GLASNIK MATEMATIČKI
Vol. 36(56)(2001), 63 – 72

SOLUTION OF THE ULAM STABILITY PROBLEM FOR CUBIC MAPPINGS

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Abstract. In 1968 S. M. Ulam proposed the general problem: When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true. In 1978 P. M. Gruber stated that this kind of stability problems are of particular interest in probability theory and in the case of functional equations of different types. In 1982-1998 we solved above Ulam problem for linear mappings and also established analogous stability problems for quadratic mappings. In this paper we introduce the new cubic mappings $C : X \to Y$, satisfying the cubic functional equation

$$C(x_1 + 2x_2) + 3C(x_1) = 3C(x_1 + x_2) + C(x_1 - x_2) + 6C(x_2)$$

for all 2-dimensional vectors $(x_1, x_2) \in X^2$, with X a linear space (Y: = a real complete linear space), and then solve the Ulam stability problem for the above-said mappings C.

1. Cubic functional equation

Definition 1.1. *Let X be a linear space and let Y be a real complete linear space. Then a mapping $C : X \to Y$, is called cubic, if the cubic functional equation*

$$C(x_1 + 2x_2) + 3C(x_1) = 3C(x_1 + x_2) + C(x_1 - x_2) + 6C(x_2) \tag{1.1}$$

holds for all 2-dimensional vectors $(x_1, x_2) \in X^2$.([14-23]).

Note that mapping C is called cubic because the following algebraic identity

$$(x_1 + 2x_2)^3 + 3x_1^3 = 3(x_1 + x_2)^3 + (x_1 - x_2)^3 + 6x_2^3$$

2000 *Mathematics Subject Classification.* 39B.
Key words and phrases. Cubic Mapping, Ulam Stability, Approximately Cubic, Approximately Odd Mapping.

holds for all $(x_1, x_2) \in R^2$ and because the functional equation

$$C(2^n x) = (2^n)^3 C(x) \tag{1.2}$$

holds for all $x \in X$ and all $n \in N$. In fact, substitution of $x_1 = x_2 = 0$ in equation (1.1) yields that

$$C(0) = 0 \tag{1.3}$$

Lemma 1.2. *Let $C : X \to Y$ be a cubic mapping satisfying equation (1.1). Then C is an odd mapping ; that is, equation*

$$C(-x) = -C(x) \tag{1.4}$$

holds for all $x \in X$.

Proof. Substituting $x_1 = x$, $x_2 = -x$ in equation (1.1) and employing (1.3) one gets that equation

$$C(-x) + 3C(x) = 3C(0) + C(2x) + 6C(-x),$$

or

$$C(2x) = 3C(x) - 5C(-x) \tag{1.5}$$

holds for all $x \in X$. Similarly substituting $x_1 = 0$, $x_2 = x$ in equation (1.1) and employing (1.3) one gets that equation

$$C(2x) + 3C(0) = 3C(x) + C(-x) + 6C(x),$$

or

$$C(2x) = 9C(x) + C(-x) \tag{1.6}$$

holds for all $x \in X$. Functional equations (1.5) and (1.6) yield the required equation (1.4), completing the proof of Lemma 1.2.

Lemma 1.3. *Let $C : X \to Y$ be a cubic mapping satisfying equation (1.1). Then C satisfies the general functional equation*

$$C(x) = 2^{-3n} C(2^n x) \tag{1.7}$$

for all $x \in X$ and all $n \in N$.

Proof. Employing equations (1.4) and (1.5) one gets that the basic equation

$$C(2x) = 8C(x),$$

or

$$C(x) = 2^{-3} C(2x) \tag{1.8}$$

holds for all $x \in X$.

Then induction on $n \in N$ with x replaced by $2^{n-1}x$ in the basic equation (1.8) yields equation (1.7). In fact, the basic equation (1.8) with x replaced by $2^{n-1}x$ yields that the functional equation

$$C(2^{n-1}x) = 2^{-3} C(2^n x) \tag{1.9}$$

holds for all $x \in X$.

Moreover replacing n by $n-1$ in the general equation (1.7) one gets that

$$C(x) = 2^{-3(n-1)} C(2^{n-1}x) \tag{1.10}$$

holds for all $x \in X$.

Thus functional equations (1.9) and (1.10) imply

$$C(x) = 2^{-3(n-1)} 2^{-3} C(2^n x),$$

or

$$C(x) = 2^{-3n} C(2^n x),$$

for all $x \in X$ and all $n \in N$ completing the proof of the required general functional equation (1.7) and hence the proof of Lemma 1.3.

2. Cubic functional inequality

DEFINITION 2.1. *Let X be a normed linear space and let Y be a real complete normed linear space. Then a mapping $f : X \to Y$, is called approximately cubic, if the cubic functional inequality*

$$\|f(x_1 + 2x_2) + 3f(x_1) - [3f(x_1 + x_2) + f(x_1 - x_2) + 6f(x_2)]\| \le c \tag{2.11}$$

holds for all 2-dimensional vectors $(x_1, x_2) \in X^2$ with a constant $c \ge 0$ (independent of x_1, x_2).

DEFINITION 2.2. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that there exists a constant $c \ge 0$ (independent of $x \in X$). Then a cubic mapping $C : X \to Y$, is said to be near an approximately cubic mapping $f : X \to Y$, if the following inequality*

$$\|f(x) - C(x)\| \le \frac{11}{42} c, \tag{2.12}$$

holds for all $x \in X$.

THEOREM 2.3. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition the above-mentioned mappings C, f and the three definitions. Then the limit*

$$C(x) = \lim_{n \to \infty} 2^{-3n} f(2^n x) \tag{2.13}$$

exists for all $x \in X$ and all $n \in N$ and $C : X \to Y$ is the unique cubic mapping near the approximately cubic mapping $f : X \to Y$.

PROOF OF EXISTENCE IN THEOREM. Substitution of $x_1 = x_2 = 0$ in inequality (2.11) yields that

$$\|4f(0) - [10f(0)]\| \le c,$$

or

$$\|f(0)\| \le \frac{c}{6} \left(= \frac{c}{3!}\right) \tag{2.14}$$

LEMMA 2.4. *Let $f : X \to Y$ be an approximately cubic mapping satisfying inequality (2.11). Then f is an approximately odd mapping ; that is, inequality*

$$\|f(-x) + f(x)\| \leq \frac{c}{3} \left(= \frac{2c}{3!}\right) \tag{2.15}$$

holds for all $x \in X$ with constant $c \geq 0$ (independent of $x \in X$).

PROOF. Substitution of $x_1 = x$, $x_2 = -x$ in inequality (2.11) yields that inequality

$$\|f(-x) + 3f(x) - [3f(0) + f(2x) + 6f(-x)]\| \leq c,$$

or

$$\| - 5f(-x) + 3f(x) - 3f(0) - f(2x)\| \leq c \tag{2.16}$$

holds for all $x \in X$. Similarly substitution of $x_1 = 0$, $x_2 = x$ in inequality (2.11) yields that inequality

$$\|f(2x) + 3f(0) - [3f(x) + f(-x) + 6f(x)]\| \leq c$$

or

$$\| - f(-x) - 9f(x) + 3f(0) + f(2x)\| \leq c \tag{2.17}$$

holds for all $x \in X$.

Functional inequalities (2.16), (2.17) and triangle inequality yield

$$\begin{aligned}
&6\|f(-x) + f(x)\| = \| - 6[f(-x) + f(x)]\| \\
&= \|[-5f(-x) + 3f(x) - 3f(0) - f(2x)] \\
&+[-f(-x) - 9f(x) + 3f(0) + f(2x)]\| \\
&\leq \| - 5f(-x) + 3f(x) - 3f(0) - f(2x)\| \\
&+\| - f(-x) - 9f(x) + 3f(0) + f(2x)\| \\
&\leq c + c = 2c,
\end{aligned}$$

or

$$\|f(-x) + f(x)\| \leq \frac{c}{3},$$

completing the proof of Lemma 2.4.

LEMMA 2.5. *Let $f : X \to Y$ be an approximately cubic mapping satisfying inequality (2.11). Then f satisfies the general functional inequality*

$$\|f(x) - 2^{-3n}f(2^n x)\| \leq \frac{11}{42}(1 - 2^{-3n})c \left(= \frac{11/7}{3!}(1 - 2^{-3n})c\right), \tag{2.18}$$

for all $x \in X$ and all $n \in N$ with constant $c \geq 0$ (independent of $x \in X$).

PROOF. Employing inequalities (2.14), (2.15), (2.17) and triangle inequality one gets that the basic inequality

$$\begin{aligned}&\|f(2x)-8f(x)\|\\&=\|[-f(-x)-9f(x)+3f(0)+f(2x)]+[f(-x)+f(x)]+[-3(f(0)]\|\\&\le\|-f(-x)-9f(x)+3f(0)+f(2x)\|+\|f(-x)+f(x)\|+3\|f(0)\|\\&\le c+\frac{c}{3}+3\frac{c}{6}=\frac{11}{6}c,\end{aligned}$$

or

$$\|f(2x)-8f(x)\|\le\frac{11}{6}c,$$

or

$$\|f(x)-2^{-3}f(2x)\|\le\frac{11}{48}c,$$

or

$$\|f(x)-2^{-3}f(2x)\|\le\frac{11}{42}(1-2^{-3})c, \tag{2.19}$$

holds for all $x\in X$ with constant $c\ge 0$ (independent of x).

Replacing now x with $2x$ in the basic inequality (2.19) one concludes that

$$\|f(2x)-2^{-3}f(2^2x)\|\le\frac{11}{42}(1-2^{-3})c,$$

or

$$\|2^{-3}f(2x)-2^{-6}f(2^2x)\|\le\frac{11}{42}(2^{-3}-2^{-6})c \tag{2.20}$$

holds for all $x\in X$.

Functional inequalities (2.19), (2.20) and the triangle inequality yield

$$\begin{aligned}\|f(x)-2^{-6}f(2^2x)\|\quad&\le\|f(x)-2^{-3}f(2x)\|+\|2^{-3}f(2x)-2^{-6}f(2^2x)\|\\&\le\tfrac{11}{42}\left[(1-2^{-3})+(2^{-3}-2^{-6})\right]c\end{aligned}$$

or that the functional inequality

$$\|f(x)-2^{-6}f(2^2x)\|\le\frac{11}{42}(1-2^{-6})c, \tag{2.21}$$

holds for all $x\in X$.

Similarly by induction on $n\in N$ with x replaced by $2^{n-1}x$ in the basic inequality (2.19) claim that the general functional inequality (2.18) holds for all $x\in X$ and all $n\in N$ with constant $c\ge 0$ (independent of $x\in X$).

In fact, the basic inequality (2.19) with x replaced by $2^{n-1}x$ yields the functional inequality

$$\|f(2^{n-1}x)-2^{-3}f(2^nx)\|\le\frac{11}{42}(1-2^{-3})c,$$

or that the functional inequality

$$\|2^{-3(n-1)}f(2^{n-1}x) - 2^{-3n}f(2^n x)\| \le \frac{11}{42}(2^{-3(n-1)} - 2^{-3n})c, \tag{2.22}$$

holds for all $x \in X$.

Moreover, by induction hypothesis with n replaced by $n-1$ in the general inequality (2.18) one gets that

$$\|f(x) - 2^{-3(n-1)}f(2^{n-1}x)\| \le \frac{11}{42}(1 - 2^{-3(n-1)})c \tag{2.23}$$

holds for all $x \in X$.

Thus the functional inequalities (2.22), (2.23) and the triangle inequality imply

$$\begin{aligned}\|f(x) - 2^{-3n}f(2^n x)\| &\le \|f(x) - 2^{-3(n-1)}f(2^{n-1}x)\| \\ &+\|2^{-3(n-1)}f(2^{n-1}x) - 2^{-3n}f(2^n x)\|,\end{aligned}$$

or

$$\begin{aligned}\|f(x) - 2^{-3n}f(2^n x)\| &\le \frac{11}{42}\Big[(1 - 2^{-3(n-1)}) \\ &+(2^{-3(n-1)} - 2^{-3n})\Big]c = \frac{11}{42}(1 - 2^{-3n})c,\end{aligned}$$

completing the proof of the required general functional inequality (2.18) and thus the proof of Lemma 2.5.

Lemma 2.6. *Let $f : X \to Y$ be an approximately cubic mapping satisfying inequality (2.11). Then the sequence*

$$\left\{2^{-3n}f(2^n x)\right\} \tag{2.24}$$

converges.

Proof. Note that from the general functional inequality (2.18) and the completeness of Y, one proves that the above-mentioned sequence (2.24) is a Cauchy sequence. In fact, if $i > j > 0$, then

$$\|2^{-3i}f(2^i x) - 2^{-3j}f(2^j x)\| = 2^{-3j}\|2^{-3(i-j)}f(2^i x) - f(2^j x)\| \tag{2.25}$$

holds for all $x \in X$, and all $i, j \in N$.

Setting $h = 2^j x$ in (2.25) and employing the afore-mentioned general functional inequality (2.18) one concludes that

$$\begin{aligned}\|2^{-3i}f(2^i x) - 2^{-3j}f(2^j x)\| &= 2^{-3j}\|2^{-3(i-j)}f(2^{i-j}h) - f(h)\| \\ &\le 2^{-3j}\tfrac{11}{42}(1 - 2^{-3(i-j)})c,\end{aligned}$$

or

$$\|2^{-3i}f(2^i x) - 2^{-3j}f(2^j x)\| \le \frac{11}{42}(2^{-3j} - 2^{-3i})c < \frac{11}{42}2^{-3j}c$$

or

$$\lim_{j\to\infty}\|2^{-3i}f(2^i x) - 2^{-3j}f(2^j x)\| = 0. \tag{2.26}$$

which yields that the sequence (2.24) is a Cauchy sequence, and thus the proof of Lemma 2.6 is complete.

LEMMA 2.7. *Let $f : X \to Y$ be an approximately cubic mapping satisfying inequality (2.11). Assume in addition a mapping $C : X \to Y$ given by the above-said formula (2.13). Then $C = C(x)$ is a well-defined mapping and that C is a cubic mapping in X.*

PROOF. Employing Lemma 2.6 and formula (2.13), one gets that C is a well-defined mapping. This means that the limit (2.13) exists for all $x \in X$.

In addition claim that C satisfies the functional equation (1.1) for all 2-dimensional vectors $(x_1, x_2) \in X^2$. In fact, it is clear from the cubic functional inequality (2.11) and the limit (2.13) that the following functional inequality

$$\begin{aligned} 2^{-3n}\|f(2^n x_1 + 2\cdot 2^n x_2) + 3f(2^n x_1) - [3f(2^n x_1 + 2^n x_2) \\ +f(2^n x_1 - 2^n x_2) + 6f(2^n x_2)]\| \le 2^{-3n}c, \end{aligned} \tag{2.27}$$

holds for all vectors $(x_1, x_2) \in X^2$ and all $n \in N$.

Therefore from inequality (2.27) one gets

$$\begin{aligned} &\| \lim_{n\to\infty} 2^{-3n} f\,[2^n(x_1 + 2x_2)] + 3\lim_{n\to\infty} 2^{-3n} f(2^n x_1) \\ &- \Big[3\lim_{n\to\infty} 2^{-3n} f\,[2^n(x_1 + x_2)] \\ &+ \lim_{n\to\infty} 2^{-3n} f\,[2^n(x_1 - x_2)] \\ &+6\lim_{n\to\infty} 2^{-3n} f(2^n x_2)\Big] \| \le \left(\lim_{n\to\infty} 2^{-3n}\right) c = 0 \end{aligned}$$

or mapping C satisfies the cubic equation (1.1) for all vectors $(x_1, x_2) \in X^2$. Thus C is a 2-dimensional cubic mapping, completing the proof of Lemma 2.7.

It is now clear from the afore-mentioned Lemmas 1.2-2.7 and especially from the general inequality (2.18), $n \to \infty$, and the formula (2.13) that inequality (2.12) holds in X, completing the existence proof in this Theorem.

PROOF OF UNIQUENESS IN THEOREM. Let $C_1 : X \to Y$ be another 2-dimensional cubic mapping satisfying the cubic functional equation (1.1), such that inequality

$$\|f(x) - C_1(x)\| \le \frac{11}{42}c \left(= \frac{11/7}{3!}c\right) \tag{2.28}$$

holds for all $x \in X$. If there exists a 2-dimensional cubic mapping $C : X \to Y$ satisfying the cubic functional equation (1.1), then

$$C(x) = C_1(x), \tag{2.29}$$

holds for all $x \in X$.

To prove the afore-mentioned uniqueness one employs equation (1.7) for C and C_1, as well, so that

$$C_1(x) = 2^{-3n} C_1(2^n x), \tag{2.30}$$

holds for all $x \in X$, and all $n \in N$. Moreover, the triangle inequality and inequalities (2.12) and (2.28) yield

$$\begin{aligned}\|C(2^n x) - C_1(2^n x)\| &\leq \|C(2^n x) - f(2^n x)\| + \|f(2^n x) - C_1(2^n x)\| \\ &\leq \frac{11}{42}c + \frac{11}{42}c = \frac{11}{21}c,\end{aligned}$$

or

$$\|C(2^n x) - C_1(2^n x)\| \leq \frac{11}{21}c \tag{2.31}$$

for all $x \in X$, and all $n \in N$. Then from equations (1.7) and (2.30), and inequality (2.31), one obtains that

$$\|C(x) - C_1(x)\| = \|2^{-3n} C(2^n x) - 2^{-3n} C_1(2^n x)\| \leq \frac{11}{21} 2^{-3n} c$$

or

$$\|C(x) - C_1(x)\| \leq \frac{11}{21} 2^{-3n} c \tag{2.32}$$

holds for all $x \in X$ and all $n \in N$. Therefore from the above inequality (2.32), and $n \to \infty$, one establishes

$$\lim_{n\to\infty} \|C(x) - C_1(x)\| \leq \frac{11}{21}\left(\lim_{n\to\infty} 2^{-3n}\right) c = 0$$

or

$$\|C(x) - C_1(x)\| = 0,$$

or

$$C(x) = C_1(x),$$

for all $x \in X$, completing the proof of uniqueness and thus the stability of this Theorem ([1-13]) and ([24-28]).

Example 2.8. Take $f : R \to R$ be a real function such that $f(x) = x^3 + k$, $k =$ constant : $|k| \leq \frac{c}{6}\left(= \frac{c}{3!}\right)$, in order that f satisfies inequality (2.11).

Moreover, there exists a unique cubic real mapping $C : R \to R$ such that from the limit (2.13) one gets

$$C(x) = \lim_{n\to\infty} 2^{-3n} f(2^n x) = \lim_{n\to\infty} 2^{-3n}\left[(2^n x)^3 + k\right] = x^3.$$

Note that inequality (2.12) holds. In fact, the above condition on k :$|k| \leq \frac{c}{6}$, implies

$$\|f(x) - C(x)\| = \|(x^3 + k) - x^3\| = |k| < \frac{11}{42}c,$$

satisfying inequality (2.12), because from inequality (2.11) one gets that

$$|k + 3k - [3k + k + 6k]| = 6|k| \leq c,$$

or

$$|k| \leq \frac{c}{6} = \frac{7}{42}c < \frac{11}{42}c.$$

References

[1] J. Aczél, "Lectures on functional equations and their applications", Academic Press, New York and London, 1966.

[2] S. Banach, Sur l' equation fonctionelle f(x+y)=f(x)+f(y), Fund. Math. 1 (1930), 123-124.

[3] I. Fenyoo, Über eine Lösungsmethode gewisser Functionalgleichungen, Acta Math. Acad. Sci. Hung. 7 (1956), 383-396.

[4] G.L. Forti, Hyers-Ulam stability of functional equations in several variables, Aequ. Math., 50 (1995), 143-190.

[5] K.G. Grosse-Erdmann, Regularity properties of functional equations and inequalities, Aequ.Math. 37 (1989), 233-251.

[6] P.M. Gruber, Stability of Isometries, Trans.Amer. Math. Soc., U.S.A., 245 (1978), 263-277.

[7] D.H. Hyers, The stability of homomorphisms and related topics, "Global Analysis-Analysis on Manifolds", Teubner - Texte zur Mathematik, 57 (1983), 140-153.

[8] A. Járai, and L. Székelyhidi, Regularization and general methods in the theory of functional equations, Aequ. Math. 52 (1996), 10-29.

[9] A.M. Kagan, Yu.V. Linnik, and C.R.C. Rao, "Characterization problems in mathematical statistics", John Wiley and Sons, New York, 1973.

[10] C.G. Khatri, and C.R. Rao, Functional equations and characterization of probability laws through linear functions of random variables, J. Multiv. Anal. 2 (1972), 162-173.

[11] M. Kuczma, "An introduction to the theory of functional equations and inequalities", Panstwowe Wydawnictwo Naukowe, Uniwersytet Slaski, Warszawa, Kraków, Katowice, 1985.

[12] S. Kurepa, A cosine functional equation in Hilbert space, Canad. J. Math. 12 (1957), 45-50.

[13] L. Paganoni, On a functional equation concerning affine transformations, J. Math. Anal. Appl. 127 (1987), 475-491.

[14] J.M. Rassias, On Approximation of Approximately Linear Mappings by Linear Mappings, J. Funct. Anal. 46 (1982), 126-130.

[15] J.M. Rassias, On Approximation of Approximately Linear Mappings by Linear Mappings, Bull. Sc. Math. 108 (1984), 445-446.

[16] J.M. Rassias, Solution of a Problem of Ulam, J. Approx. Th. 57 (1989), 268-273.

[17] J.M. Rassias, Complete Solution of the Multi-dimensional Problem of Ulam, Discuss. Math. 14 (1994), 101-107.

[18] J.M. Rassias, Solution of a Stability Problem of Ulam, Discuss. Math. 12 (1992), 95-103.

[19] J.M. Rassias, On the Stability of the Euler-Lagrange Functional Equation, Chin. J. Math. 20 (1992), 185-190.

[20] J.M. Rassias, On the Stability of the Non-linear Euler-Lagrange Functional Equation in Real Normed Linear Spaces, J. Math. Phys. Sci. 28 (1994), 231-235.

[21] J.M. Rassias, On the Stability of the Multi-dimensional Non-linear Euler-Lagrange Functional Equation, "Geometry, Analysis and Mechanics", World Sci. Publ., (1994), 275-285.

[22] J.M. Rassias, On the Stability of the General Euler-Lagrange Functional Equation, Demonstr. Math. 29 (1996), 755-766.

[23] J.M. Rassias, Solution of the Ulam Stability Problem for Euler-Lagrange quadratic mappings, J. Math. Anal. & Applications 220 (1998), 613-639.
[24] A.L. Rutkin, The solution of the functional equation of d' Alembert's type for commutative groups, Internat. J. Math. Sci. 5 (1982), No 2.
[25] L. Székelyhidi, Functional equations on Abelian groups, Acta Math. Acad. Sci. Hungar. 37 (1981), 235-243.
[26] L. Székelyhidi, Note on Hyers' theorem, C. R. Math. Rep. Sci. Canada 8 (1986), 127-129.
[27] A. Tsutsumi, and S. Haruki, On hypoelliptic functional equations, Math. Japonica 36 (3) (1991), 581-590.
[28] S.M. Ulam, "A collection of mathematical problems", Interscience Publishers, Inc., New York, 1968, p.63.

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Received: 19.03.99.
Revised: 25.11.99.
Revised: 08.02.2000.