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Asymptotic behavior of alternative Jensen and Jensen type functional equations

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Abstract

In 1941 D.H. Hyers solved the well-known Ulam stability problem for linear mappings. In 1951 D.G. Bourgin was the second author to treat the Ulam problem for additive mappings. In 1982–2005 we established the Hyers–Ulam stability for the Ulam problem of linear and nonlinear mappings. In 1998 S.-M. Jung and in 2002–2005 the authors of this paper investigated the Hyers–Ulam stability of additive and quadratic mappings on restricted domains. In this paper we improve our bounds and thus our results obtained, in 2003 for Jensen type mappings and establish new theorems about the Ulam stability of additive mappings of the second form on restricted domains. Besides we introduce alternative Jensen type functional equations and investigate pertinent stability results for these alternative equations. Finally, we apply our recent research results to the asymptotic behavior of functional equations of these alternative types. These stability results can be applied in stochastic analysis, financial and actuarial mathematics, as well as in psychology and sociology.

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Résumé

En 1941 D.H. Hyers a résolu le problème bien connu de stabilité d'Ulam pour les tracés linéaires. En 1951 D.G. Bourgin était le deuxième auteur pour traiter le problème d'Ulam pour les tracés additifs. En 1982–2005 nous avons établi la stabilité de Hyers–Ulam pour le problème d'Ulam des tracés linéaires et non-linéaires. En 1998 S.-M. Jung et en 2002–2005 les auteurs de cet article ont étudié la stabilité de Hyers–Ulam des tracés additifs et quadratiques sur des domaines restreints. Dans cet article nous améliorons nos limites et ainsi nos résultats obtenus, en 2003 pour le type tracés de Jensen et établissons de nouveaux théorèmes au sujet de la stabilité d'Ulam des tracés additifs de la deuxième forme sur des domaines restreints. Sans compter que nous présentons le type alternatif équations fonctionnelles de Jensen et étudions des résultats convenables de stabilité pour ces équations alternatives. En conclusion, nous appliquons nos résultats de la recherche récents au comportement asymptotique des équations fonctionnelles de ces types alternatifs. Ces résultats de stabilité peuvent être appliqués dans l'analyse stochastique, mathématiques financières et actuarielles,aussi bien qu'en la psychologie et la sociologie.

MSC: primary 39B; secondary 26D

Keywords: Ulam stability; Asymptoticity; Alternative Jensen; Alternative Jensen type; Restricted domain

1. Introduction

In 1940 and in 1964 S.M. Ulam [26] proposed the *general Ulam stability problem*:

> "When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

In 1941 D.H. Hyers [13] solved this problem for linear mappings. In 1951 D.G. Bourgin [3] was the second author to treat the Ulam problem for additive mappings. In 1978, according to P.M. Gruber [12], this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1980 and in 1987, I. Fenyö [7,8] established the stability of the Ulam problem for quadratic and other mappings. In 1987 Z. Gajda and R. Ger [10] showed that one can get analogous stability results for subadditive multifunctions. Other interesting stability results have been achieved also by the following authors J. Aczél [1], C. Borelli and G.L. Forti [2,9], P.W. Cholewa [4], St. Czerwik [5], and H. Drljevic [6]. In 1982–2005 J.M. Rassias [16–21,23,24] and in 2003 and 2005 the authors [22,25] solved the above Ulam problem for Jensen and Euler–Lagrange type mappings. In 1999 P. Gavruta [11] answered a question of ours [18] concerning the stability of the Cauchy equation. In 1998 S.-M. Jung [14] and in 2002–2003 the authors [21,22] investigated the Hyers–Ulam stability for additive and quadratic mappings on restricted domains. In this paper we improve our bounds and thus our results obtained, in 2003 for Jensen and Jensen type mappings and establish new theorems about the Ulam stability of additive mappings of the second form on restricted domains. Besides we introduce alternative Jensen and Jensen type functional equations and investigate pertinent stability results for these alternative functional equations. Finally, we apply our recent research results to the asymptotic behavior of functional equations of these alternative types. These stability results can be applied in stochastic analysis, financial and actuarial mathematics, as well as in psychology and sociology. In 1997, P. Malliavin [15] published an interesting reference book for stochastic analysis.

Throughout this paper, let X be a real normed space and Y be a real Banach space in the case of functional inequalities, as well as let X and Y be real linear spaces for functional equations.

Definition 1.1. A mapping $A : X \to Y$ is called *alternative additive of the first form* if A satisfies the functional equation

$$A(x_1 + x_2) + A(x_1 - x_2) = -2A(-x_1) \tag{1.1}$$

for all $x_1, x_2 \in X$. We note that equation (1.1) is equivalent to *the alternative Jensen equation*

$$A\left(-\frac{x+y}{2}\right) = -\frac{1}{2}\big[A(x) + A(y)\big], \tag{1.1a}$$

or

$$2A\left(-\frac{x+y}{2}\right) = -\big[A(x) + A(y)\big] \tag{1.1b}$$

for $x = x_1 + x_2$, $y = x_1 - x_2$. A mapping $A : X \to Y$ is called *alternative Jensen mapping* if A satisfies the functional equation (1.1a) (or (1.1b)).

Definition 1.2. A mapping $A : X \to Y$ is called *alternative additive of the second form* if A satisfies the functional equation

$$A(x_1 + x_2) - A(x_1 - x_2) = -2A(-x_2) \tag{1.2}$$

for all $x_1, x_2 \in X$. We note that (1.2) is equivalent to *the alternative Jensen type equation*

$$A\left(-\frac{x-y}{2}\right) = -\frac{1}{2}\big[A(x) - A(y)\big], \tag{1.2a}$$

or

$$2A\left(-\frac{x-y}{2}\right) = -\big[A(x) - A(y)\big] \tag{1.2b}$$

for $x = x_1 + x_2$, $y = x_1 - x_2$. A mapping $A : X \to Y$ is called *alternative Jensen type mapping* if A satisfies the functional equation (1.2a) (or (1.2b)).

Definition 1.3. A mapping $f : X \to Y$ is called *approximately odd* if f satisfies the functional inequality

$$\big\| f(x) + f(-x) \big\| \leqslant \theta \tag{1.3}$$

for some fixed $\theta \geqslant 0$ and for all $x \in X$.

2. Stability of the alternative additive equation (1.1) of the first form on unrestricted and restricted domains

We establish the following new stability Theorems 2.1–2.2 for alternative additive mappings of the first form on unrestricted and restricted domains, respectively.

Theorem 2.1. *If a mapping $f: X \to Y$ satisfies the inequalities*

$$\left\| f(x_1 + x_2) + f(x_1 - x_2) + 2f(-x_1) \right\| \leqslant \delta, \tag{2.1a}$$

$$\left\| f(-x) + f(x) \right\| \leqslant \delta/2, \tag{2.1b}$$

for some fixed $\delta \geqslant 0$ and all $x_1, x_2 \in X$ and $x \in X$, then there exists a unique alternative additive mapping $A: X \to Y$ of the first form, which satisfies $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, and the inequality

$$\left\| f(x) - A(x) \right\| \leqslant 2\delta + \left\| f(0) \right\| \ (\leqslant 9\delta/4) \tag{2.1c}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Setting $x_1 = x_2 = 0$ in inequality (2.1a), or $x = 0$ in inequality (2.1b), one gets that $\|f(0)\| \leqslant \delta/4$. Besides replacing $x_1 = x_2 = x$ in inequality (2.1a), we find $\|f(2x) + f(0) + 2f(-x)\| \leqslant \delta$, for all $x \in X$. Thus from this inequality, inequalities (2.1a)–(2.1b) and the triangle inequality, we get

$$\begin{aligned} \left\| f(2x) - 2f(x) \right\| &\leqslant \left\| f(2x) + f(0) + 2f(-x) \right\| + 2\left\| -\left[f(-x) + f(x) \right] \right\| \\ &\quad + \left\| -\left[f(0) \right] \right\| \leqslant 2\delta + \left\| f(0) \right\| \left(\leqslant 2\delta + \frac{\delta}{4} = \frac{9\delta}{4} \right). \end{aligned}$$

Thus by (or *without*) induction, one establishes the general inequality

$$\left\| f(x) - 2^{-n} f(2^n x) \right\| \leqslant \left(2\delta + \left\| f(0) \right\| \right)(1 - 2^{-n}),$$

for all $x \in X$ and all $n \in \mathbb{N} = \{1, 2, \ldots\}$. The rest of the proof is omitted as similar to the proofs of our corresponding theorems [16,25]. □

Theorem 2.2. *Let $d > 0$ and $\delta \geqslant 0$ be fixed. If a mapping $f: X \to Y$ satisfies inequalities* (2.1a)–(2.1b) *for all $x_1, x_2 \in X$ and $x \in X$, with restricted domains*: $\|x_1\| + \|x_2\| \geqslant d$, *and* $\|x\| \geqslant d$, *respectively, then there exists a unique alternative additive mapping $A: X \to Y$ of the first form, which satisfies $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, and*

$$\left\| f(x) - A(x) \right\| \leqslant 8\delta + \left\| f(0) \right\| \ (\leqslant 33\delta/4) \tag{2.1}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Assume $\|x_1\| + \|x_2\| < d$ and $\|x\| < d$. If $x_1 = x_2 = 0$ and $x = 0$, then we choose a $t \in X$ with $\|t\| = d$. Otherwise, let us choose

$$t = \left(1 + \frac{d}{\|x_1\|} \right) x_1, \quad \text{if } \|x_1\| \geqslant \|x_2\|; \qquad t = \left(1 + \frac{d}{\|x_2\|} \right) x_2, \quad \text{if } \|x_1\| \leqslant \|x_2\|.$$

Clearly, we see

$$
\begin{aligned}
&\|x_1 - t\| + \|x_2 + t\| \geqslant 2\|t\| - \big(\|x_1\| + \|x_2\|\big) \geqslant d, \quad \|x_1 - x_2\| + \|2t\| \geqslant d,\\
&\|x_1 + t\| + \|-x_2 + t\| \geqslant 2\|t\| - \big(\|x_1\| + \|x_2\|\big) \geqslant d, \quad \|x_1\| + \|t\| \geqslant d,\\
&\|t \pm x_1\| \geqslant \|t\| - \|x_1\| = \big(\|x_1\| + d\big) - \|x_1\| = d,\\
&\quad \text{because } \|t\| = \|x_1\| + d \text{ if } \|x_1\| \geqslant \|x_2\|;\\
&\|t \pm x_1\| \geqslant \|t\| - \|x_1\| = \big(\|x_2\| + d\big) - \|x_1\| \geqslant d,\\
&\quad \text{because } \|t\| = \|x_2\| + d \text{ if } \|x_1\| \leqslant \|x_2\|.
\end{aligned}
\tag{2.2}
$$

These inequalities (2.2) come from the corresponding substitutions attached between the right-hand sided parentheses of the following functional identity.

Therefore from (2.2), the triangle inequality, and the *functional identity*

$$
\begin{aligned}
&2\big[f(x_1 + x_2) + f(x_1 - x_2) + 2f(-x_1)\big]\\
&\quad = \big[f(x_1 + x_2) + f(x_1 - x_2 - 2t) + 2f\big(-(x_1 - t)\big)\big]\\
&\qquad \text{(with } x_1 - t \text{ on } x_1\text{, and } x_2 + t \text{ on } x_2\text{)}\\
&\qquad - \big[f(x_1 - x_2 - 2t) + f(x_1 - x_2 + 2t) + 2f\big(-(x_1 - x_2)\big)\big]\\
&\qquad \text{(with } x_1 - x_2 \text{ on } x_1\text{, and } 2t \text{ on } x_2\text{)}\\
&\qquad + \big[f(x_1 - x_2 + 2t) + f(x_1 + x_2) + 2f\big(-(x_1 + t)\big)\big]\\
&\qquad \text{(with } x_1 + t \text{ on } x_1\text{, and } -x_2 + t \text{ on } x_2\text{)}\\
&\qquad + 2\big[f(x_1 + t) + f(x_1 - t) + 2f(-x_1)\big] \quad \text{(with } x_1 \text{ on } x_1\text{, and } t \text{ on } x_2\text{)}\\
&\qquad + 2\big[f(x_1 - x_2) + f\big(-(x_1 - x_2)\big)\big] - 2\big[f\big(-(x_1 + t)\big) + f(x_1 + t)\big]\\
&\qquad - 2\big[f(t - x_1) + f\big(-(t - x_1)\big)\big],
\end{aligned}
$$

we get

$$
\big\| f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1)\big\| \leqslant 4\delta. \tag{2.3}
$$

Applying now Theorem 2.1 and the above inequality (2.3), one gets that there exists a unique alternative additive mapping $A : X \to Y$ of the first form that satisfies the alternative additive equation (1.1) and the inequality (2.1), such that $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$. Our last assertion is trivial according to Theorem 2.2. □

We note that, if we define $S_1 = \{x \in X\colon \|x\| < d\}$ and $S_2 = \{(x_1, x_2) \in X^2\colon \|x_i\| < d, i = 1, 2\}$, $d > 0$, then $\{x \in X\colon \|x\| \geqslant 2d\} \subset X \setminus S_1$ and $\{(x_1, x_2) \in X^2\colon \|x_1\| + \|x_2\| \geqslant 2d\} \subset X^2 \setminus S_2$.

Corollary 2.1. *If we assume that a mapping $f : X \to Y$ satisfies inequalities* (2.1a)–(2.1b) *for some fixed δ and for all $x \in X \setminus S_1$ and $(x_1, x_2) \in X^2 \setminus S_2$, then there exists a unique alternative additive mapping $A : X \to Y$ of the first form, satisfying* (2.1) *for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and all $t \in \mathbb{R}$.*

Corollary 2.2. *A mapping $f : X \to Y$ is alternative additive of the first form, if and only if the asymptotic conditions $\|f(-x) + f(x)\| \to 0$ and $\|f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1)\| \to 0$, as $\|x\| \to \infty$ and $\|x_1\| + \|x_2\| \to \infty$ hold, respectively.*

3. Stability of the alternative additive equation (1.2) of the second form

We establish the following new stability Theorem 3.1 for alternative additive mappings of the second form.

Theorem 3.1. *If a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| f(x_1 + x_2) - f(x_1 - x_2) + 2f(-x_2) \right\| \leqslant \delta \tag{3.1}$$

for some $\delta \geqslant 0$ and for all $x_1, x_2 \in X$, then there exists a unique alternative additive mapping $A : X \to Y$ of the second form, which satisfies $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, and the inequality

$$\left\| f(x) - A(x) \right\| \leqslant 3\delta + \left\| f(0) \right\| \left(\leqslant 3\delta + \frac{\delta}{2} = \frac{7}{2}\delta \right) \tag{3.2}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Replacing $x_1 = x_2 = 0$ in (3.1), we find

$$\left\| f(0) \right\| \leqslant \delta/2. \tag{3.3}$$

Thus, substituting $x_1 = x_2 = x$ in (3.1), one gets

$$\left\| f(2x) - f(0) + 2f(-x) \right\| \leqslant \delta, \tag{3.3a}$$

for all $x \in X$. Besides, replacing $x_1 = 0, x_2 = x$ in (3.1), one gets

$$\left\| f(-x) + f(x) \right\| \leqslant \delta, \tag{3.3b}$$

for all $x \in X$. Therefore from (3.3)–(3.3a)–(3.3b) and the triangle inequality, we obtain

$$\begin{aligned} &\left\| f(2x) - 2f(x) \right\| \\ &\quad \leqslant \left\| f(2x) - f(0) + 2f(-x) \right\| + 2\left\| -\left[f(-x) + f(x) \right] \right\| + \left\| f(0) \right\| \\ &\quad \leqslant 3\delta + \left\| f(0) \right\| \left(\leqslant 3\delta + \frac{\delta}{2} = \frac{7\delta}{2} \right), \end{aligned}$$

for all $x \in X$, or the inequality

$$\left\| f(x) - 2^{-1} f(2x) \right\| \leqslant \left(3\delta + \left\| f(0) \right\| \right)\left(1 - 2^{-1}\right), \tag{3.4}$$

for some $\delta \geqslant 0$, and all $x \in X$. Therefore from (3.4) and the triangle inequality, we obtain

$$\left\| f(x) - 2^{-n} f(2^n x) \right\| \leqslant \left(3\delta + \left\| f(0) \right\| \right)\left(1 - 2^{-n}\right), \tag{3.5}$$

for some $\delta \geqslant 0$, any $n \in \mathbb{N}$, and all $x \in X$.

We prove as in [22] that

$$A(x) = 2^{-n} A(2^n x) \tag{3.6}$$

holds for any $n \in \mathbb{N}$, and all $x \in X$.

By (3.5), for $n \geqslant m > 0$, we have

$$\left\| 2^{-n} f(2^n x) - 2^{-m} f(2^m x) \right\| < \left(3\delta + \left\| f(0) \right\|\right) \cdot 2^{-m} \to 0, \quad \text{as } m \to \infty. \tag{3.7}$$

Therefore we may apply a direct method to the definition of A, such that the formula

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{3.8}$$

holds for all $x \in X$ [16–19]. From this formula (3.8) and inequality (3.1), it follows that $A : X \to Y$ is an alternative additive mapping of the second form. According to the above inequality (3.5) and formula (3.8), one gets that inequality (3.2) holds.

Assume now that there is another alternative additive mapping $A' : X \to Y$ of the second form which satisfies Eq. (1.2), formula (3.6) and inequality (3.2). Therefore, as in [22], one gets

$$A(x) = A'(x) \tag{3.9}$$

for all $x \in X$, completing the proof of the first part of our Theorem 3.1.

The proof of the last assertion in our Theorem 3.1 is obvious according to the work of the first author [16], in 1982. □

4. Stability of the alternative additive equation (1.2) of the second form on a restricted domain

We establish the following new stability Theorem 4.1 for alternative additive mappings of the second form on a restricted domain.

Theorem 4.1. *Let $d > 0$ and $\delta \geqslant 0$, be fixed. If an approximately odd mapping $f : X \to Y$ satisfies inequality* (3.1) *for all $x_1, x_2 \in X$ with $\|x_1\| + \|x_2\| \geqslant d$, and inequality* (3.3b) *for all $x \in X$ with $\|x\| \geqslant d$, then there exists a unique alternative additive mapping $A : X \to Y$ of the second form such that*

$$\left\| f(x) - A(x) \right\| \leqslant 21\delta + \left\| f(0) \right\| \ (\leqslant 43\delta/2) \tag{4.1}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Assume $\|x_1\| + \|x_2\| < d$ and $\|x\| < d$. If $x_1 = x_2 = 0$ and $x = 0$, then we choose a $t \in X$ with $\|t\| = d$. Otherwise, let us choose

$$t = \left(1 + \frac{d}{\|x_1\|}\right) x_1, \quad \text{if } \|x_1\| \geqslant \|x_2\|; \qquad t = \left(1 + \frac{d}{\|x_2\|}\right) x_2, \quad \text{if } \|x_1\| \leqslant \|x_2\|.$$

We note that:

$$\|t\| = \|x_1\| + d > d, \quad \text{if } \|x_1\| \geqslant \|x_2\|; \qquad \|t\| = \|x_2\| + d > d, \quad \text{if } \|x_1\| \leqslant \|x_2\|.$$

Clearly, we see

$$\begin{aligned}
&\|x_1 - t\| + \|x_2 + t\| \geqslant 2\|t\| - \big(\|x_1\| + \|x_2\|\big) \geqslant d,\\
&\|x_1 - t\| + \|x_2 - t\| \geqslant 2\|t\| - \big(\|x_1\| + \|x_2\|\big) \geqslant d,\\
&\|x_1 - 2t\| + \|x_2\| \geqslant 2\|t\| - \big(\|x_1\| + \|x_2\|\big) \geqslant d, \quad \|t\| + \|x_2\| \geqslant d \quad \text{and}\\
&\|t - x_2\| \geqslant \|t\| - \|x_2\| = \big(\|x_2\| + d\big) - \|x_2\| = d, \quad \text{because}\\
&\|t\| = \|x_2\| + d, \text{if } \|x_1\| \leqslant \|x_2\|;\\
&\|t - x_2\| \geqslant \|t\| - \|x_2\| = \big(\|x_1\| + d\big) - \|x_2\| \geqslant d, \quad \text{because}\\
&\|t\| = \|x_1\| + d, \text{if } \|x_1\| \geqslant \|x_2\|.
\end{aligned} \tag{4.2}$$

Therefore from (3.3b), (3.1), (4.2), and the following *functional identity*

$$\begin{aligned}
&f(x_1 + x_2) - f(x_1 - x_2) + 2f(-x_2)\\
&\quad = \big[f(x_1 + x_2) - f(x_1 - x_2 - 2t) + 2f\big(-(x_2 + t)\big)\big]\\
&\qquad\quad (\text{with } x_1 - t \text{ on } x_1, \text{ and } x_2 + t \text{ on } x_2)\\
&\qquad + \big[f(x_1 + x_2 - 2t) - f(x_1 - x_2) + 2f\big(-(x_2 - t)\big)\big]\\
&\qquad\quad (\text{with } x_1 - t \text{ on } x_1, \text{ and } x_2 - t \text{ on } x_2)\\
&\qquad - \big[f(x_1 + x_2 - 2t) - f(x_1 - x_2 - 2t) + 2f(-x_2)\big]\\
&\qquad\quad (\text{with } x_1 - 2t \text{ on } x_1, \text{ and } x_2 \text{ on } x_2)\\
&\qquad + 2\big[f(t + x_2) - f(t - x_2) + 2f(-x_2)\big] \quad (\text{with } t \text{ on } x_1, \text{ and } x_2 \text{ on } x_2)\\
&\qquad - 2\big[f(t + x_2) + f\big(-(t + x_2)\big)\big] \quad (\text{with } t - x_2 \text{ on } x),
\end{aligned}$$

we get

$$\big\| f(x_1 + x_2) - f(x_1 - x_2) + 2f(-x_2)\big\| \leqslant 7\delta. \tag{4.3}$$

Therefore there exists a unique alternative additive mapping $A : X \to Y$ of the second form that satisfies Eq. (1.2) and inequality (4.1), completing the proof of this theorem. □

We note that if we define $S_1 = \{x \in X\colon \|x\| < d\}$ and $S_2 = \{(x_1, x_2) \in X^2\colon \|x_i\| < d, i = 1, 2\}$ for some fixed $d > 0$, then

$$\big\{x \in X\colon \|x\| \geqslant 2d\big\} \subset X \setminus S_1 \quad \text{and} \quad \big\{(x_1, x_2) \in X^2\colon \|x_1\| + \|x_2\| \geqslant 2d\big\} \subset X^2 \setminus S_2.$$

Corollary 4.1. *If we assume that a mapping $f : X \to Y$ satisfies inequality* (4.1) *for some fixed $\delta \geqslant 0$ and for all $(x_1, x_2) \in X^2 \setminus S_2$ and* (3.3b) *for all $x \in X \setminus S_1$, then there exists a unique alternative additive mapping $A : X \to Y$ of the second form, satisfying* (4.1) *for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.*

Corollary 4.2. *A mapping $f : X \to Y$ is alternative additive of the second form, if and only if the asymptotic conditions $\|f(-x) + f(x)\| \to 0$ and*

$$\big\| f(x_1 + x_2) - f(x_1 - x_2) - 2f(x_2)\big\| \to 0,$$

as $\|x\| \to \infty$ and $\|x_1\| + \|x_2\| \to \infty$ hold, respectively.

5. Stability of the alternative Jensen equation (1.1b)

We establish the following new stability Theorem 5.1 for Jensen mappings.

Theorem 5.1. *If a mapping $f : X \to Y$ satisfies the approximately alternative Jensen inequality*

$$\left\| 2f\left(-\frac{x_1+x_2}{2}\right) + f(x_1) + f(x_2)\right\| \leqslant \delta, \tag{5.1}$$

for some fixed $\delta \geqslant 0$, and all $x_1, x_2 \in X$, then there exists a unique alternative Jensen mapping $A : X \to Y$, satisfying $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ and the inequality

$$\left\| f(x) - A(x)\right\| \leqslant 2\delta + \left\| f(0)\right\| \quad (\leqslant 9\delta/4) \tag{5.2}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Setting $x_1 = x_2 = 0$ in inequality (5.1), we obtain

$$\left\| f(0)\right\| \leqslant \delta/4. \tag{5.3a}$$

Placing $x_1 = x_2 = x$ in (5.1), one finds

$$\left\| f(-x) + f(x)\right\| \leqslant \delta/2, \tag{5.3b}$$

for all $x \in X$. Substituting $x_1 = 2x$ and $x_2 = 0$ in (5.1), one gets

$$\left\| 2f(-x) + f(2x) + f(0)\right\| \leqslant \delta, \tag{5.3c}$$

for all $x \in X$.

Thus from inequalities (5.3a)–(5.3b)–(5.3c) and the triangle inequality, we establish

$$\begin{aligned}\left\| f(2x) - 2f(x)\right\| &\leqslant \left\| 2f(-x) + f(2x) + f(0)\right\| \\ &\qquad + \left\| -2\left[f(-x) + f(x)\right]\right\| + \left\| -\left[f(0)\right]\right\| \\ &\leqslant 2\delta + \left\| f(0)\right\| \left(\leqslant 2\delta + \frac{\delta}{4} = \frac{9}{4}\delta\right),\end{aligned} \tag{5.3d}$$

or

$$\left\| f(x) - 2^{-1} f(2x)\right\| \leqslant \left(2\delta + \left\| f(0)\right\|\right)\left(1 - 2^{-1}\right), \tag{5.3}$$

for some $\delta \geqslant 0$, and all $x \in X$. Therefore from (5.3) and the triangle inequality, we obtain

$$\left\| f(x) - 2^{-n} f(2^n x)\right\| \leqslant \left(2\delta + \left\| f(0)\right\|\right)\left(1 - 2^{-n}\right), \tag{5.4}$$

for some $\delta \geqslant 0$, any $n \in N$, and all $x \in X$. The rest of the proof is omitted as similar to the proof of Theorem 3.1. □

6. Stability of the alternative Jensen equation (1.1b) on a restricted domain

We establish the following new stability Theorem 6.1 for alternative Jensen mappings on a restricted domain.

Theorem 6.1. *Let $d > 0$ and $\delta \geqslant 0$ be fixed. If a mapping $f : X \to Y$ satisfies the approximately alternative Jensen inequality* (5.1) *for all $x_1, x_2 \in X$, with $\|x_1\| + \|x_2\| \geqslant d$, and the additional inequalities*

$$\left\| f(-x) + f(x) \right\| \leqslant \delta/2, \quad (\textit{from } (5.3b)) \tag{6.1a}$$

$$\left\| f(2x) - 2f(x) \right\| \leqslant 2\delta + \left\| f(0) \right\| \quad (\textit{from } (5.3d)) \tag{6.1b}$$

for all $x \in X$ with $\|x\| \geqslant d$, then there exists a unique alternative Jensen mapping $A : X \to Y$, such that the inequality

$$\left\| f(x) - A(x) \right\| \leqslant 20\delta + 7\left\| f(0) \right\| \left(\leqslant 20\delta + \frac{7}{4}\delta = \frac{87}{4}\delta \right) \tag{6.1}$$

holds for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. It is clear that the "*approximate odd*" inequality (6.1a) holds, if we replace $x_1 = x$, $x_2 = x$ in (5.1). Also we get (6.1b) from (5.3d). From (1.1b), the triangle inequality, and the *functional identity*

$$\begin{aligned}
&2f\left(-\frac{x_1 + x_2}{2}\right) + f(x_1) + f(x_2) \\
&\quad = 2f\left(-\frac{x_1 + x_2}{2}\right) + f(x_1 - t) + f(x_2 + t) \\
&\qquad \text{(with } x_1 - t \text{ on } x_1 \text{ and } x_2 + t \text{ on } x_2\text{)} \\
&\qquad + \frac{1}{2}\left[2f\big(-(x_2 + t)\big) + f(2x_2) + f(2t)\right] \quad \text{(with } 2x_2 \text{ on } x_1 \text{ and } 2t \text{ on } x_2\text{)} \\
&\qquad + \frac{1}{2}\left[2f\big(-(x_1 - t)\big) + f(2x_1) + f(-2t)\right] \quad \text{(with } 2x_1 \text{ on } x_1 \text{ and } -2t \text{ on } x_2\text{)} \\
&\qquad - \frac{1}{2}\left[f(2x_1) - 2f(x_1)\right] - \frac{1}{2}\left[f(2x_2) - 2f(x_2)\right] - \frac{1}{2}\left[f(-2t) + f(2t)\right] \\
&\qquad - \left[f\big(-(x_1 - t)\big) + f(x_1 - t)\right] - \left[f\big(-(x_2 + t)\big) + f(x_2 + t)\right],
\end{aligned}$$

we get

$$\begin{aligned}
&\left\| 2f\left(-\frac{x_1 + x_2}{2}\right) + f(x_1) + f(x_2) \right\| \\
&\quad \leqslant 3\delta + 3\big(2\delta + \left\| f(0) \right\|\big) + 2(\delta/2) = 10\delta + 3\left\| f(0) \right\|.
\end{aligned} \tag{6.2}$$

Applying now Theorem 5.1 and the above inequality (6.2), one gets that there exists a unique alternative Jensen mapping $A : X \to Y$ that satisfies the alternative Jensen equation

(1.1b) and inequality (6.1), such that $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ with $A(-x) = -A(x)$ (from (6.1a)). $\square$

We note that, if we define $S_1 = \{x \in X\colon \|x\| < d\}$ and $S_2 = \{(x_1, x_2) \in X^2\colon \|x_i\| < d, i = 1, 2\}$ for some $d > 0$, then $\{x \in X\colon \|x\| \geqslant 2d\} \subset X \setminus S_1$ and $\{(x_1, x_2) \in X^2\colon \|x_1\| + \|x_2\| \geqslant 2d\} \subset X^2 \setminus S_2$.

Corollary 6.1. *If we assume that a mapping $f : X \to Y$ satisfies inequality* (5.1) *for some fixed $\delta \geqslant 0$ and for all $(x_1, x_2) \in X^2 \setminus S_2$ and* (6.1a)–(6.1b) *for all $x \in X \setminus S_1$, then there exists a unique alternative Jensen mapping $A : X \to Y$, satisfying* (6.1) *for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and all $t \in \mathbb{R}$.*

Corollary 6.2. *A mapping $f : X \to Y$ is an alternative Jensen mapping, if and only if the asymptotic conditions $\|f(-x) + f(x)\| \to 0$ and $\|f(2x) - 2f(x)\| \to 0$, as $\|x\| \to \infty$ and $\|2f(-(x_1 + x_2)/2) + f(x_1) + f(x_2)\| \to 0$, as $\|x_1\| + \|x_2\| \to \infty$ hold, respectively.*

7. Stability of the alternative Jensen type equation (1.2b)

We establish the following new stability Theorem 7.1 for alternative Jensen type mappings.

Theorem 7.1. *If a mapping $f : X \to Y$ satisfies the approximately alternative Jensen type inequality*

$$\left\|2f\left(-\frac{x_1 - x_2}{2}\right) + f(x_1) - f(x_2)\right\| \leqslant \delta, \tag{7.1}$$

for some fixed $\delta \geqslant 0$, and all $x_1, x_2 \in X$, then there exists a unique alternative Jensen type mapping $A : X \to Y$, satisfying $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ and the inequality

$$\big\|f(x) - A(x)\big\| \leqslant 3\delta + \big\|f(0)\big\| \left(\leqslant \frac{7}{2}\delta\right) \tag{7.2}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Setting $x_1 = x_2 = 0$ in inequality (7.1), we obtain

$$\big\|f(0)\big\| \leqslant \delta/2. \tag{7.3a}$$

Placing $x_1 = x, x_2 = -x$ in (7.1), one finds

$$\big\|f(-x) + f(x)\big\| \leqslant \delta, \tag{7.3b}$$

for all $x \in X$. Substituting $x_1 = 2x$ and $x_2 = 0$ in (7.1), one gets

$$\big\|2f(-x) + f(2x) - f(0)\big\| \leqslant \delta, \tag{7.3c}$$

for all $x \in X$. Thus from inequalities (7.3a)–(7.3b)–(7.3c) and the triangle inequality, we establish

$$\begin{aligned} \|f(2x) - 2f(x)\| &\leqslant \|2f(-x) + f(2x) - f(0)\| \\ &\quad + \|-2[f(-x) + f(x)]\| + \|f(0)\| \\ &\leqslant \delta + 2(\delta) + \|f(0)\| = 3\delta + \|f(0)\| \left(\leqslant 3\delta + \frac{\delta}{2} = \frac{7}{2}\delta\right), \end{aligned} \tag{7.3d}$$

or

$$\|f(x) - 2^{-1}f(2x)\| \leqslant (3\delta + \|f(0)\|)(1 - 2^{-1}),$$

for some $\delta \geqslant 0$, and all $x \in X$. The rest of the proof is omitted as similar to the proof of Theorems 3.1 and 5.1. □

8. Stability of the alternative Jensen type equation (1.2b) on a restricted domain

We establish the following new stability Theorem 8.1 for alternative Jensen type mappings on a restricted domain.

Theorem 8.1. *Let $d > 0$ and $\delta \geqslant 0$ be fixed. If a mapping $f : X \to Y$ satisfies the approximately alternative Jensen type inequality* (7.1) *for all $x_1, x_2 \in X$, with $\|x_1\| + \|x_2\| \geqslant d$, and $\|f(0)\| \leqslant \delta/2$, as well as the additional inequalities*

$$\|f(-x) + f(x)\| \leqslant \delta, \tag{8.1a}$$

$$\|f(2x) - 2f(x)\| \leqslant 3\delta + \|f(0)\| \tag{8.1b}$$

for all $x \in X$ with $\|x\| \geqslant d$, then there exists a unique alternative Jensen type mapping $A : X \to Y$, such that the inequality

$$\|f(x) - A(x)\| \leqslant 24\delta + 4\|f(0)\| \ (\leqslant 24\delta + 4(\delta/2) = 26\delta) \tag{8.1}$$

holds for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. It is clear that the "*approximate odd*" inequality (8.1a) holds for all $x \in X$, if we replace $x_1 = -x, x_2 = x$ in (7.1). From (7.3) (or (7.3d)) we get (8.1b). From (1.2b), the triangle inequality, and the *functional identity*

$$\begin{aligned} &2f\left(-\frac{x_1 - x_2}{2}\right) + f(x_1) - f(x_2) \\ &\quad = 2f\left(-\frac{x_1 - x_2}{2}\right) + f(x_1 - t) - f(x_2 - t) \\ &\qquad \text{(with } x_1 - t \text{ on } x_1 \text{ and } x_2 - t \text{ on } x_2) \\ &\qquad + \frac{1}{2}\left[2f(-(-x_2 + t)) + f(-2x_2) - f(-2t)\right] \\ &\qquad \text{(with } -2x_2 \text{ on } x_1 \text{ and } -2t \text{ on } x_2) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\big[2f\big(-(x_1-t)\big)+f(2x_1)-f(2t)\big] \quad (\text{with } 2x_1 \text{ on } x_1 \text{ and } 2t \text{ on } x_2)\\
&-\frac{1}{2}\big[f(2x_1)-2f(x_1)\big]+\frac{1}{2}\big[f(2x_2)-2f(x_2)\big]+\frac{1}{2}\big[f(-2t)+f(2t)\big]\\
&-\big[f\big(-(x_1-t)\big)+f(x_1-t)\big]-\frac{1}{2}\big[f(-2x_2)+f(2x_2)\big]
\end{aligned}
$$

we get

$$
\begin{aligned}
\left\|2f\left(-\frac{x_1-x_2}{2}\right)+f(x_1)-f(x_2)\right\| &\leqslant 3\delta+2\left[\frac{1}{2}\big(3\delta+\big\|f(0)\big\|\big)\right]+2(\delta)\\
&=8\delta+\big\|f(0)\big\|. \qquad (8.2)
\end{aligned}
$$

Applying now Theorem 7.1 and the above inequality (8.2), one gets that there exists a unique alternative Jensen type mapping $A: X \to Y$ that satisfies the alternative Jensen type equation (1.2b) and inequality (8.1), such that $A(x)=\lim_{n\to\infty} 2^{-n} f(2^n x)$ with $A(-x)=-A(x)$ (from (8.1a)). $\square$

We note that, if we define $S_1=\{x\in X\colon \|x\|<d\}$ and $S_2=\{(x_1,x_2)\in X^2\colon \|x_i\|<d, i=1,2\}$ for some $d>0$, then $\{x\in X\colon \|x\|\geqslant 2d\}\subset X\setminus S_1$ and $\{(x_1,x_2)\in X^2\colon \|x_1\|+\|x_2\|\geqslant 2d\}\subset X^2\setminus S_2$.

Corollary 8.1. *If we assume that a mapping $f: X\to Y$ satisfies inequality* (7.1) *for some fixed $\delta\geqslant 0$ and for all $(x_1,x_2)\in X^2\setminus S_2$ and* (8.1a)–(8.1b) *for all $x\in X\setminus S_1$, then there exists a unique alternative Jensen type mapping $A: X\to Y$, satisfying* (8.1) *for all $x\in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x\in X$, then $A(tx)=tA(x)$ for all $x\in X$ and all $t\in\mathbb{R}$.*

Corollary 8.2. *A mapping $f: X\to Y$ is an alternative Jensen type mapping, if and only if the asymptotic conditions $\|f(-x)+f(x)\|\to 0$ and $\|f(2x)-2f(x)\|\to 0$, as $\|x\|\to\infty$ and $\|2f(-\frac{x_1-x_2}{2})+f(x_1)-f(x_2)\|\to 0$, as $\|x_1\|+\|x_2\|\to\infty$, hold, respectively.*

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