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On Three New Generalized Uniqueness Theorems Of The Tricomi Problem For Non-Linear Mixed Type Equations

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ABSTRACT

In 1923 F.G.Tricomi [5] initiated the work on boundary value problems for linear partial differential mixed type equations of second order and related equations of variable type. Then 1945 F.I.Frankl [2] drew attention to the fact that the Tricomi problem was closely connected to the study of gas flow with nearly sonic speeds. Since then has been a steady stream of new results of all kinds of linear mixed type boundary value problems only. To the best of our knowledge no serious work exists on nonlinear mixed type boundary value problems. In this paper we establish three new uniqueness theorems for quasi-regular solutions of the Tricomi problem for nonlinear mixed type equations never studied before. These results can be applied in fluid dynamics [3], [4]. Finally we quote that in 1954 L.Bers [1] stated that "to investigate problems of mixed type one must be lucky, because there are no simple methods".

1. Quasi-Regular Solutions

Consider the non-linear mixed type equation

$$\tilde{L}w = K(y).w_{xx}+w_{yy} = f(x,y,w,w_x,w_y), \tag{1}$$

where

$K=K(y)$ is a monotone increasing continuous function for any y, with a continuous derivative of second order for $y<0$,

$K(0) = 0$,

(c_1): $K'(y) > 0$ for $y<0$,

$\lim\limits_{y\to 0^-} \dfrac{K(y)}{K'(y)} = 0$, and

f, f_w, f_p, f_q,

(c_2): $(p = w_x,\ q = w_y)$: are continuous functions of x,y,w,p,q: $w,p,q \in IR^1$, and for any x,y in a simply-connected domain G

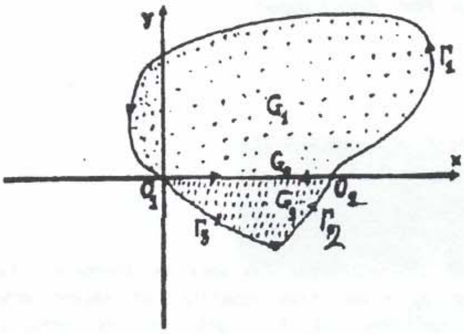

Fig.1

bounded for $y > 0$ by a non-selfintersecting (Jordan) smooth curve Γ_1 intersecting the x-axis ($:y=0$) at the points $O_1\ (0,0)$ and $O_2\ (1,0)$, and for $y < 0$ by the characteristic curves Γ_2 and Γ_3 of equation (1) emanating from the points O_1 and O_2 respectively, and intersecting at some point in the lower half-plane, such that

$$\Gamma_2: \frac{dy}{dx} = \frac{1}{\sqrt{-K}}, \text{ and } \Gamma_3: \frac{dy}{dx} = -\frac{1}{\sqrt{-K}}.$$

Denote $G_1 = G \cap \{y > 0\}$, $G_2 = G \cap \{y < 0\}$ and $G_0 = G \cap \{y = 0\}$, and boundary $\partial G = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Equation (1) is elliptic in G_1, hyperbolic in G_2, and parabolic in G_0. Assume prescribed continuous boundary values

$$w = g \text{ on } \Gamma_1 \cup \Gamma_2, \tag{2}$$

where $g = g(s_0)$, $s_0 \in \Gamma_1 \cup \Gamma_2$, and g:continuous on $\Gamma_1 \cup \Gamma_2$.

Assume two solutions w_1 and w_2, then $u = w_1 - w_2$ is a solution of equation $Lu = K(y).u_{xx} + u_{yy} - \Delta f = 0$ or

$$Lu = K(y).u_{xx} + u_{yy} - r(x,y).u - s(x,y).u_x - t(x,y).u_y = 0 \tag{3}$$

where

$$r = r(x,y) = \int_{h=0}^{h=1} f_w(x,y,w_2+h.u,\ w_{2x}+h.u_x,\ w_{2y}+h.u_y).dh,$$

$$s = s(x,y) = \int_{h=0}^{h=1} f_p(x,y,w_2+h.u,\ w_{2x}+h.u_x,\ w_{2y}+h.u_y)dh,$$

$$t = t(x,y) = \int_{h=0}^{h=1} f_q(x,y,w_2+h.u,\ w_{2x}+h.u_x,\ w_{2y}+h.u_y)dh,$$

and

$$\Delta f = f(x,y,w_1,w_{1x},w_{1y}) - f(x,y,w_2,w_{2x},w_{2y})$$

$$= f(x,y,w_2+u,w_{2x}+u_x,w_{2y}+u_y) - f(x,y,w_2,w_{2x},w_{2y})$$

(because: $u = w_1 - w_2$ or $w_1 = w_2 + u$) or

$$\Delta f = \int_{h=0}^{h=1} df(x,y,w_2+h.u,\ w_{2x}+h.u_x,\ w_{2y}+h.u_y) = r.u+s.u_x+t.u_y.$$

Besides

$$u = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2 \tag{4}$$

Mixed type boundary value problem or Problem M: consists in finding a function $u = u(x,y)$ which satisfies equation (3) in a mixed domain G and assumes boundary values (4).

Definition 1. A quasi regular solution of Problem M is a function $u = u(x,y)$ satisfying

(i) equation (3) in G,

(ii) integrals

$$\int_{G_0} u(x,0).u_y(x,0).dx, \text{ and}$$

$$\iint_{G_1} (K.u_x^2+u_y^2).dS \quad (dS=dx.dy) \text{ exist,}$$

and

(iii) Green's theorem may be applied to the integrals

$$\iint_G u.Lu.dS, \quad \iint_G u_x.Lu.dS, \quad \iint_G u_y.Lu.dS,$$

and the arising boundary integrals exist in the sense that the limits taken over corresponding inner curves exist as these inner curves approach boundary ∂G.

Denote

$$Mu = a.u + b.u_x + c.u_y,$$

where

$$a = -\frac{1}{2}.e^{\beta.x}.\cos(\gamma.y), \quad b=0, \ c=0 \ \text{ in } G_1$$

$$a = -\frac{1}{2}.e^{\beta.x}, \quad b = c.\sqrt{-K}, \quad c = \frac{4.a.K}{K'}, \ \text{ in } G_2,$$

with

$$\beta = \frac{\pi}{2.y_m.\sqrt{K(y_m)}}, \quad \gamma = \frac{\pi}{2.y_m},$$

y_m :=maximum of the ordinates of points on Γ_1.

Besides denote:

$$< Mu, Lu, >_G = \iint_G Mu.Lu.dS. \tag{5}$$

Finally denote:

$$f_0 = \frac{1}{4.K}.(f_p^2+Kf_q^2+2.K.(K.\beta^2-\gamma^2)) \ \text{ in } G_1\times IR^3$$

(<u>note</u>: $K.\beta^2-\gamma^2 = (\sqrt{\frac{K(y)}{K(y_m)}} - 1).\gamma^2 < 0$

because K is a monotone increasing function:

$$y < y_m \text{ implies } K(y) < K(y_m)),$$

and

$$R = R(y) = 1+2\left(\frac{K}{K'}\right)'-2\,\frac{K.\sqrt{-K}}{K'}.\beta \text{ in } G_2.$$

Assume additional conditions:

(c_3): $f_w \geq f_0$ for $(x,y,w,p,q) \in G_1 \times IR^3$,

(c_4); $R \geq 0$ in G_2,

(c_5): $f_p = \sqrt{-K}.f_q$ for $(x,y,w,p,q) \in G_2 \times IR^3$,

(c_6): $0 \leq f_p \leq \frac{K'}{4.\sqrt{-K}}.R$ for $(x,y,w,p,q) \in G_2 \times IR^3$,

(<u>note</u>: Conditions (c_5) and (c_6) yield:

$$0 \leq f_q \leq -\frac{K'}{4K}.R).$$

(c_7): $4.(-K).(K')^2.R.f_w \geq (4.(-K)^{3/2}.f_w+K'.f_p)^2,\quad f_w \geq 0$

or

$$f_{o1} \leq f_w \leq f_{o2} \quad \text{for} \quad (x,y,w,p,q) \in G_2 \times IR^3,$$

where

$$f_{o1} = \frac{K'}{8.K^2}.[(K'.R-2\sqrt{-K}.f_p)-\sqrt{K'.(K'.R-4\sqrt{-K}.f_p).R}\,],$$

and

$$f_{o2} = \frac{K'}{8.K^2}.[(K'.R-2.\sqrt{-K}.f_p)+\sqrt{K'.(K'.R-4\sqrt{-K}.f_p).R}].$$

Special Case: $f_p = 0$ (<u>therefore</u> $f_q = 0$):

$(c_3)'$: $f_w \geq \frac{1}{2}.(K.\beta^2-\gamma^2)$ in $G_1 \times IR^1$,

$(c_7)'$: $0 \leq f_w \leq \frac{1}{4}.(\frac{K'}{K})^2.R$ in $G_2 \times IR^1$.

Theorem 1. Assume described domain G and conditions (c_i) $(i=1,2,\ldots,7)$. Then problem $\tilde{N}$ ($\tilde{:}Lw=f$ (1), $w=g$ on $\Gamma_1 \cup \Gamma_2$ (2)) has at most one quasi-regular solution in G.

Proof. We apply the energy integral method; that is, consider integral expression (5).

Note the identities:

$$2.a.K.u.u_{xx} = 2(a.K.u.u_x)_x-2.a.K.u_x^2-(a_x.K.u^2)_x+a_{xx}.K.u^2,$$

$$2.a.u.u_{yy} = 2(a.u.u_y)_y-2.a.u_y^2-(a_y.u^2)_y+a_{yy}.u^2,$$

$$2.b.K.u_x.u_{xx} = (b.K.u_x^2)_x-b_x.K.u_x^2,$$

$$2.b.u_x.u_{yy} = 2(b.u_x.u_y)_y-2.b_y.u_x.u_y-(b.u_y^2)_x+b_x.u_y^2,$$

$$2.c.K.u_y.u_{xx} = 2(c.K.u_x.u_y)_x-(c.K.u_x^2)_y+(c.K)_y.u_x^2-2.c_x.K.u_x.u_y,$$

$$2.c.u_y.u_{yy} = (c.u_y^2)_y-c_y.u_y^2.$$

Then substitute the above identities in (5), employ operator L of (3) and apply Green's theorem:

$$0=2.<Nu,Lu>_G=2.\iint_G Nu.Lu.dS=2\iint_G (a.u+b.u_x+c.u_y).(K.u_{xx}+u_{yy}-r.u$$

$$-s.u_x-t.u_y).dS=\iint_{G_1} \{(-2.a).[(K.u_x^2+u_y^2)+(s.u.u_x+t.u.u_y)]+C_0.u^2\}.dS$$

$$+\iint_{G_2} (A.u_x^2+B.u_y^2+Cu^2+2Du_xu_y+2.E.u.u_x+2.Z.u.u_y).dS$$

$$- \int_{\Gamma_2} (b+c\,.\sqrt{-K}).(\sqrt{-K}.u_x+u_y).du + \int_{\Gamma_3} (b-c.\sqrt{-K}).(\sqrt[4]{-K}.u_x - \frac{1}{\sqrt[4]{-K}}.u_y)^2 + \int_{\Gamma_3}\left[2a\sqrt{-K}\,u\,du - (\sqrt{-K}\,da)\,u^2\right]$$

$$= I_1 + I_2 + J_1 + J_2 + J_3 \qquad (6)$$

where

$$A = -2.a.K-K.b_x+(c.K)_y-2.b.s=-e^{\beta.x}.K.(R-4.\frac{\sqrt{-K}}{K'}.s),.$$

$$B = -2a+b_x-c_y-2.c.t = e^{\beta.x}.(R+4.\frac{K}{K'}.t)$$

$$= e^{\beta.x}.(R-4.\frac{\sqrt{-K}}{K'}.s)(\text{ because } s=\sqrt{-K}.t \text{ from } (c_5)),$$

$$C = K.a_{xx}-2.a.r = e^{\beta.x}.(-\frac{K}{2}.\beta^2 + r),$$

$$D = -(K.c_x+b_y+b.t+c.s) = e^{\beta.x}.\sqrt{-K}.(R-4.\frac{\sqrt{-K}}{K'}.s)$$

(because $b.t+c.s = c(\sqrt{-K}.t+s) = 2cs$

$$E = -(a.s+b.r) = e^{\beta.x}.(\frac{1}{2}.s+2.\frac{K.\sqrt{-K}}{K'}.r),$$

$$Z = -(a.t+c.r) = \frac{e^{\beta.x}}{\sqrt{-K}}.(\frac{1}{2}.s+2.\frac{K.\sqrt{-K}}{K'}.r) \text{ in } G_2,$$

$$C_0=K.a_{xx}+a_{yy}-2.a.r=-a.(-K.\beta^2+\gamma^2+2.r)= \frac{1}{2}.e^{\beta.x}.\cos(\gamma.y).(-K.\beta^2+\gamma^2+2r)$$

in G_1.

$b-c.\sqrt{-K} = 0$ on Γ_3

(because: $b = c.\sqrt{-K}$ in G_2); thus

$$J_2 = 0 \text{ and} \qquad (7)$$

$\frac{dy}{dx} = \frac{1}{\sqrt{-K}}$ on Γ_2 with $u=0$ .on Γ_2 (from (4)) imply

$$J_1 = 0 \qquad (8)$$

Besides the symmetric quadratic form $Q_1 = Q_1(u_x, u_y, u)$ in the integrand of the integral I_1 is non-negative definite, because:

$$Q_1 = V.M_1.V^t,$$

where

$$V = (u_x, u_y, u) \quad V^t := \text{transpose of } V := \begin{pmatrix} u_x \\ u_y \\ u \end{pmatrix}$$

and

$$M_1 = \begin{bmatrix} -2a.K & 0 & -a.s \\ 0 & -2.a & -a.t \\ -a.s & -a.t & C_0 \end{bmatrix}$$

the matrix of the coefficients of the form Q_1 with non-negative all its seven principal minors in G_1:

$$M_1^{(1)} = -2a.K = e^{\beta.x}.\cos(\gamma.y).K \geq 0,$$

$$M_1^{(2)} = -2a = e^{\beta.x}.\cos(\gamma.y) > 0,$$

$$M_1^{(3)} = C_0 \geq 0 \text{ because } (c_3) \text{ holds if } -K.\beta^2+\gamma^2+2.f_w \geq \frac{f_p^2+K.f_q^2}{2.K} \quad (\geq 0),$$

$$M_1^{(4)} = (-2a.K).(-2a)-0^2 = 4a^2.K \geq 0,$$

$$M_1^{(5)} = (-2a).C_0-(-a.t)^2 = a^2.[2.(-K.\beta^2+\gamma^2+2.f)-t^2] \geq 0,$$

$$\text{because } (c_3) \text{ holds if } 2.(-K.\beta^2+\gamma^2+2.f_w)-f_q^2 \geq \frac{f_p^2}{K} \quad (\geq 0),$$

$$M_1^{(6)} = (-2a.K).C_0-(-a.s)^2 = a^2[2K.(-K.\beta^2+\gamma^2+2.r)-s^2] \geq 0,$$

because (c_3) holds if $2K.(-K.\beta^2+\gamma^2+2.f_w)-f_p^2 \geq K.f_q^2 \ (\geq 0)$, and

$$M_1^{(7)} = \det(M_1) = -2a^3.[2K.(-K.\beta^2+\gamma^2+2.r)-s^2-K.t^2] \geq 0$$

because (c_3) holds if $2.K(-K.\beta^2+\gamma^2+2.f_w)-f_p^2-K.f_q^2 \geq 0$.

Thus

$$I_1 \geq 0 \tag{9}$$

Finally the symmetric quadratic form $Q_2 = Q_2(u_x, u_y, u)$ in the integrand of the integral I_2 above is non-negative, because

$$Q_2 = V.M_2.V^t,$$

where

$$M_2 = \begin{bmatrix} A & D & E \\ D & B & Z \\ E & Z & C \end{bmatrix}$$

with non-negative all its seven principal minors in G_2:

$M_2^{(1)} = A \geq 0$ and

$M_2^{(2)} = B \geq 0$ (because of (c_4) and (c_6)),

$M_2^{(3)} = C \geq 0$ (because of $f_w \geq 0$ from (c_7)),

$M_2^{(4)} = A.B-D^2 = 0,$

$M_2^{(5)} = B.C-Z^2 \geq 0$ and

$M_2^{(6)} = A.C-E^2 \geq 0$ (because of (c_6), (c_7), and

$$(\int f_w)^2 \leq \int f_w^2, \ \int f_w . \int f_p \leq \int f_w . f_p, \ (\int f_p)^2 \leq \int f_p^2),$$

and

$$M_2^{(7)} = \det (M_2) = C.M_2^{(4)} + 2.D.E.Z-(A.Z^2+B.E^2) = 0.$$

Thus

$$I_2 \geq 0. \tag{10}$$

Therefore from (7), (8) and (10) all the integrals in (6) are non-negative; hence (6) implies $u \equiv 0$ in G, and $w_1 = w_2$ in G; thus there is at most one quasi-regular solution of Problem M in G and this completes the proof of our Theorem 1.

Theorem 1'. Assume described domain G and conditions (c_j) $(j = 1,2,3,4,5)$ and

$$(c_6)'': 0 \leq f_p \leq \frac{K'}{2.\sqrt{e_1.e_2}.\sqrt{-K}}.R, \quad (x,y,w,p,q) \in G \times IR^3,$$

where

$$e_1 = 1+e, \quad e_2 = 1 + \frac{1}{e}, \quad e > 0$$

(<u>note</u>: $0 \leq f_q \leq -\frac{K'}{2.\sqrt{e_1.e_2}.K}.R$).

$$c_7)'': \begin{cases} 4(-K).(K').(K').R.f_w \geq 16.e_1.(-K)^3.f_w^2+e_2(K')^2.f_p^2, \\ f_w \geq 0 \end{cases}$$

or

$$f'_{o1} \le f_w \le f'_{o2} \quad \text{for} \quad (x,y,w,p,q) \in G_2 \times IR^3,$$

where

$$f'_{o1} = \frac{K'}{8.e_1.K^2}.[K'.R-\sqrt{(K')^2.R^2+4.e_1.e_2.K.f_p^2}],$$

and

$$f_{o2} = \frac{K'}{8.e_1.K^2}.[K'.R+\sqrt{(K')^2.R^2+4.e_1.e_2.K.f_p^2}].$$

Then Problem $\tilde{M}$ has at most one quasi-regular solution in G.

Proof. It is the same as the proof of above Theorem 1 except here $M_2^{(i)} \ge 0$ $(i=1,2,3)$ because of (c_4) (c_6)" and (c_7)" and: $M_2^{(5)} \ge 0$ and $M^{(6)2} \ge 0$ because of (c_6)" and (c_7)" and

$$(4.(-K)^{3/2}.f_w+K'.f_p)^2 \le 16.e_1.(-K)^3.f_w^2+e_2.(K')^2.f_p^2.$$

Remarks 1. (1) Consider f for $(x,y,w,p,q) \in G_2 \times IR^3$:

$$f_w = \frac{1}{16}\left(\frac{K'}{K}\right)^2.R,$$

$$f_p = \frac{K'}{4.\sqrt{-K}}.R, \qquad f_q = -\frac{K'}{4.K}.R.$$

Then conditions (c_6) and (c_7) are satisfied.

(2). Consider f for $(x,y,w,p,q) \in G_2 \times IR^3$:

$$f_w = \frac{1}{8.e_1}.\left(\frac{K'}{K}\right)^2.R,$$

$$f_p = \frac{K'}{2.\sqrt{e_1.e_2}.\sqrt{-K}}.R, \qquad f_q = -\frac{K'}{2.\sqrt{e_1.e_2}.K}.R.$$

Then conditions $(c_6)''$ and $(c_7)''$ are satisfied.

2. Regular Solutions

Consider equation (1) and conditions (c_i) $(i=1,2,3,4,5,6,7)$, or conditions (c_j) $(j=1,2,3,4,5)$, $(c_6)''$ and $(c_7)''$, boundary conditions (2) and domain G_e :

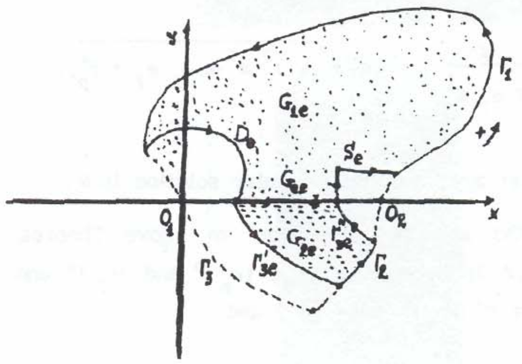

Fig.2

a simply connected domain (a part of G) bounded in the neighborhood of O_1 by a circular arc D_e with center O_1 and radius e (for $y > 0$) and near O_2 by s_e (for $y > 0$), where s_e is the line

$$x = 1-e,$$

if $x < 1$ on Γ_1 near O_2; if $x \geq 1$ on Γ_1 near O_2 then s_e consists of the two lines: $x = 1-e$, and $y = \sqrt{e}$, and by the characteristics: Γ'_{3e} , Γ''_{3e} : $\frac{dy}{dx} = -\frac{1}{\sqrt{-K}}$ as in Fig.2.

Denote $G_{1e} = G_e \cap \{y > 0\}$, $G_{2e} = G_e \cap \{y < 0\}$, and $G_{0e} = G_e \cap \{y=0\}$ and boundaries

$$\partial G_{1e} = \Gamma_1 \cup D_e \cup G_{0e} \cup s_e, \quad \partial G_{2e} = \Gamma_2 \cup \Gamma''_{3e} \cup G_{0e} \cup \Gamma'_{3e},$$

$$\partial G_e = \Gamma_1 \cup D_e \cup \Gamma'_{3e} \cup \Gamma_2 \cup \Gamma''_{3e} \cup s_e.$$

Besides assume "regularity condition"

$$(c_8): \frac{K'(y)}{\sqrt{-K(y)}} \le c_1 . (-y)^{\alpha-1} \text{ on } \Gamma_3 \text{ near } O_1,$$

where $\alpha > 0$, and c_1 :=some positive constant.

Definition 2. A regular solution of Problem M is a function $u = u(x,y)$ which satisfies

(1). equation (3) in $G \setminus G_0$ and

(2). is continuous in G,

(3). has continuous first derivatives in G and Γ_1, Γ_2 (both open), with the possible exception of the points O_1, O_2, in whose neighborhoods they may have poles of order less than 1:

$$u_x, u_y = O(|O_iP|^{-\omega_i}), \quad 0 \le \omega_i < 1,$$

$$|O_iP| \to 0 \ (i=1,2), \quad P \in G, \text{ and}$$

(4) has continuous second derivatives in G_i $(i=1,2)$ with the possible exception of points on the parabolic curve, in whose neighborhoods they may not exist.

Theorem 2. Assume above described domains G and G_e, and conditions (c_i) $(i=1,2,3,4,5,6,7,8)$ or conditions (c_j) $(j=1,2,3,4,5)$ (c_6)" and (c_7)". Then Problem M has at most one regular solution in G.

Proof. It is clear from Theorem 1 or Theorem 1' that (6) can be replaced by

$$0 = 2. <Mu, Lu>_{G_e} = 2. \iint_{G_e} Mu.Lu.dS$$

$$= \iint_{G_{1e}} \{(-2a).[(K.u_x^2+u_y^2)+(s.u.u_x+t.u.u_y)]+C_0.u^2\}.dS$$

$$+ \iint_{G_{2e}} (A.u_x^2+B.u_y^2+C.u^2+2.D.u_x.u_y+2.E.u.u_x+2.Z.u.u_y).dS$$

$$- \int_{\Gamma_2} (b+c.\sqrt{-K}).(\sqrt{-K}.u_x+u_y)du + \int_{\Gamma'_{3e}\cup\Gamma''_{3e}} (b-c.\sqrt{-K})(\sqrt[4]{-K}.u_x - \frac{1}{\sqrt[4]{-K}}.u_y)^2$$

$$= I_{1e} + I_{2e} + J_{1e} + J_{2e}, \tag{11}$$

where

$$J_{2e} = 0 \tag{12}$$

because: $b = c.\sqrt{-K}$ in G_{2e},

$$J_{1e} = 0 \tag{13}$$

because:

$$\frac{dy}{dx} = \frac{1}{\sqrt{-K}} \quad \text{on} \quad \Gamma_2$$

and $u = 0$ on Γ_2 from (4),

$$I_{ie} \geq 0 \quad (i = 1,2) \tag{14}$$

because: Q_i are non-negative definite in G_{ie} $(i=1,2)$ (in the same way as in Theorem 1).

Therefore from (12), (13) and (14) all the integrals in (11) are non-negative; hence (11) implies

$$u \equiv 0 \quad \text{in} \quad G_e. \tag{15}$$

If we take $e \to 0$, then (11), (12), (13) and (14) in G_e imply (6), (7), (8) and (9)-(10) in G respectively.

Therefore (15) yields

$$u \equiv 0 \quad \text{in} \quad G,$$

and $w_1 = w_2$ in G; hence there is at most one regular solution of Problem $\tilde{M}$ in G and this completes the proof of our Theorem 2.

Remark 2. S_e above may be replaced by a circular arc D'_e with center O_2 and radius e (for $y > 0$) and produce an analogous Theorem 2' to Theorem 2. For the proof of this new theorem employ 11, as well.

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