Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 1, Article 4, 2004

ON THE HEISENBERG-PAULI-WEYL INEQUALITY

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Received 02 January, 2003; accepted 04 March, 2003
Communicated by A. Babenko

ABSTRACT. In 1927, W. Heisenberg demonstrated the impossibility of specifying simultaneously the position and the momentum of an electron within an atom.The following result named, *Heisenberg inequality*, is not actually due to Heisenberg. In 1928, according to H. Weyl this result is due to W. Pauli.The said inequality states, as follows: Assume that $f:\mathbb{R}\to\mathbb{C}$ is a complex valued function of a random real variable x such that $f\in L^2(\mathbb{R})$. Then the product of the second moment of the random real x for $|f|^2$ and the second moment of the random real ξ for $\left|\hat{f}\right|^2$ is at least $E_{|f|^2}/4\pi$, where $\hat{f}$ is the Fourier transform of f, such that $\hat{f}(\xi)=\int_R e^{-2i\pi\xi x}f(x)\,dx$ and $f(x)=\int_R e^{2i\pi\xi x}\hat{f}(\xi)\,d\xi$, $i=\sqrt{-1}$ and $E_{|f|^2}=\int_R|f(x)|^2\,dx$. In this paper we generalize the afore-mentioned result to *the higher moments for L^2 functions f* and establish the *Heisenberg-Pauli-Weyl inequality*.

Key words and phrases: Pascal Identity, Plancherel-Parseval-Rayleigh Identity, Lagrange Identity, Gaussian function, Fourier transform, Moment, Bessel equation, Hermite polynomials, Heisenberg-Pauli-Weyl Inequality.

2000 *Mathematics Subject Classification.* Primary: 26Xxx; Secondary: 42Xxx, 60Xxx, 62Xxx.

1. INTRODUCTION

In 1927, W. Heisenberg [9] demonstrated the impossibility of specifying simultaneously the position and the momentum of an electron within an atom. In 1933, according to N. Wiener [22] *a pair of transforms cannot both be very small*. This uncertainty principle was stated in 1925 by Wiener, according to Wiener's autobiography [23, p. 105–107], in a lecture in Göttingen. In 1992, J.A. Wolf [24] and in 1997, G. Battle [1] established uncertainty principles for Gelfand pairs and wavelet states, respectively. In 1997, according to Folland et al. [6], and in 2001, according to Shimeno [14] the uncertainty principle in harmonic analysis says:

ISSN (electronic): 1443-5756

We thank the computer scientist *C. Markata* for her help toward the computation of the data of Table 9.1 for the last twenty four cases $p=10,11,\ldots,33$.

We greatly thank Prof. Alexander G. Babenko for the communication and the anonymous referee for his/her comments.

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A nonzero function and its Fourier transform cannot both be sharply localized. The following result of the *Heisenberg Inequality* is credited to W. Pauli according to H. Weyl [20, p.77, p. 393–394]. In 1928, according to Pauli [20], the following *proposition* holds: *the less the uncertainty in* $|f|^2$, *the greater the uncertainty in* $\left|\hat{f}\right|^2$ *is , and conversely*. This result does not actually appear in Heisenberg's seminal paper [9] (in 1927). According to G.B. Folland et al. [6] (in 1997) Heisenberg [9] gave an incisive analysis of the physics of the uncertainty principle but contains little mathematical precision. The following Heisenberg inequality provides a precise quantitative formulation of the above-mentioned uncertainty principle according to Pauli [20].

In what follows we will use the following notation to denote the Fourier transform of $f(x)$:

$$F(f(x)) \equiv [f(x)]\,\hat{}\,(\xi).$$

1.1. **Second Order Moment Heisenberg Inequality (**[3, 6]**).** For any $f \in L^2(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{C}$, such that

$$\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx = E_{|f|^2},$$

any fixed but arbitrary constants x_m, $\xi_m \in \mathbb{R}$, and for the second order moments (variances)

$$(\mu_2)_{|f|^2} = \sigma^2_{|f|^2} = \int_{\mathbb{R}} (x - x_m)^2 \, |f(x)|^2 \, dx$$

and

$$(\mu_2)_{\left|\hat{f}\right|^2} = \sigma^2_{\left|\hat{f}\right|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^2 \left|\hat{f}(\xi)\right|^2 d\xi,$$

the second order moment Heisenberg inequality

$$(H_1) \qquad (\mu_2)_{|f|^2} \cdot (\mu_2)_{\left|\hat{f}\right|^2} \geq \frac{E^2_{|f|^2}}{16\pi^2},$$

holds, where

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x) \, dx$$

and

$$f(x) = \int_{\mathbb{R}} e^{2i\pi\xi x} \hat{f}(\xi) \, d\xi, \quad i = \sqrt{-1}.$$

Equality holds in (H_1) *iff* (if and only if) *the Gaussians*

$$f(x) = c_0 e^{2\pi i x \xi_m} e^{-c(x-x_m)^2} = c_0 e^{-cx^2 + 2(cx_m + i\pi\xi_m)x - cx_m^2}$$

hold for some constants $c_0 \in \mathbb{C}$ and $c > 0$. We note that if $x_m \neq 0$ and $\xi_m = 0$, then $f(x) = c_0 e^{-c(x-x_m)^2}$, $c_0 \in \mathbb{C}$ and $c > 0$.

Proof. Let $x_m = \xi_m = 0$, and that the integrals in the inequality (H_1) be finite. Besides we consider both the ordinary derivative $\frac{d}{dx}|f|^2 = 2\,\mathrm{Re}\left(f\bar{f}'\right)$ and the Fourier differentiation formula

$$Ff'(\xi) = [f'(x)]\,\hat{}\,(\xi) = 2\pi i \xi \hat{f}(\xi).$$

Then we get that the finiteness of the integral

$$\int_{\mathbb{R}} \left|\xi \hat{f}(\xi)\right|^2 d\xi \quad \left(= \frac{1}{4\pi^2} \int_{\mathbb{R}} |f'(x)|^2 \, dx\right)$$

implies $f' \in L^2$. Integration by parts yields

$$2\int_{a_-}^{a_+} x\,\mathrm{Re}\left(f(x)\,\overline{f'(x)}\right)dx = \int_{a_-}^{a_+} x\frac{d}{dx}\left|f(x)\right|^2 dx = x\left|f(x)\right|^2\Big|_{a_-}^{a_+} - \int_{a_-}^{a_+}\left|f(x)\right|^2 dx, \quad -\infty < a_- < a_+ < \infty.$$

Since f, xf, $f' \in L^2$, the integrals in this equality are finite as $a_- \to -\infty$ or $a_+ \to \infty$ and thus both limits $L_- = \lim_{a_-\to-\infty} a_-\left|f(a_-)\right|^2$ and $L_+ = \lim_{a+\to\infty} a_+\left|f(a_+)\right|^2$ are finite. These two limits $L_\pm$ are equal to zero, for otherwise $\left|f(x)\right|^2$ would behave as $\frac{1}{x}$ for big x meaning that $f \notin L^2$, leading to contradiction. Therefore for the variances about the origin

$$(m_2)_{|f|^2} = s^2_{|f|^2} = \int_{\mathbb{R}} x^2\left|f(x)\right|^2 dx$$

and

$$(m_2)_{|\hat{f}|^2} = s^2_{|\hat{f}|^2} = \int_{\mathbb{R}} \xi^2\left|\hat{f}(\xi)\right|^2 d\xi,$$

one gets

$$\begin{aligned}
\frac{1}{16\pi^2}\|f\|_2^4 &= \left(\frac{1}{4\pi}E_{|f|^2}\right)^2 \\
&= \frac{1}{16\pi^2}\left(-\int_{\mathbb{R}}\left|f(x)\right|^2 dx\right)^2 \\
&= \frac{1}{16\pi^2}\left[\int_{\mathbb{R}} x\frac{d}{dx}\left|f(x)\right|^2 dx\right]^2 \\
&= \frac{1}{16\pi^2}\left[2\int_{\mathbb{R}} x\,\mathrm{Re}\left(f(x)\,\overline{f'(x)}\right)dx\right]^2 \\
&= \frac{1}{4\pi^2}\left[\frac{1}{2}\int_{\mathbb{R}}\left(xf(x)\,\overline{f'(x)} + x\overline{f(x)}f'(x)\right)dx\right]^2 \\
&\leq \frac{1}{4\pi^2}\left[\int_{\mathbb{R}}\left|xf(x)\,f'(x)\right|dx\right]^2 \\
&\leq \frac{1}{4\pi^2}\left(\int_{\mathbb{R}} x^2\left|f(x)\right|^2 dx\right)\left(\int_{\mathbb{R}}\left|f'(x)\right|^2 dx\right) = s^2_{|f|^2}\cdot s^2_{|\hat{f}|^2}.
\end{aligned}$$

Equality in these inequalities holds iff *the differential equation* $f'(x) = -2cxf(x)$ of first order holds for $c > 0$ or if the Gaussians $f(x) = c_0 e^{-cx^2}$ hold for some constants $c_0 \in \mathbb{C}$, and $c > 0$.

Assuming any fixed but arbitrary real constants x_m, ξ_m and employing the transformation

$$f_{x_m,\xi_m}(x) = e^{2\pi i x\xi_m} f(x - x_m), \quad x_\delta = x - x_m \neq 0,$$

we establish the formula

$$\hat{f}_{x_m,\xi_m}(\xi) = e^{-2\pi i x_m(\xi-\xi_m)}\hat{f}(\xi - \xi_m) = e^{2\pi i x_m\xi_m}\hat{f}_{\xi_m,-x_m}(\xi).$$

Therefore the map $f \to f_{x_m,\xi_m}$ preserves all L^{2p} ($p \in \mathbb{N}$) norms of f and $\hat{f}$ while shifting the centers of mass of f and $\hat{f}$ by real x_m and ξ_m, respectively. Therefore equality holds in (H_1) for any fixed but arbitrary constants x_m, $\xi_m \in \mathbb{R}$ iff *the general formula*

$$f(x) = c_0 e^{2\pi i x\xi_m} e^{-c(x-x_m)^2}$$

holds for some constants $c_0 \in \mathbb{C}$, and $c > 0$. One can observe that this general formula is the complete (general) solution of the following *a-differential equation* $f_a' = -2c\,(x - x_m)\,f_a$, by the method of the separation of variables, where $a = -2\pi\xi_m i,\ f_a = e^{ax}f$. We note that

$$f_a' = \frac{df_a}{dx} = \frac{df_a}{dx_\delta}\frac{dx_\delta}{dx} = \frac{df_a}{dx_\delta}, \qquad x_\delta = x - x_m.$$

In fact,

$$\ln|f_a| = -c\,(x - x_m)^2\,,$$

or

$$e^{ax}f\,(x) = f_a\,(x) = c_0 e^{-c(x-x_m)^2},$$

or

$$f\,(x) = c_0 e^{-ax}e^{-c(x-x_m)^2}.$$

This is a special case for the equality of the general formula (8.3) of our theorem in Section 8, on the generalized weighted moment Heisenberg uncertainty principle. Therefore the proof of this fundamental Heisenberg Inequality (H_1) is complete. □

We note that, if $f \in L^2(\mathbb{R})$ and the L^2-norm of f is $\|f\|_2 = 1 = \left\|\hat{f}\right\|_2$, then $|f|^2$ and $\left|\hat{f}\right|^2$ are both probability density functions. The Heisenberg inequality *in mathematical statistics* and *Fourier analysis* asserts that: The product of the variances of the probability measures $|f\,(x)|^2\,dx$ and $\left|\hat{f}\,(\xi)\right|^2 d\xi$ is larger than an absolute constant. Parts of harmonic analysis on Euclidean spaces can naturally be expressed in terms of a *Gaussian measure*; that is, a measure of the form $c_0 e^{-c|x|^2}dx$, where dx is the Lebesgue measure and c, c_0 (> 0) constants. Among these are: Logarithmic Sobolev inequalities, and Hermite expansions. Finally one [14] observes that:

$$\sigma^2_{|f|^2}\cdot\sigma^2_{\left|\hat{f}\right|^2} \geq \frac{1}{4}\,\|f\|_2^4\,,$$

if $f \in L^2(\mathbb{R})$,

$$\hat{f}\,(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi x}f\,(x)\,dx$$

and

$$f\,(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\xi x}\hat{f}\,(\xi)\,d\xi,$$

where the L^2-norm $\|f\|_2$ is defined as in (H_1) above.

In 1999, according to Gasquet et al. [8] the Heisenberg inequality *in spectral analysis* says that the product of the effective duration Δx and the effective bandwidth $\Delta\xi$ of a signal cannot be less than the value $\frac{1}{4}\pi = H^*$ (=Heisenberg lower bound), where $\Delta x^2 = \sigma^2_{|f|^2}\Big/E_{|f|^2}$ and

$$\Delta\xi^2\left(= \sigma^2_{\left|\hat{f}\right|^2}\Big/E_{\left|\hat{f}\right|^2}\right) = \sigma^2_{\left|\hat{f}\right|^2}\Big/E_{|f|^2}$$

with $f : \mathbb{R} \to \mathbb{C},\ \hat{f} : \mathbb{R} \to \mathbb{C}$ defined as in (H_1), and

$$E_{|f|^2} = \int_{\mathbb{R}}\left|\hat{f}\,(\xi)\right|^2 d\xi = E_{\left|\hat{f}\right|^2}.$$

In this paper we generalize the Heisenberg inequality to *the higher moments for L^2 functions f* and establish the *Heisenberg-Pauli-Weyl inequality*.

2. PASCAL TYPE COMBINATORIAL IDENTITY

We state and prove ***the new Pascal type combinatorial identity****.*

Proposition 2.1. *If* $0 \le \left[\frac{k}{2}\right]$ *is the greatest integer* $\le \frac{k}{2}$*, then*

$$\text{(C)}\qquad \frac{k}{k-i}\binom{k-i}{i}+\frac{k-1}{k-i}\binom{k-i}{i-1}=\frac{k+1}{k-i+1}\binom{k-i+1}{i},$$

holds for any fixed but arbitrary $k \in \mathbb{N} = \{1,2,\ldots\}$*, and* $0 \le i \le \left[\frac{k}{2}\right]$ *for* $i \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ *such that* $\binom{k}{-1} = 0$*.*

Note that *the classical Pascal identity* is

$$\binom{k-i}{i}+\binom{k-i}{i-1}=\binom{k-i+1}{i},\quad 0\le i\le\left[\frac{k}{2}\right].$$

Proof. It is clear that $k(k-2i+1)+(k-1)i=(k-i)(k+1)$. Thus

$$\begin{aligned}
&\frac{k}{k-i}\binom{k-i}{i}+\frac{k-1}{k-i}\binom{k-i}{i-1}\\
&\qquad=\frac{k}{k-i}\frac{(k-i)!}{i!\,(k-2i)!}+\frac{k-1}{k-i}\frac{(k-i)!}{(i-1)!\,(k-2i+1)!}\\
&\qquad=\frac{k}{k-i}\frac{\frac{(k-i+1)!}{k-i+1}}{i!\frac{(k-2i+1)!}{k-2i+1}}+\frac{k-1}{k-i}\frac{\frac{(k-i+1)!}{k-i+1}}{\frac{i!}{i}\,(k-2i+1)!}\\
&\qquad=\frac{(k-i+1)!}{i!\,(k-2i+1)!}\frac{1}{(k-i)\,(k-i+1)}\left[k(k-2i+1)+(k-1)i\right]\\
&\qquad=\binom{k-i+1}{i}\frac{k+1}{k-i+1},
\end{aligned}$$

completing the proof of this identity. □

Note that all of the three combinations: $\binom{k-i}{i}$, $\binom{k-i}{i-1}$, and $\binom{k-i+1}{i}$ exist and are positive numbers if $1 \le i \le \left[\frac{k}{2}\right]$ for $i \in \mathbb{N}$.

3. GENERALIZED DIFFERENTIAL IDENTITY

We state and prove *the new differential identity.*

Proposition 3.1. *If* $f:\mathbb{R}\to\mathbb{C}$ *is a complex valued function of a real variable* x*,* $0 \le \left[\frac{k}{2}\right]$ *is the greatest integer* $\le \frac{k}{2}$*,* $f^{(j)} = \frac{d^j}{dx^j} f$*, and* $\overline{(\cdot)}$ *is the conjugate of* $(\cdot)$*, then*

$$(*)\qquad f(x)\,\overline{f^{(k)}}(x)+f^{(k)}(x)\,\bar{f}(x)=\sum_{i=0}^{\left[\frac{k}{2}\right]}(-1)^i\frac{k}{k-i}\binom{k-i}{i}\frac{d^{k-2i}}{dx^{k-2i}}\left|f^{(i)}(x)\right|^2,$$

holds for any fixed but arbitrary $k \in \mathbb{N} = \{1,2,\ldots\}$*, such that* $0 \le i \le \left[\frac{k}{2}\right]$ *for* $i \in \mathbb{N}_0 = \{0,1,2,\ldots\}$*.*

Note that for $k=1$ we have

$$\begin{aligned} f(x)\overline{f^{(1)}}(x)+f^{(1)}(x)\bar{f}(x) &= (-1)^{\left[\frac{1}{2}\right](=0)}\frac{1}{1-0}\begin{pmatrix}1-0\\0\end{pmatrix}\frac{d^{1-2\cdot 0}}{dx^{1-2\cdot 0}}\left|f^{(0)}(x)\right|^2 \\ &= \frac{d}{dx}\left|f(x)\right|^2 \\ &= \left(\left|f(x)\right|^2\right)^{(1)}. \end{aligned}$$

If we denote $G_k(f)=f\overline{f^{(k)}}+f^{(k)}\bar{f}$, and

$$\begin{aligned} \left(\left|f^{(i)}\right|^2\right)^{(k-2i)} &= \frac{d^{k-2i}}{dx^{k-2i}}\left|f^{(i)}\right|^2 \\ &= \frac{d^{k-2i}}{dx^{k-2i}}\left|\frac{d^i}{dx^i}f\right|^2, \\ 0\le i &\le \left[\frac{k}{2}\right] \text{ for } k\in\mathbb{N}=\{1,2,\ldots\} \end{aligned}$$

and $i\in\mathbb{N}_0$, then (*) is equivalent to

(**) $$G_k(f)=\sum_{i=0}^{\left[\frac{k}{2}\right]}(-1)^i\frac{k}{k-i}\begin{pmatrix}k-i\\i\end{pmatrix}\left(\left|f^{(i)}\right|^2\right)^{(k-2i)}.$$

Proof. For $k=1$ (**) is trivial. Assume that (**) holds for k and claim that it holds for $k+1$. In fact,

$$\begin{aligned} G_k^{(1)}(f) &= \left(f\overline{f^{(k)}}+f^{(k)}\bar{f}\right)^{(1)} \\ &= \left(f\overline{f^{(k)}}\right)^{(1)}+\left(f^{(k)}\bar{f}\right)^{(1)} \\ &= \left(f^{(1)}\overline{f^{(k)}}+f\overline{f^{(k+1)}}\right)+\left(f^{(k+1)}\bar{f}+f^{(k)}\overline{f^{(1)}}\right) \\ &= \left(f^{(1)}\overline{f^{(k)}}+f^{(k)}\overline{f^{(1)}}\right)+\left(f\overline{f^{(k+1)}}+f^{(k+1)}\bar{f}\right) \\ &= \left(f^{(1)}\overline{\left(f^{(1)}\right)^{(k-1)}}+\left(f^{(1)}\right)^{(k-1)}\overline{f^{(1)}}\right)+G_{k+1}(f) \\ &= G_{k-1}\left(f^{(1)}\right)+G_{k+1}(f), \end{aligned}$$

or *the recursive sequence*

(R) $$G_{k+1}(f)=G_k^{(1)}(f)-G_{k-1}\left(f^{(1)}\right),$$

for $k\in\mathbb{N}=\{1,2,\ldots\}$, with $G_0\left(f^{(1)}\right)=\left|f^{(1)}\right|^2$, and $G_1(f)=\left(|f|^2\right)^{(1)}$.

From the induction hypothesis, the recursive relation (R), the fact that

$$\sum_{i=0}^{j}\lambda_{i+1}=\sum_{i=1}^{j+1}\lambda_i,\,(-1)^{i-1}=-(-1)^i$$

for $i\in\mathbb{N}_0$, and

$$\left[\frac{k-1}{2}\right]=\begin{cases}\left[\frac{k}{2}\right]-1, & k=2v \text{ for } v=1,2,\ldots\\ \left[\frac{k}{2}\right]\left(=\frac{k-1}{2}\right), & k=2\lambda+1 \text{ for } \lambda=0,1,\ldots\end{cases}$$

such that $\left[\frac{k}{2}\right] \le \left[\frac{k-1}{2}\right] + 1$, if $k \in \mathbb{N}$, we find

$$\begin{aligned}
&G_{k+1}(f)\\
&=\left\{\sum_{i=0}^{\left[\frac{k}{2}\right]}(-1)^i\frac{k}{k-i}\binom{k-i}{i}\left(\left|f^{(i)}\right|^2\right)^{(k+1-2i)}\right\}\\
&\quad-\left\{\sum_{i=0}^{\left[\frac{k-1}{2}\right]}(-1)^i\frac{k-1}{k-1-i}\binom{k-1-i}{i}\left(\left|f^{(i+1)}\right|^2\right)^{(k-1-2i)}\right\}\\
&=\left\{(-1)^0\frac{k}{k-0}\binom{k-0}{0}\left(\left|f^{(0)}\right|^2\right)^{(k+1-2\cdot 0)}\right.\\
&\quad\left.+\sum_{i=1}^{\left[\frac{k}{2}\right]}(-1)^i\frac{k}{k-i}\binom{k-i}{i}\left(\left|f^{(i)}\right|^2\right)^{(k+1-2i)}\right\}\\
&\quad-\left\{\sum_{i=1}^{\left[\frac{k-1}{2}\right]+1}(-1)^{i-1}\frac{k-1}{(k-1)-(i-1)}\binom{k-1-(i-1)}{i-1}\left(\left|f^{(i)}\right|^2\right)^{(k-1-2(i-1))}\right\}\\
&=\left\{\left(|f|^2\right)^{(k+1)}+\sum_{i=1}^{\left[\frac{k}{2}\right]}(-1)^i\frac{k}{k-i}\binom{k-i}{i}\left(\left|f^{(i)}\right|^2\right)^{(k+1-2i)}\right\}\\
&\quad+\left\{\sum_{i=1}^{\left[\frac{k-1}{2}\right]+1}(-1)^i\frac{k-1}{k-i}\binom{k-i}{i-1}\left(\left|f^{(i)}\right|^2\right)^{(k+1-2i)}\right\}\\
&=\left\{\left(|f|^2\right)^{(k+1)}+\sum_{i=1}^{\left[\frac{k}{2}\right]}(-1)^i\frac{k}{k-i}\binom{k-i}{i}\left(\left|f^{(i)}\right|^2\right)^{(k+1-2i)}\right\}\\
&\quad+\left\{\sum_{i=1}^{\left[\frac{k}{2}\right]}(-1)^i\frac{k-1}{k-i}\binom{k-i}{i-1}\left(\left|f^{(i)}\right|^2\right)^{(k+1-2i)}+S(f)\right\},
\end{aligned}$$

where

$$S(f)=\begin{cases}0, & \text{if } k=2v \text{ for } v=1,2,\ldots\\[2ex]
\dfrac{(-1)^{\left[\frac{k-1}{2}\right]+1}(k-1)}{k-\left(\left[\frac{k-1}{2}\right]+1\right)}\binom{k-\left(\left[\frac{k-1}{2}\right]+1\right)}{\left(\left[\frac{k-1}{2}\right]+1\right)-1} & \\
\qquad\times\left(\left|f^{\left(\left[\frac{k-1}{2}\right]+1\right)}\right|^2\right)^{\left(k+1-2\left(\left[\frac{k-1}{2}\right]+1\right)\right)}, & \\
\qquad\text{if } k=2\lambda+1 \text{ for } \lambda=0,1,\ldots. &
\end{cases}$$

Thus

$$G_{k+1}(f) = \left(|f|^2\right)^{(k+1)} + \sum_{i=1}^{\left[\frac{k}{2}\right]} (-1)^i \left\{ \frac{k}{k-i} \binom{k-i}{i} + \frac{k-1}{k-i} \binom{k-i}{i-1} \right\} \left(\left|f^{(i)}\right|^2\right)^{(k+1-2i)} + S(f),$$

where

$$S(f) = \begin{cases} 0, & \text{if } k = 2v \text{ for } v = 1,2,\dots \\ (-1)^{\left[\frac{k-1}{2}\right]} \frac{1-k}{k-1-\left[\frac{k-1}{2}\right]} \binom{k-1-\left[\frac{k-1}{2}\right]}{\left[\frac{k-1}{2}\right]} \left(\left|f^{\left(\left[\frac{k-1}{2}\right]+1\right)}\right|^2\right)^{\left(k-1-2\left[\frac{k-1}{2}\right]\right)}, & \text{if } k = 2\lambda+1 \text{ for } \lambda = 0,1,\dots \end{cases}$$

$$= \begin{cases} 0, & \text{if } k = 2v \text{ for } v = 1,2,\dots \\ (-1)^{\frac{k-1}{2}} \frac{1-k}{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{k-1}{2}} \left(\left|f^{\left(\frac{k+1}{2}\right)}\right|^2\right)^{(0)}, & \text{if } k = 2\lambda+1 \text{ for } \lambda = 0,1,\dots \end{cases}$$

$$= \begin{cases} 0, & \text{if } k = 2v \text{ for } v = 1,2,\dots \\ 2(-1)^{\frac{k+1}{2}} \left(\left|f^{\left(\frac{k+1}{2}\right)}\right|^2\right), & \text{if } k = 2\lambda+1 \text{ for } \lambda = 0,1,\dots \end{cases}$$

$$= \begin{cases} 0, \text{if } k = 2v \text{ for } v = 1,2,\dots \\ (-1)^{\left[\frac{k+1}{2}\right]} \frac{k+1}{k+1-\left[\frac{k+1}{2}\right]} \binom{k+1-\left[\frac{k+1}{2}\right]}{\left[\frac{k+1}{2}\right]} \left(\left|f^{\left(\left[\frac{k+1}{2}\right]\right)}\right|^2\right)^{\left(k+1-2\left[\frac{k+1}{2}\right]\right)}, & \text{if } k = 2\lambda+1 \text{ for } \lambda = 0,1,\dots, \end{cases}$$

because

$$\left[\frac{k+1}{2}\right] = \frac{k+1}{2}, \text{ if } k = 2\lambda+1 \text{ for } \lambda = 0,1,\dots$$

Besides we note that, from the above Pascal type combinatorial identity (C) and $\frac{k}{2} = \left[\frac{k}{2}\right] = \left[\frac{k+1}{2}\right]$, if $k = 2v$ for $v = 1,2,\dots$, one gets

$$(-1)^{\left[\frac{k}{2}\right]} \left\{ \frac{k}{k-\left[\frac{k}{2}\right]} \binom{k-\left[\frac{k}{2}\right]}{\left[\frac{k}{2}\right]} + \frac{k-1}{k-\left[\frac{k}{2}\right]} \binom{k-\left[\frac{k}{2}\right]}{\left[\frac{k}{2}\right]-1} \right\} \left(\left|f^{\left(\left[\frac{k}{2}\right]\right)}\right|^2\right)^{\left(k+1-2\left[\frac{k}{2}\right]\right)}$$

$$= (-1)^{\left[\frac{k}{2}\right]} \frac{k+1}{k-\left[\frac{k}{2}\right]+1} \binom{k-\left[\frac{k}{2}\right]+1}{\left[\frac{k}{2}\right]} \left(\left|f^{\left(\left[\frac{k}{2}\right]\right)}\right|^2\right)^{\left(k+1-2\left[\frac{k}{2}\right]\right)}$$

$$= (-1)^{\left[\frac{k+1}{2}\right]} \frac{k+1}{k+1-\left[\frac{k+1}{2}\right]} \binom{k+1-\left[\frac{k+1}{2}\right]}{\left[\frac{k+1}{2}\right]} \left(\left|f^{\left(\left[\frac{k+1}{2}\right]\right)}\right|^2\right)^{\left(k+1-2\left[\frac{k+1}{2}\right]\right)},$$

if $k = 2v$ for $v = 1, 2, \ldots$. From these results, one obtains

$$\begin{aligned} G_{k+1}(f) &= (-1)^0 \frac{k+1}{k+1-2\cdot 0} \begin{pmatrix} k+1-0 \\ 0 \end{pmatrix} \left(\left|f^{(0)}\right|^2\right)^{(k+1-2\cdot 0)} \\ &\quad + \sum_{i=1}^{\left[\frac{k}{2}\right]} (-1)^i \frac{k+1}{k+1-i} \begin{pmatrix} k+1-i \\ i \end{pmatrix} \left(\left|f^{(i)}\right|^2\right)^{(k+1-2i)} + S(f) \\ &= \sum_{i=1}^{\left[\frac{k+1}{2}\right]} (-1)^i \frac{k+1}{k+1-i} \begin{pmatrix} k+1-i \\ i \end{pmatrix} \left(\left|f^{(i)}\right|^2\right)^{(k+1-2i)}, \end{aligned}$$

completing the proof of (**) for $k+1$ and thus by the induction principle on k, (**) holds for any $k \in \mathbb{N}$. □

3.1. **Special cases of (**).**

$$G_1(f) = \left(|f|^2\right)^{(1)},$$

$$G_2(f) = \left(|f|^2\right)^{(2)} - 2\left|f^{(1)}\right|^2,$$

$$G_3(f) = \left(|f|^2\right)^{(3)} - 3\left(\left|f^{(1)}\right|^2\right)^{(1)},$$

$$G_4(f) = \left(|f|^2\right)^{(4)} - 4\left(\left|f^{(1)}\right|^2\right)^{(2)} + 2\left|f^{(2)}\right|^2,$$

$$G_5(f) = \left(|f|^2\right)^{(5)} - 5\left(\left|f^{(1)}\right|^2\right)^{(3)} + 5\left(\left|f^{(2)}\right|^2\right)^{(1)},$$

$$G_6(f) = \left(|f|^2\right)^{(6)} - 6\left(\left|f^{(1)}\right|^2\right)^{(4)} + 9\left(\left|f^{(2)}\right|^2\right)^{(2)} - 2\left|f^{(3)}\right|^2,$$

$$G_7(f) = \left(|f|^2\right)^{(7)} - 7\left(\left|f^{(1)}\right|^2\right)^{(5)} + 14\left(\left|f^{(2)}\right|^2\right)^{(3)} - 7\left(\left|f^{(3)}\right|^2\right)^{(1)},$$

$$G_8(f) = \left(|f|^2\right)^{(8)} - 8\left(\left|f^{(1)}\right|^2\right)^{(6)} + 20\left(\left|f^{(2)}\right|^2\right)^{(4)} - 16\left(\left|f^{(3)}\right|^2\right)^{(2)} + 2\left|f^{(4)}\right|^2.$$

We note that if one takes the above numerical coefficients of $G_i(f)$ ($i = 1, 2, \ldots, 8$) absolutely, then one establishes the pattern

1				
1	2			
1	3			
1	4	a_1= 2		
1	5	a_2= 5		
1	6	a_3= 9	b_1= 2	
1	7	a_4= 14	b_2= 7	
1	8	a_5= 20	b_3 = 16	c_1= 2

with

a_1 = 2	$b_1 = a_1$ = 2	$c_1 = b_1 = a_1$ = 2
a_2 = 2 + 3 = 5	$b_2 = a_1 + a_2$ = 7	
a_3 = 2 + 3 + 4 = 9	$b_3 = a_1 + a_2 + a_3$ = 16	
a_4 = 2 + 3 + 4 + 5 = 14		
a_5 = 2 + 3 + 4 + 5 + 6 = 20		

Following this pattern we get

$$G_9(f) = \left(|f|^2\right)^{(9)} - 9\left(\left|f^{(1)}\right|^2\right)^{(7)} + a_6\left(\left|f^{(2)}\right|^2\right)^{(5)} - b_4\left(\left|f^{(3)}\right|^2\right)^{(3)} + c_2\left(\left|f^{(4)}\right|^2\right)^{(1)},$$

where

$$\begin{aligned} a_6 &= a_5 + 7 = 27, \\ b_4 &= b_3 + a_4 = 16 + 14 = 30, \\ c_2 &= c_1 + b_2 = 2 + 7 = 9. \end{aligned}$$

Similarly, one gets

$$\begin{aligned} G_{10}(f) = \left(|f|^2\right)^{(10)} - 10\left(\left|f^{(1)}\right|^2\right)^{(8)} + a_7\left(\left|f^{(2)}\right|^2\right)^{(6)} \\ - b_5\left(\left|f^{(3)}\right|^2\right)^{(4)} + c_3\left(\left|f^{(4)}\right|^2\right)^{(2)} - d_1\left|f^{(5)}\right|^2, \end{aligned}$$

where

$$\begin{aligned} a_7 &= a_6 + 8 = 35, \\ b_5 &= b_4 + a_5 = 30 + 20 = 50, \\ c_3 &= c_2 + b_3 = 9 + 16 = 25, \\ d_1 &= c_1 = b_1 = a_1 = 2. \end{aligned}$$

3.2. **Applications of the Recursive Sequence (R).**

$$\begin{aligned} G_4(f) &= G_3^{(1)}(f) - G_2\left(f^{(1)}\right) \\ &= \left(f\overline{f^{(3)}} + f^{(3)}\overline{f}\right)^{(1)} - \left(f^{(1)}\overline{\left(f^{(1)}\right)^{(2)}} + \left(f^{(1)}\right)^{(2)}\overline{f^{(1)}}\right) \\ &= \left\{\left(|f|^2\right)^{(4)} - 3\left(\left|f^{(1)}\right|^2\right)^{(2)}\right\} - \left\{\left(\left|f^{(1)}\right|^2\right)^{(2)} - 2\left|f^{(2)}\right|^2\right\} \\ &= \left(|f|^2\right)^{(4)} - 4\left(\left|f^{(1)}\right|^2\right)^{(2)} + 2\left|f^{(2)}\right|^2 \end{aligned}$$

and

$$\begin{aligned} G_5(f) &= G_4^{(1)}(f) - G_3\left(f^{(1)}\right) \\ &= \left\{\left(|f|^2\right)^{(5)} - 4\left(\left|f^{(1)}\right|^2\right)^{(3)} + 2\left(\left|f^{(2)}\right|^2\right)^{(1)}\right\} - \left\{\left(\left|f^{(1)}\right|^2\right)^{(3)} - 3\left(\left|f^{(2)}\right|^2\right)^{(1)}\right\} \\ &= \left(|f|^2\right)^{(5)} - 5\left(\left|f^{(1)}\right|^2\right)^{(3)} + 5\left(\left|f^{(2)}\right|^2\right)^{(1)}, \end{aligned}$$

yielding also the above generalized differential identity (**) for $k = 3$ and $k = 4$, respectively.

3.3. **Generalization of the Identity (**).** We denote

$$H_{kl}(f) = f^{(l)}\overline{f^{(k)}} + f^{(k)}\overline{f^{(l)}}.$$

It is clear that

$$H_{kl}(f) = \begin{cases} G_{k-l}\left(f^{(l)}\right) = f^{(l)}\overline{\left(f^{(l)}\right)^{(k-l)}} + \left(f^{(l)}\right)^{(k-l)}\overline{f^{(l)}}, & \text{if } k > l \\ G_0\left(f^{(l)}\right) = 2\left|f^{(l)}\right|^2, & \text{if } k = l \\ G_{l-k}\left(f^{(k)}\right) = f^{(k)}\overline{\left(f^{(k)}\right)^{(l-k)}} + \left(f^{(k)}\right)^{(l-k)}\overline{f^{(k)}}, & \text{if } k < l \end{cases}$$

From these and (**) we conclude that

$$H_{kl}(f) = \begin{cases} \displaystyle\sum_{i=0}^{\left[\frac{k-l}{2}\right]} (-1)^i \frac{k-l}{k-l-i}\binom{k-l-i}{i}\left(\left|f^{(i+l)}\right|^2\right)^{(k-l-2i)}, & \text{if } k > l \\ 2\left|f^{(l)}\right|^2, & \text{if } k = l \\ \displaystyle\sum_{i=0}^{\left[\frac{l-k}{2}\right]} (-1)^i \frac{l-k}{l-k-i}\binom{l-k-i}{i}\left(\left|f^{(i+k)}\right|^2\right)^{(l-k-2i)}, & \text{if } k < l \end{cases}$$

For instance, if $k = 3 > 2 = l$, then from this formula one gets

$$\begin{aligned} H_{32}(f) &= G_{3-2}\left(f^{(2)}\right) \\ &= \sum_{i=0}^{\left[\frac{3-2}{2}\right]} (-1)^i \frac{3-2}{3-2-i}\binom{3-2-i}{i}\left(\left|f^{(i+2)}\right|^2\right)^{(3-2-2i)} \\ &= (-1)^0\frac{1}{1-0}\binom{1-0}{0}\left(\left|f^{(2)}\right|^2\right)^{(1-2\cdot 0)} \\ &= \left(\left|f^{(2)}\right|^2\right)^{(1)}. \end{aligned}$$

In fact,

$$H_{32}(f) = f^{(2)}\overline{f^{(3)}} + f^{(3)}\overline{f^{(2)}} = f^{(2)}\overline{\left(f^{(2)}\right)^{(1)}} + \left(f^{(2)}\right)^{(1)}\overline{f^{(2)}} = G_1\left(f^{(2)}\right) = \left(\left|f^{(2)}\right|^2\right)^{(1)}.$$

Another special case, if $k = 3 > 1 = l$, then one gets

$$\begin{aligned} H_{31}(f) &= G_{3-1}\left(f^{(1)}\right) \\ &= \sum_{i=0}^{\left[\frac{3-1}{2}\right]} (-1)^i \frac{3-1}{3-1-i}\binom{3-1-i}{i}\left(\left|f^{(i+1)}\right|^2\right)^{(3-1-2i)} \\ &= (-1)^0\frac{2}{2-0}\binom{2-0}{0}\left(\left|f^{(1)}\right|^2\right)^{(2-2\cdot 0)} + (-1)^1\frac{2}{2-1}\binom{2-1}{1}\left(\left|f^{(2)}\right|^2\right)^{(2-2\cdot 1)} \\ &= \left(\left|f^{(1)}\right|^2\right)^{(2)} - 2\left|f^{(2)}\right|^2. \end{aligned}$$

In fact,

$$\begin{aligned} H_{31}(f) &= f^{(1)}\overline{f^{(3)}} + f^{(3)}\overline{f^{(1)}} \\ &= f^{(1)}\overline{\left(f^{(1)}\right)^{(2)}} + \left(f^{(1)}\right)^{(2)}\overline{f^{(1)}} \\ &= G_2\left(f^{(1)}\right) = \left(\left|f^{(1)}\right|^2\right)^{(2)} - 2\left|f^{(2)}\right|^2. \end{aligned}$$

4. GENERALIZED PLANCHEREL-PARSEVAL-RAYLEIGH IDENTITY

We state and prove ***the new Plancherel-Parseval-Rayleigh identity.***

Proposition 4.1. *If f and $f_a : \mathbb{R} \to \mathbb{C}$ are complex valued functions of a real variable x, $f_a = e^{ax} f$ and $f_a^{(p)} = \frac{d^p}{dx^p} f_a$, where $a = -2\pi\xi_m i$ with $i = \sqrt{-1}$ and any fixed but arbitrary constant $\xi_m \in \mathbb{R}$, $\hat{f}$ the Fourier transform of f, such that*

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x)\, dx$$

with ξ real and

$$f(x) = \int_{\mathbb{R}} e^{2i\pi\xi x} \hat{f}(\xi)\, d\xi,$$

as well as f, $f_a^{(p)}$, and $(\xi - \xi_m)^p \hat{f}$ are in $L^2(\mathbb{R})$, then

$$\int_{\mathbb{R}} (\xi - \xi_m)^{2p} \left|\hat{f}(\xi)\right|^2 d\xi = \frac{1}{(2\pi)^{2p}} \int_{\mathbb{R}} \left|f_a^{(p)}(x)\right|^2 dx \tag{4.1}$$

holds for any fixed but arbitrary $p \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Proof. Denote

$$g(x) = e^{-2\pi i x \xi_m} f(x + x_m), \tag{4.2}$$

for any fixed but arbitrary constant $x_m \in \mathbb{R}$.

From (4.2) one gets that

$$\begin{aligned} \hat{g}(\xi) &= \int_{\mathbb{R}} e^{-2i\pi\xi x} g(x)\, dx \\ &= \int_{\mathbb{R}} e^{-2i\pi\xi x}\left(e^{-2\pi i x\xi_m} f(x + x_m)\right) dx \\ &= e^{2\pi i x_m(\xi+\xi_m)} \int_{\mathbb{R}} e^{-2\pi i(\xi+\xi_m)x} f(x)\, dx \\ &= e^{2\pi i x_m(\xi+\xi_m)} \hat{f}(\xi + \xi_m). \end{aligned} \tag{4.3}$$

Denote the Fourier transform of $g(p)$ either by $Fg^{(p)}(\xi)$, or $F\left[g^{(p)}(x)\right](\xi)$, or also as $\left[g^{(p)}(x)\right]\hat{}\ (\xi)$.

From this and Gasquet et al. [8, p. 155–157] we find

$$\xi^p \hat{g}(\xi) = \frac{1}{(2\pi i)^p} Fg^{(p)}(\xi). \tag{4.4}$$

Also denote

$$h_p(x) = g^{(p)}(x). \tag{4.5}$$

From (4.5) and the classical Plancherel-Parseval-Rayleigh identity one gets

$$\int_{\mathbb{R}} \left|\hat{h}_p(\xi)\right|^2 d\xi = \int_{\mathbb{R}} |h_p(x)|^2 dx,$$

or

$$\int_{\mathbb{R}} \left| Fg^{(p)}(\xi) \right|^2 d\xi = \int_{\mathbb{R}} \left| g^{(p)}(x) \right|^2 dx. \tag{4.6}$$

Finally denote

$$(\mu_{2p})_{|\hat{f}|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^{2p} \left| \hat{f}(\xi) \right|^2 d\xi \tag{4.7}$$

the $2p^{th}$ *moment of* ξ *for* $\left|\hat{f}\right|^2$ for any fixed but arbitrary constant $\xi_m \in \mathbb{R}$ and $p \in \mathbb{N}_0$.

Substituting ξ with $\xi_+\xi_m$ we find from (4.3) – (4.7) that

$$\begin{aligned}
(\mu_{2p})_{|\hat{f}|^2} &= \int_{\mathbb{R}} \xi^{2p} \left| \hat{f}(\xi + \xi_m) \right|^2 d\xi \\
&= \int_{\mathbb{R}} \xi^{2p} \left| e^{2\pi i x_m(\xi+\xi_m)} \hat{f}(\xi + \xi_m) \right|^2 d\xi \\
&= \int_{\mathbb{R}} \xi^{2p} \left| \hat{g}(\xi) \right|^2 d\xi \\
&= \frac{1}{(2\pi)^{2p}} \int_{\mathbb{R}} \left| Fg^{(p)}(\xi) \right|^2 d\xi \\
&= \frac{1}{(2\pi)^{2p}} \int_{\mathbb{R}} \left| g^{(p)}(x) \right|^2 dx.
\end{aligned}$$

From this and (4.2) we find

$$(\mu_{2p})_{|\hat{f}|^2} = \frac{1}{(2\pi)^{2p}} \int_{\mathbb{R}} \left| \left(e^{-2\pi i x \xi_m} f(x + x_m) \right)^{(p)} \right|^2 dx.$$

Placing $x - x_m$ on x in this identity one gets

$$\begin{aligned}
(\mu_{2p})_{|\hat{f}|^2} &= \frac{1}{(2\pi)^{2p}} \int_{\mathbb{R}} \left| \left(e^{-2\pi i (x - x_m) \xi_m} f(x) \right)^{(p)} \right|^2 dx \\
&= \frac{1}{(2\pi)^{2p}} \int_{\mathbb{R}} \left| \left(e^{ax} f(x) \right)^{(p)} \right|^2 dx.
\end{aligned}$$

Employing $e^{ax} f(x) = f_a(x)$ in this new identity we find

$$(\mu_{2p})_{|\hat{f}|^2} = \frac{1}{(2\pi)^{2p}} \int_{\mathbb{R}} \left| f_a^{(p)}(x) \right|^2 dx,$$

completing the proof of the required identity (4.1) ([16] – [24]). □

5. THE p^{TH}-DERIVATIVE OF THE PRODUCT OF TWO FUNCTIONS

We state and outline a proof for the following well-known result on the p^{th}-derivative of the product of two functions.

Proposition 5.1. *If* $f_i : \mathbb{R} \to \mathbb{C}$ $(i = 1, 2)$ *are two complex valued functions of a real variable* x*, then the* p^{th}*-derivative of the product* $f_1 f_2$ *is given, in terms of the lower derivatives* $f_1^{(m)}$, $f_2^{(p-m)}$ *by*

$$(f_1 f_2)^{(p)} = \sum_{m=0}^{p} \binom{p}{m} f_1^{(m)} f_2^{(p-m)} \tag{5.1}$$

for any fixed but arbitrary $p \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Proof. In fact, for $p = 0$ the formula (5.1) is trivial, as $(f_1 f_2)^{(0)} = f_1^{(0)} f_2^{(0)}$. When $p = 1$ the formula (5.1) is $(f_1 f_2)^{(1)} = f_1^{(1)} f_2 + f_1 f_2^{(1)}$ which holds.

Assume that (5.1) holds, as well. Differentiating this formula we get

$$\begin{aligned}(f_1 f_2)^{(p+1)} &= \sum_{m=0}^{p} \binom{p}{m} f_1^{(m+1)} f_2^{(p-m)} + \sum_{m=0}^{p} \binom{p}{m} f_1^{(m)} f_2^{(p+1-m)} \\ &= \sum_{m=1}^{p+1} \binom{p}{m-1} f_1^{(m)} f_2^{(p+1-m)} + \sum_{m=0}^{p} \binom{p}{m} f_1^{(m)} f_2^{(p+1-m)} \\ &= \binom{p}{p} f_1^{(p+1)} f_2 + \sum_{m=1}^{p} \left[\binom{p}{m-1} + \binom{p}{m} \right] f_1^{(m)} f_2^{(p+1-m)} + \binom{p}{0} f_1 f_2^{(p+1)} \\ &= \binom{p+1}{p+1} f_1^{(p+1)} f_2 + \sum_{m=1}^{p} \binom{p+1}{m} f_1^{(m)} f_2^{(p+1-m)} + \binom{p+1}{0} f_1 f_2^{(p+1)} \\ &= \sum_{m=0}^{p+1} \binom{p+1}{m} f_1^{(m)} f_2^{(p+1-m)},\end{aligned}$$

as the classical Pascal identity

$$\binom{p}{m-1} + \binom{p}{m} = \binom{p+1}{m} \tag{P}$$

holds for $m \in \mathbb{N} : 1 < m \le p$. Therefore by induction on p the proof of (5.1) is complete. $\square$

Employing the formula (5.1) with $f_1(x) = f(x)$, $f_2(x) = e^{ax}$, where $a = -2\pi\xi_m i$, $i = \sqrt{-1}$, ξ_m fixed but arbitrary real, and placing $f_a(x) = (f_1 f_2)(x) = e^{ax} f(x)$, one gets

$$f_a^{(p)}(x) = e^{ax} \sum_{m=0}^{p} \binom{p}{m} a^{p-m} f^{(m)}(x). \tag{5.2}$$

Similarly from the formula (5.1) with $f_1(x) = f(x)$, $f_2(x) = \overline{f(x)}$, $|f|^2 = f\overline{f}$ and $p = k$, $m = j$, we get the following formula

$$\left(|f|^2\right)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f^{(j)} \overline{f^{(k-j)}}, \tag{5.3}$$

for the k^{th} derivative of $|f|^2$.

Note that from (5.2) with $m = k$ one gets the modulus of $f_a^{(p)}$to be of the form

$$\left| f_a^{(p)} \right| = \left| \sum_{k=0}^{p} \binom{p}{k} a^{p-k} f^{(k)} \right|, \tag{5.4}$$

because $|e^{ax}| = 1$ by the *Euler formula* : $e^{i\theta} = \cos\theta + i\sin\theta$, with $\theta = -2\pi\xi_m x$ $(\in \mathbb{R})$.

Also note that the $2p^{th}$ *moment of the real* x *for* $|f|^2$is of the form

$$(\mu_{2p})_{|f|^2} = \int_{\mathbb{R}} (x - x_m)^{2p} |f(x)|^2 \, dx \tag{5.5}$$

for any fixed but arbitrary constant $x_m \in \mathbb{R}$ and $p \in \mathbb{N}_0$.

Placing $x + x_m$ on x in (5.5) we find

$$(\mu_{2p})_{|f|^2} = \int_{\mathbb{R}} x^{2p} |f(x + x_m)|^2 \, dx.$$

From this, (4.2) and

$$|g(x)| = \left|e^{-2\pi i x \xi_m} f(x + x_m)\right| = |f(x + x_m)|$$

one gets that

$$(\mu_{2p})_{|f|^2} = \int_{\mathbb{R}} x^{2p} |g(x)|^2 \, dx. \tag{5.6}$$

Besides (5.5) and $|f_a| = |f|$ yield

$$(\mu_{2p})_{|f|^2} = \int_{\mathbb{R}} (x - x_m)^{2p} |f_a(x)|^2 \, dx. \tag{5.7}$$

6. GENERALIZED INTEGRAL IDENTITIES

We state and outline a proof for the following well-known result on integral identities.

Proposition 6.1. *If $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a real variable x, and $h : \mathbb{R} \to \mathbb{R}$ is a real valued function of x, as well as, w, $w_p : \mathbb{R} \to \mathbb{R}$ are two real valued functions of x, such that $w_p(x) = (x - x_m)^p w(x)$ for any fixed but arbitrary constant $x_m \in \mathbb{R}$ and $v = p - 2q$, $0 \le q \le \left[\frac{p}{2}\right]$, then*

i)

$$\int w_p(x) h^{(v)}(x) \, dx = \sum_{r=0}^{v-1} (-1)^r w_p^{(r)}(x) h^{(v-r-1)}(x) + (-1)^v \int w_p^{(v)}(x) h(x) \, dx \tag{6.1}$$

holds for any fixed but arbitrary $p \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $v \in \mathbb{N}$, and all $r : r = 0, 1, 2, \ldots, v - 1$, as well as

ii)

$$\int_{\mathbb{R}} w_p(x) h^{(v)}(x) \, dx = (-1)^v \int_{\mathbb{R}} w_p^{(v)}(x) h(x) \, dx \tag{6.2}$$

holds if the condition

$$\sum_{r=0}^{v-1} (-1)^r \lim_{|x| \to \infty} w_p^{(r)}(x) h^{(v-r-1)}(x) = 0, \tag{6.3}$$

holds, and if all these integrals exist.

Proof. The proof is as follows.

i) For $v = 1$ the identity (6.1) holds, because by integration by parts one gets

$$\int \left(w_p h^{(1)}\right)(x) \, dx = \int w_p(x) \, dh(x) = (w_p h)(x) - \int \left(w_p^{(1)} h\right)(x) \, dx.$$

Assume that (6.1) holds for v. Claim that (6.1) holds for $v + 1$, as well. In fact, by integration by parts and from (6.1) we get

$$\begin{aligned}
&\int \left(w_p h^{(v+1)}\right)(x) \, dx \\
&= \int w_p(x) \, dh^{(v)}(x) \\
&= \left(w_p h^{(v)}\right)(x) - \int w_p^{(1)}(x) h^{(v)}(x) \, dx
\end{aligned}$$

$$= \left(w_p h^{(v)}\right)(x) - \left[\left(\sum_{r=0}^{v-1} (-1)^r w_p^{(r+1)} h^{(v-r-1)}\right)(x) + (-1)^v \int \left(w_p^{(v+1)} h\right)(x)\, dx\right]$$

$$= \left(w_p h^{(v)}\right)(x) - \left[\sum_{r=1}^{v} (-1)^{r-1} w_p^{(r)} h^{(v-(r-1)-1)}\right](x) - (-1)^v \int \left(w_p^{(v+1)} h\right)(x)\, dx$$

$$= \left(w_p h^{(v)}\right)(x) + \left(\sum_{r=1}^{v} (-1)^{r} w_p^{(r)} h^{((v+1)-r-1)}\right)(x) + (-1)^{v+1} \int \left(w_p^{(v+1)} h\right)(x)\, dx$$

$$= \left(\sum_{r=0}^{(v+1)-1} (-1)^{r} w_p^{(r)} h^{((v+1)-r-1)}\right)(x) + (-1)^{v+1} \int \left(w_p^{(v+1)} h\right)(x)\, dx,$$

which, by induction principle on v, completes the proof of the integral identity (6.1).

ii) The proof of (6.2) is clear from (6.1) and (6.3).

□

6.1. **Special Cases of (6.2):**

i) If $h(x) = \left|f^{(l)}(x)\right|^2$ and $v = p - 2q$, then from (6.2) one gets

$$\int_{\mathbb{R}} w_p(x) \left(\left|f^{(l)}(x)\right|^2\right)^{(p-2q)} dx = (-1)^{p-2q} \int_{\mathbb{R}} w_p^{(p-2q)}(x) \left|f^{(l)}(x)\right|^2 dx, \tag{6.4}$$

for $0 \le l \le q \le \left[\frac{p}{2}\right]$,

ii) If

$$h(x) = \operatorname{Re}\left(r_{qkj} f^{(k)}(x) \overline{f^{(j)}}(x)\right),$$

and if (6.3) holds, then from (6.2) we get

$$\int_{\mathbb{R}} w_p(x) \left(\operatorname{Re}\left(r_{qkj} f^{(k)}(x) \overline{f^{(j)}}(x)\right)\right)^{(p-2q)} dx \tag{6.5}$$
$$= (-1)^{p-2q} \int_{\mathbb{R}} w_p^{(p-2q)}(x) \left(\operatorname{Re}\left(r_{qkj} f^{(k)}(x) \overline{f^{(j)}}(x)\right)\right) dx,$$

where $r_{qkj} = (-1)^{q-\frac{k+j}{2}}$ for $0 \le k < j \le q \le \left[\frac{p}{2}\right]$, and $v = p - 2q$.

7. LAGRANGE TYPE DIFFERENTIAL IDENTITY

We state and prove ***the new Lagrange type differential identity.***

Proposition 7.1. *If $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a real variable x, and $f_a = e^{ax} f$, where $a = -\beta i$, with $i = \sqrt{-1}$ and $\beta = 2\pi\xi_m$ for any fixed but arbitrary real constant ξ_m, as well as if*

$$A_{pk} = \binom{p}{k}^2 \beta^{2(p-k)}, \qquad 0 \le k \le p,$$

and

$$B_{pkj} = s_{pk} \binom{p}{k}\binom{p}{j} \beta^{2p-j-k}, \qquad 0 \le k < j \le p,$$

where $s_{pk} = (-1)^{p-k} \quad (0 \le k \le p)$, then

$$\left|f_a^{(p)}\right|^2 = \sum_{k=0}^{p} A_{pk} \left|f^{(k)}\right|^2 + 2 \sum_{0 \le k < j \le p} B_{pkj} \operatorname{Re}\left(r_{pkj} f^{(k)} \overline{f^{(j)}}\right), \tag{7.1}$$

holds for any fixed but arbitrary $p \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, *where* $\overline{(\cdot)}$ *is the conjugate of* $(\cdot)$, *and* $r_{pkj} = (-1)^{p-\frac{k+j}{2}}$ $(0 \le k < j \le p)$, *and* $\mathrm{Re}\,(\cdot)$ *is the real part of* $(\cdot)$.

Note that $s_{pk} \in \{\pm 1\}$ and $r_{pkj} \in \{\pm 1, \pm i\}$.

Proof. In fact, *the classical Lagrange identity*

$$\left|\sum_{k=0}^{p} r_k z_k\right|^2 = \left(\sum_{k=0}^{p} |r_k|^2\right)\left(\sum_{k=0}^{p} |z_k|^2\right) - \sum_{0\le k<j\le p} |\overline{r_k} z_j - \overline{r_j} z_k|^2, \tag{7.2}$$

with r_k, $z_k \in \mathbb{C}$, such that $0 \le k \le p$, takes the new form

$$\begin{aligned}\left|\sum_{k=0}^{p} r_k z_k\right|^2 &= \left[\left(\sum_{l=0}^{p} |r_l|^2\right)\left(\sum_{k=0}^{p} |z_k|^2\right) - \sum_{0\le k<j\le p} |r_k|^2 |z_j|^2 \right.\\ &\qquad \left. - \sum_{0\le k<j\le p} |r_j|^2 |z_k|^2\right] + 2\sum_{0\le k<j\le p} \mathrm{Re}\,(r_k \overline{r_j} z_k \overline{z_j})\\ &= \left[\left(\sum_{l=0}^{p} |r_l|^2 - \sum_{k\ne 0}^{p} |r_k|^2\right)|z_0|^2 + \left(\sum_{l=0}^{p} |r_l|^2 - \sum_{k\ne 1}^{p} |r_k|^2\right)|z_1|^2\right.\\ &\qquad \left. + \cdots + \left(\sum_{l=0}^{p} |r_l|^2 - \sum_{k\ne p}^{p-1} |r_k|^2\right)|z_p|^2\right] + 2\sum_{0\le k<j\le p} \mathrm{Re}\,(r_k \overline{r_j} z_k \overline{z_j}),\end{aligned}$$

or *the new identity*

$$\left|\sum_{k=0}^{p} r_k z_k\right|^2 = \sum_{k=0}^{p} |r_k|^2 |z_k|^2 + 2\sum_{0\le k<j\le p} \mathrm{Re}\,(r_k \overline{r_j} z_k \overline{z_j}), \tag{7.3}$$

because

$$\begin{aligned}|\overline{r_k} z_j - \overline{r_j} z_k|^2 &= (\overline{r_k} z_j - \overline{r_j} z_k)(r_k \overline{z_j} - r_j \overline{z_k})\\ &= |r_k|^2 |z_j|^2 + |r_j|^2 |z_k|^2 - (r_k \overline{r_j} z_k \overline{z_j} + \overline{r_k} r_j \overline{z_k} z_j)\\ &= |r_k|^2 |z_j|^2 + |r_j|^2 |z_k|^2 - 2\,\mathrm{Re}\,(r_k \overline{r_j} z_k \overline{z_j}),\end{aligned} \tag{7.4}$$

as well as

$$\begin{aligned}&\sum_{0\le k<j\le p} |r_k|^2 |z_j|^2 + \sum_{0\le k<j\le p} |r_j|^2 |z_k|^2\\ &\quad = \sum_{0\le k\ne j\le p} |r_k|^2 |z_j|^2\\ &\quad = \left(\sum_{k\ne 0} |r_k|^2\right)|z_0|^2 + \left(\sum_{k\ne 1} |r_k|^2\right)|z_1|^2 + \cdots + \left(\sum_{k\ne p} |r_k|^2\right)|z_p|^2.\end{aligned} \tag{7.5}$$

Setting

$$r_k = \binom{p}{k} a^{p-k} = \binom{p}{k} (-\beta i)^{p-k} = (-i)^{p-k} \binom{p}{k} \beta^{p-k}, \tag{7.6}$$

one gets that

$$|r_k|^2 = \binom{p}{k}^2 \beta^{2(p-k)} = A_{pk}, \tag{7.7}$$

and

$$r_k\overline{r_j} = \binom{p}{k}\binom{p}{j}(-\beta i)^{p-k}(\beta i)^{p-j} \tag{7.8}$$
$$= i^{2p-k-j}(-1)^{p-k}\binom{p}{k}\binom{p}{j}\beta^{2p-k-j}$$
$$= r_{pkj}s_{pk}\binom{p}{k}\binom{p}{j}\beta^{2p-k-j}$$
$$= B_{pkj}r_{pkj},$$

where A_{pk}, $B_{pkj} \in \mathbb{R}$, and $s_{pk} = (-1)^{p-k} \in \{\pm 1\}$ as well as

$$r_{pkj} = i^{2p-k-j} = (-1)^{p-\frac{k+j}{2}} \in \{\pm 1, \pm i\}.$$

Thus employing (5.4) and substituting

$$z_k = f^{(k)}, \quad r_k = \binom{p}{k} a^{p-k} \quad (0 \le k \le p),$$

in (7.3), we complete from (7.6) – (7.8), the proof of the identity (7.1). □

We note that

$$s_{pk} = 1, \quad \text{if} \quad p \equiv k (\operatorname{mod} 2);$$

$$s_{pk} = -1, \quad \text{if} \quad p \equiv (k+1) \operatorname{mod} 2.$$

Similarly we have

$$r_{pkj} = 1, \quad \text{if} \quad 2p \equiv (k+j)(\operatorname{mod} 4);$$

$$r_{pkj} = i, \quad \text{if} \quad 2p \equiv (k+j+1)(\operatorname{mod} 4);$$

$$r_{pkj} = -1, \quad \text{if} \quad 2p \equiv (k+j+2)(\operatorname{mod} 4);$$

$$r_{pkj} = -i, \quad \text{if} \quad 2p \equiv (k+j+3)(\operatorname{mod} 4).$$

Finally (7.2) may be called the *Lagrange identity of first form*, and (7.3) the *Lagrange identity of second form*.

7.1. **Special cases of (7.1):**

(i)

$$\left|f_a^{(1)}\right|^2 = \left|(e^{ax}f)'\right|^2 \tag{7.9}$$
$$= \left(A_{10}|f|^2 + A_{11}|f'|^2\right) + 2B_{101}\operatorname{Re}\left(r_{101}f\overline{f'}\right)$$
$$= \beta^2|f|^2 + |f'|^2 + 2(-\beta)\operatorname{Re}\left(if\overline{f'}\right)$$
$$= \beta^2|f|^2 + |f'|^2 + 2\beta\operatorname{Im}\left(f\overline{f'}\right),$$

because

$$\operatorname{Re}(iz) = -\operatorname{Im}(z), \tag{7.10}$$

where $\operatorname{Im}(z)$ is the imaginary part of $z \in \mathbb{C}$.

Also note that *another way* to find (7.9), is to employ directly only (5.4), as follows:

$$\begin{aligned}\left|f_a^{(1)}\right|^2 &= \left|af + f'\right|^2 \\ &= \left|-i\beta f + f'\right|^2 \\ &= \left(-i\beta f + f'\right)\left(i\beta \overline{f} + \overline{f'}\right) \\ &= \beta^2 \left|f\right|^2 + \left|f'\right|^2 + \beta\left(-i\right)\left(f\overline{f'} - \overline{f}f'\right) \\ &= \beta^2 \left|f\right|^2 + \left|f'\right|^2 + 2\beta \operatorname{Im}\left(f\overline{f'}\right).\end{aligned}$$

(ii)

$$\begin{aligned}\left|f_a^{(2)}\right|^2 &= \left|\left(e^{ax}f\right)''\right|^2 \\ &= A_{20}\left|f\right|^2 + A_{21}\left|f'\right|^2 + A_{22}\left|f''\right|^2 + 2\operatorname{Re}\left(r_{201}f\overline{f'} + r_{202}f\overline{f''} + r_{212}f'\overline{f''}\right) \\ &= \beta^4\left|f\right|^2 + 4\beta^2\left|f'\right|^2 + \left|f''\right|^2 + 2\operatorname{Re}\left(-2i\beta^3 f\overline{f'} - \beta^2 f\overline{f''} - 2i\beta f'\overline{f''}\right) \\ &= \beta^4\left|f\right|^2 + 4\beta^2\left|f'\right|^2 + \left|f''\right|^2 + 4\beta^3\operatorname{Im}\left(f\overline{f'}\right) \\ &\qquad - 2\beta^2\operatorname{Re}\left(f\overline{f''}\right) + 4\beta\operatorname{Im}\left(f'\overline{f''}\right).\end{aligned} \tag{7.11}$$

Similarly, from (5.4) we get also (7.11), as follows:

$$\left|f_a^{(2)}\right|^2 = \left|a^2 f + af' + f''\right|^2 = \left(-\beta^2 f - i\beta f' + f''\right)\left(-\beta^2 \overline{f} + i\beta\overline{f'} + \overline{f''}\right),$$

leading easily to (7.11).

8. On the Heisenberg-Pauli-Weyl Inequality

We assume that $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a real variable x, and $w : \mathbb{R} \to \mathbb{R}$ a real valued weight function of x, as well as x_m, ξ_m any fixed but arbitrary real constants. Denote $f_a = e^{ax}f$, where $a = -2\pi\xi_m i$ with $i = \sqrt{-1}$, and $\hat{f}$ the Fourier transform of f, such that

$$\hat{f}\left(\xi\right) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f\left(x\right) dx$$

and

$$f\left(x\right) = \int_{\mathbb{R}} e^{2i\pi\xi x}\hat{f}\left(\xi\right) d\xi.$$

Also we denote

$$\left(\mu_{2p}\right)_{w,|f|^2} = \int_{\mathbb{R}} w^2\left(x\right)\left(x - x_m\right)^{2p}\left|f\left(x\right)\right|^2 dx$$

the $2p^{th}$ weighted moment of x for $\left|f\right|^2$ with weight function w and

$$\left(\mu_{2p}\right)_{|\hat{f}|^2} = \int_{\mathbb{R}} \left(\xi - \xi_m\right)^{2p}\left|\hat{f}\left(\xi\right)\right|^2 d\xi$$

the $2p^{th}$ moment of ξ for $\left|\hat{f}\right|^2$. Besides we denote

$$C_q = \left(-1\right)^q \frac{p}{p-q}\binom{p-q}{q}, \text{ if } 0 \le q \le \left[\frac{p}{2}\right] \quad \left(= \text{the greatest integer} \le \frac{p}{2}\right),$$

$$I_{ql} = \left(-1\right)^{p-2q}\int_{\mathbb{R}} w_p^{(p-2q)}\left(x\right)\left|f^{(l)}\left(x\right)\right|^2 dx, \quad \text{if } 0 \le l \le q \le \left[\frac{p}{2}\right],$$

$$I_{qkj} = (-1)^{p-2q} \int_{\mathbb{R}} w_p^{(p-2q)}(x) \operatorname{Re}\left(r_{qkj} f^{(k)}(x) \overline{f^{(j)}}(x) \right) dx,$$
$$\text{if } 0 \le k < j \le q \le \left[\frac{p}{2}\right],$$

where

$$r_{qkj} = (-1)^{q-\frac{k+j}{2}} \in \{\pm 1, \pm i\}$$

and

$$w_p = (x - x_m)^p w.$$

We assume that all these integrals exist. Finally we denote

$$D_q = \sum_{l=0}^{q} A_{ql} I_{ql} + 2 \sum_{0 \le k < j \le q} B_{qkj} I_{qkj},$$

if $|D_q| < \infty$ holds for $0 \le q \le \left[\frac{p}{2}\right]$, where

$$A_{ql} = \begin{pmatrix} q \\ l \end{pmatrix}^2 \beta^{2(q-l)},$$
$$B_{qkj} = s_{qk} \begin{pmatrix} q \\ k \end{pmatrix} \begin{pmatrix} q \\ j \end{pmatrix} \beta^{2q-j-k},$$

with $\beta = 2\pi \xi_m$, and $s_{qk} = (-1)^{q-k}$, and

$$E_{p,f} = \sum_{q=0}^{[p/2]} C_q D_q,$$

if $|E_{p,f}| < \infty$ holds for $p \in \mathbb{N}$.

Besides we assume *the two conditions*:

$$\sum_{r=0}^{p-2q-1} (-1)^r \lim_{|x|\to\infty} w_p^{(r)}(x) \left(\left| f^{(l)}(x) \right|^2 \right)^{(p-2q-r-1)} = 0, \tag{8.1}$$

for $0 \le l \le q \le \left[\frac{p}{2}\right]$, and

$$\sum_{r=0}^{p-2q-1} (-1)^r \lim_{|x|\to\infty} w_p^{(r)}(x) \left(\operatorname{Re}\left(r_{qkj} f^{(k)}(x) \overline{f^{(j)}}(x) \right) \right)^{(p-2q-r-1)} = 0, \tag{8.2}$$

for $0 \le k < j \le q \le \left[\frac{p}{2}\right]$. From these preliminaries we establish the following Heisenberg-Pauli-Weyl inequality .

Theorem 8.1. *If* $f \in L^2(\mathbb{R})$, *then*

$$\sqrt[2p]{(\mu_{2p})_{w,|f|^2}} \sqrt[2p]{(\mu_{2p})_{|\hat{f}|^2}} \ge \frac{1}{2\pi \sqrt[p]{2}} \sqrt[p]{|E_{p,f}|}, \tag{8.3}$$

holds for any fixed but arbitrary $p \in \mathbb{N}$.

Equality holds in (8.3) iff the a-differential equation

$$f_a^{(p)}(x) = -2c_p (x - x_m)^p f_a(x)$$

of p^{th} order holds for constants $c_p > 0$, and any fixed but arbitrary $p \in \mathbb{N}$.

In fact, if $c_p = \frac{k_p^2}{2} > 0$, $k_p \in \mathbb{R} - \{0\}$, $p \in \mathbb{N}$, then this a-differential equation holds iff

$$f(x) = e^{2\pi i x \xi_m} \sum_{j=0}^{p-1} a_j x_\delta^j \left[1 + \sum_{m=0}^{\infty} (-1)^{m+1} \frac{\left(k_p^2 x_\delta^{2p}\right)^{m+1}}{P_p^{2p+j} P_p^{4p+j} \cdots P_p^{2(m+1)p+j}} \right],$$

holds, where $x_\delta = x - x_m \neq 0$, $i = \sqrt{-1}$, x_m, ξ_m $(\in \mathbb{R})$ are any fixed but arbitrary constants, and a_j $(j = 0, 1, 2, \ldots, p-1)$ are arbitrary constants in $\mathbb{C}$, as well as

$$P_p^{2(m+1)p+j} = (2(m+1)p + j)(2(m+1)p + j - 1) \cdots ((2m+1)p + j + 1),$$

denote *permutations* for $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $j = 0, 1, 2, \ldots, p-1$.

We note that if x_m *is the mean of* x *for* $|f|^2$ then

$$x_m = \int_{\mathbb{R}} x \,|f(x)|^2 \, dx \left(= \int_0^\infty x \left(|f(x)|^2 - |f(-x)|^2 \right) dx \right).$$

Thus if f is either odd or even, then $|f|^2$ is even and $x_m = 0$. Similarly, if ξ_m *is the mean of* ξ *for* $\left|\hat{f}\right|^2$, then

$$\xi_m = \int_{\mathbb{R}} \xi \left| \hat{f}(\xi) \right|^2 d\xi.$$

Also $\xi_m = 0$ if $\hat{f}$ is either odd or even.

We also note that the conditions (8.1) – (8.2) may be replaced by *the two conditions*:

$$\lim_{|x| \to \infty} w_p^{(r)}(x) \left(\left| f^{(l)}(x) \right|^2 \right)^{(p-2q-r-1)} = 0, \tag{8.4}$$

for $0 \le l \le q \le \left[\frac{p}{2}\right]$ and $0 \le r \le p - 2q - 1$, and

$$\lim_{|x| \to \infty} w_p^{(r)}(x) \left(\operatorname{Re} \left(r_{qkj} f^{(k)}(x) \overline{f^{(j)}}(x) \right) \right)^{(p-2q-r-1)} = 0, \tag{8.5}$$

for $0 \le k < j \le q \le \left[\frac{p}{2}\right]$ and $0 \le r \le p - 2q - 1$.

Proof of the Theorem. In fact, from the generalized Plancherel-Parseval-Rayleigh identity (4.1), and the fact that $|e^{ax}| = 1$ as $a = -2\pi \xi_m i$, one gets

$$\begin{aligned} M_p &= (\mu_{2p})_{w,|f|^2} \cdot (\mu_{2p})_{|\hat{f}|^2} \\ &= \left(\int_{\mathbb{R}} w^2(x)(x - x_m)^{2p} |f(x)|^2 \, dx \right) \cdot \left(\int_{\mathbb{R}} (\xi - \xi_m)^{2p} \left| \hat{f}(\xi) \right|^2 d\xi \right) \\ &= \frac{1}{(2\pi)^{2p}} \left(\int_{\mathbb{R}} w^2(x)(x - x_m)^{2p} |f_a(x)|^2 \, dx \right) \cdot \left(\int_{\mathbb{R}} \left| f_a^{(p)}(x) \right|^2 dx \right). \end{aligned} \tag{8.6}$$

From (8.6) and *the Cauchy-Schwarz inequality*, we find

$$M_p \ge \frac{1}{(2\pi)^{2p}} \left(\int_{\mathbb{R}} \left| w_p(x) f_a(x) f_a^{(p)}(x) \right| dx \right)^2, \tag{8.7}$$

where $w_p = (x - x_m)^p w$, and $f_a = e^{ax} f$.

From (8.7) and *the complex inequality*

$$|ab| \ge \frac{1}{2} \left(a\bar{b} + \bar{a}b \right) \tag{8.8}$$

with $a = w_p(x) f_a(x)$, $b = f_a^{(p)}(x)$, we get

$$M_p \geq \frac{1}{(2\pi)^{2p}} \left[\frac{1}{2} \int_{\mathbb{R}} w_p(x) \left(f_a(x) \overline{f_a^{(p)}(x)} + \overline{f_a(x)} f_a^{(p)}(x) \right) dx \right]^2. \tag{8.9}$$

From (8.9) and the generalized differential identity (*), one finds

$$M_p \geq \frac{1}{2^{2(p+1)}\pi^{2p}} \left[\int_{\mathbb{R}} w_p(x) \left(\sum_{q=0}^{[p/2]} C_q \frac{d^{p-2q}}{dx^{p-2q}} \left| f_a^{(q)}(x) \right|^2 \right) dx \right]^2. \tag{8.10}$$

From (8.10) and the Lagrange type differential identity (7.1), we find

$$\begin{aligned} M_p \geq \frac{1}{2^{2(p+1)}\pi^{2p}} \Bigg[\int_{\mathbb{R}} w_p(x) \Bigg[\sum_{q=0}^{[p/2]} C_q \frac{d^{p-2q}}{dx^{p-2q}} \Bigg(\sum_{l=0}^{q} A_{ql} \left| f^{(l)}(x) \right|^2 \\ + 2 \sum_{0 \leq k < j \leq q} B_{qkj} \operatorname{Re}\left(r_{qkj} f^{(k)}(x) \overline{f^{(j)}}(x) \right) \Bigg) \Bigg] dx \Bigg]^2. \end{aligned} \tag{8.11}$$

From the generalized integral identity (6.2), from $f \in L^2(\mathbb{R})$, the two conditions (8.1) – (8.2) (or (8.4) – (8.5)), or from (6.4) – (6.5), and that all the integrals exist, one gets

$$\begin{aligned} \int_{\mathbb{R}} w_p(x) \frac{d^{p-2q}}{dx^{p-2q}} \left| f^{(l)}(x) \right|^2 dx &= (-1)^{p-2q} \int_{\mathbb{R}} w_p^{(p-2q)}(x) \left| f^{(l)}(x) \right|^2 dx \\ &= I_{ql}, \end{aligned} \tag{8.12}$$

as well as

$$\begin{aligned} &\int_{\mathbb{R}} w_p(x) \frac{d^{p-2q}}{dx^{p-2q}} \operatorname{Re}\left(r_{qkj} f^{(k)}(x) \overline{f^{(j)}}(x) \right) \\ &\qquad = (-1)^{p-2q} \int_{\mathbb{R}} w_p^{(p-2q)}(x) \operatorname{Re}\left(r_{qkj} f^{(k)}(x) \overline{f^{(j)}}(x) \right) = I_{qkj}. \end{aligned} \tag{8.13}$$

From (8.11) and (8.12) – (8.13) we find the generalized $2p^{th}$ order moment Heisenberg uncertainty inequality (for $p \in \mathbb{N}$)

$$\begin{aligned} M_p &\geq \frac{1}{2^{2(p+1)}\pi^{2p}} \left[\sum_{q=0}^{[p/2]} C_q \left(\sum_{l=0}^{q} A_{ql} I_{ql} + 2 \sum_{0 \leq k < j \leq q} B_{qkj} I_{qkj} \right) \right]^2 \\ &= \frac{1}{2^{2(p+1)}\pi^{2p}} E_{p,f}^2, \ (H_p), \end{aligned} \tag{H_p}$$

where

$$E_{p,f} = \sum_{q=0}^{[p/2]} C_q D_q, \quad \text{if} \quad |E_{p,f}| < \infty$$

holds, or *the general moment uncertainty formula*

$$\sqrt[2p]{M_p} \geq \frac{1}{2\pi \sqrt[p]{2}} \sqrt[p]{|E_{p,f}|}. \tag{8.14}$$

Equality holds in (8.3) iff *the a-differential equation* $f_a^{(p)}(x) = -2c_p x_\delta^p f_a(x)$ of p^{th} order with respect to x holds for some constant $c_p = \frac{1}{2}k_p^2 > 0$, $k_p \in \mathbb{R} - \{0\}$, and any fixed but arbitrary $p \in \mathbb{N}$. □

We consider the *general a-differential equation*

$$(a_p) \qquad \frac{d^p y}{dx^p} + k_p^2 x_\delta^p y = 0$$

of p^{th} *order*, where $x_\delta = x - x_m \neq 0$, $k_p \neq 0$, $y = f_a(x) = e^{ax} f(x)$, $a = -2\pi\xi_m i$, $p \in \mathbb{N}$, and the equivalent δ-differential equation

$$(\delta_p) \qquad \frac{d^p y}{dx_\delta^p} + k_p^2 x_\delta^p y = 0,$$

because

$$\frac{dy}{dx} = \frac{dy}{dx_\delta}\frac{dx_\delta}{dx} = \frac{dy}{dx_\delta}\frac{d(x - x_m)}{dx} = \frac{dy}{dx_\delta},$$

and

$$\frac{d^p y}{dx^p} = \frac{d^p y}{dx_\delta^p}, \quad p \in \mathbb{N}.$$

In order to solve equation (δ_p) one may employ the following *power series* method ([18], [21], [25]) *in* (δ_p). In fact, we consider *the power series expansion* $y = \sum_{n=0}^{\infty} a_n x_\delta^n$ about $x_{\delta_0} = 0$, converging (absolutely) in

$$|x_\delta| < \rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty, \quad x_\delta \neq 0.$$

Thus

$$k_p^2 x_\delta^p y = \sum_{n=0}^{\infty} k_p^2 a_n x_\delta^{n+p},$$

and

$$\frac{d^p y}{dx_\delta^p} = \sum_{n=p}^{\infty} P_p^n a_n x_\delta^{n-p}$$

(with permutations $P_p^n = n(n-1)(n-2)\cdots(n-p+1)$)

$$= \sum_{\substack{n+2p=p \\ (\text{or } n=-p)}}^{\infty} P_p^{n+2p} a_{n+2p} x_\delta^{n+p}$$

(with $n + 2p$) on n above and

$$\begin{aligned} P_p^{n+2p} &= (n+2p)(n+2p-1)(n+2p-2)\cdots(n+p+1)) \\ &= \sum_{n=-p}^{-1} P_p^{n+2p} a_{n+2p} x_\delta^{n+p} + \sum_{n=0}^{\infty} P_p^{n+2p} a_{n+2p} x_\delta^{n+p} \\ &= \sum_{n=0}^{\infty} P_p^{n+2p} a_{n+2p} x_\delta^{n+p} \end{aligned}$$

(with $a_{n+2p} = 0$ for all $n \in \{-p, -(p-1), \ldots, -1\}$, or equivalently $a_p = a_{p+1} = \cdots = a_{2p-1} = 0$).

Therefore from these and equation (δ_p) one gets the following *recursive relation*

$$P_p^{n+2p} a_{n+2p} + k_p^2 a_n = 0,$$

or

$$(R_p) \qquad a_{n+2p} = -\frac{k_p^2 a_n}{P_p^{n+2p}}, \quad n \in \mathbb{N}_0, \; p \in \mathbb{N}.$$

From the "*null condition*"

$$a_p = 0, a_{p-1} = 0, \dots, a_{2p-1} = 0$$

and (R_p) we get

$$a_{3p} = a_{5p} = a_{7p} = \cdots = 0, a_{3p+1} = a_{5p+1} = a_{7p+1} = \cdots = 0, \dots,$$

and

$$a_{4p-1} = a_{6p-1} = a_{8p-1} = \cdots = 0,$$

respectively. From (R_p) and $n = 2pm$ with $m \in \mathbb{N}_0$ one finds the following p sequences

$$(a_{2pm+2p}), (a_{2pm+2p+1}), (a_{2pm+2p+2}), \dots, (a_{2pm+3p-1}),$$

with fixed $p \in \mathbb{N}$ and all $m \in \mathbb{N}_0$, such that

$$a_{2p} = -\frac{k_p^2}{P_p^{2p}} a_0, \quad a_{4p} = -\frac{k_p^2}{P_p^{4p}} a_{2p} = \frac{k_p^4}{P_p^{2p} P_p^{4p}} a_0, \dots,$$

$$a_{2pm+2p} = (-1)^{m+1} \frac{k_p^{2m+2}}{P_p^{2p} P_p^{4p} \cdots P_p^{2pm+2p}} a_0$$

are the elements of the first sequence (a_{2pm+2p}) in terms of a_0, and

$$a_{2p+1} = -\frac{k_p^2}{P_p^{2p+1}} a_1, \quad a_{4p+1} = -\frac{k_p^2}{P_p^{4p+1}} a_{2p+1} = \frac{k_p^4}{P_p^{2p+1} P_p^{4p+1}} a_1, \dots,$$

$$a_{2pm+2p+1} = (-1)^{m+1} \frac{k_p^{2m+2}}{P_p^{2p+1} P_p^{4p+1} \cdots P_p^{2pm+2p+1}} a_1$$

are the elements of the second sequence $(a_{2pm+2p+1})$ in terms of $a_1, \dots$, and

$$\left(a_{2p+(p-1)} =\right) a_{3p-1} = -\frac{k_p^2}{P_p^{3p-1}} a_{p-1},$$

$$\left(a_{2p+(3p-1)} =\right) a_{5p-1} = -\frac{k_p^2}{P_p^{5p-1}} a_{3p-1} = \frac{k_p^2}{P_p^{3p-1} P_p^{5p-1}} a_{p-1}, \dots,$$

$$\left(a_{2pm+(3p-1)} =\right) a_{(2m+3)p-1} = (-1)^{m+1} \frac{k_p^{2m+2}}{P_p^{3p-1} P_p^{5p-1} \cdots P_p^{(2m+3)p-1}} a_{p-1}$$

are the elements of the last sequence $(a_{2pm+3p-1})$ in terms of a_{p-1}. Therefore we find the following p solutions

$$y_0 = y_0(x_\delta) = 1 + \sum_{m=0}^{\infty} (-1)^{m+1} \frac{\left(k_p^2 x_\delta^{2p}\right)^{m+1}}{P_p^{2p} P_p^{4p} \cdots P_p^{2(m+1)p}},$$

$$y_1 = y_1(x_\delta) = x_\delta \left[1 + \sum_{m=0}^{\infty} (-1)^{m+1} \frac{\left(k_p^2 x_\delta^{2p}\right)^{m+1}}{P_p^{2p+1} P_p^{4p+1} \cdots P_p^{2(m+1)p+1}}\right], \dots,$$

$$y_{p-1} = y_{p-1}(x_\delta) = x_\delta^{p-1} \left[1 + \sum_{m=0}^{\infty} (-1)^{m+1} \frac{\left(k_p^2 x_\delta^{2p}\right)^{m+1}}{P_p^{3p-1} P_p^{5p-1} \cdots P_p^{(2m+3)p-1}}\right],$$

or equivalently the

$$y_j = y_j(x_\delta) = x_\delta^j \left[1 + \sum_{m=0}^{\infty} (-1)^{m+1} \frac{\left(k_p^2 x_\delta^{2p}\right)^{m+1}}{P_p^{2p+j} P_p^{4p+j} \cdots P_p^{2(m+1)p+j}}\right]$$

for all $j \in \{0, 1, \ldots, p-1\}$, of the differential equation (δ_p), in the form of power series converging (absolutely) by the ratio test.

Thus an arbitrary solution of (δ_p) (and of (a_p)) is of the form

$$f(x) = e^{-ax} \left[\sum_{j=0}^{p-1} a_j y_j (x_\delta) \right], \qquad x_\delta \neq 0,$$

with arbitrary constants a_j $(j = 0, 1, 2, \ldots, p-1)$.

Choosing

$$\begin{gathered} a_0 = 1,\ \ a_1 = 0,\ \ a_2 = 0,\ \ \ldots,\ \ a_{p-2} = 0,\ \ a_{p-1} = 0; \\ a_0 = 0,\ \ a_1 = 1, a_2 = 0,\ \ \ldots,\ \ a_{p-2} = 0,\ \ a_{p-1} = 0; \\ \cdots; \end{gathered}$$

and

$$a_0 = 0,\ \ a_1 = 0,\ \ a_2 = 0,\ \ \ldots,\ \ a_{p-2} = 0,\ \ a_{p-1} = 1,$$

one gets that y_i ($i \in \{0, 1, 2, \ldots, p-1\}$) are partial solutions of (δ_p), satisfying *the initial conditions*

$$y_0(0) = 1, \quad y_0'(0) = 0, \ldots, y_0^{(p-1)}(0) = 0;$$

$$\begin{gathered} y_1(0) = 0, \quad y_1'(0) = 1, \ldots, y_1^{(p-1)}(0) = 0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots; \end{gathered}$$

and

$$y_{p-1}(0) = 0,\ y_{p-1}'(0) = 0, \ldots,\ y_{p-1}^{(p-1)}(0) = 1.$$

If $Y_p = (y_0, y_1, \ldots, y_{p-1})$, then *the Wronskian* at $x_\delta = 0$ is

$$W(Y_p(0)) = \begin{vmatrix} y_0(0) & y_1(0) & \cdots & y_{p-1}(0) \\ y_0'(0) & y_1'(0) & \cdots & {y'}_{p-1}(0) \\ \cdots & \cdots & \cdots & \cdots \\ y_0^{(p-1)}(0) & y_1^{(p-1)}(0) & \cdots & y_{p-1}^{(p-1)}(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \neq 0,$$

yielding that these p partial solutions $y_0, y_1, \ldots, y_{p-1}$of ($\delta_p$) are *linearly independent*. Thus the above formula $y = f_a(x) = \sum_{j=0}^{p-1} a_j y_j$ gives the general solution of the equation (δ_p) (and also of (a_p)).

We note that both the above-mentioned differential equations (a_p) and (δ_p) are solved completely via well-known *special functions* for $p = 1$ (with Gaussian functions) and for $p = 2$ (with Bessel functions), and via functions in terms of power series converging in $\mathbb{R}$ for $p \geq 3$. Therefore the proof of our theorem is complete.

We claim that, if $p = 1$, the function $f : \mathbb{R} \to \mathbb{C}$ given explicitly in our Introduction (with $c = c_1 = k_1^2/2 > 0, k_1 \in \mathbb{R} - \{0\}$) satisfies the equality of (H_1). In fact, the corresponding a-differential equation

$$(a_1) \qquad \frac{dy}{dx} + k_1^2 x_\delta y = 0,$$

where $x_\delta = x - x_m \neq 0$, $y = f_a(x) = e^{ax} f(x)$, $a = -2\pi\xi_m i$, is satisfied by

$$\begin{aligned} y_0 &= y_0(x) \\ &= 1 + \sum_{m=0}^{\infty} (-1)^{m+1} \frac{\left(k_1^2 x_\delta^2\right)^{m+1}}{P_1^2 P_1^4 \cdots P_1^{2(m+1)}} \\ &= 1 + \sum_{m=0}^{\infty} (-1)^{m+1} \frac{\left(c_1 x_\delta^2\right)^{m+1}}{(m+1)!} \end{aligned}$$

(by the power series method and because

$$P_1^2 P_1^4 \cdots P_1^{2(m+1)} = 2 \cdot 4 \cdot \cdots \cdot 2(m+1) = 2^{m+1}(m+1)!),$$

or

$$y_0 = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{\left(c x_\delta^2\right)^m}{m!} = 1 + \left(e^{-c x_\delta^2} - 1\right) = e^{-c x_\delta^2}$$

(because $\sum_{m=1}^{\infty} (-1)^m \frac{t^m}{m!} = e^{-t} - 1$).

Therefore the general solution of the differential equation (a_1) is of the form $y = a_0 y_0$ (with arbitrary constant $c_0 = a_0$), or

$$e^{ax} f(x) = c_0 e^{-c(x-x_m)^2}, \quad \text{or} \quad f(x) = c_0 e^{2\pi i x \xi_m} \cdot e^{-c(x-x_m)^2}.$$

However, one may establish this f much faster, by the direct application of the method of separation of variables to the differential equation (a_1).

Analogously to the proof of (H_1) in the Introduction we prove the following more general inequality (H_2).

8.1. **Fourth Order Moment Heisenberg Inequality.** For any $f \in L^2(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{C}$ and any fixed but arbitrary constants x_m, $\xi_m \in \mathbb{R}$, *the fourth order moment Heisenberg inequality*

$$(H_2) \qquad (\mu_4)_{|f|^2} \cdot (\mu_4)_{|\hat{f}|^2} \geq \frac{1}{64\pi^4} E_{2,f}^2,$$

holds, if

$$(\mu_4)_{|f|^2} = \int_{\mathbb{R}} x_\delta^4 \, |f(x)|^2 dx$$

and

$$(\mu_4)_{|\hat{f}|^2} = \int_{\mathbb{R}} \xi_\delta^4 \left|\hat{f}(\xi)\right|^2 d\xi$$

with $x_\delta = x - x_m$, and $\xi_\delta = \xi - \xi_m$, are the fourth order moments, and

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x) dx, \, f(x) = \int_{\mathbb{R}} e^{2i\pi\xi x} \hat{f}(\xi) d\xi, \qquad i = \sqrt{-1},$$

as well as

$$E_{2,f} = 2 \int_{\mathbb{R}} \left[\left(1 - 4\pi^2 \xi_m^2 x_\delta^2\right) |f(x)|^2 - x_\delta^2 \, |f'(x)|^2 \; -4\pi\xi_m x_\delta^2 \, \mathrm{Im}\left(f(x) \overline{f'(x)} \right) \right] dx,$$

if $|E_{2,f}| < \infty$ holds, where $\mathrm{Im}(\cdot)$ denotes the imaginary part of $(\cdot)$.

Equality holds in (H_2) iff the a-differential equation $f_a''(x) = -2c_2 x_\delta^2 f_a(x)$ of second order holds, for $a = -2\pi\xi_m i$, $y = f_a(x) = e^{ax} f(x)$ and a constant $c_2 = \frac{1}{2}k_2^2 > 0$, $k_2 \in \mathbb{R} - \{0\}$, $x_\delta = x - x_m \neq 0$, or equivalently

$$(a_2) \qquad \frac{d^2 y}{dx^2} + k_2^2 x_\delta^2 y = 0$$

holds iff

$$f(x) = \sqrt{|x_\delta|}e^{2\pi i x \xi_m}\left[c_{20}J_{-1/4}\left(\frac{1}{2}|k_2|x_\delta^2\right) + c_{21}J_{1/4}\left(\frac{1}{2}|k_2|x_\delta^2\right)\right]$$

holds for some constants c_{20}, $c_{21} \in \mathbb{C}$ and with $J_{\pm 1/4}$ *the Bessel functions of the first* kind *of orders* $\pm\frac{1}{4}$, [16]. We note that if $x_m \neq 0$ and $\xi_m = 0$, then

$$f(x) = \sqrt{|x_\delta|}\left[c_{20}J_{-1/4}\left(\frac{1}{2}|k_2|x_\delta^2\right) + c_{21}J_{1/4}\left(\frac{1}{2}|k_2|x_\delta^2\right)\right].$$

We claim that the above function f in terms of the Bessel functions $J_{\pm 1/4}$ is the general solution of the said a-differential equation (a_2) of second order. In fact, *the* δ*-differential equation*

$$(\delta_2) \qquad \frac{d^2y}{dx_\delta^2} + k_2^2 x_\delta^2 y = 0$$

is equivalent to the following *Bessel equation*

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + \left[z^2 - \left(\frac{1}{4}\right)^2\right]u = 0$$

of order $r = \frac{1}{4}$, with $u = \frac{y}{\sqrt{|x_\delta|}}$ and $z = \frac{1}{2}|k_2|x_\delta^2$. But the general solution of this Bessel equation is

$$\begin{aligned} u(z)\left(= y\Big/\sqrt{|x_\delta|}\right) &= c_{20}J_{-1/4}(z) + c_{21}J_{1/4}(z) \\ &= c_{20}J_{-1/4}\left(\frac{1}{2}|k_2|x_\delta^2\right) + c_{21}J_{1/4}\left(\frac{1}{2}|k_2|x_\delta^2\right) \end{aligned}$$

for some constants c_{20}, $c_{21} \in \mathbb{C}$.

In fact, if we denote

$$(8.15) \qquad S = \operatorname{sgn}(x_\delta) = \begin{cases} 1, & x_\delta > 0 \\ -1, & x_\delta < 0 \end{cases},$$

then one gets

$$\frac{du}{dz} = \frac{du}{dx_\delta}\Big/\frac{dz}{dx_\delta} = \left[\frac{S}{|x_\delta|^{3/2}}\frac{dy}{dx_\delta} - \frac{1}{2}\frac{y}{|x_\delta|^{5/2}}\right]\Big/|k_2|,$$

and

$$\frac{d^2u}{dz^2} = \frac{d}{dx_\delta}\left(\frac{du}{dz}\right)\Big/\frac{dz}{dx_\delta} = \left[\frac{S}{|x_\delta|^{5/2}}\frac{d^2y}{dx_\delta^2} - 2\frac{S}{|x_\delta|^{7/2}}\frac{dy}{dx_\delta} + \frac{5}{4}\frac{y}{|x_\delta|^{9/2}}\right]\Big/k_2^2.$$

Thus we establish

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} = \frac{1}{4}|x_\delta|^{3/2}\frac{d^2y}{dx_\delta^2} + \frac{1}{16}u.$$

But

$$\frac{d^2y}{dx_\delta^2} = -k_2^2x_\delta^2y,$$

or

$$|x_\delta|^{3/2}\frac{d^2y}{dx_\delta^2} = -k_2^2x_\delta^4\frac{y}{\sqrt{|x_\delta|}} = -4z^2u.$$

Therefore the above-mentioned Bessel equation holds.

However,

$$\frac{dy}{dx}=\frac{dy}{dx_\delta}\frac{dx_\delta}{dx}=\frac{dy}{dx_\delta}\frac{d\left(x-x_m\right)}{dx}=\frac{dy}{dx_\delta},$$

and

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx_\delta}\left(\frac{dy}{dx_\delta}\right)\frac{dx_\delta}{dx}=\frac{d^2y}{dx_\delta^2}.$$

Therefore the above two equations (a_2) and (δ_2) are equivalent. Thus one gets

$$y=f_a\left(x\right)=e^{ax}f\left(x\right)=\sqrt{\left|x_\delta\right|}\left[c_{20}J_{-1/4}\left(\frac{1}{2}\left|k_2\right|x_\delta^2\right)+c_{21}J_{1/4}\left(\frac{1}{2}\left|k_2\right|x_\delta^2\right)\right],$$

or

$$f\left(x\right)=\sqrt{\left|x_\delta\right|}e^{-ax}\left[c_{20}J_{-1/4}\left(\frac{1}{2}\left|k_2\right|x_\delta^2\right)+c_{21}J_{1/4}\left(\frac{1}{2}\left|k_2\right|x_\delta^2\right)\right],$$

establishing the function f in terms of the two Bessel functions $J_{\pm1/4}$.

However,

$$J_{1/4}\left(z\right)=\left(\frac{z}{2}\right)^{\frac{1}{4}}\sum_{n=0}^{\infty}\left(-1\right)^n\frac{\left(\frac{z}{2}\right)^{2n}}{n!\Gamma\left(\frac{1}{4}+n+1\right)},\quad z>0.$$

Thus, if $z=\frac{1}{2}\left|k_2\right|x_\delta^2>0$, then

$$\sqrt{\left|x_\delta\right|}J_{1/4}\left(\frac{1}{2}\left|k_2\right|x_\delta^2\right)=S\frac{2\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)}\sqrt[4]{\left|k_2\right|}\left[x_\delta-\frac{k_2^2}{4\cdot5}x_\delta^5+\cdots\right],$$

because

$$\Gamma\left(\frac{5}{4}\right)=\frac{1}{4}\Gamma\left(\frac{1}{4}\right),\Gamma\left(\frac{9}{4}\right)=\frac{5}{16}\Gamma\left(\frac{1}{4}\right),\dots,$$

and $S=\operatorname{sgn}\left(x_\delta\right)$, $x_\delta\neq0$, such that $\left|x_\delta\right|=Sx_\delta$. Similarly,

$$J_{-1/4}\left(z\right)=\left(\frac{z}{2}\right)^{-\frac{1}{4}}\sum_{n=0}^{\infty}\left(-1\right)^n\frac{\left(\frac{z}{2}\right)^{2n}}{n!\Gamma\left(-\frac{1}{4}+n+1\right)},z>0.$$

Therefore

$$\sqrt{\left|x_\delta\right|}J_{-1/4}\left(\frac{1}{2}\left|k_2\right|x_\delta^2\right)=\frac{\Gamma\left(\frac{1}{4}\right)}{\pi}\frac{1}{\sqrt[4]{\left|k_2\right|}}\left[1-\frac{k_2^2}{3\cdot4}x_\delta^4+\cdots\right],$$

because

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)=\frac{\pi}{\sin\frac{1}{4}\pi}=\pi\sqrt{2},$$

or

$$\Gamma\left(\frac{3}{4}\right)=\frac{\pi\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)},\ \text{ and }\ \Gamma\left(\frac{7}{4}\right)=\Gamma\left(1+\frac{3}{4}\right)=\frac{3}{4}\Gamma\left(\frac{3}{4}\right)=\frac{3\sqrt{2}\pi}{4\Gamma\left(\frac{1}{4}\right)},\dots.$$

A direct way to find the general solution of the above δ-differential equation (δ_2) is by applying *the power series method for* (δ_2). In fact, consider two arbitrary constants $a_0=c_{20}$ and $a_1=c_{21}$ such that $y=\sum\limits_{n=0}^{\infty}a_nx_\delta^n$, about $x_{\delta0}=0$, converging (absolutely) in

$$\left|x_\delta\right|<\rho=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\infty,\qquad x_\delta\neq0.$$

Thus

$$\frac{dy}{dx_\delta}=\sum_{n=1}^{\infty}na_nx_\delta^{n-1}=\sum_{n=0}^{\infty}\left(n+1\right)a_{n+1}x_\delta^n,$$

and

$$\frac{d^2y}{dx_\delta^2} = \sum_{n=1}^{\infty} (n+1)\, n a_{n+1} x_\delta^{n-1}$$
$$= \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x_\delta^{n}$$
$$= \sum_{\substack{n+2=0 \text{ or} \\ n=-2}}^{\infty} (n+4)(n+3)\, a_{n+4} x_\delta^{n+2}$$
$$= (2a_2 + 6a_3 x_\delta) + \sum_{n=0}^{\infty} (n+4)(n+3)\, a_{n+4} x_\delta^{n+2},$$

as well as

$$k_2^2 x_\delta^2 y = k_\delta^2 \sum_{n=0}^{\infty} a_n x_\delta^{n+2}.$$

Therefore from (δ_2) one gets *the recursive relation*

$$(R_2) \qquad a_{n+4} = -\frac{k_2^2 a_n}{(n+4)(n+3)},$$

with $a_2 = a_3 = 0$.

Letting $n = 4m$ with $m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ in this recursive relation, we find the following two solutions of the equation (δ_2):

$$y_0 = y_0(x_\delta)$$
$$= 1 + \sum_{m=0}^{\infty} (-1)^{m+1} \frac{(k_2^2 x_\delta^4)^{m+1}}{3 \cdot 4 \cdots (4m+3)(4m+4)}$$
$$= 1 - \frac{k_2^2}{3 \cdot 4} x_\delta^4 + \frac{k_2^4}{3 \cdot 4 \cdot 7 \cdot 8} x_\delta^8 - \frac{k_2^6}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12} x_\delta^{12} + \cdots,$$

and

$$y_1 = y_1(x_\delta)$$
$$= x_\delta \left[1 + \sum_{m=0}^{\infty} (-1)^{m+1} \frac{(k_2^2 x_\delta^4)^{m+1}}{4 \cdot 5 \cdots (4m+4)(4m+5)} \right]$$
$$= x_\delta - \frac{k_2^2}{4 \cdot 5} x_\delta^5 + \frac{k_2^4}{4 \cdot 5 \cdot 8 \cdot 9} x_\delta^9 - \frac{k_2^6}{4 \cdot 5 \cdot 8 \cdot 9 \cdot 12 \cdot 13} x_\delta^{13} + \cdots.$$

We note that each one of these two power series converges by *the ratio test.*

Thus an arbitrary solution of (δ_2) (and of (a_2)) is of the form

$$y = c_{20} y_0 + c_{21} y_1,$$

or

$$f(x) = e^{-ax} \left[c_{20} y_0(x_\delta) + c_{21} y_1(x_\delta) \right],$$

where $x_\delta = x - x_m \neq 0$,

$$y_0 = \frac{\left[\pi \sqrt[4]{|k_2|} \sqrt{|x_\delta|} J_{-1/4}\left(\frac{1}{2} |k_2|\, x_\delta^2 \right) \right]}{\Gamma\left(\frac{1}{4}\right)}, \qquad S \in \{\pm 1\},$$

and

$$y_1 = \frac{\left[S\Gamma\left(\frac{1}{4}\right)\sqrt{|x_\delta|}J_{1/4}\left(\frac{1}{2}|k_2|\,x_\delta^2\right)\right]}{2\sqrt{2}\sqrt[4]{|k_2|}},$$

where S is defined by (8.15).
Besides we note that from $a_2 = 0$, $a_3 = 0$ and the above recursive relation (R_2) one gets $a_6 = a_{10} = a_{14} = \cdots = 0$, as well as $a_7 = a_{11} = a_{15} = \cdots = 0$, respectively.

From this recursive relation (R_2) and $n = 4m$ with $m \in \mathbb{N}_0$ we get the following two sequences (a_{4m+4}), (a_{4m+5}), such that

$$a_4 = -\frac{k_2^2}{3\cdot 4}a_0, a_8 = -\frac{k_2^2}{7\cdot 8}a_4 = \frac{k_2^4}{3\cdot 4\cdot 7\cdot 8}a_0, \dots,$$
$$a_{4m+4} = (-1)^{m+1}\frac{k_2^{2m+2}}{3\cdot 4\cdot\cdots\cdot(4m+3)(4m+4)}a_0,$$

and

$$a_5 = -\frac{k_2^2}{4\cdot 5}a_1, a_9 = -\frac{k_2^2}{8\cdot 9}a_5 = \frac{k_2^4}{4\cdot 5\cdot 8\cdot 9}a_1, \dots,$$
$$a_{4m+5} = (-1)^{m+1}\frac{k_2^{2m+2}}{4\cdot 5\cdot\cdots\cdot(4m+4)(4m+5)}a_1.$$

Choosing $a_0 = c_{20} = 1$, $a_1 = c_{21} = 0$; and $a_0 = c_{20} = 0$, $a_1 = c_{21} = 1$, one gets that y_0 and y_1 are partial solutions of (δ_2), satisfying the initial conditions

$$y_0(0) = 1, \quad y_0'(0) = 0; \quad \text{and} \quad y_1(0) = 0, \quad y_1'(0) = 1.$$

Therefore *the Wronskian* of y_0, y_1 at $x_\delta = 0$ is

$$W(y_0, y_1)(0) = \begin{vmatrix} y_0(0) & y_1(0) \\ y_0'(0) & y_1'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

yielding that these $p = 2$ solutions y_0, y_1 are *linearly independent*. We note that, if we divide the above power series (expansion) solutions y_0 and y_1, we have

$$\begin{aligned}\frac{y_1(x_\delta)}{y_0(x_\delta)} &= y_1(x_\delta)(y_0(x_\delta))^{-1} \\ &= \left(x_\delta - \frac{k_2^2}{20}x_\delta^5 + \cdots\right)\left(1 - \frac{k_2^2}{12}x_\delta^4 + \cdots\right)^{-1} \\ &= x_\delta + \frac{k_2^2}{30}x_\delta^5 + \cdots,\end{aligned}$$

which obviously is nonconstant, implying also that y_0 and y_1 are linearly independent.

Thus the above formula $y = a_0y_0 + a_1y_1$ gives the general solution of (δ_2) (and also of (a_2)). Similarly we prove the following inequality (H_3).

8.2. **Sixth Order Moment Heisenberg Inequality.** For any $f \in L^2(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{C}$ and any fixed but arbitrary constants $x_m, \xi_m \in \mathbb{R}$, *the sixth order moment Heisenberg inequality*

$$(H_3) \qquad (\mu_6)_{|f|^2} \cdot (\mu_6)_{|\hat{f}|^2} \geq \frac{1}{256\pi^6}E_{3,f}^2,$$

holds, if

$$(\mu_6)_{|f|^2} = \int_{\mathbb{R}} x_\delta^6\,|f(x)|^2 dx$$

and

$$(\mu_6)_{|\hat{f}|^2} = \int_{\mathbb{R}} \xi_\delta^6 \left|\hat{f}(\xi)\right|^2 d\xi$$

with $x_\delta = x - x_m$, and $\xi_\delta = \xi - \xi_m$, and

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x) dx, f(x) = \int_{\mathbb{R}} e^{2i\pi\xi x} \hat{f}(\xi) d\xi,$$

as well as

$$E_{3,f} = -3 \int_{\mathbb{R}} \left[2\left(1 - 6\pi^2 \xi_m^2 x_\delta^2\right) |f(x)|^2 - 3x_\delta^2 |f'(x)|^2 - 12\pi\xi_m x_\delta^2 \operatorname{Im}\left(f(x)\overline{f'(x)}\right)\right] dx,$$

if $|E_{3,f}| < \infty$ holds, where $\operatorname{Im}(\cdot)$ denotes the imaginary part of $(\cdot)$.

Equality holds in (H_3) iff *the a-differential equation* $f_a'''(x) = -2c_3 x_\delta^3 f_a(x)$ of third order holds, for $a = -2\pi\xi_m i$, $i = \sqrt{-1}$, $f_a = e^{ax} f$, and a constant $c_3 = \frac{k_3^2}{2} > 0$, $k_3 \in \mathbb{R}$, or equivalently iff

$$f(x) = e^{2\pi i x\xi_m} \sum_{j=0}^{2} a_j x_\delta^j \left[1 + \sum_{m=0}^{\infty} (-1)^{m+1} \times \frac{(k_3^2 x_\delta^6)^{m+1}}{(4+j)(5+j)(6+j)\cdots(6m+4+j)(6m+5+j)(6m+6+j)} \right]$$

holds, where $x_\delta \neq 0$, and a_j ($j = 0, 1, 2$) are arbitrary constants in $\mathbb{C}$.

Consider the a-differential equation

$$(a_3) \qquad \frac{d^3 y}{dx^3} + k_3^2 x_\delta^3 y = 0,$$

with $y = f_a(x)$ and the equivalent δ-differential equation

$$(\delta_3) \qquad \frac{d^3 y}{dx_\delta^3} + k_3^2 x_\delta^3 y = 0,$$

with $x_\delta = x - x_m$ and $k_3 \in \mathbb{R} - \{0\}$, such that $d^3y/dx^3 = d^3y/dx_\delta^3$.

Employing the power series method for (δ_3), one considers the power series expansion $y = \sum_{n=0}^{\infty} a_n x_\delta^n$ about $x_{\delta_0} = 0$, converging (absolutely) in

$$|x_\delta| < \rho = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \infty, \qquad x_\delta \neq 0.$$

Thus

$$k_3^2 x_\delta^3 y = \sum_{n=0}^{\infty} k_3^2 a_n x_\delta^{n+3},$$

and

$$\begin{aligned}\frac{d^3y}{dx_\delta^3} &= \sum_{n=3}^{\infty} n\,(n-1)\,(n-2)\,a_n x_\delta^{n-3}\\ &= \sum_{n=-3}^{\infty}(n+6)\,(n+5)\,(n+4)\,a_{n+6}x_\delta^{n+3}\\ &= \left(6a_3+24a_4x_\delta+60a_5x_\delta^2\right)+\sum_{n=0}^{\infty}P_3^{n+6}a_{n+6}x_\delta^{n+3}\\ &= \sum_{n=0}^{\infty}P_3^{n+6}a_{n+6}x_\delta^{n+3}\end{aligned}$$

(with $a_3 = a_4 = a_5 = 0$), where $P_3^{n+6} = (n+6)\,(n+5)\,(n+4)$. Therefore from these and equation (δ_3) one gets *the recursive relation*

$$(R_3) \qquad a_{n+6} = -\frac{k_3^2 a_n}{P_3^{n+6}}, \qquad n \in \mathbb{N}_0$$

From "*the null condition*"

$$(N_3) \qquad a_3 = 0, \qquad a_4 = 0, \qquad \text{and} \qquad a_5 = 0$$

and the above recursive relation (R_3) we get

$$\begin{aligned} a_9 &= a_{15} = a_{21} = \cdots = 0,\\ a_{10} &= a_{16} = a_{22} = \cdots = 0, \qquad \text{and}\\ a_{11} &= a_{17} = a_{23} = \cdots = 0\end{aligned}$$

respectively.

From (R_3) and $n = 6m$ with $m \in \mathbb{N}_0$ one finds the following three sequences (a_{6m+6}), (a_{6m+7}), (a_{6m+8}), such that

$$\begin{aligned} a_6 &= -\frac{k_3^2}{4\cdot5\cdot6}a_0,\\ a_{12} &= -\frac{k_3^2}{10\cdot11\cdot12}a_6 = \frac{k_3^4}{4\cdot5\cdot6\cdot10\cdot11\cdot12}a_0,\\ &\dots,\end{aligned}$$

$$a_{6m+6} = (-1)^{m+1}\,\frac{k_3^{2m+2}}{4\cdot5\cdot6\cdot\dots\cdot(6m+4)\,(6m+5)\,(6m+6)}a_0,$$

and

$$\begin{aligned} a_7 &= -\frac{k_3^2}{5\cdot6\cdot7}a_1,\\ a_{13} &= -\frac{k_3^2}{11\cdot12\cdot13}a_7 = \frac{k_3^4}{5\cdot6\cdot7\cdot11\cdot12\cdot13}a_1,\\ &\dots,\end{aligned}$$

$$a_{6m+7} = (-1)^{m+1}\,\frac{k_3^{2m+2}}{5\cdot6\cdot7\cdot\dots\cdot(6m+5)\,(6m+6)\,(6m+7)}a_1,$$

as well as

$$\begin{aligned} a_8 &= -\frac{k_3^2}{6\cdot 7\cdot 8}a_2, \\ a_{14} &= -\frac{k_3^2}{12\cdot 13\cdot 14}a_8 = \frac{k_3^4}{6\cdot 7\cdot 8\cdot 12\cdot 13\cdot 14}a_2, \\ &\dots, \end{aligned}$$

$$a_{6m+8} = (-1)^{m+1}\frac{k_3^{2m+2}}{6\cdot 7\cdot 8\cdot\dots\cdot(6m+6)(6m+7)(6m+8)}a_2.$$

Therefore we find the following three solutions

$$\begin{aligned} y_0 &= y_0(x_\delta) \\ &= 1+\sum_{m=0}^{\infty}(-1)^{m+1}\frac{(k_3^2x_\delta^6)^{m+1}}{4\cdot 5\cdot 6\cdot\dots\cdot(6m+4)(6m+5)(6m+6)} \\ &= 1+\sum_{m=0}^{\infty}\left(\frac{a_{6m+6}}{a_0}\right)x_\delta^{6m+6} \end{aligned}$$

for $a_0 \neq 0$,

$$\begin{aligned} y_1 &= y_1(x_\delta) \\ &= x_\delta\left[1+\sum_{m=0}^{\infty}(-1)^{m+1}\frac{(k_3^2x_\delta^6)^{m+1}}{5\cdot 6\cdot 7\cdot\dots\cdot(6m+5)(6m+6)(6m+7)}\right] \\ &= x_\delta+\sum_{m=0}^{\infty}\left(\frac{a_{6m+7}}{a_1}\right)x_\delta^{6m+7} \end{aligned}$$

for $a_1 \neq 0$, and

$$\begin{aligned} y_2 &= y_2(x_\delta) \\ &= x_\delta^2\left[1+\sum_{m=0}^{\infty}(-1)^{m+1}\frac{(k_3^2x_\delta^6)^{m+1}}{6\cdot 7\cdot 8\cdot\dots\cdot(6m+6)(6m+7)(6m+8)}\right] \\ &= x_\delta^2+\sum_{m=0}^{\infty}\left(\frac{a_{6m+8}}{a_2}\right)x_\delta^{6m+8} \end{aligned}$$

for $a_2 \neq 0$, of the differential equation (δ_3), in the form of power series converging (absolutely) by the ratio test. Thus an arbitrary solution of (δ_3) (and of (a_3)) is of the form

$$y = e^{ax}f(x) = a_0y_0 + a_1y_1 + a_2y_2,$$

or

$$f(x) = e^{-ax}\left[a_0y_0(x_\delta) + a_1y_1(x_\delta) + a_2y_2(x_\delta)\right],$$

$x_\delta \neq 0$, with arbitrary constants a_i $(i = 0, 1, 2)$ in $\mathbb{C}$. Choosing

$$\begin{aligned} &a_0 = 1, \quad a_1 = 0, \quad a_2 = 0; \\ &a_0 = 0, \quad a_1 = 1, \quad a_2 = 0; \quad \text{and} \\ &a_0 = 0, \quad a_1 = 0, \quad a_2 = 1, \end{aligned}$$

one gets that $y_j\ (j = 0, 1, 2)$ are partial solutions of (δ_3), satisfying the initial conditions

$$y_0(0) = 1, \quad y_0'(0) = 0, \quad y_0''(0) = 0;$$
$$y_1(0) = 0, \quad y_1'(0) = 1, \quad y_1''(0) = 0; \qquad \text{and}$$
$$y_2(0) = 0, \quad y_2'(0) = 0, \quad y_2''(0) = 1.$$

Therefore the Wronskian of $y_j\ (j = 0, 1, 2)$ at $x_\delta = 0$ is

$$W(y_0, y_1, y_2)(0) = \begin{vmatrix} y_0(0) & y_1(0) & y_2(0) \\ y_0'(0) & y_1'(0) & y_2'(0) \\ y_0''(0) & y_1''(0) & y_2''(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0,$$

yielding that these $p = 3$ partial solutions $y_j\ (j = 0, 1, 2)$ of (δ_3) are linearly independent. Thus the above formula $y = \sum_{j=0}^{2} a_j y_j$ gives the general solution of (δ_3) (and also of (a_3)). Analogously we establish the following inequality (H_4):

8.3. **Eighth Order Moment Heisenberg Inequality.** For any $f \in L^2(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{C}$ and any fixed but arbitrary constants $x_m, \xi_m \in \mathbb{R}$, the eighth order moment Heisenberg inequality

$$(H_4) \qquad (\mu_8)_{|f|^2} \cdot (\mu_8)_{|\hat{f}|^2} \geq \frac{1}{1024\pi^8} E_{4,f}^2,$$

holds, if

$$(\mu_8)_{|f|^2} = \int_{\mathbb{R}} x_\delta^8 |f(x)|^2 dx$$

and

$$(\mu_8)_{|\hat{f}|^2} = \int_{\mathbb{R}} \xi_\delta^8 \left|\hat{f}(\xi)\right|^2 d\xi$$

with $x_\delta = x - x_m$, $\xi_\delta = \xi - \xi_m$, and

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x) dx, \ f(x) = \int_{\mathbb{R}} e^{2i\pi\xi x} \hat{f}(\xi) d\xi,$$

as well as

$$\begin{aligned} E_{4,f} = 2\int_{\mathbb{R}} \Big\{ &\left[4\left(3 - 24\pi^2\xi_m^2 x_\delta^2 + 16\pi^4\xi_m^4 x_\delta^4\right)|f(x)|^2 \right. \\ &\left. -8x_\delta^2\left(3 - 2\pi^2\xi_m^2 x_\delta^2\right)|f'(x)|^2 + x_\delta^4 |f''(x)|^2\right] \\ &-8\pi\xi_m x_\delta^2 \Big[4\left(3 - \pi^2\xi_m^2 x_\delta^2\right) \operatorname{Im}\left(f(x)\overline{f'(x)}\right) \\ &+\pi\xi_m x_\delta^2 \operatorname{Re}\left(f(x)\overline{f''(x)}\right) - x_\delta^2 \operatorname{Im}\left(f'(x)\overline{f''(x)}\right)\Big]\Big\} dx, \end{aligned}$$

if $|E_{4,f}| < \infty$ holds, where $\operatorname{Re}(\cdot)$ and $\operatorname{Im}(\cdot\cdot)$ denote the real part of $(\cdot)$ and the imaginary part of $(\cdot\cdot)$, respectively.

Equality holds in (H_4) iff the a-differential equation

$$f_a^{(4)}(x) = -2c_4 x_\delta^4 f_a(x)$$

of fourth order holds, for $a=-2\pi\xi_m i$, $i=\sqrt{-1}$, $f_a=e^{ax}f$, and a constant $c_4=\frac{k_4^2}{2}>0$, $k_4\in\mathbb{R}$, or equivalently iff

$$f(x)=e^{2\pi ix\xi_m}\sum_{j=0}^{3}a_jx_\delta^j\left[1+\sum_{m=0}^{\infty}(-1)^{m+1}\times\frac{(k_4^2x_\delta^8)^{m+1}}{(5+j)(6+j)\cdots(8m+7+j)(8m+8+j)}\right]$$

holds, where $x_\delta\neq 0$, and a_j $(j=0,1,2,3)$ are arbitrary constants in $\mathbb{C}$.

8.4. **First Four Generalized Weighted Moment Inequalities.** (i)

$$\begin{aligned}M_1&=(\mu_2)_{w,|f|^2}(\mu_2)_{|\hat{f}|^2}\\&=\left(\int_{\mathbb{R}}w^2(x)(x-x_m)^2|f(x)|^2dx\right)\left(\int_{\mathbb{R}}(\xi-\xi_m)^2\left|\hat{f}(\xi)\right|^2d\xi\right)\\&\geq\frac{1}{16\pi^2}(C_0D_0)^2\\&=\frac{1}{16\pi^2}[C_0(A_{00}I_{00})]^2\\&=\frac{1}{16\pi^2}I_{00}^2\\&=\frac{1}{16\pi^2}\left(\int_{\mathbb{R}}w_1^{(1)}(x)|f(x)|^2dx\right)^2\\&=\frac{1}{16\pi^2}E_{1,f}^2,\end{aligned}\tag{8.16}$$

because

$$I_{00}=-\int_{\mathbb{R}}\left(w_1^{(1)}|f|^2\right)(x)\,dx$$

with $w_1(x)=w(x)(x-x_m)$. We note that if $w=1$ then

$$\begin{aligned}E_{1,f}&=C_0D_0=I_{00}\\&=-\int_{\mathbb{R}}\left(w_1^{(1)}|f|^2\right)(x)\,dx\\&=-\int_{\mathbb{R}}|f(x)|^2dx\\&=-E_{|f|^2}=-1=-E_{|\hat{f}|^2}=-\int_{\mathbb{R}}\left|\hat{f}(\xi)\right|^2d\xi\end{aligned}$$

by the Plancherel-Parseval-Rayleigh identity, if $|E_{1,f}|<\infty$ holds. Thus from (8.16) one gets *the classical second order moment Heisenberg uncertainty principle* which says that the product of the variance $(\mu_2)_{|f|^2}$ of x for the probability density $|f|^2$ and the variance $(\mu_2)_{|\hat{f}|^2}$ of ξ for the probability density $\left|\hat{f}\right|^2$ is at least $\frac{E_{|f|^2}^2}{16\pi^2}$, which is the second order moment Heisenberg Inequality (H_1) in our Introduction. *The Heisenberg lower bound* $H^*=\frac{1}{4\pi}$, for $E_{|f|^2}=1$, can be different if one chooses a different formula for the Fourier transform $\hat{f}$ of f. Finally, the above *inequality* (8.16) *generalizes* (H_1) of our Introduction (there $w=1$).

(ii)

$$M_2 = (\mu_4)_{w,|f|^2} (\mu_4)_{|\hat{f}|^2} \tag{8.17}$$
$$= \left(\int_{\mathbb{R}} w^2(x)(x-x_m)^4 |f(x)|^2 dx\right)\left(\int_{\mathbb{R}} (\xi-\xi_m)^4 \left|\hat{f}(\xi)\right|^2 d\xi\right)$$
$$\geq \frac{1}{64\pi^4}(C_0 D_0 + C_1 D_1)^2$$
$$= \frac{1}{64\pi^4}\left[C_0(A_{00}I_{00}) + C_1(A_{10}I_{10} + A_{11}I_{11} + 2B_{101}I_{101})\right]^2$$
$$= \frac{1}{64\pi^4}\left[I_{00} - 2\left(\beta^2 I_{10} + I_{11} - 2\beta I_{101}\right)\right]^2$$
$$= \frac{1}{64\pi^4}\left[\int_{\mathbb{R}} w_2^{(2)}(x)|f(x)|^2 dx \right.$$
$$\left. -2\int_{\mathbb{R}} w_2(x)\left[\beta^2|f|^2 + |f'|^2 + 2\beta \operatorname{Im}\left(f\bar{f}'\right)\right](x)\,dx\right]^2,$$

because $\operatorname{Re}\left(if\bar{f}'\right)(x) = -\operatorname{Im}\left(f\bar{f}'\right)(x)$, and

$$I_{00} = \int_{\mathbb{R}} w_2^{(2)}(x)|f(x)|^2 dx,$$
$$I_{10} = \int_{\mathbb{R}} w_2(x)|f(x)|^2 dx,$$
$$I_{11} = \int_{\mathbb{R}} w_2(x)|f'(x)|^2 dx,$$

and

$$I_{101} = \int_{\mathbb{R}} w_2(x)\operatorname{Re}\left(if\bar{f}'\right)(x)\,dx,$$

with $w_2(x) = w(x)(x-x_m)^2$.

It is clear that (8.17) is equivalent to

$$M_2 \geq \frac{1}{64\pi^4}\left[\int_{\mathbb{R}}\left(w_2^{(2)} - 2\beta^2 w_2\right)(x)|f(x)|^2 dx \right. \tag{8.18}$$
$$\left. -2\int_{\mathbb{R}} w_2(x)|f'(x)|^2 dx - 4\beta\int_{\mathbb{R}} w_2(x)\operatorname{Im}\left(f\bar{f}'\right)(x)\,dx\right]^2$$
$$= \frac{1}{64\pi^4}E_{2,f}^2,$$

or

$$\sqrt[4]{M_2} \geq \frac{1}{2\pi\sqrt{2}}\sqrt{|E_{2,f}|}, \tag{8.19}$$

where

$$E_{2,f} = C_0 D_0 + C_1 D_1 \tag{8.20}$$
$$= D_0 - 2D_1$$
$$= I_{00} - 2\left(\beta^2 I_{10} + I_{11} - 2\beta I_{101}\right)$$
$$= \int_{\mathbb{R}}\left[\left(w_2^{(2)} - 2\beta^2 w_2\right)|f|^2 - 2w_2|f'|^2 - 4\beta w_2 \operatorname{Im}\left(f\bar{f}'\right)\right](x)\,dx,$$

if $|E_{2,f}| < \infty$ holds. We note that if $|E_{2,f}| = \frac{1}{2}$ holds, then from (8.19) one gets $\sqrt[4]{M_2} \geq \frac{1}{4\pi}\,(= H^*)$,while if $|E_{2,f}| = 1$, then

$$\sqrt[4]{M_2} \geq \frac{1}{2\pi\sqrt{2}} = \frac{1}{4\pi}\sqrt{2} \quad (> H^*).$$

Thus we observe that the lower bound of $\sqrt[4]{M_2}$ is greater than H^* if $|E_{2,f}| > \frac{1}{2}$; the same with H^* if $|E_{2,f}| = \frac{1}{2}$; and smaller than H^*, if $0 < |E_{2,f}| < \frac{1}{2}$. Finally, the above *inequality* (8.18) *generalizes* (H_2) of Section 8 (there $w = 1$).

(iii)

$$\begin{aligned}
M_3 &= (\mu_6)_{w,|f|^2} (\mu_6)_{|\hat{f}|^2} \\
&= \left(\int_{\mathbb{R}} w^2(x)(x-x_m)^6 |f(x)|^2 dx\right)\left(\int_{\mathbb{R}} (\xi-\xi_m)^6 \left|\hat{f}(\xi)\right|^2 d\xi\right) \\
&\geq \frac{1}{256\pi^6}(C_0D_0 + C_1D_1)^2 \\
&= \frac{1}{256\pi^6}\left[C_0(A_{00}I_{00}) + C_1(A_{10}I_{10} + A_{11}I_{11} + 2B_{101}I_{101})\right]^2 \\
&= \frac{1}{256\pi^6}\left[I_{00} - 3\left(\beta^2 I_{10} + I_{11} - 2\beta I_{101}\right)\right]^2 \\
&= \frac{1}{256\pi^6}\left[-\int_{\mathbb{R}} w_3^{(3)}(x)|f(x)|^2 dx\right. \\
&\qquad \left. + 3\int_{\mathbb{R}} w_3^{(1)}(x)\left[\beta^2|f|^2 + |f'|^2 + 2\beta\,\mathrm{Im}\left(f\overline{f'}\right)\right](x)\,dx\right]^2,
\end{aligned} \tag{8.21}$$

because

$$I_{00} = -\int_{\mathbb{R}} w_3^{(3)}(x)|f(x)|^2 dx, \qquad I_{10} = -\int_{\mathbb{R}} w_3^{(1)}(x)|f(x)|^2 dx,$$

$$I_{11} = -\int_{\mathbb{R}} w_3^{(1)}(x)|f'(x)|^2 dx,$$

and

$$\begin{aligned}
I_{101} &= -\int_{\mathbb{R}} w_3^{(1)}(x)\,\mathrm{Re}\left(if(x)\overline{f'(x)}\right)dx \\
&= \int_{\mathbb{R}} w_3^{(1)}(x)\,\mathrm{Im}\left(f(x)\overline{f'(x)}\right)dx,
\end{aligned}$$

with $w_3(x) = w(x)(x-x_m)^3$.

It is clear that (8.21) is equivalent to

$$\begin{aligned}
M_3 &\geq \frac{1}{256\pi^6}\left[\int_{\mathbb{R}}\left(-w_3^{(3)} + 3\beta^2 w_3^{(1)}\right)(x)|f(x)|^2 dx\right. \\
&\qquad \left. +3\int_{\mathbb{R}} w_3^{(1)}|f'(x)|^2 dx + 6\beta\int_{\mathbb{R}} w_3^{(1)}(x)\,\mathrm{Im}\left(f\bar{f}'\right)(x)\,dx\right]^2 \\
&= \frac{1}{256\pi^6}E_{3,f}^2,
\end{aligned} \tag{8.22}$$

or

$$\sqrt[6]{M_3} \geq \frac{1}{2\pi\sqrt[3]{2}}\sqrt[3]{|E_{3,f}|} \tag{8.23}$$

where

$$E_{3,f}=\int_{\mathbb{R}}\left[\left(-w_3^{(3)}+3\beta^2 w_3^{(1)}\right)|f|^2+3w_3^{(1)}|f'|^2+6\beta w_3^{(1)}\operatorname{Im}\left(f\bar{f}'\right)\right](x)\,dx,$$

if $|E_{3,f}|<\infty$ holds. We note that if $|E_{3,f}|=\frac{1}{4}$, then from (8.23) we find $\sqrt[6]{M_3}\geq\frac{1}{4\pi}\,(=H^*)$, while if $|E_{3,f}|=1$, then $\sqrt[6]{M_3}\geq\frac{1}{2\pi\sqrt[3]{2}}\,(>H^*)$. Thus we observe that the lower bound of $\sqrt[6]{M_3}$ is greater than H^* if $|E_{3,f}|>\frac{1}{4}$; the same with H^* if $|E_{3,f}|=\frac{1}{4}$; and smaller than H^*, if $0<|E_{3,f}|<\frac{1}{4}$. Finally, the above *inequality* (8.22) *generalizes* (H_3) of our section 8 (there $w=1$).

(iv)

(8.24)
$$\begin{aligned}
M_4&=(\mu_8)_{w,|f|^2}(\mu_8)_{|\hat{f}|^2}\\
&=\left(\int_{\mathbb{R}}w^2(x)(x-x_m)^8|f(x)|^2dx\right)\left(\int_{\mathbb{R}}(\xi-\xi_m)^8\left|\hat{f}(\xi)\right|^2d\xi\right)\\
&\geq\frac{1}{1024\pi^8}(C_0D_0+C_1D_1+C_2D_2)^2\\
&=\frac{1}{1024\pi^8}\left[C_0(A_{00}I_{00})+C_1(A_{10}I_{10}+A_{11}I_{11}+2B_{101}I_{101})\right.\\
&\qquad\left.+C_2(A_{20}I_{20}+A_{21}I_{21}+A_{22}I_{22}+2B_{201}I_{201}+2B_{202}I_{202}+2B_{212}I_{212})\right]^2\\
&=\frac{1}{1024\pi^8}\left[I_{00}-4\left(\beta^2I_{10}+I_{11}-2\beta I_{101}\right)\right.\\
&\qquad\left.+2\left(\beta^4I_{20}+4\beta^2I_{21}+I_{22}+4\beta^3I_{201}+2\beta^2I_{202}-4\beta I_{212}\right)\right]^2\\
&=\frac{1}{1024\pi^8}\left[\int_{\mathbb{R}}w_4^{(4)}(x)|f(x)|^2dx\right.\\
&\qquad-4\int_{\mathbb{R}}w_4^{(2)}(x)\left[\beta^2|f|^2+|f'|^2+2\beta\operatorname{Im}\left(f\bar{f}'\right)\right](x)\,dx\\
&\qquad+2\int_{\mathbb{R}}w_4(x)\left[\beta^4|f|^2+4\beta^2|f'|^2+|f''|^2+4\beta^3\operatorname{Im}\left(f\overline{f'}\right)\right.\\
&\qquad\left.\left.-2\beta^2\operatorname{Re}\left(f\overline{f''}\right)+4\beta\operatorname{Im}\left(f'\overline{f''}\right)\right](x)\,dx\right]^2\\
&=\frac{1}{1024\pi^8}E_{4,f}^2,
\end{aligned}$$

because

$$\begin{aligned}
I_{00}&=\int_{\mathbb{R}}w_4^{(4)}(x)|f(x)|^2dx,\\
I_{10}&=\int_{\mathbb{R}}w_4^{(2)}(x)|f(x)|^2dx,\\
I_{11}&=\int_{\mathbb{R}}w_4^{(2)}(x)|f'(x)|^2dx,\\
I_{20}&=\int_{\mathbb{R}}w_4(x)|f(x)|^2dx,\\
I_{21}&=\int_{\mathbb{R}}w_4(x)|f'(x)|^2dx,\\
I_{22}&=\int_{\mathbb{R}}w_4(x)|f''(x)|^2dx,
\end{aligned}$$

and

$$I_{101} = \int_{\mathbb{R}} w_4^{(2)}(x)\,\mathrm{Re}\left(if(x)\overline{f'(x)}\right)dx = -\int_{\mathbb{R}} w_4^{(2)}(x)\,\mathrm{Im}\left(f(x)\overline{f'(x)}\right)dx,$$

$$I_{201} = \int_{\mathbb{R}} w_4(x)\,\mathrm{Re}\left(-if(x)\overline{f'(x)}\right)dx = \int_{\mathbb{R}} w_4(x)\,\mathrm{Im}\left(f(x)\overline{f'(x)}\right)dx,$$

$$I_{202} = \int_{\mathbb{R}} w_4(x)\,\mathrm{Re}\left(-f(x)\overline{f''(x)}\right)dx = -\int_{\mathbb{R}} w_4(x)\,\mathrm{Re}\left(f(x)\overline{f''(x)}\right)dx,$$

$$I_{212} = \int_{\mathbb{R}} w_4(x)\,\mathrm{Re}\left(if'(x)\overline{f''(x)}\right)dx = -\int_{\mathbb{R}} w_4(x)\,\mathrm{Im}\left(f'(x)\overline{f''(x)}\right)dx,$$

with $w_4(x) = w(x)(x - x_m)^4$.

It is clear that (8.24) is equivalent to

$$\sqrt[8]{M_4} \geq \frac{1}{2\pi\sqrt[4]{2}}\sqrt[4]{|E_{4,f}|}, \tag{8.25}$$

where

$$E_{4,f} = \int_{\mathbb{R}} \left[\left(w_4^{(4)} - 4\beta^2 w_4^{(2)} + 2\beta^4 w_4\right)|f|^2 + 4\left(-w_4^{(2)} + 2\beta^2 w_4\right)|f'|^2 + 2w_4|f''|^2\right.$$
$$\left. -8\beta\left(w_4^{(2)} - \beta^2 w_4\right)\mathrm{Im}\left(f\overline{f'}\right) - 4\beta^2 w_4\,\mathrm{Re}\left(f\overline{f''}\right) + 8\beta w_4\,\mathrm{Im}\left(f'\overline{f''}\right)\right](x)\,dx,$$

if $|E_{4,f}| < \infty$ holds.We note that if $|E_{4,f}| = \frac{1}{8}$, then from (8.25) we find $\sqrt[8]{M_4} \geq \frac{1}{4\pi}$ $(= H^*)$, while if $|E_{4,f}| = 1$, then $\sqrt[8]{M_4} \geq \frac{1}{2\pi\sqrt[4]{2}}$ $(> H^*)$. Thus we observe that the lower bound of $\sqrt[8]{M_4}$ is greater than H^* if $|E_{4,f}| > \frac{1}{8}$; the same with H^* if $|E_{4,f}| = \frac{1}{8}$; and smaller than H^*, if $0 < |E_{4,f}| < \frac{1}{8}$. Finally, the above *inequality* (8.24) *generalizes* (H_4) of our Section 8 (there $w = 1$).

8.5. **First form of (8.3), if $w = 1$, $x_m = 0$ and $\xi_m = 0$.** We note that $\beta = 2\pi\xi_m = 0$, $w_p(x) = x^p$, and $w_p^{(p)}(x) = p!$ $(p = 1, 2, 3, 4, \ldots)$. Therefore the above-mentioned four special cases (i) – (iv) yield the four formulas:

$$E_{1,f} = -\int_{\mathbb{R}} |f(x)|^2\,dx = -E_{|f|^2}, \tag{8.26}$$

$$E_{2,f} = 2\int_{\mathbb{R}} \left[|f(x)|^2 - x^2|f'(x)|^2\right]dx, \tag{8.27}$$

$$E_{3,f} = -3\int_{\mathbb{R}} \left[2|f(x)|^2 - 3x^2|f'(x)|^2\right]dx, \tag{8.28}$$

and

$$E_{4,f} = 2\int_{\mathbb{R}} \left[12|f(x)|^2 - 24x^2|f'(x)|^2 + x^4|f''(x)|^2\right]dx, \tag{8.29}$$

respectively, if $|E_{p,f}| < \infty$ holds for $p = 1, 2, 3, 4$.

It is clear that, in general,

$$A_{ql} = \binom{q}{q}^2 \beta^0 = 1, \quad \text{if} \quad l = q, \text{ and } \quad A_{ql} = \binom{q}{l}^2 \beta^{2(q-l)}, \quad \text{if} \quad l \neq q,$$

for $0 \leq l \leq q$.

Thus, if $\beta = 0$, one gets

$$A_{ql} = \begin{cases} 1, & \text{if } l = q, \\ 0, & \text{if } l \neq q \end{cases} = \delta_{lq} \quad (= \text{the Kronecker delta}), \quad 0 \leq l \leq q. \tag{8.30}$$

It is obvious, if $\beta = 0$, that

$$B_{qkj} = (-1)^{q-k} \begin{pmatrix} q \\ k \end{pmatrix} \begin{pmatrix} q \\ j \end{pmatrix} \beta^{2q-j-k} = 0, \quad 0 \leq k < j \leq q, \tag{8.31}$$

such that $j + k < 2q$ for $0 \leq k < j \leq q$; that is, $\beta^{2q-j-k} \neq \beta^0 \, (= 1)$ for $0 \leq k < j \leq q$.

Therefore from (8.30) and (8.31) we obtain

$$D_q = A_{qq} I_{qq} = I_{qq} = (-1)^{p-2q} \int_{\mathbb{R}} w_p^{(p-2q)}(x) \left| f^{(q)}(x) \right|^2 dx, \tag{8.32}$$

if $|D_q| < \infty$, holds for $0 \leq q \leq \left[\frac{p}{2}\right]$.

We note that if $w = 1$ and $x_m = 0$, and $\xi_m = 0$ or $\beta = 0$, then $w_p(x) = x^p$ ($p = 1, 2, 3, 4, \ldots$), and

$$w_p^{(p-2q)}(x) = (x^p)^{(p-2q)} = p(p-1) \cdots (p - (p-2q) + 1) \, x^{p-(p-2q)},$$

or

$$w_p^{(p-2q)}(x) = \frac{p!}{(p - (p-2q))!} x^{2q} = \frac{p!}{(2q)!} x^{2q}, \quad 0 \leq q \leq \left[\frac{p}{2}\right]. \tag{8.33}$$

From (8.32) and (8.33) we get the formula

$$D_q = (-1)^{p-2q} \frac{p!}{(2q)!} \int_{\mathbb{R}} x^{2q} \left| f^{(q)}(x) \right|^2 dx, \tag{8.34}$$

if $|D_q| < \infty$ holds for $0 \leq q \leq \left[\frac{p}{2}\right]$.

Therefore from (8.34) one finds that

$$\begin{aligned} E_{p,f} &= \sum_{q=0}^{[p/2]} C_q D_q \\ &= \sum_{q=0}^{[p/2]} \left[(-1)^q \frac{p}{p-q} \begin{pmatrix} p-q \\ q \end{pmatrix} \right] \left[(-1)^{p-2q} \frac{p!}{(2q)!} \int_{\mathbb{R}} x^{2q} \left| f^{(q)}(x) \right|^2 dx \right], \end{aligned}$$

or the formula

$$E_{p,f} = \int_{\mathbb{R}} \sum_{q=0}^{[p/2]} (-1)^{p-q} \frac{p}{p-q} \frac{p!}{(2q)!} \begin{pmatrix} p-q \\ q \end{pmatrix} x^{2q} \left| f^{(q)}(x) \right|^2 dx, \tag{8.35}$$

if $|E_{p,f}| < \infty$ holds for $0 \leq q \leq \left[\frac{p}{2}\right]$, when $w = 1$ and $x_m = \xi_m = 0$.

Let

$$(m_{2p})_{|f|^2} = \int_{\mathbb{R}} x^{2p} |f(x)|^2 dx \tag{8.36}$$

be the $2p^{th}$ moment of x for $|f|^2$ about the origin $x_m = 0$, and

$$(m_{2p})_{|\hat{f}|^2} = \int_{\mathbb{R}} \xi^{2p} \left| \hat{f}(\xi) \right|^2 d\xi \tag{8.37}$$

the $2p^{th}$ moment of ξ for $\left|\hat{f}\right|^2$ about the origin $\xi_m = 0$. Denote

$$\varepsilon_{p,q} = (-1)^{p-q} \frac{p}{p-q} \frac{p!}{(2q)!} \begin{pmatrix} p-q \\ q \end{pmatrix}, \text{ if } p \in \mathbb{N} \text{ and } 0 \le q \le \left[\frac{p}{2}\right]. \tag{8.38}$$

Thus from (8.35) and (8.38) we find

$$E_{p,f} = \int_{\mathbb{R}} \sum_{q=0}^{[p/2]} \varepsilon_{p,q} x^{2q} \left|f^{(q)}(x)\right|^2 dx, \tag{8.39}$$

if $|E_{p,f}| < \infty$ holds for $0 \le q \le \left[\frac{p}{2}\right]$.

If $w = 1$ and $x_m = \xi_m = 0$, one gets from (8.3) and (8.35) – (8.38) the following Corollary 8.2.

Corollary 8.2. *Assume that $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a real variable x, $w = 1$, $x_m = \xi_m = 0$, and $\hat{f}$ is the Fourier transform of f, described in Theorem 8.1. Denote $(m_{2p})_{|f|^2\left(or\, |\hat{f}|^2\right)}$ and $\varepsilon_{p,q}$ as in (8.36) (or (8.37)) and (8.38), respectively for all $p \in \mathbb{N}$.*

If $f \in L^2(\mathbb{R})$, and all the above assumptions hold, then

$$\sqrt[2p]{(m_{2p})_{|f|^2}} \sqrt[2p]{(m_{2p})_{|\hat{f}|^2}} \ge \frac{1}{2\pi \sqrt[p]{2}} \sqrt[p]{\left|\sum_{q=0}^{[p/2]} \varepsilon_{p,q} (m_{2q})_{|f^{(q)}|^2}\right|}, \tag{8.40}$$

holds for any fixed but arbitrary $p \in \mathbb{N}$ and $0 \le q \le \left[\frac{p}{2}\right]$, where

$$(m_{2q})_{|f^{(q)}|^2} = \int_{\mathbb{R}} x^{2q} \left|f^{(q)}(x)\right|^2 dx. \tag{8.41}$$

Equality in (8.40) holds iff the differential equation $f^{(p)}(x) = -2c_p x^p f(x)$ of p^{th} order holds for some $c_p > 0$, and any fixed but arbitrary $p \in \mathbb{N}$.

If $q = 0$, then we note that (8.41) yields

$$(m_0)_{|f|^2} = \int_{\mathbb{R}} |f(x)|^2 dx = E_{|f|^2}.$$

We also note that if $p = 5$, then $[p/2] = 2$; $q = 0, 1, 2$. Thus from (8.39) we get

$$\begin{aligned} \varepsilon_{5,0} &= (-1)^{5-0} \frac{5}{5-0} \frac{5!}{(2\cdot 0)!} \begin{pmatrix} 5-0 \\ 0 \end{pmatrix} \\ &= -120, \\ \varepsilon_{5,1} &= (-1)^{5-1} \frac{5}{5-1} \frac{5!}{(2\cdot 1)!} \begin{pmatrix} 5-1 \\ 1 \end{pmatrix} = 300, \end{aligned}$$

and

$$\varepsilon_{5,2} = (-1)^{5-2} \frac{5}{5-2} \frac{5!}{(2\cdot 2)!} \begin{pmatrix} 5-2 \\ 2 \end{pmatrix} = -25.$$

Therefore

$$E_{5,f} = -5 \int_{\mathbb{R}} \left[24 |f(x)|^2 - 60x^2 |f'(x)|^2 + 5x^4 |f''(x)|^2\right] dx, \tag{8.42}$$

if $|E_{5,f}| < \infty$ holds.

Similarly if $p = 6$, then $\left[\frac{p}{2}\right] = 3$; $q = 0, 1, 2, 3$. Thus from (8.39) one finds

$$\varepsilon_{6,0} = 720, \quad \varepsilon_{6,1} = -2160, \quad \varepsilon_{6,2} = 270, \quad \text{and} \quad \varepsilon_{6,3} = -2.$$

Therefore

$$E_{6,f} = 2\int_{\mathbb{R}} \left[360\left|f(x)\right|^2 - 1080x^2\left|f'(x)\right|^2 + 135x^4\left|f''(x)\right|^2 - x^6\left|f'''(x)\right|^2\right]dx, \tag{8.43}$$

if $|E_{6,f}| < \infty$ holds. In the same way one gets

$$E_{7,f} = -7\int_{\mathbb{R}} \left[720\left|f(x)\right|^2 - 2520x^2\left|f'(x)\right|^2 + 420x^4\left|f''(x)\right|^2 - 7x^6\left|f'''(x)\right|^2\right]dx, \tag{8.44}$$

$$\begin{aligned} E_{8,f} = 2\int_{\mathbb{R}} \Big[&20160\left|f(x)\right|^2 - 80640x^2\left|f'(x)\right|^2 + 16800x^4\left|f''(x)\right|^2 \\ &-448x^6\left|f'''(x)\right|^2 + x^8\left|f^{(4)}(x)\right|^2\Big]\,dx, \end{aligned} \tag{8.45}$$

and

$$\begin{aligned} E_{9,f} = -9\int_{\mathbb{R}} \Big[&40320\left|f(x)\right|^2 - 181440x^2\left|f'(x)\right|^2 + 45360x^4\left|f''(x)\right|^2 \\ &-1680x^6\left|f'''(x)\right|^2 + 9x^8\left|f^{(4)}(x)\right|^2\Big]\,dx, \end{aligned} \tag{8.46}$$

if $|E_{p,f}| < \infty$ holds for $p = 7, 8, 9$. We note that the cases $E_{p,f} : p = 1, 2, 3, 4$ are given above via the four formulas (8.26) – (8.29).

8.6. **Second form of (8.3), if $\xi_m = 0$.** In general for $w_p(x) = w(x)x_\delta^p$ with $x_\delta = x - x_m$, where x_m is any fixed and arbitrary real, and $w : \mathbb{R} \to \mathbb{R}$ a real valued weight function, as well as, $\xi_m = 0$, we get from (5.1) and (8.32) that

$$\begin{aligned} D_q = I_{qq} &= (-1)^{p-2q}\int_{\mathbb{R}} \left(w(x)\,x_\delta^p\right)^{(p-2q)}\left|f^{(q)}(x)\right|^2 dx \\ &= (-1)^{p-2q}\int_{\mathbb{R}} \sum_{m=0}^{p-2q} \binom{p-2q}{m} w^{(m)}(x)\left(x_\delta^p\right)^{(p-2q-m)}\left|f^{(q)}(x)\right|^2 dx \\ &= (-1)^{p-2q}\int_{\mathbb{R}} \sum_{m=0}^{p-2q} \binom{p-2q}{m} w^{(m)}(x)\,\frac{p!}{(p-(p-2q-m))!}x_\delta^{p-(p-2q-m)}\left|f^{(q)}(x)\right|^2 dx, \end{aligned}$$

or

$$D_q = (-1)^{p-2q}\int_{\mathbb{R}} \left[\sum_{m=0}^{p-2q} \frac{p!}{(2q+m)!}\binom{p-2q}{m} w^{(m)}(x)\,x_\delta^m\right] x_\delta^{2q}\left|f^{(q)}(x)\right|^2 dx, \tag{8.47}$$

if $|D_q| < \infty$ holds for $0 \le q \le \left[\frac{p}{2}\right]$.

If $m = 0$, then one finds from (8.47) the formula (8.34). Therefore from (8.47) one gets that

$$\begin{aligned} E_{p,f} = \int_{\mathbb{R}} \sum_{q=0}^{[p/2]} (-1)^{p-q}\,\frac{p}{p-q}\binom{p-q}{q} \\ \times\left[\sum_{m=0}^{p-2q} \frac{p!}{(2q+m)!}\binom{p-2q}{m} w^{(m)}(x)\,x_\delta^m\right] x_\delta^{2q}\left|f^{(q)}(x)\right|^2 dx, \end{aligned} \tag{8.48}$$

if $|E_{p,f}| < \infty$ holds for $0 \le q \le \left[\frac{p}{2}\right]$, when $w : \mathbb{R} \to \mathbb{R}$ is a real valued weight function, x_m any fixed but arbitrary real constant and $\xi_m = 0$. If $m = 0$ and $x_m = 0$, then we find from (8.48) the formula (8.35).

If we denote

$$\varepsilon_{p,q,w}(x) = (-1)^{p-q}\frac{p}{p-q}\binom{p-q}{q}\left[\sum_{m=0}^{p-2q}\frac{p!}{(2q+m)!}\binom{p-2q}{m}w^{(m)}(x)\,x_\delta^m\right], \tag{8.49}$$

then one gets from (8.48) that

$$E_{p,f} = \int_{\mathbb{R}}\sum_{q=0}^{[p/2]}\varepsilon_{p,q,w}(x)x_\delta^{2q}\left|f^{(q)}(x)\right|^2dx = \sum_{q=0}^{[p/2]}\int_{\mathbb{R}}\varepsilon_{p,q,w}(x)\,x_\delta^{2q}\left|f^{(q)}(x)\right|^2dx, \tag{8.50}$$

if $|E_{p,f}| < \infty$ holds for $0 \le q \le \left[\frac{p}{2}\right]$.

It is clear that the formula

$$E_{p,f} = \sum_{q=0}^{[p/2]}(-1)^{p-q}\frac{p}{p-q}\binom{p-q}{q} \times \int_{\mathbb{R}}\left[\sum_{m=0}^{p-2q}\frac{p!}{(2q+m)!}\binom{p-2q}{m}w^{(m)}(x)\,x_\delta^m\right]x_\delta^{2q}\left|f^{(q)}(x)\right|^2dx, \tag{8.51}$$

if $|E_{p,f}| < \infty$ holds for $0 \le q \le \left[\frac{p}{2}\right]$.

Therefore from (8.3) and (8.49) – (8.50) we get the following Corollary 8.3.

Corollary 8.3. *Assume that $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a real variable x, $w : \mathbb{R} \to \mathbb{R}$ a real valued weight function, x_m any fixed but arbitrary real number, $\xi_m = 0$, and $\hat{f}$ is the Fourier transform of f, described in our above theorem. Denote $(\mu_{2p})_{w,|f|^2}$, $(m_{2p})_{|\hat{f}|^2}$ and $\varepsilon_{p,q,w}(x)$ as in the preliminaries of the above theorem, (8.37) and (8.49), respectively, for all $p \in \mathbb{N}$.*

If $f \in L^2(\mathbb{R})$, and all the above assumptions hold, then

$$\sqrt[2p]{(\mu_{2p})_{w,|f|^2}}\,\sqrt[2p]{(m_{2p})_{|\hat{f}|^2}} \ge \frac{1}{2\pi\sqrt[p]{2}}\left[\left|\sum_{q=0}^{[p/2]}\int_{\mathbb{R}}\varepsilon_{p,q,w}(x)\,x_\delta^{2q}\left|f^{(q)}(x)\right|^2dx\right|\right]^{\frac{1}{p}}, \tag{8.52}$$

holds for any fixed but arbitrary $p \in \mathbb{N}$ and $0 \le q \le \left[\frac{p}{2}\right]$.

Equality in (8.52) holds iff the differential equation

$$f^{(p)}(x) = -2c_px_\delta^pf(x)$$

of p^{th} order holds for some $c_p > 0$, and any fixed but arbitrary $p \in \mathbb{N}$.

We note that for $p = 2$; $q = 0, 1$ and $w_2(x) = w(x)\,x_\delta^2$, with $x_\delta = x - x_m$; $\xi_m = 0$, we get from (8.49) that

$$\varepsilon_{2,0,w}(x) = 2w(x) + 4w'(x)\,x_\delta + w''(x)\,x_\delta^2,$$

and

$$\varepsilon_{2,1,w}(x) = -2w(x)\,.$$

Therefore from (8.48) one obtains

$$E_{2,f} = \int_{\mathbb{R}}\left[\left(2w(x) + 4w'(x)\,x_\delta + w''(x)\,x_\delta^2\right)|f(x)|^2 - 2w(x)\,x_\delta^2\,|f'(x)|^2\right]dx, \quad \text{if } |E_{2,f}| < \infty. \tag{8.53}$$

This result (8.53) can be found also from (8.20), where $\beta = 2\pi\xi_m = 0$ and thus

$$E_{2,f} = \int_{\mathbb{R}} \left[w_2^{(2)}(x) \left| f(x) \right|^2 - 2w_2(x) \left| f'(x) \right|^2 \right] dx, \tag{8.54}$$

if $|E_{2,f}| < \infty$ holds, with

$$w_2(x) = w(x)(x - x_m)^2 = w(x) x_\delta^2,$$
$$w_2^{(1)}(x) = 2w(x) x_\delta + w'(x) x_\delta^2$$

and

$$w_2^{(2)}(x) = 2w(x) + 4w'(x) x_\delta + w''(x) x_\delta^2.$$

8.7. **Third form of (8.3), if** $w = 1$**.** In general with any fixed but arbitrary real numbers x_m, ξ_m, $x_\delta = x - x_m$ and $\xi_\delta = \xi - \xi_m$, one finds

$$w_p = x_\delta^p,$$
$$w_p^{(r)} = (x_\delta^p)^{(r)} = \frac{p!}{(p-r)!} x_\delta^{p-r},$$

and

$$w_p^{(p-2q)} = \frac{p!}{(2q)!} x_\delta^{2q}.$$

Therefore the integrals I_{ql}, I_{qkj} of the above theorem take the form

$$I_{ql} = (-1)^{p-2q} \frac{p!}{(2q)!} (\mu_{2q})_{\left|f^{(l)}\right|^2}, \quad 0 \le l \le q \le \left[\frac{p}{2}\right], \tag{8.55}$$

and

$$I_{qkj} = (-1)^{p-2q} \frac{p!}{(2q)!} (\mu_{2q})_{f_{kj}}, \quad 0 \le k < j \le q \le \left[\frac{p}{2}\right], \tag{8.56}$$

where $f_{kj} : \mathbb{R} \to \mathbb{R}$ are real valued functions of x, such that

$$f_{kj}(x) = \operatorname{Re}\left(r_{qkj} f^{(k)}(x) \overline{f^{(j)}}(x) \right), \quad 0 \le k < j \le q \le \left[\frac{p}{2}\right], \tag{8.57}$$

with $r_{qkj} = (-1)^{q - \frac{k+j}{2}}$, and

$$(\mu_{2q})_{\left|f^{(l)}\right|^2} = \int_{\mathbb{R}} x_\delta^{2q} \left| f^{(l)}(x) \right|^2 dx, \quad 0 \le l \le q \le \left[\frac{p}{2}\right], \tag{8.58}$$

is the $2q^{th}$ moment of x for $\left|f^{(l)}\right|^2$, and

$$(\mu_{2q})_{f_{kj}} = \int_{\mathbb{R}} x_\delta^{2q} f_{kj}(x) dx, \quad 0 \le k < j \le q \le \left[\frac{p}{2}\right], \tag{8.59}$$

is the $2q^{th}$ moment of x for f_{kj}.

We note that if $0 \le k = j = l \le q \le \left[\frac{p}{2}\right]$, then

$$f_{ll}(x) = s_{ql} \left| f^{(l)}(x) \right|^2, \tag{8.60}$$

and thus

$$(\mu_{2q})_{f_{ll}} = s_{ql} (\mu_{2q})_{\left|f^{(l)}\right|^2}, 0 \le l \le q \le \left[\frac{p}{2}\right], \tag{8.61}$$

where $s_{ql} = (-1)^{q-l}$.

We consider A_{ql} and B_{qkj} and β as in the theorem, and $\varepsilon_{p,q}$ as in (8.38). Therefore

$$D_q = (-1)^{p-2q} \frac{p!}{(2q)!} \left[\sum_{l=0}^{q} A_{ql} \left(\mu_{2q}\right)_{\left|f^{(l)}\right|^2} + 2 \sum_{0 \le k < j \le q} B_{qkj} \left(\mu_{2q}\right)_{f_{kj}} \right], \tag{8.62}$$

if $|D_q| < \infty$ holds for $0 \le q \le \left[\frac{p}{2}\right]$. From (8.55) – (8.62) one gets

$$E_{p,f} = \sum_{q=0}^{[p/2]} \left[\sum_{l=0}^{q} A_{ql} \left(\mu_{2q}\right)_{\left|f^{(l)}\right|^2} + 2 \sum_{0 \le k < j \le q} B_{qkj} \left(\mu_{2q}\right)_{f_{kj}} \right], \tag{8.63}$$

if $|E_{p,f}| < \infty$ holds, for any fixed but arbitrary $p \in \mathbb{N}$.

If $w = 1$ one gets from (8.3) and (8.55) – (8.63) the following Corollary 8.4.

Corollary 8.4. *Assume that $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a real variable x, $w = 1$, x_m and ξ_m any fixed but arbitrary real numbers, $x_\delta = x - x_m$ and $\xi_\delta = \xi - \xi_m$, $w_p(x) = x_\delta^p$, and $\hat{f}$ is the Fourier transform of f, described in our theorem. Let*

$$\left(\mu_{2p}\right)_{|f|^2} = \int_{\mathbb{R}} x_\delta^{2p} \left|f(x)\right|^2 dx,$$

and

$$\left(\mu_{2p}\right)_{\left|\hat{f}\right|^2} = \int_{\mathbb{R}} \xi_\delta^{2p} \left|\hat{f}(\xi)\right|^2 d\xi$$

be the $2p^{th}$ moment of x for $|f|^2$, and the $2p^{th}$ moment of ξ for $\left|\hat{f}\right|^2$, respectively. Denote $\left(\mu_{2q}\right)_{\left|f^{(l)}\right|^2}$, $\left(\mu_{2q}\right)_{f_{kj}}$ (with $f_{kj} : \mathbb{R} \to \mathbb{R}$ as in (8.57)) , $\varepsilon_{p,q}$ and A_{ql}, B_{qkj} via (8.58), (8.59), (8.38) and the preliminaries of the theorem, respectively for all $p \in \mathbb{N}$. Also denote

$$U_p = \sqrt[2p]{\left(\mu_{2p}\right)_{|f|^2}} \sqrt[2p]{\left(\mu_{2p}\right)_{\left|\hat{f}\right|^2}}.$$

If $f \in L^2(\mathbb{R})$, and all the above assumptions hold, then

$$U_p \ge H_p^* \left[\left| \sum_{q=0}^{[p/2]} \varepsilon_{p,q} \left[\sum_{l=0}^{q} A_{ql} \left(\mu_{2q}\right)_{\left|f^{(l)}\right|^2} + 2 \sum_{0 \le k < j \le q} B_{qkj} \left(\mu_{2q}\right)_{f_{kj}} \right] \right| \right]^{\frac{1}{p}} \tag{8.64}$$

holds for any fixed but arbitrary $p \in \mathbb{N}$ and $0 \le q \le \left[\frac{p}{2}\right]$, where $H_p^* = 1/\,2\pi \sqrt[p]{2}$ (for $p \in \mathbb{N}$) is the generalized Heisenberg constant.

Equality in (8.64) holds iff the a-differential equation

$$f_a^{(p)}(x) = -2c_p x_\delta^p f_a(x), a = -2\pi\xi_m i,$$

holds for some $c_p > 0$, and any fixed but arbitrary $p \in \mathbb{N}$.

We call U_p the *uncertainty product* due to the Heisenberg uncertainty principle (8.64).

We note that if $f : \mathbb{R} \to \mathbb{R}$ is a real valued function of a real variable, in the above Corollary 8.4, then

$$f_{kj} = \left(f^{(k)} f^{(j)}\right) \operatorname{Re}\left(r_{qkj}\right), \quad \text{for} \;\; 0 \le k < j \le q \le \left[\frac{p}{2}\right], \tag{8.65}$$

where $r_{qkj} \in \{\pm 1, \pm i\}$, such that

$$r_{qkj} = \begin{cases} 1, & \text{if } 2q \equiv (k+j) \,(\mathrm{mod}\, 4) \\ -1, & \text{if } 2q \equiv (k+j+2) \,(\mathrm{mod}\, 4) \end{cases} ; \text{ and}$$

$$r_{qkj} = \begin{cases} i, & \text{if } 2q \equiv (k+j+1) \,(\mathrm{mod}\, 4) \\ -i, & \text{if } 2q \equiv (k+j+3) \,(\mathrm{mod}\, 4). \end{cases}$$

Thus

(8.66)
$$f_{kj} = \left(f^{(k)} f^{(j)}\right) \begin{cases} 1, & \text{if } 2q \equiv (k+j) \,(\mathrm{mod}\, 4) \\ -1, & \text{if } 2q \equiv (k+j+2) \,(\mathrm{mod}\, 4) \\ 0, & \text{if } 2q \equiv (k+j+1) \,(\text{or } (k+j+3)) \,(\mathrm{mod}\, 4) \end{cases} ,$$

for $0 \le k < j \le q \le \left[\frac{p}{2}\right]$.

Therefore

(8.67)
$$(\mu_{2q})_{f_{kj}} = (\mu_{2q})_{f^{(k)} f^{(j)}} \begin{cases} 1, & \text{if } 2q \equiv (k+j) \,(\mathrm{mod}\, 4) \\ -1, & \text{if } 2q \equiv (k+j+2) \,(\mathrm{mod}\, 4) \\ 0, & \text{if } 2q \equiv (k+j+1) \,(\text{or } (k+j+3)) \,(\mathrm{mod}\, 4) \end{cases} ,$$

where

$$(\mu_{2q})_{f^{(k)} f^{(j)}} = \int_{\mathbb{R}} x_\delta^{2q} \left(f^{(k)} f^{(j)}\right)(x)\, dx,$$

for $0 \le k < j \le q \le \left[\frac{p}{2}\right]$.

Similarly if $f^{(k)} \overline{f^{(j)}} : \mathbb{R} \to \mathbb{R}$, for $0 \le k < j \le q \le \left[\frac{p}{2}\right]$, are real valued functions of a real variable x, we get analogous results.

9. GAUSSIAN FUNCTION

Consider $w = 1$, x_m and ξ_m means and *the Gaussian function* $f : \mathbb{R} \to \mathbb{C}$, such that $f(x) = c_0 e^{-cx^2}$, where c_0, c are constants and $c_0 \in \mathbb{C}$, $c > 0$. It is easy to prove *the integral formula*

(9.1)
$$\int_{\mathbb{R}} x^{2p} e^{-2cx^2} dx = \frac{\Gamma\left(p + \frac{1}{2}\right)}{(2c)^{p+\frac{1}{2}}}, c > 0,$$

for all $p \in \mathbb{N}$ and $p = 0$, where Γ is *the Euler gamma function* [21], such that

$$\Gamma\left(p + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot \cdots \cdot (2p-1)}{2^p} \sqrt{\pi}$$

for $p \in \mathbb{N}$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ for $p = 0$. Note that the mean x_m of x for $|f|^2$is given by

$$x_m = \int_{\mathbb{R}} x\, |f(x)|^2\, dx = 0.$$

Also from Gasquet et al [8, p. 159-161], by applying differential equations [25], one gets that the Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{C}$ is of the form

(9.2)
$$\hat{f}(\xi) = c_0 \sqrt{\frac{\pi}{c}} e^{-\frac{\pi^2}{c}\xi^2}, \quad c_0 \in \mathbb{C}, \quad c > 0.$$

In this case the mean ξ_m of ξ for $\left|\hat{f}\right|^2$ is given by

$$\xi_m = \int_{\mathbb{R}} \xi \left|\hat{f}(\xi)\right|^2 d\xi = 0.$$

Therefore from (8.36) – (8.37) with means $x_m = 0$, $\xi_m = 0$ and from (9.1) – (9.2) one finds that the $2p^{th}$ power of the left-hand side of the inequality (8.40) of the Corollary 8.2 is

$$\begin{aligned}(9.3)\qquad &(m_{2p})_{|f|^2} \cdot (m_{2p})_{|\hat{f}|^2} \\ &= \left(\int_{\mathbb{R}} x^{2p} |f(x)|^2 dx\right)\left(\int_{\mathbb{R}} \xi^{2p} \left|\hat{f}(\xi)\right|^2 d\xi\right) \\ &= |c_0|^4 \frac{\pi}{c}\left(\int_{\mathbb{R}} x^{2p} e^{-2cx^2} dx\right)\left(\int_{\mathbb{R}} \xi^{2p} e^{-2c^*\xi^2} d\xi\right)\left(\text{where } c^* = \frac{\pi^2}{c}\right) \\ &= \pi \frac{|c_0|^4}{c} \frac{\Gamma\left(p+\frac{1}{2}\right)}{(2c)^{p+\frac{1}{2}}} \frac{\Gamma\left(p+\frac{1}{2}\right)}{(2c^*)^{p+\frac{1}{2}}} = \left(H_p^*\right)^{2p} 2\Gamma^2\left(p+\frac{1}{2}\right)\frac{|c_0|^4}{c},\end{aligned}$$

for all fixed but arbitrary $p \in \mathbb{N}$, $c_0 \in \mathbb{C}$, and $c > 0$ $\left(\text{where } H_p^* = \frac{1}{2\pi \sqrt[p]{2}}\right)$. We note that

$$\begin{aligned}(9.4)\qquad E_{|f|^2} &= \int_{\mathbb{R}} |f(x)|^2 dx \\ &= |c_0|^2 \int_{\mathbb{R}} e^{-2cx^2} dx \\ &= \frac{|c_0|^2}{\sqrt{2c}} \Gamma\left(\frac{1}{2}\right) \\ &= |c_0|^2 \sqrt{\frac{\pi}{2c}}, \quad \text{where } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.\end{aligned}$$

If we denote

$$(2p-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2p-1), 0!! = (-1)!! = 1,$$

for $p \in \mathbb{N}$ and $p = 0$, respectively, then

$$(9.5)\qquad \Gamma\left(p+\frac{1}{2}\right) = \frac{(2p-1)!!}{2^p}\sqrt{\pi} \ \text{ for } \ p \in \mathbb{N}, \ \text{ and } \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \ \text{ for } \ p = 0.$$

We consider *the Legendre duplication formula for* Γ ([18], [21])

$$(9.6)\qquad \Gamma(2p) = \frac{2^{2p-1}}{\sqrt{\pi}} \Gamma(p) \Gamma\left(p+\frac{1}{2}\right), \quad p \in \mathbb{N}$$

and *the factorial formula*

$$(9.7)\qquad \Gamma(p+1) = p!, \ p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

We take *the Hermite polynomial* ([18], [21])

$$\begin{aligned}(9.8)\quad H_q(x) = (2x)^q - \frac{q(q-1)}{1!}(2x)^{q-2} + \frac{q(q-1)(q-2)(q-3)}{2!}(2x)^{q-4} \\ - \cdots + (-1)^{\left[\frac{q}{2}\right]} \frac{q!}{\left[\frac{q}{2}\right]!}(2x)^{q-2\left[\frac{q}{2}\right]}, \qquad q \in \mathbb{N}_0,\end{aligned}$$

where $\left[\frac{q}{2}\right]=\frac{q}{2}$ if q is even and $\left[\frac{q}{2}\right]=\frac{q-1}{2}$ if q is odd. We consider *the Rodrigues* formula ([18], [21])

$$H_q(x)=(-1)^q e^{x^2}\frac{d^q}{dx^q}\left(e^{-x^2}\right),q\in\mathbb{N}_0. \tag{9.9}$$

If one places $\sqrt{c}x$ on x into (9.8) and employs

$$\frac{d}{d(\sqrt{c}x)}(\cdot)=\frac{d}{dx}(\cdot)\frac{dx}{d(\sqrt{c}x)}=\frac{1}{\sqrt{c}}\frac{d}{dx}(\cdot), \tag{9.10}$$

then he proves *the generalized Rodrigues formula*

$$H_q\left(\sqrt{c}x\right)=(-1)^q c^{-\frac{q}{2}}e^{cx^2}\frac{d^q}{dx^q}\left(e^{-cx^2}\right),\quad c>0,\quad q\in\mathbb{N}_0. \tag{9.11}$$

In this paper we have $0\le q\le\left[\frac{p}{2}\right]$, $p\in\mathbb{N}$. From (9.11) with $f(x)=c_0e^{-cx^2}$, $c_0\in\mathbb{C}$, $c>0$, we get

$$\frac{d^q}{dx^q}f(x)=(-1)^q c^{\frac{q}{2}}f(x)H_q\left(\sqrt{c}x\right), \tag{9.12}$$

and thus the moment

$$\begin{aligned}(m_{2q})_{\left|f^{(q)}\right|^2}&=\int_{\mathbb{R}}x^{2q}\left|f^{(q)}(x)\right|^2dx\\&=\int_{\mathbb{R}}x^{2q}\left|(-1)^q c^{\frac{q}{2}}f(x)H_q\left(\sqrt{c}x\right)\right|^2dx\\&=|c_0|^2c^q\int_{\mathbb{R}}x^{2q}e^{-2cx^2}\left|H_q\left(\sqrt{c}x\right)\right|^2dx.\end{aligned} \tag{9.13}$$

Substituting $y=\sqrt{c}x$, $c>0$ into (9.13) one gets

$$(m_{2q})_{\left|f^{(q)}\right|^2}=\frac{|c_0|^2}{\sqrt{c}}\int_{\mathbb{R}}y^{2q}e^{-2y^2}H_q^2(y)\,dy. \tag{9.14}$$

We consider the Hermite polynomial

$$H_q(y)=\sum_{k=0}^{\left[\frac{q}{2}\right]}(-1)^k\frac{q!}{k!(q-2k)!}(2y)^{q-2k},\quad 0\le q\le\left[\frac{p}{2}\right], \tag{9.15}$$

and the Lagrange identity of the second form (7.3). Setting

$$r_k=(-1)^k\frac{q!}{k!(q-2k)!}2^{q-2k}=(-1)^k\frac{(2k)!}{k!}\binom{q}{2k}2^{q-2k}$$

with

$$\frac{(2k)!}{k!}\binom{q}{2k}=\frac{q!}{k!(q-2k)!},$$

and denoting

$$\begin{aligned}A_{qk}^*&=\left(\frac{(2k)!}{k!}\right)^2\binom{q}{2k}^2 2^{2(q-2k)}\in\mathbb{R},\\r_{qkj}^*&=4^{q-(k+j)}\in\mathbb{R},\\s_{qkj}^*&=(-1)^{k+j}\in\mathbb{R},\end{aligned}$$

and

$$B_{qkj}^*=s_{qkj}^*\frac{(2k)!(2j)!}{k!j!}\binom{q}{2k}\binom{q}{2j}\in\mathbb{R},$$

one gets that $r_k^2 = A_{qk}^*$ and $r_k r_j = r_{qkj}^* B_{qkj}^*$.

Thus employing (9.15) and substituting $z_k = y^{q-2k}$ in (7.3) we find

$$z_k z_j = y^{2(q-k-j)}, r_k z_k = (-1)^k \frac{q!}{k!\,(q-2k)!} (2y)^{q-2k},$$

and

$$H_q^2(y) = \left(\sum_{k=0}^{[q/2]} r_k z_k \right)^2 = \sum_{k=0}^{[q/2]} A_{qk}^* y^{2q-4k} + 2 \sum_{0 \le k < j \le \left[\frac{q}{2}\right]} r_{qkj}^* B_{qkj}^* y^{2(q-k-j)}, \tag{9.16}$$

for $0 \le q \le \left[\frac{p}{2}\right]$, $p \in \mathbb{N}$. Let us denote

$$J_{qk}^* = \int_{\mathbb{R}} y^{4(q-k)} e^{-2y^2} dy, J_{qkj}^* = \int_{\mathbb{R}} y^{2(2q-k-j)} e^{-2y^2} dy, \tilde{A}_{qk} = A_{qk}^* J_{qk}^*,$$

and $\tilde{B}_{qkj} = r_{qkj}^* B_{qkj}^* J_{qkj}^*$. Therefore from (9.14) and (9.16) one gets

$$(m_{2q})_{\left|f^{(q)}\right|^2} = \frac{|c_0|^2}{\sqrt{c}} \left(\sum_{k=0}^{[q/2]} \tilde{A}_{qk} + 2 \sum_{0 \le k < j \le [q/2]} \tilde{B}_{qkj} \right), \quad 0 \le q \le \left[\frac{p}{2}\right]. \tag{9.17}$$

From (9.1) and (9.5) we find

$$J_{qk}^* = \frac{(4(q-k)-1)!!}{16^{q-k}} \sqrt{\frac{\pi}{2}}, \text{ and } J_{qkj}^* = \frac{(2(2q-k-j)-1)!!}{4^{2q-k-j}} \sqrt{\frac{\pi}{2}}. \tag{9.18}$$

From (9.18) one gets

$$\tilde{A}_{qk} = \frac{1}{2^{2q}} \sqrt{\frac{\pi}{2}} \binom{q}{2k}^2 \left(\frac{(2k)!}{k!} \right)^2 (4q-4k-1)!!, \tag{9.19}$$

$$\tilde{B}_{qkj} = (-1)^{k+j} \frac{1}{2^{2q}} \sqrt{\frac{\pi}{2}} \binom{q}{2k} \binom{q}{2j} \frac{(2k)!\,(2j)!}{k!j!} (4q-2k-2j-1)!!, \tag{9.20}$$

$0 \le k < j \le \left[\frac{q}{2}\right]$,$0 \le q \le \left[\frac{p}{2}\right]$, $p \in \mathbb{N}$. From (9.5) one finds

$$(4q-4k-1)!! = \frac{2^{2q-2k}}{\sqrt{\pi}} \Gamma\left(2q-2k+\frac{1}{2}\right), \tag{9.21}$$

$$(4q-2k-2j-1)!! = \frac{2^{2q-k-j}}{\sqrt{\pi}} \Gamma\left(2q-k-j+\frac{1}{2}\right). \tag{9.22}$$

Also from (9.6) – (9.7) we get

$$\frac{(2p)!}{p!} = \frac{2^{2p}}{\sqrt{\pi}} \Gamma\left(p+\frac{1}{2}\right), \quad p \in \mathbb{N}. \tag{9.23}$$

Therefore from (9.19) – (9.22) and placing k, j on p into (9.23) we find

$$\tilde{A}_{qk} = \frac{1}{\pi\sqrt{2}} 2^{2k} \binom{q}{2k}^2 \Gamma^2\left(k+\frac{1}{2}\right) \Gamma\left(2q-2k+\frac{1}{2}\right), \tag{9.24}$$

$$\tilde{B}_{qkj} = \frac{1}{\pi\sqrt{2}} (-1)^{k+j} 2^{k+j} \binom{q}{2k} \binom{q}{2j} \times \Gamma\left(k+\frac{1}{2}\right) \Gamma\left(j+\frac{1}{2}\right) \Gamma\left(2q-k-j+\frac{1}{2}\right), \tag{9.25}$$

for all $0 \le k < j \le \left[\frac{q}{2}\right]$, $0 \le q \le \left[\frac{p}{2}\right]$, $p \in \mathbb{N}$. Let us denote

$$\begin{aligned}(9.26)\quad \Gamma_q = \sum_{k=0}^{[q/2]} 2^{2k} \binom{q}{2k}^2 \Gamma^2\left(k+\frac{1}{2}\right)\Gamma\left(2q-2k+\frac{1}{2}\right) \\ + 2\sum_{0\le k<j\le[q/2]} (-1)^{k+j}\, 2^{k+j} \binom{q}{2k}\binom{q}{2j} \\ \times \Gamma\left(k+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(2q-k-j+\frac{1}{2}\right).\end{aligned}$$

From (9.26) one gets

$$(9.27)\qquad \sum_{k=0}^{[q/2]} \tilde{A}_{qk} + 2\sum_{0\le k<j\le[q/2]} \tilde{B}_{qkj} = \frac{1}{\pi\sqrt{2}}\Gamma_q.$$

Thus from (9.17) and (9.27) we find

$$(9.28)\qquad (m_{2q})_{\left|f^{(q)}\right|^2} = \frac{|c_0|^2}{\sqrt{c}}\frac{1}{\pi\sqrt{2}}\Gamma_q,$$

for all $0 \le q \le \left[\frac{p}{2}\right]$, $p \in \mathbb{N}$, and $c_0 \in \mathbb{C}$, $c > 0$. Let us denote

$$(9.29)\qquad \Gamma_p^* = \left|\sum_{q=0}^{[p/2]} \varepsilon_{p,q}\Gamma_q\right|,$$

where $\varepsilon_{p,q}$ is given as in (8.38). Therefore from the $2p^{th}$ power of the right-hand side of the inequality (8.40) of the Corollary 8.2 and (9.28) – (9.29) we get

$$(9.30)\qquad \left(H_p^*\right)^{2p}\left(\sum_{q=0}^{[p/2]} \varepsilon_{p,q}\left(m_{2q}\right)_{\left|f^{(q)}\right|^2}\right)^2 = \left(H_p^*\right)^{2p}\frac{1}{2\pi^2}\left(\Gamma_p^*\right)^2\frac{|c_0|^4}{c},$$

for all $p \in \mathbb{N}$, $c_0 \in \mathbb{C}$, and $c > 0$ $\left(\text{where } H_p^* = \frac{1}{2\pi\sqrt[p]{2}}\right)$.

If $\hat{f} : \mathbb{R} \to \mathbb{C}$ is the Fourier transform of f of the form $f(x) = c_0 e^{-cx^2}$ ($c_0 \in \mathbb{C}$, $c > 0$), given as in the abstract, x_m the mean of x for $|f|^2$, and ξ_m the mean of ξ for $\left|\hat{f}\right|^2$, then

$$(9.31)\qquad |E_{p,f}| = \frac{|c_0|^2}{\sqrt{c}}\frac{1}{\pi\sqrt{2}}\Gamma_p^*\left(=\frac{E_{|f|^2}}{\pi\sqrt{\pi}}\Gamma_p^*\right),$$

for any fixed but arbitrary $p \in \mathbb{N}$. For instance, if $c_0 = 1$ and $c = \frac{1}{2}$, then

$$\Gamma_p^* = \pi\left|E_{p,f}\right|,\qquad p \in \mathbb{N}.$$

Therefore from (8.40), (9.3) and (9.30) we get the following Corollary 9.1.

Corollary 9.1. *Assume that* Γ *is the Euler gamma function defined by the formula* [21]

$$(9.32)\qquad \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt,\quad \operatorname{Re}(z) > 0,$$

whenever the complex variable $z = \operatorname{Re}(z) + i\operatorname{Im}(z)$, $i = \sqrt{-1}$, *has a positive real part* $\operatorname{Re}(z)$. *Denote* $\varepsilon_{p,q}$, Γ_q *and* Γ_q^* *as in (8.38), (9.26) and (9.29), respectively. Let us consider the* *non-negative real function* $\mathbb{R} : \mathbb{N} \to \mathbb{R}$*, such that*

$$R(p) = \frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma_p^*}, \quad p \in \mathbb{N} = \{1, 2, 3, \ldots\}.$$

Then the extremum principle

$$R(p) \geq \frac{1}{2\pi}, \tag{9.33}$$

holds for any fixed but arbitrary $p \in \mathbb{N}$*. Equality holds for* $p = 1$.

For instance, if $p \in \mathbb{N}_9 = \{1, 2, 3, \ldots, 9\}$, then

$$\frac{1}{2\pi} \leq R(p) \leq \frac{429}{23} \cdot \frac{1}{2\pi}.$$

9.1. **First nine cases of (9.33).**

i) If $p = 1$, then $q = 0$. Thus $\Gamma_0 = \Gamma^3\left(\frac{1}{2}\right) = \pi\sqrt{\pi}$, $\varepsilon_{1,0} = -1$, and $\Gamma_1^* = |\varepsilon_{1,0}\Gamma_0| = \pi\sqrt{\pi}$.
 But $\Gamma\left(1 + \frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$. Hence $R(1) = \frac{1}{2\pi}$. Therefore the *equality* in (9.33) holds for $p = 1$.

ii) If $p = 2$, then $q = 0, 1$.Thus

$$\begin{aligned} \Gamma_0 &= \pi\sqrt{\pi}, \quad \varepsilon_{2,0} = 2; \\ \Gamma_1 &= \Gamma^2\left(\frac{1}{2}\right)\Gamma\left(2 + \frac{1}{2}\right) = \frac{3}{4}\pi\sqrt{\pi}, \\ \varepsilon_{2,1} &= -2, \end{aligned}$$

and

$$\Gamma_2^* = |\varepsilon_{2,0}\Gamma_0 + \varepsilon_{2,1}\Gamma_1| = \frac{1}{2}\pi\sqrt{\pi}.$$

But $\Gamma\left(2 + \frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$. Hence $R(2) = 3 \cdot \frac{1}{2\pi}$.
 Therefore the inequality in (9.33) holds for $p = 2$.

iii) If $p = 3$, then $q = 0, 1$.Thus $\Gamma_0 = \pi\sqrt{\pi}$, $\varepsilon_{3,0} = -6$; $\Gamma_1 = \frac{3}{4}\pi\sqrt{\pi}$, $\varepsilon_{3,1} = 9$, and

$$\Gamma_3^* = |\varepsilon_{3,0}\Gamma_0 + \varepsilon_{3,1}\Gamma_1| = \frac{3}{4}\pi\sqrt{\pi}.$$

But $\Gamma\left(3 + \frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$. Hence $R(3) = 5 \cdot \frac{1}{2\pi}$.
 Therefore the inequality in (9.33) holds for $p = 3$.

iv) If $p = 4$, then $q = 0, 1, 2$. Thus $\Gamma_0 = \pi\sqrt{\pi}$, $\varepsilon_{4,0} = 24$; $\Gamma_1 = \frac{3}{4}\pi\sqrt{\pi}$, $\varepsilon_{4,1} = -48$; $\varepsilon_{4,2} = 2$,

$$\begin{aligned}\Gamma_2 &= \sum_{k=0}^{1} 2^{2k} \begin{pmatrix} 2 \\ 2k \end{pmatrix}^2 \Gamma^2 \left(k + \frac{1}{2}\right) \Gamma \left(4 - 2k + \frac{1}{2}\right) \\ &\quad + 2 \sum_{0 \le k < j \le 1} (-1)^{k+j} \, 2^{k+j} \begin{pmatrix} 2 \\ 2k \end{pmatrix} \begin{pmatrix} 2 \\ 2j \end{pmatrix} \\ &\quad \times \Gamma \left(k + \frac{1}{2}\right) \Gamma \left(j + \frac{1}{2}\right) \Gamma \left(4 - k - j + \frac{1}{2}\right) \\ &= \Gamma^2 \left(\frac{1}{2}\right) \Gamma \left(4 + \frac{1}{2}\right) + 2^2 \Gamma^2 \left(1 + \frac{1}{2}\right) \Gamma \left(2 + \frac{1}{2}\right) \\ &\quad + 2\,(-1)\,2\Gamma \left(\frac{1}{2}\right) \Gamma \left(1 + \frac{1}{2}\right) \Gamma \left(3 + \frac{1}{2}\right) \\ &= \frac{105 + 12 - 60}{16} \pi\sqrt{\pi} = \frac{57}{16} \pi\sqrt{\pi},\end{aligned}$$

because $(k, j) \in \{(0, 1)\}$, and

$$\begin{aligned}\Gamma_4^* &= |\varepsilon_{4,0}\Gamma_0 + \varepsilon_{4,1}\Gamma_1 + \varepsilon_{4,2}\Gamma_2| \\ &= \left| 24 + (-48) \frac{3}{4} + 2 \left(\frac{57}{16}\right) \right| \pi\sqrt{\pi} = \frac{39}{8} \pi\sqrt{\pi}.\end{aligned}$$

But $\Gamma\left(4 + \frac{1}{2}\right) = \frac{105}{16}\sqrt{\pi}$. Hence $R\,(4) = \frac{35}{13} \cdot \frac{1}{2\pi}$.

Therefore the inequality in (9.33) holds for $p = 4$.

v) If $p = 5$, then $q = 0, 1, 2$. Thus $\Gamma_5^* = \frac{255}{16}\pi\sqrt{\pi}$. But $\Gamma\left(5 + \frac{1}{2}\right) = \frac{945}{32}\sqrt{\pi}$. Hence $R\,(5) = \frac{63}{17} \cdot \frac{1}{2\pi}$. Therefore the inequality in (9.33) holds for $p = 5$.

vi) If $p = 6$, then $q = 0, 1, 2, 3$. Thus $\Gamma_6^* = \frac{855}{32}\pi\sqrt{\pi}$. But $\Gamma\left(6 + \frac{1}{2}\right) = \frac{10395}{64}\sqrt{\pi}$. Hence $R\,(6) = \frac{231}{19} \cdot \frac{1}{2\pi}$. Therefore the inequality in (9.33) holds for $p = 6$.

vii) If $p = 7$, then $q = 0, 1, 2, 3$. Thus $\Gamma_7^* = \frac{7245}{64}\pi\sqrt{\pi}$. But $\Gamma\left(7 + \frac{1}{2}\right) = \frac{135135}{128}\sqrt{\pi}$. Hence $R\,(7) = \frac{429}{23} \cdot \frac{1}{2\pi}$. Therefore the inequality in (9.33) holds for $p = 7$.

viii) If $p = 8$, then $q = 0, 1, 2, 3, 4$. Thus $\Gamma_8^* = \frac{192465}{128}\pi\sqrt{\pi}$. But $\Gamma\left(8 + \frac{1}{2}\right) = \frac{2027025}{256}\sqrt{\pi}$. Hence $R\,(8) = \frac{495}{47} \cdot \frac{1}{2\pi}$. Therefore the inequality in (9.33) holds for $p = 8$.

ix) If $p = 9$, then $q = 0, 1, 2, 3, 4$. Thus $\Gamma_9^* = \frac{2344545}{256}\pi\sqrt{\pi}$. But $\Gamma\left(9 + \frac{1}{2}\right) = \frac{34459425}{512}\sqrt{\pi}$.

Hence $R\,(9) = \frac{12155}{827} \cdot \frac{1}{2\pi}$. Therefore the inequality in (9.33) holds for $p = 9$.

In fact ,

$$\begin{aligned}&\varepsilon_{9,0} = -9!, \qquad \varepsilon_{9,1} = 9 \cdot \frac{9!}{2!}, \qquad \varepsilon_{9,2} = -\frac{9}{7}\frac{9!}{4!} \begin{pmatrix} 7 \\ 2 \end{pmatrix}, \\ &\varepsilon_{9,3} = \frac{9}{6}\frac{9!}{6!} \begin{pmatrix} 6 \\ 3 \end{pmatrix}, \qquad \varepsilon_{9,4} = -81\end{aligned}$$

and

$$\Gamma_9^* = |\varepsilon_{9,0}\Gamma_0 + \varepsilon_{9,1}\Gamma_1 + \varepsilon_{9,2}\Gamma_2 + \varepsilon_{9,3}\Gamma_3 + \varepsilon_{9,4}\Gamma_4|\,,$$

where $\Gamma_0 = \pi\sqrt{\pi}$, $\Gamma_1 = \frac{3}{4}\pi\sqrt{\pi}$, and $\Gamma_2 = \frac{57}{16}\pi\sqrt{\pi}$ from the above case iv). Besides

$$\begin{aligned}\Gamma_3 &= \sum_{k=0}^{1} 2^{2k} \binom{3}{2k}^2 \Gamma^2\left(k+\frac{1}{2}\right)\Gamma\left(6-2k+\frac{1}{2}\right) \\ &\quad + 2\sum_{0\le k<j\le 1} (-1)^{k+j}\, 2^{k+j} \binom{3}{2k}\binom{3}{2j} \\ &\qquad \times\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(6-k-j+\frac{1}{2}\right) \\ &= \Gamma^2\left(\frac{1}{2}\right)\Gamma\left(6+\frac{1}{2}\right) + 2^2\binom{3}{2}^2\Gamma^2\left(1+\frac{1}{2}\right)\Gamma\left(4+\frac{1}{2}\right) \\ &\quad + 2(-1)\,2\binom{3}{0}\binom{3}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\right)\Gamma\left(5+\frac{1}{2}\right) \\ &= \frac{2835}{64}\pi\sqrt{\pi},\end{aligned}$$

because $(k,j) \in \{(0,1)\}$, and

$$\begin{aligned}\Gamma_4 &= \sum_{k=0}^{2} 2^{2k} \binom{4}{2k}^2 \Gamma^2\left(k+\frac{1}{2}\right)\Gamma\left(8-2k+\frac{1}{2}\right) \\ &\quad + 2\sum_{0\le k<j\le 2} (-1)^{k+j}\, 2^{k+j} \binom{4}{2k}\binom{4}{2j} \\ &\qquad \times\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(8-k-j+\frac{1}{2}\right) \\ &= \frac{273105}{256}\pi\sqrt{\pi},\end{aligned}$$

because $(k,j) \in \{(0,1),(0,2),(1,2)\}$.We note that if one denotes $R^*(p) = 2\pi R(p)$, then he easily gets $R^*(p) \ge 1$ for any $p \in \mathbb{N}$.

Corollary 9.2. *Assume that Γ is defined by (9.32). Consider the Gaussian $f : \mathbb{R} \to \mathbb{C}$ such that $f(x) = c_0 e^{-cx^2}$, where c_0, c are fixed but arbitrary constants and $c_0 \in \mathbb{C}$, $c > 0$. Assume that x_m is the mean of x for $|f|^2$. Consider $\hat{f} : \mathbb{R} \to \mathbb{C}$ the Fourier transform of f, given as in the abstract and ξ_m the mean of ξ for $\left|\hat{f}\right|^2$. Denote $(m_{2q})_{\left|f^{(q)}\right|^2}$, the $2q^{th}$ moment of x for $\left|f^{(q)}\right|^2$ about the origin, as in (8.41), and the real constants $\varepsilon_{p,q}$ as in (8.38). Denote*

$$E_{|f|^2} = \int_{\mathbb{R}} |f(x)|^2\, dx$$

and

$$E_{p,f} = \sum_{q=0}^{[p/2]} \varepsilon_{p,q} (m_{2q})_{\left|f^{(q)}\right|^2}, \tag{9.34}$$

if $|E_{p,f}| < \infty$ holds for $0 \le q \le \left[\frac{p}{2}\right]$ and any fixed but arbitrary $p \in \mathbb{N}$.

Then the extremum principle

$$R_f(p) = \frac{\Gamma\left(p+\frac{1}{2}\right)}{|E_{p,f}|}\left(= \frac{\sqrt{\pi}}{2E_{|f|^2}} R^*(p)\right) \ge \frac{1}{|c_0|^2}\sqrt{\frac{c}{2}}\left(=\frac{\sqrt{\pi}}{2E_{|f|^2}}\right), \tag{9.35}$$

holds for any $p \in \mathbb{N}$. Equality holds for $p = 1$.

For instance, if $p \in \mathbb{N}_9 = \{1, 2, 3, \ldots, 9\}$, then $E_{p,f} > 0$ for $p = 2, 3, 5, 8$; and < 0 for $p = 1, 4, 6, 7, 9$. Besides

$$\frac{1}{|c_0|^2}\sqrt{\frac{c}{2}} \leq R_f(p) \leq \frac{429}{23} \cdot \frac{1}{|c_0|^2}\sqrt{\frac{c}{2}}, \text{ if } p \in \mathbb{N}_9.$$

The proof of Corollary 9.2 is a direct application of the above-mentioned formula (9.31) and the Corollary 9.1 (or (9.33)). In fact,

$$\begin{aligned} R_f(p) &= \Gamma\left(p + \frac{1}{2}\right) \Big/ |E_{p,f}| \\ &= \Gamma\left(p + \frac{1}{2}\right) \Big/ \frac{1}{\pi\sqrt{2}}\Gamma_p^* \frac{|c_0|^2}{\sqrt{c}} \\ &= \pi\sqrt{2}\frac{\sqrt{c}}{|c_0|^2} R(p) \\ &\geq \pi\frac{\sqrt{2c}}{|c_0|^2} \cdot \frac{1}{2\pi} = \frac{1}{|c_0|^2}\sqrt{\frac{c}{2}}. \end{aligned}$$

Besides from (8.26) one gets

$$E_{1,f} = -\int_{\mathbb{R}} |f(x)|^2 dx = -|c_0|^2 \int_{\mathbb{R}} e^{-2cx^2} dx = -\frac{|c_0|^2}{\sqrt{2c}}\sqrt{\pi} = -\frac{|c_0|^2}{\sqrt{2c}} 2\Gamma\left(1 + \frac{1}{2}\right),$$

or $R_f(1) = \frac{1}{|c_0|^2}\sqrt{\frac{c}{2}}$, completing the proof of Corollary 9.2. We note that if $c_0 = 1$, $c = \frac{1}{2}$, or $f(x) = e^{-\frac{1}{2}x^2}$, then $R_f(p) \geq \frac{1}{2}$.

Also we note that the formula (9.28) is an interesting formula on moments for Gaussians.

9.2. First nine cases of (9.35).

i) If $p = 1$, then

$$E_{|f|^2} = \int_{\mathbb{R}} |f(x)|^2 dx = |c_0|^2 \int_{\mathbb{R}} e^{-2cx^2} dx = \frac{|c_0|^2}{\sqrt{2c}}\sqrt{\pi}.$$

Thus from (8.26) we get $E_{1,f} = -E_{|f|^2}$. But $\Gamma\left(1 + \frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$. Hence

$$R_f(1) = \Gamma\left(1 + \frac{1}{2}\right) \Big/ |E_{1,f}| = \frac{1}{|c_0|^2}\sqrt{\frac{c}{2}} = \sqrt{\pi} \Big/ 2E_{|f|^2}.$$

Therefore the *equality* in (9.35) holds for $p = 1$.

We note from (9.28) and (9.31) that $q = 0$ such that

$$\Gamma_0 = \pi\frac{\sqrt{2c}}{|c_0|^2}(m_0)_{|f|^2} = \pi\frac{\sqrt{2c}}{|c_0|^2}E_{|f|^2} = \pi\sqrt{\pi},$$

and

$$\Gamma_1^* = \pi\frac{\sqrt{2c}}{|c_0|^2}|E_{1,f}| = \pi\frac{\sqrt{2c}}{|c_0|^2}E_{|f|^2} = \Gamma_0 = \pi\sqrt{\pi},$$

respectively.

ii) If $p=2$, then from (8.27) we get

$$E_{2,f}=2\left[E_{|f|^2}-\int_{\mathbb{R}}x^2\left|f'(x)\right|^2dx\right]=2\left[E_{|f|^2}-\frac{3}{4}E_{|f|^2}\right]=\frac{1}{2}E_{|f|^2}.$$

But $\Gamma\left(2+\frac{1}{2}\right)=\frac{3}{4}\sqrt{\pi}$. Hence

$$R_f(2)=\Gamma\left(2+\frac{1}{2}\right)\Big/\left|E_{2,f}\right|=3\cdot\frac{1}{\left|c_0\right|^2}\sqrt{\frac{c}{2}}.$$

Therefore the inequality in (9.35) holds for $p=2$. We note from (9.28) and (9.31) that $q=0,1$ such that $\Gamma_0=\pi\sqrt{\pi}$ as in the above case i),

$$\Gamma_1=\pi\frac{\sqrt{2c}}{\left|c_0\right|^2}(m_2)_{|f'|^2}=\pi\frac{\sqrt{2c}}{\left|c_0\right|^2}\int_{\mathbb{R}}x^2\left|f'(x)\right|^2dx=\pi\frac{\sqrt{2c}}{\left|c_0\right|^2}\frac{3}{4}E_{|f|^2}=\frac{3}{4}\pi\sqrt{\pi},$$

and

$$\Gamma_2^*=\pi\frac{\sqrt{2c}}{\left|c_0\right|^2}\left|E_{2,f}\right|=\pi\frac{\sqrt{2c}}{\left|c_0\right|^2}\frac{1}{2}E_{|f|^2}=\frac{1}{2}\Gamma_0=\frac{1}{2}\pi\sqrt{\pi},$$

respectively.

iii) If $p=3$, then from (8.28) we find

$$E_{3,f}=-3\left[2E_{|f|^2}-3\int_{\mathbb{R}}x^2\left|f'(x)\right|^2dx\right]=-3\left[2E_{|f|^2}-3\cdot\frac{3}{4}E_{|f|^2}\right]=\frac{3}{4}E_{|f|^2}.$$

But $\Gamma\left(3+\frac{1}{2}\right)=\frac{15}{8}\sqrt{\pi}$. Hence

$$R_f(3)=\Gamma\left(3+\frac{1}{2}\right)\Big/\left|E_{3,f}\right|=5\cdot\frac{1}{\left|c_0\right|^2}\sqrt{\frac{c}{2}}.$$

Therefore the inequality in (9.35) holds for $p=3$.

iv) If $p=4$, then from (8.29) one finds

$$\begin{aligned}E_{4,f}&=2\left[12E_{|f|^2}-24\int_{\mathbb{R}}x^2\left|f'(x)\right|^2dx+\int_{\mathbb{R}}x^4\left|f''(x)\right|^2dx\right]\\&=2\left[12-24\cdot\frac{3}{4}+\frac{57}{16}\right]E_{|f|^2}=-\frac{39}{8}E_{|f|^2}<0.\end{aligned}$$

Hence

$$R_f(4)=\frac{35}{13}\cdot\frac{1}{\left|c_0\right|^2}\sqrt{\frac{c}{2}}.$$

Therefore the inequality in (9.35) holds for $p=4$.

v) If $p=5$, then from (8.42) one gets

$$\begin{aligned}E_{5,f}&=-5\left[24E_{|f|^2}-60\int_{\mathbb{R}}x^2\left|f'(x)\right|^2dx+5\int_{\mathbb{R}}x^4\left|f''(x)\right|^2dx\right]\\&=-5\left[24-60\cdot\frac{3}{4}+5\cdot\frac{57}{16}\right]E_{|f|^2}=\frac{255}{16}E_{|f|^2}.\end{aligned}$$

Hence

$$R_f(5)=\frac{63}{17}\cdot\frac{1}{\left|c_0\right|^2}\sqrt{\frac{c}{2}}.$$

Therefore the inequality in (9.35) holds for $p=5$.

vi) If $p = 6$, then from (8.43) we get

$$\begin{aligned} E_{6,f} &= 2\Big[360 E_{|f|^2} - 1080 \int_{\mathbb{R}} x^2 \left|f'(x)\right|^2 dx \\ &\qquad + 135 \int_{\mathbb{R}} x^4 \left|f''(x)\right|^2 dx - \int_{\mathbb{R}} x^6 \left|f'''(x)\right|^2 dx\Big] \\ &= 2\left[360 - 1080 \cdot \frac{3}{4} + 135 \cdot \frac{57}{16} - \frac{2835}{64}\right] E_{|f|^2} \\ &= -\frac{855}{32} E_{|f|^2} < 0. \end{aligned}$$

Hence

$$R_f(6) = \frac{231}{19} \cdot \frac{1}{|c_0|^2} \sqrt{\frac{c}{2}}.$$

Therefore the inequality in (9.35) holds for $p = 6$.

vii) If $p = 7$, then from (8.44) one obtains

$$\begin{aligned} E_{7,f} &= -7\Big[720 E_{|f|^2} - 2520 \int_{\mathbb{R}} x^2 \left|f'(x)\right|^2 dx \\ &\qquad + 420 \int_{\mathbb{R}} x^4 \left|f''(x)\right|^2 dx - 7 \int_{\mathbb{R}} x^6 \left|f'''(x)\right|^2 dx\Big] \\ &= -7\left[720 - 2520 \cdot \frac{3}{4} + 420 \cdot \frac{57}{16} - 7 \cdot \frac{2835}{64}\right] E_{|f|^2} \\ &= -\frac{7245}{64} E_{|f|^2} < 0. \end{aligned}$$

Hence

$$R_f(7) = \frac{429}{23} \cdot \frac{1}{|c_0|^2} \sqrt{\frac{c}{2}}.$$

Therefore the inequality in (9.35) holds for $p = 7$.

viii) If $p = 8$, then from (8.45) we obtain

$$\begin{aligned} E_{8,f} &= 2\Big[20160 E_{|f|^2} - 80640 \int_{\mathbb{R}} x^2 \left|f'(x)\right|^2 dx + 16800 \int_{\mathbb{R}} x^4 \left|f''(x)\right|^2 dx \\ &\qquad - 448 \int_{\mathbb{R}} x^6 \left|f'''(x)\right|^2 dx + \int_{\mathbb{R}} x^8 \left|f^{(4)}(x)\right|^2 dx\Big] \\ &= 2\left[20160 - 80640 \cdot \frac{3}{4} + 16800 \cdot \frac{57}{16} - 448 \cdot \frac{2835}{64} + \frac{273105}{256}\right] E_{|f|^2} \\ &= \frac{192465}{128} E_{|f|^2}. \end{aligned}$$

Hence

$$R_f(8) = \frac{495}{47} \cdot \frac{1}{|c_0|^2} \sqrt{\frac{c}{2}}.$$

Therefore the inequality in (9.35) holds for $p = 8$.

ix) If $p = 9$, then from (8.46) one finds

$$E_{9,f} = -9\left[40320E_{|f|^2} - 181440\int_{\mathbb{R}} x^2 |f'(x)|^2 dx + 45360\int_{\mathbb{R}} x^4 \left|f''(x)\right|^2 dx \right.$$
$$\left. -1680\int_{\mathbb{R}} x^6 \left|f'''(x)\right|^2 dx + 9\int_{\mathbb{R}} x^8 \left|f^{(4)}(x)\right|^2 dx\right]$$
$$= 9\left[40320 - 181440\cdot\frac{3}{4} + 45360\cdot\frac{57}{16} - 1680\cdot\frac{2835}{64} + 9\cdot\frac{273105}{256}\right]E_{|f|^2}$$
$$= -\frac{2344545}{256}E_{|f|^2} < 0.$$

Hence

$$R_f(9) = \frac{12155}{827}\cdot\frac{1}{|c_0|^2}\sqrt{\frac{c}{2}}.$$

Therefore the inequality in (9.35) holds for $p = 9$.

We note that, from the Corollary 9.1,

$$R_f(p) = 2\pi R(p) = R^*(p) \geq 1$$

for any $p \in \mathbb{N}$, if

$$f(x) = e^{-2x^2} \quad \left(\text{or } E_{|f|^2} = \frac{\sqrt{\pi}}{2}\right),$$

because $|E_{p,f}| = \frac{1}{2\pi}\Gamma_p^*$, from (9.31).

Corollary 9.3. *Assume that the Euler gamma function* Γ *is defined by (9.32). Consider the Gaussian function* $f : \mathbb{R} \to \mathbb{C}$ *of the form* $f(x) = c_0 e^{-c(x-x_0)^2}$*, where* c_0*,* c*,*x_0 *are fixed but arbitrary constants and* $c_0 \in \mathbb{C}$*,* $c > 0$*,* $x_0 \in \mathbb{R}$*. Assume that the mathematical expectation* $E(x - x_0)$ *of* $x - x_0$ *for* $|f|^2$ *equals to*

$$x_m = \int_{\mathbb{R}} (x - x_0)|f(x)|^2 dx = 0.$$

Consider the Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{C}$ *of* f*, given as in the abstract of this paper, and* ξ_m *the mean of* ξ *for* $\left|\hat{f}\right|^2$*. Denote by* $(m_{2q})_{\left|f^{(q)}\right|^2}$ *the* $2q^{th}$ *moment of* x *for* $\left|f^{(q)}\right|^2$ *about the origin, as in (8.41), and the constants* $\varepsilon_{p,q}$ *as in (8.38). Consider*

$$E_{p,f} = \sum_{q=0}^{[p/2]} \varepsilon_{p,q}(m_{2q})_{\left|f^{(q)}\right|^2},$$

if $|E_{p,f}| < \infty$ *for* $0 \leq q \leq \left[\frac{p}{2}\right]$*, and any fixed but arbitrary* $p \in \mathbb{N}$*. If* $\frac{d^{2p}}{dx_0^{2p}}e^{2cx_0^2}$ (> 0) *denotes the* $2p^{th}$ *order derivative of* $e^{2cx_0^2}$ *with respect to* x_0*, then the extremum principle*

$$(9.36)\qquad |E_{p,f}| \leq \frac{\sqrt[4]{\pi}}{2^{\frac{3p-1}{2}}}\cdot\Gamma^{\frac{1}{2}}\left(p+\frac{1}{2}\right)\cdot\frac{|c_0|^2}{c^{\frac{p+1}{2}}}\cdot e^{-cx_0^2}\cdot\left(\frac{d^{2p}}{dx_0^{2p}}e^{2cx_0^2}\right)^{\frac{1}{2}},$$

holds for any fixed but arbitrary $p \in \mathbb{N}$*. Equality holds for* $p = 1$ *and* $x_0 = 0$*.*

We note that $x_m = 0$ even if $x_0 \neq 0$, while in the following Corollary 9.4 we have $x_m = 0$ only if $x_0 = 0$.

Proof. At first, we claim that *the general integral formula*

$$\int_{\mathbb{R}} x^{2p} e^{-2c(x-x_0)^2} dx = \frac{\sqrt{\pi}}{2^{4p+\frac{1}{2}}} \cdot \frac{1}{c^{2p+\frac{1}{2}}} \cdot e^{-2cx_0^2} \cdot \frac{d^{2p}}{dx_0^{2p}} e^{2cx_0^2}, \; c > 0, \; x_0 \in \mathbb{R}, \tag{9.37}$$

holds for all $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We note that, if $x_0 = 0$, then (9.1) follows. For example, if $p = 1$ and $x_0 = 0$, then

$$\frac{d^2}{dx_0^2} e^{2cx_0^2} = \frac{d}{dx_0} \left(4cx_0 e^{2cx_0^2} \right) = 4c \left(1 + 4cx_0^2 \right) e^{2cx_0^2} = 4c.$$

Thus (9.37) yields

$$\int_{\mathbb{R}} x^2 e^{-2cx^2} dx = \frac{1}{(4c)^2} \sqrt{\frac{\pi}{2c}} \cdot 1 \cdot 4c = \frac{1}{4c} \sqrt{\frac{\pi}{2c}}.$$

This equals to $\frac{\Gamma\left(1+\frac{1}{2}\right)}{(2c)^{1+\frac{1}{2}}}$ because

$$\Gamma \left(\frac{3}{2} \right) = \frac{1}{2} \Gamma \left(\frac{1}{2} \right) = \frac{\sqrt{\pi}}{2}$$

implying (9.1). A direct proof for this goes, as follows:

$$\begin{aligned} \int_{\mathbb{R}} x^2 e^{-2cx^2} dx &= \frac{1}{-4c} \int_{\mathbb{R}} x d \left(e^{-2cx^2} \right) \\ &= \frac{1}{-4c} \left[x e^{-2cx^2} \left.\right|_{\mathbb{R}} - \int_{\mathbb{R}} e^{-2cx^2} dx \right] \\ &= \frac{1}{4c} \int_{\mathbb{R}} e^{-2cx^2} dx \\ &= \frac{1}{4c\sqrt{2c}} \int_{\mathbb{R}} e^{-\left(\sqrt{2c}x\right)^2} d \left(\sqrt{2c}x \right) \\ &= \frac{1}{4c\sqrt{2c}} \sqrt{\pi}, \end{aligned}$$

because $|x| \, e^{-2cx^2} \to 0$, as $|x| \to \infty$. It is easy to prove *the integral formula* [21]

$$\int_{\mathbb{R}} x^n e^{-(x-x_0)^2} dx = (2i)^{-n} \sqrt{\pi} H_n \left(i x_0 \right), \quad i = \sqrt{-1}, \; x_0 \in \mathbb{R}, \tag{9.38}$$

for all $n \in \mathbb{N}_0$, where H_n is *the Hermite polynomial* ([18], [21]).

We note that if x_m is *the mean* of x for $|f|^2$and $x_0 \in \mathbb{R} - \{0\}$, then

$$\begin{aligned} x_m &= \int_{\mathbb{R}} x \, |f(x)|^2 \, dx \\ &= |c_0|^2 \int_{\mathbb{R}} x e^{-2c(x-x_0)^2} dx \\ &= -\frac{|c_0|^2}{4c} \int_{\mathbb{R}} d \left(e^{-2c(x-x_0)^2} \right) + x_0 \, |c_0|^2 \int_{\mathbb{R}} e^{-2c(x-x_0)^2} dx, \end{aligned}$$

or

$$x_m = x_0 E_{|f|^2} = \sqrt{\frac{\pi}{2}} \frac{|c_0|^2}{\sqrt{c}} x_0, \tag{9.39}$$

because

$$E_{|f|^2} = \int_{\mathbb{R}} |f(x)|^2 \, dx = |c_0|^2 \int_{\mathbb{R}} e^{-2c(x-x_0)^2} dx = \frac{|c_0|^2}{\sqrt{2c}} \int_{\mathbb{R}} e^{-\left(\sqrt{2c}(x-x_0)\right)^2} d\left(\sqrt{2c}\,(x-x_0)\right),$$

or

$$E_{|f|^2} = \sqrt{\frac{\pi}{2}} \frac{|c_0|^2}{\sqrt{c}}. \tag{9.40}$$

On the other hand, if *the mathematical expectation* of $x - x_0$ for $|f|^2$ is

$$E(x - x_0) = x_m = \int_{\mathbb{R}} (x - x_0) |f(x)|^2 \, dx,$$

then from (9.39) – (9.40) one gets

$$x_m = \int_{\mathbb{R}} x |f(x)|^2 \, dx - x_0 \int_{\mathbb{R}} |f(x)|^2 \, dx = x_0 E_{|f|^2} - x_0 E_{|f|^2},$$

or

$$x_m = 0. \tag{9.41}$$

In this case the mathematical expectation x_m is the mean of x for $|f|^2$only if $x_0 = 0$.

We note that if one places $q = 2p$ and ix_0 $\left(i = \sqrt{-1}\right)$ on x into (9.9) and employs

$$\frac{d}{d(ix_0)}(\cdot) = \frac{d}{dx_0}(\cdot) \frac{dx_0}{d(ix_0)} = -i \frac{d}{dx_0}(\cdot),$$

and thus

$$\frac{d^{2p}}{d(ix_0)^{2p}}(\cdot) = (-1)^p \frac{d^{2p}}{dx_0^{2p}}(\cdot) \tag{9.42}$$

then he proves

$$H_{2p}(ix_0) = (-1)^p e^{-x_0^2} \frac{d^{2p}}{dx_0^{2p}} e^{x_0^2}, \quad p \in \mathbb{N}_0. \tag{9.43}$$

Therefore from (9.38) with $n = 2p$, and (9.43) one gets that the integral formula

$$\int_{\mathbb{R}} x^{2p} e^{-(x-x_0)^2} dx = \frac{\sqrt{\pi}}{2^{2p}} \cdot e^{-x_0^2} \cdot \frac{d^{2p}}{dx_0^{2p}} e^{x_0^2}, \tag{9.44}$$

holds for $x_0 \in \mathbb{R}$, and all $p \in \mathbb{N}_0$.

If one substitutes $\frac{s}{\sqrt{2c}}$, $s \in \mathbb{R}$ on x into the following general integral he finds from (9.44) that

$$\int_{\mathbb{R}} x^{2p} e^{-2c(x-x_0)^2} dx = \frac{1}{(2c)^{p+\frac{1}{2}}} \cdot \int_{\mathbb{R}} s^{2p} e^{-\left(s-\sqrt{2c}x_0\right)^2} ds$$
$$= \frac{1}{(2c)^{p+\frac{1}{2}}} \cdot \frac{\sqrt{\pi}}{2^{2p}} \cdot e^{-\left(\sqrt{2c}x_0\right)^2} \cdot \frac{d^{2p}}{d\left(\sqrt{2c}x_0\right)^{2p}} e^{\left(\sqrt{2c}x_0\right)^2}$$
$$= \frac{\sqrt{\pi}}{2^{3p+\frac{1}{2}}} \frac{1}{c^{p+\frac{1}{2}}} \cdot e^{-2cx_0^2} \cdot \frac{d^{2p}}{d\left(\sqrt{2c}x_0\right)^{2p}} e^{2cx_0^2}, \quad c > 0, \quad x_0 \in \mathbb{R}, \tag{9.45}$$

holds for all $p \in \mathbb{N}_0$.

However,

$$\frac{d}{d\left(\sqrt{2c}x_0\right)}\left(\cdot\right)=\frac{d}{dx_0}\left(\cdot\right)\frac{dx_0}{d\left(\sqrt{2c}x_0\right)}=\frac{1}{\left(2c\right)^{\frac{1}{2}}}\cdot\frac{d}{dx_0}\left(\cdot\right),$$

and

$$\frac{d^{2p}}{d\left(\sqrt{2c}x_0\right)^{2p}}\left(\cdot\right)=\frac{1}{\left(2c\right)^{p}}\frac{d^{2p}}{dx_0^{2p}}\left(\cdot\right),\quad c>0, x_0\in\mathbb{R}, \tag{9.46}$$

hold for all $p\in\mathbb{N}_0$. Therefore from (9.45) and (9.46) we complete the proof of (9.37).

Second, from Gasquet et al. [8, p.157-161] we claim that the Fourier transform $\hat{f}:\mathbb{R}\to\mathbb{C}$ is of the form

$$\hat{f}\left(\xi\right)=c_0\sqrt{\frac{\pi}{c}}e^{-\frac{\pi^2}{c}\xi^2-i2\pi x_0\xi},\ \ c_0\in\mathbb{C},\ \ c>0,\ \ x_0\in\mathbb{R}. \tag{9.47}$$

In fact, differentiating the Gaussian function $f:\mathbb{R}\to\mathbb{C}$ of the form $f\left(x\right)=c_0e^{-c\left(x-x_0\right)^2}$ with respect to x, one gets

$$f'\left(x\right)=-2c\left(x-x_0\right)f\left(x\right)=-2cxf\left(x\right)+2cx_0f\left(x\right).$$

Thus the Fourier transform of f' is

$$Ff'\left(\xi\right)=F\left[f'\left(x\right)\right]\left(\xi\right)=\left[f'\left(x\right)\right]^{\wedge}\left(\xi\right)=\left[-2cxf\left(x\right)\right]^{\wedge}\left(\xi\right)+\left[2cx_0f\left(x\right)\right]^{\wedge}\left(\xi\right),$$

or

$$2i\pi\xi\hat{f}\left(\xi\right)=\frac{-2c}{-2i\pi}\left[\left(-2i\pi x\right)f\left(x\right)\right]^{\wedge}\left(\xi\right)+2cx_0\hat{f}\left(\xi\right), \tag{9.48}$$

by standard formulas on differentiation, from Gasquet et al [8, p. 157]. Thus

$$2i\pi\xi\hat{f}\left(\xi\right)=\frac{c}{i\pi}\left(\hat{f}\left(\xi\right)\right)'+2cx_0\hat{f}\left(\xi\right),$$

or

$$-2\pi^2\xi\hat{f}\left(\xi\right)=c\hat{f}'\left(\xi\right)+2i\pi cx_0\hat{f}\left(\xi\right),$$

or

$$\left(\hat{f}\left(\xi\right)\right)'=\hat{f}'\left(\xi\right)=-\frac{2\pi}{c}\left(\pi\xi+icx_0\right)\hat{f}\left(\xi\right). \tag{9.49}$$

Solving the first order ordinary differential equation (9.49) by the method [25] of the separation of variables we get the general solution

$$\hat{f}\left(\xi\right)=K\left(\xi\right)e^{-\frac{\pi^2}{c}\xi^2-i2\pi x_0\xi}, \tag{9.50}$$

such that $\hat{f}\left(0\right)=K\left(0\right)$. Differentiating the formula (9.50) with respect to ξ one finds

$$\hat{f}'\left(\xi\right)=e^{-\frac{\pi^2}{c}\xi^2-i2\pi x_0\xi}\left[K'\left(\xi\right)+K\left(\xi\right)\left(-\frac{2\pi^2}{c}\xi-i2\pi x_0\right)\right]. \tag{9.51}$$

From (9.48), (9.49), (9.50) and (9.51) we find $0=K'\left(\xi\right)e^{-\frac{\pi^2}{c}\xi^2-i2\pi x_0\xi}$, or $K'\left(\xi\right)=0$, or

$$K\left(\xi\right)=K, \tag{9.52}$$

which is a constant. But from (9.50) and (9.52) one gets

$$\hat{f}\left(0\right)=K\left(0\right)=K. \tag{9.53}$$

Besides from the definition of the Fourier transform we get

$$\begin{aligned}\hat{f}(0) &= \int_{\mathbb{R}} e^{-2i\pi\cdot 0\cdot x} f(x)\,dx \\ &= \int_{\mathbb{R}} f(x)\,dx \\ &= c_0 \int_{\mathbb{R}} e^{-c(x-x_0)^2}dx \\ &= \frac{c_0}{\sqrt{c}} \int_{\mathbb{R}} e^{-\left[\sqrt{c}(x-x_0)\right]^2} d\left(\sqrt{c}\,(x-x_0)\right),\end{aligned}$$

or

(9.54) $$\hat{f}(0) = c_0\sqrt{\frac{\pi}{c}},\;\; c_0\in\mathbb{C},\;\; c>0.$$

From (9.53) and (9.54) one finds

(9.55) $$K = c_0\sqrt{\frac{\pi}{c}},\;\; c_0\in\mathbb{C},\;\; c>0.$$

Therefore from (9.50) and (9.55) we complete the proof of the formula (9.47). *Another proof* of (9.47) is by employing the formula (9.2) for *the special Gaussian* $\phi(x) = c_0 e^{-cx^2}$, such that

$$\hat{\phi}(\xi) = c_0\sqrt{\frac{\pi}{c}} e^{-\frac{\pi^2}{c}\xi^2}, \quad c_0\in\mathbb{C},\;\; c>0.$$

In fact, $f(x) = \phi(x-x_0)$, or

$$\hat{f}(\xi) = \left[\phi(x-x_0)\right]^{\wedge}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x}\phi(x-x_0)\,dx = \int_{\mathbb{R}} e^{-2i\pi\xi(x+x_0)}\phi(x)\,dx$$

(with $x+x_0$ on x)

$$= e^{-2i\pi\xi x_0}\int_{\mathbb{R}} e^{-2i\pi\xi x}\phi(x)\,dx = e^{-2i\pi\xi x_0}\hat{\phi}(\xi), \quad \text{or} \quad \hat{f}(\xi) = e^{-2i\pi x_0\xi} c_0\sqrt{\frac{\pi}{c}} e^{-\frac{\pi^2}{c}\xi^2},$$

establishing (9.47).

Therefore from (8.36) – (8.37) with $x_m = 0$, from (9.41), and the mean of ξ for $\left|\hat{f}\right|^2$ of the form

$$\xi_m = \int_{\mathbb{R}} \xi\left|\hat{f}(\xi)\right|^2 d\xi = |c_0|^2\frac{\pi}{c}\int_{\mathbb{R}} \xi\cdot e^{-2\frac{\pi^2}{c}\xi^2}d\xi = 0,$$

as well as from (9.1), (9.37) and (9.47), one finds that the left-hand side of the inequality (8.40) of Corollary 8.2 is

$$\begin{aligned}(m_{2p})_{|f|^2}(m_{2p})_{\left|\hat{f}\right|^2} &= \left(\int_{\mathbb{R}} x^{2p}|f(x)|^2 dx\right)\cdot\left(\int_{\mathbb{R}} \xi^{2p}\left|\hat{f}(\xi)\right|^2 d\xi\right) \\ &= |c_0|^4\frac{\pi}{c}\left(\int_{\mathbb{R}} x^{2p}e^{-2c(x-x_0)^2}dx\right)\cdot\left(\int_{\mathbb{R}} \xi^{2p}e^{-2c^*\xi^2}d\xi\right)\left(\text{where } c^* = \frac{\pi^2}{c}\right) \\ &= |c_0|^4\frac{\pi}{c}\left(\frac{\sqrt{\pi}}{2^{4p+\frac{1}{2}}}\cdot\frac{1}{c^{2p+\frac{1}{2}}}\cdot e^{-2cx_0^2}\cdot\frac{d^{2p}}{dx_0^{2p}}e^{2cx_0^2}\right)\left(\frac{\Gamma\left(p+\frac{1}{2}\right)}{(2c^*)^{p+\frac{1}{2}}}\right) \\ &= \left(H_p^*\right)^{2p}\frac{\sqrt{\pi}}{2^{3p-1}}\Gamma\left(p+\frac{1}{2}\right)\cdot\frac{|c_0|^4}{c^{p+1}}\cdot e^{-2cx_0^2}\frac{d^{2p}}{dx_0^{2p}}e^{2cx_0^2},\end{aligned}$$

$\left(\text{with } H_p^* = 1\left/2\pi\sqrt[p]{2}\right.\right)$ holds for all $p\in\mathbb{N}$, $c_0\in\mathbb{C}$, $c>0$, and $x_0\in\mathbb{R}$.

Finally from the right-hand side of the inequality (8.40) of Corollary 8.2 with

$$E_{p,f} = \sum_{q=0}^{[p/2]} \varepsilon_{p,q} \left(\int_{\mathbb{R}} x^{2q} \left| f^{(q)} \right|^2 dx \right)$$

such that $|E_{p,f}| < \infty$ and

$$(m_{2p})_{|f|^2} (m_{2p})_{|\hat{f}|^2} \geq \left(H_p^* \right)^{2p} E_{p,f}^2,$$

for any fixed but arbitrary $p \in \mathbb{N}$, one completes the proof of the extremum principle (9.36). □

Corollary 9.4. *Assume that the Euler gamma function* Γ *is defined by (9.32). Consider the Gaussian function* $f : \mathbb{R} \to \mathbb{C}$ *of the form* $f(x) = c_0 e^{-c(x-x_0)^2}$, *where* c_0, c,x_0 *are fixed but arbitrary constants and* $c_0 \in \mathbb{C}$, $c > 0$, $x_0 \in \mathbb{R}$. *Assume that* x_m *is the mean of* x *for* $|f|^2$. *Consider the Fourier transform* $\hat{f} : \mathbb{R} \to \mathbb{C}$ *of* f, *given as in the abstract, and* ξ_m *the mean of* ξ *for* $\left| \hat{f} \right|^2$. *Consider the* $2q^{th}$ *moment of* x *for* $\left| f^{(q)} \right|^2$*by*

$$(\mu_{2q})_{\left| f^{(q)} \right|^2} = \int_{\mathbb{R}} (x - x_m)^{2q} \left| f^{(q)}(x) \right|^2 dx,$$

the constants $\varepsilon_{p,q}$ *as in (8.38), and*

$$E_{p,f} = \sum_{q=0}^{[p/2]} \varepsilon_{p,q} (\mu_{2q})_{\left| f^{(q)} \right|^2},$$

if $|E_{p,f}| < \infty$ *holds for* $0 \leq q \leq \left[\frac{p}{2} \right]$, *and any fixed but arbitrary* $p \in \mathbb{N}$. *If*

$$E_{|f|^2}^* = 1 - E_{|f|^2} \left(= 1 - \sqrt{\frac{\pi}{2}} \frac{|c_0|^2}{\sqrt{c}} \right)$$

and $x_0^* = x_0 E_{|f|^2}^*$, *then the extremum principle*

$$|E_{p,f}| \leq \frac{\sqrt[4]{\pi}}{2^{\frac{3p-1}{2}}} \cdot \Gamma^{\frac{1}{2}} \left(p + \frac{1}{2} \right) \cdot \frac{|c_0|^2}{c^{\frac{p+1}{2}}} \cdot \left(E_{|f|^2}^* \right)^{-p} \cdot e^{-c x_0^{*2}} \cdot \left(\frac{d^{2p}}{dx_0^{2p}} e^{2c x_0^{*2}} \right)^{\frac{1}{2}}, \tag{9.56}$$

holds for any fixed but arbitrary $p \in \mathbb{N}$.

Equality holds for $p = 1$; $x_0 = 0$, *or for* $p = 1$; $E_{|f|^2} = 1$.

We note that $x_m = 0$ only if $x_0 = 0$, while in the previous Corollary 9.3 we have $x_m = 0$ even if $x_0 \neq 0$.We may call $E_{|f|^2}$ and $E_{|f|^2}^*$ *complementary probability* (*or energy*) *integrals*.

Proof. It is clear, that from (9.1), (9.37), (9.39), (9.40) and (9.47) that

$$\begin{aligned}
&(\mu_{2p})_{|f|^2} (m_{2p})_{|\hat{f}|^2} \\
&= \left(\int_{\mathbb{R}} \left(x - x_0 E_{|f|^2} \right)^{2p} |f(x)|^2 dx \right) \cdot \left(\int_{\mathbb{R}} \xi^{2p} \left| \hat{f}(\xi) \right|^2 d\xi \right) \left(\text{where } \ x_m = x_0 E_{|f|^2} \right) \\
&= \frac{|c_0|^4}{c} \pi \left(\int_{\mathbb{R}} \left(x - x_0 E_{|f|^2} \right)^{2p} e^{-2c(x-x_0)^2} dx \right) \cdot \left(\int_{\mathbb{R}} \xi^{2p} e^{-2c^* \xi^2} d\xi \right) \left(\text{where } \ c^* = \frac{\pi^2}{c} \right).
\end{aligned}$$

By placing $x+x_0E_{|f|^2}$on x and letting $x_0^*=x_0\left(1-E_{|f|^2}\right)=x_0E^*_{|f|^2}$ we have

$$\left(\mu_{2p}\right)_{|f|^2}\left(m_{2p}\right)_{\left|\hat{f}\right|^2}=\pi\frac{|c_0|^4}{c}\left(\int_{\mathbb{R}}x^{2p}e^{-2c\left(x-x_0^*\right)^2}dx\right)\left(\frac{\Gamma\left(p+\frac{1}{2}\right)}{(2c^*)^{p+\frac{1}{2}}}\right)$$
$$=\pi\frac{|c_0|^4}{c}\left(\frac{\sqrt{\pi}}{2^{4p+\frac{1}{2}}}\frac{1}{c^{2p+\frac{1}{2}}}\cdot e^{-2cx_0^{*2}}\cdot\frac{d^{2p}}{dx_0^{*2p}}e^{2cx_0^{*2}}\right)\left(\frac{\Gamma\left(p+\frac{1}{2}\right)}{(2\pi^2/c)^{p+\frac{1}{2}}}\right).$$

However,

$$\frac{d^{2p}}{dx_0^{*2p}}(\cdot)=\frac{1}{\left(E^*_{|f|^2}\right)^{2p}}\frac{d^{2p}}{dx_0^{2p}}(\cdot),$$

and

$$\left(H_p^*\right)^{2p}=\frac{1}{2^{2(p+1)}\pi^{2p}}.$$

Therefore

$$(9.57)\qquad \left(m_{2p}\right)_{|f|^2}\left(m_{2p}\right)_{\left|\hat{f}\right|^2}$$
$$=\left(H_p^*\right)^{2p}\frac{\sqrt{\pi}}{2^{3p-1}}\Gamma\left(p+\frac{1}{2}\right)\cdot\frac{|c_0|^4}{c^{p+1}}\left(E^*_{|f|^2}\right)^{-2p}\cdot e^{-2cx_0^{*2}}\cdot\frac{d^{2p}}{dx_0^{2p}}e^{2cx_0^{*2}}$$
$$\left(\geq\left(H_p^*\right)^{2p}\left|E_{p,f}\right|^2\ \text{from our above theorem}\right),$$

completing the proof of Corollary 9.4. □

9.3. **Two Special Cases of (9.56).**

(i) If $p=1$, then

$$\frac{d^2}{dx_0^2}e^{2cx_0^{*2}}=\frac{d}{dx_0}\left(4cx_0^*e^{2cx_0^{*2}}\frac{dx_0^*}{dx_0}\right)$$
$$=4c\frac{d}{dx_0^*}\left(x_0^*e^{2cx_0^{*2}}\right)\left(\frac{dx_0^*}{dx_0}\right)^2$$
$$=4cE^{*2}_{|f|^2}\left(1+4cx_0^{*2}\right)e^{2cx_0^{*2}},\quad\text{where }\frac{dx_0^*}{dx_0}=E^*_{|f|^2}.$$

Therefore at $x_0=0$ (or $x_0^*=0$), $d^2e^{2cx_0^{*2}}/dx_0^2=4cE^{*2}_{|f|^2}$. If we denote by R.H.S. the right hand side of (9.56), then

$$(9.58)\qquad \text{R.H.S. }(\text{for }p=1;\ x_0=0\text{ or }x_0^*=0)=\frac{\sqrt[4]{\pi}}{2}\sqrt{\Gamma\left(\frac{3}{2}\right)}\frac{|c_0|^2}{c}\left(E^*_{|f|^2}\right)^{-1}\sqrt{4cE^{*2}_{|f|^2}}$$
$$=\sqrt{\frac{\pi}{2}}\frac{|c_0|^2}{\sqrt{c}}=E_{|f|^2}.$$

We note at $x=x_0=0$ one can get from (9.39) that $x_m=x_0E_{|f|^2}=0$.

But we have $\xi_m=0$. Therefore from (8.26) one finds that

$$(9.59)\qquad \left|E_{1,f}\right|=E_{|f|^2}.$$

Thus from (9.58) and (9.59) we establish the equality in (9.56) for $p=1$ and $x_0=0$.This corresponds to the equality of (9.36), as well.

Besides we note from (9.56) at $x_0 \neq 0$ one gets that

$$\text{(9.60)}\quad \text{R.H.S. } (\text{for } p=1;\ x_0 \neq 0) = \frac{\sqrt[4]{\pi}}{2}\sqrt{\Gamma\left(\frac{3}{2}\right)\frac{|c_0|^2}{c}\left(E^*_{|f|^2}\right)^{-1}\left[4cE^{*^2}_{|f|^2}\left(1+4cx_0^{*^2}\right)\right]^{\frac{1}{2}}}$$

$$= E_{|f|^2}\left[1+4c\left(1-E_{|f|^2}\right)^2 x_0^2\right]^{\frac{1}{2}}.$$

In this case from (9.39) we have that $x_m = x_0 E_{|f|^2} \neq 0$. But $\xi_m = 0$. Therefore from (8.51) one finds for $p = 1$; $q = 0$, and $w = 1$ that

$$E_{1,f} = -\int_{\mathbb{R}} |f(x)|^2\, dx = -E_{|f|^2}$$

satisfying (8.26).

Thus from (9.59) and (9.60) one establishes the inequality in (9.56) for $p = 1$ and both $x_0 \neq 0$ and $E_{|f|^2} \neq 1$, such that

$$\text{(9.61)}\quad |E_{1,f}| = E_{|f|^2} \leq E_{|f|^2}\left[1+4c\left(1-E_{|f|^2}\right)^2 x_0^2\right]^{\frac{1}{2}}.$$

If either $x_0 = 0$, or $E_{|f|^2} = 1$, then the equality in (9.56) holds for $p = 1$.

(ii) If $p = 2$, then one gets

$$\text{(9.62)}\quad \frac{d^4}{dx_0^4}e^{2cx_0^{*^2}} = 4cE^{*^2}_{|f|^2}\frac{d^2}{dx_0^2}\left(\left(1+4cx_0^{*^2}\right)e^{2cx_0^{*^2}}\right)$$

$$= 16c^2E^{*^3}_{|f|^2}\frac{d}{dx_0}\left(\left(3x_0^*+4cx_0^{*^3}\right)e^{2cx_0^{*^2}}\right)$$

$$= 16c^2E^{*^4}_{|f|^2}\left(3+24cx_0^{*^2}+16c^2x_0^{*^4}\right)e^{2cx_0^{*^2}}.$$

Therefore at $x_0 = 0$ (or $x_0^* = 0$), we have

$$\text{(9.63)}\quad \frac{d^4}{dx_0^4}e^{2cx_0^{*^2}} = 48c^2E^{*^4}_{|f|^2}.$$

Thus

$$\text{R.H.S. } (\text{for } p=2; x_0 = 0 \text{ (or } x_0^* = 0)) = \frac{\sqrt[4]{\pi}}{2^{\frac{5}{2}}}\sqrt{\Gamma\left(\frac{5}{2}\right)\frac{|c_0|^2}{c^{\frac{3}{2}}}\left(E^*_{|f|^2}\right)^{-2}\sqrt{48c^2E^{*^4}_{|f|^2}}},$$

or

$$\text{(9.64)}\quad \text{R.H.S.} = \frac{3}{2}\sqrt{\frac{\pi}{2}}\frac{|c_0|^2}{\sqrt{c}} = \frac{3}{2}E_{|f|^2}, \text{ for } p=2; x_0 = 0.$$

We note at $x = x_0 = 0$ one can get from (9.39) that $x_m = 0$. But we have $\xi_m = 0$. Therefore from (8.27) one finds that

$$E_{2,f} = 2\int_{\mathbb{R}}\left[|f(x)|^2 - x^2|f'(x)|^2\right]dx = 2\left(E_{|f|^2} - \frac{3}{4}E_{|f|^2}\right),$$

or

$$\text{(9.65)}\quad |E_{2,f}| = \frac{1}{2}E_{|f|^2}.$$

Thus from (9.64) and (9.65) we establish the inequality in (9.56) for $p = 2$ and $x_0 = 0$, because

$$\text{(9.66)}\quad |E_{2,f}| = \frac{1}{2}E_{|f|^2} < \frac{3}{2}E_{|f|^2}.$$

Finally we note from (9.56) at $x_0 \neq 0$ one gets that

(9.67) R.H.S. (for $p = 2; x_0 \neq 0$)

$$= \frac{3}{2} E_{|f|^2} \left[3 + 24c \left(1 - E_{|f|^2}\right)^2 x_0^2 + 16c^2 \left(1 - E_{|f|^2}\right)^4 x_0^4 \right]^{\frac{1}{2}}.$$

We note that

$$\begin{aligned}
\int_{\mathbb{R}} (x - x_m)^2 |f'(x)|^2 dx &= 4c^2 \int_{\mathbb{R}} \left(x - x_0 E_{|f|^2}\right)^2 (x - x_0)^2 |f(x)|^2 dx \\
&= 4c^2 \left[\int_{\mathbb{R}} x^4 |f(x)|^2 dx - 2\left(1 + E_{|f|^2}\right) x_0 \int_{\mathbb{R}} x^3 |f(x)|^2 dx \right. \\
&\quad + \left(1 + 4E_{|f|^2} + E_{|f|^2}^2\right) x_0^2 \int_{\mathbb{R}} x^2 |f(x)|^2 dx \\
&\quad \left. - 2\left(E_{|f|^2} + E_{|f|^2}^2\right) x_0^3 \int_{\mathbb{R}} x |f(x)|^2 dx + E_{|f|^2}^2 x_0^4 \int_{\mathbb{R}} |f(x)|^2 dx \right].
\end{aligned}$$

But (9.37) holds even if we replace $2p$ with any fixed but arbitrary $n \in \mathbb{N}_0$ (from (9.38)). Then one gets that,

$$\begin{aligned}
\int_{\mathbb{R}} |f(x)|^2 dx &= E_{|f|^2}, \\
\int_{\mathbb{R}} x |f(x)|^2 dx &= x_m = x_0 E_{|f|^2}, \\
\int_{\mathbb{R}} x^2 |f(x)|^2 dx &= (m_2)_{|f|^2} = \frac{1 + 4cx_0^2}{4c} E_{|f|^2}, \\
\int_{\mathbb{R}} x^3 |f(x)|^2 dx &= (m_3)_{|f|^2} = \frac{3x_0 + 4cx_0^3}{4c} E_{|f|^2},
\end{aligned}$$

and

$$\int_{\mathbb{R}} x^4 |f(x)|^2 dx = (m_4)_{|f|^2} = \frac{3 + 24cx_0^2 + 16c^2 x_0^4}{16c^2} E_{|f|^2}$$

hold, if $f(x) = c_0 e^{-c(x - x_0)^2}$, $c_0 \in \mathbb{C}$, $c > 0$, $x_0 \in \mathbb{R}$. Therefore

$$\begin{aligned}
&\int_{\mathbb{R}} (x - x_m)^2 |f'(x)|^2 dx \\
&= \frac{1}{4} E_{|f|^2} \left[\left(3 + 24cx_0^2 + 16c^2 x_0^4\right) - 8c\left(1 + E_{|f|^2}\right)\left(3x_0^2 + 4cx_0^4\right) \right. \\
&\quad + 4c\left(1 + 4E_{|f|^2} + E_{|f|^2}^2\right)\left(x_0^2 + 4cx_0^4\right) \\
&\quad \left. -32c^2\left(E_{|f|^2} + E_{|f|^2}^2\right) x_0^4 + 16c^2 E_{|f|^2}^2 x_0^4 \right] \\
&= \frac{1}{4} E_{|f|^2} \left\{ 3 + 4c\left[6 - 6\left(1 + E_{|f|^2}\right) + \left(1 + 4E_{|f|^2} + E_{|f|^2}^2\right)\right] x_0^2 \right. \\
&\quad + 16^2 \left[1 - 2\left(1 + E_{|f|^2}\right) + \left(1 + 4E_{|f|^2} + E_{|f|^2}^2\right) \right. \\
&\quad \left. \left. - 2\left(E_{|f|^2} + E_{|f|^2}^2\right) + E_{|f|^2}^2\right] x_0^4 \right\},
\end{aligned}$$

or

$$\int_{\mathbb{R}} (x - x_m)^2 |f'(x)|^2 dx = \frac{3 + 4c\left(1 - E_{|f|^2}\right)^2 x_0^2}{4} E_{|f|^2},$$

holds, if $c > 0$, $x_0 \in \mathbb{R}$.

In this case from (9.39) we have that $x_m = x_0 E_{|f|^2} \neq 0$. But $\xi_m = 0$. Therefore from (8.51) one finds for $p = 2$; $q = 0, 1$ and $w = 1$ that

$$\begin{aligned} E_{2,f} = (-1)^{2-0} \frac{2}{2-0} \binom{2-0}{0} \int_{\mathbb{R}} \frac{2!}{0!} \binom{2}{0} (1)^{(0)} \\ \times (x - x_m)^0 (x - x_m)^{2\cdot 0} \left|f^{(0)}(x)\right|^2 dx \\ + (-1)^{2-1} \frac{2}{2-1} \binom{2-1}{1} \int_{\mathbb{R}} \frac{2!}{2!} \binom{0}{0} (1)^{(0)} \\ \times (x - x_m)^0 (x - x_m)^{2\cdot 1} \left|f^{(1)}(x)\right|^2 dx \\ = 2 \int_{\mathbb{R}} \left[|f(x)|^2 dx - (x - x_m)^2 |f'(x)|^2 dx \right] \\ = 2 \left[E_{|f|^2} - \frac{3 + 4c\left(1 - E_{|f|^2}\right)^2 x_0^2}{4} E_{|f|^2} \right], \end{aligned}$$

or

$$E_{2,f} = \frac{1}{2} E_{|f|^2} \left[1 - 4c\left(1 - E_{|f|^2}\right)^2 x_0^2 \right]. \tag{9.68}$$

Thus from (9.67) and (9.68) one establishes the inequality in (9.56) for $p = 2$ such that

$$\begin{aligned} |E_{2,f}| &= \frac{1}{2} E_{|f|^2} \left| 1 - 4c\left(1 - E_{|f|^2}\right)^2 x_0^2 \right| \\ &< \frac{3}{2} E_{|f|^2} \left[3 + 24c\left(1 - E_{|f|^2}\right)^2 x_0^2 + 16c^2\left(1 - E_{|f|^2}\right)^4 x_0^4 \right]^{\frac{1}{2}}, \end{aligned} \tag{9.69}$$

because the condition

$$4\left[4c\left(1 - E_{|f|^2}\right)^2 x_0^2\right]^2 + 28\left[4c\left(1 - E_{|f|^2}\right)^2 x_0^2\right] + 13 > 0,$$

or

$$64c^2\left(1 - E_{|f|^2}\right)^4 x_0^4 + 112c\left(1 - E_{|f|^2}\right)^2 x_0^2 + 13 > 0 \tag{9.70}$$

holds for $p = 2$ and for fixed but arbitrary constants $c > 0$, and $x_0 \in \mathbb{R}$.

If either $x_0 = 0$, or $E_{|f|^2} = 1$, then we still have inequality in (9.56) for $p = 2$.

Corollary 9.5. *Assume that the Gamma function Γ is defined by (9.32). Consider the general Gaussian function $f : \mathbb{R} \to \mathbb{C}$ of the form $f(x) = c_0 e^{-c_1 x^2 + c_2 x + c_3}$, where c_i, $(i = 0, 1, 2, 3)$ are fixed but arbitrary constants and $c_0 \in \mathbb{C}$, $c_1 > 0$, and $c_2, c_3 \in \mathbb{R}$. Assume that the mathematical expectation $E\left(x - \frac{c_2}{2c_1}\right)$ of $x - \frac{c_2}{2c_1}$ for $|f|^2$ equals to*

$$x_m = \int_{\mathbb{R}} \left(x - \frac{c_2}{2c_1}\right) |f(x)|^2 dx = 0.$$

Consider the Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{C}$ *of* f*, given as in the abstract, and* ξ_m *the mean of* ξ *for* $\left|\hat{f}\right|^2$*. Denote by* $(m_{2q})_{\left|f^{(q)}\right|^2}$ *the* $2q^{th}$ *moment of* x *for* $\left|f^{(q)}\right|^2$ *about the origin, as in (8.41), and the constants* $\varepsilon_{p,q}$ *as in (8.38). Consider*

$$E_{p,f} = \sum_{q=0}^{[p/2]} \varepsilon_{p,q} (m_{2q})_{\left|f^{(q)}\right|^2}, \quad p \in \mathbb{N}.$$

If $\varepsilon_0 = c_0 e^{\left(c_2^2+4c_1c_3\right)/4c_1} \in \mathbb{C}$ *and* $t_0 = \frac{c_2}{2c_1} \in \mathbb{R}$*, then the extremum principle*

$$|E_{p,f}| \leq \frac{\sqrt[4]{\pi}}{2^{\frac{3p-1}{2}}} \cdot \Gamma^{\frac{1}{2}}\left(p + \frac{1}{2}\right) \cdot \frac{|\varepsilon_0|^2}{c_1^{\frac{p+1}{2}}} \cdot e^{-c_1 t_0^2} \cdot \left(\frac{d^{2p}}{dt_0^{2p}} e^{2c_1 t_0^2}\right)^{\frac{1}{2}}, \tag{9.71}$$

holds for any fixed but arbitrary $p \in \mathbb{N}$*.*

Equality holds for $p = 1$*;* $t_0 = 0$ *(or* $c_2 = 0$*).*

We note that $x_m = 0$ even if $t_0 \neq 0$ or $c_2 \neq 0$, while in the following Corollary 9.6 we have $x_m = 0$ only if $t_0 = 0$ or $c_2 = 0$.

Also we observe that if $g(x) = c_0 e^{-c_1(x-t_0)^2}$, and $d_0 = e^{\left(c_2^2+4c_1c_3\right)/4c_1}$ (> 0), then

$$f(x) = d_0 g(x) = \varepsilon_0 e^{-c_1(x-t_0)^2}, \quad \varepsilon_0 \in \mathbb{C}. \tag{9.72}$$

Proof. In fact, from (9.36) and (9.72) one gets that

$$\begin{aligned} x_m &= E\left(x - \frac{c_2}{2c_1}\right) \\ &= \int_{\mathbb{R}} \left(x - \frac{c_2}{2c_1}\right) |f(x)|^2 \, dx \\ &= d_0^2 \int_{\mathbb{R}} \left(x - \frac{c_2}{2c_1}\right) |g(x)|^2 \, dx \\ &= d_0^2 \left[\int_{\mathbb{R}} x |g(x)|^2 \, dx - \frac{c_2}{2c_1} \int_{\mathbb{R}} |g(x)|^2 \, dx\right] \\ &= d_0^2 \left[\frac{c_2}{2c_1} E_{|g|^2} - \frac{c_2}{2c_1} E_{|g|^2}\right], \end{aligned}$$

or

$$x_m = 0. \tag{9.73}$$

In this case the mathematical expectation x_m is the mean of x for $|f|^2$ if $t_0 = 0$ or $c_2 = 0$.

We note that from (9.40) and (9.72) one establishes

$$E_{|f|^2} = d_0^2 E_{|g|^2} = d_0^2 \sqrt{\frac{\pi}{2}} \frac{|c_0|^2}{\sqrt{c_1}} = \sqrt{\frac{\pi}{2}} \frac{|c_0|^2}{\sqrt{c_1}} e^{\frac{c_2^2+4c_1c_3}{2c_1}}. \tag{9.74}$$

This result can be computed directly from (9.40), as follows:

$$E_{|f|^2} = \sqrt{\frac{\pi}{2}} \frac{|\varepsilon_0|^2}{\sqrt{c_1}}$$

which leads to (9.74).

Similarly from (9.39) we get the mean of f for $|f|^2$ of the form

$$\int_{\mathbb{R}} x |f(x)|^2 \, dx = t_0 E_{|f|^2} = \sqrt{\frac{\pi}{2}} \frac{|\varepsilon_0|^2}{\sqrt{c_1}} t_0 = \frac{1}{2} \sqrt{\frac{\pi}{2}} \frac{|c_0|^2 c_2}{c_1 \sqrt{c_1}} e^{\left(c_2^2+4c_1c_3\right)/2c_1}. \tag{9.75}$$

Also from (9.47) one finds the Fourier transform $\hat{f}$ of f of the form

$$\hat{f}(\xi) = \varepsilon_0 \sqrt{\frac{\pi}{c_1}} e^{-\frac{\pi^2}{c_1}\xi^2 - i2\pi t_0 \xi} = c_0 \sqrt{\frac{\pi}{c_1}} e^{\left(c_2^2+4c_1c_3\right)/4c_1} e^{-\left(\pi^2\xi^2+i\pi c_2\xi\right)/c_1}, \tag{9.76}$$

$c_0 \in \mathbb{C}$, $c_1 > 0$, and $c_2, c_3 \in \mathbb{R}$.

Finally we find the mean of ξ for $\left|\hat{f}\right|^2$, as follows:

$$\xi_m = \int_{\mathbb{R}} \xi \left|\hat{f}(\xi)\right|^2 d\xi = |c_0|^2 e^{\left(c_2^2+4c_1c_3\right)/2c_1} \frac{\pi}{c_1} \int_{\mathbb{R}} \xi \cdot e^{-2\frac{\pi^2}{c_1}\xi^2} d\xi = 0. \tag{9.77}$$

The rest of the proof is similar to the proof of the Corollary 9.3. □

Corollary 9.6. *Assume the gamma function* Γ,*given as in (9.32). Consider the general Gaussian function* $f : \mathbb{R} \to \mathbb{C}$ *of the form* $f(x) = c_0 e^{-c_1x^2+c_2x+c_3}$, *where* c_i, $(i = 0, 1, 2, 3)$ *are fixed but arbitrary constants and* $c_0 \in \mathbb{C}$, $c_1 > 0$, *and* c_2, $c_3 \in \mathbb{R}$. *Assume that* x_m *is the mean (or the mathematical expectation* $E(x)$) *of* x *for* $|f|^2$. *Consider the Fourier transform* $\hat{f} : \mathbb{R} \to \mathbb{C}$ *of* f, *given as in the abstract, and* ξ_m *the mean of* ξ *for* $\left|\hat{f}\right|^2$.

Consider $(\mu_{2q})_{\left|f^{(q)}\right|^2}$ *the* $2q^{th}$ *moment of* x *for* $\left|f^{(q)}\right|^2$ *by*

$$(\mu_{2q})_{\left|f^{(q)}\right|^2} = \int_{\mathbb{R}} (x - x_m)^{2q} \left|f^{(q)}(x)\right|^2 dx,$$

the constants $\varepsilon_{p,q}$ *as in (8.38), and*

$$E_{p,f} = \sum_{q=0}^{[p/2]} \varepsilon_{p,q} (\mu_{2q})_{\left|f^{(q)}\right|^2}, \quad \text{if } |E_{p,f}| < \infty$$

holds for $0 \le q \le \left[\frac{p}{2}\right]$, *and any fixed but arbitrary* $p \in \mathbb{N}$. *If*

$$\varepsilon_0 = c_0 e^{\left(c_2^2+4c_1c_3\right)/4c_1} \in \mathbb{C},$$

and

$$t_0 = \frac{c_2}{2c_1} \in \mathbb{R},$$

$$E^*_{|f|^2} = 1 - E_{|f|^2} \left(= 1 - \sqrt{\frac{\pi}{2}} \frac{|\varepsilon_0|^2}{\sqrt{c_1}} \right)$$

and $t_0^* = t_0 E^*_{|f|^2}$, *then the following extremum principle*

$$|E_{p,f}| \le \frac{\sqrt[4]{\pi}}{2^{\frac{3p-1}{2}}} \cdot \Gamma^{\frac{1}{2}}\left(p + \frac{1}{2}\right) \cdot \frac{|\varepsilon_0|^2}{c_1^{\frac{p+1}{2}}} \cdot \left(E^*_{|f|^2}\right)^{-p} \cdot e^{-c_1 t_0^{*}2} \cdot \left(\frac{d^{2p}}{dt_0^{2p}} e^{2c_1 t_0^{*2}}\right)^{\frac{1}{2}}, \tag{9.78}$$

holds for any fixed but arbitrary $p \in \mathbb{N}$.

Equality holds for $p = 1$; $t_0 = 0$ *(or* $c_2 = 0$*), or for* $p = 1$; $E_{|f|^2} = 1$.

We note that $x_m = 0$ only if $t_0 = 0$, while in the previous Corollary 9.5 we have $x_m = 0$ even if $t_0 \neq 0$.

From (9.74) – (9.75) one gets that $x_m = t_0 E_{|f|^2}$, where $E_{|f|^2} = \sqrt{\frac{\pi}{2}} \frac{|\varepsilon_0|^2}{\sqrt{c_1}}$. Thus we get from (9.76) – (9.77) the proof of Corollary 9.6, in a way similar to the proof of the Corollary 9.4, because from (9.72) we have $f(x) = \varepsilon_0 e^{-c_1(x-t_0)^2}$, and x_m is the mean of x for $|f|^2$.

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Table 9.1: The first thirty-three cases of (9.33) and $R^*(p) = 2\pi R(p) \geq 1$

p	***R*(*p*)**	***R*(p)***
1	0.16	1.00
2	0.48	3.00
3	0.80	5.00
4	0.43	2.69
5	0.59	3.71
6	1.93	12.16
7	2.97	18.65
8	1.68	10.53
9	2.34	14.70
10	7.80	48.98
11	11.63	73.06
12	6.65	41.81
13	9.33	58.61
14	31.30	196.66
15	46.04	289.30
16	26.52	166.61
17	37.26	234.09
18	125.48	788.41
19	183.10	1150.43
20	105.83	664.95
21	148.89	935.48
22	502.68	3158.42
23	729.57	4584.05
24	422.69	2655.83
25	595.18	3739.60
26	2012.88	12647.30
27	2910.37	18286.41
28	1688.95	10611.98
29	2379.65	14951.80
30	8058.08	50630.40
31	11617.84	72997.06
32	6750.36	42413.73
33	9515.51	59787.71