The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 2, Issue 2, Article 11, pp. 1-9, 2005

SOLUTION OF THE HYERS-ULAM STABILITY PROBLEM FOR QUADRATIC TYPE FUNCTIONAL EQUATIONS IN SEVERAL VARIABLES

JOHN MICHAEL RASSIAS

Received 30 March 2005; accepted 9 June 2005; published 19 December 2005.

PEDAGOGICAL DEPARTMENT, E.E., NATIONAL AND CAPODISTRIAN UNIVERSITY OF ATHENS, SECTION OF MATHEMATICS AND INFORMATICS, 4, AGAMEMNONOS STR., AGHIA PARASKEVI, ATHENS 15342, GREECE
jrassias@primedu.uoa.gr
URL: http://www.primedu.uoa.gr/~jrassias/

ABSTRACT. In 1940 (and 1968) S. M. Ulam proposed the well-known Ulam stability problem. In 1941 D. H. Hyers solved the Hyers-Ulam problem for linear mappings. In 1951 D. G. Bourgin has been the second author treating the Ulam problem for additive mappings. In 1978 according to P. M. Gruber this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1982-2004 we established the Hyers-Ulam stability for the Ulam problem for different mappings. In this article we solve the Hyers-Ulam problem for quadratic type functional equations in several variables. These stability results can be applied in stochastic analysis, financial and actuarial mathematics, as well as in psychology and sociology.

Key words and phrases: Hyers-Ulam stability, Quadratic functional equation, Several variables.

2000 *Mathematics Subject Classification.* Primary 39B. Secondary 26D..

ISSN (electronic): 1449-5910

1. INTRODUCTION

In 1940 (and 1968) S. M. Ulam [24] proposed the *Ulam stability problem*:

"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true ?"

In 1941 D. H. Hyers [13] solved this problem for linear mappings. In 1951 D. G. Bourgin [3] was the second author to treat the Ulam problem for additive mappings. In 1978, according to P. M. Gruber [12], this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1980 and in 1987, I. Fenyö [7, 8] established the stability of the Ulam problem for quadratic and other mappings. In 1987 Z. Gajda and R. Ger [10] showed that one can get analogous stability results for subadditive multifunctions. Other interesting stability results have been achieved also by the following authors : J. Aczél [1], C. Borelli and G. L. Forti [2, 9], P. W. Cholewa [4], St. Czerwik [5], and H. Drljevic [6], and Pl. Kannappan [15]. In 1982-2004 J. M. Rassias ([16, 17, 18, 19, 20, 21, 23]) and in 2003 M. J. Rassias and the author [22] solved the above Ulam problem for different mappings. In 1999 P. Gavruta [11] answered a question of ours [18] concerning the stability of the Cauchy equation. In 1998 S.-M. Jung [14] and in 2002-2003 M. J. Rassias and the author [21, 22] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains.

In this article we solve the Hyers-Ulam stability problem for quadratic type functional equations in several variables.

Throughout this paper, let X be a real normed space and Y be a real Banach space in the case of functional inequalities, as well as let X and Y be real linear spaces for functional equations. In this paper we introduce the following quadratic type functional equation in several variables $x_1, x_2, \dots, x_p \in X$

$$\sum_{\varepsilon_j\in\{-1,1\}} Q\left(x_1+\sum_{j=2}^{p}\varepsilon_j x_j\right) = 2^{p-1}\sum_{i=1}^{p} Q(x_i) \tag{1.1}$$

for p arbitrary but fixed and equal to $2, 3, 4, \dots$, with mappings $Q : X \to Y$.

Definition 1.1. A mapping $Q : X \to Y$ is called *quadratic type*, if the above-mentioned quadratic type functional equation (1.1) holds for every p-dimensional vector $(x_1, x_2, \dots, x_p) \in X^p$ with an arbitrary but fixed $p = 2, 3, 4, \dots$.

We consider the approximately quadratic type functional inequality

$$\left\|\sum_{\varepsilon_j\in\{-1,1\}} f\left(x_1+\sum_{j=2}^{p}\varepsilon_j x_j\right) - 2^{p-1}\sum_{i=1}^{p} f(x_i)\right\| \le cK_r\left(x_1, x_2, \dots, x_p\right) \tag{1.2}$$

with approximately quadratic type mappings $f : X \to Y$, where

$$K_r\left(x_1, x_2, \dots, x_p\right) = \sum_{\varepsilon_j\in\{-1,1\}} \left\|x_1+\sum_{j=2}^{p}\varepsilon_j x_j\right\|^r,$$

and a constant $c \ge 0$ (independent of $x_1, x_2, \dots, x_p \in X$), $r \in \mathbb{R} - \{2\}$. If we denote $Q_k = Q_k(x_1, x_2, \dots, x_p)$ for $k = 0, 1, 2, 3, \dots, p-2, p-1$ with an arbitrary but fixed $p = 2, 3, 4, \dots,$ $E_p = x_1 + \sum\limits_{j=2}^{p} \varepsilon_j x_j$, where for $j = 2, 3, \dots, p$, such that

$Q_0 = Q(x_1 + x_2 + \cdots + x_p)$: with all of the $\varepsilon_j = 1$ in E_p,

$Q_1 = Q(x_1 - x_2 + x_3 + \cdots + x_p) + Q(x_1 + x_2 - x_3 + \cdots + x_p) + \cdots + Q(x_1 + x_2 + x_3 +$

$\cdots - x_{p-1} + x_p) + Q(x_1 + x_2 + x_3 + \cdots + x_{p-1} - x_p)$: with any one of the $\varepsilon_j = -1$ and the rest of the $\varepsilon_j = 1$ in E_p,
$Q_2 = Q(x_1 - x_2 - x_3 + \cdots + x_p) + \cdots + Q(x_1 - x_2 + x_3 + \cdots - x_{p-1} + x_p) + Q(x_1 - x_2 + x_3 + \cdots + x_{p-1} - x_p) + Q(x_1 + x_2 - x_3 - x_4 + \cdots + x_p) + \cdots + Q(x_1 + x_2 - x_3 + x_4 + \cdots + x_{p-1} - x_p) + \cdots + Q(x_1 + x_2 + \cdots + x_{p-2} - x_{p-1} - x_p)$: with any two of the $\varepsilon_j = -1$ and the rest of the $\varepsilon_j = 1$ in E_p,
$Q_3 = Q(x_1 - x_2 - x_3 - x_4 + \cdots + x_{p-1} + x_p) + \cdots + Q(x_1 - x_2 - x_3 + \cdots + x_{p-1} - x_p) + \cdots + Q(x_1 + x_2 + \cdots + x_{p-3} - x_{p-2} - x_{p-1} - x_p)$: with any three of the $\varepsilon_j = -1$ and the rest of the $\varepsilon_j = 1$ in $E_p, \ldots,$
$Q_{p-2} = Q(x_1 - x_2 - x_3 - x_4 - \cdots - x_{p-1} + x_p) + Q(x_1 - x_2 - x_3 - x_4 - \cdots - x_{p-2} + x_{p-1} - x_p) + Q(x_1 - x_2 + x_3 - x_4 - \cdots - x_{p-1} - x_p) + \cdots + Q(x_1 + x_2 - x_3 - x_4 - \cdots - x_{p-1} - x_p)$: with any $p-2$ of the $\varepsilon_j = -1$ and the rest of the $\varepsilon_j = 1$ in E_p,
$Q_{p-1} = Q(x_1 - x_2 - \cdots - x_p)$: with all of the $\varepsilon_j = -1$ in E_p,
then the functional equation (1.1) is equivalent to the following functional equation

$$\sum_{k=0}^{p-1} Q_k(x_1, x_2, \ldots, x_p) = 2^{p-1} \sum_{i=1}^{p} Q(x_i) \tag{1.3}$$

Also if $f_k = f_k(x_1, x_2, \ldots, x_p)$ $(k = 0, 1, 2, 3, \ldots, p-2, p-1)$ is given as a sum of $\binom{p-1}{k}$ terms of the form $f(x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \cdots + \varepsilon_p x_p) : \varepsilon_j \in \{-1, 1\}$ $(j = 2, 3, \ldots, p)$, in the same way as the afore-mentioned $Q_k = Q_k(x_1, x_2, \ldots, x_p)$ in terms of $Q(x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \cdots + \varepsilon_p x_p)$, then the functional inequality (1.2) is equivalent to the following functional inequality

$$\left\| \sum_{k=0}^{p-1} f_k(x_1, x_2, \ldots, x_p) - 2^{p-1} \sum_{i=1}^{p} f(x_i) \right\| \leq c K_r\left(x_1, x_2, \ldots, x_p\right). \tag{1.4}$$

Note that $K_r = K_r(x_1, x_2, \ldots, x_p) = \sum_{k=0}^{p-1} R_k(x_1, x_2, \ldots, x_p)$, where $R_k(x_1, x_2, \ldots, x_p)$ are equal to the above-mentioned $Q_k (k = 0, 1, 2, \ldots, p-1)$ with Q replaced by $\|\cdot\|^r$. Thus
$R_0 = \|x_1 + x_2 + \cdots + x_p\|^r$,
$R_1 = \|x_1 - x_2 + x_3 + \cdots + x_p\|^r + \|x_1 + x_2 - x_3 + \cdots + x_p\|^r + \ldots$
$+ \|x_1 + x_2 + x_3 + \cdots + x_{p-1} - x_p\|^r, \ldots, R_{p-1} = \|x_1 - x_2 - \cdots - x_p\|^r$, and
$f_0 = f(x_1 + x_2 + \cdots + x_p)$,
$f_1 = f(x_1 - x_2 + x_3 + \cdots + x_p) + f(x_1 + x_2 - x_3 + \cdots + x_p) + \cdots + f(x_1 + x_2 + x_3 \cdots + x_{p-1} - x_p)$,
$\ldots, f_{p-1} = f(x_1 - x_2 - \cdots - x_p)$.
It is useful for the following, to observe that, from (1.3) with $x_i = 0 (i = 1, 2, 3, \ldots, p-2, p-1, p)$, we get

$$Q_k(0, 0, \ldots, 0) = \binom{p-1}{k} Q(0). \tag{1.5}$$

For $k = 0, 1, 2, 3, \ldots, p-1$ with $p = 2, 3, 4, \ldots$ and $\left[\sum_{k=0}^{p-1} \binom{p-1}{k} - p 2^{p-1}\right] Q(0) = 0$, or $(1-p) 2^{p-1} Q(0) = 0$, because $\sum_{k=0}^{p-1} \binom{p-1}{k} = (1+1)^{p-1} = 2^{p-1}$, or

$$Q(0) = 0. \tag{1.6}$$

Now claim that for $n \in \mathbb{N}_0 = \{0\} \bigcup \mathbb{N} = \{0, 1, 2, \ldots\}$

$$Q\left(2^n x\right) = \left(2^n\right)^2 Q(x) \;\; (n \in \mathbb{N}_0) \tag{1.7}$$

For $n=0$, it is trivial. From (1.3), with $x_1=x_2=x, x_j=0 (j=3,4,\dots,p)$, we obtain

$$Q_k(x,x,0,\dots,0)=\binom{p-2}{k-1}Q(0)+\binom{p-2}{k}Q(2x) \tag{1.8}$$

for $k=0,1,2,\dots,p-1$ with

$$\binom{p-2}{-1}=\binom{p-2}{p-1}=0 \tag{1.9}$$

for $p=2,3,4,\ldots$ Therefore from (1.3), (1.6), (1.8) and (1.9), we get

$$\sum_{k=0}^{p-1}\binom{p-2}{k-1}Q(0)+\sum_{k=0}^{p-1}\binom{p-2}{k}Q(2x)=2^{p-1}\left[2Q(x)+(p-2)Q(0)\right],$$

or $2^{p-2}Q(2x)=2^{p-1}\left[2Q(x)\right]$, or

$$Q(2x)=2^2Q(x) \tag{1.10}$$

which is (1.7) for $n=1$. Assume (1.7) is true. From (1.10), with $2^n x$ on place of x, we get

$$Q\left(2^{n+1}x\right)=2^2Q\left(2^n x\right)=2^2\left(2^n\right)^2Q(x)=\left(2^{n+1}\right)^2Q(x).$$

This by induction, proves the formula (1.7). Similarly from (1.3), (1.5) and (1.6), with $x_1=x_2=\frac{x}{2}$, $x_j=0$ $(j=3,4,\dots,p)$, we get $Q_k\left(\frac{x}{2},\frac{x}{2},0,\dots,0\right)=\binom{p-2}{k}Q\left(x\right)$ $(k=0,1,2,\dots,p-1)$ and $\sum\limits_{k=0}^{p-1}\binom{p-2}{k}Q\left(x\right)=2^{p-1}\left[2Q\left(2^{-1}x\right)\right]=2^pQ\left(2^{-1}x\right)$, or $Q\left(2^{-1}x\right)=2^{-2}Q(x)$.

By induction one gets that

$$Q\left(2^{-n}x\right)=\left(2^{-n}\right)^2Q(x)\ \ (n\in\mathbb{N}_0)\,. \tag{1.11}$$

In fact, $Q\left(2^{-(n+1)}x\right)=2^{-2}Q\left(2^{-n}x\right)=2^{-2}\left(2^{-n}\right)^2Q(x)=\left(2^{-(n+1)}\right)^2Q(x)$.

2. HYERS-ULAM QUADRATIC TYPE STABILITY

Theorem 2.1. *Let X and Y be normed linear spaces. Assume that Y is complete. Assume in addition that $f:X\to Y$ is a mapping for which there exists a constant $c\geq 0$ (independent of $x_1,x_2,\dots,x_p\in X$) and $r\in\mathbb{R}-\{2\}$, such that the above-mentioned quadratic type functional inequality (1.2) holds for every p-dimensional vector $(x_1,x_2,\dots,x_p)\in X^p$ with an arbitrary but fixed $p=2,3,4,\ldots$. Denote*

$$F_n\left(x\right)=\begin{cases}2^{-2n}f\left(2^n x\right), & if\ r<2\\ 2^{2n}f\left(2^{-n}x\right), & if\ r>2\end{cases}\ \ (n\in\mathbb{N}_0)\,. \tag{2.1}$$

Then the limit $Q\left(x\right)=\lim\limits_{n\to\infty}F_n\left(x\right)$ exists for every $x\in X$ and $Q:X\to Y$ is the unique quadratic type mapping, such that the inequality

$$\left\|f(x)-Q(x)\right\|\leq c_r\left\|x\right\|^r, \tag{2.2}$$

holds for every $x\in X$, where $c_r=\frac{c}{\left|1-2^{2-r}\right|}=\begin{cases}\frac{c}{2^{2-r}-1}, & if\ r<2\\ \frac{c}{1-2^{2-r}}, & if\ r>2\end{cases}$.

Proof. It is useful for the following, to observe that, with $x_i = 0$ $(i = 1, 2, \ldots, p)$, we get

$$f_k(0, 0, \ldots, 0) = \left(\begin{array}{c} p-1 \\ k \end{array} \right) f(0). \tag{2.3}$$

From (1.4) for $k = 0, 1, 2, 3, \ldots, p-1$ with an arbitrary but fixed $p = 2, 3, 4, \ldots$, we get $\left\| \left[\sum\limits_{k=0}^{p-1} \left(\begin{array}{c} p-1 \\ k \end{array} \right) - p2^{p-1} \right] f(0) \right\| \leq cK_r\,(0, 0, \ldots, 0) = 0$, or $(p-1)2^{p-1}\,\|f(0)\| \leq 0$, or

$$f(0) = 0. \tag{2.4}$$

From (1.9), with $x_1 = x_2 = x, x_j = 0$ $(j = 3, 4, \ldots, p)$, we obtain

$$f_k(x, x, 0, \ldots, 0) = \left(\begin{array}{c} p-2 \\ k-1 \end{array} \right) f(0) + \left(\begin{array}{c} p-2 \\ k \end{array} \right) f(2x) \tag{2.5}$$

and $R_k(x, x, 0, \ldots, 0) = \left(\begin{array}{c} p-2 \\ k \end{array} \right) \|2x\|^r = \left(\begin{array}{c} p-2 \\ k \end{array} \right) 2^r\,\|x\|^r$ for $k = 0, 1, 2, \ldots, p-1$. Therefore from (1.4), (1.9), (2.4) and (2.5) we get that

$$K_r = K_r(x, x, 0, \ldots, 0) = \sum_{k=0}^{p-1} R_k\,(x, x, 0, \ldots, 0) = 2^{p+r-2}\,\|x\|^r\,,$$

and

$$\begin{aligned} & \left\| \sum_{k=0}^{p-1} \left(\begin{array}{c} p-2 \\ k-1 \end{array} \right) f(0) + \sum_{k=0}^{p-1} \left(\begin{array}{c} p-2 \\ k \end{array} \right) f(2x) - 2^{p-1}\,[2f(x) + (p-2)f(0)] \right\| \\ \leq \;& cK_r\,(x, x, 0, \ldots, 0)\,, \end{aligned}$$

or

$$\left\| 2^{p-2} f(2x) - 2^p f(x) \right\| \leq c \left\{ \sum_{k=0}^{p-1} \left(\begin{array}{c} p-2 \\ k \end{array} \right) \right\} \|2x\|^r = c2^{p+r-2}\,\|x\|^r\,,$$

or

$$\left\| f(x) - 2^{-2} f(2x) \right\| \leq c2^{r-2}\,\|x\|^r = c_r\left(1 - 2^{r-2}\right)\|x\|^r\,, \tag{2.6}$$

where $c_r = \frac{c}{2^{2-r}-1}, r < 2$. Claim that inequality

$$\left\| f(x) - 2^{-2n} f(2^n x) \right\| \leq c_r(1 - 2^{(r-2)n})\,\|x\|^r\,, \tag{2.7}$$

holds for every $x \in X, n \in \mathbb{N}_0$ with $p = 2, 3, 4, 5, \ldots$, and $c_r = \frac{c}{2^{2-r}-1}$, $r<2$.
For $n = 0$, it is trivial. Note that (2.6) yields (2.7) for $n = 1$. Assume (2.7) is true and from (2.6), with $2^{n-1}x$ on place of x, we get

$$\begin{aligned} & \left\| f(x) - 2^{-2n} f(2^n x) \right\| \\ \leq \;& \left\| f(x) - 2^{-2(n-1)} f(2^{n-1}x) \right\| + \left\| 2^{-2(n-1)} f(2^{n-1}x) - 2^{-2n} f(2^n x) \right\| \\ \leq \;& c_r \left\{ \left(1 - 2^{(r-2)(n-1)}\right) + 2^{-2(n-1)}\left(1 - 2^{r-2}\right) 2^{r(n-1)} \right\} \|x\|^r = c_r\left(1 - 2^{(r-2)n}\right)\|x\|^r\,. \end{aligned}$$

Similarly from (1.9), with $x_1 = x_2 = \frac{x}{2}$, $x_j = 0$ $(j = 3, 4, \ldots, p)$, we get $f_k\left(\frac{x}{2}, \frac{x}{2}, 0\right) = \left(\begin{array}{c} p-2 \\ k-1 \end{array} \right) f\,(0) + \left(\begin{array}{c} p-2 \\ k \end{array} \right) f\,(x)$, and $R_k\left(\frac{x}{2}, \frac{x}{2}, 0\right) = \left(\begin{array}{c} p-2 \\ k \end{array} \right) \|x\|^r$ for $k = 0, 1, 2, \ldots, p-1$. Therefore from these and (1.4), (1.9), and (2.4) we obtain that

$$K_r = K_r\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right) = \sum_{k=0}^{p-1} R_k\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right) = 2^{p-2}\,\|x\|^r\,,$$

and

$$\left\| \sum_{k=0}^{p-1} \binom{p-2}{k-1} f(0) + \sum_{k=0}^{p-1} \binom{p-2}{k} f(x) - 2^{p-1}\left(2f(2^{-1}x) + (p-2)f(0)\right) \right\|$$
$$\leq \; cK_r,$$

or

$$\left\| 2^{p-2} f(x) - 2^p f(2^{-1}x) \right\| \leq c2^{p-2} \left\| x \right\|^r ,$$

or

$$\left\| f(x) - 2^2 f(2^{-1}x) \right\| \leq c \left\| x \right\|^r = c_r \left(1 - 2^{2-r}\right) \left\| x \right\|^r , \tag{2.8}$$

where $c_r = \frac{c}{1-2^{2-r}}$, $r > 2$. Claim that inequality

$$\left\| f(x) - 2^{2n} f(2^{-n}x) \right\| \leq c_r (1 - 2^{(2-r)n}) \left\| x \right\|^r , \tag{2.9}$$

holds for every $x \in X, n \in \mathbb{N}_0$ with $p = 2, 3, 4, 5, \ldots$, and $c_r = \frac{c}{1-2^{2-r}}$, $r>2$.
For $n = 0$, it is trivial. Note that (2.8) yields (2.9) for $n = 1$. Assume (2.9) is true and from (2.8), with $2^{-(n-1)}x$ on place of x, we obtain :

$$\begin{aligned}
& \left\| f(x) - 2^{2n} f(2^{-n}x) \right\| \\
\leq \; & \left\| f(x) - 2^{2(n-1)} f(2^{-(n-1)}x) \right\| + \left\| 2^{2(n-1)} f(2^{-(n-1)}x) - 2^{2n} f(2^{-n}x) \right\| \\
\leq \; & c_r \left\{ \left(1 - 2^{(2-r)(n-1)}\right) + 2^{2(n-1)} \left(1 - 2^{2-r}\right) 2^{-r(n-1)} \right\} \left\| x \right\|^r \\
= \; & c_r \left(1 - 2^{(2-r)n}\right) \left\| x \right\|^r , \; r > 2.
\end{aligned}$$

Claim now that the sequence $\{F_n(x)\} : F_n(x) = 2^{-2n} f(2^n x)$, converges if $r<2$. To do this it suffices to prove that $\{F_n(x)\}$ is a Cauchy sequence. Inequality (2.7) is involved. In fact, if $i>j>0$ and $h_1 := 2^j x$, we have

$$\begin{aligned}
& \left\| F_i(x) - F_j(x) \right\| = \left\| 2^{-2i} f(2^i x) - 2^{-2j} f(2^j x) \right\| = 2^{-2j} \left\| 2^{-2(i-j)} f(2^{i-j} h_1) - f(h_1) \right\| \\
\leq \; & 2^{-2j} c_r \left(1 - 2^{(i-j)(r-2)}\right) \left\| h_1 \right\|^r = 2^{-2j} c_r \left(1 - 2^{(i-j)(r-2)}\right) 2^{jr} \left\| x \right\|^r \\
= \; & 2^{j(r-2)} c_r \left(1 - 2^{i(r-2)}\right) \left\| x \right\|^r \\
< \; & c_r 2^{(r-2)j} \left\| x \right\|^r \to 0, \text{as } j \to \infty, r<2.
\end{aligned}$$

Similarly claim that the sequence $\{F_n(x)\} : F_n(x) = 2^{2n} f(2^{-n}x)$ converges if $r>2$. To do this it suffices to prove that $\{F_n(x)\}$ is a Cauchy sequence. Inequality (2.9) is involved. In fact, if $i>j>0$ and $h_2 := 2^{-j}x$, we get

$$\begin{aligned}
& \left\| F_i(x) - F_j(x) \right\| = \left\| 2^{2i} f(2^{-i} x) - 2^{2j} f(2^{-j} x) \right\| = 2^{2j} \left\| 2^{2(i-j)} f(2^{-(i-j)} h_2) - f(h_2) \right\| \\
\leq \; & 2^{2j} c_r \left(1 - 2^{(i-j)(2-r)}\right) \left\| h_2 \right\|^r = 2^{2j} c_r \left(1 - 2^{(i-j)(2-r)}\right) 2^{-jr} \left\| x \right\|^r \\
= \; & 2^{j(2-r)} c_r \left(1 - 2^{i(2-r)}\right) \left\| x \right\|^r \\
< \; & 2^{j(2-r)} c_r \left\| x \right\|^r \underset{j \to \infty}{\longrightarrow} 0.
\end{aligned}$$

Also claim that formula (2.1), with $r<2$, yields a quadratic type mapping $Q : X \to Y$. Note that from (1.4), (2.1) with $r<2$ and the fact that $\lim\limits_{n\to\infty} 2^{-2n} f_k(2^n x_1, 2^n x_2, \ldots, 2^n x_p) = Q_k(x_1, x_2, \ldots, x_p)$, as well as $K_r(2^n x_1, 2^n x_2, \ldots, 2^n x_p) = \sum\limits_{k=0}^{p-1} R_k(2^n x_1, 2^n x_2, \ldots, 2^n x_p) =$

$2^{nr}K_r(x_1, x_2, \ldots, x_p)$ we get

$$\begin{aligned} &\left\| \sum_{k=0}^{p-1} \lim_{n\to\infty} 2^{-2n} f_k(2^n x_1, 2^n x_2, \ldots, 2^n x_p) - 2^{p-1} \sum_{i=1}^{p} \lim_{n\to\infty} 2^{-2n} f(2^n x_i) \right\| \\ = &\left\| \sum_{k=0}^{p-1} Q_k(x_1, x_2, \ldots, x_p) - 2^{p-1} \sum_{i=1}^{p} Q(x_i) \right\| \leq c \lim_{n\to\infty} 2^{-2n} K_r(2^n x_1, 2^n x_2, \ldots, 2^n x_p) \\ = &\left(\lim_{n\to\infty} 2^{n(r-2)} \right) cK_r(x_1, x_2, \ldots, x_p) = 0, \end{aligned}$$

which is (1.3). Similarly claim that the formula (2.1), with $r>2$, yields a quadratic type mapping $Q : X \to Y$. Note that from (1.4), (2.1) with $r>2$ and the fact that

$$\lim_{n\to\infty} 2^{2n} f_k(2^{-n}x_1, 2^{-n}x_2, \ldots, 2^{-n}x_p) = Q_k(x_1, x_2, \ldots, x_p)$$

as well as

$$\begin{aligned} &K_r\left(2^{-n}x_1, 2^{-n}x_2, \ldots, 2^{-n}x_p\right) = \sum_{k=0}^{p-1} R_k\left(2^{-n}x_1, 2^{-n}x_2, \ldots, 2^{-n}x_p\right) \\ = \ &2^{-nr}K_r(x_1, x_2, \ldots, x_p) \end{aligned}$$

we obtain

$$\begin{aligned} &\left\| \sum_{k=0}^{p-1} \lim_{n\to\infty} 2^{2n} f_k(2^{-n}x_1, 2^{-n}x_2, \ldots, 2^{-n}x_p) - 2^{p-1} \sum_{i=1}^{p} \lim_{n\to\infty} 2^{2n} f(2^{-n}x_i) \right\| \\ = &\left\| \sum_{k=0}^{p-1} Q_k(x_1, x_2, \ldots, x_p) - 2^{p-1} \sum_{i=1}^{p} Q(x_i) \right\| \leq c \lim_{n\to\infty} 2^{2n} K_r\left(2^{-n}x_1, 2^{-n}x_2, \ldots, 2^{-n}x_p\right) \\ = &\left(\lim_{n\to\infty} 2^{n(2-r)} \right) cK_r(x_1, x_2, \ldots, x_p) = 0, \end{aligned}$$

which is (1.3). It is now clear from (2.7) and (2.9) with $n \to \infty$, and the formula (2.1) that inequality (2.2) holds in X. This completes *the existence proof* of our Theorem 2.1. It remains to prove *the uniqueness* for this Theorem. *Let* $Q' : X \to Y$ *be another quadratic type mapping satisfying (2.2). Then we have to prove that* $Q' = Q$. In fact, remember that both Q and Q' satisfy (1.7) for $r<2$. Then from the triangle inequality and (2.2) with $r<2$ one gets that

$$\begin{aligned} &\|Q(x) - Q'(x)\| = \left\|2^{-2n}Q(2^n x) - 2^{-2n}Q'(2^n x)\right\| \\ \leq \ &2^{-2n}\left\{\|Q(2^n x) - f(2^n x)\| + \|Q'(2^n x) - f(2^n x)\|\right\} \\ \leq \ &2^{-2n} \cdot 2 \cdot 2^{nr} c_r \|x\|^r = 2 \cdot 2^{n(r-2)} \cdot c_r \cdot \|x\|^r \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

for every$x \in X$ and $n \in \mathbb{N}_0$. Thus

$$Q(x) = Q'(x) \tag{2.10}$$

for every $x \in X$ and $r<2$. Similarly both Q and Q' satisfy (1.11), as well. Then from the triangle inequality and (2.2) with $r>2$ one obtains that

$$\begin{aligned} &\|Q(x) - Q'(x)\| = \left\|2^{2n}Q\left(2^{-n}x\right) - 2^{2n}Q'\left(2^{-n}x\right)\right\| \\ \leq \ &2^{2n}\left\{\left\|Q(2^{-n}x) - f(2^{-n}x)\right\| + \left\|Q'\left(2^{-n}x\right) - f\left(2^{-n}x\right)\right\|\right\} \\ \leq \ &2^{2n} \cdot 2 \cdot 2^{-nr} c_r \|x\|^r = 2 \cdot 2^{n(2-r)} \cdot c_r \cdot \|x\|^r \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

for every $x \in X$ and $n \in \mathbb{N}_0$. Thus (2.10) holds for every $x \in X$ and $r>2$. This completes the proof of *the uniqueness* of our theorem and thus of *the stability* for the quadratic type equation (1.1) in several variables $x_1, x_2, \ldots, x_p \in X$. ∎

Corollary 2.2. *Let X and Y be normed linear spaces. Assume that Y is complete. Assume in addition that $f : X \to Y$ is a mapping for which there exists a constant $c \geq 0$ (independent of $x_1, x_2 \in X$) and $r \in \mathbb{R} - \{2\}$, such that*

$$\|f(x_1 + x_2) + f(x_1 - x_2) - 2(f(x_1) + f(x_2))\| \leq c(\|x_1 + x_2\|^r + \|x_1 - x_2\|^r).$$

Then the limit of the formula (2.1) exists and $Q : X \to Y$ is the unique quadratic type mapping, such that (2.2) holds.

References

[1] J. ACZEL, *Lectures on Functional Equations and their applications,* Academic Press, New York and London, 1966.

[2] C. BORELLI and G. L. FORTI, On a general Hyers-Ulam stability result, *Internat. J. Math. Sci.*, **18** (1995), 229-236.

[3] D. G. BOURGIN, Classes of transformations and bordering transformations, *Bull. Amer. Math. Soc.*, **57** (1951), 223-237.

[4] P. W. CHOLEVA, Remarks on the stability of functional equations, *Aequationes Math.*, **27** (1984), 76- 86.

[5] ST. CZERWIK, On the stability of the quadratic mapping in normed spaces, *Abh. Math. Sem. Univ. Hamburg,* **62** (1992), 59-64.

[6] H. DRLJEVIC, On the stability of the functional quadratic on A-orthogonal vectors, *Publ. Inst. Math.* (*Beograd*) (*N.S.*), **36**(50) (1984), 111-118.

[7] I. FENYO, Osservazioni su alcuni teoremi di D. H. Hyers, *Istit. Lombardo Accad. Sci. Lett. Rend.*, **A 114** (1980), (1982), 235-242.

[8] I. FENYO, On an inequality of P. W. Cholewa. In: *General Inequalities,* **5**. [Internat. Schriftenreiche Numer. Math., Vol. 80]. Birkhäuser, Basel-Boston, MA, 1987, 277-280.

[9] G. L. FORTI, Hyers-Ulam stability of functional equations in several variables, *Aequationes Math.*, **50** (1995), 143-190.

[10] Z. GAJDA and R. GER, Subadditive multifunctions and Hyers-Ulam Stability. In: *General Inequalities*, **5**.[Internat. Schriftenreiche Numer. Math., Vol. 80]. Birkhäuser, MA, 1987.

[11] P. GAVRUTA, An answer to a question of John M. Rassias concerning the stability of Cauchy equation. In: *Advances in Equations and Inequalities,* Hadronic Math. Series, U.S.A., 1999, 67-71.

[12] P. M. GRUBER, Stability of isometries, *Trans. Amer. Math. Soc.,* U.S.A., **245** (1978), 263-277.

[13] D. H. HYERS, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.* **27** (1941), 222-24: The stability of homomorphisms and related topics, "Global Analysis-Analysis on Manifolds", *Teubner - Texte zur Mathematik*, **57** (1983), 140-153.

[14] S. M. JUNG, On the Hyers-Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. & Appl.*, **222** (1998), 126-137.

[15] PL. KANNAPPAN, Quadratic functional equation and inner product spaces, *Results Math.*, **27** (1995), 368-372.

[16] J. M. RASSIAS, On approximation of approximately linear mappings by linear mappings, *J. Funct. Anal.* **46** (1982), 126-130.

[17] J. M. RASSIAS, On approximation of approximately linear mappings by linear mappings, *Bull. Sc. Math*. **108** (1984), 445-446.

[18] J. M. RASSIAS, Solution of a problem of Ulam, *J. Approx. Th*. **57** (1989), 268-273.

[19] J. M. RASSIAS, On the stability of a multi-dimensional Cauchy type functional equation, *Geometry, Analysis and Mechanics*, World Sci. Publ. Co., 1994, 365-376.

[20] J. M. RASSIAS, Solution of the Ulam stability problem for Euler-Lagrange quadratic mappings, *J. Math. Anal. Appl.* **220** (1998), 613-639.

[21] J. M. RASSIAS, On the Ulam stability of mixed type mappings on restricted domains, *J. Math. Anal. Appl.* **276**(2002),747-762.

[22] J. M. RASSIAS and M. J. RASSIAS, On the Ulam stability of Jensen and Jensen type mappings on restricted domains, *J. Math. Anal. Appl.* **281**(2003) 516-524.

[23] J. M. RASSIAS, Asymptotic behavior of mixed type functional equations, *Austral. J. Math. Anal. Applications*, **1**(2004), Art. 10, Issue 1, 1-21.

[24] S. M. ULAM, *A collection of mathematical problems*, Interscience Publishers, Inc., New York, 1968, p.63.