Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 1, Article 11, 2005

ON THE HEISENBERG-WEYL INEQUALITY

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Received 20 September, 2004; accepted 25 November, 2004
Communicated by G. Anastassiou

ABSTRACT. In 1927, W. Heisenberg demonstrated the impossibility of specifying simultaneously the position and the momentum of an electron within an atom.The well-known *second moment Heisenberg-Weyl inequality* states: Assume that $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a random real variable x such that $f \in L^2(\mathbb{R})$. Then the product of the second moment of the random real x for $|f|^2$ and the second moment of the random real ξ for $\left|\hat{f}\right|^2$ is at least $E_{|f|^2}\Big/4\pi$, where $\hat{f}$ is the Fourier transform of f, such that $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x)\, dx$ and $f(x) = \int_{\mathbb{R}} e^{2i\pi\xi x} \hat{f}(\xi)\, d\xi$, $i = \sqrt{-1}$ and $E_{|f|^2} = \int_{\mathbb{R}} |f(x)|^2\, dx$. In 2004, the author generalized the afore-mentioned result to *the higher order absolute moments for L^2 functions f* with orders of moments in the set of natural numbers . In this paper, a new generalization proof is established with orders of absolute moments in the set of non-negative real numbers. Afterwards, an application is provided by means of the well-known Euler gamma function and the Gaussian function and an open problem is proposed on some pertinent extremum principle. This inequality can be applied in harmonic analysis and quantum mechanics.

Key words and phrases: Heisenberg-Weyl Inequality, Uncertainty Principle, Absolute Moment, Gaussian, Extremum Principle.

2000 *Mathematics Subject Classification.* 26Dxx, 30Xxx, 33Xxx, 42Xxx, 43Xxx, 60Xxx, 62Xxx, 81Xxx.

1. INTRODUCTION

The serious question of certainty in science was high-lighted by Heisenberg (1901-1976), in 1927, via his "uncertainty principle" [7]. He demonstrated, for instance, the impossibility of specifying simultaneously the position and the speed (or the momentum) of an electron within

ISSN (electronic): 1443-5756

We are grateful to Professors George Anastassiou and Bill Beckner for their great suggestions.

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an atom. In 1933, according to Wiener (1894-1964) [10] *a pair of transforms cannot both be very small*.

This uncertainty principle was stated in 1925 by Wiener, according to Wiener's autobiography [11, p. 105–107], in a lecture in Göttingen. In 1997, according to Folland and Sitaram [5] the uncertainty principle in harmonic analysis says: *A nonzero function and its Fourier transform cannot both be sharply localized.* The following result of the *Heisenberg-Weyl Inequality* is credited to Pauli (1900 – 1958) according to Weyl [9, p. 77, p. 393–394]. In 1928, according to Pauli [9], *the less the uncertainty in* $|f|^2$, *the greater the uncertainty in* $\left|\hat{f}\right|^2$, *and conversely*. This result does not actually appear in Heisenberg's seminal paper [7] (in 1927). In 1997 Battle [1] proved a number of excellent uncertainty results for wavelet states. Coifman et al. [3] established important results in signal processing and compression with wavelet packets. For fundamental accounts of the construction of orthonormal wavelets we refer the reader to Daubechies [4]. In 1998, Burke Hubbard [2] wrote a remarkable book on wavelets. According to her, most people first learn the Heisenberg uncertainty principle in connection with quantum mechanics, but it is also a central statement of information processing. According to Folland and Sitaram [5] (in 1997), Heisenberg gave an incisive analysis of the physics of the uncertainty principle but contains little mathematical precision. The following second order moment Heisenberg-Weyl inequality provides a precise quantitative formulation of the above-mentioned uncertainty principle according to W. Pauli.

1.1. **Second Moment Heisenberg-Weyl Inequality ([2] – [5]).** *For any* $f \in L^2(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{C}$, *such that* $\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 dx = E_{|f|^2}$, *any fixed but arbitrary constants* x_m, $\xi_m \in \mathbb{R}$, *and for the second order moments (variances)*

$$(\mu_2)_{|f|^2} = \sigma^2_{|f|^2} = \int_{\mathbb{R}} (x - x_m)^2 |f(x)|^2 dx$$

and

$$(\mu_2)_{\left|\hat{f}\right|^2} = \sigma^2_{\left|\hat{f}\right|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^2 \left|\hat{f}(\xi)\right|^2 d\xi,$$

the second order moment Heisenberg-Weyl inequality

$$(H_1) \qquad \sigma^2_{|f|^2} \cdot \sigma^2_{\left|\hat{f}\right|^2} \geq \frac{\|f\|_2^4}{16\pi^2},$$

holds. Equality holds in (H_1) *if and only if the generalized Gaussians*

$$f(x) = c_0 \exp(2\pi i x \xi_m) \exp\left(-c(x - x_m)^2\right)$$

hold for some constants $c_0 \in \mathbb{C}$ *and* $c > 0$.

The *Heisenberg-Weyl inequality* in mathematical statistics and Fourier analysis asserts that: The product of the variances of the probability measures $|f(x)|^2 dx$ and $\left|\hat{f}(\xi)\right|^2 d\xi$ is larger than an absolute constant. Parts of harmonic analysis on euclidean spaces can naturally be expressed in terms of a *Gaussian measure*; that is, a measure of the form $c_0 e^{-c|x|^2} dx$, where dx is the Lebesgue measure and c, c_0 (> 0) constants. Among these are: Logarithmic Sobolev inequalities, and Hermite expansions. In 1999, according to Gasquet and Witomski [6] the Heisenberg-Weyl inequality in *spectral analysis* says that the product of the effective duration Δx and the effective bandwidth $\Delta\xi$ of a signal cannot be less than the value $1/4\pi$ =Heisenberg lower bound, where $\Delta x^2 = \sigma^2_{|f|^2} \Big/ E_{|f|^2}$ and $\Delta\xi^2 \left(= \sigma^2_{\left|\hat{f}\right|^2} \Big/ E_{\left|\hat{f}\right|^2}\right) = \sigma^2_{\left|\hat{f}\right|^2} \Big/ E_{|f|^2}$ with f :

$\mathbb{R} \to \mathbb{C}$, $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined as in (H_1), and

(PPR) $$E_{|f|^2} = \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} \left|\hat{f}(\xi)\right|^2 d\xi = E_{|\hat{f}|^2},$$

according to the Plancherel-Parseval-Rayleigh identity [6].

1.2. **Fourth Moment Heisenberg-Weyl Inequality (** [8, p. 26]**).** *For any $f \in L^2(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{C}$, such that $\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 dx = E_{|f|^2}$, any fixed but arbitrary constants x_m, $\xi_m \in \mathbb{R}$, and for the fourth order moments*

$$(\mu_4)_{|f|^2} = \int_{\mathbb{R}} (x - x_m)^4 |f(x)|^2 dx$$

and

$$(\mu_4)_{|\hat{f}|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^4 \left|\hat{f}(\xi)\right|^2 d\xi,$$

the fourth order moment Heisenberg - Weyl inequality

(H_2) $$(\mu_4)_{|f|^2} \cdot (\mu_4)_{|\hat{f}|^2} \geq \frac{1}{64\pi^4} \mathrm{E}_{2,f}^2,$$

holds, where

$$E_{2,f} = 2 \int_{\mathbb{R}} \left[(1 - 4\pi^2 \xi_m^2 x_\delta^2) |f(x)|^2 - x_\delta^2 |f'(x)|^2 - 4\pi \xi_m x_\delta^2 \operatorname{Im}(f(x)\overline{f'(x)}) \right] dx,$$

with $x_\delta = x - x_m$, $\xi_\delta = \xi - \xi_m$, $\operatorname{Im}(\cdot)$ is the imaginary part of $(\cdot)$, and $|E_{2,f}| < \infty$.

The "inequality" (H_2) holds, unless $f(x) = 0$.

We note that if the ordinary differential equation of second order

(ODE) $$f_\alpha''(x) = -2c_2 x_\delta^2 f_\alpha(x)$$

holds, with $\alpha = -2\pi\xi_m i$, $f_\alpha(x) = e^{\alpha x} f(x)$, and a constant $c_2 = \frac{1}{2}k_2^2 > 0, k_2 \in \mathbb{R}$ and $k_2 \neq 0$, then "equality" in (H_2) seems to occur. However, the solution of this differential equation (ODE), given by the function

$$f(x) = \sqrt{|x_\delta|} e^{2\pi i x \xi_m} \left[c_{20} J_{-1/4}\left(\frac{1}{2}|k_2| x_\delta^2\right) + c_{21} J_{1/4}\left(\frac{1}{2}|k_2| x_\delta^2\right) \right],$$

in terms of the Bessel functions $J_{\pm 1/4}$ of the first kind of orders $\pm 1/4$, leads to a contradiction, because this $f \notin L^2(\mathbb{R})$. Furthermore, a limiting argument is required for this problem. For the proof of this inequality see [8]. In 2004, we [8] generalized the Heisenberg-Weyl inequality with orders of moments in the set of natural numbers. In this paper we establish a new generalization proof with orders of absolute moments in the set of non-negative real numbers. It is *open* to investigate cases, where the integrand on the right-hand side of integrals of $E_{2,f}$ will be nonnegative. For instance, for $x_m = \xi_m = 0$, this integrand is:$= |f(x)|^2 - x^2 |f'(x)|^2 \ (\geq 0)$.

2. HEISENBERG-WEYL INEQUALITY

If $\int_{\mathbb{R}} |f(x)|^2 dx = E_{|f|^2}$, then we state and prove the following new theorem.

Theorem 2.1. *If $f \in L^2(\mathbb{R})$ and $\rho \geq 2$, then the Heisenberg-Weyl inequality*

(2.1) $$\left(\mu_\rho^*\right)_{|f|^2}^{1/\rho} \left(\mu_\rho^*\right)_{|\hat{f}|^2}^{1/\rho} \geq E_{|f|^2}^{2/\rho} \Big/ 4\pi,$$

holds for any fixed but arbitrary real constants x_m, ξ_m and the higher order absolute moments

$$\left(\mu_\rho^*\right)_{|f|^2} = \int_{\mathbb{R}} |x_\delta|^\rho |f(x)|^2 dx$$

with $x_\delta = x - x_m$ *and*

$$\left(\mu_\rho^*\right)_{|\hat{f}|^2} = \int_{\mathbb{R}} |\xi_\delta|^\rho \left|\hat{f}\left(\xi\right)\right|^2 d\xi$$

with $\xi_\delta = \xi - \xi_m$. *The "inequality" (2.1) holds, unless* $f(x) = 0$. *Equality in (2.1) holds for* $\rho = 2$ *and all the Gaussian mappings of the form* $f\left(x\right) = c_0 \exp\left(-cx^2\right)$, *where* c_0, c *are constants and* $c_0 \in \mathbb{C}$, $c > 0$, *or for* $\rho \geq 2$ *and all mappings* $f \in L^2(\mathbb{R})$, *such that* $|x_\delta| = |\xi_\delta| = \sqrt{1/4\pi}$.

Proof. Applying the inequality (H_1), *the Hölder inequality* and *the Plancherel-Parseval-Rayleigh identity* one gets

$$\begin{aligned}
\left(\mu_\rho^*\right)_{|f|^2}^{\frac{2}{\rho}} \left(E_{|f|^2}\right)^{1-\frac{2}{\rho}} &= \left(\int_{\mathbb{R}} |x_\delta|^\rho \left|f\left(x\right)\right|^2 dx\right)^{\frac{2}{\rho}} \left(\int_{\mathbb{R}} \left|f\left(x\right)\right|^2 dx\right)^{1-\frac{2}{\rho}} \\
&= \left[\int_{\mathbb{R}} \left(|x_\delta|^2 \left|f\left(x\right)\right|^{4/\rho}\right)^{\rho/2} dx\right]^{\frac{2}{\rho}} \left[\int_{\mathbb{R}} \left(\left|f\left(x\right)\right|^{2\left(1-\frac{2}{\rho}\right)}\right)^{1/\left(1-\frac{2}{\rho}\right)} dx\right]^{1-\frac{2}{\rho}} \\
&\geq \int_{\mathbb{R}} \left[\left(x_\delta^2 \left|f\left(x\right)\right|^{4/\rho}\right) \left(\left|f\left(x\right)\right|^{2\left(1-\frac{2}{\rho}\right)}\right)\right] dx \\
&= \int_{\mathbb{R}} x_\delta^2 \left|f\left(x\right)\right|^2 dx = \sigma_{|f|^2}^2,
\end{aligned}$$

or

$$\left(\mu_\rho^*\right)_{|f|^2}^{1/\rho} \geq \sigma_{|f|^2} \Big/ \left(E_{|f|^2}\right)^{\left(1-\frac{2}{\rho}\right)/2}. \tag{2.2}$$

Equality in (2.2) holds if and only if

$$|x_\delta|^\rho E_{|f|^2} = \left(\mu_\rho^*\right)_{|f|^2}.$$

Similarly, we prove from (2.2) and (PPR) that

$$\left(\mu_\rho^*\right)_{|\hat{f}|^2}^{2/\rho} \left(E_{|\hat{f}|^2}\right)^{1-\frac{2}{\rho}} \geq \sigma_{|\hat{f}|^2}^2,$$

or

$$\left(\mu_\rho^*\right)_{|\hat{f}|^2}^{1/\rho} \geq \sigma_{|\hat{f}|^2} \Big/ \left(E_{|f|^2}\right)^{\left(1-\frac{2}{\rho}\right)/2}. \tag{2.3}$$

Equality in (2.3) holds if and only if

$$|\xi_\delta|^\rho E_{|f|^2} = \left(\mu_\rho^*\right)_{|\hat{f}|^2}.$$

Multiplying (2.2) and (2.3) one finds

$$M_\rho^* = \left(\mu_\rho^*\right)_{|f|^2}^{1/\rho} \left(\mu_\rho^*\right)_{|\hat{f}|^2}^{1/\rho} \geq \sigma_{|f|^2} \cdot \sigma_{|\hat{f}|^2} \Big/ \left(E_{|f|^2}\right)^{1-\frac{2}{\rho}}. \tag{2.4}$$

It is now clear, from (2.4) and the classical Heisenberg-Weyl inequality (H_1), the complete proof of the above theorem. □

2.1. **Euler gamma function and Gaussian function.** Assume *the Gaussian function* of the form

$$f(x) = c_0 \exp\left(-cx^2\right), \tag{2.5}$$

where c_0, c are constants and $c_0 \in \mathbb{C}$, $c > 0$. Besides consider that x_m, ξ_m, are *means* of x for $|f|^2$ and of ξ for $\left|\hat{f}\right|^2$, respectively. If Γ is *the Euler gamma function* and $\rho = 2, 3, 4, \ldots$, then $x_m = \int_{\mathbb{R}} x\,|f(x)|^2\,dx = 0$ .We claim that *the Fourier transform* $\hat{f} : \mathbb{R} \to \mathbb{C}$ is of the form

$$\hat{f}(\xi) = c_0 \left(\frac{\pi}{c}\right)^{\frac{1}{2}} \exp\left(-\frac{\pi^2}{c}\xi^2\right), \tag{2.6}$$

by applying a direct computation using a differential equation ([6, p. 159–161]).

In fact, differentiating the Gaussian function $f : \mathbb{R} \to \mathbb{C}$ of the form $f(x) = c_0 e^{-cx^2}$ with respect to x, one gets the ordinary differential equation $f'(x) = -2cxf(x)$. Thus the Fourier transform of f' is

$$Ff'(\xi) = F[f'(x)](\xi) = [f'(x)]^{\wedge}(\xi) = [-2cxf(x)]^{\wedge}(\xi),$$

or

$$2i\pi\xi\hat{f}(\xi) = \frac{-2c}{-2i\pi}\left[(-2i\pi x) f(x)\right]^{\wedge}(\xi),$$

by standard formulas on differentiation. Thus $2i\pi\xi\ \hat{f}(\xi) = \frac{c}{i\pi}\left(\hat{f}(\xi)\right)'$, or $-2\pi^2\xi\ \hat{f}(\xi) = c\hat{f}'(\xi)$, or $\left(\hat{f}(\xi)\right)' = \hat{f}'(\xi) = -\frac{2\pi}{c}(\pi\xi)\hat{f}(\xi)$.

Solving this *first order differential equation* by the method of the separation of variables we get the general solution

$$\hat{f}(\xi) = K(\xi)\, e^{-\frac{\pi^2}{c}\xi^2}, \tag{2.7}$$

such that $\hat{f}(0) = K(0)$. Differentiating the above formula with respect to ξ one finds

$$\hat{f}'(\xi) = e^{-\frac{\pi^2}{c}\xi^2}\left[K'(\xi) + K(\xi)\left(-\frac{2\pi^2}{c}\xi\right)\right].$$

Therefore we find $0 = K'(\xi)\, e^{-\frac{\pi^2}{c}\xi^2}$, or $K'(\xi) = 0$, or

$$K(\xi) = K, \tag{2.8}$$

which is a constant. But from (2.7) and (2.8) one gets

$$\hat{f}(0) = K(0) = K. \tag{2.9}$$

Besides from the definition of the Fourier transform we get

$$\hat{f}(0) = \int_{\mathbb{R}} e^{-2i\pi\cdot 0\cdot x} f(x)\,dx = \int_{\mathbb{R}} f(x)\,dx = c_0 \int_{\mathbb{R}} e^{-cx^2} dx = \frac{c_0}{\sqrt{c}} \int_{\mathbb{R}} e^{-\left[\sqrt{c}x\right]^2} d\left(\sqrt{c}x\right),$$

or

$$\hat{f}(0) = c_0\sqrt{\frac{\pi}{c}}, \quad c_0 \in \mathbb{C}, \quad c > 0. \tag{2.10}$$

From (2.9) and (2.10) one finds $K = c_0\sqrt{\frac{\pi}{c}}$, $c_0 \in \mathbb{C}$, $c > 0$.

Therefore we complete the proof of the formula (2.6). Moreover,

$$\xi_m = \int_{\mathbb{R}} \xi \left|\hat{f}(\xi)\right|^2 d\xi = |c_0|^2 \frac{\pi}{c} \int_{\mathbb{R}} \xi \cdot e^{-2\frac{\pi^2}{c}\xi^2} d\xi = 0.$$

Therefore

$$\left(M_\rho^*\right)^\rho = \left(\mu_\rho^*\right)_{|f|^2} \cdot \left(\mu_\rho^*\right)_{|\hat{f}|^2} = \left(H_{\rho/2}^*\right)^\rho 2\Gamma^2\left(\frac{\rho+1}{2}\right)\left(\frac{|c_0|^4}{c}\right),$$

or

$$M_\rho^* = H_{\rho/2}^*\left[\frac{4}{\pi}\Gamma^2\left(\frac{\rho+1}{2}\right)\right]^{\frac{1}{\rho}} E_{|f|^2}^{2/\rho},$$

because $E_{|f|^2} = |c_0|^2\left(\pi/2c\right)^{1/2}$, $H_{\rho/2}^* = 1\big/\left((2\pi)\,4^{1/\rho}\right)$, and

$$\int_{\mathbb{R}} |x|^\rho \exp\left(-2cx^2\right)dx = \frac{\Gamma\left(\frac{\rho+1}{2}\right)}{(2c)^{\frac{\rho+1}{2}}}, \quad c>0,\ \rho\in N_0.$$

But we have for $\rho = 2p$, $p\in\mathbb{N}$ that

$$\Gamma\left(\frac{\rho+1}{2}\right) = (\rho-1)!!\left(\frac{\pi}{2^\rho}\right)^{\frac{1}{2}} \geq \left(\frac{\pi}{2^\rho}\right)^{\frac{1}{2}},$$

where $(\rho-1)!! = 1\cdot 3\cdot 5\cdot\cdots\cdot(\rho-1)$ (for $\rho=2p$, $p\in\mathbb{N}$). It is clear that this holds as well for $\rho = 2q+1$, $q\in\mathbb{N}$. Thus one gets

$$M_\rho^* \geq \left(\frac{1}{(2\pi)\,4^{1/\rho}}\right)\left[\frac{4}{\pi}\left(\frac{\pi}{2^\rho}\right)\right]^{\frac{1}{\rho}} E_{|f|^2}^{2/\rho} = E_{|f|^2}^{2/\rho}\Big/4\pi,$$

verifying (2.1) for all $\rho = 2,3,4,\ldots$. We note that if $\rho=2$, $p=1$ then the equality in (2.1) holds for these Gaussian mappings.

Queries. *Concerning our Section 8.1 on pp. 26-27 of* [8]*, further investigation is needed for the case of the fundamental "equality" in (H_2). As a matter of fact, our function f is not in $L^2(\mathbb{R})$, leading the left-hand side to be infinite in that "equality". A limiting argument is required for this problem. On the other hand, why doesn't the corresponding "inequality" (H_2) attain an extremal in $L^2(\mathbb{R})$?*

Here are some of our old results [8] related to the above *Queries*. In particular, if we take into account these results contained in Section 9 on pp. 46-70 [8], where the Gaussian function and the Euler gamma function Γ are employed, then via Corollary 9.1 on pp. 50-51 [8] we conclude that "equality" in (H_p), $p\in\mathbb{N}=\{1,2,3,\ldots\}$, holds only for $p=1$. Furthermore, employing the above Gaussian function, we established the following *extremum principle* (via (9.33) on p. 51 [8]):

$$\text{(R)} \qquad R(p) \geq \frac{1}{2\pi}, \quad p\in\mathbb{N}$$

for the corresponding "inequality" (H_p), $p\in\mathbb{N}$, where the constant $1/2\pi$ "on the right-hand side" is the best lower bound for $p\in\mathbb{N}$. Therefore "equality" in (H_p), $p\in\mathbb{N}-\{1\}$, in Section 8.1 on pp. 19-46 [8] cannot occur under the afore-mentioned well-known functions. On the other hand, there is a lower bound "on the right-hand side" of the corresponding "inequality" in (H_2) on p. 26 and pp. 54-55 [8] if we employ the above Gaussian function, which equals to $\frac{1}{64\pi^4}E_{2,f}^2 = \frac{1}{512\pi^3}\cdot\frac{|c_0|^4}{c}$, with c_0, c constants and $c_0\in\mathbb{C}$, $c>0$, because $E_{|f|^2} = |c_0|^2\sqrt{\frac{\pi}{2c}}$ and $E_{2,f} = \frac{1}{2}E_{|f|^2}$.

Analogous pertinent results are investigated via our Corollaries 9.2-9.6 on pp. 53-68 [8].

Open Problem And Extremum Principle. Employing our Theorem 8.1 on p. 20 [8], the Gaussian function, the Euler gamma fuction Γ, and other related *"special functions"*, we established and explicitly proved *the extremum principle* (R): $R(p) \geq 1/2\pi$, $p \in \mathbb{N}$, where

$$R(p) = \frac{\Gamma\left(p+\frac{1}{2}\right)}{\left|\sum_{q=o}^{\left[\frac{p}{2}\right]} (-1)^{p-q} \frac{p}{p-q} \cdot \frac{p!}{(2q)!} \binom{p-q}{q} \Gamma_q\right|},$$

with

$$\Gamma_q = \sum_{k=0}^{\left[\frac{q}{2}\right]} 2^{2k} \binom{q}{2k}^2 \Gamma^2\left(k+\frac{1}{2}\right) \Gamma\left(2q-2k+\frac{1}{2}\right) + 2 \sum_{0 \leq k \leq j \leq \left[\frac{q}{2}\right]} (-1)^{k+j} 2^{k+j} \binom{q}{2k} \binom{q}{2j} \Gamma\left(k+\frac{1}{2}\right) \Gamma\left(j+\frac{1}{2}\right) \Gamma\left(2q-k-j+\frac{1}{2}\right),$$

$0 \leq \left[\frac{q}{2}\right]$ is the greatest integer $\leq \frac{q}{2}$ for $q \in \mathbb{N} \cup \{0\} = \mathbb{N}_0$, $\binom{p}{q} = \frac{p!}{q!(p-q)!}$ for $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $0 \leq q \leq p$, $p! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (p-1) \cdot p$ and $0! = 1$, as well as

$$\Gamma\left(p+\frac{1}{2}\right) = \frac{1}{2^{2p}} \cdot \frac{(2p)!}{p!}\sqrt{\pi}, \quad p \in \mathbb{N}$$

and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

In addition, we [8] analytically verified this extremum principle for $p = 1, 2, \ldots, 9$ by carrying out all the involved operations. In particular, if we denote $L = 1/2\pi (\cong 0.159)$, then the first nine exact values of $R(p)$ are, as follows: $\mathbb{R}(1) = L$, $\mathbb{R}(2) = 3L$, $\mathbb{R}(3) = 5L$, $\mathbb{R}(4) = \frac{35}{13}L$, $\mathbb{R}(5) = \frac{63}{17}L$, $\mathbb{R}(6) = \frac{231}{19}L$, $\mathbb{R}(7) = \frac{429}{23}L$, $\mathbb{R}(8) = \frac{495}{47}L$, $\mathbb{R}(9) = \frac{12155}{827}L$.

Furthermore, by employing computer techniques, this principle was verified for $p = 1, 2, 3, \ldots, 32, 33$, as well. It now remains *open* to give an explicit second proof of verification for the extremum principle (R) through a much shorter and more elementary method, without applying our Heisenberg-Pauli-Weyl inequality [8].

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