Discussiones Mathematicae
T. IX, 1988, 255-267

TWO NEW CRITERIA ON CHARACTERIZATIONS OF INNER PRODUCTS

by

JOHN M. RASSIAS (GREECE)

ABSTRACT

It is known that S.Banach (1932) introduced normed linear spaces and that most "natural" geometric properties may fail to hold in a general normed space unless the space is an inner (or: scalar) product space. Later P.Jordan and J.von Neuman (1935) showed that a normed linear space V is an inner product space if the parallelogram equality: $\|v-w\|^2+\|v+w\|^2=2\|v\|^2+2\|w\|^2$ $(v,w\in V)$ *holds. Then G.Birkhoff (1935), M.M.Day (1947), and R.C.James (1947) gave the basic characterizations of orthogonality relations and by the duality map. Later another major improvement was I.J.Schoenberg's idea of using Loewner ellipsoids to replace the parallelogram equality by an inequality in either direction (1952). An interesting book on normed linear spaces is due to M.M.Day (1973). Among the most interesting developments of this topic in the last five years are due to M.Baronti (1981), W.L.Zeng (1982), P.L.Papini (1982-1983), D.Amir and C.Franchetti (1983), C.Benitez and M.del Rio (1984), M.M.Day (1984), T.M.Rassias (1984), and R.W.Freese, C.R.Diminnie and E.Z.Andalafte (1985). In particular, M.M.Day (1984) established the sufficiency of the condition:* $R_2=\|2u-v-w\|^2+\|u-2v+w\|^2+\|u+v-2w\|^2+3\|u+v+w\|^2=27$ $(\|u\|=\|v\|=\|w\|=1)$. *An interesting collection of some 350 linear isometric and geometric characterizations of inner product spaces among normed linear spaces is due to D.Amir*

(1986). In this paper we extend condition $R_2=27$ to $R_n=\|(k-1)u_1-u_2-\ldots-u_k\|^n+\ldots+\|u_1+u_2+\ldots+u_{k-1}-(k-1)u_k\|^n+$ $+k\|u_1+u_2+\ldots+u_k\|^n$ (n: positive real number) by imposing and proving two criteria on characterizations of inner products.

Criterion 1. Given

$$R_n=\|(k-1)u_1-u_2-\ldots-u_k\|^n+\ldots+\|u_1+u_2+\ldots+u_{k-1}-(k-1)u_k\|^n+$$

$$+k\|u_1+u_2+\ldots+u_k\|^n,$$

$$A_n=|(k-1)\|u_1\|-\|u_2\|-\ldots-\|u_k\||^n+\ldots+|\|u_1\|+\|u_2\|+\ldots+$$

$$+\|u_{k-1}\|-(k-1)\|u_k\||^n+k(\|u_1\|+\|u_2\|+\ldots+\|u_k\|)^n,$$

$$B_n=\sum_{i=1}^{k}\|u_i\|^n,$$

$$C_n=\prod_{i=1}^{k}\|u_i\|^{n/k},$$

$$D_n=\min_{i\in I}\|u_i\|^n,\quad I=\{1,2,\ldots,k\},\ \text{for } u_i\in V,$$

and n=positive real number, $k\in\mathbb{N}$, then necessary and sufficient conditions that the norm defined over a real vector space V be induced by an inner product are that:

(1). $R_n\le A_n$, $n>2$,
(2). $R_n\ge A_n$, $0<n<2$,
(3). $R_n\le k^nB_n$, $n>2$,
(4). $R_n\ge k^nB_n$, $1<n<2$,
(5). $R_n\ge k^{n+1}C_n$, $1<n<2$,
(6). $R_n\ge k^{n+1}D_n$, $1<n<2$.

Proof. *Conditions are necessary.*

We assume that the norm $\|\cdot\|$ defined over V is induced by an inner product $<\cdot,\cdot>$, $\|f\|^2=<f,f>$, $f\in V$, and p is such that $<f,g>=\|f\|\cdot\|g\|\cdot\cos p$, $f,g\in V$.

Consider the function R_n: $\mathbb{R}^m\to\mathbb{R}$, $m=k(k-1)/2$, $k\geq 2$, $k\in\mathbb{N}$, $R_n=R_n(p)$, $p=(p_1,\dots,p_m)\in\mathbb{R}^n$, $p_1=(u_1\hat{,}u_2)$, $p_2=(u_1\hat{,}u_3),\dots$, $p_m=(u_{k-1}\hat{,}u_k)$. Then,

$$R_n=(\|(k-1)u_1-u_2-\dots-u_k\|^2)^{n/2}+\dots+(\|u_1+u_2+\dots+$$

$$+u_{k-1}-(k-1)u_k\|^2)^{n/2}+k(\|u_1+u_2+\dots+u_k\|^2)^{n/2},$$

or

$$R_n(p)=(a_1-(k-1)b_1\cos p_1-(k-1)b_2\cos p_2-\dots+b_m\cos p_m)^{n/2}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+(a_k+b_1\cos p_1+b_2\cos p_2+\dots-(k-1)b_m\cos p_m)^{n/2}$$

$$+k(a_{k+1}+b_1\cos p_1+b_2\cos p_2+\dots+b_m\cos p_m)^{n/2},$$

where

$$a_1=(k-1)^2\|u_1\|^2+\|u_2\|^2+\dots+\|u_k\|^2,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_k=\|u_1\|^2+\|u_2\|^2+\dots+(k-1)^2\|u_k\|^2,$$

$$a_{k+1}=\sum_{i=1}^{k}\|u_i\|^2 \quad (=B_2),$$

and

$$b_1=2\|u_1\|\|u_2\|,\ b_2=2\|u_1\|\|u_3\|,\ \dots,\ b_{k-1}=2\|u_1\|\|u_k\|,$$

$$b_k=2\|u_2\|\|u_3\|,\ \ldots\ ,b_m=2\|u_{k-1}\|\|u_k\|.$$

Denote

$$c_1=a_1-(k-1)b_1\cos p_1-(k-1)b_2\cos p_2-\ldots+b_m\cos p_m,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$c_k=a_k+b_1\cos p_1+b_2\cos p_2+\ldots-(k-1)b_m\cos p_m,$$

$$c_{k+1}=a_{k+1}+b_1\cos p_1+b_2\cos p_2+\ldots+b_m\cos p_m.$$

Then

$$R_n(p)=c_1^{n/2}+c_2^{n/2}+\ldots+c_k^{n/2}+kc_{k+1}^{n/2}.$$

We optimize $R_n(p)$. By differentiating R_n we get:

$$\frac{\partial R_n}{\partial p_1}=\frac{\partial R_n}{\partial c_1}\cdot\frac{\partial c_1}{\partial p_1}+\ldots+\frac{\partial R_n}{\partial c_k}\cdot\frac{\partial c_k}{\partial p_1}+\frac{\partial R_n}{\partial c_{k+1}}\cdot\frac{\partial c_{k+1}}{\partial p_1}$$

or

$$\frac{\partial R}{\partial p_1}=\frac{n}{2}Q_1(c)b_1\sin p_1,\ Q_1=Q_1(c),\ Q_1:\ \mathbb{R}^{k+1}\to\mathbb{R},$$

$$c=(c_1,c_2,\ldots,c_k,c_{k+1}),$$

$$Q_1(c)=(k-1)c_1^{(n-2)/2}+(k-1)c_2^{(n-2)/2}+\ldots-c_k^{(n-2)/2}-kc_{k+1}^{(n-2)/2}.$$

In general,

$$\frac{\partial R}{\partial p_i}=\frac{n}{2}Q_i(c)b_i\sin p_i,\ Q_i:\ \mathbb{R}^{k+1}\to\mathbb{R},$$

for $p_i\neq0$: $Q_i(c)(\neq0)<\infty$, $Q_i(c)<0$ for $n>2$, $Q_i(c)>0$ for $0<n<2$, $(i=1,\ldots,m)$.

Denote

$$d_i = \frac{n}{2} Q_i(c) b_i \qquad (i=1,2,\dots,m).$$

Then

$$\frac{\partial R_n}{\partial p_i} = d_i \sin p_i \qquad (i=1,2,\dots,m).$$

By differentiating $\frac{\partial R_n}{\partial p_1}$ with respect to p_1 we get:

$$\frac{\partial^2 R_n}{\partial p_1^2} = \frac{\partial}{\partial p_1}(d_1 \sin p_1) = \frac{\partial d_1}{\partial p_1} \sin p_1 + d_1 \cos p_1 =$$

$$= \left[\frac{\partial d_1}{\partial c_1} \cdot \frac{\partial c_1}{\partial p_1} + \dots + \frac{\partial d_1}{\partial c_k} \cdot \frac{\partial c_k}{\partial p_1} + \frac{\partial d_1}{\partial c_{k+1}} \cdot \frac{\partial c_{k+1}}{\partial p_1} \right] \sin p_1 +$$

$$+ d_1 \cos p_1$$

or

$$\frac{\partial^2 R_n}{\partial p_1^2} = \frac{n(n-2)}{4} \phi_1(c) b_1^2 \sin^2 p_1 + d_1 \cos p_1,$$

$$\phi_1 = \phi_1(c), \ \phi_1 : \mathbb{R}^{k+1} \to \mathbb{R}, \ c = (c_1, c_2, \dots, c_k, c_{k+1}),$$

$$\phi_1(c) = (k-1)^2 c_1^{(n-4)/2} + (k-1)^2 c_2^{(n-4)/2} + \dots + c_k^{(n-4)/2} + k c_{k+1}^{(n-4)/2}.$$

In general,

$$\frac{\partial^2 R_n}{\partial p_i^2} = \frac{n(n-2)}{4} \phi_i(c) b_i^2 \sin^2 p_i + d_i \cos p_i, \ \phi_i : \mathbb{R}^{k+1} \to \mathbb{R},$$

for $p_i = 0$: $\phi_i(c) < \infty$, $\phi_i > 0$ $(i=1,\dots,m)$.

By continuing differentiation we get:

$$\frac{\partial^2 R_n}{\partial p_i \partial p_j} = \frac{\partial^2 R_n}{\partial p_j \partial p_i} = \frac{n(n-2)}{4}\,\psi_{ij}(c) b_i b_j \sin p_i \sin p_j,$$

ψ_{ij}: $\mathbb{R}^{k+1} \to \mathbb{R}$, $\psi_{ij}(c) < \infty$ for $p_i = p_j = 0$ $(i \neq j := 1,2,\ldots,m)$.

Then from

$$\frac{\partial R_n}{\partial p_i} = 0 \quad (i=1,2,\ldots,m)$$

we get critical values

$p_i = 2K_i\pi$, and $p_i = (2K_i+1)\pi$ $(i=1,2,\ldots,m)$, $K_i = 0, \pm 1, \pm 2, \ldots,$

for any $u_i \in V$ $(i=1,2,\ldots,m)$.

At these critical values we have:

$$c_1 = ((k-1)\|u_1\| + \|u_2\| + \ldots + \|u_k\|)^2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$c_k = (\|u_1\| + \|u_2\| + \ldots + \|u_{k-1}\| - (k-1)\|u_k\|)^2,$$

$$c_{k+1} = \left[\sum_{i=1}^{k} \|u_i\|\right]^2,$$

$$\frac{\partial^2 R_n}{\partial p_i^2} = d_i \quad (i=1,2,\ldots,m),$$

and

$$\frac{\partial^2 R_n}{\partial p_i \partial p_j} = \frac{\partial^2 R_n}{\partial p_j \partial p_i} = 0 \quad (i \neq j := 1,2,\ldots,m).$$

To optimize $R_n(p)$ at these critical values we consider the determinants

$$D_1=D_1(p)=\frac{\partial^2 R_n(p)}{\partial p_1^2}\ ,\quad D_2=D_2(p)=\begin{vmatrix} \dfrac{\partial^2 R_n(p)}{\partial p_1^2} & \dfrac{\partial^2 R_n(p)}{\partial p_1 \partial p_2} \\ \dfrac{\partial^2 R_n(p)}{\partial p_2 \partial p_1} & \dfrac{\partial^2 R_n(p)}{\partial p_2^2} \end{vmatrix},$$

$$\dots, D_m=D_m(p)=\begin{vmatrix} \dfrac{\partial^2 R_n(p)}{\partial p_1^2} & \dfrac{\partial^2 R_n(p)}{\partial p_1 \partial p_2} & \dots & \dfrac{\partial^2 R_n(p)}{\partial p_1 \partial p_m} \\ \dfrac{\partial^2 R_n(p)}{\partial p_2 \partial p_1} & \dfrac{\partial^2 R_n(p)}{\partial p_2^2} & \dots & \dfrac{\partial^2 R_n(p)}{\partial p_2 \partial p_m} \\ \dots & \dots & \dots & \dots \\ \dfrac{\partial^2 R_n(p)}{\partial p_m \partial p_1} & \dfrac{\partial^2 R_n(p)}{\partial p_m \partial p_2} & \dots & \dfrac{\partial^2 R_n(p)}{\partial p_m^2} \end{vmatrix},$$

where $p=(p_1,p_2,\dots,p_m)$, $p=2K\pi$, and $p=(2K+1)\pi$, $K=0,\pm1,\pm2,\dots$.

Therefore

$$D_1=d_1,\ D_2=d_1 d_2,\ \dots\ .\ D_m=\prod_{i=1}^{m} d_i .$$

We observe that $d_i<0$ $(i=1,2,\dots,m)$ for $n>2$, and therefore $(-1)^i D_i>0$ $(i=1,2,\dots,m)$ for $n>2$. Then

$$\max_{p\in\mathbb{R}} R_n(p)=R_n(2K\pi)=R_n((2K+1)\pi)=A_n,\quad n>2,$$

which yields (1). Similarly $d_i>0$ $(i=1,2,\dots,m)$ for $0<n<2$, and therefore $D_i>0$ $(i=1,2,\dots,m)$ for $0<n<2$. Then

$$\min_{p\in R} R_n(p)=A_n, \quad 0<n<2$$

which yields (2).

Moreover Clarkson's inequalities ([1; 36-37], [11; 224--225]):

$$|x_1-x_2|^n+(x_1+x_2)^n\le 2^{n-1}(x_1^n+x_2^n), \quad n>2,$$

and

$$|x_1-x_2|^n+(x_1+x_2)^n\ge 2^{n-1}(x_1^n+x_2^n), \quad 1<n<2, \quad (x_1, x_2\ge 0),$$

are equivalent to

$$|x_1-y|^n+y^n\le \frac{x_1^n+x_2^n}{2}, \quad n>2,$$

and

$$|x_1-y|^n+y^n\ge \frac{x_1^n+x_2^n}{2}, \quad 1<n<2,$$

where $y=\frac{x_1+x_2}{2}$, $(x_1, x_2\ge 0)$. Similarly we prove that

$$|x_1-z|^n+z^n\le \frac{\sum_{i=1}^{k} x_i^n}{k}, \quad n>2,$$

and

$$|x_1-z|^n+z^n\ge \frac{\sum_{i=1}^{k} x_i^n}{k}, \quad 1<n<2,$$

where $z=\frac{1}{k}\sum_{i=1}^{k} x_i$, $(x_i\ge 0,\ i=1,2,\ldots,k)$, or equivalently

$$|(k-1)x_1-x_2-\ldots-x_k|^n+\left[\sum_{i=1}^{k} x_i\right]^n\le k^{n-1}\left[\sum_{i=1}^{k} x_i^n\right], \quad n>2,$$

and

$$|(k-1)x_1-x_2-\dots-x_k|^n+\left(\sum_{i=1}^{k}x_i\right)^n \ge k^{n-1}\left(\sum_{i=1}^{k}x_i^n\right),\quad 1<n<2,$$

$$(x_i\ge 0,\quad i=1,2,\dots,k),$$

If we replace $x_i=\|u_i\|$, $(i=1,2,\dots,k)$, then

$$A_n=\left[|(k-1)x_1-x_2-\dots-x_k|^n+\left(\sum_{i=1}^{k}x_i\right)^n\right]+\dots+$$

$$+\left[|x_1+x_2+\dots+x_{k-1}-(k-1)x_k|^n+\left(\sum_{i=1}^{k}x_i\right)^n\right]\le k^nB_n,\quad n>2,$$

and

$$A_n\ge k^nB_n,\quad 1<n<2.$$

These inequalities combined with (1), and (2) yield (3), and (4), respectively.

From Hölder's inequality

$$\left(\sum_{i=1}^{k}x_i^{p_o}\right)^{q_o}\left(\sum_{i=1}^{k}y_i^{q_o}\right)^{p_o} \ge \left(\sum_{i=1}^{k}x_iy_i\right)^{p_oq_o}$$

for $x_i,y_i\ge 0$, $(i=1,2,\dots,k)$, $p_0,q_0\in\mathbb{R}^+$, $1/p_0+1/q_0=1$, $k\in\mathbb{N}$.

If we set:

$$p_0=n,\quad q_0=\frac{n}{n-1},\quad (n>1),\quad y_i=1,\quad (i=1,2,\dots,k)$$

we get:

$$\frac{\sum_{i=1}^{k}x_i^n}{k}\ge\left[\frac{\sum_{i=1}^{k}x_i}{k}\right]^n$$

for every $n\in\mathbb{R}^+$, $n>1$, $k\in\mathbb{N}$, $x_i\ge 0$, $(i=1,2,\dots,k)$.

Besides

$$\frac{\sum_{i=1}^{k} x_i}{k} \geq \sqrt[k]{\prod_{i=1}^{k} x_i}$$

for every $k \in \mathbb{N}$, $x_i \geq 0$, $(i=1,2,\ldots,k)$. Therefore

$$\sum_{i=1}^{k} x_i^n \geq k \prod_{i=1}^{k} x_i^{n/k}$$

for every $n \in \mathbb{R}^+$, $n>1$, $k \in \mathbb{N}$, $x_i \geq 0$, $(i=1,2,\ldots,k)$. Then $B_n \geq kC_n$ for every $n \in \mathbb{R}^+$, $n>1$, $k \in \mathbb{N}$. Therefore (4) yields (5).

Moreover

$$\sqrt[k]{\prod_{i=1}^{k} z_i} = \min_{i \in I} z_i$$

for every $k \in \mathbb{N}$, $I=\{1,2,\ldots,k\}$, $z_i \geq 0$, $i \in I$.

If we set:

$$z_i = \|u_i\|^n \quad (=x_i^n), \quad i \in I$$

we get

$$C_n \geq D_n .$$

Therefore (5) implies (6).

Conditions are sufficient.

If we suppose conditions (1), and (3), then by the continuity of the function $n \mapsto \|\cdot\|^n$ the following conditions hold

$$R_2 \leq A_2, \quad R_2 \leq k^2 B_2$$

where $k \in \mathbb{N}$ and $A_2 = k^3$, $B_2 = k$, $(u_i \in V,\ \|u_i\|=1,\ i \in I=\{1,2,\ldots,k\})$.

From the sufficient condition of M. Day it can be proved the following "condition":

The necessary and sufficient condition for a norm defined over a vector space V to spring from an inner product is that $R_2 \le k^3$, $k \in \mathbb{N}$, $(u_i \in V,\ \|u_i\|=1,\ i \in I)$.

But this "condition" holds.

Hence the norm is induced by an inner product $\langle\cdot,\cdot\rangle$ ([2]-[20]).

Similarly if we assume conditions (2), (4), (5), and (6), then by the continuity of the above function the following conditions hold

$$R_2 \ge A_2,\quad R_2 \ge k^2 B_2,\quad R_2 \ge k^3 C_2,\quad R_2 \ge k^4 D_2,$$

where $k \in \mathbb{N}$ and $C_2 = D_2 = 1$ in V $(\|u_i\|=1,\ i \in I)$. The same statement as above "condition" except here $R_2 \ge k^3$. But this "new condition" holds.

Hence the norm is induced by an inner product $\langle\cdot,\cdot\rangle$, and the proof of this Criterion 1 is complete.

Criterion 2. Given R_n as in Criterion 1, then necessary and sufficient conditions that the norm defined over a real vector space V $(u_i \in V\colon \|u_i\|=1,\ i \in I=\{1,2,\dots,k\},\ k \in \mathbb{N})$ be induced by an inner product are that:

(i) $R_n \le k^{n+1}$, $n>2$,

(ii) $R_n \ge k^{n+1}$, $1<n<2$.

Proof. This criterion 2 follows criterion 1. In particular, (i) implies from criterion 1 (1), and (3), and (ii) from criterion 1 (2), (4), (5), and (6), where $A_n = k^{n+1}$, $B_n = k$, $C_n = D_n = 1$ in V $(u_i \in V\colon \|u_i\|=1,\ i \in \mathbb{N})$. Therefore the proof of this criterion 2 is complete.

Remark. Define

$$R_{1^+} = \lim_{n \to 1^+} R_n,\quad R_{2^-} = \lim_{n \to 2^-} R_n,\quad R_{2^+} = \lim_{n \to 2^+} R_n \text{ in } V\ (\|u_i\|=1,\ i \in I).$$

Then

$$R_{1_+} \geq k^2 \ , \ R_{2_-} \geq k^3 \ , \ R_{2_+} \geq k^3 .$$

REFERENCES

[1] R.A. Adams, *Sobolev Spaces*, Acad. Press, 1975.

[2] D. Amir, *Characterizations of Inner Product Spaces*, *OT 20*: Operator Theory: *Advances and Applications*, Vol. 20, Birkhäuser Verlag, Basel/Boston/Stuttgart, 1986.

[3] D. Amir and C. Franchetti, *A note on characterizations of Hilbert spaces*, Bol. Unione Mat. Ital. (6) 2A, 1983, 305-309.

[4] M. Baronti, *Su alcuni parametri degli spazi normati*, Bol. Unione Mat. Ital. B (5) 18, 1981, 1065-1085.

[5] C. Benitez and M. Del Rio, *Characterization of inner product spaces through rectangle and square inequalities*, Revue Roum. Math. Pures Appl. 29, 1984, 543-546.

[6] G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. Jour., 1, 1935, 169-172.

[7] M.M. Day, *Some characterizations of inner product spaces*, Trans. Amer. Math. Soc., 62, 1947, 320-337.

[8] M.M. Day, *Normed Linear Spaces*, 3rd edition, Springer, New York, 1973.

[9] M.M. Day, *New characterizations of inner product spaces*, by T.M. Rassias, Bull. Sci. Math. (2), 108, 1984, 97-98.

[10] R.W. Freese, C.R. Diminnie and E.Z. Andalafte, *A study of generalized orthogonality relations in normed linear spaces*, Math. Nachrichten, Vol. 122, 1985, 197-204.

[11] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, 1965.

[12] R.C. James, *Inner products in normed linear spaces*, Bull. Amer. Math. Soc., Vol. 53, 1947, 559-566.

[13] R.C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc., Vol. 61, 1947, 265-292.

[14] P.Jordan and J.Von Neumann, *On inner products in linear metric speces*", Ann. of Math. (2), 36, 1935, 719-723.

[15] P.L.Papini, *Approximation and norm dervatives in real normed spaces*, Resultate der Math. Vol. 5, 1982, 81-94.

[16] P.L.Papini, *Minimal and closest points nonexpansive and quasi-non-expansive retractions in real Banach spaces. Convexity and its Applications*, Birkhäuser, 148-253, 1983.

[17] P.L.Papini, *Inner products and norm derivatives*, Jour. Math. Anal. Appl., Vol. 91, 1983, 592-598.

[18] T.M.Rassias, *New characterizations of inner product spaces*, Bull. Sci. Math. (2) 108, 1984, 95-99.

[19] I.J.Schoenberg, *A remark on M.M.Day's characterization of inner product speces and a conjecture of L.M. Blumenthal*", Proc. Amer. Math. Soc. Vol. 3, 1952, 961-964.

[20] W.L.Ze'ng, *An extension of the parallelogram characterization of inner product spaces*, Jour. Math. Res. Exposition, Vol. 2, 1982, 23-25.

Received JULY 02, 1986.
Revised DECEMBER 10, 1986.