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Uniqueness of Quasi-Regular Solutions for a Bi-Parabolic Elliptic Bi-Hyperbolic Tricomi Problem

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The *Tricomi equation* $yu_{xx} + u_{yy} = 0$ was established in 1923 by Tricomi who is the pioneer of parabolic elliptic and hyperbolic boundary value problems and related problems of variable type. In 1945 Frankl established a generalization of these problems for the well-known *Chaplygin equation* $K(y)u_{xx} + u_{yy} = 0$ subject to the Frankl condition $1 + 2(K/K')' > 0$, $y < 0$. In 1953 and 1955 Protter generalized these problems even further by improving the above Frankl condition. In 1977 we generalized these results in R^n $(n > 2)$. In 1986 Kracht and Kreyszig discussed the Tricomi equation and transition problems. In 1993 Semerdjieva considered the hyperbolic equation $K_1(y)u_{xx} + (K_2(y)u_y)_y + ru = f$ for y < 0. In this paper we establish uniqueness of quasi-regular solutions for the Tricomi problem concerning the more general mixed type partial differential equation $K_1(y)(M_2(x)u_x)_x + M_1(x)(K_2(y)u_y)_y + ru = f$ which is parabolic on both lines $x = 0$; $y = 0$, elliptic in the first quadrant $x > 0$, $y > 0$ and hyperbolic in both quadrants $x < 0$, $y > 0$; $x > 0$, $y < 0$. In 1999 we proved existence of weak solutions for a particular Tricomi problem. These results are interesting in fluid mechanics.

Keywords: Quasi-regular solution; Tricomi equation; Chaplygin equation; Bi-parabolic equation; Bi-hyperbolic equation; Tricomi problem

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1. INTRODUCTION

In 1904 Chaplygin [11] noticed that the equation of a perfect gas was $K(y)u_{xx} + u_{yy} = 0$. In 1923 Tricomi [17] initiated the work on boundary value problems for linear partial differential mixed type equations of second order and related equations of variable type. In 1945 Frankl [3] drew attention to the fact that the Tricomi problem was closely connected to the study of gas flow with nearly sonic speeds. In 1953 and 1955 Protter [7] generalized and improved the aforementioned results in the euclidean plane. In 1977

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we [8] generalized these results in R^n ($n > 2$). In 1982 we [9] established a maximum principle of the Cauchy problem for hyperbolic equations in R^{n+1} ($n \geq 2$). In 1983 we [10] solved the Tricomi problem with two parabolic lines of degeneracy and, in 1992, we [12] established the well-posedness of the Tricomi problem in euclidean regions. Interesting results for the Tricomi problem were achieved by Barantsev [1] in 1986, and Kracht and Kreyszig [4] in 1986, as well. Related information was reported by Fichera [2] in 1985, and Kreyszig [5,6] in 1989 and 1994. Our [11,14,15] work, in 1990 and 1999, was in analogous areas of mixed type equations. In 1993 Semerdjieva introduced the hyperbolic equation $K_1(y)u_{xx} + (K_2(y)u_y)_y + ru = f$ in the lower half-plane. In 1997 we [13] considered the more general case of the above hyperbolic equation, so that it was elliptic in the upper half-plane and parabolic on the line $y = 0$. In this paper we consider the more general Tricomi problem with partial differential equation the new *bi-parabolic elliptic bi-hyperbolic equation*

$$Lu \equiv K_1(y)(M_2(x)u_x)_x + M_1(x)(K_2(y)u_y)_y + r(x,y)u = f(x,y), \qquad (*)$$

which is parabolic on both segments $x = 0,\ 0 < y \leq 1;\ y = 0,\ 0 < x \leq 1$, elliptic in the euclidean region $G_e = \{(x,y) \in G(\subset R^2):\ x > 0,\ y > 0\}$ and hyperbolic in both euclidean regions $G_{h_1} = \{(x,y) \in G(\subset R^2):\ x > 0,\ y < 0\}$; $G_{h_2} = \{(x,y) \in G(\subset R^2):\ x < 0,\ y > 0\}$, with G the mixed domain of (*). In 1999 we [15] proved existence of weak solutions for a particular Tricomi problem. Then we establish uniqueness of quasi-regular solutions [3,7,8,10–13] for the Tricomi problem, concerning the above mixed type Eq. (*). However, the question about the uniqueness of quasi-regular solutions and the existence of weak solutions for this general Tricomi problem in several variables is still *open*. These results are interesting in Aerodynamics and Hydrodynamics.

2. THE TRICOMI PROBLEM

Consider the bi-parabolic elliptic bi-hyperbolic equation (*) in a bounded simply-connected mixed domain G with a piecewise smooth boundary $\partial G = g_1 \cup g_2 \cup g_3 \cup \gamma_2 \cup \gamma_3$, where $f = f(x,y)$ is continuous in G, $r = r(x,y)$ is once-continuously differentiable in $G, K_i = K_i(y)$ $(i = 1,2)$ are once-continuously differentiable for $y \in [-k_1, k_2]$ with $-k_1 = \inf\{y:\ (x,y) \in G\}$ and $k_2 = \sup\{y:\ (x,y) \in G\}$, and $M_i = M_i(x)$ $(i = 1,2)$ are once-continuously differentiable for $x \in [-m_1, m_2]$ with $-m_1 = \inf\{x: (x,y) \in G\}$ and $m_2 = \sup\{x:\ (x,y) \in G\}$. Besides

$$K_1(y)\begin{cases} > 0 & \text{for } y > 0, \\ = 0 & \text{for } y = 0, \\ < 0 & \text{for } y < 0, \end{cases} \quad \text{and} \quad M_1(x)\begin{cases} > 0 & \text{for } x > 0, \\ = 0 & \text{for } x = 0, \\ < 0 & \text{for } x < 0, \end{cases}$$

as well as $K_2(y) > 0$ and $M_2(x) > 0$ everywhere in G, so that

$$K(y) = K_1(y)/K_2(y)\begin{cases} > 0 & \text{for } y > 0, \\ = 0 & \text{for } y = 0, \\ < 0 & \text{for } y < 0, \end{cases} \quad \text{and} \quad M(x) = M_1(x)/M_2(x)\begin{cases} > 0 & \text{for } x > 0 \\ = 0 & \text{for } x = 0\,. \\ < 0 & \text{for } x < 0 \end{cases}$$

We assume that the following two limits $\lim_{y\to 0} K(y)$ and $\lim_{x\to 0} M(x)$ exist in G.

In this paper we also assume

$$K(y)M(x)\begin{cases} > 0 & \text{for } x > 0, y > 0 \\ = 0 & \text{for } x = 0; y = 0 \\ < 0 & \text{for } x > 0, y < 0;\ x < 0, y > 0. \end{cases}$$

We note that the case $KM > 0$ for $x < 0, y < 0$ is not considered here. The above Eq. (*) degenerates its order at the origin $O(0,0)$. The boundary ∂G of the domain G is formed by the following curves:

(1) A curve g_1 which is the elliptic arc lying in the first quadrant $x > 0$, $y > 0$ and connecting the points $A(1,0)$ and $B(0,1)$; (2) two hyperbolic characteristic arcs g_2 and g_3:

$$g_2\colon \int_1^x (M(t))^{1/2}\,dt = \int_0^y (-K(t))^{1/2}\,dt, \quad g_3\colon \int_0^x (M(t))^{1/2}\,dt = -\int_0^y (-K(t))^{1/2}\,dt,$$

descending from the points $A(1,0)$ and $O(0,0)$ until they terminate at a common point of intersection $P_1(x_{p_1}, y_{p_1})$ in the fourth quadrant $x > 0$, $y < 0$; and (3) two other hyperbolic characteristic arcs γ_2 and γ_3:

$$\gamma_2 : \int_0^x (-M(t))^{1/2}\,dt = \int_1^y (K(t))^{1/2}\,dt, \qquad \gamma_3 : \int_0^x (-M(t))^{1/2} dt = -\int_0^y (K(t))^{1/2}\,dt,$$

emanating from the points $B(0,1)$ and $O(0,0)$ until they terminate at a common point of intersection $P_2(x_{p_2}, y_{p_2})$ in the second quadrant $x < 0$, $y > 0$. Assume *the boundary condition*

$$u = 0 \quad \text{on } g_1 \cup g_2 \cup \gamma_2. \tag{**}$$

The Tricomi problem, or *Problem* (T) consists in finding a function $u = u(x,y)$ which satisfies the Eq. (*) in G and the boundary condition (**) on $g_1 \cup g_2 \cup \gamma_2$.

Definition A function $u = u(x,y)$ is *a quasi-regular solution* [7,8,10–13] of Problem (T) if (i) $u \in C^2(G) \cap C(\bar{G})$, $\bar{G} = G \cup \partial G$; (ii) the Green's theorem (of the integral calculus) is applicable to the integrals

$$\iint_G u_x Lu\,dx\,dy, \qquad \iint_G u_y Lu\,dx\,dy;$$

(iii) the boundary and region integrals, which arise, exist; and (iv) u satisfies the mixed type Eq. (*) in G and the boundary condition (**) on $g_1 \cup g_2 \cup \gamma_2$.

THEOREM *Consider the bi-parabolic elliptic bi-hyperbolic Eq.* (*) *and the boundary condition* (**). *Also consider the afore-described simply-connected mixed domain G of* *the xy euclidean plane. Besides let us assume the conditions:*

(R_1) : $r<0$ *on* $g_3\cup\gamma_3$,

(R_2) : *the elliptic arc* g_1 *is star-like in the sense that* $x\,dy - y\,dx\geq 0$,

$$(\mathrm{R}_3):\quad \begin{cases} 2r+xr_x+yr_y<0 & \textit{for } x\geq 0, y\geq 0\\ r+xr_x<0 & \textit{for } x\geq 0, y\leq 0 \quad \textit{and} \quad r+yr_y<0 \quad \textit{for } x\leq 0, y\geq 0, \end{cases}$$

$$(\mathrm{R}_4):\quad \begin{cases} K_1(y)>0 & \textit{for } y>0;\ K_1(y)<0 \quad \textit{for } y<0;\ K_1(0)=0\\ K_2(y)>0 & \textit{in } G \quad \textit{and } K_2(y)-yK_2'(y)>0 \quad \textit{for } y\geq 0\ , \end{cases}$$

$$(\mathrm{R}_5):\quad \begin{cases} M_1(x)>0 & \textit{for } x>0; M_1(x)<0 \quad \textit{for } x<0;\ M_1(0)=0\\ M_2(x)>0 & \textit{in } G \quad \textit{and } M_2(x)-x\overset{\bullet}{M}_2(x)>0 \quad \textit{for } x\geq 0\ , \end{cases}.$$

(R_6) : $K_i'(y)>0$, *in* G, *and*

(R_7) : $\overset{\bullet}{M}_i(x)>0$, *in* G, *for* $i=1,2$, *with symbols*

$$()_x=\partial()/\partial x, \quad ()^{\bullet}=d()/dx, \quad ()_y=\partial()/\partial y, \quad ()'=d()/dy,$$

where $f=f(x,y)$ *is continuous in* G, $r=r(x,y)$ *is once-continuously differentiable in* G, $K_i=K_i(y)$ $(i=1,2)$ *are once-continuously differentiable for* $y\in[-k_1,k_2]$ *with* $-k_1=\inf\{y\colon (x,y)\in G\}$ *and* $k_2=\sup\{y\colon (x,y)\in G\}$, *and* $M_i=M_i(x)$ $(i=1,2)$ *are once-continuously differentiable for* $x\in[-m_1,m_2]$ *with* $-m_1=\inf\{x\colon (x,y)\in G\}$ *and* $m_2=\sup\{x\colon (x,y)\in G\}$. *Then the Problem* (T) *has at most one quasi-regular solution in* G.

Proof We apply the well-known *a–b–c* energy integral method with $a=0$, and use the above mixed type Eq. (*) as well as the boundary condition (**). First, we assume two quasi-regular solutions u_1, u_2 of the Problem (T).

Then we claim that $u=u_1-u_2=0$ holds in the domain G. In fact, we investigate

$$0=J=2\langle lu, Lu\rangle_0=\iint_G 2\,lu\,Lu\,dx\,dy, \tag{1}$$

where $lu=b(x)\,u_x+c(y)u_y$, and $Lu=L(u_1-u_2)=Lu_1-L\,u_2=f-f=0$ in G, with *choices*

$$b=b(x)=\begin{cases} x & \text{in } G,\ x\geq 0, y\geq 0\\ x & \text{in } G,\ x\geq 0, y\leq 0\ ,\\ 0 & \text{in } G,\ x\leq 0, y\geq 0\end{cases} \qquad c=c(y)=\begin{cases} y & \text{in } G,\ x\geq 0, y\geq 0\\ 0 & \text{in } G,\ x\geq 0, y\leq 0\\ y & \text{in } G,\ x\leq 0, y\geq 0\end{cases} \tag{2}$$

We consider the new *differential identities*

$$2bK_1M_2u_xu_{xx}=\left(bK_1M_2u_x^2\right)_x-(bM_2)^{\bullet}K_1u_x^2,$$

$$2bK_2M_1u_xu_{yy} = (2bK_2M_1u_xu_y)_y - 2bM_1K_2'u_xu_y - \left(bK_2M_1u_y^2\right)_x + (bM_1)^\bullet K_2u_y^2,$$

$$2cK_1M_2u_yu_{xx} = (2cK_1M_2u_xu_y)_x - 2cK_1\overset{\bullet}{M}_2u_xu_y - \left(cK_1M_2u_x^2\right)_y + (cK_1)'M_2u_x^2,$$

$$2cK_2M_1u_yu_{yy} = \left(cK_2M_1u_y^2\right)_y - (cK_2)'M_1u_y^2,$$

$$2bruu_x = (bru^2)_x - (br)_xu^2,\ 2cruu_y = (cru^2)_y - (cr)_yu^2,$$

as well as t_1 is the coefficient of u_x in Lu, or

$$t_1 = t_1(x, y) = K_1(y)\overset{\bullet}{M}_2(x), \tag{3}$$

and t_2 is the coefficient of u_y in Lu, or

$$t_2 = t_2(x, y) = K_2'(y)M_1(x). \tag{4}$$

Employing these identities and the classical Green's theorem of the integral calculus we obtain from (*), (1), (3), and (4) that

$$\begin{aligned} 0 = J &= \iint_G 2(bu_x + cu_y)\left[K_1(M_2u_x)_x + M_1(K_2u_y)_y + ru\right] dx\,dy \\ &= \iint_G 2(bu_x + cu_y)\left[K_1M_2u_{xx} + K_2M_1u_{yy} + t_1u_x + t_2u_y + ru\right] dx\,dy = I_G + I_{\partial G}, \end{aligned} \tag{5}$$

where

$$I_G = \iint_G \left(Au_x^2 + Bu_y^2 + \Gamma u^2 + 2\Delta u_xu_y\right) dx\,dy,$$

and

$$I_{\partial G} = \int_{\partial G} \left(\tilde{A}u_x^2 + \tilde{B}u_y^2 + \tilde{\Gamma}u^2 + 2\tilde{\Delta}u_xu_y\right) ds$$

with

$$\begin{aligned} &A = -K_1(bM_2)^\bullet + (cK_1)'M_2 + 2bt_1, B = K_2(bM_1)^\bullet - (cK_2)'M_1 + 2ct_2, \\ &\Gamma = -\left[(br)_x + (cr)_y\right], \\ &\Delta = -[bK_2'M_1 + cK_1\overset{\bullet}{M}_2 - bt_2 - ct_1] \\ &\quad = -[b(K_2'M_1 - t_2) + c(K_1\overset{\bullet}{M}_2 - t_1)] = 0 \text{ (because of (3) and (4)) in } G, \end{aligned}$$

and

$$\begin{aligned} &\tilde{A} = (bv_1 - cv_2)K_1M_2, \qquad \tilde{B} = (-bv_1 + cv_2)K_2M_1, \\ &\tilde{\Gamma} = (bv_1 + cv_2)r, \qquad \tilde{\Delta} = bK_2M_1v_2 + cK_1M_2v_1 \quad \text{on } \partial G, \end{aligned}$$

where

$$v = (v_1, v_2) = (dy/ds, -dx/ds) \tag{6}$$

is *the outer unit normal vector* on the boundary ∂G of the domain G such that

$$ds^2 = dx^2 + dy^2 > 0, \qquad |v| = \left(v_1^2 + v_2^2\right)^{1/2} = 1,$$

and

$$\iint_G ()_x dx\, dy = \int_{\partial G} () v_1\, ds, \qquad \iint_G ()_y dx\, dy = \int_{\partial G} () v_2\, ds,$$

are the *Green's integral formulas*.

Note that in G, $x \geq 0$, $y \geq 0$ with $b = x$, $c = y$ (from (2)) one gets, from (3) and (4), that

$$\begin{aligned} A &= -K_1(xM_2)^\bullet + (yK_1)'M_2 + 2xt_1 = -K_1(M_2 + x\dot{M}_2) + (K_1 + yK_1')M_2 + 2xK_1\dot{M}_2 \\ &= xK_1\dot{M}_2 + yK_1'M_2 \geq 0 \quad \text{(from conditions (R}_6\text{) and (R}_7\text{))}, \end{aligned}$$

$$\begin{aligned} B &= K_2(xM_1)^\bullet - (yK_2)'M_1 + 2yt_2 = K_2(M_1 + x\dot{M}_1) - (K_2 + yK_2')M_1 + 2yK_2'M_1 \\ &= xK_2\dot{M}_1 + yK_2'M_1 \geq 0 \quad \text{(from conditions (R}_6\text{) and (R}_7\text{))}, \end{aligned}$$

$$\Gamma = -[(xr)_x + (yr)_y] = -(2r + xr_x + yr_y) > 0 \quad \text{(from condition (R}_3\text{)} : x \geq 0, y \geq 0),$$

and

$$\begin{aligned} AB - \Delta^2 &= (xK_1\dot{M}_2 + yK_1'M_2)(xK_2\dot{M}_1 + yK_2'M_1) \\ &= x\left(K_1K_2\dot{M}_1\dot{M}_2\right) + xy\left(K_1K_2'M_1\dot{M}_2 + K_1'K_2\dot{M}_1M_2\right) + y\left(K_1'K_2'M_1M_2\right) \geq 0 \\ &\quad \text{(from conditions (R}_6\text{) and (R}_7\text{))}. \end{aligned}$$

Similarly in G, $x \geq 0$, $y \leq 0$ with $b = x$, $c = 0$ (from (2)) we find, from (3) and (4), that

$$\begin{aligned} A &= -K_1(xM_2)^\bullet + (0 \cdot K_1)'M_2 + 2xt_1 = -K_1(M_2 + x\dot{M}_2) + 2xK_1\dot{M}_2 \\ &= (-K_1)(M_2 - x\dot{M}_2) > 0 \quad \text{(from conditions (R}_4\text{) and (R}_5\text{))}, \end{aligned}$$

$$B = K_2(xM_1)^\bullet - (0 \cdot K_2)'M_1 + 2 \cdot 0 \cdot K_2'M_1 = K_2(M_1 + x\dot{M}_1) \geq 0 \quad \text{(from (R}_4\text{) and (R}_7\text{))},$$

$$\Gamma = -[(xr)_x + (0 \cdot r)_y] = -(r + xr_x) > 0 \quad \text{(from condition (R}_3\text{)}: x \geq 0, y \leq 0),$$

and

$$AB - \Delta^2 = (-K_1)K_2(M_1 + x\overset{\bullet}{M}_1)(M_2 - x\overset{\bullet}{M}_2) > 0 \text{ (from conditions (R}_4\text{) and (R}_5\text{)).}$$

Finally in G, $x \le 0$, $y \ge 0$ with $b = 0$, $c = y$ (from (2)) we find, from (3) and (4), that

$$A = -K_1(0 \cdot M_2)^{\bullet} + (yK_1)' M_2 + 2 \cdot 0 \cdot K_1 \overset{\bullet}{M}_2 = (yK_1)' M_2 = (K_1 + yK_1')M_2 \ge 0,$$
(from conditions (R_5) and (R_6)),

$$B = K_2(0 \cdot M_1)^{\bullet} - (yK_2)' M_1 + 2yK_2' M_1 = -K_2 M_1 - yK_2' M_1 + 2yK_2' M_1$$
$$= (-M_1)(K_2 - yK_2') > 0 \text{ (from conditions (R}_4\text{) and (R}_5\text{))}$$

$$\Gamma = -\Big[(0 \cdot r)_x + (yr)_y\Big] = -(r + yr_y) > 0 \text{ (from condition } (R_3)\text{: } x \le 0, y \ge 0),$$

and

$$AB - \Delta^2 = (-M_1)M_2(K_2 - yK_2') > 0 \quad \text{(from conditions } (R_4) \text{ and } (R_5)).$$

Therefore the region first integral I_G (of (5)) is

$$I_G = I_{\overline{G}_e} + I_{\overline{G}_{h_1}} + I_{\overline{G}_{h_2}} + I_0 > 0, \tag{7}$$

where $\overline{G}_e = G_e \cup \partial G_e$, such that $\partial G_e = g_1 \cup (OA)$, and $\overline{G}_{h_i} = G_{h_i} \cup \partial G_{h_i} (i = 1, 2)$, such that $\partial G_{h_1} = (AO) \cup g_3 \cup g_2$ and $\partial G_{h_2} = (OB) \cup \gamma_2 \cup \gamma_3$.

In fact,

$$Q = Au_x^2 + Bu_y^2 + 2\Delta u_x u_y = Q(u_x, u_y),$$

where

$$I_{\overline{G}_e} = \iint_{G,\, x\ge 0,\, y\ge 0} Q(u_x, u_y)\, dx\, dy$$
$$= \iint_{G,\, x\ge 0,\, y\ge 0} \Big[\Big(xK_1 \overset{\bullet}{M}_2 + yK_1' M_2\Big)u_x^2 + \Big(xK_2 \overset{\bullet}{M}_1 + yK_2' M_1\Big)u_y^2\Big]\, dx\, dy \ge 0$$

(from conditions (R_6) and (R_7)),

$$I_{\overline{G}_{h_1}} = \iint_{G,\, x\ge 0,\, y\le 0} Q(u_x, u_y)\, dx\, dy$$
$$= \iint_{G,\, x\ge 0,\, y\le 0} \Big[(-K_1)\Big(M_2 - x\overset{\bullet}{M}_2\Big)u_x^2 + K_2\Big(M_1 + x\overset{\bullet}{M}_1\Big)u_y^2\Big]\, dx\, dy \ge 0$$

(from conditions (R_5) and (R_7)),

$$I_{\overline{G}_{h_2}} = \iint\limits_{G,\, x\leq 0,\, y\geq 0} Q(u_x, u_y)\, dx\, dy$$
$$= \iint\limits_{G,\, x\leq 0,\, y\geq 0} \left[(K_1 + yK_1')M_2u_x^2 + (K_2 - yK_2')(-M_1)u_y^2\right] dx\, dy \geq 0$$

(from conditions (R_4) and (R_6)),

and

$$I_0 = \iint\limits_G \Gamma u^2\, dx\, dy = -\begin{cases} \iint\limits_{G,\, x\geq 0,\, y\geq 0} \left(2r + xr_x + yr_y\right)u^2\, dx\, dy > 0 \\ \iint\limits_{G,\, x\geq 0,\, y\leq 0} (r + xr_x)u^2\, dx\, dy > 0 \\ \iint\limits_{G,\, x\leq 0,\, y\geq 0} \left(r + yr_y\right)u^2\, dx\, dy > 0 \end{cases} \quad \text{(from condition (R}_3\text{)).}$$

We note that on g_1 with $b = x\ (> 0)$, $c = y\ (\geq 0)$ (from (2)) we find that

$$\tilde{A} = (xv_1 - yv_2)K_1M_2, \qquad \tilde{B} = (-xv_1 + yv_2)K_2M_1,$$
$$\tilde{\Gamma} = (xv_1 + yv_2)r, \qquad \tilde{\Delta} = xK_2M_1v_2 + yK_1M_2v_1.$$

From the boundary condition (**) we get on g_1 that $0 = du = u_x\, dx + u_y\, dy$, or

$$u_x = Nv_1, \qquad u_y = Nv_2, \tag{8}$$

on g_1 where N is a normalizing factor. We denote

$$\tilde{Q} = \tilde{Q}\ (u_x, u_y) = \tilde{A}u_x^2 + \tilde{B}u_y^2 + 2\tilde{\Delta}u_xu_y, \tag{9}$$

a quadratic form on ∂G with respect to u_x, u_y. Also we denote

$$H = K_1M_2v_1^2 + K_2M_1v_2^2, \tag{10}$$

on the boundary ∂G of the mixed domain G. From (8) and (10) the form (9) is

$$\tilde{Q} = N^2(xv_1 + yv_2)H. \tag{11}$$

From the star-likeness condition (R_2) on g_1, the fact that $H > 0$ on g_1, and (**) on g_1 as well as from (11) we get

$$I_{g_1} = \int_{g_1} \tilde{Q}(u_x, u_y)\, ds + \int_{g_1} \tilde{\Gamma}u^2 ds = \int_{g_1} N^2(xv_1 + yv_2)H\, ds + \int_{g_1} (xv_1 + yv_2)ru^2\, ds,$$

or

$$I_{g_1} = \int_{g_1} N^2(x\,dy - y\,dx)H \geq 0. \tag{12}$$

Similarly on g_2 with $b = x$ (> 0), $c = 0$ (from (2)) we get

$$I_{g_2} = \int_{g_2} \tilde{Q}(u_x, u_y)\; ds + \int_{g_2} \tilde{\Gamma}u^2\, ds = \int_{g_2} N^2(xv_1)H\, ds + \int_{g_2} (xv_1)ru^2\, ds, \quad \text{or} \\ I_{g_2} = 0, \tag{13}$$

because $u = 0$ on g_2 (from (**) and thus from (8) on g_2) and $H = 0$ on the characteristic g_2 of (*) (from (10)). On g_3 also with $b = x$ (> 0), $c = 0$ (from (2)) one gets

$$I_{g_3} = \int_{g_3} \tilde{Q}(u_x, u_y)\, ds + \int_{g_3} \tilde{\Gamma}u^2\, ds$$

$$= \int_{g_3} \Big[(xK_1M_2v_1)u_x^2 + (-xK_2M_1v_1)u_y^2 + 2(xK_2M_1v_2)u_xu_y\Big]\, ds + \int_{g_3} [(xv_1)r]u^2\, ds, \text{ or}$$

$$I_{g_3} = \int_{g_3} \Big[(K_1M_2)(xv_1)u_x^2 + (-K_2M_1)(xv_1)u_y^2 + 2(K_2M_1)(xv_2)u_xu_y\Big]\, ds$$

$$+ \int_{g_3} [r(xv_1)]u^2\, ds > 0, \tag{14}$$

because on g_3 we have $v_1 = -(M/(M-K))^{1/2} < 0$, $v_2 = -(-K/(M-K))^{1/2} < 0$, and $r < 0$ (from (R_1)), as well as

$$\tilde{A} = (K_1M_2)(xv_1) = x(-K_1)M_2(M/(M-K))^{1/2} > 0,$$

$$\tilde{B} = (-K_2M_1)(xv_1) = xK_2M_1(M/(M-K))^{1/2} > 0, \quad \text{and}$$

$$\tilde{A}\tilde{B} - (\tilde{\Delta})^2 = [(K_1M_2)(xv_1)][(-K_2M_1)(xv_1)] - [(K_2M_1)(xv_2)]^2$$

$$= -x^2K_1K_2M_1M_2v_1^2 - x^2(K_2M_1)^2v_2^2 = -x^2K_2M_1H = 0$$

because $H = 0$ on the characteristic g_3 of (*) (from (10)). Besides on γ_2 with $b = 0$, $c = y$ (from (2)) we get

$$I_{\gamma_2} = \int_{\gamma_2} \tilde{Q}\big(u_x, u_y\big)\, ds + \int_{\gamma_2} \tilde{\Gamma}u^2\, ds = \int_{\gamma_2} N^2(yv_2)H\, ds + \int_{\gamma_2} (yv_2)ru^2\, ds, \text{ or} \\ \mathrm{I}_{\gamma_2} = 0, \tag{15}$$

because $u=0$ on γ_2 (from (**) and thus from (8) on γ_2) and $H=0$ on the characteristic γ_2 of (*) (from (10)). Finally on γ_3 also with $b=0$, $c=y$ (>0) (from (2)) one gets

$$I_{\gamma_3} = \int_{\gamma_3} \tilde{Q}(u_x, u_y)\, ds + \int_{\gamma_3} \tilde{\Gamma} u^2\, ds$$
$$= \int_{\gamma_3} \left[(-yK_1M_2v_2)u_x^2 + (yK_2M_1v_2)u_y^2 + 2(yK_1M_2v_1)u_xu_y \right] ds + \int_{\gamma_3} [(yv_2)r]u^2\, ds,$$

or

$$I_{\gamma_3} = \int_{\gamma_3} \left[(-K_1M_2)(yv_2)u_x^2 + (K_2M_1)(yv_2)u_y^2 + 2(K_1M_2)(yv_1)u_xu_y \right] ds$$
$$+ \int_{\gamma_3} [r(yv_2)]u^2\, ds > 0, \tag{16}$$

because on γ_3 we have $v_1 = -(-M/(K-M))^{1/2} < 0$, $v_2 = -(K/(K-M))^{1/2} < 0$, and $r < 0$ (from (R_1)). Therefore from (12) to (16)

$$I_{\partial G} = I_{g_1} + I_{g_2} + I_{g_3} + I_{\gamma_2} + I_{\gamma_3} = I_{g_1} + I_{g_3} + I_{\gamma_3} > 0. \tag{17}$$

From (5), (7), and (17) we claim that

$$u = 0 \tag{18}$$

in G. In fact, from (5), (7), and (17) we get $0 = I_G + I_{\partial G} > 0$ with $I_G > 0$, $I_{\partial G} > 0$. These relations yield

$$I_G = I_{\partial G} = 0. \tag{19}$$

From (19): $I_G=0$ and the fact that $I_{\overline{G}_e} \geq 0$, $I_{\overline{G}_{h_i}} \geq 0$ ($i=1, 2$), $I_0 > 0$, we find that

$$I_{\overline{G}_e} = \iint_{G,\, x\geq 0,\, y\geq 0} \left[\left(xK_1\dot{M}_2 + yK_1'M_2 \right) u_x^2 + \left(xK_2\dot{M}_1 + yK_2'M_1 \right) u_y^2 \right] dx\, dy = 0,$$

yielding $u_x = u_y = 0$ in G, $x \geq 0$, $y \geq 0$ since $K_i' > 0$ and $\dot{M}_i > 0$ ($i=1,2$) from conditions (R_6) and (R_7), respectively. Thus $u=$constant in G, $x \geq 0$, $y \geq 0$, and $u=0$ on g_1 (from (**)) it will follow that

$$u(x, y) = 0 \quad \text{in } G, x \geq 0, y \geq 0. \tag{20}$$

We find also the same result as (20) if we employ

$$I_{\bar{G}_{h_1}} = 0, \quad \text{or } I_{\bar{G}_{h_2}} = 0, \quad \text{or } I_0 = 0 \quad (\text{with } r > 0 \text{ and } 2r + xr_x + yr_y > 0\colon x \geq 0, y \geq 0).$$

Similarly from (19): $I_{\partial G} = 0$ and the fact that $I_{g_1} \geq 0$, $I_{g_2} = 0$, $I_{g_3} > 0$, $I_{\gamma_2} = 0$, $I_{\gamma_3} > 0$ we get that

$$I_{g_3} = \int_{g_3} \Big[(-K_1)M_2(M/(M-K))^{1/2}\,u_x^2 + K_2M_1(M/(M-K))^{1/2}\,u_y^2$$
$$-2K_2M_1(-K/(M-K))^{1/2}u_xu_y\Big]x\,ds + \int_{g_3}(-r)(M/(M-K))^{1/2}u_x^2\,ds$$
$$= \int_{g_3}\Big[(-K_1)M_2M^{1/2}u_x^2 + K_2M_1M^{1/2}u_y^2 - 2K_2M_1(-K)^{1/2}u_xu_y$$
$$+(-r)M^{1/2}u^2\Big]x(M-K)^{-1/2}\,ds, \quad \text{or}$$
$$I_{g_3} = \int_{g_3}\Big[K_2M_2\big((-K)^{1/2}u_x - M^{1/2}u_y\big)^2 + (-r)u^2\Big]x(-dy) = 0, \tag{21}$$

yielding that

$$u = 0 \quad \text{on } g_3, \tag{22}$$

as $r<0$ on g_3 from condition (R_1). Similarly

$$I_{\gamma_3} = \int_{\gamma_3}\Big[K_2M_2\big(K^{1/2}u_x - (-M)^{1/2}u_y\big)^2 + (-r)u^2\Big]y(-dy) = 0, \tag{23}$$

yielding

$$u = 0 \quad \text{on } \gamma_3, \tag{24}$$

as $r<0$ on γ_3 from condition (R_1).

Thus by a well-known theorem on hyperbolic equations if $u=0$ on g_2 (from (**)) and $u=0$ on g_3 (from (22)) then $u=0$ in G, $x \geq 0$, $y \leq 0$. (Another reasoning is that, in particular, $u(x,0)=0$ and $u_y(x,0)=0$, so that $u=0$ in G, $x\geq 0$, $y \leq 0$, because of the uniqueness of the solution of the Cauchy problem for hyperbolic Eq. (*)). Similarly if $u=0$ on γ_2 (from (**)) and $u=0$ on γ_3 (from (24)) then $u=0$ throughout G, $x\leq 0$, $y \geq 0$. Thus

$$u(x,y) = 0,$$

everywhere in G, completing the proof of the uniqueness theorem.

Note that the case: $r=0$ in G and $K_i'(0)=M_i(0)=0$ $(i=1,2)$, yields also uniqueness results for the Problem (T).

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