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Solution of the Ulam Stability Problem for an Euler Type Quadratic Functional Equation

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Abstract. In 1968 S.M. Ulam proposed the *problem*: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?". In 1978 according to P.M. Gruber this kind of problems is of particular interest in *probability theory* and in the case of *functional equations* of different types. In 1997 W. Schuster established *a new vector quadratic identity* on the basis of the well-known *Euler type theorem on quadrilaterals*: If ABCD is a quadrilateral and M, N are the mid-points of the diagonals AC, BD as well as A′, B′, C′, D′ are the mid-points of the sides AB, BC, CD, DA, then $|AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 = 2|A'C'|^2 + 2|B'D'|^2 + 4|MN|^2$. Employing in this paper the above geometric identity we introduce *the new Euler type quadratic functional equation*

$$2[Q(x_0 - x_1) + Q(x_1 - x_2) + Q(x_2 - x_3) + Q(x_3 - x_0)] = Q(x_0 - x_1 - x_2 + x_3) + Q(x_0 + x_1 - x_2 - x_3) + 2Q(x_0 - x_1 + x_2 - x_3)$$

for all vectors $(x_0, x_1, x_2, x_3) \in X^4$, with X and Y linear spaces. For every $x \in R$ set $Q(x) = x^2$. Then the mapping $Q : X \to Y$ is quadratic. Note also that if $Q : R \to R$ is quadratic, then we have $Q(x) = Q(1)x^2$. Besides note that the geometric interpretation of the special example

$$2[(x_0 - x_1)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_0)^2] = (x_0 - x_1 - x_2 + x_3)^2 + (x_0 + x_1 - x_2 - x_3)^2 + 2(x_0 - x_1 + x_2 - x_3)^2$$

leads to the above-mentioned Euler type theorem on quadrilaterals ABCD with position vectors x_0, x_1, x_2, x_3 of vertices A, B, C, D, respectively. Then we solve the Ulam stability problem for the afore-mentioned equation, with non-linear Euler type quadratic mappings $Q : X \to Y$.

Keywords: Euler type quadratic, functional equation, Ulam stability problem

1. The Euler Type Quadratic Functional Equation

Definition 1. *Let X and Y be linear spaces. Then a* 4*-dimensional non-linear mapping* $Q : X \to Y$*, is called* Euler type quadratic*, if the new* 4*-dimensional Euler type qua-*

dratic functional equation

$$2[Q(x_0 - x_1) + Q(x_1 - x_2) + Q(x_2 - x_3) + Q(x_3 - x_0)]$$
$$= Q(x_0 - x_1 - x_2 + x_3) + Q(x_0 + x_1 - x_2 - x_3) + 2Q(x_0 - x_1 + x_2 - x_3) \quad (1)$$

holds for all 4*-dimensional vectors* $(x_0, x_1, x_2, x_3) \in X^4$ ([17–28]).

Note that Q is called *Euler type quadratic* because the following identity

$$2[(x_0 - x_1)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_0)^2]$$
$$= (x_0 - x_1 - x_2 + x_3)^2 + (x_0 + x_1 - x_2 - x_3)^2 + 2(x_0 - x_1 + x_2 - x_3)^2$$

holds for all real x_0, x_1, x_2, x_3, whose geometric interpretation leads to the Euler type theorem on quadrilaterals ABCD with position vectors x_0, x_1, x_2, x_3 of vertices A, B, C, D, respectively, and because the functional equation

$$Q(2^n x) = (2^n)^2 Q(x), \quad (2)$$

holds for all $x \in X$ and all $n \in N$ ([2], [4], [35]).

In fact, substitution of $x_i = 0$ $(i = 0, 1, 2, 3)$ in Equ. (1) yields that

$$Q(0) = 0. \quad (1a)$$

Lemma 1. *Let* $Q : X \to Y$ *be a* 4*-dimensional non-linear Euler type quadratic mapping satisfying Equ.* (1)*. Then* Q *is* ***an even mapping****; that is, equation*

$$Q(-x) = Q(x) \quad (3)$$

holds for all $x \in X$.

Proof. Substituting $x_0 = x_1 = x_2 = 0$ and $x_3 = x$ in Equ. (1) and employing (1a) one gets that equation

$$2[2Q(0) + Q(-x) + Q(x)] = Q(x) + Q(-x) + 2Q(-x),$$

or

$$Q(-x) - Q(x) = 4Q(0),$$

or the required equation (3), completing the proof of Lemma 1. ■

Lemma 2. *Let* $Q : X \to Y$ *be a* 4*-dimensional non-linear Euler type quadratic mapping satisfying Equ.* (1)*. Then* Q *satisfies* ***the general functional equation***

$$Q(x) = 2^{-2n} Q(2^n x) \quad (2a)$$

for all $x \in X$ *and all* $n \in N$.

Proof. Substituting $x_0 = x$, $x_1 = 0$, $x_2 = x$, $x_3 = 0$ in Equ. (1) and employing Eqs. (1a) and (3) one gets *the basic equation*

$$2[2Q(x) + 2Q(-x)] = 2Q(0) + 2Q(2x),$$

or

$$2Q(2x) - 4Q(-x) - 4Q(x) = -2Q(0) = 0,$$

or

$$2Q(2x) - 8Q(x) = 0,$$

or

$$Q(x) = 2^{-2}Q(2x) \tag{4}$$

for all $x \in X$.

Then induction on $n \in N$ with $x \to 2^{n-1}x$ in the basic equation (4) yields Equ. (2a). In fact, the basic equation (4) with $x \to 2^{n-1}x$ yield that the functional equation

$$Q(2^{n-1}x) = 2^{-2}Q(2^n x) \tag{4a}$$

holds for all $x \in X$.

Moreover by induction hypothesis with $n \to n - 1$ in the general equation (2a) one gets that

$$Q(x) = 2^{-2(n-1)}Q(2^{n-1}x) \tag{4b}$$

holds for all $x \in X$.

Thus functional equations (4a)–(4b) imply

$$Q(x) = 2^{-2(n-1)}2^{-2}Q(2^n x),$$

or

$$Q(x) = 2^{-2n}Q(2^n x),$$

for all $x \in X$ and all $n \in N$, completing the proof of the required general functional equation (2a) and hence the proof of Lemma 2. ■

2. The Euler Type Quadratic Functional Inequality

Definition 2. *Let X be a normed linear space and let Y be a real complete normed linear space. Then a* 4*-dimensional non-linear mapping $f : X \to Y$, is called* approximately Euler type quadratic, *if the new* 4*-dimensional Euler type quadratic functional inequality*

$$\|2[f(x_0 - x_1) + f(x_1 - x_2) + f(x_2 - x_3) + f(x_3 - x_0)] - [f(x_0 - x_1 - x_2 + x_3) + f(x_0 + x_1 - x_2 - x_3) + 2f(x_0 - x_1 + x_2 - x_3)]\| \le c \qquad (1)'$$

holds for all 4*-dimensional vectors* $(x_0, x_1, x_2, x_3) \in X^4$ *with a constant* $c \ge 0$ (*independent of* x_0, x_1, x_2, x_3).

Definition 3. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that there exists a constant* $c \ge 0$ (*independent of* $x \in X$). *Then a* 4*-dimensional non-linear Euler type quadratic mapping* $Q : X \to Y$, *is said that exists* near *an approximately* 4*-dimensional non-linear Euler type quadratic mapping* $f : X \to Y$, *if the following inequality*

$$\|f(x) - Q(x)\| \le \frac{11}{12}c, \qquad (1)''$$

holds for all $x \in X$.

Example. Take $f : R \to R$ be a real function such that $f(x) = lx^2 + k$, $l =$ real constant ($\neq 0$), and $k =$ constant : $|k| \le \frac{1}{4}c$, in order that f satisfies Inequ. $(1)'$.

Moreover, there exists a *unique* 4-dimensional non-linear Euler type quadratic mapping $Q : R \to R$ such that one gets

$$Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x) = \lim_{n\to\infty} 2^{-2n}[l(2^n x)^2 + k] = lx^2.$$

Finally claim that Inequ. $(1)''$ holds. In fact, the above-mentioned condition on $k : |k| \le \frac{1}{4}c$, implies

$$\|f(x) - Q(x)\| = \|(lx^2 + k) - lx^2\| = |k| \le \frac{1}{4}c < \frac{11/3}{4}c = \frac{11}{12}c,$$

satisfying Inequ. $(1)''$, because from Inequ. $(1)'$ one gets that

$$\|2\{[l(x_0 - x_1)^2 + k] + [l(x_1 - x_2)^2 + k] + [l(x_2 - x_3)^2 + k] + [l(x_3 - x_0)^2 + k]\} - \{[l(x_0 - x_1 - x_2 + x_3)^2 + k] + [l(x_0 + x_1 - x_2 - x_3)^2 + k] + 2[l(x_0 - x_1 + x_2 - x_3)^2 + k]\}\| \le c$$

or

$$|2\{k+k+k+k\}-\{k+k+2k\}|=4|k|\le c$$

or

$$|k|\le \frac{1}{4}c.$$

In general, we have the following conclusion.

Theorem. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition the above-mentioned* 4*-dimensional non-linear mappings Q, f and the three definitions. Then the limit*

$$Q(x)=\lim_{n\to\infty} 2^{-2n}f(2^n x) \tag{5}$$

exists *for all $x\in X$ and all $n\in N$ and $Q:X\to Y$ is the* ***unique*** 4*-dimensional non-linear Euler type quadratic mapping near an approximately* 4*-dimensional non-linear Euler type quadratic mapping $f:X\to Y$.*

We shall first prove the existence in Theorem, which is a consequence of the following Lemmas 3–6. Substitution of $x_i=0$ $(i=0,1,2,3)$ in Inequ. $(1)'$ yields that

$$\|4[4f(0)]-[2f(0)+2f(0)]\|\le c,$$

or

$$\|f(0)\|\le \frac{c}{4}. \tag{1a$'$}$$

Lemma 3. *Let $f:X\to Y$ be an approximately* 4*-dimensional non-linear Euler type quadratic mapping satisfying Inequ.* $(1)'$*. Then f is* ***an approximately even mapping****; that is, inequality*

$$\|f(-x)-f(x)\|\le c \tag{3$'$}$$

holds for all $x\in X$ with constant $c\ge 0$ (independent of $x\in X$).

Proof. Substituting $x_0=x_1=x_2=0$, $x_3=x$ in Inequ. $(1)'$ one gets that inequality

$$\|2[2f(0)+f(-x)+f(x)]-[f(x)+3f(-x)]\|\le c,$$

or

$$\|-f(-x)+f(x)+4f(0)\|\le c,$$

or

$$\|f(-x) - f(x) - 4f(0)\| \leq c \tag{3$''$}$$

holds for all $x \in X$.

Similarly substituting $x_0 = x_1 = x_2 = x$, $x_3 = 0$ in Inequ. $(1)'$ we establish inequality

$$\|2[2f(0) + f(x) + f(-x)] - [f(-x) + 3f(x)]\| \leq c,$$

or

$$\|f(-x) - f(x) + 4f(0)\| \leq c \tag{3$'''$}$$

holds for all $x \in X$. Note that substitution of x with $-x$ in Inequ. $(3)''$ also yields Inequ. $(3)'''$ for all $x \in X$.

Thus employing Ineqs. $(3)''$–$(3)'''$ and triangle inequality one finds the required inequality

$$\begin{aligned} 2\|f(-x) - f(x)\| &\leq \|f(-x) - f(x) - 4f(0)\| + \|f(-x) - f(x) + 4f(0)\| \\ &\leq c + c = 2c, \end{aligned}$$

or

$$\|f(-x) - f(x)\| \leq c,$$

completing the proof of Lemma 3. ∎

Lemma 4. *Let $f : X \to Y$ be an approximately 4-dimensional non-linear Euler type quadratic mapping satisfying Inequ. $(1)'$. Then f satisfies* ***the general functional inequality***

$$\|f(x) - 2^{-2n}f(2^n x)\| \leq \frac{11}{12}(1 - 2^{-2n})c, \tag{2a$'$}$$

for all $x \in X$ and all $n \in N$ with constant $c \geq 0$ (independent of $x \in X$).

Proof. Substituting $x_0 = x$, $x_1 = 0$, $x_2 = x$, $x_3 = 0$ in Inequ. $(1)'$ one gets that inequality

$$\|2[2f(x) + 2f(-x)] - [2f(0) + 2f(2x)]\| \leq c,$$

or

$$\|-2f(2x) + 4f(-x) + 4f(x) - 2f(0)\| \leq c,$$

or

$$\|f(2x) - 2f(-x) - 2f(x) + f(0)\| \le \frac{c}{2} \tag{3a)''}$$

holds for all $x \in X$.

Inequ. (3)′ yields functional inequality

$$\|2f(-x) - 2f(x)\| \le 2c \tag{3b)''}$$

for all $x \in X$.

Applying Ineqs. (1a)′–(3a)″–(3b)″ and triangle inequality we find that *the basic inequality*

$$\begin{aligned}
&4\|f(x) - 2^{-2}f(2x)\| \\
&\quad = \|f(2x) - 4f(x)\| = \|[f(2x) - 4f(x) + f(0)] + [-f(0)]\| \\
&\quad \le \|f(2x) - 4f(x) + f(0)\| + \|-f(0)\| \\
&\quad \le \|[f(2x) - 2f(-x) - 2f(x) + f(0)] + [2f(-x) - 2f(x)]\| + \|f(0)\| \\
&\quad \le \|f(2x) - 2f(-x) - 2f(x) + f(0)\| + \|2f(-x) - 2f(x)\| + \|f(0)\| \\
&\quad \le \frac{c}{2} + 2c + \frac{c}{4} = \frac{2c + 8c + c}{4} = \frac{11}{4}c,
\end{aligned}$$

or

$$\|f(x) - 2^{-2}f(2x)\| \le \frac{11}{16}c \left(= \frac{11}{12}(1 - 2^{-2})c \right), \tag{4)'}$$

holds for all $x \in X$ with constant c (independent of $x \in X$) ≥ 0.

Replacing now x with $2x$ in the basic inequality (4)′ one concludes that

$$\|f(2x) - 2^{-2}f(2^2x)\| \le \frac{11}{12}(1 - 2^{-2})c,$$

or

$$\|2^{-2}f(2x) - 2^{-4}f(2^2x)\| \le \frac{11}{12}(2^{-2} - 2^{-4})c \tag{6}$$

holds for all $x \in X$.

Functional inequalities (4)′–(6) and the triangle inequality yield

$$\begin{aligned}
\|f(x) - 2^{-4}f(2^2x)\| &\le \|f(x) - 2^{-2}f(2x)\| + \|2^{-2}f(2x) - 2^{-4}f(2^2x)\| \\
&\le \frac{11}{12}[(1 - 2^{-2}) + (2^{-2} - 2^{-4})]c,
\end{aligned}$$

or that the functional inequality

$$\|f(x) - 2^{-4}f(2^2x)\| \leq \frac{11}{12}(1 - 2^{-4})c, \tag{6a}$$

holds for all $x \in X$.

Similarly by induction on $n \in N$ with $x \to 2^{n-1}x$ in the basic Inequ. $(4)'$ claim that *the general functional inequality* $(2a)'$ holds for all $x \in X$ and all $n \in N$ with constant c (independent of $x \in X$) ≥ 0. In fact, the basic inequality $(4)'$ with $x \to 2^{n-1}x$ yield the functional inequality

$$\|f(2^{n-1}x) - 2^{-2}f(2^nx)\| \leq \frac{11}{12}(1 - 2^{-2})c,$$

or that the functional inequality

$$\|2^{-2(n-1)}f(2^{n-1}x) - 2^{-2n}f(2^nx)\| \leq \frac{11}{12}(2^{-2(n-1)} - 2^{-2n})c, \tag{7a}$$

holds for all $x \in X$.

Moreover, by induction hypothesis with $n \to n-1$ in the general inequality $(2a)'$ one gets that

$$\|f(x) - 2^{-2(n-1)}f(2^{n-1}x)\| \leq \frac{11}{12}(1 - 2^{-2(n-1)})c, \tag{7b}$$

holds for all $x \in X$.

Thus functional inequalities (7a)–(7b) and the triangle inequality imply

$$\begin{aligned}\|f(x) - 2^{-2n}f(2^nx)\| &\leq \|f(x) - 2^{-2(n-1)}f(2^{n-1}x)\| \\ &\quad + \|2^{-2(n-1)}f(2^{n-1}x) - 2^{-2n}f(2^nx)\| \\ &\leq \frac{11}{12}[(1 - 2^{-2(n-1)}) + (2^{-2(n-1)} - 2^{-2n})]c\end{aligned}$$

or

$$\|f(x) - 2^{-2n}f(2^nx)\| \leq \frac{11}{12}(1 - 2^{-2n})c,$$

for all $x \in X$ and all $n \in N$, completing the proof of the required general functional inequality $(2a)'$, and thus the proof of Lemma 4. ∎

Lemma 5. *Let* $f : X \to Y$ *be an approximately* 4*-dimensional non-linear Euler type quadratic mapping satisfying Inequ.* $(1)'$. *Then the sequence* $\{f_n(x)\}$ *of functions* f_n:

$$f_n(x) = 2^{-2n}f(2^nx) \tag{8}$$

converges.

Proof. Note that from the general functional inequality (2a)$'$ and the completeness of Y, one proves that the above-mentioned sequence (8) is *a Cauchy sequence*.

In fact, if $i > j > 0$, then

$$\|f_i(x) - f_j(x)\| = \|2^{-2i}f(2^i x) - 2^{-2j}f(2^j x)\| \\ = 2^{-2j}\|2^{-2(i-j)}f(2^i x) - f(2^j x)\|, \tag{9}$$

holds for all $x \in X$, and all $i, j \in N$. Setting $h = 2^j x$ in (9) and employing the general functional inequality (2a)$'$ one concludes that

$$\|f_i(x) - f_j(x)\| = 2^{-2j}\|2^{-2(i-j)}f(2^{i-j}h) - f(h)\| \\ \leq 2^{-2j}\frac{11}{12}(1 - 2^{-2(i-j)})c,$$

or

$$\|f_i(x) - f_j(x)\| \leq \frac{11}{12}(2^{-2j} - 2^{-2i})c < \frac{11}{12}2^{-2j}c = \frac{11}{3}2^{-2(j+1)}c,$$

or

$$\lim_{j\to\infty}\|f_i(x) - f_j(x)\| < \frac{11}{3}\left(\lim_{j\to\infty} 2^{-2(j+1)}\right)c = 0$$

or

$$\lim_{j\to\infty}\|f_i(x) - f_j(x)\| = 0, \tag{9a}$$

which yields that the sequence of functions $f_n : f_n(x) = 2^{-2n}f(2^n x)$ or (8) is *a Cauchy sequence*, completing the proof of Lemma 5. ■

Lemma 6. *Let $f : X \to Y$ be an approximately* 4*-dimensional non-linear Euler type quadratic mapping satisfying Inequ.* (1)$'$. *Assume in addition a mapping $Q : X \to Y$ given by the above-said formula* (5). *Then $Q = Q(x)$ is* ***a well-defined mapping*** *and that Q is a* 4*-dimensional non-linear* ***Euler type quadratic mapping*** *in X.*

Proof. Employing Lemma 5 and formula (5), one gets that Q is *a well-defined mapping*. This means that the limit (5) exists for all $x \in X$.

In addition we can claim that Q *satisfies* the functional equation (1) for all 4-dimensional vectors $(x_0, x_1, x_2, x_3) \in X^4$. In fact, it is clear from the 4-dimensional Euler type quadratic functional inequality (1)$'$ and the limit (5) that the following functional inequality

$$2^{-2n}\|2[f(2^n x_0 - 2^n x_1) + f(2^n x_1 - 2^n x_2) + f(2^n x_2 - 2^n x_3) + f(2^n x_3 - 2^n x_0)] \\ - [f(2^n x_0 - 2^n x_1 - 2^n x_2 + 2^n x_3) + f(2^n x_0 + 2^n x_1 - 2^n x_2 - 2^n x_3) \\ + 2f(2^n x_0 - 2^n x_1 + 2^n x_2 - 2^n x_3)]\| \leq 2^{-2n}c, \tag{10}$$

holds for all vectors $(x_0, x_1, x_2, x_3) \in X^4$ and all $n \in N$. Therefore from (10) one gets

$$\Big\| 2\Big\{ \lim_{n\to\infty} 2^{-2n} f[2^n(x_0 - x_1)] + \lim_{n\to\infty} 2^{-2n} f[2^n(x_1 - x_2)] + \lim_{n\to\infty} 2^{-2n} f[2^n(x_2 - x_3)] + \lim_{n\to\infty} 2^{-2n} f[2^n(x_3 - x_0)] \Big\} - \Big\{ \lim_{n\to\infty} 2^{-2n} f[2^n(x_0 - x_1 - x_2 + x_3)] + \lim_{n\to\infty} 2^{-2n} f[2^n(x_0 + x_1 - x_2 - x_3)] + 2 \lim_{n\to\infty} 2^{-2n} f[2^n(x_0 - x_1 + x_2 - x_3)] \Big\} \Big\| \leq \Big(\lim_{n\to\infty} 2^{-2n} \Big) c = 0,$$

or

$$\begin{aligned} \|2[Q(x_0 - x_1) + Q(x_1 - x_2) + Q(x_2 - x_3) + Q(x_3 - x_0)] \\ - [Q(x_0 - x_1 - x_2 + x_3) + Q(x_0 + x_1 - x_2 - x_3) \\ + 2Q(x_0 - x_1 + x_2 - x_3)]\| = 0 \end{aligned} \tag{10a}$$

or mapping Q satisfies equation (1) for all vectors $(x_0, x_1, x_2, x_3) \in X^4$. Thus Q is a 4-dimensional non-linear *Euler type quadratic mapping*, completing the proof of Lemma 6.

It is now clear from Lemmas 1–6 and especially from the general inequality $(2a)'$, $n \to \infty$, and formula (5) that inequality $(1)''$ holds in X. Hence *the existence proof in this Theorem* is complete. ■

Proof of Uniqueness in Theorem. Let $Q' : X \to Y$ be another 4-dimensional non-linear Euler type quadratic mapping satisfying the new 4-dimensional Euler type quadratic functional equation (1), such that inequality

$$\|f(x) - Q'(x)\| \leq \frac{11}{12} c, \tag{1a)''$$

holds for all $x \in X$. If there exists a 4-dimensional non-linear Euler type quadratic mapping $Q : X \to Y$ satisfying the new 4-dimensional Euler type quadratic functional equation (1), then

$$Q(x) = Q'(x), \tag{11}$$

holds for all $x \in X$; that is, Q is *unique* in X.

To prove the afore-mentioned *uniqueness* one employs the general functional Equ. (2a) for Q and Q', as well, so that

$$Q'(x) = 2^{-2n} Q'(2^n x), \tag{2a)''$$

holds for all $x \in X$, and all $n \in N$. Moreover, the triangle inequality and Ineqs. (1)″–(1a)″ yield inequality

$$\|Q(2^n x) - Q'(2^n x)\| \le \|Q(2^n x) - f(2^n x)\| + \|f(2^n x) - Q'(2^n x)\|$$
$$\le \frac{11}{12}c + \frac{11}{12}c = \frac{11}{6}c,$$

or

$$\|Q(2^n x) - Q'(2^n x)\| \le \frac{11}{6}c, \tag{12}$$

for all $x \in X$, and all $n \in N$. Then from Eqs. (2a)–(2a)″, and Inequ. (12), one proves that

$$\|Q(x) - Q'(x)\| = \|2^{-2n}Q(2^n x) - 2^{-2n}Q'(2^n x)\| = 2^{-2n}\|Q(2^n x) - Q'(2^n x)\|$$
$$\le \frac{11}{6}2^{-2n}c = \frac{11}{3}2^{-(2n+1)}c,$$

or

$$\|Q(x) - Q'(x)\| \le \frac{11}{3}2^{-(2n+1)}c \tag{12a}$$

holds for all $x \in X$ and all $n \in N$. Therefore from above-mentioned Inequ. (12a), and $n \to \infty$, one establishes

$$\lim_{n\to\infty} \|Q(x) - Q'(x)\| \le \frac{11}{3}\left(\lim_{n\to\infty} 2^{-(2n+1)}\right)c = 0,$$

or

$$Q(x) = Q'(x),$$

for all $x \in X$, completing the proof of *uniqueness* and thus the *stability* of this Theorem ([1]–[16]) and ([29]–[35]). ■

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