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Solution of the Ulam Stability Problem for Euler–Lagrange Quadratic Mappings

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In 1940 S. M. Ulam proposed at the University of Wisconsin the *problem*: "Give conditions in order for a linear mapping near an approximately linear mapping to exist." In 1968 S. U. Ulam proposed the more *general problem*: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" In 1978 P. M. Gruber proposed the *Ulam type problem*: "Suppose a mathematical object satisfies a certain property approximately. Is it then possible to approximate this object by objects, satisfying the property exactly?" According to P. M. Gruber this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1982–1996 we solved the above Ulam problem, or equivalently the Ulam type problem for linear mappings and established analogous stability problems. In this paper we first introduce new *quadratic weighted means* and *fundamental functional equations* and then solve the *Ulam stability problem* for *non-linear Euler–Lagrange quadratic mappings* Q: $X \to Y$, satisfying a mean equation and functional equation

$$m_1 m_2 Q(a_1 x_1 + a_2 x_2) + Q(m_2 a_2 x_1 - m_1 a_1 x_2) = \left(m_1 a_1^2 + m_2 a_2^2\right)\left[m_2 Q(x_1) + m_1 Q(x_2)\right]$$

for all 2-dimensional vectors $(x_1, x_2) \in X^2$, with X a normed linear space ($Y :=$ a real complete normed linear space), and any fixed pair (a_1, a_2) of reals a_i and any fixed pair (m_1, m_2) of positive reals m_i $(i = 1, 2)$,

$$0 < m = \frac{m_1 + m_2}{m_1 m_2 + 1}\left(m_1 a_1^2 + m_2 a_2^2\right).$$

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1. FUNDAMENTAL FUNCTIONAL EQUATION OF FIRST TYPE

Let X be a normed linear space and let Y be a real complete normed linear space. Then consider a non-linear mapping $Q\colon X \to Y$ satisfying the *fundamental functional equation*

$$m_1^2 m_2 Q(a_1 x) + m_1 Q(m_2 a_2 x) = m_0^2 m_2 Q\left(\frac{m_1}{m_0} a_1 x\right) + m_0^2 m_1 Q\left(\frac{m_2}{m_0} a_2 x\right), \tag{*}$$

with

$$m_0 = \frac{m_1 m_2 + 1}{m_1 + m_2}$$

for all $x \in X$, and any fixed reals a_i and positive reals m_i $(i = 1, 2)$:

$$m = \frac{m_1 a_1^2 + m_2 a_2^2}{m_0} > 1.$$

Note that if $m_1 = 1$, $m_2 > 0$, then $m_0 = 1$, $m = a_1^2 + m_2 a_2^2$, and (*) is an *identity* in X. In this case (*) is *not* required.

Moreover this mapping Q may be called *quadratic* because if $Q(x) = x^2$, then (*) holds.

Definition 1. Let X be a normed linear space and let Y be a real complete normed linear space. Then a *non-linear mapping* $Q\colon X \to Y$ is called *Euler–Lagrange quadratic* if (*) and

$$m_1 m_2 Q(a_1 x_1 + a_2 x_2) + Q(m_2 a_2 x_1 - m_1 a_1 x_2) = \left(m_1 a_1^2 + m_2 a_2^2\right)\left[m_2 Q(x_1) + m_1 Q(x_2)\right] \tag{1}$$

hold for all 2-dimensional vectors $(x_1, x_2) \in X^2$, and any fixed reals a_i and positive reals m_i $(i = 1, 2)$: $m > 1$ [4–13].

Note that mapping Q may be called *quadratic*, as well, because the following *Euler–Lagrange identity*

$$m_1 m_2 (a_1 x_1 + a_2 x_2)^2 + (m_2 a_2 x_1 - m_1 a_1 x_2)^2 = \left(m_1 a_1^2 + m_2 a_2^2\right)\left[m_2 x_1^2 + m_1 x_2^2\right]$$

holds with any fixed reals a_i and m_i $(i = 1, 2)$, and because the functional equation

$$Q(m^n x) = (m^n)^2 Q(x), \tag{2}$$

holds for all $x \in X$, all $n \in N$, and any fixed reals a_i and positive reals m_i $(i = 1, 2)$: $m > 1$.

In fact, substitution of $x_1 = x_2 = 0$ in Eq. (1) yields

$$(m_1 m_2 + 1)(1 - m)Q(0) = 0,$$

or

$$Q(0) = 0, \qquad m > 1 \text{ (and } m_1, m_2 > 0). \tag{1a}$$

Substituting $x_1 = x$, $x_2 = 0$ in (1) and employing (1a) one gets that

$$m_1 m_2 Q(a_1 x) + Q(m_2 a_2 x) = \frac{m_1 m_2 + 1}{m_1 + m_2} m\left[m_2 Q(x) + m_1 Q(0)\right],$$

or

$$\frac{m_1}{m_0} Q(a_1 x) + \frac{1}{m_0 m_2} Q(m_2 a_2 x) = mQ(x), \tag{2a}$$

holds for all $x \in X$, and any fixed real m: $m > 1$.

Moreover substitution of $x_1 = (m_1 a_1 / m_0)x$, $x_2 = (m_2 a_2 / m_0)x$ in (1) and using (1a) one finds that

$$m_1 m_2 Q(mx) + Q(0) = m_0 m\left[m_2 Q\left(\frac{m_1 a_1}{m_0} x\right) + m_1 Q\left(\frac{m_2 a_2}{m_0} x\right)\right],$$

or

$$\frac{m_0}{m_1 m_2}\left[m_2 Q\left(\frac{m_1 a_1}{m_0} x\right) + m_1 Q\left(\frac{m_2 a_2}{m_0} x\right)\right] = m^{-1} Q(mx),$$

or

$$\frac{m_0}{m_1} Q\left(\frac{m_1}{m_0} a_1 x\right) + \frac{m_0}{m_2} Q\left(\frac{m_2}{m_0} a_2 x\right) = m^{-1} Q(mx), \qquad m > 1, \tag{2b}$$

holds for all $x \in X$.

Functional Eqs. (2a)–(2b) and (*) yield

$$Q(mx) = (m)^2 Q(x), \tag{2c}$$

for all $x \in X$, and any fixed real m: $m > 1$.

Then induction on $n \in N$ with $x \to m^{n-1} x$ yields Eq. (2).

DEFINITION 2. Let X be a normed linear space and let Y be a real complete normed linear space. Then we call the non-linear mappings $\overline{Q}$: $X \to Y$, and $\overline{\overline{Q}}$: $X \to Y$ 2-*dimensional quadratic weighted means of first and second form* if

$$\overline{Q}(x) = m_0^2 \frac{m_2 Q((m_1/m_0)a_1 x) + m_1 Q((m_2/m_0)a_2 x)}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)} \tag{3$_1$}$$

and

$$\overline{\overline{Q}}(x) = \frac{m_1 m_2 Q(a_1 x) + Q(m_2 a_2 x)}{m_2 (m_1 a_1^2 + m_2 a_2^2)} \tag{3$_2$}$$

hold for all $x \in X$ and any fixed real $m > 1$, respectively.

Note that the fundamental functional equation (*) is equivalent to *the mean functional equation*,

$$\overline{\overline{Q}}(x) = \overline{Q}(x), \tag{[*]}$$

for all $x \in X$, and any fixed real m: $m > 1$.

Moreover note that in the case of Eqs. (*) and (1), formulas $(3)_i$ $(i = 1, 2)$, from [*] and (2a), are of the form

$$\overline{\overline{Q}}(x) = \overline{Q}(x) = Q(x), \tag{3a}$$

for all $x \in X$, and any fixed real m: $m > 1$ [2].

THEOREM 1. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that f: $X \to Y$ is a mapping for which there exists a constant $c' \geq 0$ such that the fundamental functional inequality*

$$\left\| \bar{\bar{f}}(x) - \bar{f}(x) \right\| \leq \frac{c'}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)} \tag{4$_1$}$$

holds for all $x \in X$, c' $(:=$ const. indep. of $x)$ ≥ 0, and any fixed reals a_1, a_2 and positive reals m_1, m_2: $m > 1$, where

$$\bar{f}(x) = m_0^2 \frac{m_2 f((m_1/m_0)a_1 x) + m_1 f((m_2/m_0)a_2 x)}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)},$$

$$\text{with } m_0 = \frac{m_1 m_2 + 1}{m_1 + m_2},$$

and

$$\bar{\bar{f}}(x) = \frac{m_1 m_2 f(a_1 x) + f(m_2 a_2 x)}{m_2(m_1 a_1^2 + m_2 a_2^2)},$$

are 2*-dimensional quadratic-weighted means of first and second form, respectively, for fixed real* $m > 1$.

Assume in addition that f: $X \to Y$ *is a mapping for which there exists a constant* c (*independent of* x_1, x_2) ≥ 0 *such that the Euler–Lagrange functional inequality*

$$\begin{aligned}\|m_1 m_2 f(a_1 x_1 + a_2 x_2) + f(m_2 a_2 x_1 - m_1 a_1 x_2) \\ -(m_1 a_1^2 + m_2 a_2^2)[m_2 f(x_1) + m_1 f(x_2)]\| \leq c \qquad (4)_2\end{aligned}$$

holds for all 2*-dimensional vectors* $(x_1, x_2) \in X^2$ *and any fixed reals* a_1, a_2 *and positive reals* m_1, m_2:

$$m = \frac{m_1 a_1^2 + m_2 a_2^2}{m_0} = \frac{m_1 + m_2}{m_1 m_2 + 1}(m_1 a_1^2 + m_2 a_2^2) > 1.$$

Then the limit

$$Q(x) = \lim_{n \to \infty} m^{-2n} f(m^n x) \qquad (5)$$

exists for all $x \in X$, *all* $n \in N$, *and any fixed real* m: $m > 1$ *and* Q: $X \to Y$ *is the unique* 2*-dimensional quadratic mapping satisfying functional equation* (1) *and mean equation* [*] *or equivalently* (1) *and* (*), *such that*

$$\|f(x) - Q(x)\| \leq c_1, \qquad m > 1 \qquad (6)$$

holds for all $x \in X$ *with constant*

$$\begin{aligned}c_1 = \{[m_1(2m_1 + m_2)m^2 + (1 - m_1^2)m - m_1 m_2]c \\ + [(m_1 + m_2)(m-1)m]c'\}/[m_1 m_2(m_1 m_2 + 1) \\ \times (m-1)^2(m+1)].\end{aligned}$$

Moreover, identity

$$Q(x) = m^{-2n} Q(m^n x) \qquad (6a)$$

holds for all $x \in X$, *all* $n \in N$, *and any real* a_1, a_2 *and fixed positive reals* m_1, m_2: $m > 1$.

Note that if one replaces $x = 0$ in $\bar{f}(x)$, and $\bar{\bar{f}}(x)$ for $m > 1$, then

$$\bar{f}(0) = \frac{m_1 m_2 + 1}{m_1 m_2} \frac{1}{m} f(0), \qquad \bar{\bar{f}}(0) = \frac{m_1 + m_2}{m_2} \frac{1}{m} f(0),$$

or

$$\bar{\bar{f}}(0) - \bar{f}(0) = \frac{m_1^2 - 1}{m_1 m_2} \frac{1}{m} f(0),$$

or

$$\left(\| f(0) \| \leq \frac{c}{(m_1 m_2 + 1)(m - 1)}, \right.$$

$$\left. m > 1 \text{ after substitution of } x_1 = x_2 = 0 \text{ in } (4)_2 \right)$$

$$\begin{aligned} &\left\| \bar{\bar{f}}(0) - \bar{f}(0) \right\| \\ &\leq \frac{|m_1^2 - 1|}{m_1 m_2 (m_1 m_2 + 1)(m - 1)} \frac{1}{m} c \leq \frac{m_1 + m_2}{m_1 m_2 (m_1 m_2 + 1)} \frac{1}{m} c' \\ &= \frac{c'}{m_1 m_2 \left(m_1 a_1^2 + m_2 a_2^2 \right)}, \end{aligned}$$

$$\text{if } c' \geq \frac{|m_1^2 - 1|}{(m_1 + m_2)(m - 1)} c, \text{ for } m > 1, m_i > 0 \ (i = 1, 2).$$

Moreover note that if $m_1 = m_2 = 1$, then $m_0 = 1$, and $m = a_1^2 + a_2^2 > 1$. In this case

$$\bar{f}(x) = \frac{f(a_1 x) + f(a_2 x)}{a_1^2 + a_2^2} = \bar{\bar{f}}(x).$$

Thus the fundamental functional inequality $(4)_1$ (or constant c') is *not* required (because $\bar{f}(x) = \bar{\bar{f}}(x)$), yielding

$$c_1 = \frac{3m^2 - 1}{2(m - 1)^2 (m + 1)} c.$$

Therefore one gets from Theorem 1 the following Theorem 1a.

THEOREM 1a. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that f: $X \to Y$ is a* *mapping for which there exists a constant* $c \geq 0$ *such that the Euler–Lagrange functional inequality.*

$$\left\| f(a_1 x_1 + a_2 x_2) + f(a_2 x_1 - a_1 x_2) - \left(a_1^2 + a_2^2\right)\left[f(x_1) + f(x_2)\right] \right\| \leq c \tag{4a}$$

holds for all 2*-dimensional vectors* $(x_1, x_2) \in X^2$ *and* c ($:=$ *const. indep. of* x_1, x_2) ≥ 0, *and any fixed reals* a_1, a_2: $m = a_1^2 + a_2^2 > 1$.

Then the limit

$$Q(x) = \lim_{n \to \infty} m^{-2n} f(m^n x),$$

exists for all $x \in X$, *all* $n \in N$, *and any fixed real* m: $m > 1$ *and* Q: $X \to Y$ *is the unique* 2*-dimensional quadratic mapping satisfying the functional equation*

$$Q(a_1 x_1 + a_2 x_2) + Q(a_2 x_1 - a_1 x_2) = \left(a_1^2 + a_2^2\right)\left[Q(x_1) + Q(x_2)\right],$$

such that

$$\|f(x) - Q(x)\| \leq \frac{1}{2} \frac{3m^2 - 1}{(m-1)^2 (m+1)} c,$$

and

$$Q(x) = m^{-2n} Q(m^n x)$$

for all $x \in X$, *all* $n \in N$, *and any fixed reals* a_1, a_2: $m > 1$.

Note that if $m_1 = 1$, $m_2 > 0$, then $m_0 = 1$, and $m = a_1^2 + m_2 a_2^2 > 1$. In this case

$$\bar{f}(x) = \frac{m_2 f(a_1 x) + f(m_2 a_2 x)}{m_2\left(a_1^2 + m_2 a_2^2\right)} = \bar{\bar{f}}(x).$$

Thus the fundamental inequality $(4)_1$ (or constant c') is not required (because $\bar{f}(x) = \bar{\bar{f}}(x)$), yielding

$$c_1 = \frac{(m_2 + 2)m^2 - m_2}{m_2(m_2 + 1)(m-1)^2(m+1)} c.$$

Therefore one gets from Theorem 1 the following Theorem 1b.

THEOREM 1b. *Let* X *be a normed linear space and let* Y *be a real complete normed linear space. Assume in addition that* f: $X \to Y$ *is a mapping for which there exists a constant* $c \geq 0$ *such that the Euler–Lagrange* *functional inequality*

$$\begin{aligned}\|m_2 f(a_1x_1 + a_2x_2) &+ f(m_2a_2x_1 - a_1x_2)\\ &-(a_1^2 + m_2a_2^2)[m_2f(x_1) + f(x_2)]\| \leq c \end{aligned}\tag{4b}$$

holds for all 2*-dimensional vectors* $(x_1, x_2) \in X^2$ *and* c ($:=$ *const. indep. of* x_1, x_2) ≥ 0, *and any fixed reals* a_1, a_2 *and positive real* m_2: $m = a_1^2 + m_2a_2^2 > 1$.

Then the limit

$$Q(x) = \lim_{n\to\infty} m^{-2n} f(m^n x),$$

exists for all $x \in X$ *and any fixed real* m: $m > 1$ *and* Q: $X \to Y$ *is the unique* 2*-dimensional quadratic mapping satisfying the functional equation*

$$\begin{aligned}m_2Q(a_1x_1 + a_2x_2) &+ Q(m_2a_2x_1 - a_1x_2)\\ &= (a_1^2 + m_2a_2^2)[m_2Q(x_1) + Q(x_2)],\end{aligned}$$

such that

$$\|f(x) - Q(x)\| \leq \frac{(m_2+2)m^2 - m_2}{m_2(m_2+1)(m-1)^2(m+1)}c,$$

and

$$Q(x) = m^{-2n}Q(m^n x)$$

for all $x \in X$, *all* $n \in N$, *and any fixed reals* a_1, a_2 *and positive real* m_2: $m > 1$.

Proof of Existence in Theorem 1. Substitution of, $x_1 = x_2 = 0$, in inequality $(4)_2$ with $m_1a_1^2 + m_2a_2^2 = mm_0$, where $m_0 = (m_1m_2+1)/(m_1 + m_2)$ yields that

$$\|m_1m_2f(0) + f(0) - mm_0[(m_2+m_1)f(0)]\| \leq c,$$

or

$$\|f(0)\| \leq \frac{c}{(m_1m_2+1)(m-1)}, \qquad m > 1. \tag{7}$$

Moreover substituting $x_1 = x$, $x_2 = 0$ in inequality $(4)_2$ and employing (7) and the triangle inequality one concludes the functional inequality

$$\|m_1m_2f(a_1x) + f(m_2a_2x) - m_0m[m_2f(x) + m_1f(0)]\| \leq c,$$

or

$$\left\| \bar{\bar{f}}(x) - f(x) \right\| \leq \frac{c}{m_0 m_2 m} + \frac{m_1}{m_2} \frac{c}{(m_1 m_2 + 1)(m - 1)},$$

or

$$\left\| \bar{\bar{f}}(x) - f(x) \right\| \leq \frac{(2m_1 + m_2)m - (m_1 + m_2)}{m_2(m_1 m_2 + 1)(m - 1)m} c, \qquad m > 1, \quad (8)$$

where

$$\bar{\bar{f}}(x) = \frac{m_1 m_2 f(a_1 x) + f(m_2 a_2 x)}{m_2\left(m_1 a_1^2 + m_2 a_2^2\right)}, \qquad m > 1 \qquad (8a)$$

is the 2-dimensional *quadratic weighted mean of second form* (for $m > 1$).

In addition replacing

$$x_1 = \frac{m_1 a_1}{m_0} x, \qquad x_2 = \frac{m_2 a_2}{m_0} x$$

in inequality $(4)_2$ and using (7) and (8a), and the triangle inequality, one gets the functional inequality

$$\left\| m_1 m_2 f(mx) + f(0) - m_0 m \left[m_2 f\left(\frac{m_1 a_1}{m_0} x \right) + m_1 f\left(\frac{m_2 a_2}{m_0} x \right) \right] \right\| \leq c,$$

or

$$\left\| \bar{f}(x) - m^{-2} f(mx) \right\| \leq \frac{(m_1 m_2 + 1)(m - 1) + 1}{m_1 m_2 (m_1 m_2 + 1)(m - 1)m^2} c, \qquad m > 1, \qquad (9)$$

where

$$\bar{f}(x) = m_0^2 \frac{m_2 f((m_1/m_0) a_1 x) + m_1 f((m_2/m_0) a_2 x)}{m_1 m_2 \left(m_1 a_1^2 + m_2 a_2^2\right)}, \qquad m > 1, \quad (9a)$$

is the 2-*dimensional quadratic weighted mean of first form* (for $m > 1$).

Moreover

$$\left\| \bar{\bar{f}}(x) - \bar{f}(x) \right\| = \left\| \frac{m_1 m_2 f(a_1 x) + f(m_2 a_2 x)}{m_2 m_0 m} - \frac{m_0 [m_2 f((m_1/m_0) a_1 x) + m_1 f((m_2/m_0) a_2 x)]}{m_1 m_2 m} \right\|,$$

or

$$\left\| \bar{\bar{f}}(x) - \bar{f}(x) \right\| = \frac{\| m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_2 f((m_1/m_0) a_1 x) - m_0^2 m_1 f((m_2/m_0) a_2 x) \|}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)}.$$

Employing the *fundamental functional inequality* $(4)_1$ one gets the equivalent *inequality*

$$\left\| m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_2 f\left(\frac{m_1}{m_0} a_1 x \right) - m_0^2 m_1 f\left(\frac{m_2}{m_0} a_2 x \right) \right\| \le c', \tag{4a$_1$}$$

for all $x \in X$, and any fixed real $m > 1$.

Functional inequalities (8)–(9) and $(4)_1$ (or $(4\text{a})_1$), and the triangle inequality yield *the basic inequality*,

$$\begin{aligned} \| f(x) - m^{-2} f(mx) \| &\le \left\| f(x) - \bar{\bar{f}}(x) \right\| + \left\| \bar{\bar{f}}(x) - \bar{f}(x) \right\| + \left\| \bar{f}(x) - m^{-2} f(mx) \right\| \\ &\le \frac{(2m_1 + m_2)m - (m_1 + m_2)}{m_2 (m_1 m_2 + 1)(m - 1)m} c + \frac{m_1 + m_2}{m_1 m_2 (m_1 m_2 + 1) m} c' \\ &\quad + \frac{(m_1 m_2 + 1)(m - 1) + 1}{m_1 m_2 (m_1 m_2 + 1)(m - 1) m^2} c \\ &= \frac{[m_1 (2m_1 + m_2) m^2 - m_1 (m_1 + m_2) m + (m_1 m_2 + 1)(m - 1) + 1] c + (m_1 + m_2)(m - 1) m c'}{m_1 m_2 (m_1 m_2 + 1)(m - 1) m^2} \end{aligned}$$

or

$$\|f(x) - m^{-2}f(mx)\| \leq \frac{\left[m_1(2m_1 + m_2)m^2 + \left(1 - m_1^2\right)m - m_1m_2\right]c + (m_1 + m_2)(m - 1)mc'}{m_1m_2(m_1m_2 + 1)(m - 1)m^2}$$

or

$$\|f(x) - m^{-2}f(mx)\| \leq c_1(1 - m^{-2}), \qquad m > 1, \tag{10}$$

where

$$c_1 = \frac{\left[m_1(2m_1 + m_2)m^2 + \left(1 - m_1^2\right)m - m_1m_2\right]c + \left[(m_1 + m_2)(m - 1)m\right]c'}{m_1m_2(m_1m_2 + 1)(m - 1)^2(m + 1)}. \tag{10a}$$

For instance, if $m_1 = m_2 = 1$ and $a_1 = a_2 = 1$, then $m_0 = 1$, and $\underline{m} = 2 > 1$. In this case there is *no c'-part* in c_1 (formula (10a)) because $\bar{\bar{f}}(x) = \bar{f}(x)$. Hence $c_1 = (11/6)c$.

Note that in this case a *better constant* $c_1 = (1/2)c(< (11/6)c)$ may be found if *new* substitution $x_1 = x_2 = x$ is applied into Eq. $(4)_1$ with $m_i = a_i = 1$ $(i = 1, 2)$. In fact, $\|f(2x) + f(0) - 4f(x)\| \leq c$ with $\|f(0)\| \leq c/2$, or

$$\|f(2x) - 4f(x)\| \leq c + \|f(0)\| \leq \frac{3}{2}c,$$

or

$$\|f(x) - 2^{-2}f(2x)\| \leq c_1(1 - 2^{-2}), \qquad c_1 = \frac{1}{2}c. \tag{11}$$

Replacing now x with mx in (10) one concludes that

$$\|f(mx) - m^{-2}f(m^2x)\| \leq c_1(1 - m^{-2}),$$

or

$$\|m^{-2}f(mx) - m^{-4}f(m^2x)\| \leq c_1(m^{-2} - m^{-4}) \tag{10b}$$

holds for all $x \in X$ and any fixed real m: $m > 1$.

Functional inequalities (10)–(10b) and the triangle inequality yield

$$\begin{aligned}\|f(x) - m^{-4}f(m^2x)\| &\leq \|f(x) - m^{-2}f(mx)\| + \|m^{-2}f(mx) - m^{-4}f(m^2x)\| \\ &\leq c_1\left[(1 - m^{-2}) + (m^{-2} - m^{-4})\right],\end{aligned}$$

or

$$\| f(x) - m^{-4} f(m^2 x) \| \le c_1 (1 - m^{-4}), \qquad m > 1, \tag{10c}$$

holds for all $x \in X$.

Similarly by induction on $n \in N$ with $x \to m^{n-1} x$ in (10) claim that *the general functional inequality*

$$\| f(x) - m^{-2n} f(m^n x) \| \le c_1 (1 - m^{-2n}), \qquad m > 1, \tag{12}$$

holds for all $x \in X$, all $n \in N$, and any fixed real m: $m > 1$.

In fact, the basic inequality (10) with $x \to m^{n-1} x$ yields inequality

$$\| f(m^{n-1} x) - m^{-2} f(m^n x) \| \le c_1 (1 - m^{-2}),$$

or

$$\begin{aligned} &\| m^{-2(n-1)} f(m^{n-1} x) - m^{-2n} f(m^n x) \| \\ &\qquad \le c_1 (m^{-2(n-1)} - m^{-2n}), \qquad m > 1, \end{aligned} \tag{12a}$$

for all $x \in X$.

By induction hypothesis with $n \to n - 1$ in (12) inequality

$$\| f(x) - m^{-2(n-1)} f(m^{n-1} x) \| \le c_1 (1 - m^{-2(n-1)}), \qquad m > 1 \tag{12b}$$

holds for all $x \in X$.

Thus functional inequalities (12a)–(12b) and the triangle inequality imply

$$\begin{aligned} &\| f(x) - m^{-2n} f(m^n x) \| \\ &\qquad \le \| f(x) - m^{-2(n-1)} f(m^{n-1} x) \| \\ &\qquad\quad + \| m^{-2(n-1)} f(m^{n-1} x) - m^{-2n} f(m^n x) \|, \end{aligned}$$

or

$$\begin{aligned} &\| f(x) - m^{-2n} f(m^n x) \| \\ &\qquad \le c_1 \left[(1 - m^{-2(n-1)}) + (m^{-2(n-1)} - m^{-2n}) \right] \\ &\qquad = c_1 (1 - m^{-2n}), \qquad m > 1 \end{aligned}$$

completing the proof of the required functional inequality (12).

Claim now that the sequence

$$\{ m^{-2n} f(m^n x) \}$$

converges.

Note that from the general inequality (12) and the completeness of Y, one proves that the above sequence is a *Cauchy sequence*.

In fact, if $i > j > 0$, then

$$\|m^{-2i}f(m^i x) - m^{-2j}f(m^j x)\| = m^{-2j}\|m^{-2(i-j)}f(m^i x) - f(m^j x)\|, \tag{13}$$

for all $x \in X$, all $i, j \in N$, and any fixed real $m > 1$.

Setting $h = m^j x$ in (13) and employing general inequality (12) one concludes that

$$\begin{aligned}\|m^{-2i}f(m^i x) - m^{-2j}f(m^j x)\| &= m^{-2j}\|m^{-2(i-j)}f(m^{i-j}h) - f(h)\| \\ &\le m^{-2j}c_1(1 - m^{-2(i-j)}),\end{aligned}$$

or

$$\|m^{-2i}f(m^i x) - m^{-2j}f(m^j x)\| \le c_1(m^{-2j} - m^{-2i}) < c_1 m^{-2j},$$

or

$$\lim_{j\to\infty} \|m^{-2i}f(m^i x) - m^{-2j}f(m^j x)\| = 0 \tag{13a}$$

completing the proof that the sequence $\{m^{-2n}f(m^n x)\}$ converges. Hence $Q = Q(x)$ is a *well-defined mapping* via the formula (5). This means that the limit (5) exists for all $x \in X$.

In addition claim that mapping Q satisfies the functional equation (1) for all vectors $(x_1, x_2) \in X^2$.

In fact, it is clear from functional inequality $(4)_2$ and the limit (5) that inequality

$$\begin{aligned}m^{-2n}\|m_1 m_2 f(a_1 m^n x_1 + a_2 m^n x_2) + f(m_2 a_2 m^n x_1 - m_1 a_1 m^n x_2) \\ - (m_1 a_1^2 + m_2 a_2^2)[m_2 f(m^n x_1) + m_1 f(m^n x_2)]\| \le m^{-2n}c\end{aligned} \tag{14}$$

holds for all $x_1, x_2 \in X$, all $n \in N$, and any fixed real $m > 1$.

Therefore from inequality (14) one gets

$$\begin{aligned}&\Big\| m_1 m_2 \lim_{n\to\infty} m^{-2n}f[m^n(a_1x_1 + a_2x_2)] \\ &\quad + \lim_{n\to\infty} m^{-2n}f[m^n(m_2a_2x_1 - m_1a_1x_2)] - (m_1a_1^2 + m_2a_2^2) \\ &\quad \times\Big[m_2 \lim_{n\to\infty} m^{-2n}f(m^n x_1) + m_1 \lim_{n\to\infty} m^{-2n}f(m^n x_2)\Big]\Big\| \\ &\qquad \le \Big(\lim_{n\to\infty} m^{-2n}\Big)c = 0, \qquad m > 1,\end{aligned}$$

or

$$\|m_1 m_2 Q(a_1 x_1 + a_2 x_2) + Q(m_2 a_2 x_1 - m_1 a_1 x_2) - (m_1 a_1^2 + m_2 a_2^2)[m_2 Q(x_1) + m_1 Q(x_2)]\| = 0,$$

or mapping Q *satisfies* the functional equation (1) for all $x_1, x_2 \in X$, and $m > 1$. Thus Q is a 2-dimensional *quadratic mapping*. It is clear now from general inequality (12), $n \to \infty$, and formula (5) that inequality (6) holds in X, completing *the existence proof* of this Theorem 1.

Proof of Uniqueness in Theorem 1. Let Q': $X \to Y$ be *another* 2-dimensional quadratic mapping satisfying functional equation (1), such that

$$\|f(x) - Q'(x)\| \le c_1, \tag{6$'$}$$

for all $x \in X$, and any fixed real $m > 1$.

If there exists a 2-dimensional quadratic mapping Q: $X \to Y$ satisfying Eq. (1), then

$$Q(x) = Q'(x), \tag{15}$$

for all $x \in X$, and any fixed real $m > 1$.

To prove the above-mentioned *uniqueness* employ (6a) for Q and Q', as well, so that

$$Q'(x) = m^{-2n} Q'(m^n x) \tag{6a$'$}$$

holds for all $x \in X$, all $n \in N$, and any fixed $m > 1$.

Moreover the triangle inequality and functional inequalities (6)–(6)$'$ yield

$$\|Q(m^n x) - Q'(m^n x)\| \le \|Q(m^n x) - f(m^n x)\| + \|f(m^n x) - Q'(m^n x)\|,$$

or

$$\|Q(m^n x) - Q'(m^n x)\| \le 2c_1, \tag{16}$$

for all $x \in X$, all $n \in N$, and any fixed real $m > 1$.

Then from (6a)–(6a)$'$, and (16), one proves that

$$\|Q(x) - Q'(x)\| = \|m^{-2n} Q(m^n x) - m^{-2n} Q'(m^n x)\|,$$

or

$$\|Q(x) - Q'(x)\| \le 2m^{-2n} c_1, \tag{16a}$$

holds for all $x \in X$, all $n \in N$, and any fixed real $m > 1$.

Therefore from (16a), and $n \to \infty$, one establishes

$$\lim_{n\to\infty} \|Q(x) - Q'(x)\| \leq 2\Big(\lim_{n\to\infty} m^{-2n}\Big)c_1 = 0, \qquad m > 1,$$

or

$$\|Q(x) - Q'(x)\| = 0,$$

or

$$Q(x) = Q'(x), \qquad m > 1, \tag{17}$$

for all $x \in X$, completing the proof of *uniqueness* and thus the *stability* of Theorem 1.

2. FUNDAMENTAL FUNCTIONAL EQUATION OF SECOND TYPE

We note that *an analogous definition* to Definition 1 holds for quadratic mapping Q (for $0 < m < 1$) if we replace $m > 1$ in Definition 1 with $0 < m < 1$ and keep the rest of Definition 1 unchanged.

Moreover the functional equation

$$Q(m^{-n}x) = (m^{-n})^2 Q(x), \qquad 0 < m < 1, \tag{2$'$}$$

holds for all $x \in X$, all $n \in N$, and any fixed real m: $0 < m < 1$.

Similarly substitution of $x_1 = x_2 = 0$ in (1) yields

$$Q(0) = 0, \qquad 0 < m < 1 \text{ (and fixed } m_1, m_2 > 0). \tag{1a$'$}$$

Substituting $x_1 = x/m$, $x_2 = 0$ in (1) and employing (1a)$'$ one finds that

$$m_1 m_2 Q\left(\frac{a_1}{m}x\right) + Q\left(m_2 \frac{a_2}{m}x\right) = m_0 m_2 m Q(m^{-1}x),$$

or

$$\frac{m_1}{m_0} Q\left(\frac{a_1}{m}x\right) + \frac{1}{m_0 m_2} Q\left(m_2 \frac{a_2}{m}x\right) = mQ(m^{-1}x), \tag{2a$'$}$$

holds for all $x \in X$ and any fixed real a_i and positive real m_i ($i = 1, 2$): $0 < m < 1$.

In addition substituting $x_1 = (m_1 a_1/m_0 m)x$, $x_2 = (m_2 a_2/m_0 m)x$ in (1) and employing (1a)′ one gets that

$$m_1 m_2 Q(x) = m_0 m\left[m_2 Q\left(\frac{m_1 a_1}{m_0 m}x\right) + m_1 Q\left(\frac{m_2 a_2}{m_0 m}x\right)\right],$$

or

$$\frac{m_0}{m_1} Q\left(\frac{m_1 a_1}{m_0 m}x\right) + \frac{m_0}{m_2} Q\left(\frac{m_2 a_2}{m_0 m}x\right) = m^{-1} Q(x), \qquad (2b)'$$

holds for all $x \in X$ and any fixed real a_i and positive real m_i $(i = 1, 2)$: $0 < m < 1$.

Let X be a normed linear space and let Y be a real complete normed linear space. Then consider a non-linear mapping Q: $X \to Y$ satisfying the *fundamental functional equation*

$$m_1^2 m_2 Q\left(\frac{a_1}{m}x\right) + m_1 Q\left(m_2 \frac{a_2}{m}x\right) = m_0^2 m_2 Q\left(\frac{m_1 a_1}{m_0 m}x\right) + m_0^2 m_1 Q\left(\frac{m_2 a_2}{m_0 m}x\right), \qquad (**)$$

for all $x \in X$ and any fixed reals a_i and positive reals m_i $(i = 1, 2)$: $0 < m < 1$.

Note that if $m_1 = 1$, $m_2 > 0$, then $m_0 = 1$, $m = a_1^2 + m_2 a_2^2$, and (**) is an identity in X. In this case (**) is not required.

Moreover this mapping Q may be called *quadratic* because (**) holds for $Q(x) = x^2$.

Functional equations (2a)′–(2b)′ and (**) yield

$$Q(m^{-1}x) = (m^{-1})^2 Q(x) \qquad (2c)'$$

for all $x \in X$, and any fixed real m: $0 < m < 1$.

Then induction on $n \in N$ with $x \to m^{-(n-1)}x$ yields equation

$$Q(m^{-n}x) = (m^{-n})^2 Q(x), \qquad 0 < m < 1 \qquad (2d)'$$

completing the proof for Eq. (2)′.

DEFINITION 3. Let X be a normed linear space and let Y be a real complete normed linear space. Then we call *the non-linear mappings* $\overline{Q}$:

$X \to Y$, and $\bar{\bar{Q}}$: $X \to Y$ 2-dimensional *quadratic weighted means of first and second form*, if

$$\bar{Q}(x) = m_0^2 m^2 \frac{m_2 Q((m_1 a_1/m_0 m)x) + m_1 Q((m_2 a_2/m_0 m)x)}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)} \qquad (3)_1'$$

and

$$\bar{\bar{Q}}(x) = m^2 \frac{m_1 m_2 Q((a_1/m)x) + Q(m_2(a_2/m)x)}{m_2(m_1 a_1^2 + m_2 a_2^2)} \qquad (3)_2'$$

hold for all $x \in X$, and any real fixed m: $0 < m < 1$, respectively. Note that (*) is equivalent to the *mean functional equation*

$$\bar{\bar{Q}}(x) = \bar{Q}(x), \qquad [**]$$

for all $x \in X$, and any fixed real m: $0 < m < 1$.

Note that functional equation (2a)′ comes from Eq. (2a) if we replace x by x/m. But this *x-substitution* (x by x/m) does not yield Eq. (2)′ directly from Eq. (2). Also note that the x-substitution or the *a-substitution* (a_i by a_i/m: $i = 1, 2$) does not yield Eq. (2b)′ directly from Eq. (2b). Such problems in the transition from the first section to the second section arise many times in this paper. These reasons forced us to add this second section separately.

THEOREM 2. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that* f: $X \to Y$ *is a mapping for which there exists a constant* $c'' \geq 0$ *such that the fundamental functional inequality*

$$\left\|\bar{\bar{f}}(x) - \bar{f}(x)\right\| \leq \left(\frac{m_1 + m_2}{m_1 m_2 + 1}\right)^2 \left(\frac{m_1 a_1^2 + m_2 a_2^2}{m_1 m_2}\right) c'' \qquad (4)_1'$$

holds for all $x \in X$, c'' ($:=$ *const. indep. of* x) ≥ 0, *and any fixed reals* a_1, a_2 *and positive reals* m_1, m_2: $0 < m < 1$, *and*

$$\bar{f}(x) = m_0^2 m^2 \frac{m_2 f((m_1 a_1/m_0 m)x) + m_1 f((m_2 a_2/m_0 m)x)}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)}$$

and

$$\bar{\bar{f}}(x) = m^2 \frac{m_1 m_2 f((a_1/m)x) + f(m_2(a_2/m)x)}{m_2(m_1 a_1^2 + m_2 a_2^2)}$$

are 2*-dimensional quadratic weighted means of first and second form, respectively, for* $0 < m < 1$.

Assume in addition that f: $X \to Y$ *is a mapping for which there exists a constant* c (*independent of* x_1, x_2) ≥ 0 *such that the Euler–Lagrange functional inequality*

$$\begin{aligned}\big\| m_1 m_2 f(a_1 x_1 + a_2 x_2) &+ f(m_2 a_2 x_1 - m_1 a_1 x_2) \\ &- \left(m_1 a_1^2 + m_2 a_2^2\right)\left[m_2 f(x_1) + m_1 f(x_2)\right]\big\| \leq c \end{aligned} \qquad (4)'_2$$

holds for all 2*-dimensional vectors* $(x_1, x_2) \in X^2$ *and any fixed reals* a_1, a_2, *and positive reals* m_1, m_2: $0 < m < 1$, *where* $m = ((m_1 + m_2)/(m_1 m_2 + 1)) \times (m_1 a_1^2 + m_2 a_2^2)$.

Then the limit

$$Q(x) = \lim_{n \to \infty} m^{2n} f(m^{-n} x), \qquad (5)'$$

exists for all $x \in X$, *all* $n \in N$, *and any fixed real* m: $0 < m < 1$ *and* Q: $X \to Y$ *is the unique* 2*-dimensional quadratic mapping satisfying functional equation* (1) *and mean equation* [**] *or equivalently* (1) *and* (**), *such that*

$$\|f(x) - Q(x)\| \leq c_2, \qquad (6)''$$

holds for all $x \in X$ *with constant*

$$c_2 = \frac{\left[(-m_1 m_2) m^2 + \left(m_1^2 - 1\right) m + (m_1 m_2 + 2)\right] c + \left[(m_1 + m_2)(1 - m) m\right] c''}{m_1 m_2 (m_1 m_2 + 1)(1 - m)^2 (1 + m)}.$$

Moreover, identity

$$Q(x) = m^{2n} Q(m^{-n} x), \qquad (6a)''$$

holds for all $x \in X$, *all* $n \in N$, *and any fixed real* a_1, a_2 *and positive reals* m_1, m_2: $0 < m < 1$.

Note that if one replaces $x = 0$ in $\bar{f}(x)$, and $\bar{\bar{f}}(x)$ for $0 < m < 1$, then

$$\bar{f}(0) = \frac{m_1 m_2 + 1}{m_1 m_2} m f(0), \qquad \bar{\bar{f}}(0) = \frac{m_1 + m_2}{m_2} m f(0),$$

or

$$\bar{\bar{f}}(0) - \bar{f}(0) = \frac{m_1^2 - 1}{m_1 m_2} m f(0),$$

or

$$\left\| \bar{\bar{f}}(0) - \bar{f}(0) \right\|$$

$$\leq \frac{|m_1^2 - 1|}{m_1 m_2 (m_1 m_2 + 1)(1 - m)} mc \leq \frac{m_1 + m_2}{m_1 m_2 (m_1 m_2 + 1)} mc'',$$

$$= \left(\frac{m_1 + m_2}{m_1 m_2 + 1} \right)^2 \left(\frac{m_1 a_1^2 + m_2 a_2^2}{m_1 m_2} \right) c'',$$

if

$$c'' \geq \frac{|m_1^2 - 1|}{(m_1 + m_2)(1 - m)} c,$$

for

$$0 < m < 1, \qquad m_i > 0 \; (i = 1, 2),$$

and

$$\| f(0) \| \leq \frac{c}{(m_1 m_2 + 1)(1 - m)}, \qquad 0 < m < 1$$

(after substitution, $x_1 = x_2 = 0$ in inequality $(4)_2'$).

Moreover if $m_1 = m_2 = 1$, then

$$\bar{f}(x) = \left(a_1^2 + a_2^2 \right) \left[f\left(\frac{a_1}{a_1^2 + a_2^2} x \right) + f\left(\frac{a_2}{a_1^2 + a_2^2} x \right) \right] = \bar{\bar{f}}(x).$$

Thus in this case *the fundamental functional inequality* $(4)_1'$ (or constant c'') is *not* required (because $\bar{f}(x) = \bar{\bar{f}}(x)$) yielding

$$c_2 = \frac{3 - m^2}{2(1 - m)^2 (1 + m)} c.$$

Therefore one gets from Theorem 2 the following Theorem 2a.

THEOREM 2a. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that f: $X \to Y$ is a mapping for which there exists a constant $c \geq 0$ such that the Euler–Lagrange functional inequality*

$$\left\| f(a_1 x_1 + a_2 x_2) + f(a_2 x_1 - a_1 x_2) - \left(a_1^2 + a_2^2 \right) [f(x_1) + f(x_2)] \right\| \leq c \tag{$(4)_2''$}$$

holds for all 2-dimensional vectors $(x_1, x_2) \in X^2$ *and c* (:= *const. indep. of* x_1, x_2) ≥ 0 *and any fixed reals* a_1, a_2: $0 < m = a_1^2 + a_2^2 < 1$.

Then the limit

$$Q(x) = \lim_{n \to \infty} m^{2n} f(m^{-n} x)$$

exists for all $x \in X$, *and any fixed real m*: $0 < m < 1$ *and* $Q\colon X \to Y$ *is the unique 2-dimensional quadratic mapping satisfying the functional equation*

$$Q(a_1 x_1 + a_2 x_2) + Q(a_2 x_1 - a_1 x_2) = \left(a_1^2 + a_2^2\right)\left[Q(x_1) + Q(x_2)\right]$$

such that

$$\|f(x) - Q(x)\| \leq \frac{1}{2} \frac{3 - m^2}{(1-m)^2(1+m)} c,$$

and

$$Q(x) = m^{2n} Q(m^{-n} x),$$

for all $x \in X$, *all* $n \in N$, *and any fixed reals* a_1, a_2: $0 < m < 1$.

Note that if $m_1 = 1$, $m_2 > 0$, then $m_0 = 1$, $m = a_1^2 + m_2 a_2^2$, and

$$\bar{f}(x) = \left(a_1^2 + m_2 a_2^2\right) \frac{m_2 f\left((a_1/m)x\right) + f\left(m_2(a_2/m)x\right)}{m_2} = \bar{\bar{f}}(x).$$

In this case fundamental inequality $(4)'_1$ (or constant c'') is not required (because $\bar{f}(x) = \bar{\bar{f}}(x)$), yielding

$$c_2 = \frac{(-m_2)m^2 + (m_2 + 2)}{m_2(m_2 + 1)(1-m)^2(1+m)} c.$$

Therefore one gets from Theorem 2 the following Theorem 2b.

THEOREM 2b. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that* $f\colon X \to Y$ *is a mapping for which there exists a constant* $c \geq 0$ *such that the Euler–Lagrange functional inequality*

$$\begin{aligned}\big\|m_2 f(a_1 x_1 + a_2 x_2) &+ f(m_2 a_2 x_1 - a_1 x_2) \\ &- \left(a_1^2 + m_2 a_2^2\right)\left[m_2 f(x_1) + f(x_2)\right]\big\| \leq c\end{aligned}$$

holds for all 2-dimensional vectors $(x_1, x_2) \in X^2$ *and c* (:= *const. indep. of* x_1, x_2) ≥ 0 *and any fixed reals* a_1, a_2 *and positive real* m_2: $0 < m = a_1^2 + m_2 a_2^2 < 1$.

Then the limit

$$Q(x) = \lim_{n\to\infty} m^{2n} f(m^{-n}x)$$

exists for all $x \in X$, *all* $n \in N$ *and any fixed real* m: $0 < m < 1$ *and* Q: $X \to Y$ *is the unique* 2-*dimensional quadratic mapping satisfying the functional equation*

$$m_2 Q(a_1x_1 + a_2x_2) + Q(m_2a_2x_1 - a_1x_2)$$
$$= (a_1^2 + m_2a_2^2)[m_2Q(x_1) + Q(x_2)],$$

such that

$$\|f(x) - Q(x)\| \leq \frac{(-m_2)m^2 + (m_2 + 2)}{m_2(m_2 + 1)(1 - m)^2(1 + m)} c,$$

and

$$Q(x) = m^{2n} Q(m^{-n}x),$$

for all $x \in X$, *all* $n \in N$, *and any fixed reals* a_1, a_2, *and positive real* m_2: $0 < m < 1$.

Proof of Existence in Theorem 2. Claim first that *the following general inequality* $(12)'$ holds. In fact, substitution of $x_1 = x_2 = 0$ in inequality $(4)'_2$ yields that

$$\|f(0)\| \leq \frac{c}{(m_1m_2 + 1)(1 - m)}, \qquad 0 < m < 1. \qquad (7)'$$

Moreover substituting $x_1 = x/m$, $x_2 = 0$ in inequality $(4)'_2$ and employing $(7)'$ and the triangle inequality one concludes functional inequality

$$\left\| m_1m_2 f\left(\frac{a_1}{m}x\right) + f\left(\frac{m_2a_2}{m}x\right) - mm_0\left[m_2 f\left(\frac{x}{m}\right) + m_1 f(0)\right]\right\| \leq c,$$

or

$$\left\| \frac{m}{m_0m_2}\left[m_1m_2 f\left(\frac{a_1}{m}x\right) + f\left(m_2\frac{a_2}{m}x\right)\right] - \frac{m^2}{m_2}\left[m_2 f\left(\frac{x}{m}\right) + m_1 f(0)\right]\right\|$$
$$\leq \frac{m}{m_0m_2} c,$$

or

$$\left\| \bar{\bar{f}}(x) - m^2 f(m^{-1}x) \right\| \leq \frac{(m_1 m_2 + 1)(1-m)m + m_1 m_0 m^2}{m_0 m_2 (m_1 m_2 + 1)(1-m)} c, \qquad 0 < m < 1, \quad (8)'$$

where

$$\bar{\bar{f}}(x) = m^2 \frac{m_1 m_2 f\big((a_1/m)x\big) + f\big(m_2(a_2/m)x\big)}{m_2\big(m_1 a_1^2 + m_2 a_2^2\big)}, \qquad 0 < m < 1, \quad (8a)'$$

is *the 2-dimensional quadratic weighted mean of second form* (for $0 < m < 1$).

In addition replacing

$$x_1 = \frac{m_1 a_1}{m_0 m} x, \qquad x_2 = \frac{m_2 a_2}{m_0 m} x$$

in inequality $(4)'_2$ and using $(7)'$ and $(8a)'$, and the triangle inequality, one gets the functional inequality

$$\left\| m_1 m_2 f(x) + f(0) - m_0 m \left[m_2 f\left(\frac{m_1 a_1}{m_0 m} x \right) + m_1 f\left(\frac{m_2 a_2}{m_0 m} x \right) \right] \right\| \leq c,$$

or

$$\left\| f(x) - \frac{m_0 m}{m_1 m_2} \left[m_2 f\left(\frac{m_1 a_1}{m_0 m} x \right) + m_1 f\left(\frac{m_2 a_2}{m_0 m} x \right) \right] \right\| \leq \frac{c}{m_1 m_2} + \frac{1}{m_1 m_2} \| f(0) \|,$$

or

$$\| f(x) - \bar{f}(x) \| \leq \frac{(m_1 m_2 + 1)(1-m) + 1}{m_1 m_2 (m_1 m_2 + 1)(1-m)} c, \qquad 0 < m < 1, \quad (9)'$$

where

$$\bar{f}(x) = m_0^2 m^2 \frac{m_2 f\big((m_1 a_1/m_0 m)x\big) + m_1 f\big((m_2 a_2/m_0 m)x\big)}{m_1 m_2 \big(m_1 a_1^2 + m_2 a_2^2\big)}, \qquad 0 < m < 1, \quad (9a)'$$

is *the 2-dimensional quadratic weighted mean of first form* (for $0 < m < 1$).

Moreover

$$\left\| \bar{\bar{f}}(x) - \bar{f}(x) \right\| = \left\| \frac{m}{m_0 m_2} \left[m_1 m_2 f\left(\frac{a_1}{m} x \right) + f\left(m_2 \frac{a_2}{m} x \right) \right] - \frac{m m_0}{m_1 m_2} \left[m_2 f\left(\frac{m_1 a_1}{m_0 m} x \right) + m_1 f\left(\frac{m_2 a_2}{m_0 m} x \right) \right] \right\|,$$

or

$$\left\| \bar{\bar{f}}(x) - \bar{f}(x) \right\| = m^2 \frac{\| m_1^2 m_2 f((a_1/m)x) + m_1 f(m_2(a_2/m)x) - m_0^2 m_2 f((m_1 a_1/m_0 m)x) - m_0^2 m_1 f((m_2 a_2/m_0 m)x) \|}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)}.$$

Employing *the fundamental functional inequality* $(4)_1'$ or the *equivalent* inequality

$$\left\| \bar{\bar{f}}(x) - \bar{f}(x) \right\| \le m^2 \frac{c''}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)}, \qquad (4)_1''$$

one gets

$$\left\| m_1^2 m_2 f\left(\frac{a_1}{m} x \right) + m_1 f\left(m_2 \frac{a_2}{m} x \right) - m_0^2 m_2 f\left(\frac{m_1 a_1}{m_0 m} x \right) - m_0^2 m_1 f\left(\frac{m_2 a_2}{m_0 m} x \right) \right\| \le c'' \qquad (4\text{a})_1'$$

for all $x \in X$, and any fixed real m: $0 < m < 1$.

Functional inequalities $(8)'$–$(9)'$ and $(4)_1'$ (or $(4\text{a})_1''$ or $(4\text{a})_1'$), and the triangle inequality *yield the basic inequality*

$$\begin{aligned} \| f(x) - m^2 f(m^{-1}x) \| &\le \left\| f(x) - \bar{f}(x) \right\| + \left\| \bar{f}(x) - \bar{\bar{f}}(x) \right\| + \left\| \bar{\bar{f}}(x) - m^2 f(m^{-1}x) \right\| \\ &\le \frac{(m_1 m_2 + 1)(1 - m) + 1}{m_1 m_2 (m_1 m_2 + 1)(1 - m)} c + \frac{m}{m_0 m_1 m_2} c'' \\ &\quad + \frac{(m_1 m_2 + 1)(1 - m) m + m_1 m_0 m^2}{m_0 m_2 (m_1 m_2 + 1)(1 - m)} c \end{aligned}$$

or

$$\| f(x) - m^2 f(m^{-1}x) \| \leq c_2(1 - m^2), \qquad 0 < m < 1, \qquad (10)'$$

where

$$c_2 = \frac{\left[(-m_1 m_2)m^2 + \left(m_1^2 - 1\right)m + (m_1 m_2 + 2)\right]c + \left[(m_1 + m_2)(1 - m)m\right]c''}{m_1 m_2 (m_1 m_2 + 1)(1 - m)^2 (1 + m)}. \qquad (10a)'$$

For instance, if $m_1 = m_2 = 1$ and $a_1 = a_2 = 1/2$, then $m_0 = 1$, and $0 < m = 1/2 < 1$. In this case there is no c''-part in c_2 (formula (10a)$'$) because $\bar{\bar{f}}(x) = \bar{f}(x)$. Hence $c_2 = (11/3)c$. By induction on $n \in N$ with $x \to m^{-(n-1)}x$ in (10)$'$ claim that *the general inequality*

$$\|f(x) - m^{2n} f(m^{-n}x)\| \leq c_2(1 - m^{2n}) \qquad (12)'$$

holds for all $x \in X$, all $n \in N$, and any fixed real m: $0 < m < 1$.

In fact, (10)$'$–(10a)$'$ with $x \to m^{-(n-1)}x$ yield inequality

$$\| f(m^{-(n-1)}x) - m^2 f(m^{-n}x) \| \leq c_2(1 - m^2),$$

or

$$\| m^{2(n-1)} f(m^{-(n-1)}x) - m^{2n} f(m^{-n}x) \| \leq c_2(m^{2(n-1)} - m^{2n}), \quad (12a)'$$

for all $x \in X$, and any fixed real m: $0 < m < 1$.

By the induction hypothesis with $n \to n - 1$ in (12)$'$ inequality

$$\| f(x) - m^{2(n-1)} f(m^{-(n-1)}x) \| \leq c_2(1 - m^{2(n-1)}), \qquad (12b)'$$

for all $x \in X$, and any fixed real m: $0 < m < 1$.

Thus functional inequalities (12a)$'$–(12b)$'$ and the triangle inequality imply

$$\| f(x) - m^{2n} f(m^{-n}x) \| \leq \| f(x) - m^{2(n-1)} f(m^{-(n-1)}x) \| + \| m^{2(n-1)} f(m^{-(n-1)}x) - m^{2n} f(m^{-n}x) \|,$$

or

$$\| f(x) - m^{2n} f(m^{-n}x) \| \leq c_2\left[(1 - m^{2(n-1)}) + (m^{2(n-1)} - m^{2n})\right],$$

or

$$\| f(x) - m^{2n} f(m^{-n}x) \| \leq c_2(1 - m^{2n}), \qquad 0 < m < 1,$$

completing the proof of the required functional inequality (12)$'$.

The rest of the proof of Theorem 2 (*uniqueness*, etc.) is omitted as similar to the corresponding proof of Theorem 1 [1, 3, 14].

EXAMPLE. Take f: $R \to R$ to be a real function such that $f(x) = x^2 + k$, k = constant: $|k| \le c/(m_1 m_2 + 1)(1 - m)$, for any fixed reals a_1, a_2 and positive reals m_1, m_2: $0 < m = ((m_1 + m_2)/(m_1 m_2 + 1))(m_1 a_1^2 + m_2 a_2^2) < 1$.

Moreover there exists a *unique* quadratic mapping Q: $R \to R$ such that

$$Q(x) = \lim_{n \to \infty} m^{2n}\left[(m^{-n}x)^2 + k\right] = x^2, \qquad 0 < m < 1.$$

Therefore inequality (6)″ holds. In fact the *condition* on k,

$$|k| \le \frac{c}{(m_1 m_2 + 1)(1 - m)}, \qquad 0 < m < 1,$$

implies

$$\left\|(x^2 + k) - x^2\right\| = |k| \le \frac{1}{m_1 m_2 + 1} \frac{1}{1 - m} c.$$

But

$$\frac{1}{1 - m} < \frac{(-m_1 m_2)m^2 + \left(m_1^2 - 1\right)m + (m_1 m_2 + 2)}{m_1 m_2 (1 - m)^2 (1 + m)}, \qquad 0 < m < 1.$$

Hence

$$\| f(x) - Q(x) \| = \left\|(x^2 + k) - x^2\right\| < c_2, \qquad 0 < m < 1, \text{ satisfying } (6)''.$$

Note that if $m > 1$, then take any real k = constant:

$$|k| \le \frac{c}{(m_1 m_2 + 1)(m - 1)}.$$

THEOREM 3. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that f: $X \to Y$ is a mapping for which there exist constants c, and c' (≥ 0) such that the Euler–Lagrange functional inequality*

$$\left\|f(a(x_1 + x_2)) + f(a(x_1 - x_2)) - [f(x_1) + f(x_2)]\right\| \le c$$

holds for all 2-dimensional vectors $(x_1, x_2) \in X^2$, $\|f(0)\| \le c'$, and c, c' ($:=$ const. indep. of x_1, x_2) ≥ 0 and $a = 1/\sqrt{2}$ (or $:= -1/\sqrt{2}$).

Then the limit

$$Q(x) = \lim_{n\to\infty} 2^{-n} f\big((2a)^n x\big)$$

exists for all $x \in X$, *and all* $n \in N$ *any fixed real* m: $m = 2a^2$ $(= 1)$ *and* Q: $X \to Y$ *is the unique* 2-*dimensional quadratic mapping satisfying the functional equation*

$$Q\big(a(x_1 + x_2)\big) + Q\big(a(x_1 - x_2)\big) = Q(x_1) + Q(x_2), \qquad m = 1,$$

and $Q(0) = 0$, *such that*

$$\|f(x) - Q(x)\| \le c + c', \qquad m = 1,$$

and

$$Q(x) = 2^{-n} Q\big((2a)^n x\big),$$

for all $x \in X$, *and all* $n \in N$, *and* $m = 1$.

Note that in this Theorem 3, $a_1 = a_2 = a$, $m_1 = m_2 = 1$, and thus $m_0 = 1$, and $m = a_1^2 + a_2^2 = 2a^2 = 1$. Thus Theorem 3 is *a singular case* of Theorems 1–2.

Also substitution of $x_1 = x_2 = x$ in the Euler–Lagrange inequality of this Theorem 3 yields *that the basic inequality*

$$\|f(x) - 2^{-1} f(2ax)\| \le (c + c')(1 - 2^{-1})$$

$$\big(\text{from the condition } \|f(0)\| \le c'\big)$$

holds for all $x \in X$.

Then induction on n with $x \to (2a)^{n-1} x$ in the above basic inequality yields *the general functional inequality*

$$\big\|f(x) - 2^{-n} f\big((2a)^n x\big)\big\| \le (c + c')(1 - 2^{-n}),$$

for all $x \in X$, and all $n \in N$, and $a = \pm 1/\sqrt{2}$.

Note that substitution of $x_1 = x_2 = x$ in the Euler–Lagrange equation of this Theorem 3, the fact that $Q(0) = 0$, and induction on n, yield

$$Q(x) = 2^{-n} Q\big((2a)^n x\big),$$

for all $x \in X$, all $n \in N$, and $a = \pm 1/\sqrt{2}$.

The rest of the proof of Theorem 3 is omitted as it is similar to the proof of Theorem 1.

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