AN APPLICATION OF THE THEORY OF POSITIVE SYMMETRIC SYSTEMS TO A DEGENERATE MULTIDIMENSIONAL HYPERBOLIC EQUATION IN $\mathbb{R}^3$

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In this paper we generalize the results of Karatoprakliev (1973) and Popivanov (1975) for the equation

$$Lu \equiv K_1(z)u_{xx}+K_2(z)u_{yy}-u_{zz}+a(x, y, z)u_x$$

$$+b(x, y, z)u_y+c(x, y, z)u_z+\lambda(x, y, z)u=f(x, y, z) \in L_2(\bar{D}).$$

Both the above mentioned investigators have studied this equation for $K_1=K_2=K$, while here $K_i=K_i(z) \in C^2(\bar{D})$ such that $K_i(0)=0$ and $K_i(z)>0$, for $z>0$, $K_i'(0)>0$, $i=1, 2$. The equation is, therefore, a hyperbolic degenerate equation in $\mathbb{R}^3$. Also $a, b, c \in C^1(\bar{D})$, $\lambda \in C^0(\bar{D})$ and $f \in C^0(\bar{D})$. We study it applying the well-known theory of Positive-Symmetric Systems established by K. O. Friedrichs (1958).

We shall consider the hyperbolic degenerate equation

$$(*) \qquad Lu \equiv K_1(z)u_{xx}+K_2(z)u_{yy}-u_{zz}+a(x, y, z)u_x+b(x, y, z)u_y$$
$$+c(x, y, z)u_z+\lambda(x, y, z)u=f(x, y, z) \in L_2(\bar{D}),$$

$K_i=K_i(z) \in C^2(\bar{D})$ such that $K_i(0)=0$ and $K_i(z)>0$ for $z>0$, $K_i'(0)>0$ $(i=1, 2)$; $a, b, c \in C^1(\bar{D})$, $\lambda \in C^0(\bar{D})$ and $f \in C^0(\bar{D})$. D is a simply connected region in $\mathbb{R}^3$, bounded by the characteristic surfaces S_3 and S_4 defined below and the parabolic plane of degeneracy $z=0$. $\bar{D}$ is the closure of the domain D of the solution $u=u(x, y, z) \in C^2(\bar{D})$. In order to find the equations of the characteristic surfaces S_3 and S_4 see [9] and [10].

Equation (*) is obviously a hyperbolic degenerate equation in $\mathbb{R}^3$.

Since the solutions of elliptic and hyperbolic degenerate partial differential equations have many different properties, and since quite different types of data must be imposed to determine such solutions it would seem unnatural to attempt a unified treatment of these equations. Still such a unified treatment — up to a certain point is given in K. O. Friedrichs [2].

While most of the treatment of these equations naturally employ completely different tools, some of them employ variants of the same positive definite quadratic forms, the so-called "Energy Integrals" K. O. Friedrichs has shown that this tool can be adapted to a large class of differential equations which include the classical elliptic and hyperbolic equations of the second order. The main motivation for this approach was not the desire for a unified treatment of elliptic and hyperbolic equations, but the desire to handle equations of mixed type which are partly elliptic, partly hyperbolic, and partly parabolic.

Many problems of classical partial differential equations, such as Cauchy problems, mixed problems for hyperbolic equations, and certain boundary value problems for equations of mixed type, can be considered as special cases of the above theory of positive symmetric systems.

For systems of equations of this kind, K. O. Friedrichs [3] established a set of standard approaches in order to determine suitable assumptions for the boundary conditions. According to these conditions, we can assure uniqueness of a solution and existence of certain generalized solution.

Corresponding to his semi-admissible condition, there exists a so-called "weak solution", while corresponding to his admissible condition, there exists a so-called "strong solution". Chao-hao Gu [4], however, has found sufficient conditions for the existence of a differentiable solution of higher order.

An application of the theory of positive symmetric systems to a degenerate multidimensional hyperbolic equation in $\mathbf{R}^3$. Let us compute S_4, and then we get S_3 immediately. To do that we reduce the first order nonlinear partial differential equation of S_4 to a system of first order ordinary differential equations, as follows:

Let $\Phi=\Phi(x, y, z)=0$ be the non-parametric equation of S_4 such that, by definition, Φ must satisfy the characteristic partial differential equation of first order of (*); that is,

$$(1) \qquad F(x, y, z;\ p, q, r)\equiv K_1(z)p^2+K_2(z)q^2-r^2=0,$$

where $p=\Phi_x$, $q=\Phi_y$, $r=\Phi_z$. We choose as initial data: $x_0=0$, $y_0=0$, $z_0=0$, and therefore by the compatibility condition: $K_1(z_0)p_0^2+K_2(z_0)q_0^2-r_0^2=0$, where $K_i(z_0)=K_i(0)=0$ $(i=1, 2)$, we choose $p_0=\cos\theta$, $q_0=\sin\theta$, $r_0=0$, $\theta\in[0, 2\pi]$.

We now reduce (1) to the following equivalent system of first order ordinary differential equations, the so-called system of equations of the characteristic strip of the equation (*), as follows:

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{dz}{F_r}=\frac{d\Phi}{pF_p+qF_q+rF_r}=\frac{dp}{-F_x-F_\Phi q}=\frac{dq}{-F_y-F_\Phi q}=\frac{dr}{-F_z-F_\Phi r}=ds,$$

or

$$(2) \qquad \frac{dx}{2K_1 p}=\frac{dy}{2K_2 q}=\frac{dz}{-2r}=\frac{d\Phi}{2F}=\frac{dp}{0}=\frac{dq}{0}=\frac{dr}{-K_1' p^2-K_2' q^2}=ds.$$

Therefore by (2) and by integrating we find that

$$p=p_0=\cos\theta,\ q=q_0=\sin\theta,\ x=2\left[\int_0^z K_1(z)\,ds\right]\cos\theta,\ y=2\left[\int_0^z K_2(z)\,ds\right]\sin\theta.$$

$$(3) \qquad dr/ds=-K_1'(z)\cos^2\theta-K_2'(z)\sin^2\theta,\ r=-(1/2)(dz/ds).$$

(3) implies that

$$(4) \qquad d^2z/ds^2-2[K_1'(z)\cos^2\theta+K_2'(z)\sin^2\theta]=0.$$

We now change differentiation:

$$(5) \qquad d^2z/ds^2=-(ds/dz)^{-3}d^2s/dz^2.$$

Therefore, (4) and (5) imply that

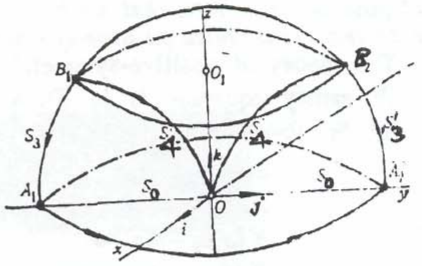
(6) $$d^2s/dz^2+2(ds/dz)^3[K_1'(z)\cos^2\theta+K_2'(z)\sin^2\theta]=0.$$

To solve the ordinary differential equation (6) with respect to ds/dz we work, as follows:

Let us set $ds/dz=\varphi(z)$ in (6). Then

$$\varphi'(z)+2\varphi^3(z)[K_1'(z)\cos^2\theta+K_2'(z)\sin^2\theta]=0,$$

which implies by integration that $\varphi=ds/dz=\pm[K_1(z)\cos^2\theta+K_2(z)\sin^2\theta]^{-1/2}/2$. Hence, we find that the characteristic conic surface S_4 with vertex at $(x_0, y_0, z_0)=(0, 0, 0)$ (origin of the coordinate (x, y, z) — Cartesian orthogonal system in R^3, is given parametrically by the following equations:



$$x=\cos\theta\left[\int_0^z\frac{K_1(t)dt}{[K_1(t)\cdot\cos^2\theta+K_2(t)\cdot\sin^2\theta]^{1/2}}\right],$$

$$y=\sin\theta\left[\int_0^z\frac{K_2(t)dt}{[K_1(t)\cos^2\theta+K_2(t)\sin^2\theta]^{1/2}}\right],$$

$$z=z,$$

where $\theta\in[0, 2\pi]$.

Similarly, we find that the characteristic conic surface S_3 is given parametrically by the following equations:

$$x=\cos\theta\left[1-\int_0^z\frac{K_1(t)dt}{[K_1(t)\cos^2\theta+K_2(t)\sin^2\theta]^{1/2}}\right],$$

$$y=\sin\theta\left[1-\int_0^z\frac{K_2(t)dt}{[K_1(t)\cos^2\theta+K_2(t)\sin^2\theta]^{1/2}}\right],$$

$$z=z,$$

where $\theta\in[0, 2\pi]$.

In D, we consider the linear, first-order system of partial differential equations:

(7) $$Mu=(A_1\partial_1u+A_2\partial_2u+A_3\partial_3u)+Bu=h, \text{ in } D\subset R^3,$$

where $h=(h_1, h_2, h_3, h_4)$; $u=(u_1, u_2, u_3, u_4)$; $\partial_1=\partial/\partial x$, $\partial_2=\partial/\partial y$, $\partial_3=\partial/\partial z$; A_i $(i=1, 2, 3)$ and B are (4×4)-matrices such that A_i $(i=1, 2, 3)$ are continuously differentiable and B is continuous in $\bar{D}$.

We also use the matrix $k=B-(1/2)\Sigma_{i=1}^3\partial_iA_i$.

In the present work we consider the formulation of several boundary value problems for equations of mixed type in a bounded multidimensional domain $D\subset R^3$. On the other hand, without imposing any other restrictions on the coefficients of (*), we derive a-priori estimates for general boundary value problems. Then by applying these estimates we study weak, and strong solutions. Finally, we note that the theory of positive-symmetric systems is applied in the search for regularly formulated boundary value problems.

D e f i n i t i o n [2—4]. *If the* $A_i\,(i=1, 2, 3)$ *are symmetric and* k^t *(the transpose of* k*) is such that* $k+k^t$ *is a positive-definite matrix in* $\bar{D}$, *then the system* (7) *is called positive-symmetric.*

The theory of positive-symmetric systems. Let us assume $u=u(x, y, z)\in C^2(\bar{D})$ satisfy equation (*) in $D\subset R^3$, then the auxiliary functions: $u_0=u$, $u_1=\partial_1 u$, $u_2=\partial_2 u$, $u_3=\partial_3 u$, satisfy the following system in $\bar{D}$:

$$
(8)\qquad
\begin{aligned}
&u_3-\partial_3 u_0=0,\\
&K_1(\partial_1 u_1)+K_2(\partial_2 u_2)-\partial_3 u_3+au_1+bu_2+cu_3+\lambda u=f,\\
&K_1(\partial_1 u_3-\partial_3 u_1)=0,\\
&K_2(\partial_2 u_3-\partial_3 u_2)=0.
\end{aligned}
$$

This system can be written as:

$$
(9)\qquad \hat{L}u=\hat{A}_1(\partial_1\hat{u})+\hat{A}_2(\partial_2\hat{u})+\hat{A}_3(\partial_3\hat{u})+\hat{B}\hat{u}=\hat{f},
$$

where $\hat{f}=(0, f, 0, 0)$, and $\hat{u}=(u_0, u_3, u_1, u_2)$. The system (8) is symmetric, but not positive. Left multiplication by the diagonal matrix

$$
E=\begin{bmatrix} \alpha & 0 & 0 & 0\\ 0 & \gamma & 0 & 0\\ 0 & 0 & \gamma & 0\\ 0 & 0 & 0 & \gamma \end{bmatrix}=\{\alpha,\ \gamma,\ \gamma,\ \gamma\},
$$

where $\alpha=\alpha(x, y, z)$, and $\gamma=\gamma(x, y, z)\in C^1(\bar{D})$ are for the moment arbitrary functions, yields:

$$
(10)\qquad M\hat{u}=A_1(\partial_1\hat{u})+A_2(\partial_2\hat{u})+A_3(\partial_3\hat{u})+B\hat{u}=h \text{ in } D\subset R^3,
$$

$$
A_1=E\hat{A}_1=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & \gamma K_1 & 0\\ 0 & \gamma K_1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix},\quad
A_2=E\hat{A}_2=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \gamma K_2\\ 0 & 0 & 0 & 0\\ 0 & \gamma K_2 & 0 & 0 \end{bmatrix},
$$

$$
A_3=E\hat{A}_3=\begin{bmatrix} \alpha & 0 & 0 & 0\\ 0 & -\gamma & 0 & 0\\ 0 & 0 & -\gamma K_1 & 0\\ 0 & 0 & 0 & -\gamma K_2 \end{bmatrix}=\{-\alpha,\ -\gamma,\ -\gamma K_1,\ -\gamma K_2\}.
$$

$$
B=E\hat{B}=\begin{bmatrix} 0 & a & 0 & 0\\ 0 & \gamma c & \gamma a & \gamma b\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix},\quad
h=E\hat{f}=\begin{bmatrix} 0\\ \gamma f\\ 0\\ 0 \end{bmatrix}
$$

The two systems (9) and (10) are equivalent if det $(E)=\alpha\gamma^3\neq 0$ in D. We introduce the matrix

$$k=B-1/2(\partial_1 A_1+\partial_2 A_2+\partial_3 A_3)$$

$$=\begin{bmatrix} \alpha_z/2 & \alpha & 0 & 0 \\ 0 & \gamma c+(\gamma_z/2) & \gamma a+[(\gamma K_1)_x/2] & \gamma b+(\gamma K_2)_y/2] \\ 0 & (\gamma K_1)_x/2 & (\gamma K_1)_z/2 & 0 \\ 0 & (\gamma K_2)_y/2 & 0 & (\gamma K_2)_z/2 \end{bmatrix},$$

therefore,

$$k+k^t=\begin{bmatrix} \alpha_z & \alpha & 0 & 0 \\ \alpha & 2\gamma c+\gamma_z & \gamma a+(\gamma K_1)_x & \gamma b+(\gamma K_2)_y \\ 0 & \gamma a+(\gamma K_1)_x & (\gamma K_1)_z & 0 \\ 0 & \gamma b+(\gamma K_2)_y & 0 & (\gamma K_2)_z \end{bmatrix},$$

which is a symmetric matrix, while k is not.

Let us consider the principal minors Δ_i of $k+k^t$ and apply the Sylvester's Criterium.

The $k+k^t$ symmetric matrix Δ_i $(i=1, 2, 3, 4)$ are given by the following simplified formulas:

$$\Delta_1=\alpha_z\,;\ \Delta_2=\alpha_z(2\gamma c+\gamma_z)-\alpha_z^2\,;\ \Delta_3=(\gamma K_1)_z\Delta_2-ab\gamma^2\alpha_z\,;$$

$$\Delta_4=(\gamma k_2)_z\Delta_3+\gamma b[(-\gamma b)(\gamma K_1)_z]=(\gamma K_2)_z\Delta_3-(\gamma b)^2(\gamma K_1)_z\alpha_z$$

According to Popivanov [7] there exist $\alpha=\alpha(x, y, z)$ and $\gamma=\gamma(x, y, z)$ {try, for instance, $\alpha=\exp(\varepsilon_1 z)$, and $\gamma=\exp(\varepsilon_2 z)$, where ε_1 is a fixed constant (>1) and ε_2 is for the moment an arbitrary positive constant$(\geqq 1)$} such that $\Delta_i>0$ in D $(i=1, 2, 3, 4)$. This implies that the matrix $k+k^t$ is positive definite symmetric, and therefore the system (10) is positive-symmetric.

We now consider the following characteristic matrix on $D\subset R^3$:

$$(11)\qquad \beta=A_1v_1+A_2v_2+A_3v_3=\begin{bmatrix} -\alpha v_3 & 0 & 0 & 0 \\ 0 & -\gamma v_3 & \gamma K_1 v_1 & \gamma K_2 v_2 \\ 0 & \gamma K_1 v_1 & -\gamma K_1 v_3 & 0 \\ 0 & \gamma K_2 v_2 & 0 & -\gamma K_2 v_3 \end{bmatrix}.$$

By computing the determinant of the matrix β we find:

$$\det(\beta)=-\alpha v_3\{\gamma K_2 v_2\,.\,\gamma K_1 v_3\,.\,\gamma K_2 v_2-\gamma K_2 v_3\,.\,\gamma^2 K_1 v_3^2+\gamma K_2 v_3\,.\,\gamma^2 K_1^2 v_1^2\}$$

and because $K_1\,.\,v_1^2+K_2\,.\,v_2^2-v_3^2=0$ on $S_4\cup S_3\subset\partial D$ we find $\det(\beta)=-\alpha\gamma^3 K_1 K_2 v_3^2[K_2\,.\,v_2^2+K_1\,.\,v_1^2-v_3^2]=0$ on $S_3\cup S_4$. On the other hand, $\det(\beta)=-\alpha\gamma^3 K_1 K_2 v_3^2[K_1\,.\,v_1^2+K_2 v_2^2-v_3^2]=0$ on $S_0\subset\partial D$, because $v_1=v_2=0$ on S_0 and $v_3<0$ on S_0; therefore $K_1v_1^2+K_3v_2^2-v_3^2\leqq 0$. Besides $K_i(0)=0$ $(i=1, 2)$ on S_0.

We note that by computing the unit outer normal vector on S_4 and S_3 we find:

$$v=v_1 i+v_2 j+v_3 k=(1+K_1\cos^2\theta+K_2\sin^2\theta)^{-1/2}\{(\cos\theta)i+(\sin\theta)j$$

$$+(K_1\cos^2\theta+K_2\sin^2\theta)^{1/2}k\}:\ \text{on } S_4$$

and

$$v=v_1i+v_2j+v_3k=(1+K_1\cos^2\theta+K_2\sin^2\theta)^{-1/2}\{(-\cos\theta)i+(-\sin\theta)j$$
$$+(K_1\cos^2\theta+K_2\sin^2\theta)^{1/2}k\}: \text{ on } S_3$$

such that $v_3>0$ on $S_4\cup S_3\subset\partial D$.

We consider also the following quadratic form $v_3\neq0$ the matrix of representation of which is (11):

$$(12)\qquad \hat{u}\beta\hat{u}=(u_0,u_3,u_1,u_2)\begin{bmatrix}-\alpha v_3 & 0 & 0 & 0\\ 0 & -\gamma v_3 & \gamma K_1 v_1 & \gamma K_2 v_2\\ 0 & \gamma K_1 v_1 & -\gamma K_1 v_3 & 0\\ 0 & \gamma K_2 v_2 & 0 & -\gamma K_2 v_3\end{bmatrix}\cdot\begin{bmatrix}u_0\\ u_3\\ u_1\\ u_2\end{bmatrix}$$

$$=-\alpha v_3u_0^2-\frac{\gamma}{v_3}[K_1(v_3u_1-v_1u_3)^2+K_2(v_3u_2-v_2u_3)^2-Hu_3^2],$$

where

$$(13)\qquad H=K_1v_1^2+K_2v_2^2-v_3^2\ (v_3\neq0).$$

We can now formulate a Boundary Value Problem for the system (10) with admissible boundary conditions. We know that $H=K_1v_1^2+K_2v_2^2-v_3^2\leqq0$ on $S_0\subset\partial D$ such that $v_3<0$ on S_0. Therefore, (12) and (13) imply that $\hat{u}\beta\hat{u}\geq0$ on S_0: hence we can put $\beta_-=0$: and $\beta_+=\beta$, where

$$\beta=A_10+A_20+A_3v_3=\begin{bmatrix}-\alpha v_3 & 0 & 0 & 0\\ 0 & -\gamma v_3 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\text{ on } S_0.$$

On the other hand, $H=K_1v_1^2+K_2v_2^2-v_3^2=0$ on $S_3\cup S_4\subset\partial D$ such that $v_3>0$ on $S_3\cup S_4$. Therefore, (12) and (13) imply that

$$\hat{u}\beta\hat{u}=-\alpha v_3u_0^2-\frac{\gamma}{v_3}[K_1(v_3u_1-v_1u_3)^2+K_2(v_3u_2-v_2u_3)^2]\leq0.$$

Hence, we can put: $\beta_-=\beta$; $\beta_+=0$, where β is given by (11).

Since the quadratic forms $\hat{u}\beta_\pm\hat{u}$ are sign-constant on ∂D; that is, $\hat{u}\beta_\pm\hat{u}\geq0$ on S_0 and $\hat{u}\beta_\pm\hat{u}\leq0$ on $S_3\cup S_4$ then Ker $(\beta_\pm)=\{\hat{u}:\hat{u}\beta_\pm\hat{u}=0\}$. Hence, for arbitrary $\hat{u}=(u_0,u_3,u_1,u_2)\in U^4$ implies Ker$(\beta_+)+$Ker $(\beta_-)=U^4$ and $\hat{u}_+=(0,u_3,\frac{v_1}{v_3}u_3,\frac{v_2}{v_3}u_3)$; $\hat{u}_-=\hat{u}-\hat{u}_+$. Therefore, the boundary condition $\beta_-\hat{u}=0$ is admissible; it has the form

$$(14)\qquad u_0=0;\ v_3u_1-v_1u_3=0;\ v_3u_2-v_2u_3=0 \text{ on } S_3\cup S_4\subseteq\partial D$$

and no boundary conditions are given on S_0.

But since $u_0=u=0$ on $S_3\cup S_4$, and since $H=0$ on $S_3\cup S_4$ implies

$$du=u_xdx+u_ydy+u_zdz=(\partial_1u)dx+(\partial_2u)dy+(\partial_3u)dz=u_1dx+u_2dy+u_3dz=0,$$

therefore

$$(15)\qquad u_1=N_0v_1;\ u_2=N_0v_2;\ u_3=N_0v_3,$$

where N_0 is a normalizing factor.

Hence, (14) and (15) imply that the following boundary condition $\beta_-\widehat{u}=0$ is admissible; it has the form

$$(^{**})\qquad u_0|_{S_3\cup S_4}=0$$

We note that since $v_3>0$ and $H=0$ on $S_3\cup S_4$ there are no conjugate boundary conditions.

Therefore, we formulate the following

Theorem. *If there exists a function $u=u(x,y,z)\in C^2(\bar{D})$ satisfying equation (*) under the boundary condition (**), then it is unique (i. e. $u\equiv 0$ in D).*

Remarks. In general, in order to formulate correctly posed Boundary value problems in different domains $D\subset\mathbf{R}^3$, we note the following general conditions:

Consider the parts of ∂D on which $H=K_1v_1^2+K_2v_2^2-v_3^2\leqq 0$. Therefore, the expression in square brackets in (12) is non-negative; hence, $\widehat{u}\beta\widehat{u}\leqq 0$ if $v_3>0$, and we can put $\beta_-=\beta$; $\beta_+=0$. Similarly, since

$$\widehat{u}\beta\widehat{u}\geqq 0 \text{ if } v_3<0, \text{ we can put } \beta_-=0;\ \beta_+=\beta.$$

On the other hand, on those parts of ∂D on which $H>0$, we define the symmetric matrices β_+ and β_- by the following relations:

$$\widehat{u}\beta_-\widehat{u}=\left(\frac{\gamma}{v_3}\right)Hu_3^2;\ \beta_+=\beta-\beta_- \text{ if } v_3<0;$$

$$\widehat{u}\beta_+\widehat{u}=\left(\frac{\gamma}{v_3}\right)Hu_3^2;\ \beta_-=\beta-\beta_+ \text{ if } v_3>0.$$

Since for either $H\leqq 0$, or $H>0$, both the quadratic forms $\widehat{u}\beta_-\widehat{u}$ and $\widehat{u}\beta_+\widehat{u}$ are sign-constant, $\operatorname{Ker}(\beta_\pm)=\{\widehat{u}:\widehat{u}\beta_\pm\widehat{u}=0\}$.

Hence, for arbitrary $\widehat{u}=(u_0,\ u_3,u_1,u_2)\in U^4$, $\widehat{u}_+=\left(0,\ u_3,\ \frac{v_1}{v_3}\ u_3,\ \frac{v_2}{v_3}\ u_3\right)$; $u_-=\widehat{u}-\widehat{u}_+$.

Therefore, on ∂D, the boundary condition $\beta_-u=0$ is admissible; it has the form:

$$u_0=0;\ v_3u_1-v_1u_3=0;\ v_3u_2-v_2u_3=0, \text{ if } v_3>0,\ H\geqq 0;$$

$$u_0=0;\ u_1=u_2=u_3=0, \text{ if } v_3>0,\ H<0;$$

$$u_3=0, \text{ if } v_3<0,\ H>0.$$

We note that no boundary conditions are given on the parts of ∂D on which $v_3<0$, $H\leqq 0$.

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