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ON THE TRICOMI PROBLEM WITH TWO PARABOLIC LINES OF DEGENERACY

BY

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Abstract. Tricomi and, to the best of my knowledge, all of the recent research workers have worked the Tricomi problem only with one parabolic line of degeneracy. The problem with more than one parabolic lines of degeneracy becomes more complicated. In this paper we present one of these cases with two parabolic lines of degeneracy and establish a certain uniqueness theorem by applying the classical energy integral method in $\mathbf{R}^2$. Finally we mention certain remarks connecting this work with Frankl's condition, or Protter's condition, and the "non-characteristic" problem.

Consider equation.

$$Lu = y \cdot (y-1) \cdot u_{xx} + u_{yy} = 0, \tag{$*$}$$

where $(*)$ is elliptic for $y > 1$, and $y < 0$, hyperbolic for $0 < y < 1$, and parabolic for $y = 0$, and $y = 1$. Assume D is a simply connected domain of $\boldsymbol{R}^2$, bounded by an "elliptic" simple curve Γ_0 which emanates from the points $A_1(0, 1)$ and $B_1(1, 1)$ and lies in the upper plane $y > 1$, by an "elliptic" simple curve Γ_0' which emanates from the points $A_2(0, 0)$ and $B_2(1, 0)$ and lies in the lower plane $y < 0$, and by the following "real" characteristic surves of $(*)$ which lie in the region $G_2 = \{(x, y) : x > 0,\ 0 < y < 1\} \subset \boldsymbol{R}^2$.

$$\Gamma_1(= A_1P_1) : x = -\int_1^y \sqrt{y \cdot (1-y)} \cdot dy$$
$$= \frac{(1-2y) \cdot \sqrt{y - y^2}}{4} - \frac{1}{8} \cdot \sin^{-1}(2y-1) + \frac{\pi}{16}$$

$$\Gamma_1'(= A_2P_1) : x = \int_0^y \sqrt{y(1-y)} \cdot dy$$
$$= \frac{(2y-1) \cdot \sqrt{y - y^2}}{4} + \frac{1}{8} \sin^{-1}(2y-1) \qquad - \frac{3\pi}{16}$$

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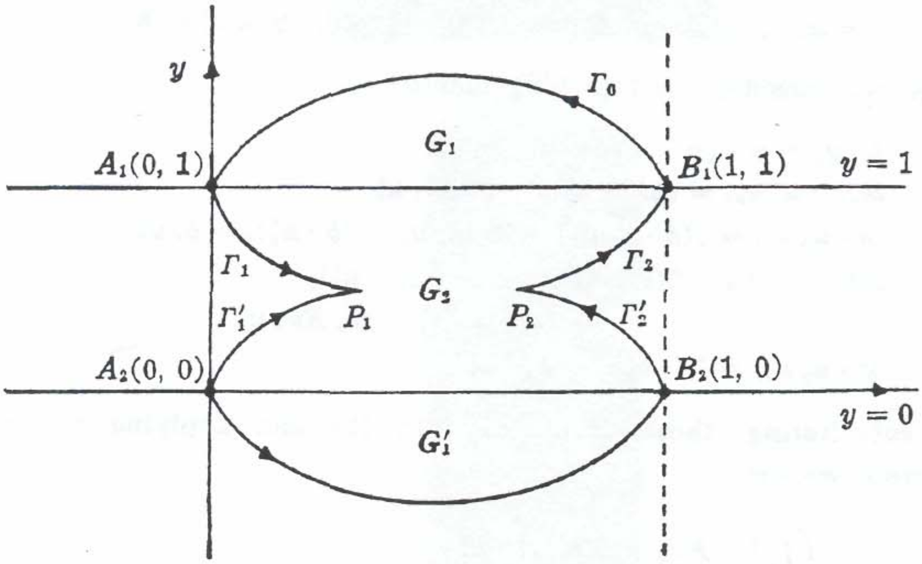
$$\Gamma_2(=B_1P_2) : x = 1 + \int_1^y \sqrt{y\cdot(1-y)}\cdot dy$$

$$= \frac{(2y-1)\cdot\sqrt{y-y^2}}{4} + \frac{1}{8}\cdot\sin^{-1}(2y-1) + 1 - \frac{\pi}{16}$$

$$\Gamma_2'(=B_2P_2) : x = 1 - \int_0^y \sqrt{y\cdot(1-y)}\cdot dy$$

$$= \frac{(1-2y)\cdot\sqrt{y-y^2}}{4} - \frac{1}{8}\cdot\sin^{-1}(2y-1) + 1 + \frac{3\pi}{16}$$



The Tricomi problem in R^2

If we consider equation (*) in $D \subset R^2$, then the Tricomi problem consists in finding a function $u = u(x, y)$ which satisfies equation (*) and the following boundary condition:

$$(**) \qquad u/_{(\Gamma_0 \cup \Gamma_1) \cup \vee (\Gamma_0' \cup \Gamma_1')} = 0$$

A UNIQUENES THEOREM. *Assume the above-mentioned domain* $D \subset R^2$, *and the star-likedness conditions*:

(S1): $\quad x\cdot dy - (y-1)\cdot dx \geq 0 \quad$ *on* Γ_0

(S2): $\quad x\cdot dy - y\cdot dx \geq 0 \quad$ *on* Γ_0'

Then the Tricomi problem (*), *and* (**) *has at most one quasi-regular solution* u.

Proof. We apply the classical energy integral method and use (**), and (S_i) $(i=1, 2)$. At first, we *investigate* the *integral* expression: ([1]−[5])

$$J = 2\iint_D (b\cdot u_x + c\cdot u_y)\cdot Lu\cdot dx\,dy, \tag{1}$$

where b, and c are *chosen* as follows:

$$b = x,\quad c = y-1 \quad \text{in} \quad G_1 = \{(x, y): x>0,\ y>1\}\subset \mathbf{R}^2, \tag{2}$$

$$b = x,\quad c = 0 \quad \text{in} \quad G_2 \quad \text{and} \tag{3}$$

$$b = x,\quad c = y \quad \text{in} \quad G_1' = \{(x, y): x>0,\ y<0\}\subset \mathbf{R}^2. \tag{4}$$

Then we consider the following identities:

$$\begin{aligned}
&K(y) = y\cdot(y-1)\\
&2bK\cdot u_x u_{xx} = (bK\cdot u_x^2)_x - b_x K\cdot u_x^2,\\
&2b\cdot u_x u_{yy} = 2(b\cdot u_x u_y) - 2b_y u_x u_y - (b\cdot u_y^2)_x + b_x u_y^2,\\
&2cK\cdot u_y u_{xx} = 2(cK\cdot u_x u_y)_x - (cK\cdot u_x^2)_y\\
&\qquad\qquad + (cK)_y\cdot u_x^2 - 2c_x K u_x u_y,\\
&2c\cdot u_y u_{yy} = (c\cdot u_y^2)_y - c_y\cdot u_y^2
\end{aligned} \tag{5}$$

By substituting these identities into (1) and applying Green's theorem we get:

$$\begin{aligned}
0 = &\iint_D [-Kb_x + (cK)_y]\cdot u_x^2\\
&- 2(Kc_x + b_y)\cdot u_x u_y + (b_x - c_y)\cdot u_y^2\, dx\,dy\\
&+ \oint_{\partial D} [(-2b\cdot u_x u_y + K\cdot c\cdot u_x^2 - c\cdot u_y^2)\cdot dx\\
&+ (Kb\cdot u_x^2 - b\cdot u_y^2 + 2Kc\cdot u_x u_y)\cdot dy]
\end{aligned} \tag{6}$$

Therefore, (**), (S_i) $(i=1, 2)$, (2)−(4), (5), (6), and the maximum principle ([1]−[5]) imply $u=0$ in D. In fact, if $K(y) = y(y-1)$ and if

$$\begin{aligned}
\tilde{Q}(u_x, u_y) = &[-Kb_x + (cK)_y]\cdot u_x^2\\
&- 2(Kc_x + b_y)\cdot u_x u_y + (b_x - c_y)\cdot u_y^2
\end{aligned}$$

in D, then

$$\iint_{G_1} \tilde{Q}(u_x, u_y)\cdot dx\,dy > 0,$$

because

$$\tilde{Q}(u_x, u_y) = [(y-1)(2y-1)] \cdot u_x^2$$

in G_1, and $(y-1)\ (2y-1) > 0$ in G_1.
Similarly,

$$\iint_{G_1'} \tilde{Q}(u_x, u_y) \cdot dx\, dy > 0,$$

because

$$\tilde{Q}(u_x, u_y) = [y \cdot (2y-1)] \cdot u_x^2$$

in G_1', and $y \cdot (2y-1) > 0$ in G_1'. Finally,

$$\iint_{G_2} \tilde{Q}(u_x, u_y) \cdot dx\, dy > 0,$$

because

$$\tilde{Q}(u_x, u_y) = [y(1-y)] \cdot u_x^2 + u_y^2$$

in G_2, and $y \cdot (1-y) > 0$ in G_2. On the other hand, if $v = (v_1, v_2)$ is an outer normal vector on the boundary ∂D of D, and if

$$Q(u_x, u_y) = (Kb \cdot v_1 - Kc \cdot v_2) \cdot u_x^2 + (-b \cdot v_1 + c \cdot v_2) \cdot u_y^2 + 2(Kc \cdot v_1 + b \cdot v_2) \cdot u_x u_y$$

on the boundary $\partial D = (\Gamma_0 \cup \Gamma_1 \cup \Gamma_2) \cup (\Gamma_0' \cup \Gamma_1' \cup \Gamma_2')$, then

$$\int_{\Gamma_0 \cup \Gamma_0'} Q(u_x, u_y) \cdot ds \geq 0,$$

because

$$Q(u_x, u_y) = N^2 \cdot (b \cdot v_1 + c \cdot v_2)(K \cdot v_1^2 + v_2^2)$$

on $\Gamma_0 \subset \Gamma_0'$, $b \cdot v_1 + c \cdot v_2 \geq 0$ on $\Gamma_0 \cup \Gamma_0'$ (by (S_i), $i = 1, 2$),

$$K = y \cdot (y-1) > 0 \qquad \text{on} \quad \Gamma_0 \cup \Gamma_0',$$

and

$$u_x = N \cdot v_1, \qquad u_y = N \cdot v_2$$

(by (**)): N = normalizing factor. Similarly,

$$\int_{\Gamma_2 \cup \Gamma_2'} Q(u_x, u_y) \cdot ds = 0$$

because

$$Q(u_x, u_y) = N^2 \cdot (x \cdot v_1) \cdot (K \cdot v_1^2 + v_2^2) = 0$$

on $\Gamma_2 \cup \Gamma_2'$, $K \cdot v_1^2 + v_2^2 = 0$ (by the geometrical structure of Γ_2 and Γ_2'), and $u_x = N \cdot v_1$, $u_y = N \cdot v_2$ (by (**)): N = normalizing factor. Finally

$$\int_{\Gamma_1 \cup \Gamma_1'} Q(u_x, u_y) \cdot ds > 0$$

because

$$Q(u_x, u_y) = x \cdot [(K \cdot v_1) \cdot u_x^2 + (-v_1) \cdot u_y^2 + (2 \cdot v_2) \cdot u_x u_y]$$

on $\Gamma_1 \cup \Gamma_1'$, $K = y \cdot (y-1) < 0$ on $\Gamma_1 \cup \Gamma_1'$, $x > 0$ on $\Gamma_1 \cup \Gamma_1'$, $v_1 < 0$ on $\Gamma_1 \cup \Gamma_1'$ and

$$\begin{vmatrix} x \cdot K \cdot v_1 & x \cdot v_2 \\ x \cdot v_2 & -x \cdot v_1 \end{vmatrix} = -x^2 \cdot (K \cdot v_1^2 + v_2^2) = 0 \qquad \text{on} \quad \Gamma_1 \cup \Gamma_1'$$

(That is, all the principles minors of the matrix of representation of Q are non-negative definite).

Remarks. If we apply the techniques of the above theorem then it is easy to work the case:

$$[*] \qquad Lu = y \cdot (y-1) \cdot M(y) \cdot u_{xx} + u_{yy} = 0,$$

where $M(y)$ is a continuously differentiable non-negative function of y. However, if we consider the case when Γ_2 and Γ_2' are any two noncharacteristic curves such that

$$K \cdot v_1^2 + v_2^2 > 0 \qquad \text{on} \quad \Gamma_2 \cup \Gamma_2'$$

we establish corresponding results without any difficulty. Finally, if we work the above theorem by applying Frankl's choice, or the more general one by Protter we don't need to assume conditions (S_i) $(i = 1, 2)$ or any other condition on $K(y)$ for the case of $(*)$, but we do need to assume additional conditions on $K(y)$ (i. e. Frankl's or Protter's) for [*].

References

1. F. I. Frank, *Problems for mixed subsonic and supersonic flows*, Izv. Akad. Nauk SSSR Ser. Mat. 9, 2 (1945), 121-143.

2. M. H. Protter, *Uniqueness theorems for the tricomi problem*, J. Rat. Mech. Analysis 2 (1953), 107-144.

3. J. M. Rassias, *Mixed type equations in fluid dynamics between 1923 and 1981*, Post-doctoral (Dozent) Dissertation), (In English and Greek), Athens, Greece.

4. M. M. Smirnov, ***Equations of mixed type*** (in English) 56, Transl. Math. Monographs, Vol. 51, A. M. S., Provid., R. I., 1978.

5. F. Tricomi, ***Sulle equazioni lineari alle derivate parziali di 2° ordine, di tipc misto***, Atti. Accad. Nazionale dei Lincei (5), Vol. 14 (1923), 133–247.

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