \): with any one of the \( \varepsilon_j = -1 \) and the rest of the \( \varepsilon_j = 1 \) in \( E_p \), 
\( Q_2 = Q(x_1 - x_2 - x_3 + \cdots + x_p) + \cdots + Q(x_1 - x_2 + x_3 + \cdots + x_{p-1} - x_p) + Q(x_1 + x_2 - x_3 - x_4 + \cdots + x_p) + \cdots + Q(x_1 + x_2 - x_3 + x_4 + \cdots + x_{p-1} - x_p) + \cdots + Q(x_1 + x_2 + \cdots + x_{p-2} - x_{p-1} - x_p) \): with any two of the \( \varepsilon_j = -1 \) and the rest of the \( \varepsilon_j = 1 \) in \( E_p \), 
\( Q_3 = Q(x_1 - x_2 - x_3 - x_4 + \cdots + x_{p-1} + x_p) + \cdots + Q(x_1 - x_2 - x_3 + \cdots + x_{p-1} - x_p) + \cdots + Q(x_1 + x_2 + \cdots + x_{p-3} - x_{p-2} - x_{p-1} - x_p) : \) with any three of the \( \varepsilon_j = -1 \) and the rest of the \( \varepsilon_j = 1 \) in \( E_p \), 
\( Q_{p-2} = Q(x_1 - x_2 - x_3 - x_4 - \cdots - x_{p-1} + x_p) + Q(x_1 - x_2 - x_3 - x_4 - \cdots - x_{p-2} + x_{p-1} - x_p) + Q(x_1 - x_2 + x_3 - x_4 - \cdots - x_{p-1} - x_p) + \cdots + Q(x_1 + x_2 - x_3 - x_4 - \cdots - x_{p-1} - x_p) \) : with any \( p - 2 \) of the \( \varepsilon_j = -1 \) and the rest of the \( \varepsilon_j = 1 \) in \( E_p \), 
\( Q_{p-1} = Q(x_1 - x_2 - \cdots - x_p) : \) with all of the \( \varepsilon_j = -1 \) in \( E_p \), 
then the functional equation (1.1) is equivalent to the following functional equation

\[
\sum_{k=0}^{p-1} Q_k(x_1, x_2, \ldots, x_p) = 2^{p-1} \sum_{i=1}^{p} Q(x_i)
\]

Also if \( f_k = f_k(x_1, x_2, \ldots, x_p) \) \( (k = 0, 1, 2, 3, \ldots, p - 2, p - 1) \) is given as a sum of \( \binom{p - 1}{k} \) terms of the form \( f(x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \cdots + \varepsilon_p x_p) : \) \( \varepsilon_j \in \{-1, 1\} \) \( (j = 2, 3, \ldots, p) \), in the same way as the afore-mentioned \( Q_k = Q_k(x_1, x_2, \ldots, x_p) \) in terms of \( Q(x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \cdots + \varepsilon_p x_p) \), then the functional inequality (1.2) is equivalent to the following functional inequality

\[
\left\| \sum_{k=0}^{p-1} f_k(x_1, x_2, \ldots, x_p) - 2^{p-1} \sum_{i=1}^{p} f(x_i) \right\| \leq cK_r(x_1, x_2, \ldots, x_p).
\]

Note that \( K_r = K_r(x_1, x_2, \ldots, x_p) = \sum_{k=0}^{p-1} R_k(x_1, x_2, \ldots, x_p) \), where \( R_k(x_1, x_2, \ldots, x_p) \) are equal to the above-mentioned \( Q_k(k = 0, 1, 2, \ldots, p - 1) \) with \( Q \) replaced by \( \| \cdot \| \). Thus

\[
R_0 = \| x_1 + x_2 + \cdots + x_p \|, \\
R_1 = \| x_1 - x_2 + x_3 + \cdots + x_p \| + \| x_1 + x_2 - x_3 + \cdots + x_p \| + \cdots + \| x_1 + x_2 + x_3 + \cdots + x_{p-1} - x_p \|, \\
\vdots \\
R_{p-1} = \| x_1 - x_2 - \cdots - x_p \|, \\
f_0 = f(x_1 + x_2 + \cdots + x_p), \\
f_1 = f(x_1 - x_2 + x_3 + \cdots + x_p) + f(x_1 + x_2 - x_3 + \cdots + x_2) + \cdots + f(x_1 + x_2 + x_3 + \cdots + x_{p-1} - x_p), \\
\vdots \\
f_{p-1} = f(x_1 - x_2 - \cdots - x_p).
\]

It is useful for the following, to observe that, from (1.3) with \( x_i = 0 (i = 1, 2, 3, \ldots, p - 2, p - 1, p) \), we get

\[
Q_k(0, 0, \ldots, 0) = \binom{p - 1}{k} Q(0).
\]

For \( k = 0, 1, 2, 3, \ldots, p - 1 \) with \( p = 2, 3, 4, \ldots \) and \( \left[ \sum_{k=0}^{p-1} \binom{p - 1}{k} \right] - p2^{p-1} \) \( Q(0) = 0 \), or \( (1 - p)2^{p-1}Q(0) = 0 \), because \( \sum_{k=0}^{p-1} \binom{p - 1}{k} = (1 + 1)^{p-1} = 2^{p-1} \), or

\[
Q(0) = 0.
\]

Now claim that for \( n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\} \)

\[
Q(2^n x) = (2^n)^2 Q(x) (n \in \mathbb{N}_0)
\]
For \( n = 0 \), it is trivial. From (1.3), with \( x_1 = x_2 = x, x_j = 0 (j = 3, 4, \ldots, p) \), we obtain
\[
Q_k(x, x, 0, \ldots, 0) = \left( \frac{p - 2}{k - 1} \right) Q(0) + \left( \frac{p - 2}{k} \right) Q(2x)
\]
for \( k = 0, 1, 2, \ldots, p - 1 \) with
\[
\begin{pmatrix}
p - 2 \\
-1
\end{pmatrix} = \begin{pmatrix} p - 2 \\ p - 1 \end{pmatrix} = 0
\]
for \( p = 2, 3, 4, \ldots \). Therefore from (1.3), (1.6), (1.8) and (1.9), we get
\[
\sum_{k=0}^{p-1} \left( \frac{p - 2}{k - 1} \right) Q(0) + \sum_{k=0}^{p-1} \left( \frac{p - 2}{k} \right) Q(2x) = 2^{p-1} [2Q(x) + (p - 2)Q(0)],
\]
or \( 2^{p-2}Q(2x) = 2^{p-1} [2Q(x)] \), or
\[
Q(2x) = 2^2Q(x)
\]
which is (1.7) for \( n = 1 \). Assume (1.7) is true. From (1.10), with \( 2^n x \) on place of \( x \), we get
\[
Q(2^{n+1}x) = 2^2Q(2^nx) = 2^2 (2^n)^2Q(x) = (2^{n+1})^2Q(x).
\]
This by induction, proves the formula (1.7). Similarly from (1.3), (1.5) and (1.6), with \( x_1 = x_2 = \frac{x}{2}, x_j = 0 (j = 3, 4, \ldots, p) \), we get
\[
Q_k \left( \frac{x}{2}, \frac{x}{2}, 0, \ldots, 0 \right) = \left( \frac{p - 2}{k} \right) Q(x)
\]
\( (k = 0, 1, 2, \ldots, p - 1) \) and
\[
\sum_{k=0}^{p-1} \left( \frac{p - 2}{k} \right) Q(x) = 2^{p-1} [2Q(2^{-1}x)] = 2^pQ(2^{-1}x), \text{ or}
\]
\[
Q(2^{-1}x) = 2^{-2}Q(x).
\]
By induction one gets that
\[
Q(2^{-n}x) = (2^{-n})^2Q(x) \quad (n \in \mathbb{N}_0).
\]
In fact, \( Q(2^{-(n+1)}x) = 2^{-2}Q(2^{-n}x) = 2^{-2} (2^{-n})^2Q(x) = (2^{-(n+1)})^2Q(x) \).

2. HYERS-ULAM QUADRATIC TYPE STABILITY

**Theorem 2.1.** Let \( X \) and \( Y \) be normed linear spaces. Assume that \( Y \) is complete. Assume in addition that \( f : X \to Y \) is a mapping for which there exists a constant \( c \geq 0 \) (independent of \( x_1, x_2, \ldots, x_p \in X \)) and \( r \in \mathbb{R} - \{2\} \), such that the above-mentioned quadratic type functional inequality (1.2) holds for every \( p \)-dimensional vector \( (x_1, x_2, \ldots, x_p) \in X^p \) with an arbitrary but fixed \( p = 2, 3, 4, \ldots \). Denote
\[
F_n(x) = \begin{cases} 
2^{-2n}f(2^n x), & \text{if } r < 2 \\
2^{2n}f(2^{-n}x), & \text{if } r > 2
\end{cases} \quad (n \in \mathbb{N}_0).
\]
Then the limit \( Q(x) = \lim_{n \to \infty} F_n(x) \) exists for every \( x \in X \) and \( Q : X \to Y \) is the unique quadratic type mapping, such that the inequality
\[
\|f(x) - Q(x)\| \leq c_r \|x\|^r,
\]
holds for every \( x \in X \), where \( c_r = \frac{c}{|1-2^r|} = \begin{cases} 
\frac{c}{2^{2r-1}}, & \text{if } r < 2 \\
\frac{c}{1-2^{-r}}, & \text{if } r > 2
\end{cases} \).
Proof. It is useful for the following, to observe that, with \( x_i = 0 \) \((i = 1, 2, \ldots, p)\), we get
\[
(2.3) \quad f_k(0, 0, \ldots, 0) = \left( \frac{p - 1}{k} \right) f(0).
\]
From (1.4) for \( k = 0, 1, 2, 3, \ldots, p - 1 \) with an arbitrary but fixed \( p = 2, 3, 4, \ldots \), we get
\[
\left\| \sum_{k=0}^{p-1} \left( \frac{p - 1}{k} \right) - p2^{p-1} \right\| f(0) \leq cK_r(0, 0, \ldots, 0) = 0, \text{ or } (p - 1)2^{p-1} \| f(0) \| \leq 0, \text{ or }
\]
\[
(2.4) \quad f(0) = 0.
\]
From (1.9), with \( x_1 = x_2 = x, x_j = 0 \) \((j = 3, 4, \ldots, p)\), we obtain
\[
(2.5) \quad f_k(x, x, 0, \ldots, 0) = \left( \frac{p - 2}{k - 1} \right) f(0) + \left( \frac{p - 2}{k} \right) f(2x)
\]
and \( R_k(x, x, 0, \ldots, 0) = \left( \frac{p - 2}{k} \right) \| 2x \|^r = \left( \frac{p - 2}{k} \right) 2^r \| x \|^r \) for \( k = 0, 1, 2, \ldots, p - 1 \).

Therefore from (1.4), (1.9), (2.4) and (2.5) we get that
\[
(2.6) \quad K_r = K_r(x, x, 0, \ldots, 0) = \sum_{k=0}^{p-1} R_k(x, x, 0, \ldots, 0) = 2^{p+r-2} \| x \|^r,
\]
and
\[
\left\| \sum_{k=0}^{p-1} \left( \frac{p - 2}{k - 1} \right) f(0) + \sum_{k=0}^{p-1} \left( \frac{p - 2}{k} \right) f(2x) - 2^{p-1} [2f(x) + (p - 2)f(0)] \right\|
\leq cK_r(x, x, 0, \ldots, 0),
\]
or
\[
\left\| 2^{p-2}f(2x) - 2^p f(x) \right\| \leq c \left\{ \sum_{k=0}^{p-1} \left( \frac{p - 2}{k} \right) \right\} \| 2x \|^r = c2^{p+r-2} \| x \|^r,
\]
or
\[
(2.7) \quad \| f(x) - 2^{-2n}f(2^nx) \| \leq c_r \left( 1 - 2^{(r-2)n} \right) \| x \|^r,
\]
holds for every \( x \in X, n \in \mathbb{N}_0 \) with \( p = 2, 3, 4, \ldots \), and \( c_r = \frac{c}{2^{2r-2n}}, r < 2 \).

For \( n = 0 \), it is trivial. Note that (2.6) yields (2.7) for \( n = 1 \). Assume (2.7) is true and from (2.6), with \( 2^{n-1} x \) on place of \( x \), we get
\[
\left\{ \left\| f(x) - 2^{-2n}f(2^nx) \right\| \right.
\leq \left. \left\| f(x) - 2^{-2(n-1)}f(2^{n-1}x) \right\| + \left\| 2^{-2(n-1)}f(2^{n-1}x) - 2^{-2n}f(2^nx) \right\| \right.
\leq \left. c_r \left\{ \left( 1 - 2^{(r-2)(n-1)} \right) + 2^{2(n-1)} \left( 1 - 2^{(r-2)} \right) 2^{r(n-1)} \right\} \| x \|^r \right. = c_r \left( 1 - 2^{(r-2)n} \right) \| x \|^r.
\]
Similarly from (1.9), with \( x_1 = x_2 = \frac{x}{2}, x_j = 0 \) \((j = 3, 4, \ldots, p)\), we get
\[
f_k \left( \frac{x}{2}, \frac{x}{2}, 0 \right) = \left( \frac{p - 2}{k - 1} \right) f(0) + \left( \frac{p - 2}{k} \right) f(x), \text{ and } R_k \left( \frac{x}{2}, \frac{x}{2}, 0 \right) = \left( \frac{p - 2}{k} \right) \| x \|^r \text{ for } k = 0, 1, 2, \ldots, p - 1.
\]
Therefore from these and (1.4), (1.9), and (2.4) we obtain that
\[
K_r = K_r \left( \frac{x}{2}, \frac{x}{2}, 0, \ldots, 0 \right) = \sum_{k=0}^{p-1} R_k \left( \frac{x}{2}, \frac{x}{2}, 0, \ldots, 0 \right) = 2^{p-2} \| x \|^r,
\]
and
\[
\left\| \sum_{k=0}^{p-1} \left( \frac{p-2}{k-1} \right) f(0) + \sum_{k=0}^{p-1} \left( \frac{p-2}{k} \right) f(x) - 2^{p-1} \left( 2f(2^{-1}x) + (p-2)f(0) \right) \right\| \\
\leq cK_r,
\]
or
\[
\left\| 2^{p-2}f(x) - 2^{p}f(2^{-1}x) \right\| \leq c2^{p-2} \|x\|^r,
\]
or
\[
(2.8) \quad \left\| f(x) - 2^{2}f(2^{-1}x) \right\| \leq c \|x\|^r = c_r \left( 1 - 2^{-r} \right) \|x\|^r,
\]
where \(c_r = \frac{c}{1 - 2^{-r}}, r > 2\). Claim that inequality
\[
(2.9) \quad \left\| f(x) - 2^{2n}f(2^{-n}x) \right\| \leq c_r \left( 1 - 2^{(2-r)n} \right) \|x\|^r,
\]
holds for every \(x \in X, n \in \mathbb{N}_0\) with \(p = 2, 3, 4, 5, \ldots\), and \(c_r = \frac{c}{1 - 2^{-r}}, r > 2\).

For \(n = 0\), it is trivial. Note that (2.8) yields (2.9) for \(n = 1\). Assume (2.9) is true and from (2.8), with \(2^{-n}x\) on place of \(x\), we obtain:
\[
\left\| f(x) - 2^{2n}f(2^{-n}x) \right\| \\
\leq \left\| f(x) - 2^{2(n-1)}f(2^{-n+1}x) \right\| + 2^{2(n-1)}f(2^{-n+1}x) - 2^{2n}f(2^{-n}x) \left\| x \right\|^r \\
\leq c_r \left\{ (1 - 2^{(2-r)(n-1)}) + 2^{2(n-1)} \left( 1 - 2^{-2r} \right) 2^{-r(n-1)} \right\} \left\| x \right\|^r \\
= c_r \left( 1 - 2^{(2-r)n} \right) \|x\|^r, \quad r > 2,
\]
Claim now that the sequence \(\{F_n(x)\} : F_n(x) = 2^{-2n}f(2^n x)\), converges if \(r < 2\). To do this it suffices to prove that \(\{F_n(x)\}\) is a Cauchy sequence. Inequality (2.7) is involved. In fact, if \(i > j > 0\) and \(h_1 := 2^j x\), we have
\[
\left\| F_i(x) - F_j(x) \right\| = \left\| 2^{-2i}f(2^i x) - 2^{-2j}f(2^j x) \right\| = 2^{-2j} \left\| 2^{2(i-j)}f(2^{-i}h_1) - f(h_1) \right\| \\
\leq 2^{-2j} c_r \left( 1 - 2^{(2-r)(i-j)} \right) \left\| h_1 \right\|^r = 2^{-2j} c_r \left( 1 - 2^{(2-r)(i-j)} \right) 2^{-rj} \left\| x \right\|^r \\
= 2^{j(r-2)}c_r \left( 1 - 2^{(r-2)(i-j)} \right) \left\| x \right\|^r \\
< c_r 2^{(r-2j)} \left\| x \right\|^r \to 0, \quad \text{as } j \to \infty, r < 2.
\]
Similarly claim that the sequence \(\{F_n(x)\} : F_n(x) = 2^{2n}f(2^{-n}x)\) converges if \(r > 2\). To do this it suffices to prove that \(\{F_n(x)\}\) is a Cauchy sequence. Inequality (2.9) is involved. In fact, if \(i > j > 0\) and \(h_2 := 2^{-j} x\), we get
\[
\left\| F_i(x) - F_j(x) \right\| = \left\| 2^{2i}f(2^{-i}x) - 2^{2j}f(2^{-j}x) \right\| = 2^{2j} \left\| 2^{2(i-j)}f(2^{-(i-j)}h_2) - f(h_2) \right\| \\
\leq 2^{2j} c_r \left( 1 - 2^{(2-r)(i-j)} \right) \left\| h_2 \right\|^r = 2^{2j} c_r \left( 1 - 2^{(2-r)(i-j)} \right) 2^{-jr} \left\| x \right\|^r \\
= 2^{j(2-r)}c_r \left( 1 - 2^{(2-r)} \right) \left\| x \right\|^r \\
< 2^{j(2-r)}c_r \left\| x \right\|^r \xrightarrow{j \to \infty} 0.
\]
Also claim that formula (2.1), with \(r < 2\), yields a quadratic type mapping \(Q : X \to Y\).

Note that from (1.4), (2.1) with \(r < 2\) and the fact that \(\lim_{n \to \infty} 2^{-2n}f_k(2^n x_1, 2^n x_2, \ldots, 2^n x_p) = Q_k(x_1, x_2, \ldots, x_p)\), as well as \(K_r(2^n x_1, 2^n x_2, \ldots, 2^n x_p) = \sum_{k=0}^{p-1} R_k(2^n x_1, 2^n x_2, \ldots, 2^n x_p)\) =
2^{nr}K_r (x_1, x_2, \ldots, x_p) we get
\begin{align*}
\left\lVert \sum_{k=0}^{p-1} \lim_{n \to \infty} 2^{-2n} f_k(2^n x_1, 2^n x_2, \ldots, 2^n x_p) - 2^{p-1} \sum_{i=1}^{p} \lim_{n \to \infty} 2^{-2n} f(2^n x_i) \right\rVert \\
= \left\lVert \sum_{k=0}^{p-1} Q_k(x_1, x_2, \ldots, x_p) - 2^{p-1} \sum_{i=1}^{p} Q(x_i) \right\rVert \leq c \lim_{n \to \infty} 2^{-2n} K_r (2^n x_1, 2^n x_2, \ldots, 2^n x_p) \\
= \left( \lim_{n \to \infty} 2^{n(2-r)} \right) c K_r (x_1, x_2, \ldots, x_p) = 0,
\end{align*}
which is (1.3). Similarly claim that the formula (2.1), with \( r > 2 \), yields a quadratic type mapping
\( Q : X \to Y \). Note that from (1.4), (2.1) with \( r > 2 \) and the fact that
\begin{align*}
\lim_{n \to \infty} 2^{-2n} f_k(2^{-n} x_1, 2^{-n} x_2, \ldots, 2^{-n} x_p) = Q_k(x_1, x_2, \ldots, x_p)
\end{align*}
as well as
\begin{align*}
K_r (2^{-n} x_1, 2^{-n} x_2, \ldots, 2^{-n} x_p) = \sum_{k=0}^{p-1} R_k (2^{-n} x_1, 2^{-n} x_2, \ldots, 2^{-n} x_p)
= 2^{-nr} K_r (x_1, x_2, \ldots, x_p)
\end{align*}
we obtain
\begin{align*}
\left\lVert \sum_{k=0}^{p-1} \lim_{n \to \infty} 2^{2n} f_k(2^{-n} x_1, 2^{-n} x_2, \ldots, 2^{-n} x_p) - 2^{p-1} \sum_{i=1}^{p} \lim_{n \to \infty} 2^{2n} f(2^n x_i) \right\rVert \\
= \left\lVert \sum_{k=0}^{p-1} Q_k(x_1, x_2, \ldots, x_p) - 2^{p-1} \sum_{i=1}^{p} Q(x_i) \right\rVert \leq c \lim_{n \to \infty} 2^{-2n} K_r (2^{-n} x_1, 2^{-n} x_2, \ldots, 2^{-n} x_p) \\
= \left( \lim_{n \to \infty} 2^{n(2-r)} \right) c K_r (x_1, x_2, \ldots, x_p) = 0,
\end{align*}
which is (1.3). It is now clear from (2.7) and (2.9) with \( n \to \infty \), and the formula (2.1) that
inequality (2.2) holds in \( X \). This completes the existence proof of our Theorem 2.1. It remains
to prove the uniqueness for this Theorem. Let \( Q' : X \to Y \) be another quadratic type mapping
satisfying (2.2). Then we have to prove that \( Q' = Q \). In fact, remember that both \( Q \) and \( Q' \)
satisfy (1.7) for \( r < 2 \). Then from the triangle inequality and (2.2) with \( r < 2 \) one gets that
\begin{align*}
\|Q(x) - Q'(x)\| = \|2^{-2n}Q(2^n x) - 2^{-2n}Q'(2^n x)\| \\
\leq 2^{-2n} \left\{ \|Q(2^n x) - f(2^n x)\| + \|Q'(2^n x) - f(2^n x)\| \right\} \\
\leq 2^{-2n} \cdot 2 \cdot 2^{nr} c_r \|x\|^r = 2 \cdot 2^{n(r-2)} \cdot c_r \cdot \|x\|^r \xrightarrow{n \to \infty} 0,
\end{align*}
for every \( x \in X \) and \( n \in \mathbb{N}_0 \). Thus
\begin{align*}
(2.10) \quad Q(x) = Q'(x)
\end{align*}
for every \( x \in X \) and \( r < 2 \). Similarly both \( Q \) and \( Q' \) satisfy (1.11), as well. Then from the
triangle inequality and (2.2) with \( r > 2 \) one obtains that
\begin{align*}
\|Q(x) - Q'(x)\| = \|2^{2n}Q(2^{-n} x) - 2^{2n}Q'(2^{-n} x)\| \\
\leq 2^{2n} \left\{ \|Q(2^{-n} x) - f(2^{-n} x)\| + \|Q'(2^{-n} x) - f(2^{-n} x)\| \right\} \\
\leq 2^{2n} \cdot 2 \cdot 2^{-nr} c_r \|x\|^r = 2 \cdot 2^{n(2-r)} \cdot c_r \cdot \|x\|^r \xrightarrow{n \to \infty} 0,
\end{align*}
for every \( x \in X \) and \( n \in \mathbb{N}_0 \). Thus (2.10) holds for every \( x \in X \) and \( r>2 \). This completes the proof of the uniqueness of our theorem and thus of the stability for the quadratic type equation (1.1) in several variables \( x_1, x_2, \ldots, x_p \in X \).

**Corollary 2.2.** Let \( X \) and \( Y \) be normed linear spaces. Assume that \( Y \) is complete. Assume in addition that \( f : X \to Y \) is a mapping for which there exists a constant \( c \geq 0 \) (independent of \( x_1, x_2 \in X \)) and \( r \in \mathbb{R} - \{2\} \), such that

\[
\|f(x_1 + x_2) + f(x_1 - x_2) - 2(f(x_1) + f(x_2))\| \leq c(\|x_1 + x_2\|^r + \|x_1 - x_2\|^r).
\]

Then the limit of the formula (2.1) exists and \( Q : X \to Y \) is the unique quadratic type mapping, such that (2.2) holds.

**References**


