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UNIQUENESS OF QUASI-REGULAR SOLUTIONS FOR A PARABOLIC ELLIPTIC-HYPERBOLIC TRICOMI PROBLEM

BY

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Abstract. The Tricomi equation: $yu_{xx} + u_{yy} = 0$ was established in 1923 by F.G. Tricomi who is the pioneer of mixed type boundary value problems. In 1981, A.V. Bitsadze and then R.I. Semerdjieva (1992) introduced the hyperbolic equation:

$$Lu \equiv K_1(y)u_{xx} + (K_2(y)u_y)_y + ru = f.$$

In this paper we consider the more general case of above equation: $Lu \equiv f$, so that it is also **elliptic** in the upper half-plane and **parabolic** on the line $y = 0$, and then prove **the uniqueness of quasi-regular solutions** for the corresponding Tricomi problem by employing the well-known a-b-c energy integral method. This result is interesting in fluid mechanics. (S.A. Chaplygin (1904) considered the equation of a perfect gas, which was of mixed type).

The Tricomi Problem. Consider the parabolic elliptic-hyperbolic equation

$$(*) \qquad Lu \equiv K_1(y)u_{xx} + (K_2(y)u_y)_y + r(x,y)u = f(x,y)$$

([1], [5]), in a bounded simply-connected domain D of the xy plane with a piecewise-smooth boundary $G = \partial D = g_1 \cup g_2 \cup g_3$, where $f = f(x,y)$ is continuous, $r = r(x,y)$ and $K_1 = K_1(y)$ are once-continuously differentiable

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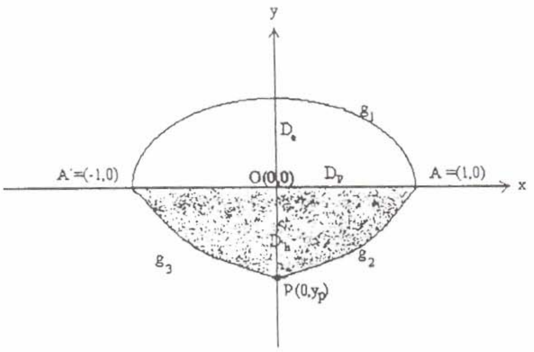
for $x \in [-1,1]$ and $y \in [-m,M]$ with $-m = \inf\{y : (x,y) \in D\}$, and $M = \sup\{y : (x,y) \in D\}$, as well as K_1 satisfies

$$K_1(y)\begin{cases} > 0, & \text{for} \quad y > 0 \\ = 0, & \text{for} \quad y = 0 \\ < 0, & \text{for} \quad y < 0 \end{cases}$$

Also $K_2 = K_2(y)$ is once-continuously differentiable in $[-m,M]$, and $K_2(y) > 0$ everywhere in D. Besides $\lim_{y\to 0} K(y)$ exists, if

$$K = K(y) = K_1(y)/K_2(y)\begin{cases} > 0, & \text{for} \quad y > 0 \\ = 0, & \text{for} \quad y = 0 \\ < 0, & \text{for} \quad y < 0 \end{cases}$$

The boundary of D is formed by the following curves:



(1) a curve g_1 which is the elliptic arc lying in the half-plane $y > 0$ and connecting points $A(1,0)$ and $A'(-1,0)$;

(2) two hyperbolic characteristic arcs g_2 and g_3:

$$g_2 : x = \int_0^y \sqrt{-K(t)}dt + 1, \quad g_3 : x = -\int_0^y \sqrt{-K(t)}dt - 1$$

descending from $A(1,0)$ and $A'(-1,0)$ until they terminate at a common point of intersection $P(0,y_p)$ in the lower half-plane.

Denote the elliptic subregion of D by

$$D_e (=\text{the space bounded by } g_1 \text{ and } A'A),$$

the hyperbolic subregion of D by

$$D_h(=\text{the space bounded by } g_2, g_3 \text{ and } AA'),$$

and the parabolic arc of D by

$$D_p(\equiv A'A) = \{(x,y) \in D : -1 < x < 1, y = 0\}.$$

Note that the **order** of equation (*) does not **degenerate** on the line $y = 0$. But (*) is parabolic for $y = 0$ because $K_1(0) = 0$ and $K_2(0) > 0$.

Assume **boundary condition**

$$u = 0 \quad \text{on} \quad g_1 \cup g_2. \tag{**}$$

The Tricomi problem, or **Problem (T)** consists in finding a function $u = u(x,y)$ which satisfies equation (*) in D and boundary condition (**) on $g_1 \cup g_2$([3], [4], [6]).

Definition. A function $u{=}u(x,y)$ is a **quasi-regular solution** ($[3]_{(2)}$) of Problem (T) if

i) $u \in C^2(D) \cap C(\overline{D})$, $\overline{D} = D \cup G$, $G = \partial D$,

ii) Green's theorem is applicable to the integrals

$$\iint_D u_x Lu dx dy, \quad \iint_D u_y Lu dx dy,$$

iii) the boundary and region (surface) integrals which arise exist,

iv) u satisfies equation (*) in D and boundary condition (**) on $g_1 \cup g_2$.

Uniqueness Theorem. *Consider parabolic elliptic-hyperbolic equation (*) and boundary condition (**). Also consider the afore described simply-connected domain D of the xy plane.*

Assume **conditions***:*

(R_1): $r < 0$ *on* g_3,

(R_2): *the boundary arc* g_1 *is* **star-like** *in the sense that* $(x+1)dy - ydx \geq 0$

(R_3): $\begin{cases} 2r + (x+1)r_x + yr_y < 0 & \textit{for} \quad y \geq 0 \\ r + (x+1)r_x < 0 & \textit{for} \quad y \leq 0, \end{cases}$

(R_4): $\begin{cases} K_1(y) > 0 \quad \textit{whenever } y > 0, K_1(y) = 0 \textit{ whenever } y = 0, \\ \qquad\qquad \textit{and } K_1(y) < 0 \textit{ whenever } y < 0 \\ K_2(y) > 0 \quad \textit{everywhere in } D, \end{cases}$

(R_5): *$K_i'(y) > 0 (i = 1, 2)$ in D,*
where $K_i (i = 1, 2)$, and r are once-continuously differentiable and f is continuous.

Then Problem (T) has at most one **quasi-regular solution** *in D.*

Proof. We apply the a-b-c classical energy integral method with $a = 0$ and use boundary condition (**). First, we assume two quasi-regular solutions u_1, u_2 of Problem (T). Then **claim** that

$$u = u_1 - u_2 = 0 \quad \text{in} \quad D.$$

In fact, we investigate

$$0 = J = 2\langle Mu, Lu\rangle_0 = \iint_D 2MuLudxdy, \tag{1}$$

where $Mu = b(x, y)u_x + c(x, y)u_y$, and

$$Lu = L(u_1 - u_2) = Lu_1 - Lu_2 = f - f = 0 \quad \text{in} \quad D,$$

with **choices:**

$$b = x + 1 \quad \text{in} \quad D,\ c = \begin{cases} y, & y \geq 0 \\ 0, & y \leq 0. \end{cases} \tag{2}$$

Consider **the ordinary identities:**

$$2bK_1u_xu_{xx} = (bK_1u_x^2)_x - b_xK_1u_x^2$$
$$2bK_2u_xu_{yy} = (2bK_2u_xu_y)_y - (bK_2u_y^2)_x + b_xK_2u_y^2 - 2(bK_2)_yu_xu_y$$
$$2cK_1u_yu_{xx} = (2cK_1u_xu_y)_x - (cK_1u_x^2)_y + (cK_1)_yu_x^2 - 2K_1c_xu_xu_y$$
$$2cK_2u_yu_{yy} = (cK_2u_y^2)_y - (cK_2)_yu_y^2,$$
$$2bruu_x = (bru^2)_x - (br)_xu^2,$$
$$2cruu_y = (cru^2)_y - (cr)_yu^2,$$
$$2btu_xu_y = 2btu_xu_y,$$
$$2ctu_yu_y = 2ctu_y^2$$

where $t(\equiv$ coefficient of u_y in $Lu)$, or

$$t = K_2'(y). \tag{3}$$

Then employing above identities and Green's theorem, and setting $t = K_2'(y)$ we obtain from (1) and (*) that

$$0 = J = \iint_D 2\Big(bu_x + cu_y\Big)\Big[K_1(y)u_{xx} + K_2(y)u_{yy} + ru + tu_y\Big]dxdy = I_D + I_G, \tag{4}$$

where

$$I_D = \iint_D \Big(Au_x^2 + Bu_y^2 + Cu^2 + 2Du_xu_y\Big)dxdy,$$
$$I_G = \oint_{G(=\partial D)} \Big(\widetilde{A}u_x^2 + \widetilde{B}u_y^2 + \widetilde{C}u^2 + 2\widetilde{D}u_xu_y\Big)ds,$$

with

$$A = -b_xK_1 + \Big(cK_1\Big)_y,\ B = b_xK_2 - \Big(cK_2\Big)_y + 2ct,$$
$$C = -\Big[(br)_x + (cr)_y\Big], \quad \text{and}$$
$$D = -\Big[K_1c_x + (bK_2)_y - bt\Big],$$
$$\widetilde{A} = \Big(bv_1 - cv_2\Big)K_1, \quad \widetilde{B} = \Big(-bv_1 + cv_2\Big)K_2,$$
$$\widetilde{C} = \Big(bv_1 + cv_2\Big)r, \quad \widetilde{D} = bK_2v_2 + cK_1v_1, \quad \text{where}$$
$$v = \Big(v_1, v_2\Big) = \Big(\frac{dy}{ds}, -\frac{dx}{ds}\Big), \quad (ds > 0), \tag{5}$$

is the outer unit normal vector on the boundary G of the mixed domain D. **Note** that in $D, y \geq 0$(if $b = x + 1,\ c = y$) :

$$A = -K_1 + \Big(yK_1\Big)' = yK_1' \geq 0 \quad (\text{from condition } (\mathrm{R}_5) : i = 1)$$
$$B = K_2 - \Big(yK_2\Big)' + 2yt = -yK_2' + 2yK_2' = yK_2' \geq 0$$
$$(\text{from condition } (\mathrm{R}_5) : i = 2)$$
$$C = -((x+1)r)_x - (yr)_y = -\Big(2r + (x+1)r_x + yr_y\Big) \geq 0$$
$$(\text{from condition } (\mathrm{R}_3) : y \geq 0)$$
$$D = -\Big((x+1)K_2\Big)_y + (x+1)t = (x+1)\Big(-K_2' + t\Big) = 0 \quad (\text{from } (3)),$$

and

$$AB - D^2 = y^2 K_1' K_2' \geq 0 \quad \text{(from condition } (\mathrm{R}_5)),$$

Similarly in $D, y \leq 0$ (if $b = x + 1,\ c = 0$) :

$A = -K_1 \geq 0,\ B = K_2 \geq 0$

$C = -((x+1)r)_x = -(r + (x+1)r_x) \geq 0$ (from condition $(\mathrm{R}_3) : y \leq 0$)

$D = -(x+1)K_2' + (x+1)t = (x+1)(-K_2' + t) = 0$ (from (3)),

and

$$AB - D^2 = (-K_1)K_2 \geq 0 \text{ (from condition } (\mathrm{R}_4) : y \leq 0).$$

Therefore the region (surface) first integral I_D (of (4)) is nonnegative. In fact,

$$I_D = I_{1D} + I_{2D} + I_0 > 0, \tag{6}$$

because, if $Q = Au_x^2 + Bu_y^2 + 2Du_x u_y = Q(u_x, u_y)$, then

$$I_{1D} = \iint_{D, y \geq 0} Q(u_x, u_y) dx dy = \iint_{D, y \geq 0} y\Big(K_1' u_x^2 + K_2' u_y^2\Big) dx dy \geq 0,$$

(from (R_5))

$$I_{2D} = \iint_{D, y \leq 0} Q(u_x, u_y) dx dy = \iint_{D, y \leq 0} \Big(-K_1 u_x^2 + K_2 u_y^2\Big) dx dy \geq 0,$$

(from (R_4))

and

$$I_0 = \iint_D Cu^2 dx dy = \begin{cases} -\displaystyle\iint_{D, y \geq 0} \Big(2r + (x+1)r_x + y r_y\Big) u^2 dx dy > 0 \\ -\displaystyle\iint_{D, y \leq 0} \Big(r + (x+1)r_x\Big) u^2 dx dy > 0, \end{cases}$$

from condition (R_3).

Note that on g_1 (if $b = x + 1, c = y$):

$$\widetilde{A} = \Big[(x+1)v_1 - y v_2\Big] K_1,\ \widetilde{B} = \Big[-(x+1)v_1 + y v_2\Big] K_2,$$
$$\widetilde{C} = \Big[(x+1)v_1 + y v_2\Big] r,\ \widetilde{D} = (x+1)K_2 v_2 + y K_1 v_1.$$

From boundary condition (**) we get $0 = du|_{g_1} = u_x dx + u_y dy$, or

$$u_x = Nv_1, \; u_y = Nv_2, \tag{6a}$$

where N is a normalizing factor. Therefore

$$\begin{aligned} I_{g_1} &= \int_{g_1} \widetilde{Q}(u_x, u_y)ds + \int_{g_1} \widetilde{C}u^2 ds \\ &= \int_{g_1} N^2 \Big[(x+1)v_1 + yv_2\Big]\Big(K_1 v_1^2 + K_2 v_2^2\Big)ds \\ &\qquad + \int_{g_1} \Big[(x+1)v_1 + yv_2\Big] ru^2 ds, \end{aligned} \tag{7}$$

where

$$\widetilde{Q} = \widetilde{Q}(u_x, u_y) = \widetilde{A}u_x^2 + \widetilde{B}u_y^2 + 2\widetilde{D}u_x u_y$$

is a quadratic form on G with respect to u_x, u_y.

It is clear from (**), (5), (7) and (R_2) that

$$I_{g_1} = \int_{g_1} N^2[(x+1)dy - ydx]H \geq 0, \tag{8}$$

where

$$H = K_1 v_1^2 + K_2 v_2^2 \; (> 0 \text{ on } g_1). \tag{9}$$

Similarly on g_2 (if $b = x+1, \; c = 0$) :

$$\begin{aligned} I_{g_2} &= \int_{g_2} \widetilde{Q}(u_x u_y)ds + \int_{g_2} \widetilde{C}u^2 ds \\ &= \int_{g_2} N^2[(x+1)v_1]Hds + \int_{g_2} [(x+1)v_1]ru^2 ds, \text{ or} \end{aligned}$$

$$I_{g_2} = 0, \tag{10}$$

because of (**) and $H = 0$ on g_2 (as g_2 is characteristic of (*)).

Finally claim that on g_3 (if $b = x+1, \; c = 0$) :

$$I_{g_3} = \int_{g_3} \widetilde{Q}(u_x, u_y)ds + \int_{g_3} \widetilde{C}u^2 ds > 0. \tag{11}$$

In fact, $\widetilde{C} = [(x+1)v_1]r > 0$ on g_3, because of condition (R_1). Also $\widetilde{Q} = \widetilde{Q}(u_x, u_y)$ is non-negative definite on g_3. It is clear that $\widetilde{A} = [(x+1)v_1]K_1 > 0$ on g_3, because

$$(x+1)|_{g_3} = -\int_0^y \sqrt{-K(t)}dt > 0 \text{ on } g_3,\ v_1|_{g_3} < 0, \text{ and } K_1|_{g_3} < 0.$$

Also

$$\widetilde{B} = -\Big[(x+1)v_1\Big]K_2 > 0 \text{ on } g_3,$$

because $K_2|_{g_3} > 0$ and above facts.

Besides

$$\widetilde{D} = (x+1)K_2 v_2.$$

Finally

$$\begin{aligned}\widetilde{A}\widetilde{B} - (\widetilde{D})^2 &= -\Big[(x+1)v_1\Big]^2 K_1 K_2 - \Big[(x+1)v_2\Big]^2 K_2^2 \\ &= -(x+1)^2 K_2\Big[K_1 v_1^2 + K_2 v_2^2\Big] \\ &= -(x+1)^2 K_2 H = 0 \text{ on } g_3,\end{aligned}$$

because $H = 0$ on g_3 (as g_3 is characteristic of (*)). Thus $I_{g_3} > 0$ and the proof of (11) is complete. Therefore

$$I_G = I_{g_1} + I_{g_2} + I_{g_3} = I_{g_1} + I_{g_3} > 0. \tag{12}$$

Hence from (4), (6), (12) and the fact that $I_D > 0$ and $I_G > 0$ we get that $u = 0$ in D.

In fact, from (4) yields that the sum J of $I_D (= I_{1D} + I_{2D} + I_0)$ and $I_G (= I_{g_1} + I_{g_2} + I_{g_3})$ vanishes in D.

Also $I_{1D} \geq 0$, $I_{2D} \geq 0$, $I_0 > 0$, and $I_{g_1} \geq 0$, $I_{g_2} = 0$, $I_{g_3} > 0$. Therefore $I_D = 0$ and $I_G = 0$, or $I_{1D} = 0$ and $I_{g_3} = 0$.

First $I_{1D} \doteq \iint_{D, y\geq 0} y(K_1' u_x^2 + K_2' u_y^2)dxdy = 0$ yielding $u_x = u_y = 0$ in $D, y \geq 0$ since $K_i' > 0 (i = 1, 2)$ from condition (R_5). Thus $u = c$ in $D, y \geq 0$, and as $u = 0$ on g_1 (from (**)) it will follow that $u(x, y) = 0$ in $D, y \geq 0$.

Second

$$I_{g_3} = \int_{g_3} \left[K_2\Big(\sqrt{-K}u_x - u_y\Big)^2 + (-r)u^2\right]\Big(\int_0^y \sqrt{-K(t)}dt\Big)dy = 0$$

(because on

$$g_3 : dx = -\sqrt{-K}dy = -\left(\sqrt{-K_1}/\sqrt{K_2}\right)dy,\ x+1 = -\int_0^y \sqrt{-K(t)}dt > 0$$

and $v_1 = dy/ds < 0$) yielding that $u = 0$ on g_3 (as $r < 0$ on g_3 from condition (R_1)).

Thus by a well known theorem for hyperbolic equations if u vanishes on g_2 (which holds from (**)) and g_3 then it vanishes throughout $D, y \leq 0$. (Another reasoning is that, in particular, $u(x,0) = 0$ and $u_y(x,0) = 0$, so that $u(x,y) = 0$ in $D, y \leq 0$, because of the uniqueness of the solution of the Cauchy problem for equation (*)). Thus $u(x,y) = 0$ everywhere in D, completing the proof of the uniqueness theorem.

Note that the afore-mentioned theorem is interesting in Aerodynamics and Hydrodynamics ([2]).

Special Uniqueness Theorem 1. *Assume a new Tricomi equation*

$$Lu \equiv yu_{xx} + u_{yy} + ru = f(x,y), \tag{13}$$

where $r = const. < 0$, in mixed domain D of xy plane bounded by characteristic arcs g_2, g_3 (for $y < 0$):

$$g_2(\equiv PA) : x = -\frac{2}{3}(-y)^{3/2} + 1 \ \textit{and}$$
$$g_3(\equiv A'P) : x = \frac{2}{3}(-y)^{3/2} - 1,$$

so that they intersect at point $P = \left(0, -\left(\frac{3}{2}\right)^{2/3}\right)$, and by the arc g_1 (for $y > 0$) connecting points $A = (1,0)$ and $A' = (-1,0)$.

Also assume that the boundary arc g_1 is **star-like** *in the sense that*

$$(R_{21}) : (x+1)dy - ydx \geq 0.$$

(Take e.g. the upper semi-ellipse $g_1 : x^2 + \lambda y^2 = 1$, $\lambda(= \text{const.}) > 0$, $y > 0$, satisfying (R_{21})).

Then Problem (T) for equation (13) has at most one **quasi-regular solution** *in corresponding special domain D.*

Special Uniqueness Theorem 2. *Assume a parabolic elliptic-hyperbolic equation*

$$Lu \equiv yu_{xx} + ((y - ky_p)u_y)_y + ru = f(x, y), \tag{14}$$

where $r = const. < 0$, $k = const. > 2$, $y_p = 1 \Big/ \left(\sqrt{k-1} - k\tan^{-1}\frac{1}{\sqrt{k-1}}\right) (< 0)$ *in mixed domain* D *of* xy *plane bounded by characteristic arcs* g_2, g_3 *(for* $y < 0$*):*

$$g_2(\equiv PA) : x = ky_p \tan^{-1}\sqrt{-\frac{y}{y - ky_p}} + \sqrt{-y(y - ky_p)} + 1 \text{ and}$$

$$g_3(\equiv A'P) : x = -ky_p \tan^{-1}\sqrt{-\frac{y}{y - ky_p}} - \sqrt{-y(y - ky_p)} - 1,$$

so that they intersect at point

$$P = \left(0, 1 \Big/ \left(\sqrt{k-1} - k\tan^{-1}\frac{1}{\sqrt{k-1}}\right)\right),$$

and by the arc g_1 *(for* $y > 0$*) connecting points* $A = (1, 0)$ *and* $A' = (-1, 0)$.

Also assume that the boundary arc g_1 *is* **star-like** *in the sense that*

$$(R_{22}) : (x + 1)dy - ydx \geq 0.$$

Then Problem (T) for equation (14) has at most **one quasi-regular solution** *in corresponding special domain* D.

Note 1. Substituting $\sqrt{-t/(t - ky_p)} = \varphi$, we get that

$$\int_0^y \sqrt{-\frac{t}{t - ky_p}}dt = ky_p \tan^{-1}\sqrt{-\frac{y}{y - ky_p}} + \sqrt{-y(y - ky_p)},$$

where

$$\int \frac{2\varphi^2}{(1 + \varphi^2)^2}d\varphi = \tan^{-1}\varphi - \frac{\varphi}{1 + \varphi^2} + c.$$

Note 2. The cases $r = 0$ and $1 < k = \text{const.} \leq 2$ in D yield also uniqueness results for quasi-regular solutions in above theorems.

Note 3. No compatibility relations exist about $u = u(x, y)$ on the parabolic arc D_p.

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