Доклады Болгарской академии наук
Comptes rendus de l'Académie bulgare des Sciences
Tome 38, №° 1, 1985

MATHEMATIQUES
Équations différentielles

EXTENDED BITSADZE-LAVRENT'EV PROBLEM WITH TWO PARABOLIC LINES OF DEGENERACY AND TWO ELLIPTIC ARCS IN EUCLIDEAN PLANE

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(Submitted by Academician L. Iliev on June 26, 1984)

As known, Lavrent'ev a. Bitsadze [1] established a uniqueness theorem for the mixed type equation: $\operatorname{sgn}(y).u_{xx}+u_{yy}=0$ with special boundary conditions. In this paper we establish a uniqueness theorem for the extended equation

$$(+) \qquad Lu=\operatorname{sgn}(y.(y-1)).u_{xx}+u_{yy}+r(x,y).u=f(x,y)$$

with new boundary conditions, the two parabolic lines of degeneracy: $y=0$, $y=1$, and two 'elliptic' arcs g_0 and g_0', such that: $x.dy-(y-1).dx\geq 0$ on g_0, and $x.dy-y.dx\geq 0$ on g_0'.

Consider equation (+) in the region $G\subset\mathbb{R}^2$ bounded by the curves: A piecewise smooth curve g_0 in the region G_1: $x>0$, $y>1$ and intersecting the line $y=1$ at the points $A(0,1)$, and $B(2,1)$, another smooth curve g_0' in the region G_1': $x>0$, $v<0$ and intersecting the line $y=0$ at the points $A'(0,0)$, and $B'(2,0)$, the characteristic curve g_1: $x=-y+1$ of (+) through $A(0,1)$ meeting the characteristic g_1': $y=x$ of (+) issued from $A(0,0)$ at the point $C(1/2, 1/2)$ in G_2: $x>0$, $0<y<1$, and the characteristic g_2: $y=x-1$ of (+) through $B(2,1)$ meeting the characteristic g_2': $y=-x+2$ of (+) issued from $B'(2,0)$ at the point $D(3/2, 1/2)$. Therefore, the boundary ∂G of G is given by

$$\partial G=(g_0)\cup(AC)\cup(CA')(g_0')\cup(B'D)\cup(DB).$$

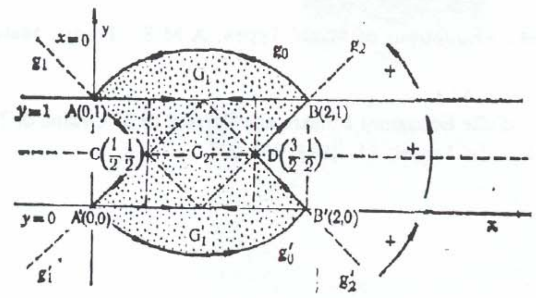

Fig. 1

We assume the following conditions

(1): $K = \operatorname{sgn}(y.(y-1))$ is a discontinuous function in $\bar{G}$ (=closure of $G := G \cup \partial G$), such that $K=1$ for $y<0$ and $y>1$, $=-1$ for $0<y<1$, $=0$ for $y=0$, $y=1$,

(2): $r = r(x, y) \in C^1(\bar{G})$, $f = f(x, y) \in C^\alpha(\bar{G})$,

(3): $r \leq 0$ on $(AC) \cup (CA')$, $2.r + x.r_x + (y-1).r_y \leq 0$ in $\bar{G}_1$,

$2.r + x.r_x + y.r_y \leq 0$ in $\bar{G}_1'$, $r + x.r_x \leq 0$ in $\bar{G}_2$,

$x.dy - (y-1).dx \geq 0$ on g_0 'star-likeness on g_0'

$x.dy - y.dx \geq 0$ on g_0' 'star-likeness on g_0'',

where

$$\bar{G}_1 = G_1 \cup \partial G_1 : \partial G_1 = (g_0) \cup (AB),$$

$$\bar{G}_2 = G_2 \cup \partial G_2 : \partial G_2 = (AC) \cup (CA') \cup (A'B') \cup (B'D) \cup (DB) \cup (BA),$$

$$\bar{G}_1' = C_1' \cup \partial G_1' : \partial G_1' = (g_0') \cup (B'A').$$

In addition, we assume boundary condition

(4) $$u = 0 \text{ on } (g_0') \cup (B'D) \cup (DB) \cup (g_0)$$

The Extended Bitsadze-Lavrent'ev Problem consists in finding a solution $u = u(x, y)$ of (+) satisfying (4).

Theorem. Assume domain $G \subset \mathbb{R}^2$ described above. If we assume conditions (1—4), then the extended Bitsadze-Lavrent'ev Problem has at most one quasi-regular solution u (i. e. if there exists a solution $u = u(x, y)$ for the extended Bitsadze-Lavrent'ev Problem and if Green's theorem holds, then $u=0$ in G).

Proof. Assume u_1 and u_2 two solutions of (+) satisfying boundary condition (4). Denote $u = u_1 - u_2$. Then claim that

(5) $$u = 0 \text{ in } G.$$

The proof of (5) follows if we apply the energy integral method in each region G_1, G_2, G_1' separately (because of the discontinuity of K in G, Green's theorem cannot be applied directly in the whole region G) and then the maximum principle for elliptic and hyperbolic equations [2].

Note the integral expressions

(6) $$2.(lu, Lu)_{G_i} = 2.\iint_{G_i} lu.Lu.dxdy \quad (i=1, 2),$$

(7) $$2.(lu, Lu)_{G_1'} = 2.\iint_{G_1'} lu.Lu.dxdy,$$

where

(8) $lu = x.u_x + (y-1).u_y$ in $\bar{G}_1$, $= x.u_x$ in $\bar{G}_2$, $= x.u_x + y.u_y$ in $\bar{G}_1'$.

By applying Green's theorem in (6) and (7), using (8) and if $v = (v_1, v_2)$ is the outer normal vector on ∂G, then

(9) $$2.(lu, Lu)_{G_1} = -\iint_{G_1} [(2.r + x.r_x + (y-1).r_y).u^2].dxdy + \int_{g_0} [N_1^2.(x.dy - (y-1).dx).(v_1^2 + v_2^2)].ds + 2.\int_{AB} [(x.v_2).u_x u_y].ds = I_1 + I_2 + I_3$$

(N_1=normalizing factor: $u_x = N_1 . v_1$, $u_y = N_1 . v_2$ on g_0, because $u=0$ on g_0 by (4)),

(10)
$$2.(lu,\ Lu)_{G_1'} = -\iint_{G_1'} [(2.r + x.r_x + y.r_y).u^2].dxdy$$
$$\int_{B'A'} [2.(x.v_2).u_x u_y].ds$$
$$+\int_{g_0'} [(N_1')^2.(x.dy - y.dx).(v_1^2 + v_2^2)].ds$$
$$= I_1' + I_2' + I_3'$$

(N_1'=normalizing factor: $u_x = N_1' . v_1$, $u_y = N_1' . v_2$ on g_0', because $u=0$ on g_0' by (4)),

(11)
$$2.(lu,\ Lu)_{G_2} = -\iint_{G_2} [(r + x.r_x).u^2].dxdy + \iint_{G_2} (u_x^2 + u_y^2).dxdy$$
$$+ \int_{(AC)\cup(CA')} \{x.[(-v_1).u_x^2 + (-v_1).u_y^2 + 2.v_2.u_x u_y]\}.ds$$
$$+ \int_{(B'D)\cup DB)} [N_2^2.(x.v_1).(-v_1^2 + v_2^2)].ds$$
$$+2.\int_{(BA)} [(x.v_2).u_x u_y].ds$$
$$+\int_{A'B'} [2.(x.v_2).u_x u_y].ds$$
$$+ \int_{(AC)\cup(CA')} [r.(x.v_1).u^2].ds = \sum_{i=1}^{7} J_i$$

(N_2=normalizing factor: $u_x = N_2 . v_1$, $u_y = N_2 . v_2$ on $(B'D)\cup(D'B)$, because $u=0$ on $(B'D)\cup(D'B)$ by (4)).

It is clear that

$I_1 \geq 0$, because: $2.r + x.r_x + (y-1).r_y \leq 0$ in $\bar{G}_1$ by (3),

$I_2 \geq 0$, because: $x.dy - (y-1).dx \geq 0$ on g_0 by (3),

$I_3 + J_5 = 0$, because: v_2 (on AB) $= -v_2$ (on BA),

$I_1' \geq 0$, because: $2.r + x.r_x + y.r_y \leq 0$ in $\bar{G}_1'$ by (3).

$I_2' + J_6 = 0$, because: v_2 (on $A'B'$) $= -v_2$ (on $B'A'$),

$I_3' \geq 0$, because: $x.dy - y.dx \geq 0$ on g_0' by (3),

$J_1 \geq 0$, because: $r + x.r_x \leq 0$ in $\bar{G}_2$ by (3),

$J_2 \geq 0$, obvious,

$J_3 \geq 0$, because: $v_1 < 0$ on $(AC)\cup(CA')$, and $\begin{vmatrix} -v_1 & v_2 \\ v_2 & -v_1 \end{vmatrix} = v_1^2 - v_2^2 = 0$

on $(AC)\cup(CA')$, because AC, and CA' are characteristic segments of (+),

$J_4 = 0$, because: $-v_1^2 + v_2^2 = 0$ on $(B'D)\cup(DB)$, because $B'D$, and DB are characteristic segments of (+),

$J_7 \geq 0$, because: $r \leq 0$ on $(AC)\cup(CA')$ by (3), and $v_1 < 0$ on $(AC)\cup(CA')$.

Therefore, by adding (9), (10), and (11), and by taking into account the above results on the integrals I_i, I_i' $(i=1, 2, 3)$, $J_k (k=1, 2, \ldots, 7)$, we conclude that $u=0$ on ∂G.

Thus, by employing the maximum principle, we prove that $u=0$ in G (that is, (5) holds) and this completes the proof of the theorem.

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