The Well-posed Tricomi-Bitsadze-Lavrentjev Problem in the Euclidean Plane

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Summary. *S.A. Chaplygin (1904) considered a two-dimensional adiabatic potential flow of a perfect gas the equation of which was transformed to a linear mixed type equation. F. G. Tricomi (1923), F. I. Frankl (1945), M. H. Protter (1953), S. Nocilla (1957), C. Ferrari (1959) and most of the recent workers in the field of mixed type boundary value problems have restricted their attention to the Chaplygin equation:* $K(y)\cdot u_{xx}+u_{yy}=0$, $K\in C^2(\cdot)$ *and not considered «the generalized equation»:*

$$K(y)\cdot u_{xx}+u_{yy}+\tau(x,y)\cdot u=f(x,y) \quad \text{when} \quad K=sgn(y):$$

discontinuous *and* $\tau :=$ non-trivial ($\neq 0$). *In this paper I consider the generalized case afore-mentioned and establish* well-posedness *in the sense that:* there is at most one quasi-regular solution and a weak solution exists. *This boundary value problem of Tricomi-Bitsadze-Lavrentjev type is important in fluid dynamics.*

The Tricomi-Bitsadze-Lavrentjev problem

Consider

$$(*) \qquad Lu=K(y)\cdot u_{xx}+u_{yy}+\tau(x,y)\cdot u=f(x,y)$$

$$K=sgn(y):=1,\ y>0; \quad =-1,\ y<0; \quad =0,\ y=0,$$

$$\tau\in C^1(\cdot)\ ,\ f\in C^0(\cdot)\ .$$

Consider a mixed domain D which is a simply-connected region and contains the parabolic arc of degeneracy: $A'A$ with endpoints:

$$A'=(-1,\ 0), \quad A=(1,0)$$

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and has boundary

$$G := \partial G = g_1 \cup g_2 \cup g_3$$

with boundary curves:

g_1: piece-wise smooth «elliptic arc» for $y > 0$ connecting points: A', A,

g_2: characteristic for $y < 0$ emanating from point A:

$$\int_1^x dx = \int_0^y \sqrt{-K}\cdot dy, \quad \text{or} \quad g_2 : x = y + 1 \ ,$$

g_3: characteristic for $y < 0$ emanating from point A':

$$\int_{-1}^x dx = -\int_0^y \sqrt{-K}\cdot dy, \quad \text{or} \quad g_3 . x = -y - 1 \ .$$

Denote

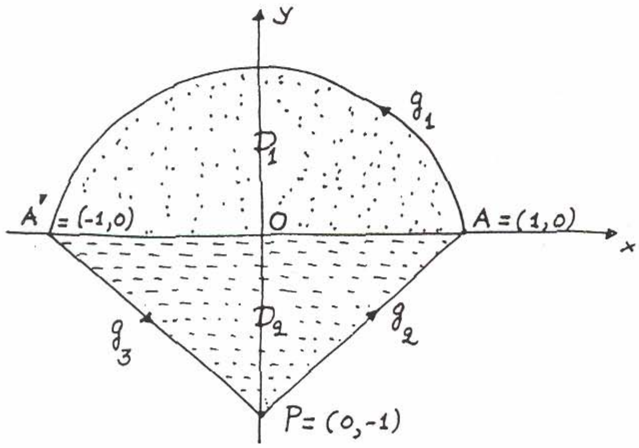
D_1: elliptic region: $= \{(x, y) \in D, \ -1 < x < 1, \ y > 0\}$,

D_2: hyperbolic region: $= \{(x, y) \in D, \ -1 < x < 1, \ y < 0\}$

with boundary

$$G_1 . = \partial D_1 = g_1 \cup (A'A)$$

$$G_2 : = \partial D_2 = g_2 \cup g_3 \cup (AA') \ .$$



Then

$$D = D_1 \cup D_2 \cup (A'A) = D_1 \cup D_2 \cup (AA') .$$

The above mentioned two characteristic curves g_2, g_3 intersect at the point $P=(0,-1)$ in D_2.

Assume now boundary condition

(**) $$u = 0 \quad \text{on} \quad g_1 \cup g_3 .$$

The Tricomi-Bitsadze-Lavrentjev problem, or *problem* (TBL): consists in finding a function $u = u(x, y)$ which satisfies equation (*) and boundary condition (**).

DEFINITION 1. A function $u = u(x, y)$ is a *quasi-regular solution* of Problem (TBL) if

i) $$u \in C^2(D) \cap C(D),$$

ii) the integral

$$\int_{-1}^{1} (\cdot) \cdot ds \quad \text{exists},$$

iii) Green's theorem is applicable to the integrals

$$\iint_D u \cdot Lu \cdot dxdy, \quad \iint_D u_x \cdot Lu \cdot dxdy, \quad \iint_D u_y \cdot Lu \cdot dxdy,$$

iv) the boundary integrals which arise exist in the sense that: *the limits taken over corresponding interior curves exist as these interior curves approach the boundary*,

v) u satisfies equation (*) in D, and

vi) u satisfies boundary condition (**).

Uniqueness Theorem

Assume the afore-mentioned domain $D \subset \mathbf{R}^2$, and the conditions:

(R_1): $\tau \leqq 0 \quad on \quad g_2$

(R_2): $(x-1)\cdot dy - y\cdot dx \geqq 0 \quad on \quad g_1$

(R_3): $\begin{cases} 2\cdot\tau + (x-1)\cdot\tau_x + y\cdot\tau_y \leqq 0 & in \quad D_1 \\ \tau + (x-1)\cdot\tau_x \leqq 0 & in \quad D_2. \end{cases}$

The Problem (TBL) *has at most one quasi-regular solution in the mixed domain D.*

To prove this theorem we apply the *b, c* energy integral method *separately* in D_1, D_2 because K is discontinuous. First, we assume u_1, u_2: two quasi-regular solutions satisfying Problem (TBL). Then claim that $u = u_1 - u_2 = 0$ in D. To do it I need to show that $u = 0$ on g_2. Second, investigate

$$0 = J^{(i)} = 2\cdot \iint_{D_i} (b\cdot u_x + c\cdot u_y)\cdot Lu\cdot dxdy, \qquad i = 1,2 ,$$

where

$$b = x-1 \quad \text{in} \quad D, \qquad c = \begin{cases} y & \text{in} \quad D_1 \\ o & \text{in} \quad D_2 . \end{cases}$$

Then employ Green's theorem and prove that all integrals are non-negative. Finally apply a classical maximum principle.

Preliminaries

Denote

$$\alpha = (\alpha_1, \alpha_2), \quad \alpha_1, \alpha_2 > 0, \quad |\alpha| = \alpha_1 + \alpha_2$$

$$p = (x, y) \in \mathbf{R}^2, \ q = (\tilde{x}, \tilde{y}) \in \mathbf{R}^2, \ \langle p, q\rangle = x\cdot\tilde{x} + y\cdot\tilde{y} ,$$

$$p^\alpha = x^{\alpha_1} y^{\alpha_2}, \ q^\alpha = \tilde{x}^{\alpha_1}\tilde{y}^{\alpha_2} ,$$

$$D_1 = \frac{\partial}{\partial x},\ D_2 = \frac{\partial}{\partial y},\ (D^{\alpha} u)(p) = (D_1^{\alpha_1} D_2^{\alpha_2} u)(p)$$

for sufficiently smooth functions $u = u(p)$.

Consider *adjoint equation*

[*] $$L^* w \equiv k(y) \cdot w_{xx} + w_{yy} + \tau(x, y) \cdot u = f(x, y),$$

where

$L^*(=L)$: formal adjoint operator of L.

Note: In general, if:

$$Lu = \sum_{i,j=1}^{2} a_{ij}(p) \cdot D_i D_j u + \sum_{i=1}^{2} a_i(p) \cdot D_i u + a(p) \cdot u,$$

then

$$L^* w = \sum_{i,j=1}^{2} D_i D_j (a_{ij}(p) \cdot w) - \sum_{i=1}^{2} D_i (a_i(p) \cdot w) + a(p) \cdot w\ .$$

Assume *adjoint boundary conditions*

[**] $$w = 0 \quad \text{on} \quad g_1 \cup g_2\ .$$

Denote

$$W_2^m(D) = W^{m,2}(D) = \{u(p) : p \in D,\ u(p) \in L^2(D),\ D^{\alpha} u(p) \in L^2(D),\ |\alpha| \leqslant m\} :$$

the *Sobolev space* with inner product

$$\langle u, w \rangle_m = \langle u, w \rangle_{W_2^m(D)} = \langle u, w \rangle_{L^2(D)} + \sum_{|\alpha| = m} \langle D^{\alpha} u, D^{\alpha} w \rangle_{L^2(D)}$$

and norm

$$\| u \|_m = \| u \|_{W_2^m(D)} = \left(\| u \|_{L^2(D)}^2 + \sum_{|\alpha| = m} \| D^{\alpha} u \|_{L^2(D)}^2 \right)^{1/2}.$$

Also denote

$$D(L)=\{u\in C^2(D): u=0 \quad \text{on} \quad g_1\cup g_3\}\ ,$$

$$W_2^2(D,bd)=\overline{D(L)}\Big|_{\|\cdot\|_2} \quad : \text{closure of } D(L) \text{ with norm } \|\cdot\|_2\ ,$$

$$D(L^*)=\{w\in C^2(D): w=0 \quad \text{on} \quad g_1\cup g_2\},$$

$$W_2^2(D,b^*d)=\overline{D(L^*)}\Big|_{\|\cdot\|_2} \quad : \text{closure of } D(L^*) \text{ with norm } \|\cdot\|_2,$$

or equivalently

$$W_2^2(D,b^*d)=\{w\in W_2^2(D): \langle Lu,w\rangle_0=\langle u,L^*w\rangle_0 \quad \text{for all} \quad u\in W_2^2(D,bd)\}\ .$$

DEFINITION 2. A function $u=u(p)\in L^2(D)$ is a *weak solution* of Problem (*) and (**) if

$$\langle f,w\rangle_0=\langle u,L^*w\rangle_0 \quad \text{for all} \quad w\in W_2^2(D,b^*d)\ .$$

CRITERION. *A necessary and sufficient condition for the existence of a weak solution of Problem* (*) *and* (**) *is that the following α-priori estimate*

(AP) $\quad \|w\|_0\leqq C\cdot\|L^*w\|_0,\quad C:=const.>0$

holds for all $w\in W_2^2(D,b^*d)$, *or*

[AP] $\quad \|w\|_1\leqq C\cdot\|L^*w\|_0\ ,\quad C:=const.>0,$

where

$$\|w\|_0^2=\iint_D w^2\cdot dxdy\leqq\iint_D (w^2+w_x^2+w_y^2)\cdot dxdy=\|w\|_1^2$$

([2], p. 79-80).

Existence Theorem

Assume the afore-mentioned domain $D \subset \mathbf{R}^2$, and the conditions

$[R_1]$: $\tau \leqq 0$ *on* g_3

$[R_2]$: $(x+1)\cdot dy - y\cdot dx \geqq 0$ *on* g_1

$$[R_3]: \quad \begin{cases} 5\cdot\tau + 2\cdot(x+1)\cdot\tau_x + 2\cdot y\cdot\tau_y + \dfrac{1}{8}\cdot\mu_1 \leqq -\delta_1^{(1)} < 0 & in\ D_1 \\[2ex] 3\cdot\tau + 2\cdot(x+1)\cdot\tau_x + \dfrac{1}{8}\cdot\mu_1 \leqq -\delta_1^{(2)} < 0 & in\ D_2 \end{cases}$$

$[R_4]$: $1 - 2\cdot\mu_2\cdot(x+1)^2 \geqq \delta_2 > 0$ *in* D

$[R_5]$: $1 - 2\cdot\mu_3\cdot y^2 \geqq \delta_3 > 0$ *in* D_1 ,

where $\mu_1,\ \mu_2,\ \mu_3 := const. > 0$.

Then a weak solution exists for Problem (TBL).

PROOF. We apply the $a^*,\ b^*,\ c^*$ energy integral method separately in D_1, D_2 and use adjoint boundary conditions [**].

Denote

$$M^*w = a^*\cdot w + b^*\cdot w_x + c^*\cdot w_y, \qquad \text{where}$$

$$[\mathrm{c}] \qquad a^* = -\frac{1}{4}\ , \quad b^* = x+1, \quad c^* = \begin{cases} y & \text{in} \quad D_1 \\ 0 & \text{in} \quad D_2 \end{cases} .$$

Then employing Green's theorem we get

$$J^{(i)*} = \iint_{D_i} 2\cdot M^*w\cdot L^*w\cdot dxdy$$

$$= I_1^{(i)*} + I_2^{(i)*} + J_1^{(i)*} + J_2^{(i)*} + J_3^{(i)*}, \quad i = 1, 2,$$

where

$$I_1^{(i)*} = \iint_{D_i} A_1^* \cdot w^2 \cdot dxdy$$

$$I_2^{(i)*} = \iint_{D_i} (A_2^* \cdot w_x^2 - 2 \cdot B^* \cdot w_x \cdot w_y + A_3^* \cdot w_y^2) \cdot dx \cdot dy \ ,$$

$$J_1^{(i)*} = \oint_G B_1^* \cdot w^2 \cdot ds \ , \quad J_2^{(i)*} = \oint_G B_2^* \cdot ds \ ,$$

$$J_3^{(i)*} = \oint_G Q^* (w_x, w_y) \cdot ds \qquad i = 1, 2, \quad \text{such that}$$

$$A_1^* = (2 \cdot a^* - b_x^* - c_y^*) \cdot \tau - (b^* \cdot \tau_x + c^* \cdot \tau_y) + (K \cdot a_{xx}^* + a_{yy}^*) \ ,$$

$$A_2^* = -2 \cdot a^* \cdot K - b_x^* \cdot K + (c^* \cdot K)_y, \quad A_3^* = -2a^* + b_x^* - c_y^* \ ,$$

$$B^* = b_y^* + c_x^* \cdot K \ (: = 0 \text{ by considering choices [c] above}),$$

$$B_1^* = (b^* \cdot \nu_1 + c^* \cdot \nu_2) \cdot \tau \ ,$$

$$B_2^* = 2 \cdot a^* \cdot w \cdot (K \cdot w_x \cdot \nu_1 + w_y \cdot \nu_2) - (K \cdot a_x^* \cdot \nu_1 + a_y^* \cdot \nu_2) \cdot w^2,$$

$$Q^* = (b^* \cdot \nu_1 - c^* \cdot \nu_2) \cdot K \cdot w_x^2 + 2 \cdot (b^* \cdot \nu_2 + c^* \cdot K \cdot \nu_1) \cdot$$
$$\cdot w_x \cdot w_y + (-b^* \cdot \nu_1 + c^* \cdot \nu_2) \cdot w_y^2.$$

From

$$\left(\sqrt{\mu} \cdot |a| - \frac{1}{\sqrt{\mu}} \cdot |b| \right)^2 \geqq 0, \qquad \mu > 0$$

we get

$$\mu \cdot a^2 + \frac{1}{\mu} \cdot b^2 \geqq 2 \cdot |a \cdot b|, \ \ \mu > 0.$$

Therefore

$$J^{(i)*} \leqslant \iint_{D_i} 2\cdot|M^* w\cdot L^* w|\cdot dxdy \leqslant$$

$$\leqslant \iint_{D_i} [2\cdot|a^* w|\cdot|L^* w|+2\cdot|b^*\cdot w_x|+2\cdot|c^*\cdot w_y|]\cdot$$

$$\cdot dxdy \leqslant \iint_{D_i} \left\{\left[\mu_1\cdot(a^*\cdot w)^2+\frac{1}{\mu_1}\cdot(L^* w)^2\right]+\right.$$

$$+\left[\mu_2\cdot(b^*\cdot w_x)^2+\frac{1}{\mu_2}\cdot(L^* w)^2\right]$$

$$\left.+\left[\mu_3\cdot(c^*\cdot w_y)^2+\frac{1}{\mu_3}\cdot(L^* w)^2\right]\right\}\cdot dxdy\ ,$$

or

$$J^{(i)*} \leqslant \iint_{D_i} [\mu_1\cdot(a^*)^2\cdot w^2+\mu_2\cdot(b^*)^2\cdot(w_x)^2+\mu_3\cdot(c^*)^2\cdot(w_y)^2]\cdot dxdy$$
$$+C_1^2\cdot\|L^* w\|_0^2\ ,$$

where

$$C_1=\sqrt{\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}}:=\text{const.}>0, \qquad \mu_1,\mu_2,\mu_3:=\text{const.}>0.$$

But it is clear from conditions $[R_1]-[R_2]$ that

$$J_1^{(i)*},\ J_2^{(i)*},\ J_3^{(i)*}:\geqq 0,\ \text{ or }\ J^{(i)*}\geqq I_1^{(i)*}+I_2^{(i)*}, \qquad i=1,2.$$

Therefore

$$I_1^{(i)*}+I_2^{(i)*} \leqq \iint_{D_i} [\mu_1\cdot(a^*)^2\cdot w^2+\mu_2\cdot(b^*)^2\cdot(w_x)^2+\mu_3\cdot(c^*)^2\cdot$$
$$\cdot(w_y)^2]\cdot dxdy+C_1^2\cdot\|L^* w\|_0^2,$$

or

$$\left(I_1^{(i)*}-\iint_{D_i}\mu_1\cdot(a^*)^2\cdot w^2\cdot dxdy\right)+\left[I_2^{(i)*}-\iint_{D_i}(\mu_2(b^*)^2\cdot w_x^2+\right.$$
$$\left.+\mu_3\cdot(c^*)^2\cdot w_y^2)\cdot dxdy\right]\leqq C_1^2\cdot\|L^* w\|_0^2\ ,$$

or

$$\iint_{D_i} \beta_1 \cdot w^2 \cdot dxdy + \iint_{D_i} (\beta_2 \cdot w_x^2 + \beta_3 \cdot w_y^2) \cdot dxdy \leqq C_1^2 \cdot \| L^* w \|_0^2 ,$$

where

$$\beta_1 = A_1^* - \mu_1 \cdot (a^*)^2, \quad \beta_2 = A_2^* - \mu_2 \cdot (b^*)^2, \quad \beta_3 = A_3^* - \mu_3 \cdot (c^*)^2 .$$

From conditions $[R_3] - [R_5]$ we get that

$$C_2^2 \cdot \| w \|_1^2 \leqq C_1^2 \cdot \| L^* w \|_0^2, \quad \text{or} \quad \| w \|_1 \leqslant C \cdot \| L^* w \|_0 ,$$

where

$$C = C_1 / C_2 := \text{const.} > 0, \qquad C_1, C_2 := \text{const.} > 0 ,$$

completing the proof of the α-priori estimate [AP] and therefore (from Criterion above) the proof of the Existence theorem.

To elucidate more the fact that $J_1^{(i)*}, J_2^{(i)*}, J_3^{(i)*} : \geqq 0$ it is enough to show that the following conditions hold on G:

$(R_1 T)$ $\qquad b^* \cdot dy - c^* \cdot dx \geqq 0 \quad \text{on} \quad g_1 ,$

$(R_2 T)$
$$\begin{cases} (b^* + c^* \cdot \sqrt{-K}) \cdot \tau \leqq 0 & \text{on} \quad g_3 , \\ b^* - c^* \cdot \sqrt{-K} \geqq 0 & \text{on} \quad g_3, \\ a_x^* \cdot \sqrt{-K} - a_y^* + \dfrac{a^* \cdot K'}{4 \cdot (-\mathrm{K})} \leqq 0 & \text{on} \quad g_3 . \end{cases}$$

REFERENCES

[1] BARANTSEV R.G., «Lectures on transonic gas dynamics», Izdat. Leningrad Univ., Leningrad, 1965.

[2] BEREZANSKII, Ju. M., «Expansion in eigenfunctions of selfadjoint operators», Naukova Dumkova, Kiev, 1965; A.M.S. transl., R.I., 1968.

[3] BITSADZE A.V. and LAVRENTJEV M.A., «On the Problem of Equations of the Mixed Type», Dokl, Akad. Nauk S.S.S.R. 70, 1950, 373-376.

[4] CHAPLYGIN S.A., «On Gas Jets», Scientific Annals of the Imperial University of Moscow, Publ. No. 21, 104; transl. Brown University, R.I., 1944.

[5] FERRARI C., «Sul flusso transonico con onda d'urto...», Part II, Atti Acc. Naz. Lincei, 26, 1959, p. 313.

[6] FRANKL F.I., «On the Problems of Chaplygin for Mixed Sub- and Supersonic Flows», Bull. de l'Acad. des Sciences de l'U.R.S.S., 9, 2, 1945, 121-143.

[7] NOCILLA S., «Flussi transonici ...», Atti Acc. Sc. Torino, 92, 1957-58, P. 282; and idem part II, 93, 1958-59 , p. 124.

[8] PROTTER M.H., «Unigueness Theorems for the Tricomi Problem», J. Rat. Mech. and Anal., 2, No. 1, 1953, 107-114.

[9] RASSIAS J.M., «Mixed Type Partial Differential Equations in $\mathbf{R}^n$», Doct. Diss., U.C.-Berkeley, 1977.

[10] ——, «Mixed Type Equations», Teubner-Texte zur Mahematik, Leipzig, 90, 1986.

[11] SMYRNOV M.M., «Equations of Mixed Type», A.M.S., Transl. Math. Mon. 51, 1978.

[12] TRICOMI F.G., «Sulle Equazioni Lineare alle Parziali di 2° Ordine di Tipo Misto», Atti Accad. Naz. dei Lincei, 14, 1923, 133-247.