Solution of a Problem of Ulam

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THEOREM. Let X be a normed linear space with norm ∥·∥ and let Y be a Banach space with norm ∥·∥2. Assume in addition that f: X → Y is a mapping such that f(t·x) is continuous in t for each fixed x. If there exist a, b, 0 ≤ a + b < 1, and c2 ≥ 0 such that

\[ \|f(x + y) - [f(x) + f(y)]\|_2 \leq c_2 \cdot \|x\|^a \cdot \|y\|^b \] (1)

for all x, y ∈ X, then there exists a unique linear mapping L: X → Y such that

\[ \|f(x) - L(x)\|_2 \leq c \cdot \|x\|^a \] (2)

for all x ∈ X, where c = c2/(2 - 2a+b).

If one takes a = b = 0 in this theorem and follows our proof, one obtains an additive functional L such that ∥f(x) - L(x)∥2 ≤ c2, for all x in X. This is Hyer's result [3].

Proof of Existence. Inequality (1) and y = x imply

\[ \|f(2x) - 2f(x)\|_2 \leq c_2 \cdot \|x\|^a \]
More generally, the following lemma holds.

**Lemma 1.** In the space $X$,

$$\|f(2^n x)/2^n - f(x)\|_2 \leq c_2 \cdot \sum_{i=0}^{n-1} 2^{i(a+b-1)-1} \cdot \|x\|_1^{a+b}$$

for some $c_2 \geq 0$ and for any integer $n$.

To prove Lemma 1 we proceed by induction on $n$.

For $n = 1$, the result is obvious from (3). We assume then that (4) holds for $n = k$ and prove that (4) is true for $n = k + 1$. Indeed, from (4) and $n = k$ and $2 \cdot x = z$ we find:

$$\|f(2^{k+1} z)/2^{k+1} - f(z)\|_2 \leq c_2 \cdot \sum_{i=0}^{k-1} 2^{i(a+b-1)-1} \cdot \|z\|_1^{a+b},$$

or

$$\|f(2^{k+1} \cdot x)/2^{k+1} - f(2x)/2\|_2 \leq c_2 \cdot \sum_{i=0}^{k-1} 2^{(i+1)(a+b-1)-1} \cdot \|x\|_1^{a+b},$$

or

$$\|f(2^{k+1} \cdot x)/2^{k+1} - f(2x)/2\|_2 \leq c_2 \cdot \sum_{i=1}^{k} 2^{(a+b+1)-1} \cdot \|x\|_1^{a+b}.$$
But
\[
\sum_{i=0}^{n-1} 2^{i(a+b-1)} < \sum_{i=0}^{\infty} 2^{i(a+b-1)} = \frac{1}{1-2^{a+b-1}} = c_0. \tag{7}
\]
Set
\[c = c_0 \cdot \frac{c_2}{2}. \tag{7}'\]
It is clear that (3) and (6) yield (4), completing the proof of Lemma 1.
Then Lemma 1, (7), and (7)' imply
\[
\|f(2^n \cdot x)/2^n - f(x)\|_2 \leq 2^{-n} \cdot \|f(h)\|_2 \left(1 + 2^{-n} \cdot \|f(h)\|_2\right) \tag{8}
\]
for any \(x \in X\), any positive integer \(n\), and some \(c_2 \geq 0\).

**Lemma 2.** The sequence \(\{f(2^n \cdot x)/2^n\}\) converges.

We first use (8) and the completeness of \(Y\) to prove that the sequence \(\{f(2^n \cdot x)/2^n\}\) is a Cauchy sequence. In fact, if \(i > j > 0\), then
\[
\|f(2^i \cdot x)/2^i - f(2^j \cdot x)/2^j\|_2 = 2^{-j} \cdot \|f(2^i \cdot x)/2^i - f(2^j \cdot x)/2^j\|_2 \tag{9}
\]
and if we set \(2^j \cdot x = h\) in (9) and employ (8) we get
\[
\|f(2^i \cdot x)/2^i - f(2^j \cdot x)/2^j\|_2 = \frac{1}{2^j} \cdot \frac{1}{2^j} \cdot 2^{-j} \cdot \|f(h)\|_2 < 2^{j-1} \cdot c \cdot \|x\|_1 \cdot 2^{j-1} \cdot \|x\|_1 \tag{10}
\]
or
\[
\lim_{j \to \infty} \|f(2^j \cdot x)/2^j - f(2^j \cdot x)/2^j\|_2 = 0 \tag{10}
\]
because \(a, b: 0 \leq a + b < 1\).

It is obvious now from (10) and the completeness of \(Y\) that the sequence \(\{f(2^n \cdot x)/2^n\}\) converges and therefore the proof of Lemma 2 is complete.

We set
\[
L(x) = \lim_{n \to \infty} \frac{f(2^n \cdot x)}{2^n}. \tag{11}
\]
It is clear from (1) and (11) that
\[
\|f(2^n \cdot x + 2^n \cdot y) - [f(2^n \cdot x) + f(2^n \cdot y)]\|_2 \leq c_2 \cdot \|2^n \cdot x\|_1 \cdot \|2^n \cdot y\|_1 \tag{12}
\]
or
\[
2^{-n} \cdot \|f(2^n \cdot x + 2^n \cdot y) - [f(2^n \cdot x) + f(2^n \cdot y)]\|_2 \leq c_2 \cdot 2^{(a+b-1)n} \cdot \|x\|_1 \cdot \|y\|_1. \tag{13}
\]
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or

\[ \lim_{n \to \infty} \left[ f(2^n \cdot (x + y))/2^n \right] - \lim_{n \to \infty} \left[ f(2^n \cdot x)/2^n \right] - \lim_{n \to \infty} \left[ f(2^n \cdot y)/2^n \right] = 0, \]

or

\[ \| L(x + y) - L(x) - L(y) \|_2 = 0 \quad \text{for any } x, y \in X, \]

or

\[ L(x + y) = L(x) + L(y) \quad \text{for any } x, y \in X. \quad (12) \]

From (12) we get

\[ L(q \cdot x) = q \cdot L(x) \quad (13) \]

for any \( q \in \mathbb{Q} \), where \( \mathbb{Q} \) is the set of rationals.

**Lemma 3.** Let \( Y^+ \) be the space of continuous linear functionals and consider the mapping

\[ T: t \to g(L(t \cdot x)), \quad \text{or} \quad T: \mathbb{R} \to \mathbb{R} \quad (14) \]

such that

\[ T(t) = g(Ut \cdot x), \quad (15) \]

where \( g \in Y^+ \), \( t \in \mathbb{R} \), and \( x \in X \), \( x := \text{fixed} \). Then \( T \) is a continuous mapping.

To prove Lemma 3 we proceed as follows: Let

\[ T_n(t) = g \left( \frac{f(2^n \cdot x \cdot t)}{2^n} \right) \quad (16) \]

such that

\[ T(t) = \lim_{n \to \infty} T_n(t), \quad (17) \]

where \( x \in X \), \( x := \text{fixed} \) and \( t \in \mathbb{R} \), \( g \in Y^+ \).

Then \( T_n(t) \) are continuous and therefore \( T \) is measurable as the pointwise limit of continuous mappings \( T_n \). Moreover, \( T \) is a homomorphism with respect to addition "+," that is,

\[ T(x + y) = T(x) + T(y) \quad (18) \]

for any \( x, y \in \mathbb{R} \). It is clear now that (18) and the measurability of \( T \) imply
that $T$ is a continuous mapping and thus the proof of Lemma 3 is complete.

Then Lemma 3 and the fact that $Y^+$ separates points of $Y$ yield the linearity of $L$. However, if we take limits on both sides of (8) as $n \to \infty$ we obtain (2). Therefore, we have proved the existence of a linear mapping $L : X \to Y$ which also satisfies (2).

Uniqueness. It remains to show the uniqueness part of our theorem.

Let $M$ be a linear mapping $M : X \to Y$, such that

$$
\|f(x) - M(x)\|_2 \leq c' \cdot \|x\|_1^{a' + b'}, \quad c' \geq 0,
$$

(19)

for any $x \in X$ where $a', b' : 0 \leq a' + b' < 1$ and $c'$ is a constant. If there exists a linear mapping $L : X \to Y$ such that (2) holds, then

$$
L(x) = M(x)
$$

(20)

for any $x \in X$.

To prove (20) we must prove the following

**Lemma 4.** If (2) and (19) hold, then

$$
\|L(x) - M(x)\|_2 \leq m^{a' + b' - 1} \cdot c \cdot \|x\|_1^{a' + b' + m^{a' + b' - 1} \cdot c' \cdot \|x\|_1^{a' + b'}}
$$

(21)

for any $x \in X$.

The required result (21) follows immediately if we use inequalities (2) and (19), the linearity of $L$ and $M$, as well as the triangle inequality. In fact,

$$
L(x) = \frac{L(m \cdot x)}{m}, \quad M(x) = \frac{M(m \cdot x)}{m},
$$

$$
\|L(m \cdot x) - M(m \cdot x)\|_2 \leq L(m \cdot x) - f(m \cdot x)\|_2 + \|M(m \cdot x) - f(m \cdot x)\|_2.
$$

Then if we apply (2) and (19) we obtain inequality (21) and the proof of Lemma 4 is complete.

It is clear now that (21) implies $\lim_{m \to \infty} \|L(x) - M(x)\|_2 = 0$ for any $x \in X$, completing the proof of (20). Thus the uniqueness part of our Theorem is complete, as well.

**Remark.** A Banach space $Y$ is said to have the approximation property if for any compact set $K \subset Y$ and any $\varepsilon > 0$, there exists $P \in L(Y, Y)$ depending on $K$ and $\varepsilon$, with finite-dimensional range such that

$$
\|P(x) - x\| \leq \varepsilon
$$

for any $x \in K$. 
The approximation problem states: Is every compact operator $T$ in $L(X, Y)$ a limit in the norm of operators with finite dimensional range?

The approximation problem has a negative solution in Banach spaces (which are not Hilbert spaces) and was solved in the negative by Per Enflo (1973) via an example of a separable reflexive Banach space that does not have the approximation property.

Query. What is the situation in the above theorem in case $a + b = 1$?

REFERENCES