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Dynamical Systems & Integro-Differential Equations

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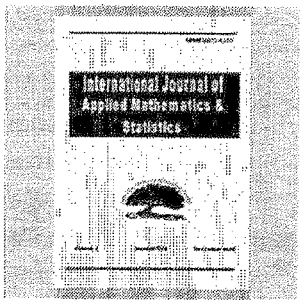

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PREFACE

This Euler's commemorating volume entitled :

Functional Equations , Integral Equations, Differential Equations and Applications **(F. I. D. A)**,

is a forum for exchanging ideas among eminent mathematicians and physicists, from many parts of the world, as a tribute to the tri-centennial birthday anniversary of Leonhard Paul Euler (April 15,1707 A.D., b. in Basel – September 18, 1783 A.D., d. in St. Petersburg).

This 998 pages long collection is composed of outstanding contributions in mathematical and physical equations and inequalities and other fields of mathematical, physical and life sciences.

In addition, this anniversary volume is unique in its target, as it strives to represent a broad and highly selected participation from across and beyond the scientific and technological country regions. It is intended to boost the cooperation among mathematicians and physicists working on a broad variety of pure and applied mathematical areas.

Moreover, this new volume will provide readers and especially researchers with a detailed overview of many significant insights through advanced developments on Euler's mathematics and physics. This transatlantic collection of mathematical ideas and methods comprises a wide area of applications in which equations, inequalities and computational techniques pertinent to their solutions play a core role.

Euler's influence has been tremendous on our everyday life, because new tools have been developed, and revolutionary research results have been achieved , bringing scientists of exact sciences even closer, by fostering the emergence of new approaches, techniques and perspectives.

The central scope of this commemorating 300birthday anniversary volume is broad, by deeper looking at the impact and the ultimate role of mathematical and physical challenges, both inside and outside research institutes, scientific foundations and organizations.

We have recently observed a more rapid development in the areas of research of Euler worldwide. Leonhard P. Euler (1707-1783) was actually the most influential mathematician and prolific writer of the eighteenth century, by having contributed to almost all the fundamental fields of mathematics and mathematical physics. In calculus of variations, according to C. Caratheodory, Euler's work: *Methodus inveniendi lineas curves*...(1740 A.D.) was one of the most beautiful works ever written. Euler was dubbed *Analysis Incarnate* by his peers for his incredible ability. He was especially great from his writings and that produced more academic work on mathematics than anyone. He could produce an entire new mathematical paper in about thirty minutes and had huge piles of his works lying on his desk. It was not uncommon to find Analysis Incarnate ruminating over a new subject with a child on his lap.

This volume is suitable for graduate students and researchers interested in functional equations, integral equations and differential equations and would make an ideal supplementary reading or independent study research text.

These issues will also be of interest to those working in other areas of mathematics and physics. It is a work of great interest and enjoyable read as well as unique in market.

This **Euler's volume (F. I. D. A.)** consists of six (6) issues containing various parts of contemporary pure and applied mathematics with emphasis to Euler's mathematics and physics.

It contains sixty eight (68) fundamental research papers of one hundred one (101) outstanding research contributors from twenty seven (27) different countries. In particular, these contributors come from:

Algerie (1 contributor); Belgique (2); Bosnia and Herzegovina (2); Brazil (2); Bulgaria (3); China (9); Egypt (1); France (3); Greece (2); India (8); Iran (3); Italy (1); Japan (7); Korea (7); Morocco (3); Oman (2); Poland (3); R. O. Belarus (8); Romania (2); Russia (3); Saudi Arabia (1); Serbia and Montenegro (5); The Netherlands (3); U. A. Emirates (1);U. K. (2); U. S. A. (15); Uzbekistan (2).

First Issue (F. E. I.) contains various parts of ***Functional Equations and Inequalities,*** namely: Euler's Life and Work, Ulam stability, Hyers – Ulam stability and Ulam – Gavruta – Rassias stability of functional equations, Euler – Lagrange type and Euler – Lagrange – Rassias quadratic mappings in Banach and Hilbert spaces, Aleksandrov and isometry Ulam stability problems, stability of Pexider and Drygas functional equations, alternative of fixed point, and Hyers - Ulam stability of differential equations.

Second Issue (MT. PDE) contains various parts of ***Mixed Type Partial Differential Equations,*** namely: Tricomi - Protter problem of nD mixed type partial differential equations, solutions of generalized Rassias' equation, degenerated elliptic equations, mixed type oblique derivative problem, Cauchy problem for Euler – Poisson - Darboux equation, non - local boundary value problems, non-uniqueness of transonic flow past a flattened airfoil, multiplier methods for mixed type equations.

Third Issue (F . D . E.) contains various parts of ***Functional and Differential Equations,*** namely: Iterative method for singular Sturm - Liouville problems, Euler type boundary value problems in quantum mechanics, positive solutions of boundary value problems, controllability of impulsive functional semi-linear differential inclusions in Frechet spaces, asymptotic properties of solutions of the Emden-Fowler equation, comparison theorems for perturbed half-linear Euler differential equations, almost sure asymptotic estimations for solutions of stochastic differential delay equations, , difference equations inspired by Euler's discretization method, extended oligopoly models.

Fourth Issue (D. E. I.) contains various parts of ***Differential Equations and Inequalities,*** namely:

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Sixth Issue (DS. IDE.), contains various parts of ***Dynamical Systems and Integro - Differential Equations,*** namely: Semi-global analysis of dynamical systems, nonlinear functional-differential and integral equations, optimal control of dynamical systems, analytical and numerical solutions of singular integral equations, chaos control of classes of complex dynamical systems, second order integro-differential equation, integro-differential equations with variational derivatives generated by random partial integral equations, inequalities for positive operators, strong convergence for a family of non-expansive mappings.

Deep gratitude is due to all those Guest Editors and Contributors who helped me to carry out this intricate project. My warm thanks to my family:

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Finally I express my special appreciation to: The *Executive Editor* of the *International Journal of Applied Mathematics and Statistics* (IJAMAS.) *Dr. Tanuja Srivastava* for her nice cooperation and great patience.

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Organising Centres in the Semi-global Analysis of Dynamical Systems

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ABSTRACT

The notion of organising centre of a bifurcation diagram is used as an ordering principle in the analysis of nonlinear problems in pure and applied dynamical systems theory. When considering a given, generic n-parameter family of dynamical systems the codimension n bifurcations are isolated in the parameter space, generating a more global array of lower codimension bifurcations. It often makes sense to add one extra parameter to the system, e.g., by varying a 'constant' coefficient. In such cases, semi-global parts of the given n-dimensional bifurcation set can be understood as generic slices in versal unfoldings of codimension $n+1$ singularities. This can give great insight in the structure of the given system, as we shall illustrate in two extensive case studies.

Keywords: Organising Centre, Bifurcations, Predator-prey Model, Normal-internal Resonance.

2000 Mathematics Subject Classification: 58K45, 37C10, 34C23.

1 Introduction

Singularity Theory and Catastrophe Theory as originated by Whitney, Thom and Arnold [82, 94, 7, 8, 92] are known to have wide applications in the field of dynamical systems and beyond.

In particular this holds for modelling problems where 'system parameters' play a role. An important example, which also is the focus of the present work, concerns bifurcation theory. The theoretical studies of bifurcations deal with so-called universal problems. This means that sufficiently many parameters are available for (uni-) versality of generic families of dynamical systems in the context at hand, under a relevant equivalence relation. This has led to the classification of generic, local bifurcations as reported in, e.g., [5, 52, 63, 43].

In many applications models have a certain number, say n, of parameters. Moreover, the bifurcation analysis, taking place in the product of phase space and parameter space, is not restricted to local features only. On the contrary, often the interest is the global organisation of the parameter space regarding bifurcations, which can be both local and global. Here global bifurcations refer to homoclinic and heteroclinic tangencies [72], or to periodic and quasi-periodic bifurcations [16, 23]. The present work focuses on the (semi-) global organisation of the parameter space in terms of local bifurcations. Generically we only meet bifurcations of codimension not exceeding the given number n of parameters. The bifurcations of codimension n correspond to isolated points in the parameter space and these act as organising centres of the bifurcation diagram, according to the principle of Singularity Theory. With V.I. Arnold ([8], Part II. *Catastrophe Theory*) we say: "*It may be that the most astounding conclusion from the large number of 'classification' problems in singularity and bifurcation theory is that a comparatively small list of standard forms (the cusp (semi-cubical), the swallowtail, ...) turns out to be universal, and serves in a large number of different theories, between which, at first glance, no connection at all is present.*"

The main interest of this research is the following. In many examples the (semi-) global array of codimension n points (and their connections) is generated by one or more codimension $n+1$ bifurcation points outside the given family, but in a certain sense 'near' to it. This can be made clear by adding a suitable extra parameter to the family (often one of the coefficients of the model). The given family then can be viewed as a generic n-dimensional section or slice in a $(n+1)$-dimensional parameter space. Compare with the so-called 'path formalism' [49, 50]. As we shall illustrate by our examples, the codimension $n+1$ points then generate the array of codimension n bifurcations in a semi-global or even global way; in this way the bifurcations of codimension less than or equal to n are subordinate to one or more codimension $n+1$ singularities. Colloquially we say that they act as 'organising centres', also compare with [31, 32, 33, 34, 95, 61, 85]. To fix thoughts consider the following example.

(Organising Centre) A two-parameter family with cusps and folds, as they occur in the catastrophe set of the two-parameter family of functions [7, 82, 94]

$$f_{\kappa,\eta}(x) = x^5 + bx^3 + \kappa x^2 + \eta x,$$

where $x \in \mathbb{R}$ and where $\eta, \kappa \in \mathbb{R}$ are parameters. Recall that the catastrophe set of a family of functions is given by the projection of the set of singular points (i.e. points for which $f'_{\kappa,\eta}(x) = 0$) on the parameter space. Figure 1(a) shows the catastrophe set for the case that $b > 0$: two

cusp points are joined by three fold curves. In the case $b < 0$ (not shown), there is just a single fold curve in the form of a parabola.

If we consider b to be a parameter as well, then Figure 1(a) is a slice of the three-dimensional catastrophe set of the family, which has a swallowtail bifurcation occurring at $(\kappa, \eta, b) = (0, 0, 0)$ as organising centre; see Figure 1(b).

Catastrophe theory deals with critical points of real functions, but the above idea applies just as well within the context of local bifurcations of vector fields, where the codimension 0 strata contain Morse-Smale dynamics, e.g., compare with [71, 43]. For similar examples see below. For an interpretation of the present example in an optimisation problem, see §5.

A second message that we shall touch upon more briefly is the complication of the bifurcation geometry by resonances. To fix thoughts, consider an autonomous model, where the coefficients are made time-periodic or quasi-periodic in a perturbative way. In practice this situation often is attained by a normalisation or averaging process. Our claim is that the dynamics and bifurcation geometry of the autonomous 'principal part' in the product of phase- and parameter space is complicated and fine-structured by perturbative (quasi-) periodicity. Generically, this occurs in a way that is theoretically predictable. Here key notions are Cantorisation of bifurcation diagrams in the product of phase- and parameter space, and in particular the fraying of certain bifurcation boundaries, c.f. [15, 24, 23, 20, 18, 22, 54, 88, 75].

2 A Predator-prey Model

The first example consists of a particular family of planar vector fields, which models the dynamics of the populations of predators and their prey in a given ecosystem, and which is a variation on the classical Volterra-Lotka system [65, 86]. For the latter system the change of the prey density per unit of time per predator, called the *response function* [76], is proportional to the prey density. This means that there is no saturation of the predator when the amount of available prey is large. However, it is more realistic to consider a nonlinear and bounded response function, and in fact different response functions have been used in the literature to model the predator response, see [3, 12, 55].

For instance, Zhu et al. [95] have studied recently the following predator-prey model

$$\begin{cases} \dot{x} &= x(a - \lambda x) - yP(x) \qquad \text{(prey)}, \\ \dot{y} &= -\delta y + yQ(x) \qquad \text{(predator)}. \end{cases} \tag{2.1}$$

The variables x and y denote the density of the prey and predator populations respectively, while $P(x)$ is a non-monotonic response function [2] given by

$$P(x) = \frac{mx}{\alpha x^2 + \beta x + 1}, \tag{2.2}$$

where α, m are positive and where $\beta > -2\sqrt{\alpha}$. Observe that in the absence of predators, The number of prey increases according to a logistic growth law. The coefficient a represents the intrinsic growth rate of the prey, while $\lambda > 0$ is the rate of competition or resource limitation of prey. The natural death rate of the predator is given by $\delta > 0$. In Gause's model [48] the function $Q(x)$ is given by $Q(x) = cP(x)$, where $c > 0$ is the rate of conversion between prey and predator. For further discussion on the biological relevance of the model see [75, 29, 95, 74].

In this section, we investigate the following family

$$\begin{cases} \dot{x} &= x\left(1 - \lambda x - \dfrac{y}{\alpha x^2 + \beta x + 1}\right) \qquad \text{(prey)}, \\ \dot{y} &= y\left(-\delta - \mu y + \dfrac{x}{\alpha x^2 + \beta x + 1}\right) \qquad \text{(predator)}, \end{cases} \tag{2.3}$$

where $\alpha \geq 0$, $\delta > 0$, $\lambda > 0$, $\mu \geq 0$ and $\beta > -2\sqrt{\alpha}$ are parameters. We note that (2.3) is obtained from (2.1) by adding the term $-\mu y^2$ to the second equation and after scaling x and y, as well as the parameters and the time t. In this way we take into account competition between predators for resources other than prey, see [10, 11]. The non-negative coefficient μ is the rate of competition amongst predators.

Our goal is to understand the structurally stable dynamics of (2.3) and in particular the attractors with their basins where we have a special interest for multi-stability. We investigate the bifurcations between the open regions of the parameter space that concern such dynamics, thereby giving a better understanding of the family.

We also address the modification of this system where a small parametric forcing is applied in the parameter λ, as suggested by Rinaldi et al. [73]. We put

$$\lambda = \lambda(t) = \lambda_0\left(1 + \varepsilon \sin\left(\frac{2\pi}{w}t\right)\right), \tag{2.4}$$

where $\varepsilon < 1$ is a perturbation parameter and w is a constant. We shall discuss how this modification effects the results for the autonomous system (2.3), which then serves as principal part of the forced system. We are particularly interested in the occurrence of 'large scale' strange attractors.

Remark 2.1. In addition to (2.4), we could also have set

$$a = a(t) = a_0\left(1 + \varepsilon \sin\left(\frac{2\pi}{w}t\right)\right),$$

which models seasonal growth rate fluctuations, see for instance [73]. More generally, the periodic forcing terms may be replaced by quasi-periodic terms.

2.1 Sketch of Results

In what follows we describe the results on the autonomous system (2.3).

Trapping domains and Reduced Morse-Smale portraits. The investigation concerns the dynamics of (2.3) in the closed first quadrant $\mathrm{clos}(\mathcal{Q})$ where $\mathcal{Q} = \{(x, y) \in \mathbb{R}^2 | x > 0,\ y > 0\}$ with boundary $\partial\mathcal{Q}$, which are both invariant under the flow. We shall show that system (2.3) has a compact *trapping domain* $\mathcal{B}_p \subset clos(\mathcal{Q})$: all orbits in $clos(\mathcal{Q})$ enter $\mathcal{B}_p$ after finite time and do not leave it again. For the moment we restrict the attention to structurally stable (or Morse-Smale) dynamics. In the interior of $\mathcal{B}_p$, there can be at most two stable equilibria and possibly one saddle-point. We study these singular points using algebraic tools, occasionally supported by computer algebra. Also we numerically detect several cases with one or two limit cycles. Here we often use numerical continuation, where algebraic detection of Hopf or Bogdanov-Takens bifurcations helps to initiate the continuation process.

Since limit cycles are hard to detect mathematically, our approach is to reduce, by surgery, the structurally stable phase portraits to new portraits without limit cycles. During surgery we change a part of the phase space by smooth cutting and pasting [66, 69]. Next, with the help of topological means (Poincaré-Hopf Index Theorem, Poincaré-Bendixson Theorem [66, 71]) we find a complete classification that is of great help to understand the original system (2.3). Compare with Figure 2.

Notation	Name	Incidence
TC_1	Transcritical	
TC_2	Degenerate transcritical	$SN_1 + TC_1$
TC_3	Doubly degenerate transcritical	$SN_2 + TC_2$
PF_1	Pitchfork	
PF_2	Degenerate pitchfork	$PF_1 + SN_1$
SDZ_2	Symmetric double zero	$PF_1 + H_1$
SN_1	Saddle-node	
SN_2	Cusp	$SN_1 + SN_1$
BT_2	Bogdanov-Takens	$SN_1 + H_1 + L_1$
BT_3	Degenerate Bogdanov-Takens	$BT_2 + H_2 + DL_2$
NF_3	Singularity of nilpotent-focus type	$SN_2 + BT_2 + L_2 + H_2$
NE_3	Singularity of nilpotent-elliptic type	$SN_2 + BT_2 + L_2 + H_2$
H_1	Hopf	
H_2	Degenerate Hopf	$H_1 + SNLC_1$
L_1	Homoclinic (or Blue Sky)	
L_2	Homoclinic at saddle-node	$L_1 + SN_1$
DL_2	Degenerate homoclinic	$L_1 + SNLC_1$
$SNLC_1$	Saddle-node of limit cycles	
HE_1	Heteroclinic	

Table 1: List of bifurcations occurring in systems (2.3), (3.6) and (3.7). This notation will be kept throughout. All bifurcations are local except the latter five, which are global. In all cases the subscript indicates the codimension of the bifurcation. In the column 'Incidence' we put the subordinate bifurcations of the highest codimension. See [1, 43, 44, 52, 63] for details concerning the terminology and fine structure.

Organization of the parameter space. Our main interest is the dense-open subset of the parameter space $^5 = \{\alpha, \beta, \mu, \delta, \lambda\}$ with structurally stable dynamics. The complement of this

set is the bifurcation set, which consists of strata of different codimension.

It turns out that the parameters δ and λ play a minor role and that we can describe the bifurcation set as follows. Let $\Delta = \{(\delta, \lambda) \in \mathbb{R}^2 | \delta > 0, \lambda > 0\}$ and $\mathcal{W} = \{(\alpha, \beta, \mu) \in \mathbb{R}^3 | \alpha \geq 0, \beta > -2\sqrt{\alpha}, \mu \geq 0\}$. We fix $(\delta, \lambda) \in \Delta$ and bifurcations of (2.3) are described in the space $\mathcal{W}$. To discuss this we introduce the projection

$$\Pi : \mathcal{W} \times \Delta \to \Delta, \ (\alpha, \beta, \mu, \delta, \lambda) \mapsto (\delta, \lambda),$$

studying all fibers $\Pi^{-1}(\delta, \lambda)$. This argument works as long as the fibers are transversal to the bifurcation set consisting of singularities of nilpotent-focus type (NF_3), in which case we only have to consider bifurcations of codimension less than or equal to 3. It turns out that this is the case in the complement of a smooth curve $\mathcal{C}$ in Δ, compare with Figure 3(a). Indeed the bifurcation set in $\mathcal{W}$, up to diffeomorphisms, is constant above each open region Δ_1 and Δ_2, separated by $\mathcal{C}$, see [27, 28, 29]. Above the curve $\mathcal{C}$ there is a folding of the bifurcation set whenever the fiber $\Pi^{-1}(\delta, \lambda)$ is tangent to it.

When restricting to Δ_1 and Δ_2 the codimension 3 bifurcations in $\mathcal{W}$ act as organising centres. This means that when taking two-dimensional sections in $\mathcal{W}$ we see a semi-global picture organized by the trace of the codimension 3 bifurcations, compare with [29]. For each region Δ_1 and Δ_2 the associated bifurcation set in $\mathcal{W}$ is depicted in Figure 3(b) and Figure 3(c), respectively. Figure 3(c) shows that the bifurcation set possesses several codimension 2 curves subordinate to four codimension 3 points which act as organising centres. Now we explain how to understand the bifurcations up to codimension 1.

Organising centres and two-dimensional bifurcation diagrams. Given the organising centres of the bifurcation sets in $\mathcal{W}$, we take two-dimensional sections $\mathcal{S}_i$, $i = 1, \ldots, 6$, transversal to the codimension 2 curves as indicated in Figure 3. Each two-dimensional section intersects codimension 0 strata in several open regions separated by codimension 1 curves. The two-dimensional bifurcation diagrams are depicted, emphasising the basins of attraction and the possible multi-stability.

We illustrate our strategy in Figure 4 by presenting one of the two-dimensional bifurcation diagrams (in $\mathcal{S}_1$). For the other two-dimensional diagrams see [29, 74].

Limit cycles and homoclinic loops. Limit cycles may be created by Hopf bifurcations (H_1) (see for instance regions 1 and 7 of Figure 4, by saddle-node bifurcations of limit cycles ($SNLC_1$) (see regions 9 and 11 of the same Figure 4 and by homoclinic bifurcations (L_1) (or Blue Sky catastrophe [1], see regions 1 and 12 in Figure 4. The occurrence of limit cycles is investigated numerically (continuation) with help of Matlab [53], Matcont [51] and AUTO [42].

As said before, all local bifurcations can be detected algebraically, which is not the case for the global bifurcations L_1 and $SNLC_1$. Again we resort to numerical continuation methods, using various codimension 2 bifurcations to create initial data. For example, in Figure 4, the

degenerate Hopf bifurcation H_2 'generates' the curve $SNLC_1$, while the Bogdanov-Takens bifurcation BT_2 'generates' the curve L_1^a.

2.2 Parametric forcing

Dynamical properties of system (2.3) with parametric forcing (2.4) can be expressed in terms of the stroboscopic map

$$\mathcal{P}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto \varphi_\varepsilon^1(x, y), \tag{2.5}$$

where φ_ε^t denotes the flow of the time-periodic system written as a three-dimensional vector field $X_\varepsilon = X_\varepsilon(x, y, t; \alpha, \beta, \mu, \delta, \lambda)$. Fixed points of $\mathcal{P}_\varepsilon$ correspond to periodic solutions of X_ε with period w, and similarly invariant circles of $\mathcal{P}_\varepsilon$ correspond to invariant 2-tori of X_ε.

We discuss here the general relationship between the autonomous system X_0, which here plays the role of principal part, and the time-periodic perturbation X_ε for $|\varepsilon|$ small. A number of dynamical properties of $\mathcal{P}_\varepsilon$ are considered, as these follow from more or less classical perturbation theory [4, 30, 31, 52]. First of all hyperbolic periodic points (including fixed points) of $\mathcal{P}_0$ persist for $\mathcal{P}_\varepsilon$, for $\varepsilon \ll 1$, as well as their local stable and unstable manifolds. We note that globally the stable and unstable manifolds generically will behave different by separatrix splitting, giving rise to (transversal) homoclinic and heteroclinic tangles. Secondly, local bifurcations are persistent. In particular, this holds for saddle-node and cusp bifurcations of periodic points but also for the Hopf bifurcations of these periodic points. In the latter case (for which the three-dimensional vector field X_ε gives Neĭmark-Sacker bifurcations), we encounter resonances due to the interaction of internal ('natural') and forcing frequencies. The strong resonances are more involved [4, 30, 31, 59, 73, 81], but in the case of weak resonances, near the Hopf curve, the limit cycle turns into a $\mathcal{P}_\varepsilon$-invariant circle. In a corresponding three-dimensional section in $\mathcal{W}$, the associated rotation number is rational in a dense-open array of Arnol'd tongues emanating from the Hopf curve. Here the circle dynamics is of Kupka-Smale type [71], which corresponds to frequency locking with periodic forcing. For a large measure set outside the tongues the invariant circles are quasi-periodic with Diophantine rotation number. The invariant circles break up further away from the Hopf curve in a complicated way, compare with [30]. The saddle-node bifurcation of limit cycles in X_0 turns into a quasi-periodic saddle-node bifurcation for $\mathcal{P}_\varepsilon$ [23, 24] with all the ensuing dynamical complexity [38, 39, 40], also compare with [26]. In a systematic study of the attractors of $\mathcal{P}_\varepsilon$ as a function of the parameters, we expect the same complexity as described in [25, 30, 31], for more background also compare with [67, 72]. In this investigation the present study of the autonomous system provides a skeleton.

In this section we restrict to the numerical detection of a few attractors of $\mathcal{P}_\varepsilon$ near homoclinic connections in the autonomous system X_0. More precisely we consider the two-dimensional bifurcation diagrams of X_0, looking for the loci of homoclinic orbits (L_1 in Figure 4 and L_1^a, L_1^b in Figure 4. These loci can be continued in the ε-direction for $\varepsilon \geq 0$. In particular, we look at a neighbourhood of L_1 where complicated dynamics related to homoclinic tangencies are to be expected. As an example, we plot a few attractors for $\mathcal{P}_\varepsilon$ in Figures 5 and 6, for parameter values near the homoclinic curve L_1^a in region 9 of Figure 4. We have numerical evidence for

the following statements. Figure 5 shows a strange attractor that consists of 11 connected components mapped by $\mathcal{P}_\varepsilon$ to one another in a cyclic way. These components in Figure 6 'connect' in a scenario called heteroclinic tangency (or boundary crisis), compare [30]. A more systematic approach of this and related time-periodic systems is subject of future research.

Remark 2.2. When applying quasi-periodic forcing to system (2.3), local bifurcations in (2.3) give rise to quasi-periodic bifurcations [24, 23]. In the case of Hopf bifurcation, resonances are encountered between internal and forcing frequencies [75, 85]. See §3 and §5 for details on resonant dynamics.

3 Quasi-periodic Response Solutions at A Strong Resonance

Our second case originates from Stoker's problem [79], which asks for response solutions in quasi-periodically forced damped oscillators. To be definite, as a leading example we consider the forced Duffing – Van der Pol oscillator

$$\ddot{y} + \left(a + cy^2\right)\dot{y} + by + dy^3 = \varepsilon f(\omega_1 t, \ldots, \omega_m t, y, \dot{y}, a, b, c, d, \varepsilon).$$

Here the perturbation f is assumed to be periodic in its first m arguments with period 2π. This non-autonomous second order differential equation can be written as a vector field (in system form)

$$\begin{cases} \dot{x}_j &= \omega_j, \qquad j = 1, \ldots, m, \\ \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -\left(a + cy_1^2\right) y_2 - by_1 - dy_1^3 + \varepsilon f(x_1, \ldots, x_m, y_1, y_2, a, b, c, d, \varepsilon), \end{cases} \tag{3.1}$$

defined on the phase space ${}^m\times^2 = \{(x_1, \ldots, x_m), (y_1, y_2)\}$. Here ${}^m = {}^m/(2\pi^m)$ denotes the standard m-torus and $x = (x_1, \ldots, x_m) \in {}^m$ is called *internal* variable. Moreover, $y_1 = y$ and $y_2 = \dot{y}$ are called *normal* variables. To begin with, we may take f independent of (c, d), including only (a, b) as parameters. Accordingly, for the moment we abbreviate $\sigma = (a, b)$. However, we note that for later purposes and as an illustration of our main message we shall take (a, b, c, d) as parameters. Throughout we fix the frequency vector $\omega \in {}^m$, but in certain other cases ω may be included as an extra multi-parameter. Finally $\varepsilon \in$, with $|\varepsilon| \ll 1$, is a perturbation parameter. We assume f and all other functions to depend infinitely differentiably (or smoothly) on all their arguments.

Note that (3.1) is a skew system in the sense that the internal dynamics are independent of the normal variables. This allows one to focus on the influence of the internal dynamics on the normal dynamics, without having to take care of internal resonances. It is assumed that the frequencies $\omega_1, \ldots, \omega_m$ are rationally independent, in which case the forcing is called quasi-periodic.

The existence of a quasi-periodic response solution of frequency vector ω reduces to the existence of an invariant m-torus of (3.1) in the phase space ${}^m\times^2$, which is a graph $y = y(x)$ over ${}^m \times \{0\}$; each such m-torus corresponds to an m-parameter family of response solutions

$y = y(x(t))$. In the case of strong damping $|a| \gg 1$ (with a either positive or negative) the problem is solved in [79]: in a more contemporary language, it reduces to the persistence of normally hyperbolic invariant m-tori close to $^m \times \{0\} \subset^n \times^2$; the corresponding region in the parameter space often is called the 'Stoker domain'. The problem is more involved in the case of small damping $|a|$, as we shall discuss now. For details see [15,24,23,41,20,75].

3.1 Quasi-periodic Hopf Bifurcation

In the case that $|a| \ll 1$ small divisors enter into the problem, as was already observed by Moser [68], which makes application of KAM theory necessary. This, among other things, involves *Diophantine conditions* on the frequency vector $\omega = (\omega_1, \ldots, \omega_m)$. Given positive constants τ and γ, these conditions require that

$$|k\omega - \ell\alpha(\sigma)| \geq \gamma |k|^{-\tau},$$

for all $k \in^m \setminus\{0\}$ and for $\ell = 0, 1, 2$. Here is the standard inner product and $|k| = \Sigma|k_j|$. Moreover $\alpha(\sigma) = \sqrt{4b - a^2}$ is the normal frequency, also see below. For $\tau > m - 1$ and $\gamma > 0$ small, the corresponding set of Diophantine frequency vectors has positive measure; for details see, e.g., [23, 24, 41]. Note that a Diophantine frequency vector surely has rationally independent components, which brings us in the above quasi-periodic set-up.

Generically this setting involves a Hopf bifurcation of the corresponding free oscillator. Fixing a Diophantine frequency vector ω, one needs at least two parameters to describe the phenomenon well; we first restrict to cases where we can take $\sigma = (a, b)$, for a near 0.

Normal-internal resonance. To explain what is going on, we generalize the above set-up by considering the family of vector fields

$$\begin{aligned} X_{\sigma,\varepsilon}(x, y) &= \omega + Z_{\sigma,\varepsilon}(x, y) \\ &= \omega + (A(\sigma)y + B(y, c, d) + \varepsilon F(x, y, \sigma, c, d, \varepsilon))\, y, \end{aligned} \tag{3.2}$$

where $(x, y) \in^m \times^2$, using the shorthand notation $\omega = \sum_{j=1}^m \omega_j \frac{\partial}{\partial x_j}$. Note that in the above example (3.1)

$$A(\sigma) = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}, \qquad B(y, c, d) = \begin{pmatrix} 0 \\ -cy_1^2 y_2 - dy_1^3 \end{pmatrix},$$

recalling that $\sigma = (a, b)$. Also we consider the vector field

$$N_\sigma(x, y) = \omega \frac{\partial}{\partial x} + A(\sigma) y \frac{\partial}{\partial y},$$

which is the normal linear part of the unperturbed invariant m-torus $^m \times \{0\} \subset^m \times^2$, see [24]. We note that N_σ is of Floquet form, i.e., independent of the internal variable $x \in^m$. Let $\mu(\sigma) \pm i\alpha(\sigma)$ denote the complex conjugate eigenvalues of $A(\sigma)$. In the Duffing – Van der Pol

example (3.1) we assume that $a^2 < 4b$, noting that $\mu(\sigma) = -a$ and $\alpha(\sigma) = \sqrt{4b - a^2}$. In general we call $\omega_1, \ldots, \omega_m$ the internal frequencies and α the normal frequency of the unperturbed m-torus.

A normal-internal $k : \ell$ resonance of $^m \times \{0\}$ occurs if

$$k\omega + \ell\alpha(\sigma) = 0, \tag{3.3}$$

for some $k \in^m$ and $\ell \in$ not both equal to zero. The smallest absolute value $|\ell|$, where ℓ ranges over all integers for which there is a $k \in^m$ such that (3.3) holds, is called the order of the resonance. Resonances of order up to 4 are called strong resonances [81, 5]; all other resonances are called weak. If for $\sigma = \sigma_0$ the torus $^m \times \{0\}$ is respectively non-resonant, weakly resonant or strongly resonant, the value σ_0 is called a non-degenerate, weakly resonant or strongly resonant quasi-periodic Hopf bifurcation value.

The weakly resonant case. The non-degenerate quasi-periodic Hopf bifurcation, as well as its weakly resonant analogue, have been investigated by Braaksma and Broer [15] for Diophantine internal frequency vectors ω. For small positive values of , system (3.2) has in its parameter space a codimension 1 submanifold $\mathcal{H}$, carrying a quasi-periodic Hopf bifurcation set $\mathcal{H}_c$ that has positive measure in $\mathcal{H}$. The set $\mathcal{H}_c$ is found by KAM theory. At every point $\sigma \in \mathcal{H}_c$, two open disk-shaped regions $\mathcal{A}_\sigma$ and $\mathcal{R}_\sigma$ exist in the complement of $\mathcal{H}$ and separated by $\mathcal{H}$ meet with infinite order of contact at $\mathcal{H}_c$. For parameter values in $\mathcal{A}_\sigma$, the vector field has an attracting normally hyperbolic m-dimensional invariant torus close to $^m \times \{0\}$; for values in $\mathcal{R}_\sigma$, a repelling one. Moreover, there are two possibilities regarding invariant $(m+1)$-dimensional tori of (3.2). This distinction depends on the type (supercritical or subcritical) of the autonomous Hopf bifurcation as it occurs in the free oscillator given by (3.2) for $\varepsilon = 0$. Indeed, there exists a subset $\mathcal{H}_c^+ \subset \mathcal{H}_c$ of positive (relative) measure, such that for $\sigma \in \mathcal{H}_c^+$ there is another disk $\mathcal{A}_\sigma^+ \subset \mathcal{A}_\sigma$ in the supercritical case, or $\mathcal{R}_\sigma^+ \subset \mathcal{R}_\sigma$ in the subcritical case. In the first case, for parameter values in $\mathcal{A}_\sigma^+$ there exists an invariant attracting $(m+1)$-torus in the system; in the second case, there is an invariant repelling $(m+1)$-torus for parameter values in $\mathcal{R}_\sigma^+$. All these invariant tori are of finite differentiability; the size of sets $\mathcal{A}_\sigma, \mathcal{A}_\sigma^+$ etc. shrinks if we impose a higher degree of differentiability. The disks $\mathcal{A}_\sigma^+$, $\mathcal{A}_\sigma$, $\mathcal{R}_\sigma^+$ and $\mathcal{R}_\sigma$ are all found by centre manifold theory. For more details see [15, 23, 24].

In the context of the Duffing – Van der Pol example (3.1), we sketch in Figure 7 the situation in the parameter space for the case where $\sigma = (a, b)$, so in a two-dimensional parameter space. At the present level the quasi-periodic Hopf bifurcation has codimension 2. Loosely speaking, μ serves as bifurcation parameter, while α keeps track of the normal frequency. For a discussion how a one-parameter Hopf-Landau family relates to a two-parameter quasi-periodic Hopf bifurcation, see [23]. Now we shall consider cases of strong normal-internal resonances, a proper discussion of which requires the availability of more parameters.

Remark 3.1.

1. Central for the above results is a quasi-periodic normalisation process [15, 24], also see below, which leads to a principal part based on the standard Hopf bifurcation for planar vector fields (and close to the one in the underlying free oscillator). After normalization and truncation a $^{m+1}$-symmetry can be factored out. Here the normal linear part $N = N_\sigma(x, y)$, for $\sigma \in \mathcal{H}_c$, plays a fundamental role. The (relative) equilibria then roughly correspond to m-tori and the periodic solutions to $(m+1)$-tori. For more details, generalisations and references see [15, 18, 24, 23, 41, 20, 57, 58].

2. We recall that in the periodic case $m = 1$ the boundary between the attracting and repelling m-tori (i.c., periodic solutions), away from the strong resonances, is smooth; this case is known by the names Hopf-Neĭmark-Sacker [52, 63]. The weak resonances do play a role in the occurence of Arnold tongues [5, 80, 21] related to phase-lock dynamics on the normally hyperbolic 2-tori that branch off. In the present case $m \geq 2$ the boundary between the attracting and repelling m-tori is expected to become 'frayed' by the normal-internal resonances (3.3).

   It is to be noted that in [15, 24, 23] the resonance holes have a somewhat proof generated character. In [75, 87] the expectation of a 'frayed' (and non-smooth) boundary is given further ground by showing the existence of resonant dynamics.

3.2 Normal-internal Resonance: Setting of the Problem

We are interested in the complement of the parameter domains with normally hyperbolic invariant m- and $(m + 1)$-tori. This complement consists of countably many resonance holes (or bubbles) related to the normal-internal resonances (3.3), in particular for the cases with $\ell \geq 3$. Compare with Figure 7. For persistent case studies in the context of a quasi-periodic saddle-node bifurcation, see Chenciner [38, 39, 40].

We extend the research initiated by Gambaudo [45] and Wagener [87] on the strong resonances with $\ell = 1, 2$, both for the cases $m = 1$ and $m = 2$, compare with [75]. After appropriate averaging [19, 16, 77] and truncation, in [45, 87] a ^{m}-symmetry can be divided out and a bifurcation analysis of the resulting principal part is held, centered around the (relative) equilibria. The corresponding 'local' bifurcations have been tracked down, and also certain 'semi-global' phenomena occur in the analysis. This is exactly what makes the example fit in the present discussion.

To be precise, for the cases $\ell = 1$ and $\ell = 2$, the bifurcation diagrams are completed. For the former case, the bifurcation diagram of the principal part is understood completely by taking the codimension 3 singularity of nilpotent-elliptic type [43], found in [87] as organising centre. As a direct consequence, we also find a codimension 2 degenerate Hopf bifurcation, which was still absent from [45, 87]. For both the cases $\ell = 1, 2$ we add curves of homoclinic and heteroclinic global bifurcations to the bifurcation diagram, with help of AUTO [42] and Matlab [53].

The first step in the analysis of the quasi-periodic case amounts to understanding the periodic case. Though these two scenario's are strongly related the quasi-periodic case is more

involved than the periodic one. The fact that the resonating normal frequencies are dense in the set of all normal frequencies implies that the overall quasi-periodic picture has a lot more fine-structure.

3.3 Further Background

For a systematic bifurcation analysis of the periodic resonances see Takens [81]. There, with help of normal form theory the generic strong ($1:\ell$ for $\ell = 1,\ldots,4$) and weak ($\ell \geq 5$) resonances of periodic response solutions were studied by their Poincaré mappings and a corresponding vector field principal parts. For the $1:4$ resonance see [59, 60].

In this respect, the $1:1$ resonance, also see [5, 6, 13, 14], gives rise to the Bogdanov-Takens bifurcation. However, from a practical point of view, the Bogdanov-Takens bifurcation does not fully describe what happens generically to the response solutions of a driven damped oscillator at resonance. Indeed, for certain parameter values, there are two response solutions, while for others there are none. However under natural hypotheses, like e.g. the friction coefficient growing sufficiently quickly with amplitude, a topological argument can be employed to show the existence of at least one response solution in the oscillator [37]. With help of index theory it is shown that at least one equilibrium exists in the oscillator dynamics.

This implies that simultaneously to the response solutions related to the Bogdanov-Takens bifurcation, there is yet another response solution, which, for parameter values close to the Bogdanov-Takens bifurcation value, is 'far away' and thus is not captured by a local analysis. Holmes and Rand [56, 52] gave a semi-global analysis of the $1:1$ resonance in a special case where all response solutions are captured. The term 'semi-global' here means that after averaging and scaling, we consider phenomena in a given 'large' region of the phase space, rather than in the infinitesimally small regions related for purely local bifurcation analysis. Compare with, e.g., [87, 62, 70, 61].

The analysis of Holmes and Rand was extended by Gambaudo [45] to a semi-global bifurcation analysis of the generic codimension 2 cases of strong resonances. Note that recently a beginning has been made of analysing resonances of periodic response solutions at degenerate resonant bifurcations [21]. In that case resonances $1:\ell$ are strong if $\ell \leq 6$ and weak otherwise. As we find families of degenerate Hopf bifurcations in our analysis, the phenomena reported in [21] will probably occur in our system as well.

3.4 Sketch of Results

In [87] Wagener has taken up the semi-global analysis of driven damped anharmonic oscillators at resonant bifurcation in case that the driving is quasi-periodic instead of periodic, see the set-up (3.2). In this context, the $1:1$ resonance gives rise to quasi-periodic Bogdanov-Takens bifurcation. As mentioned before, in the first step of such an investigation normalising or averaging techniques are applied [6, 80, 35, 36, 19, 60], that permit to write the system as a periodically forced system with a small quasi-periodic perturbation term. Working in a different

parametrisation from [56, 52, 45], and considering a third natural parameter, analysis of the periodic part yields that in the $1:1$ resonance, the two generic cases reported in [45] can be seen as subfamilies of a generic three-dimensional bifurcation diagram that is organised by a singularity of nilpotent-elliptic type (see [43]).

The analysis initiated in [87] is completed in [75], also see [76], by taking the singularity of nilpotent-elliptic type as organising centre of the bifurcation diagram and exploring the implications of this. In this way, the occurrence of degenerate Hopf bifurcations in the $1:1$ resonance case is shown. We compute all global bifurcation curves that are known from the analysis of the singularity of nilpotent-elliptic type, except those relating to 'boundary bifurcations'. The latter concerns tangencies of the vector field to the boundary of any small neighbourhood containing the singularity [44]. Note that this concept is not related to the transcritical bifurcation. Further, a complete two-dimensional bifurcation diagram for the case $1:2$ resonance is given, also see [61, 62, 70, 45, 84]. In the quasi-periodic context, this gives rise to quasi-periodic $1:2$ resonant bifurcation.

Moreover, the differences between periodicity and quasi-periodicity are spelled out more fully. For instance, as said before, in the periodic case the Hopf bifurcation set is a smooth manifold. In the quasi-periodic case, it is shown in [87] that every resonance between the perturbing quasi-periodic frequency ω and the Floquet exponent α of the free oscillator generically gives rise to a resonance hole. This gives further ground to the expectation that the Hopf bifurcation set is frayed. We shall invoke quasi-periodic bifurcation theory [38, 24, 91] to investigate which portions of the local bifurcation diagram persist under small perturbations. In the bifurcation diagram there are quasi-periodic Hopf bifurcation sets for which the present analysis can be applied repeatedly: in this way resonance within resonances are found, as in, e.g., [9]. For a Hamiltonian analogue of the present program see [22].

Resonant normal forms. Let $k\omega + \ell\alpha_0 = 0$ for either $\ell = 1$ or $\ell = 2$. For λ close to $i\alpha_0$, performing successive 'normal form' or 'averaging' transformations [19, 16, 36, 22] and scaling of phase phase and parameter space, yields a coordinate system relative to which the vector field X is of the form

$$X = \omega x + \delta^2\, \Re\Big(\lambda z + {}^{i\theta}\,|z|^2 z + \bar{z}^{\ell-1}\Big)\, z + \delta^3 \tilde{Z}\,.$$

See [75] for details. We put

$$Z_0 = \Re(\lambda z + {}^{i\theta}\,|z|^2 z + \bar{z}^{\ell-1})z\,. \tag{3.4}$$

From the more precise results derived in [76, 29] it follows that X can actually be put in the form

$$X = \omega x + \delta^2\left(Z_0 + \delta Z_1 + \delta^N Z_2\right), \tag{3.5}$$

where Z_1 is an integrable vector field, and where N can be choosen arbitrarily large. The present semi-global context motivates the study of this equation for $\ell = 1$ and $\ell = 2$.

Bifurcation analysis of the $k:1$ resonance. Now we perform a bifurcation analysis of the vector field Z_0, see (3.4), for the case $\ell = 1$. The analysis is complete with respect to local bifurcations, which are obtained analytically. Using numerical tools like Matlab [51, 53] and AUTO [42], we detect global bifurcations, i.e., homoclinic loops and saddle-node bifurcations of limit cycles.
Consider the principal part

$$Z_0 = \Re\left(\lambda z + e^{i\theta}|z|^2 z + 1\right) z\,. \tag{3.6}$$

We shall see that the bifurcation diagram of this family has a codimension 3 singularity of nilpotent-elliptic type as organising centre 'outside this family'. Observe that Z_0 is symmetric with respect to the group generated by the involutions

$$(t, z, \lambda, \theta) \mapsto (t, \bar{z}, \bar{\lambda}, -\theta) \qquad \text{and} \qquad (t, z, \lambda, \theta) \mapsto (-t, -z, -\lambda, \theta + \pi).$$

Because of this symmetry, we can restrict our attention to the part of parameter space for which $0 \leq \theta \leq \pi/2$.

Figure 8 gives a graphical overview of our results on the position of the bifurcation manifolds that correspond to local bifurcations of codimension 1 and 2. The codimension 2 bifurcation curves corresponding to cusp and Bogdanov-Takens bifurcations are indicated in Figure 8(a). Figure 8(b) shows the relative positions of the codimension 1 bifurcation manifolds in the vicinity of a NE_3 point.

We extend the description of the bifurcation diagram of equation (3.6) given in [87]. In Figure 9, we plot all singularities of system (3.6) for the two-dimensional section $\mathcal{S}_2$ of Figure 8(a), as well as the generic phase portraits of the system for parameters in different regions of the bifurcation diagram. For the two-dimensional section $\mathcal{S}_1$ see [76, 75].

We find all local bifurcations of (3.6): saddle-node (SN_1), Hopf (H_1), cusp (SN_2), Bogdanov-Takens (BT_2) and degenerate Hopf (H_2) bifurcations, which are expected from the bifurcation diagram of a singularity of nilpotent-elliptic type [44]. We also retrieve homoclinic (L_1) and saddle-node of limit cycles bifurcations ($SNLC_1$); these are found using the numerical packages AUTO [42] and Matlab [53, 51] respectively.

Remark 3.2. System (3.6) also contains a global feature, namely, a large homoclinic loop of a hyperbolic saddle-point, which is not explained by the NE_3 point. This large homoclinic loop is detected by AUTO [42]. See Figure 9.

Bifurcation analysis of the $k:2$ resonance. As in our previous investigation, here we perform a bifurcation analysis of the vector field Z_0, see (3.4), for the case $\ell = 2$. As remarked before, in this case it is sufficient to consider two bifurcation parameters. The local bifurcations in the $\ell = 2$ case have already been given in [45, 87]; in this section, after briefly recalling those results, global bifurcations are determined numerically whose existence follows from our knowledge of the local bifurcation diagram.

Consider the principal part

$$Z_0 = \Re\left(\lambda z + {}^{i\theta}|z|^2 z + \bar{z}\right) z\,. \tag{3.7}$$

As before, this is a two-parameter family of planar vector fields (parameterised by $\lambda \in \mathbb{C}$) as we shall treat θ as a generic constant. Observe that Z_0 is symmetric with respect to the group generated by the two involutions

$$(z, \lambda, \theta) \mapsto (\bar{z}, \bar{\lambda}, -\theta) \qquad \text{and} \qquad (z, \lambda, \theta) \mapsto (-z, \lambda, \theta).$$

Because of this symmetry, we restrict our attention to those values of θ satisfying $0 \leq \theta \leq \pi$.

The bifurcation diagram of (3.7) is depicted in Figure 10. From the occurrence of a Bogdanov-Takens bifurcation, we infer the existence of a homoclinic bifurcation curve L_1. One symmetric double zero SDZ_2 bifurcation gives rise to curves of saddle-nodes of limit cycles $SNLC_1$, Hopf bifurcations H_1 and homoclinic loops L_1. This last curve ends on the curve of pitchfork bifurcation PF_1; at that point, a curve of heteroclinic bifurcation points HE_1 departs that ends on a line of saddle-nodes of equilibria SN_1. At the other SDZ_2 point, we have a second curve of heteroclinic bifurcations HE_1, also ending on a saddle-node line.

For $\ell \geq 3$ techniques from KAM theory (see [16]) can be applied to show the existence of invariant tori close to $\mathbb{T}^m \times \{0\}$, also compare with [31, 32, 24].

4 Related Settings

The approaches in §3 and §4 both fit into a larger frame of related research, including many applications. For details we refer to [29, 75] and their bibliographies. In this section we present a few further illustrations of the organising centre principle, though we do not attempt to be complete.

4.1 Meteorological Dynamics

The Lorenz84 climate model [64] is a three-dimensional vector field depending on four 'physical' parameters a, b, F and G. The model is obtained from the so-called 'primitive equations' (which are Navier-Stokes based) of the atmosphere by a procedure involving a Galerkin projection, *i.e.*, truncation to low modes of a Fourier expansion of the solution, see [83]. Regarding the numbers of degrees-of-freedom and of control parameters the Lorenz84 system is the simplest model describing atmosphere dynamics at midlatitudes of the northern hemisphere. Due to the severe truncation, however, the model is not suitable for quantitative predictions or statistical studies. Its main climatological appeal lies in the simple representation of a dominant physical process in the atmosphere, namely the interaction between westerly jet stream and baroclinic waves.

In [78] a bifurcation analysis is given of a principal part vector field, in dependence of the two parameters F, G; in most research on the subject a and b are kept constant. This analysis forms the basis of many ensuing studies on Lorenz84. For the present purposes we just

mention [31,32,83,85]; referring for further reading to their extensive bibliographies. In [83], the effect on the (F, G) bifurcation diagram of varying a third parameter b is studied. Next, in [31,85] the parameters (F, G) are made periodic in a perturbative way, thereby modelling seasonal effects. This amounts to a three-dimensional Poincaré return map analysis, depending on three parameters.

In all these cases, it turns out that quite a few local codimension 2 bifurcations act as organising centers, among which a Hopf saddle-node (HSN) bifurcation [52, 63] as well as several strong resonances [5,81]. Moreover, behind the semi-global array of bifurcations a few singularities of higher codimension act as organising centers; among those are certain degenerate Bogdanov-Takens and Hopf bifurcations [43] and degenerate versions of the HSN bifurcation. Many of the latter have not yet been studied theoretically and certain models are recently being investigated [34]. It should be mentioned that in the HSN model for maps, in the analysis of a normal-internal resonance, once more underlying degenerate Bogdanov-Takens and Hopf bifurcations act as organising centers. Compare with §4 above and with, e.g., [75]. For a theoretical link of the climate models with chaos see [32, 33].

4.2 Economic Dynamics

The Brock-Hommes [17] model for price dynamics of risky assets sketches a simplified picture of a stock market. There are many agents on this market, who can invest in either a risky asset (a stock) or a riskless asset (e.g. a bond). All prices are expressed in units of the riskless asset, which pays a known fixed interest each period. The risky asset pays a random dividend; moreover, the market price of this asset in the next period is unknown and therefore subject to speculation.

Though there are many traders, in the simplest set-up it is assumed that their beliefs (or predictions) on the next-period price of the risky asset is of either of two types: there is the fundamental belief, which holds that the 'true' or 'fundamental' value of the risky asset is equal to the discounted stream of future dividends, and which therefore expects that the next-period price will be closer to the fundamental value than the current price. The other group of traders is more pragmatic and expects, for example, that the next-period price can be predicted by some linear extrapolation of recent prices. This prediction method is called 'trend-chasing' or 'technical analysis', as it mimics the kind of price chart analysis which is actually performed by stock traders. Every period, a trader decides on the prediction rule to be used to determine his demand for the risky asset on the basis of the past success of the two rules.

In a recent set of papers [47, 46], a variant of this model has been subjected to bifurcation analysis, where three parameters were considered: the switching rate β ('intensity of choice'), which determines how precisely the traders perceive which strategy is the better of the two; the adaptation rate v of the fundamentalist belief, determining the speed at which the price is assumed to converge to the fundamental price; and the trend parameter g, which controls the price change that trend chasers expect from the recent price data.

Keeping first the parameters v and β at some 'typical' values, the following bifurcation scenario is observed: for values of the trend parameter g smaller than 1, the fundamental steady state is stable, and both fundamentalists and trend chasers make the same predictions: the market is homogeneous. Then, as g increases, a pair of invariant circles are generated in a saddle-node bifurcation far away from the fundamental state. The unstable circle then shrinks in size until it vanishes in a subcritical Hopf bifurcation involving the fundamental steady state.

The economic interest of this scenario is the following: the deterministic dynamical system is seen as an approximation to a stochastic situation, where for instance the supply of the risky asset on the market is taken as a random quantity. Consider now parameter values such that there are a stable fundamental steady state and a stable invariant circle in the system, each having a basin of attraction. The stochastic dynamics follow the deterministic dynamics most of the time, but if the shocks are great enough, the system can jump from one basin of attraction to the other. These jumps will by necessity be uncorrelated in time. It follows that the resulting market will have 'quiet' periods, for which the price is close to the fundamental, and 'lively' periods, for which the price is tracking the dynamics on the invariant circle.

The two bifurcations, saddle-node of invariant circles and subcritical Hopf bifurcation of the fundamental steady state, seem to be unrelated in a one-parameter bifurcation study. But they are seen to be connected when the system is studied with three parameters, for there is a curve of degenerate Hopf bifurcations (H_2 in Table 1.1) acting as organising centres of this system.

4.3 Dynamic Optimisation

In a typical dynamic optimisation problem an evolution law of a system variable is given, for instance the amount of capital available to a firm, the stock of pollutant in a lake or the position of a ship with respect to the harbour. It is assumed that this evolution can be influenced by exercising a control, like investing in capital in the first example, reducing pollution in the second and changing speed or direction of the course of the ship in the last example. The object of controlling the system is either to minimise some total cost or to maximise some total benefit derived from the system. There is always a trade-off between several conflicting interests at work. For instance, the manager of the firm tries to reach high production and high profits through an accumulation of firm capital, but investment is costly; the local government dealing with the lake has to weigh the interests of the farmers, whose use of artificial fertiliser pollutes the lake, against those of tourists and water companies, who prefer very much a clean lake. Finally, the ship's captain tries to reach the harbour quickly, but without using too much fuel.

The optimal choice for the control is obtained from the solution of a certain dynamical system, the state-costate system, that has twice as many variables as the dimension of the system's state. For the economic applications that we are interested in, this system is a perturbation of an autonomous Hamiltonian system, where the perturbation term is related to the amount of discounting of future costs relative to present costs. In this case, the autonomous system acts as a principal part.

Pollution of a lake. In the lake system as investigated in [89], there are three basic parameters: the sedimentation rate b, which determines the speed at which the pollutant moves from the water in the lake to the sediment at the lake's bottom; the economic parameter c, which determines the relative weight of the tourists' interest relative to those of the farmers; and the discount rate ρ described above.

As the sedimentation rate is a physical characteristic of the lake, in a first analysis this is kept fixed. Also the discount rate is fixed, giving us a one-dimensional 'slice' or 'probe' of the three-dimensional parameter space. For typical values of b and ρ, there are then three different solution structures: for low values of c, it is always optimal to let the lake serve as a waste dump, leading to a steady state of the lake where it is highly polluted. For high values of c, it is likewise always optimal to restrict severely the use of fertiliser on the farmland surrounding the lake, such that the lake can reach a clean steady state at which there is but little pollutant in the lake. Finally, for intermediate values of c, one of the two options described is optimal, but now this depends on the initial state of the lake.

It turns out that the separate regions are bounded by two codimension one heteroclinic bifurcations of the state-costate system. If we now consider the system in the full three-dimensional parameter space, the corresponding heteroclinic bifurcation surfaces end on two saddle-node surfaces of equilibria of the state-costate system, that in turn meet in a curve of cusp bifurcations. All these bifurcation curves are finally seen to originate in a Hamiltonian cusp bifurcation, which, in this special set-up, is a codimension three organising centre. Note that the situation here is different from a 'generic' dynamical system, where a Hamiltonian bifurcation would be of infinite codimension; the difference is due to the special structure of economic optimisation problems.

Optimal investment. The optimal investment problem of a monopolistic firm investigated in [90] is analysed similarly. Here there are four parameters; the discount rate ρ and three more parameters that model the 'firm structure', that is the structure of the firm's costs and revenues. There are now three different possible steady states, corresponding to a firm of small, medium and large size, which may be reached by optimal investment schedules. The structure of the bifurcation diagram that summarises which firm size will be obtained optimally is suitably complicated, but it is as in the Lake Pollution problem organised by a single bifurcation, in this case a Hamiltonian swallowtail that has codimension 4 in the set-up of an economic optimisation problem. From the codimension 4 point, there is a curve of 'ordinary' codimension 3 swallowtail catastrophes projecting into our three-dimensional space of parameters. See Figure 1.1 for the ordinary swallowtail catastrophe.

5 Concluding Remarks

We briefly summarize the program of this research, which mainly sits within the 'general dissipative' setting without preservation of structure. In other cases (Hamiltonian, volume preserving, reversible, equivariant) the program has to be adapted, e.g., compare [22, 20].

A general and widely used approach to dynamical systems is the following, as reflected in the textbooks [52, 5]; also compare with, e.g., [92, 6, 19, 35, 49, 50, 77, 81, 80, 59, 60, 87].

The first step is to identify and analyse a relevant principal part of the system, obtained by projections and truncations of the Taylor-Fourier series and in such a way that the original system can be viewed as a perturbation of this principal part. Next, in the dissipative context, the attention goes to the structurally stable or Morse-Smale 'strata' and a corresponding bifurcation analysis, beginning with the principal part.

Our main message is that the analysis of the principal part often benefits from considering more parameters. Certain semi-global arrays of bifurcations turn out to be subordinate to one or more bifurcations of higher codimension.

Next, conclusions from the principal part analysis to bifurcations of the original systems run by perturbation theory, often containing an application of Singularity Theory and an underlying universal problem.

A second message then is how this bifurcation analysis of the principal part gets complicated when periodic and quasi-periodic perturbations are applied. This complication and fine-structuring is due to resonances and, generically, the effects on the bifurcation diagrams can be well-described, [15, 24, 23, 20, 18, 22, 38, 39, 40, 54, 88, 75]. We note that by the information thus obtained, it is often possible to identify parameter regions where chaos might be found, compare with §3.2 and, e.g., [29, 32, 33].

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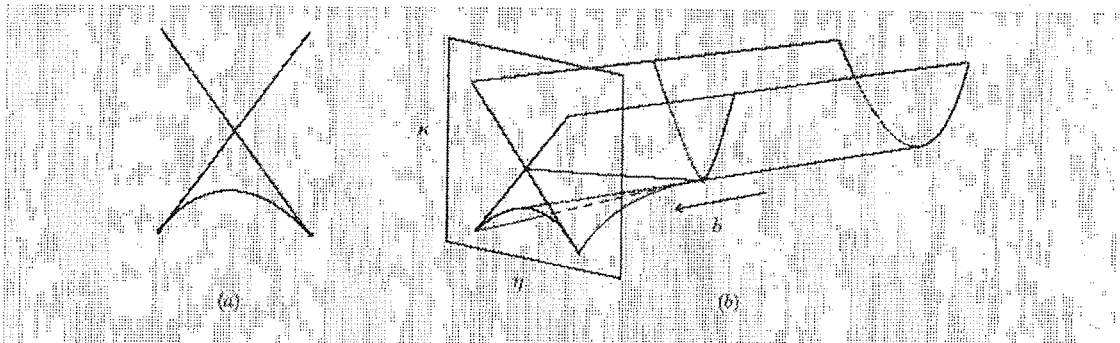

Figure 1: (*a*): Two-dimensional diagram with semi-global array of cusps and folds. (*b*): Three-dimensional diagram with swallowtail as organising centre.

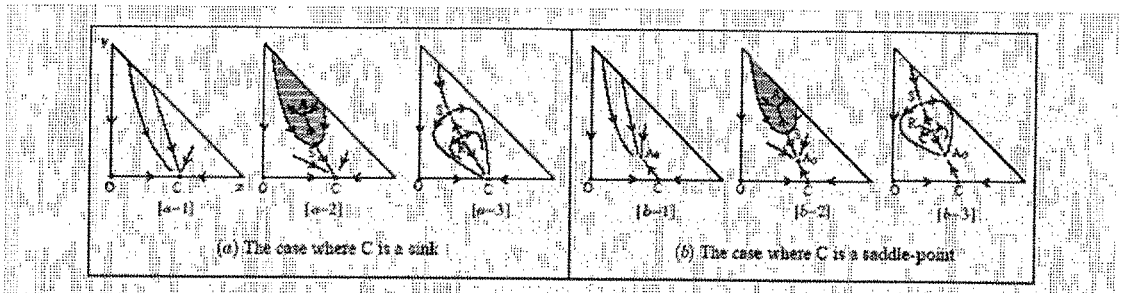

Figure 2: Reduced Morse-Smale portraits occurring in system (2.3): A is a sink (the corresponding basin is dashed), S is a saddle-point and R a source. (*a*): The case where C is a sink (with corresponding basin in white). (*b*): The case where C is a saddle-point. In the latter case the interior of the trapping domain always contains an attractor denoted by A_0 with basin in white. Bi-stability only occurs in portraits [*a*-2] and [*b*-2].

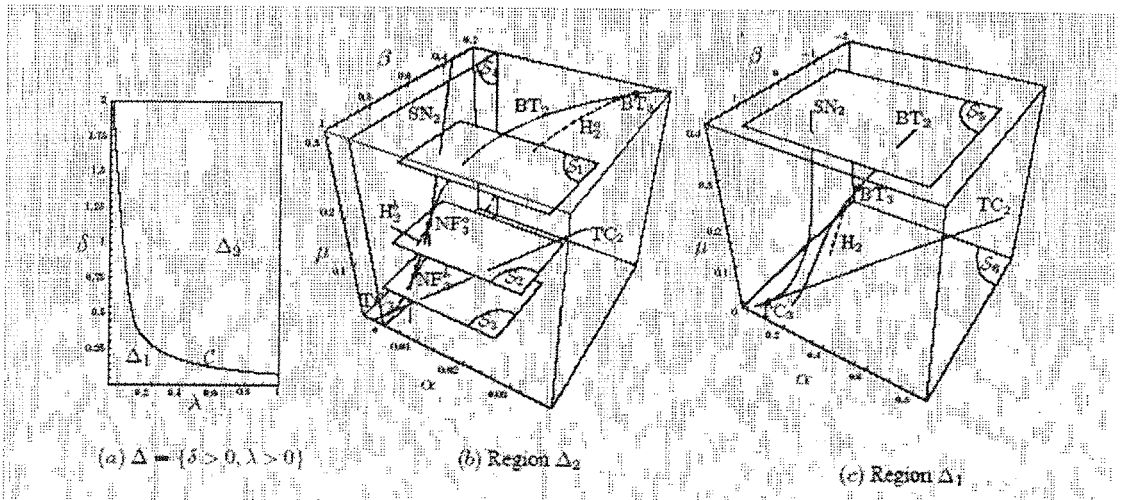

Figure 3: (*a*): Region $\Delta = \{\delta > 0, \lambda > 0\}$. (*b*): Bifurcation set in $\mathcal{W} = \{\alpha \geq 0, \beta > -2\sqrt{\alpha}, \mu \geq 0\}$ when $(\delta, \lambda) \in \Delta_1$. Section $\mathcal{S}_4 = \{\alpha = 0\}$ is the two-dimensional section associated to the bifurcation diagram of Bazykin's model [63]. (*c*): Similar to (*b*) for the case $(\delta, \lambda) \in \Delta_2$. Section $\mathcal{S}_6 = \{\mu = 0\}$ covers the case of Zhu's model [95]. For terminology see Table 1.

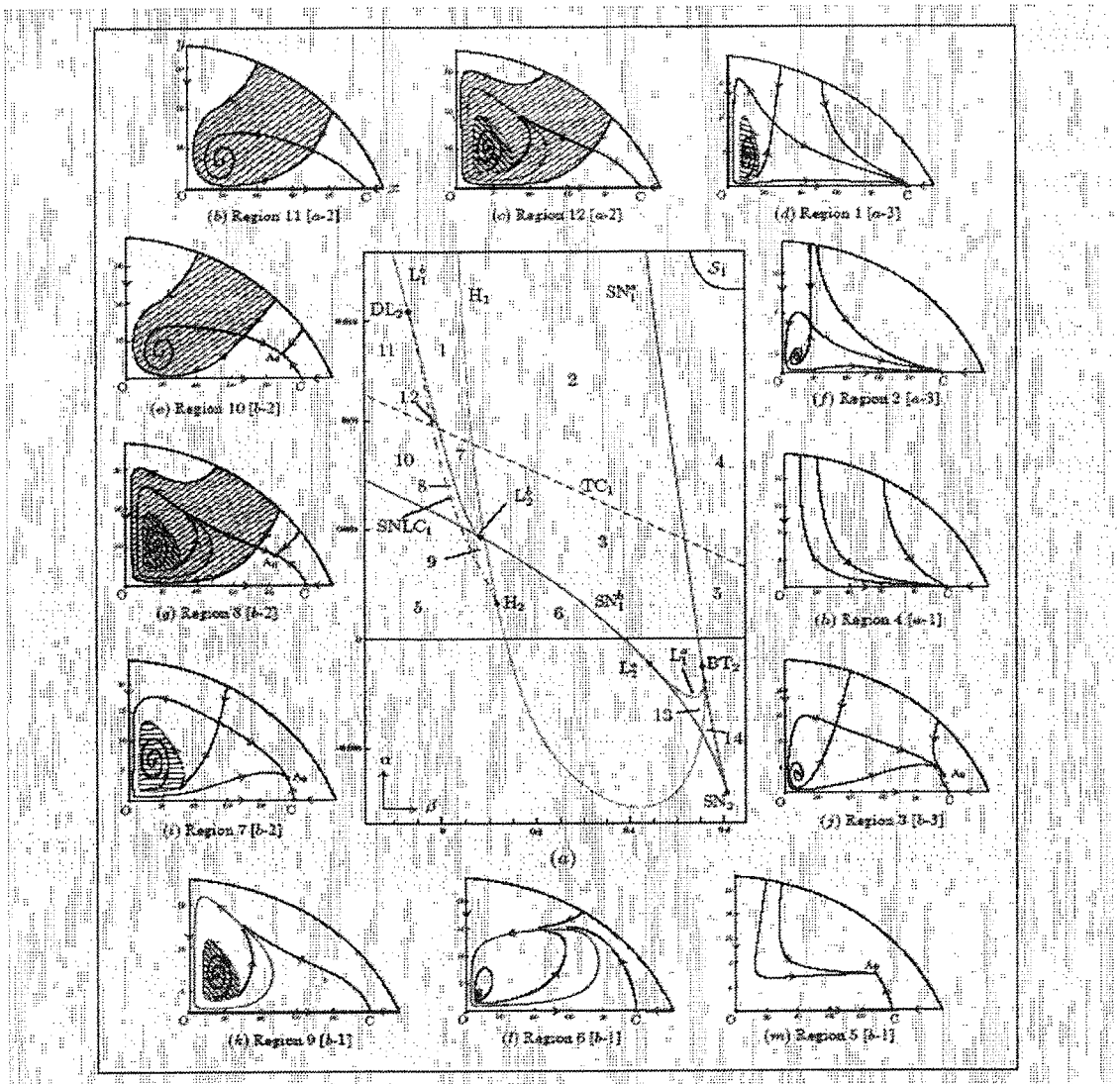

Figure 4: (*a*). Bifurcation diagram in two-dimensional section $\mathcal{S}_1 \subset \{\mu = 0.1\}$ of Figure 3-(b), $(\delta, \lambda) = (1.01, 0.01) \in \Delta_1$. A codimension 2 transcritical point lies on the curve TC_1 but is not mentioned since it occurs for $\beta \ll -\sqrt{\alpha}$. Note the presence of a cusp point (SN_2) and a Bogdanov-Takens point (BT_2) below $\{\alpha = 0\}$ both acting as organising centres. These points are depicted for a better understanding of the bifurcation set. (*b*)-(*m*). Associated phase portraits referring to the corresponding Reduced Morse-Smale portraits of Figure 2. Bi-stability in regions 8 and 9 both correspond to [*b*-2] in Figure 2. Bi-stability also holds in region 10 (corresponding to [*b*-1], which does not show bi-stability). The white basin of attraction is either for C or for A_0, while the shaded ones are for other attractors. For terminology see Table 1.

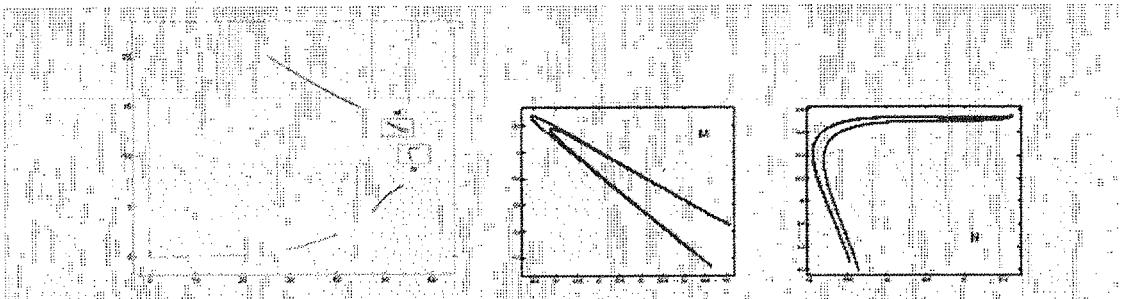

Figure 5: Left: Strange attractor for P_ε for parameter values in region 9 of Figure 4 using 500 000 iterations, $(\alpha, \beta, \mu, \delta, \lambda) = (0.007, 0.036, 0.1, 1.01, 0.01)$, $w = 1$ and $\varepsilon = 0.6$. The attractor consists of 11 connected components mapped by P_ε to one another in a cyclic way. Middle and Right: Magnifications of boxes M and N in the figure on the left.

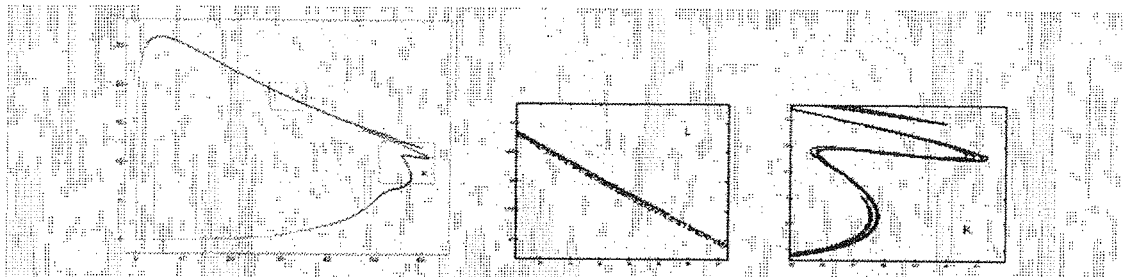

Figure 6: Left: Viana-like strange attractor for P_ε for parameter values in region 9 of Figure 4 using 500 000 iterations, $(\alpha, \beta, \mu, \delta, \lambda) = (0.007, 0.036, 0.1, 1.01, 0.01)$, $\omega = 1$ and $\varepsilon = 0.99$, compare [30]. Middle and Right: Magnifications of boxes L and K in the figure on the left, respectively.

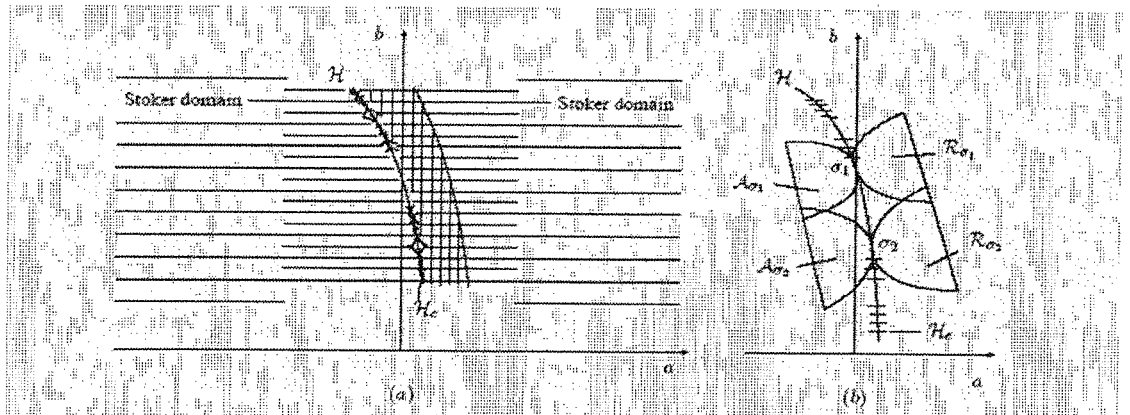

Figure 7: (a): Sketch of global bifurcation set $\mathcal{H}_c \subset \mathcal{H}$ of the quasi-periodic Hopf bifurcation in the context of the Duffing – Van der Pol example (3.1) with Stoker m-tori (shaded) and $(m+1)$-tori (doubly shaded) using $\sigma = (a, b)$, compare with [15]. (b): Detail with resonance hole or bubble between discs $\mathcal{A}_\sigma$ and $\mathcal{R}_\sigma$ attached to different $\sigma \in \mathcal{H}_c$.

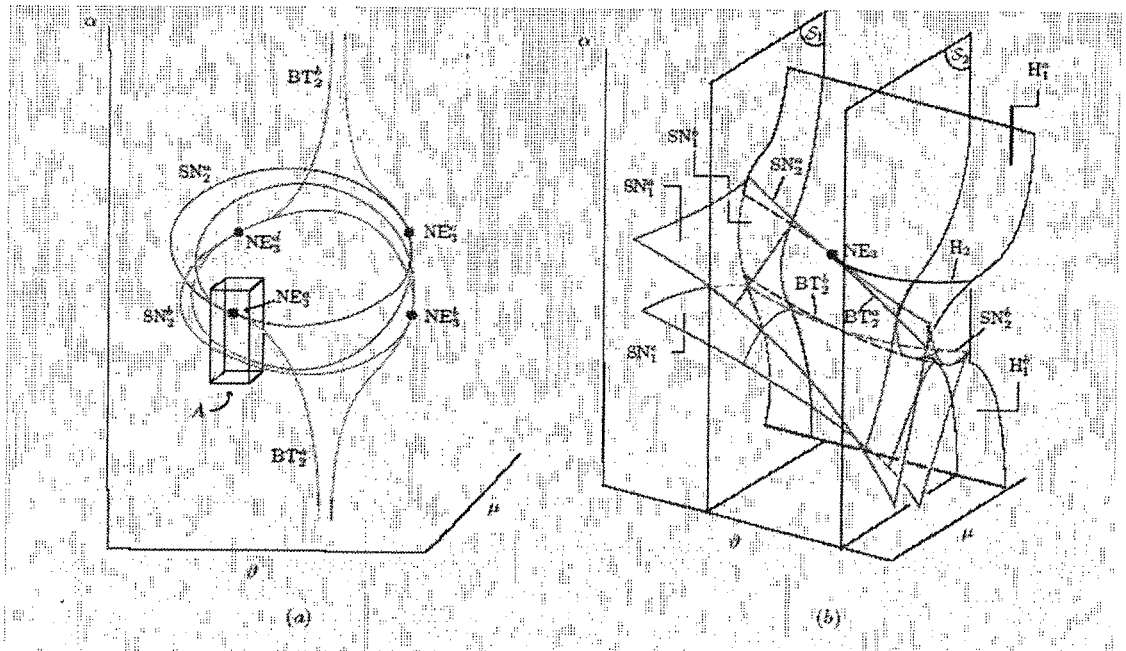

Figure 8: (a): Sketch of the global bifurcation diagram of $\dot{z} = (\mu + i\alpha)z + e^{i\theta}|z|^2 z + 1$ in the (θ, μ, α)-parameter space. All singularity of nilpotent-elliptic points (NE_3) are connected by cusp (SN_2) and Bogdanov-Takens (BT_2) lines. The curves BT_2^a and BT_2^b tend to $\pm\infty$, respectively, when θ goes to $\pi/2$. (b): Detail of the bifurcation set in box A of figure (a). At the singularity of nilpotent-elliptic type (NE_3) point, curves of Bogdanov-Takens (BT_2^b), cusp (SN_2^b), and degenerate Hopf (H_2) bifurcations meet tangently. The curves BT_2^a and SN_2^a do not intersect. For terminology see Table 1.

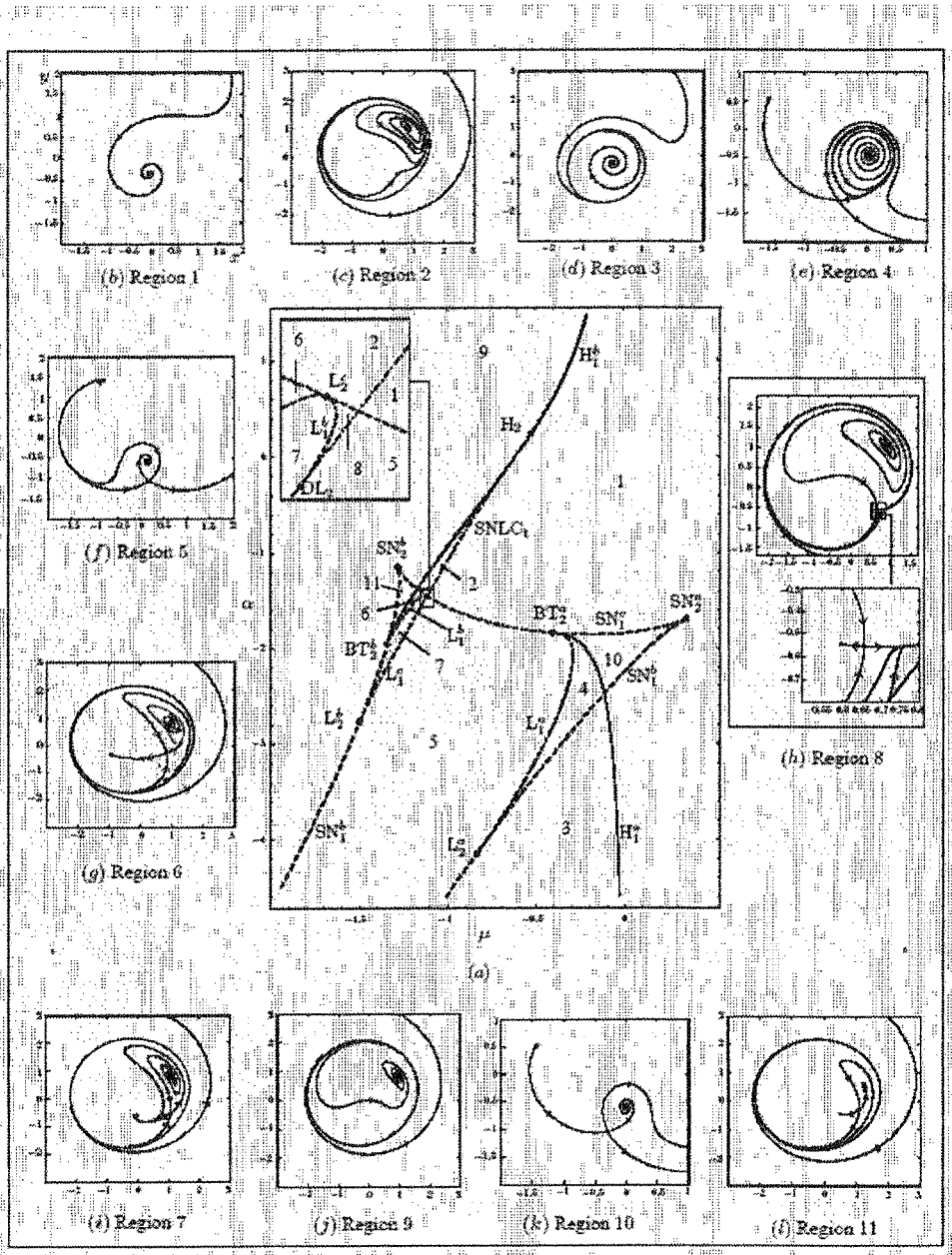

Figure 9: (a): Bifurcation diagram of (3.6) for $\vartheta = 2\pi/5$. (b)-(l): Phase portraits for parameters in the different regions of the bifurcation diagram of the family (3.6). For regions 1, 2, 3 and 9, single orbits, leaving an unstable equilibrium or unstable periodic orbits are shown. For all other regions, stable and unstable manifolds of saddle points are drawn. For terminology see Table 1.

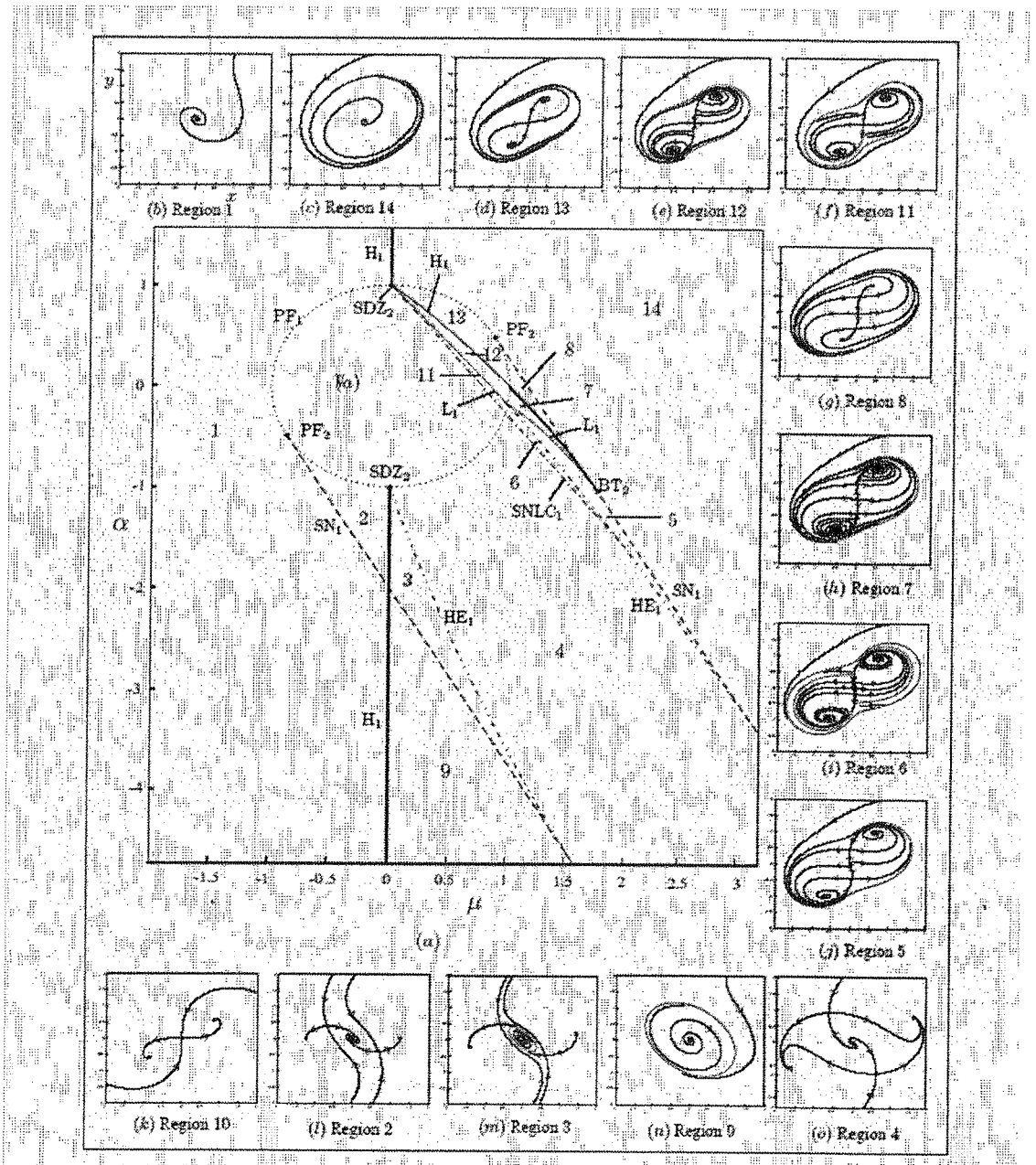

Figure 10: (a): Bifurcation diagram of $\dot{z} = (\lambda_1 + i\lambda_2)z + e^{i\theta}|z|^2 z + \bar{z}$ for $\theta = 2\pi/3$, compare with [45, 61, 62, 70, 84]. (b) – (o): Generic phase portraits for different regions in the parameter space. For terminology see Table 1.

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The existence and the uniqueness of solutions to nonlinear functional-differential and integral equations

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ABSTRACT

The aim of this paper is to present a collection of recent results, especially concerning the problems of the existence and the uniqueness of solutions to some classes of functional-differential and integral equations. In Section 1 we present results concerning differential equations (especially concerning the equation of n-th order) formulated in terms of approximate derivatives. This type of a derivative seems to be one of the most important generalizations of the classical one. The second section contains the results concerning the existence and the uniqueness of solutions to nonlinear Hammerstein as well as to nonlinear Volterra - Hammerstein integral equation in the classes of functions of bounded variation in the sense of Jordan and Young. In Section 3 we present the results concerning the existence and the uniqueness of solutions to some equations of neutral type, especially concerning the Darboux problem. In the next section we deal with almost periodic solutions to differential and integral equations in Fréchet spaces. Moreover, we present there Bohr - Neugebauer type theorems for almost periodic solutions to linear equations in such spaces. Finally, in the Appendix we present fixed point theorems for locally convex spaces as well as for Banach spaces, which can be tools in investigations concerning the existence of solutions to nonlinear differential and integral equations.

Keywords: α-nonexpansive mapping, almost periodic function, Aronszajn property, asymptotically almost periodic function, Cauchy problem, Darboux problem of neutral type, Denjoy integral, Denjoy - Perron integral, differential equations with maxima, existence and uniqueness of solutions, fixed point theorems, function of bounded variation in the sense of Jordan, function of bounded variation in the sense of Young, functional partial differential equation, Hammerstein integral equation, Henstock - Kurzweil integral, Kuratowski measure of noncompactness, measure of weak noncompactness, nonexpansive mapping, perfect Fréchet space, spectral radius, Volterra integral equation, Volterra - Hammerstein integral equation, ω-nonexpansive mapping, weakly F-contractive mapping.

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1 Differential equations and approximate derivatives. Applications of nonabsolute convergent integrals

The theory of an integral introduced by Denjoy (Denjoy, 1912) and Perron (Perron, n.d.) allows us to integrate an arbitrary derivative, that is for such an integral the following formula

$$\int_a^b f'(s)ds = f(b) - f(a)$$

holds for any differentiable function $f : [a, b] \to \mathbb{R}$. Over fifty years later Kurzweil (Kurzweil, 1957) in 1957 and, independly, Henstock (Henstock, 1961) in 1961 proved that the Denjoy - Perron integral can be defined as a modification of the Riemann original definition.
In (Kurzweil, 1957) Kurzweil, as the first, has applied this type of an integral to investigations of the existence of solutions to classical Cauchy problem

$$x' = f(t, x), \qquad x(t_0) = x_0. \tag{1.1}$$

In this case these solutions are usually called generalized ones. These investigations have been conducted for example by Bullen, Výborný (Bullen and Výborný, 1991), Chew, Flordeliza (Chew, 1991), Henstock (Henstock, 1988) and many others.
W 1993 Schwabik (Schwabik, 1993) proved that all known conditions which guarantee the existence of generalized solutions to the problem (1.1), concern the case in which the right - hand side of the equation (1.1) is the sum of a Carathéodory function and a function integrable in the Denjoy-Perron sense.
However, there where not known papers which would describe the topological structure of solution sets to the problem (1.1) and to other ones, to investigations of which, nonabsolute convergent integrals would have been applied. Moreover, in equations of such a type, the methods of the fixed point theory were applied rather rarely.
In (Bugajewski and Szufla, 1995) we proved that under the assumptions considered by Schwabik, an Aronszajn property holds for the problem (1.1) (see (Aronszajn, 1942)), that is , there exists an interval J such that the set of all generalized solutions to the problem (1.1), defined on J and considered as a subset of the space $C(J, \mathbb{R})$ of all continuous functions $J \to \mathbb{R}$ with the topology of uniform convergence, is an R_δ. This means that it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. In particular, it is nonempty, compact and connected.
Moreover, in this paper we extended an application of Denjoy- Perron integrals (more precisely the Perron-Ward integral ((Kurzweil, 1980), Def. 4.12)) to the case of partial - differential equations. More precisely, let us consider the following Darboux problem

$$\frac{\partial^2 z}{\partial x \partial y} = f(x, y, z),$$
$$z(x, 0) = 0, \qquad z(0, y) = 0, \tag{1.2}$$

in which the right - hand side satisfies assumptions of the Schwabik type. We proved that for generalized solutions to the problem (1.2), the Aronszajn property holds, too.

On the other hand, in the late 80-ties there were published papers concerning the existence of solutions to the problem (1.1), in which the usual derivative was replaced by the approximate one (see (Saks, 1937) for the definition).

This type of a derivative seems to be one of the most essential generalization of the classical one (see (Saks, 1937)). Recall that it is strongly connected with the notion of the integral introduced by Denjoy (Denjoy, 1916) in 1916.

For example Bullen and Sarkhel (Bullen and Sarkhel, 1987) obtained a generalization of the classical Picard result to the problem

$$x'_{ap} = f(t,x), \qquad x(t_0) = x_0, \tag{1.3}$$

where x'_{ap} (or $(\frac{dx}{dt})_{ap}$) stands for the approximate derivative. Moreover, they gave an example of the function f such that the problem (1.3) has a locally bounded unique solution at $(0,0)$, but the problem (1.1), generated by the same function f, has no solution passing through $(0,0)$.

The investigations started by Bullen and Sarkhel were continued e.g. in the paper (Bugajewski and Szufla, 1995). In particular, we proved in this paper that if f satisfies the Carathéodory conditions and there exist two real valued functions g, h, which are integrable in the Denjoy sense and such that

$$g(t) \leq f(t,x) \leq h(t), \tag{1.4}$$

then the set of all generalized solutions to the problem (1.3), defined on a common interval, is an R_δ. In the paper (Bugajewski and Wójtowicz (Bugajewska), 1996), we considered the following initial value problem for the equation of $n-th$ order with the approximate derivative

$$\left(\frac{dx}{dt}\right)^{(n)}_{ap} = f(t,x), \qquad x^{(i)}_{ap}(t_0) = x_i, \qquad i = 0,\ldots,n-1, \tag{1.5}$$

where $I = [t_0, t_0+a]$, $B = \{x \in \mathbb{R} : \|x - x_0\| \leq b\}$, $a, b > 0$ and $f : I \times B \to \mathbb{R}$.

By a generalized solution to the problem (1.5), defined on an interval $J \subset I$, we understand a function $x : J \to \mathbb{R}$ such that $x(t) \in B$ for $t \in J$, $x^{(n-1)}_{ap}$ is an ACG (see (Saks, 1937)), $\left(\frac{dx}{dt}\right)^{(n)}_{ap}(t) = f(t,x(t))$ for almost all $t \in J$ and $x^{(i)}_{ap}(t_0) = x_i$, $i = 0,1,\ldots,n-1$. Such a solution can be equivalently defined in an integral form, that is as a continuous solution to the following Volterra-Hammerstein integral equation

$$x(t) = \sum_{i=0}^{n-1} \frac{(t-t_0)^i}{i!} + \frac{1}{(n-1)!}(D)\int_{t_0}^{t} (t-s)^{n-1} f(s,x(s))ds, \quad t \in J, \tag{1.6}$$

where the sign "$(D)\int_{t_0}^{t}$" stands for the Denjoy integral.

The main result proved in (Bugajewski and Wójtowicz (Bugajewska), 1996) is the following

Theorem 1. Let $f : I \times B \to \mathbb{R}$ be a function such that

(i) $t \to f(t,x)$ is measurable in the Lebesgue sense for every $x \in B$;

(ii) $x \to f(t,x)$ is continuous for almost all $t \in I$;

(iii) there exist two functions $m : I \to \mathbb{R}$, $M : I \to \mathbb{R}$ integrable in the Denjoy sense and such that

$$m(t) \leq f(t,x) \leq M(t) \quad for\ every\ pair\ (t,x) \in I \times B.$$

Under the above assumptions there exists an interval $J \subset I$ such that the set of all generalized solutions to the problem (1.5), defined on J, is an R_δ.

The proof of this result is based on the Vidossich connectness principle ((Vidossich, 1971)). This pinciple allows to obtain information about the existence of a generalized solution to a problem (1.5) as well as about the topological structure of solutions under consideration. However, note that to obtain the existence of a generalized solution, it would be enough to apply the Schauder fixed point theorem.

The proof of Theorem 1 relies on investigation of the set of fixed points of the Volterra type operator, generated by the right-hand side of the equation (1.6). In particular, this proof shows how to omit one of the basic difficulties which appear when one applies the Denjoy integral to equations formulated in terms of approximate derivatives: namely, for the Denjoy integral does not hold an analogue of the classical Fubini theorem.

A natural question concerns the existence of generalized solutions to equations with partial approximate derivatives (see (Celidze and Dzvarsheishvili, 1987) for the definition). One of the first papers concerning this problem seems to be the article (Bugajewski and Szufla, 1995), in which we investigated the existence and the topological structure of generalized solutions sets to the problem (1.2), in which the second mixed derivative was replaced by the second mixed approximate one.

Also in this case the problem of the existence of a generalized solution was replaced by investigation of properties of fixed points to a certain nonlinear integral operator of Volterra type, defined by the appropriate Denjoy integral.

The paper (Bugajewski and Wójtowicz (Bugajewska), 1996) has inspired other authors for further investigations. For example, in the paper (Bugajewska and O'Regan, n.d.) the authors investigated the problem of the existence of a generalized solution to the equation (1.5), which is included between two functions: s^1 (a lower solution) and s^0 (an upper solution).

Recall also that in the paper (Bugajewski, 1998) the problem of the existence as well as the topological structure of generalized solutions to the nonlinear Volterra integral equation with the Denjoy-Perron integral (equivalently with the Henstock-Kurzweil integral) has been investigated.

In connection with Schwabik's article (Schwabik, 1993) concerning the problem (1.1), the following question still remains open:

How to characterize the class of functions, which is described by the assumptions of the main result from (Bugajewski, 1998)?

2 Solutions to nonlinear integral equations in classes of functions of different types bounded variation

Consider the nonlinear Hammerstein integral equation

$$x(t) = g(t) + \lambda \int_I K(t,s) f(x(s)) ds \quad \text{for } t \in I \text{ and } \lambda \in \mathbb{R}, \tag{2.1}$$

and the nonlinear Volterra-Hammerstein integral equation

$$x(t) = g(t) + \int_0^t K(t,s) f(x(s)) ds \quad \text{for } t \in I, \tag{2.2}$$

where $I = [0, a]$ is a compact interval in $\mathbb{R}$.

We are interested in the problems of the existence and the uniqueness of solutions to the equations (2.1) and (2.2), in the classes of functions of bounded variation in the sense of Jordan and Young. Up to now these solutions have not been widely investigated in the literature. There exist at least two motivations to deal with such solutions.

First, let us recall that if $h : I \to \mathbb{R}$ is a function integrable in the Denjoy-Perron sense, then $h\phi$ is integrable in this sense, if $\phi : I \to \mathbb{R}$ is a function of bounded variation in the sense of Jordan. Hence it is natural to investigate solutions or, in particular, continuous solutions to the equations (2.1) and (2.2) with the additional requirement that these solutions are functions of bounded variation on the sense of Jordan.

The other motivation to consider such solutions comes from the real world. Namely, it appears that solutions to many integral equations which describe concrete physical phenomena are very often functions of bounded variation in the sense of Jordan.

The superposition operator in the space of functions of bounded variation in the sense of Jordan has been investigated e.g. in (Appell and Zabrejko, 1990), (Josephy, 1981) and (Ljamin, n.d.). Since there are not known general results concerning the superposition operator in the nonautonomous case, that is in the case $f = f(s,x)$, apart from trivial suffficient conditions, we have considered only the autonomous case, that is the case in which the function f depends on one variable only.

In the paper (Bugajewski, 2003b), we formulated the general conditions, which guarantee the existence and the uniqueness of local and global solutions to the equations (2.1) i (2.2) in the space of functions of bounded variation in the sense of Jordan or in the space of continuous functions of bounded variation in the sense of Jordan. In this paper we considered the equations with the Lebesgue integral as well as with the Lebesgue-Bochner integral. One of the basic results from the paper (Bugajewski, 2003b) is the following result concerning the Hammerstein equation (2.1):

Theorem 2. Assume that

1^0 $g : I \to \mathbb{R}$ is a function of bounded variation in the sense of Jordan;

2^0 $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function;

3^0 $K : I \times I \to \mathbb{R}$ is a funtion such that $\bigvee_0^a(K(\cdot, s)) \leq M(s)$ for almost all $s \in I$, where $M : I \to \mathbb{R}_+$ is a function integrable in the Lebesgue sense and $K(t, \cdot)$ is a function integrable in the Lebesgue sense for every $t \in I$.

Then there exists a number $\rho > 0$ such that for every λ with $|\lambda| < \rho$, the equation (2.1) has a unique solution of bounded variation in the sense of Jordan, defined on I.

For the Volterra-Hammerstein equation (2.2) we proved the following result about the existence and the uniqueness of solutions in the class of functions of bounded variation in the sense of Jordan.

Theorem 3. Suppose 1^0 and 2^0 are satisfied. Moreover, assume that

4^0 $T = \{(t, s) : 0 \leq t \leq a, 0 \leq s \leq t\}$ and $K : T \to \mathbb{R}$ is a function such that $|K(s, s)| + \bigvee_s^a(K(\cdot, s)) \leq m(s)$ for almost all $s \in I$, where $m : I \to \mathbb{R}_+$ is a function integrable in the Lebesgue sense and $K(t, \cdot)$ is a function integrable in the Lebesgue sense on the interval $[0, t]$ for every $t \in I$.

Under the above assumptions there exists an interval $J \subset I$ such that the equation (2.2) has exactly one solution of bounded variation in the sense of Jordan, defined on J.

If, under the assumptions of Theorem 2, we assume additionally that

5^0 $g : I \to \mathbb{R}$ is a continuous function;

6^0 for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t, \tau, s \in I$ the following implication holds:

$$|\tau - t| < \delta \Longrightarrow |K(\tau, s) - K(t, s)| < \varepsilon,$$

then the solution described in Theorem 2, is also a continuous function on the interval I. In a similar manner, we can deal with Theorem 3.

In the paper (Bugajewski and D. O'Regan, 2003) we dealt with the existence of global solutions to the Volterra-Hammerstein as well as to the Hammerstein integral equation (without the parameter), in the class of functions of bounded variation in the sense of Jordan. We considered there the equations with the Lebesgue as well as with the Denjoy-Perron integral. We proved the following result concerning the equation (2.1) (without the parameter) with the Lebesgue integral.

Theorem 4. Assume that the assumptions 1^0 and 3^0 are satisfied. Moreover, assume that

7^0 $f : \mathbb{R} \to \mathbb{R}$;

8^0 there exists a function $\Psi : [0, +\infty) \to [0, +\infty)$ such that $\Psi(u) > 0$ for $u > 0$ and $\sup_{s \in [0,a]} |f(x(s))| \leq \Psi(\|x\|_{BV})$ for every $x \in BV(I)$, where $BV(I)$ denotes the space of all functions x defined on I such that the variation in the sense of Jordan $\bigvee_0^a(x) < +\infty$, with the norm $\|x\|_{BV} = |x(0)| + \bigvee_0^a(x)$;

9^0 there exists $M_0 > 0$ such that $M_0(\|g\|_{BV} + \Psi(M_0) \cdot T_0) > 1$, where $T_0 = \int_0^a (|K(0,s)| + M(s))ds$;

10^0 there exists a continuous and nondecreasing function $\phi_{M_0} : [0, +\infty) \to [0, +\infty)$ such that $T_0\phi_{M_0}(z) < z$ for $z > 0$ and $|f(x) - f(y)| < \phi_{M_0}(|x - y|)$ for $|x|, |y| \leq M_0$.

Under the above assumptions the equation (2.1) has a solution of bounded variation in the sense of Jordan, defined on the interval I.

A similar result we obtained also for the Volterra- Hammerstein equation (2.2). We also proved results concerning the existence of continuous solutions (2.1) (without parameter) and to the equation (2.2), defined on the interval I.

It is interesting to note that the proofs of the results from the paper (Bugajewski and D. O'Regan, 2003) are based on the following Leray- Schauder alternative for contraction (see (O'Regan, 1996)):

Let U be an open subset of a Banach space $(E, \|\cdot\|)$ such that $0 \in U$ and let $\bar{U}$ denote the closure of the set U in the space E. Assume that $F : \bar{U} \to E$ and there exists a nondecreasing continuous function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the inequality $\phi(z) < z$ for $z > 0$ and such that for $x, y \in \bar{U}$ we have $\|F(x) - F(y)\| \leq \phi(\|x - y\|)$. Moreover, assume that the image $F(\bar{U})$ is bounded and $x \neq \lambda F(x)$ for $x \in \partial U$ and $\lambda \in (0, 1]$, where ∂U denotes the boundary of U in E. Then F has exactly one fixed point in U.

In the paper (Bugajewski and D. O'Regan, 2003) we have investigated also the equations (2.1) (without parameter) and (2.2), defined via the Denjoy-Perron integral. We proved a collection of theorems concerning global solutions, in particular continuous solutions to these equations in the class of functions of bounded variation in the sense of Jordan. Note that the proofs of these results are more complicated from technical point of view than in the case of equations defined via the Lebesgue integral. In particular, it follows from the fact that nonabsolute convergent integrals, for example the Denjoy-Perron integral do not have some nice properties which characterize the classical Lebesgue integral. In particular, the absolute value of a function integrable in the Denjoy-Perron sense, does not have to be integrable in this sense.

Recall that the classical variation in the sense of Jordan has many generalizations. One of its more important generalizations is the so-called ϕ-variation introduced by Young (Young, 1937) in connection with his investigations concerning the behaviour of Fourier series. Moreover, recall that the space of functions of bounded ϕ-variation was investigated by Musielak and Orlicz (Musielak and Orlicz, 1959), and by Leśniewicz and Orlicz (Leśniewicz and Orlicz, 1973). These investigations are still extensively continued (see e.g. (Cai and Hudzik, 2001)).

In the paper (Bugajewska, Bugajewski and Hudzik, 2003) we have investigated solutions, in particular continuous solutions to nonlinear Hammerstein as well as to Volterra-Hammerstein integral equation in the class of functions of bounded ϕ-variation (we have considered solutions defined locally and globally).

Let $\phi : [0, +\infty) \to [0, +\infty)$ be a ϕ-function (that is ϕ is continuous, unbounded, nondecreasing and $\phi(u) = 0$ if and only if $u = 0$). It is well-known that if ϕ is a convex function, $\phi(0) = 0$ and ϕ

is nontrivial, then the following class of functions

$$BV_\phi(I) = \{x \in X : \text{var}_\phi(\nu x) < +\infty \text{ for some } \nu > 0\}$$

with the norm $\|x\|_{V_\phi} = \inf\{\varepsilon > 0 : \text{var}_\phi(\frac{x}{\varepsilon}) \leq 1\}$ is a Banach space (X denotes here the vector space of all real valued functions, defined on the interval I and such that $x(0) = 0$, and $\text{var}_\phi(x)$ denotes the ϕ-variation of a function x on the interval I; for simplicity we put here $a = 1$). Elements of the space $BV_\phi(I)$ are called BV_ϕ-functions, and solutions to the equations (2.1) and (2.2), belonging to the space $BV_\phi(I)$, are called BV_ϕ-solutions.
Assume that

11^0 $g \in X$ is a BV_ϕ-function;

12^0 $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function;

13^0 $K : I \times I \to \mathbb{R}$ is a function such that $K(t, \cdot)$ is a function integrable in the Lebesgue sense for every $t \in I$, $K(0, s) = 0$ and there exists a number $\alpha > 0$ such that $\text{var}_\phi(\frac{K(\cdot,s)}{\alpha}) \leq M(s)$ for almost all $s \in I$, where $M : I \to \mathbb{R}_+$ is a function integrable in the Lebesgue sense.

One of the basic results proved in the paper (Bugajewska et al., 2003) is the following

Theorem 5. Under the above assumptions there exists a number $\rho > 0$ such that for every λ with $|\lambda| < \rho$, the equation (2.1) has a unique BV_ϕ- solution, defined on I.

Now, suppose that a ϕ-function is convex and satisfies the classical Δ_2-condition. By $\text{var}_\phi|_s^1(x)$, where $x \in X$, we will denote the ϕ-variation of the function x on the interval $[s, 1]$, where $0 \leq s \leq 1$.
Assume that

14^0 $T = \{(t, s) : 0 \leq t \leq 1,\ 0 \leq s \leq t\}$ and $K : T \to \mathbb{R}$ is a function such that $K(t, \cdot)$ is integrable in the Lebesgue sense on the interval $[0, t]$ for every $t \in I$ and there exists a number $\alpha > 0$ such that $\phi(\frac{|K(s,s)|}{\alpha}) + \text{var}_\phi|_s^1(\frac{K(\cdot,s)}{\alpha}) \leq m(s)$ for almost all $s \in I$, where $m : I \to \mathbb{R}_+$ is a function integrable in the Lebesgue sense.

Theorem 6. Suppose the conditions 11^0, 12^0 and 14^0 are fulfield. Then there exists an interval $J \subset I$ such that the equation (8) has a unique BV_ϕ-solution, defined on the interval J.

Let us emphasize that the results from the paper (Bugajewska et al., 2003) are connected with the theory of integral equations in modular spaces (see e.g. (Jedryka and Musielak, 1976) and (Musielak, 1983), Chap. V, §21).

3 The existence and the uniqueness of solutions to some equations of neutral type

The differential equations of neutral type are an important class of equations with a deviated argument. These equations appeared in mathematics many years ago, however, for last forty years this theory has focused a particular attension and it is developed by many specialists (let us emphasize that one of the basic articles concerning this topic is Myszkis' paper (Myszkis,

1950)). It is closely connected with wide applications of these equations especially in the theory of optimal control, mathematical biology and mathematical economy.
In papers (Bugajewski and Zima, 1996) and (Bugajewski, 1997), we investigated the existence and the uniqueness of global solutions to the Darboux problem of neutral type

$$\begin{aligned} &z_{xy} = f(x,y,z(h(x,y)),z_{xy}(H(x,y))) \quad \text{for } (x,y)\in I^2,\\ &z(x,0)=0 \qquad\qquad \text{dla } x\in I,\\ &z(0,y)=0 \qquad\qquad \text{dla } y\in I, \end{aligned} \tag{3.1}$$

where z_{xy} denotes the second mixed derivative and $I=[0,a]$, $a>0$.
The most classical tool which is applied in the existence and the uniqueness results for problems of different types, is the Banach contraction principle. However, note that direct applications of this nice theorem, frequently lead to quite restricted assumptions (see e.g. (Bugajewski, 1997), Remark 3). Therefore in the papers (Bugajewski and Zima, 1996), (Bugajewski, 1997) and (Bugajewski and Zima, 2003) we utilized other method which relies on application of a certain extension of the Banach principle for quasi-ordered Banach spaces endowed with a quasi-modul, which was proved in the paper (Zima, 1992) (Th. 1).
One of the basic assumptions of this theorem is the inequality $r(A)<1$, where $A: E\to E$ is a suitable linear operator, E- a Banach space and $r(A)$- the spectral radius of A. In view of the fact that the right-hand side of the equation (9) depends on $z(h(x,y))$ and $z_{xy}(H(x,y))$, so to apply Th. 1 ((Zima, 1992)) one has to estimate the spectral radius of the sum of suitable linear operators A_1 and A_2.
It is well known that if $A_1, A_2: E\to E$ are linear bounded commutative operators, then

$$r(A_1+A_2)\le r(A_1)+r(A_2). \tag{3.2}$$

Classical examples show that in the above inequality the commutativity assumption is essential (see (Krasnosielski, 1972), Chap. III). However, there exist results which give sufficient conditions (different from the global commutativity) under which the inequality (3.2) holds (see e.g. (Esayan, 1966), (Zima, 1993)). For example, an application of the following theorem from the paper (Esayan, 1966):

Let K be a normal and generating cone in the space E and let linear operators $A_1, A_2: E\to E$ be positive on K, semi-commutative and μ_0- upper bounded. Then the inequality (3.2) *holds.*

allowed us to prove the following result ((Bugajewski, 1997), Th. 4) concerning the existence and the uniqueness of solutions (more precisely: such solutions which have a continuous second mixed derivative) to the problem (3.1), namely:

Theorem 7. Assume that

15^0 $h=(h_1,h_2): I^2\to I^2$ is a continuous function and $h(x,y)\le(x,y)$ for every pair $(x,y)\in I^2$, where the inequality $(x_1,y_1)\le(x_2,y_2)$ stands for $x_1\le x_2$ and $y_1\le y_2$;

16^0 $H: I^2\to I^2$ is a continuous function such that $H(\text{int}I^2)=\text{int}I^2$ and $H|_{\text{int}I^2}$ is a diffeomorphism;

17^0 $|H'(x,y)| \leq 1$ and $D(h(H(x,y))) \subset H(D(h(x,y)))$ for every pair $(x,y) \in \text{int}I^2$, where $|H'(x,y)|$ denotes the absolute value of the jacobian of the mapping H at (x,y) and $D(x,y) = \{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq x,\ 0 \leq t \leq y\}$;

18^0 $(x,y,u,v) \to f(x,y,u,v)$ is a continuous function $(f : I^2 \times \mathbb{R}^2 \to \mathbb{R})$ satisfying the Lipschitz condition

$$|f(x,y,u_1,v_1) - f(x,y,u_2,v_2)| \leq L_1|u_1 - u_2| + L_2|v_1 - v_2|$$

for $(x,u,u_1,v_1),\ (x,y,u_2,v_2) \in I^2 \times \mathbb{R}^2$, where $L_1 > 0$ and $0 < L_2 < 1$.

Then the problem (3.1) has a unique solution, defined on I^2.

Note also that in the paper (Bugajewski, 1997) we gave the examples which show that applications of different theorems which guarantee that the inequality (3.2) is satisfied, leads to different results on the existence and the uniqueness of solutions to the neutral type equations (3.1).

In connection with the equations in the Carathéodory sense, which where discussed in Section 1, a natural question concerning the existence and the uniqueness of such solutions to the problem (3.1) arises here. Obviously, we require now that the equation (3.1) is satisfied almost everywhere. The theorems which give an answer to this question were proved in (Bugajewski and Zima, 1996). We considered there solutions of two types: first, such solutions z, for which $z_{xy} \in L^\infty(I^2)$, and next such solutions for which $z_{xy} \in L^1(I^2)$. Solutions of first type are more easy to deal with, since the cone of nonnegative functions in the space $L^\infty(I^2)$ is normal and its interior is nonempty.

Let us note also that Theorem 2 from (Bugajewski and Zima, 1996) was generalized in the paper (Zima, 1999) by Zima, under application of the local spectral radius theory.

In the paper (Bugajewski and Zima, 2003) we formulated conditions, different from the global commutativity, under which one can estimate the spectral radius of the composition of linearly bounded operators. Let us recall that an operator $A : E \to E$ is said to be linearly bounded, if there exists a number $M > 0$ such that $\|Ax\| \leq M\|x\|$ for every $x \in E$. An example of such a theorem which concerns the estimation of the spectral radius of the superposition of linearly bounded operators is the following result from (Bugajewski and Zima, 2003).

Theorem 8. Let K be a normal cone in a Banach space E and let $A, B : E \to E$ be linearly bounded, positively increasing operators acting in the space E. Suppose there exists $x_0 \in K$ such that

(j) $r(AB) = \lim_{n\to\infty} \|(AB)^n x_0\|^{1/n}$;

(jj) $BA^jB^jx_0 \prec A^jB^{j+1}x_0$ for every $j \in \mathbb{N}$.

Then the following inequality holds:

$$r(AB) \leq r(A)r(B).$$

In particular, we applied the above theorem to investigation of the existence and the uniqueness of solutions to the problem (3.1), in which $z(h(x,y))$ was replaced by $\max_{D(h(x,y))}\{z(t,s)\}$ and $I = [0,1]$ (we will denote this problem by (3.1')). Assume that

19^0 $f : I^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying the Lipschitz condition from the assumption 18^0, where $L_1(1-\alpha)^2 + L_2 < 1$ for some $\alpha > 0$ and $L_1, L_2 > 0$;

20^0 $H(D(h(x,y))) \subset D(H(x,y))$ and $|H'(x,y)| \geq 1$ for every pair $(x,y) \in \text{Int}I^2$;

21^0 $h : I^2 \to I^2$ is a continuous function such that $(x,y) \leq h(x,y) \leq (x^\alpha, y^\alpha)$ for every pair $(x,y) \in I^2$.

Theorem 9. Under the assumptions 16^0, $19^0 - 21^0$ the problem (3.1') has a unique solution, defined on I^2.

A similar result as above, was proved for a system of functional-differential equations, too.

4 Almost periodic solutions to differential and integral equations in Fréchet spaces

The theory of almost periodicity developed by Harald Bohr in 1923-1926 lead to essential development of the harmonic analysis on group as well as topological compact semigroups of linear operators. For many years this theory has focused attention of many mathematicians. Recall that almost periodic functions in Banach spaces were investigated by S. Bochner, and next by C. Corduneanu, S. Zaidman, J. A. Goldstein and many others. The theory of almost periodic functions in Fréchet spaces was investigated by G. M. N'Guerekata e.g. in papers (N'Guérékata, 1984), (N'Guérékata, 1986). A series of theorems concerning almost periodic solutions to differential equations in Fréchet spaces one can find in the book (N'Guérékata, 2001).

In the paper (Bugajewski and G.M. N'Guerekata, 2004) we have investigated the properties of almost periodic functions with values in a Fréchet space E. In particular, we proved that the set $AP(E)$ of all almost periodic functions $\mathbb{R} \to E$ is a Fréchet space.

Recall that the Fréchet space is said to be perfect, if every bounded function $f : \mathbb{R} \to E$ with an almost periodic derivative f', is almost periodic itself. For many years, an answer to the question about the existence of a perfect Fréchet space which is not a Banach one, still remained open.

In the paper (Bugajewski and G.M. N'Guerekata, 2004) we proved that the answer to this question is positive. An example of such a space is the space s of all bounded sequences of real numbers: $s = \{x = (x_n) : x_n \in \mathbb{R} \text{ for } n \in \mathbb{N}\}$ with a family of seminorms $q_n = p_1 \vee p_2 \vee \ldots \vee p_n$, $p_n(x) = |x_n|$ for $x \in s$ and $n \in \mathbb{N}$.

Moreover, in this paper we investigated almost periodic solutions to differential and integral equations in perfect Fréchet spaces. In particular, we investigated the almost periodicity of solutions to a linear equation

$$x'(t) = Ax(t) + f(t), \qquad t \in \mathbb{R}, \tag{4.1}$$

in such a space. Assume that

22^0 A is a compact linear operator in the space E;

23^0 the family A^k, $k \in \mathbb{N}$ is equicontinuous.

Denote by $\mathcal{P} = \{p_n\}$ a family of seminorms which generates the topology of the space E. Further, assume that

24^0 for every seminorm $p_n \in \mathcal{P}$, there exists a seminorm $q_n \in \mathcal{P}$ such that

$$p_n(e^{tA}x) \leq q_n(x), \qquad \text{for all } t \in \mathbb{R} \text{ and } x \in E;$$

25^0 $f \in AP(E)$ and for every seminorm $p_n \in \mathcal{P}$, there exists a function $\psi_{p_n} : \mathbb{R} \to \mathbb{R}_+$ such that

$$p_n(f(s)) \leq \psi_{p_n}(s), \quad \text{for every } s \in \mathbb{R} \text{ and } \int_{-\infty}^{+\infty} \psi_{p_n}(s)ds < +\infty.$$

Theorem 10. Under the above asumptions every solution to the equation (4.1) is an almost periodic function in E.

One of more important results from the paper (Bugajewski and G.M. N'Guerekata, 2004) is the below given theorem concerning almost periodic solutions to the Cauchy problem

$$x' = f(t,x), \quad x(0) = x_0, \quad t \in \mathbb{R}, \tag{4.2}$$

where x_0 is an element of a given Fréchet space E and $f : \mathbb{R} \times E \to E$. Denote by $(\beta_{p_n}(\cdot))_{n\in\mathbb{N}}$ the Sadovski measure of noncompactness (see (Sadovskii, 1972) for the definition and the basic properties). Assume that

26^0 f is a continuus function such that for every seminorm $p_n \in \mathcal{P}$ there exists a function $\psi_{p_n} : \mathbb{R} \to \mathbb{R}_+$ and

$$p_n(f(s,x)) \leq \psi_{p_n}(s), \qquad \int_{-\infty}^{+\infty} \psi_{p_n}(s)ds < +\infty;$$

27^0 for every seminorm $p_n \in \mathcal{P}$, every bounded set $M \subset E$ and every interval $[\min(0,a), \max(0,a)]$ there exists a continuous nondecreasing function $h_{n,M,a} : \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality

$$u(t) \leq \int_{\min(0,t)}^{\max(0,t)} h_{n,M,a}(u(s))ds, \quad \text{for } t \in [\min(0,a), \max(0,a)],$$

has the trivial solution $u \equiv 0$ only and

$$\beta_{p_n}(f(A \times X)) \leq h_{n,M,a}(\beta_{p_n}(X))$$

for $A \subset [\min(0,a), \max(0,a)]$ and $X \subset M$;

28^0 $(s,x) \to f(s,x)$ is almost periodic in s for every $x \in E$ and satisfies the Lipschitz condition in x, uniformly in $s \in \mathbb{R}$;

29^0 for every $x \in AP(E)$ the image $R_{F(x)} = \{\int_0^t f(s,x(s))ds : t \in \mathbb{R}\}$ is relatively compact.

Theorem 11. Under the assumptions $26^0 - 29^0$, the equation (4.2) has a solution in the space $AP(E)$. Moreover, the set S of such solutions is compact.

The proof of Th.11 relies on an application of the generalization of the Schauder fixed point theorem from (Bugajewski, 1994). Moreover, we extended this result to the case of almost periodic solutions to the nonlinear Volterra integral equation

$$x(t) = g(t) + \int_0^t K(s)f(s, x(s))ds, \quad t \in \mathbb{R}. \tag{4.3}$$

In the paper (Bugajewski and G.M. N'Guerekata, 2004) we introduced also so called asymptotically almost periodic functions in Fréchet spaces. Recall that a continuous function $f : \mathbb{R}_+ \to E$ is said to be asymptotically almost periodic, if it can be expressed as

$$f(t) = g(t) + h(t), \quad t \in \mathbb{R}_+,$$

where $g : \mathbb{R} \to E$ is almost periodic function and $h : \mathbb{R}_+ \to E$ is a continuous function such that $\lim_{t\to+\infty} h(t) = 0$. In particular, we proved that a decomposition of an asymptotically almost periodic function is unique as well as the vector space of asymptotically almost periodic functions is a Fréchet one.

Let us consider again the equation (4.1), where $t \in \mathbb{R}_+$. Assume that

30^0 A is a infinitesimal generator of a group of equicontinuous operators $(T(t))_{t\in\mathbb{R}}$ such that for every $x \in E$ the mapping $T(t)x : \mathbb{R} \to E$ is almost periodic;

31^0 $f = f_0 + f_1$, where $f_0 : \mathbb{R} \to E$ is an almost periodic function, the set $\{\int_0^t T(-s)f_0(s)ds : t \in \mathbb{R}\}$ is relatively compact and $f_1 \in C(\mathbb{R}_+, E)$ has the following property: for every $p_n \in \mathcal{P}$ there exists a function $\psi_{p_n} : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

$$p_n(f_1(s)) \leq \psi_{p_n}(s), \quad \text{and} \quad \int_0^{+\infty} \psi_{p_n}(s)ds < +\infty.$$

Theorem 12. Under the above assumptions, every solution to the equation (4.1) is asymptotically almost periodic.

Moreover, applying again the fixed point theorem from (Bugajewski, 1994) we proved existence results concerning asymptotically almost periodic solutions to the Cauchy problem (4.2) as well as to the nonlinear Volterra integral equation (4.3) (the autonomous cases).

5 Fixed point theorems in Banach spaces and in locally convex spaces

The Banach contraction pronciple as well as the Schauder fixed point theorem belong to the basic result in the fixed point theory. These results have had many generalizations, what is also strongly connected with their applications in the theory of differential and integral equations.

Let us recall the definition of weakly F-contractive mapping, introduced by Furi and Vignoli in (Furi and Vignoli, 1969).

Let X be a topological space and F be a real-valued lower semicontinuous function, defined on the product $X \times X$. A mapping $f : X \to X$ is a weak F-contraction, if the inequality $F(f(x), f(y)) < F(x, y)$ is satisfied for every $x, y \in X$, $x \neq y$.

In particular, if X is a metric space with a metric d and $F(x, y) = d(x, y)$, then f is called a weak contraction.

For a weak F-contraction, we have the following result on the existence and the uniqueness of a fixed point, proved in (Bugajewski, 2003a), namely

Theorem 13. Let X be a topological space, $x_0 \in X$ and $f : X \to X$ be a continuous weak F-contraction. If the implication

$$V = f(V) \cup \{x_0\} \Longrightarrow V \text{ is relatively compact}$$

holds for any countable subset V of X, then f has a unique fixed point.

Note that the above theorem contain the Banach contraction principle, the fixed point theorem of Browder (Browder, 1968), of Edelstein (Edelstein, 1962), of Furi and Vignoli (Furi and Vignoli, 1969) and others.

Now, let (X, d) be a metric space and $f : X \to X$. It is well-known that f has a fixed point if and only if the following two conditions are satisfied:

$$\inf\{d(x, f(x)) : x \in X\} = 0 \tag{5.1}$$

$$\text{the functional } x \to d(x, f(x)) \text{ has a minimum.} \tag{5.2}$$

In the paper (Gőhde, 1965) Göhde proved that if $f : D \to D$ is a nonexpansive mapping defined on a bounded, closed and convex subset D of a Banach space, then the property (5.1) holds. Next in the paper (Furi, and Vignoli, 1970) Furi and Vignoli established that the condition (5.1) is satisfied , if $f : D \to D$ is α-nonexpansive, where α denotes the Kuratowski measure of noncompactness. Emmanuele (Emmanuele, 1981) proved that (5.1) is also satisfied, if $f : D \to D$ is ω-nonexpansive, where ω denotes the measure of weak noncompactness, introduced by De Blasi (De Blasi, 1977).

These three results mentioned above are special cases of the following theorem, proved in (Bugajewski, 2003a).

Theorem 14. Let D be a closed, convex and bounded subset of a locally convex space, $x_0 \in D$ and let $f : D \to D$ be a continuous mapping such that for every $\lambda \in (0, 1)$ the implication

$$\left.\begin{array}{l} V = (conv\lambda f(V)) + (1 - \lambda)x_0 \\ \text{or} \\ V = (\lambda f(V) + (1 - \lambda)x_0) \cup \{x_0\} \end{array}\right\} \Longrightarrow V \text{ is relatively compact}$$

holds for every subset V of D. Then for every $p \in \mathcal{P}$, where $\mathcal{P}$ denotes the family of seminorms, which generate the topology of the space under consideration, we have

$$\inf\{p(x - f(x)) : x \in D\} = 0.$$

In particular, in the case of a Banach space, we have:

$$\inf\{\|x - f(x)\| : x \in D\} = 0.$$

A similar result one can prove for star-shaped subsets of a locally convex space.
In the proof of Theorem 14 it is established that there exists a sequence (x_n), $x_n \in D$ for $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty}(x_n - f(x_n)) = 0.$$

Hence it is clear that the assumptions of the above results together with the Palais-Smale condition ensure the existence of a fixed point of the mapping f in D. At the first glance this theorem seems to be similar to the Mönch type fixed point theorem proved in (Bugajewski, 1994). However, the examples given in (Bugajewski, 2003a) explain that these results are independent. Despite of this, in many problems in the theory of nonlinear differential as well as integral equations, these two results can be applied simultanously.

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Real-Time Optimal Control of Dynamical Systems

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ABSTRACT

A problem of synthesis of optimal feedbacks for linear and nonlinear control systems is under consideration. At first, based on fast algorithms of constructing optimal open-loop controls, an algorithm of calculating current values of optimal feedback in linear optimal control problem in real time is described. The idea of closed-loop solution is based on constructing a realization of optimal feedback in any concrete control process under unknown but bounded disturbances. The realization of the algorithm of closed-loop solution is oriented on fast corrections of optimal open-loop control subject to small variations of initial state. This is possible due to storage a small amount of additional information allowing to avoid the complete integration of primal or adjoint system. Then, this result is generalized for piecewise linear systems and for control systems with nonlinear dynamics. In the case of piecewise linear systems, algorithms are supplemented by procedures of parametric optimization. In addition, asymptotic methods of correcting solutions for linear or piecewise linear optimal control problems are used for nonlinear systems. Efficiency of described algorithms is illustrated by examples.

Keywords: optimal control, real-time optimization, open-loop and closed-loop controls, optimal controller, asymptotic method.

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... nihil omnino in mundo contingit,
in quo non maximi minimive ratio quæpiam cluceat ...

(... nothing at all takes place in the Universe in which
some rule of maximum or minimum does not appear ...)
L.Euler

1 Introduction

Being a "father" of the calculus of variations, Leonhard Euler put a keystone to the foundation of optimal control (OC) theory. The Euler equation and the rules of its constructing are still

significant for OC theory which lays within both the non-classical calculus of variations and control theory.

It is generally recognized that the synthesis of optimal systems is a central problem of OC theory. However, well-developed classical tools of OC theory (the Pontriagin maximum principle and the Bellman dynamic programming) do not provide suitable technique for this problem.

During the last decades, outstanding advances in computer technology and urgent applied problems stimulate investigations in the field of real-time optimization of control systems (Büskens and Maurer, 2000; Diehl, Bock and Schlöder, 2005). A lot of knowledge on methods of constructing open-loop controls have been collected but these methods mostly fail in actual control processes because of inaccuracies of mathematical models and unknown disturbances affecting the system. So the general problem is how to treat information on current states and parameters of control system in order to adapt the control law to deviation from the model in short time between sampling instants. The ideas of real-time optimization are widely used within the methodology of Model Predictive Control (Balan, Matie and Stan, 2005; Maciejowski, 2002; Mayne, Rawlings, Rao and Scokaert, 2000; Morari and Lee, 1999) which assumes to optimize the behavior of the system over fixed future horizon taking into account available data on the model and correcting the system's model and the control strategy as soon as new information is received.

In the paper, a new approach to synthesis of optimal systems based on the up-to-date optimization methods (Gabasov, Kirillova, Kostyukova and Pokataev, 1998) and the advances of computer technology is suggested. This approach throws a bridge between the classical principle of feedback control and the present-day principle of real-time control. The basic components of the developed technique are a new parametrization of the optimal control and fast algorithms of correcting optimal open-loop solutions which generalize special linear programming methods to optimal control problems (Gabasov, Kirillova and Balashevich, 2000). High rate of computation is provided due to special procedures of parallel integration of the adjoint system over short intervals on iterations of the method.

The paper has the following structure:

1. Introduction.
2. The scheme of the approach.
3. Synthesis of optimal feedback for linear systems.
4. Optimization of piecewise linear control systems.
5. Optimal Controller for nonlinear dynamical systems.
6. Conclusion.

In the paper a method of constructing an optimal feedback for finite time OC problem with input and endpoint state constraints is suggested. The proposed approach is based on online correction of an optimal open-loop control with the use of auxiliary information formed and stored at a previous sampling instant. This strategy proved to be effective for real-time optimization of control systems.

The details of the suggested approach have been elaborated by the way of successive complication of studied problems: at first, the algorithm of solving a linear endpoint OC problem (Gabasov et al., 2000) was generalized to OC problem for a linear system with intermediate

state constraints (Balashevich, Gabasov and Kirillova, 2001a); then it was supplemented with the scheme of storing and transforming an auxiliary information and with the procedure of optimization in parameters for solving OC problem of a piecewise linear system (Balashevich, Gabasov and Kirillova, 2001b); at last, the asymptotic method of optimization of quasilinear systems was generalized to piecewise quasilinear control systems and the latest result is connected with the following OC problem for a nonlinear system

$$c'x(t^*) \to \max,\ \dot{x} = f(x) + bu,\ x(0) = x_0,\ Hx(t^*) = g,\ |u(t)| \le 1,\ t \in T = [0, t^*]. \quad (1.1)$$

Here $x = x(t)$ is an n-vector of system state at an instant t; $u = u(t)$ is a value of a scalar control; $f(x)$, $x \in R^n$, is enough times differentiable function; $g \in R^m$, $\operatorname{rank} H = m < n$.
In OC theory, a function

$$u = u^0(t, x), \quad |u^0(t, x)| \le 1, \quad x \in X_t, \quad t \in T,$$

where X_t is a set of admissible states at instant $t \in T$, is called the (classical) optimal state feedback if the closed control system

$$\dot{x} = f(x) + bu^0(t, x), \quad x(\tau) = z,$$

moves along the optimal trajectory $x^0(t|\tau, z)$, $t \in T^\tau = [\tau, t^*]$, starting from any position (τ, z).
Although the problem of synthesis of optimal systems, which served as the initial impulse for arising OC theory, was stated in the late 40s of XXth century, it remains unsolved up to now even for linear systems with geometric constraints on control. Neither of two main methodologies of OC theory — the Pontriagin maximum principle and the Bellman dynamic programming — gives the satisfactory solution to this problem.
The main idea of our approach to the optimal synthesis problem consists in the use of opportunity to perform on-line computations in the course of control of processes with finite velocities of development.

2 The scheme of the approach

Let us present the suggested scheme of solving problem (1.1). This scheme contains two procedures:
1) synthesis of OC for a piecewise linear (PL) approximation of problem (1.1);
2) asymptotic correction of a positional solution of a PL problem.
To construct a PL approximation, a domain X, in which processes of problem (1.1) proceed, is divided into convex polyhedral sets X_i, $i = \overline{1,p}$: $X = \bigcup_{i=1}^{p} X_i$, $\quad X_i \cap X_j = \emptyset$, $i \ne j$.
As a PL approximation of the function $f(x)$, $x \in X$, we introduce a function $\hat{f}(x)$, $x \in X$, continuous on X and linear on each domain X_i, $i = \overline{1,p}$:

$$\hat{f}(x) = \hat{f}_i(x),\ x \in X_i,\ i = \overline{1,p}; \quad \hat{f}_i(x) = A_i x + a_i,\ x \in X_i,\ i = \overline{1,p}.$$

A number

$$\delta = \max_{x \in X} \|f(x) - \hat{f}(x)\| / \|f(x)\| \quad (2.1)$$

is said to be an accuracy of approximation. The accuracy of approximation can be made as high as is wished by the choice of a big enough p and an appropriate function $\hat{f}(x)$, $x \in X$.
A problem

$$c'x(t^*) \to \max, \quad \dot{x} = \hat{f}(x) + bu, \quad x(0) = x_0, \quad Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T, \qquad (2.2)$$

is a PL approximation of problem (1.1).
OC in problem (2.2) is constructed by solving a linear OC problem with intermediate state constraints and by optimization with respect to instants of transition between domains of linearity of function $\hat{f}(x)$, $x \in X$.
An algorithm of synthesis of optimal feedback in a linear problem with intermediate state constraints is a generalization of the algorithm elaborated for a linear OC problem

$$c'x(t^*) \to \max, \quad \dot{x} = Ax + bu, \quad x(0) = x_0, \quad Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T. \qquad (2.3)$$

The second procedure for solving nonlinear problem (1.1) (asymptotic correction of the solution to PL problem (2.2)) consists in the following. Problem (1.1) is rewritten in the equivalent form

$$c'x(t^*) \to \max, \quad \dot{x} = \hat{f}(x) + \delta g(x) + bu, \quad x(0) = x_0, \quad Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T, \ (2.4)$$

where $g(x) = (f(x) - \hat{f}(x))/\delta$.
Then problem (2.4) is imbedded into the family

$$c'x(t^*) \to \max, \quad \dot{x} = \hat{f}(x) + \mu g(x) + bu, \quad x(0) = x_0, \quad Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T, \ (2.5)$$

depending on a small parameter μ.
To construct an asymptotic solution $u^s(t, x, \mu)$, $x \in X_t$, $t \in T$, $\mu \ge 0$, for problem (2.5) with any degree of accuracy s fast algorithms based on solutions of linear and PL problems (2.2), (2.3) have been elaborated.
The function $u^s(t, x, \delta)$, $x \in X_t$, $t \in T$, is taken as a solution $u^0(t, x)$, $x \in X_t$, $t \in T$, to problem (1.1).
From the presented scheme of synthesis of optimal system (1.1) follows that the basis of the scheme is synthesis of OC in linear problem (2.3).

3 Synthesis of optimal feedback for linear systems

As solutions to mentioned problems cannot be constructed without the use of computer devices of discrete type, admissible controls are defined in the class of discrete functions

$$u(t) = u(\tau), \ t \in [\tau, \tau + h[, \ \tau \in T_h = \{0, h, \dots, (N-1)h\},$$

where $h = t^*/N$ is a quantization step.
To introduce a concept of optimal feedback, we imbed problem (2.3) into the family

$$c'x(t^*) \to \max, \quad \dot{x} = Ax + bu, \quad x(\tau) = z, \quad Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T^\tau, \qquad (3.1)$$

depending on a scalar $\tau \in T_h$ and an n-vector z.
Let $u^0(t|\tau, z)$, $t \in T^\tau$, be an optimal open-loop control of problem (3.1) for a position (τ, z); X_τ be a set of all $z \in R^n$ for which there exists $u^0(t|\tau, z)$, $t \in T^\tau$.

Definition 3.1. A function

$$u^0(t,x) = u^0(t|t,x),\ x \in X_t,\ t \in T_h, \tag{3.2}$$

is called an optimal (discrete) feedback for problem (2.3).

The classical statement of the optimal synthesis problem consists in constructing an explicit formula for (3.2), dynamic programming demands its tabulation. These requirements are cannot be satisfied even for linear problem (2.3). To overcome the deadlock let's analyse what is required in fact for actual control (Gabasov, Kirillova and Prischepova, 1995).
Suppose that function (3.2) has been constructed. Let us close by this feedback an actual system for which system (2.3) is a mathematical model. The behavior of the closed system is described in the form

$$\dot{x} = Ax + bu(t) + w,\ x(0) = x_0,$$

where an unknown but bounded function w reflects inaccuracies of modelling and external disturbances; $u(t) = u^0(\tau, x(\tau))$, $t \in [\tau, \tau + h[$, $\tau \in T_h$.
In a certain control process under a realized disturbance $w^*(t)$, $t \in T$, only the values

$$u^*(\tau) = u^0(\tau, x^*(\tau)), \quad \tau \in T_h, \tag{3.3}$$

along a realized isolated continuous trajectory $x^*(t)$, $t \in T$, are used. Moreover, it is enough to know how to calculate the current value $u^*(\tau)$ at any current instant $\tau \in T_h$, on the base of the current state $x^*(\tau)$, in time not exceeding h, i.e. in real time.

Definition 3.2. Function (3.3) is said to be a realization of optimal feedback in a particular control process. If at each current instant $\tau \in T_h$ the time of calculating the value $u^*(\tau)$ does not exceed h, one can say that the realization of the optimal feedback is constructed in real time. Any device able to perform this work is said to be *Optimal Controller* for problem (2.3).

Thus, the problem of synthesis of optimal systems is reduced to constructing an algorithm of Optimal Controller. This algorithm is based on a special dual method which calculates current values of optimal feedback in time less than h. This dual method, in turn, is a modification of the adaptive method (Gabasov et al., 1995) of linear programming elaborated in order to take into account specific features of OC problems. Details of the algorithm are published in (Gabasov et al., 2000). Let us give only the basic concepts.
The mentioned dual algorithm operates with supports of problem (2.3) defined as follows.
Choose from T_h an arbitrary subset of m elements $T_{sup} = \{t_l,\ l = \overline{1,m}\}$. Let $\chi_l(s)$, $s \in T$, $l = \overline{1,m}$, be a solution to the equation $\dot{x} = Ax + bu$ with an initial condition $x(0) = 0$ and control:

$$u(t) = \begin{cases} 1, & t \in [t_l, t_l + h[; \\ 0, & t \in T \backslash [t_l, t_l + h[\quad t_l \in T_{sup},\ l = \overline{1,m}. \end{cases}$$

Form the matrix $D_{sup} = \left(H\chi_l(t^*),\ l = \overline{1,m}\right)$.
A set T_{sup} is said to be a support of problem (2.3) if $\det D_{sup} \neq 0$. A support characterizes controllability of the output signal $z = Hx(t^*)$ by impulses applied at support instants $t \in T_{sup}$.

Put in correspondence to a support T_{sup} following (accompanying) elements:
1) the Lagrange vector $\nu = \nu(I)$, $I = \{1, 2, \ldots, m\}$, that is a solution of the equation

$$\nu' D_{sup} = c'_{sup}, \quad c_{sup} = (c(t) = \int_t^{t+h} \psi_c(\vartheta) b d\vartheta, \ t \in T_{sup}),$$

where $\psi_c(t)$, $t \in T$, is a solution to the adjoint equation

$$\dot{\psi} = -A'\psi, \quad \psi(t^*) = c;$$

2) a cocontrol (a switching function) $\Delta(t) = \int_t^{t+h} (\psi_c'(\vartheta) - \nu' G(\vartheta)) b d\vartheta$, $t \in T_h$, where $G(t)$, $t \in T$, is an $m \times n$-matrix function, a solution to the equation

$$\dot{G} = -GA, \quad G(t^*) = H;$$

3) a pseudocontrol $\omega(t)$, $t \in T$. Nonsupport values $\omega(t)$, $t \in T_n = T_h \backslash T_{sup}$, are defined as

$$\omega(t) = -1 \text{ if } \Delta(t) < 0; \quad \omega(t) = 1 \text{ if } \Delta(t) > 0; \quad \omega(t) \in [-1, 1] \text{ if } \Delta(t) = 0, \quad t \in T_n.$$

Support values $\omega(t)$, $t \in T_{sup}$, of pseudocontrol are found from the equation

$$\sum_{t \in T_{sup}} d(t)\omega(t) + \sum_{t \in T_n} d(t)\omega(t) = g - G(0)x_0,$$

where $d(t) = \int_t^{t+h} G(\vartheta) b d\vartheta$, $t \in T_h$.

Theorem 3.1. *If $|\omega(t)| \leq 1$, $t \in T_{sup}$, then $\omega(t)$, $t \in T$, is OC of problem (2.3) and a corresponding support T_{sup} is said to be an optimal support of problem (2.3). Problem (2.3) has an optimal support if it has admissible controls.*

The dual method allows to construct the optimal support T^0_{sup} for any initial support T_{sup} in a finite number of iterations.
In the procedure of changing the support the following information is used:
1) a set of nonsupport zeroes $T_{n0} = \{t \in T_n \backslash t_* : \Delta(t-h)\Delta(t) < 0\}$;
2) a set $T_{sn} = T_{sup} \cup T_{n0} \cup \{t_*, t^*\} = \{t_k, \ k \in K \cup k^* + 1\}$, $K = \{0, 1, \ldots, k^*\}$; and intervals of constant sign of a cocontrol:

$$T_k = \{t_{*k} = t_k, t_k + h, \ldots, t_k^* = t_{k+1} - h\}, \text{ if } t_k \notin T_{sup};$$
$$T_k = \{t_{*k} = t_k + h, t_k + 2h, \ldots, t_k^* = t_{k+1} - h\}, \text{ if } t_k \in T_{sup};$$

3) a value $\gamma = \begin{cases} \operatorname{sign} \Delta(t_*), & \text{if } t_* \notin T_{sup}; \\ \operatorname{sign} \Delta(t_* + h), & \text{if } t_* \in T_{sup}; \end{cases}$
4) a vector $p = \gamma \sum_{k=0}^{k^*} (-1)^k \sum_{t \in T_k} d(t) = \gamma \sum_{k=0}^{k^*} (-1)^k \int_{t_{*k}}^{t_k^* + h} G(\vartheta) b(\vartheta) d\vartheta$.
In the course of iteration the following information is stored and transformed: 1) a support T_{sup}; 2) a set of nonsupport zeroes T_{n0}; 3) a support matrix D_{sup}; 4) values $G(t)$, $\psi_c(t)$, $t \in T_{sn} \backslash t^*$; 5) a vector p; 6) a number γ, support values $\omega(t)$, $t \in T_{sup}$; 7) the Lagrange vector ν.
Let $t^0 \in T_{sup}$ be such an instant that $|\omega(t^0)| = \max_{t \in T_{sup}} |\omega(t)|$.
A general scheme of an algorithm of constructing a new support $\bar{T}_{sup}$ consists in the following:
1. A variation $\Delta\nu$ of the Lagrange vector is calculated;

2. A rate of change of the dual performance index is calculated;
3. A short step along the direction $\Delta\nu$ providing a new zero of the varied cocontrol is calculated. This step causes a positive jump of the rate of change of the dual performance index;
4. The information stored in the computer memory is transformed;
5. Operations 2—4 are repeated till the value of the rate of change of the dual performance index reaches a nonnegative value;
6. The element t^0 of the support T_{sup} is changed by the last obtained zero $\tilde{t}^{K_0}$ of the varied cocontrol.

The main idea of the proposed realization relates to operations 3, 4 of the above scheme.
The variation (direction of changing) $\Delta\nu$ of the Lagrange vector is calculated as a solution to the system of linear equations

$$-D'_{sup}\Delta\nu = (\Delta\delta(t), t \in T_{sup}),$$

where $\Delta\delta(t^0) = \operatorname{sign}\omega(t^0)$; $\Delta\delta(t) = 0$, $t \in T_{sup}\backslash t^0$. Set

$$\Delta\delta(t) = -\Delta\nu' d(t) = -\int_t^{t+h} \Delta\nu' G(\vartheta) b(\vartheta) d\vartheta, \ t \in T_n \text{ (variation of the cocontrol);}$$
$$\Delta(t,\sigma) = \Delta(t) + \sigma\Delta\delta(t), \ \ t \in T_h; \quad \sigma \geq 0 \text{ (varied cocontrol).} \tag{3.4}$$

Further assume that $\Delta\delta(t_{*k})\Delta\delta(t^*_{k-1}) > 0$, if $t_{*k} \in T_{n0}$, $k = \overline{1,k^*}$.
Denote by $\varphi(\sigma)$ a value of the performance index of the problem dual to (2.3) at function (3.4). According to (Gabasov et al., 1995), in a small right-side neighborhood of the point $\sigma = 0$ the function $\varphi(\sigma)$, $\sigma \geq 0$, is linear and decreases with the rate $\alpha^1 = 1 - |\omega(t^0)| < 0$. Such behavior remains while σ increases from $\sigma = 0$ till the value $\sigma^1 > 0$ when a new zero $t(\sigma^1)$ of function (3.4) arises: $\Delta(t(\sigma^1), \sigma^1) = 0$. When $\sigma = \sigma^1$, the rate of the change of the dual performance index makes a positive jump. For changing a support, a sequence of short steps σ^k, $k = \overline{1,K_0}$, when the rate makes jumps, is formed. We calculate a long step $\sigma^* = \sigma^{K_0}$, after which the rate of the change of the dual performance index becomes nonnegative. With the direct realization of the adaptive method (Gabasov et al., 1995) for detecting σ^k, $k = \overline{1,K_0}$, it is necessary to calculate steps $\sigma(t)$ for each $t \in T_n$. This requires enormous computational resources. In the proposed realization it is taken into account that with increasing σ a new zero of the varied cocontrol can appear only by one of the following ways:
1. At the point $t^0 + h$ or $t^0 - h$ as a result of movement of $t^0(\sigma)$, $\sigma \geq 0$: $\Delta(t^0(\sigma), \sigma) = 0$, $t^0(0) = t^0$.
2. At a point t or $t - h$ as a result of movement of a point $t(\sigma)$, $\sigma \geq 0$, $t \in T_{n0}$.
3. At the point t_* as a result of arising at the left end of the interval T.
4. At the point $t^* - h$ as a result of arising at the right end of the interval T.
5. Inside the interval T from a stationary point of the function

$$\tilde{\Delta}(t) = (\psi'_c(t) - \nu' G(t))b(t), t \in T.$$

Thus, it is sufficient to investigate the behavior of the function $\Delta(t,\sigma)$ not on the whole set T_n but only for points 1–5.
Details of an iteration are given in (Gabasov et al., 2000).
Under certain conditions of nonsingularity the described method is finite (Gabasov et al., 1995).

The problem of effectiveness of methods is principal in constructive theory of OC. It is not always reasonable to estimate the effectiveness of methods by the amount of iterations because different methods consist of different types of iterations. It is more natural to use as a measure of effectiveness the quantity of complete integrations of primal and adjoint systems which are used to construct the optimal solution (Fedorenko, 1978). In this case, as a unit of complexity, one integration of a primal or adjoint system on the whole interval T is taken.

It is impossible to estimate in advance the complexity of solving a particular problem but computer experiments can give a certain representation on characteristics of the methods and on their properties.

Now describe a scheme of operating Optimal Controller. Before the beginning of control process the controller constructs an optimal support $T^0_{sup}(0)$ by solving problem (2.3) and put $u^*(0) = u^0(0, x^*_0)$.

Suppose that Optimal Controller has operated on the interval $[0, \tau[$, $\tau \in T_h$, and the control process is in a position $(\tau, x^*(\tau))$, $x^*(\tau) \in X_\tau$, $\tau \in T_h$. To calculate a signal $u^*(\tau)$ the controller has to solve problem (3.1) for the state $z = x^*(\tau)$.

The system reaches the state $x^*(\tau)$ from the state $x^*(\tau - h)$ under action of the control $u^*(t) = u^*(\tau - h)$, $t \in [\tau - h, \tau[$, and the disturbance $w^*(t)$, $t \in [\tau - h, \tau[$. By assumption, at the previous instant $\tau - h$ Optimal Controller solved problem (3.1) for the position $(\tau - h, x^*(\tau - h))$. Denote by $x^0(t) = x^0(t|\tau - h, x^*(\tau - h))$, $t \in T^{\tau - h}$, the optimal trajectory of problem (3.1) for the position $(\tau - h, x^*(\tau - h))$.

It is not difficult to show that in the case of bounded $w^*(s)$, $s \in [\tau - h, \tau[$, the states $x^*(\tau)$, $x^0(\tau)$ are the closer to each other, the smaller h. For small h zeros of the cocontrols $\Delta(t|\tau - h, x^*(\tau - h))$, $t \in T^{\tau - h}$; $\Delta(t|\tau, x^*(\tau))$, $t \in T^\tau$, are close too. So the distances required to move zeros of the first cocontrol to coincide with zeros of the second cocontrol are very small. It is these distances that determine intervals of integrating the adjoint system while transiting from $x^*(\tau - h)$ to $x^*(\tau)$.

Example. As an example, a problem of damping a two-mass oscillating system in finite time with minimal fuel consumption is considered:

$$\begin{gathered} \int_0^{25} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = -x_1 + x_2 + u, \quad \dot{x}_4 = 0.1x_1 - 1.02x_2, \\ x_1(0) = x_2(0) = 0,\ x_3(0) = 2,\ x_4(0) = 1, \\ x_1(25) = x_2(25) = x_3(25) = x_4(25) = 0, \quad 0 \le u(t) \le 1,\ t \in [0, 25[, \end{gathered} \tag{3.5}$$

where $x_1 = x_1(t)$, $x_2 = x_2(t)$ are deviations of the masses from the equilibrium; $x_3 = dx_1/dt$, $x_4 = dx_2/dt$; $u = u(t)$ is fuel consumption per second.

Assume that in the course of control process the oscillating system is affected by disturbance presented in the fourth equation $\dot{x}_4 = 0.1x_1 - 1.02x_2 + w$ and the realizing disturbance has the form $w^*(t) = 0.3 \sin 4t$, $t \in [0, 9.75[$; $w^*(t) \equiv 0$, $t \ge 9.75$. This disturbance is unknown to Optimal Controller but at any instant $\tau \in T_h$ a current state $x^*(\tau)$ is measured.

Following the described approach, to generate the initial value $u^*(0)$, preliminary work on constructing the optimal support $T^0_{sup}(0)$ in problem (3.5) has been done before the beginning of the control. Although the time required for constructing $T^0_{sup}(0)$ does not play significant part in solving the optimal synthesis problem, let us give some numerical results to illustrate the

effectiveness of the dual method.

The problem was solved with the quantization step $h = 25/1000$. As an initial support a set $T_{sup} = \{5, 10, 15, 20\}$ was taken. With this support 4 additional nonsupport cocontrol zeros $T_{n0} = \{3.725, 9.725, 15.3, 21.3\}$ were obtained.

Movements of cocontrol zeros at iterations are pictured in Figure 1. Numbers near each jump of curves show movements of the corresponding instants during the iteration in h units.

As a unit of complexity one complete integration of a primal or adjoint system on interval T in taken. In doing so, one assumes that there are enough microprocessors at our disposal to organize calculations in parallel. One integration is required for constructing the support matrix D_{sup}, the second integration is necessary for detecting nonsupport cocontrol zeros. The complexity of iterations of the dual method equals 0.433.

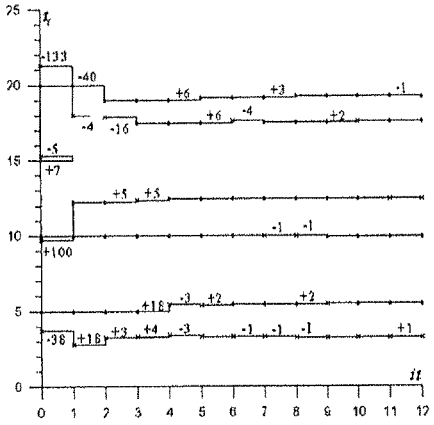

Figure 1: Movement of cocontrol zeros

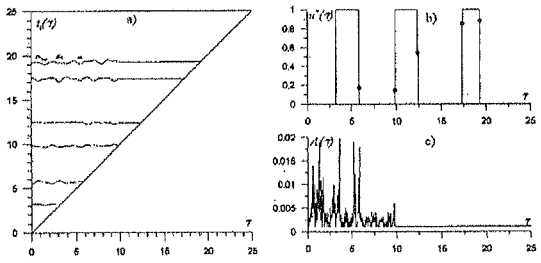

Figure 2: Closed-loop control

Behaviour of the cocontrol zeros in the course of control process is presented in Figure 2a. The realization $u^*(t)$, $t \in T$, is given in Figure 2b. Realized support values of the control are marked by dots. In Figure 2c the complexity $A(\tau)$ of correction of current supports by the dual method at each instant $\tau \in T_h$ is given: $A(\tau) \leq 0.02$.

Let α be the time required for a microprocessor to integrate a system on interval T. Then using these microprocessors one can implement the feedback if $0.02\alpha \leq h$. Based on this inequality one can choose a quantization step h for a given microprocessor, and, on the other hand, take a suitable microprocessor for a given h.

4 Optimization of piecewise linear control systems

Let us give a scheme of constructing an optimal open-loop control and an optimal feedback in problem (2.2). Details of the algorithm are contained in (Balashevich et al., 2001b).

Each admissible control $u(t)$, $t \in T$, of problem (2.2) generates a trajectory which successively passes through some (admissible) sequence X_{i_1}, X_{i_2}, ..., X_{i_k} of sets from the totality X_1, ..., X_p, crossing through the borders of neighboring sets at (admissible) instants θ_{i_1}, θ_{i_2}, ..., $\theta_{i_k-1} \in T_h$. Denote by X^0_1, ..., $X^0_{j^*}$ and θ^0_1, ... $\theta^0_{j^*-1}$ sequences of sets and instants of

intersection of the borders corresponding to the optimal trajectory $x^0(t)$, $t \in T$. Assume that the optimal trajectory $x^0(t)$, $t \in T$, intersects the borders of sets X_j^0, X_{j+1}^0, $j = \overline{1, j^*-1}$, on hyperplanes $H_j x = g_j$ where $H_j \in R^{m_j \times n}$, $g_j \in R^{m_j}$, $j = \overline{1, j^*-1}$. On a set X_j^0 the function $\hat{f}(x)$ has the form $\hat{f}(x) = A_j x + a_j$ where $A_j \in R^{n \times n}$, $a_j \in R^n$; $A_j x + a_j = A_{j+1} x + a_{j+1}$ if $H_j x = g_j$, $j = \overline{1, j^*-1}$; $\operatorname{rank}\binom{A_j - A_{j+1}}{H_j} < n$, $j = \overline{1, j^*-1}$. Denote $m_{j^*} = \hat{m}$; $H_{j^*} = H$, $g_{j^*} = g$; set $m = \sum_{j=1}^{j^*} m_j$.

In many problems (2.2) on the base of a priori information the set $X_1^0, \ldots, X_{j^*}^0$ could be formed before solving the problem. Then this problem can be reduced to the OC problem for a totality of linear systems

$$\begin{gathered} J(\theta, u) = c' x(\theta_{j^*}) \to \max, \\ \dot{x}(t) = A_j x(t) + a_j + bu(t), \quad t \in [\theta_{j-1}, \theta_j[, \; j = \overline{1, j^*}, \quad x(\theta_0) = x_0, \\ H_j x(\theta_j) = g_j, \; j = \overline{1, j^*}, \quad |u(t)| \le 1, \; t \in T, \quad \theta_0 < \theta_1 < \cdots < \theta_{j^*-1} < \theta_{j^*}, \end{gathered} \tag{4.1}$$

$$\left(g_j \in R^{m_j}, j = \overline{1, j^*}, \; \operatorname{rank} H_j = m_j < n, \; \theta_j \in T_h, j = \overline{1, j^*-1}, \; \theta_0 = 0, \; \theta_{j^*} = t^* \right).$$

In problem (4.1) instants $\theta = (\theta_1, \theta_2, \ldots \theta_{j^*-1})$ of transition between domains of linearity are chosen together with a control $u(t)$, $t \in T$.

The solution to problem (4.1) consists of two procedures:

1) linearization of problem (4.1) along an admissible trajectory and solution to the linearized problem;

2) correction of solution to the linearized problem by the choice of the optimal positions of instants of transition between domains of linearity.

Linearization of problem (4.1) consists in fixing a vector θ corresponding to an admissible trajectory. The linearized problem has form (4.1) but now it is assumed that the vector θ is fixed. Thus we get a linear OC problem with intermediate state constraints

$$\begin{gathered} c' x(t^*) \to \max, \quad \dot{x} = A(t)x + a(t) + bu, \quad x(\theta_0) = x_0, \\ H(s)x(s) = g(s), \; s \in S = \{\theta_1, \ldots, \theta_{j^*}\}, \quad |u(t)| \le 1, \quad t \in T. \end{gathered} \tag{4.2}$$

Here $A(t) = A_j$, $a(t) = a_j$, $t \in T_j = [\theta_{j-1}, \theta_j[$, $j = \overline{1, j^*}$; $H(s) = H_j$, $g(s) = g_j$, $s = \theta_j$, $j = \overline{1, j^*}$.

An algorithm of solving problem (4.2) is a generalization of the method of section 3 and described in details in (Balashevich et al., 2001a). New principal elements of algorithm are caused by the presence of inequality constraints and by possible discontinuities of cocontrol at instants $s \in S$.

As a result of solving the linearized problem we get an optimal support $T_{sup}^0(\theta)$ and the corresponding optimal value $J^0(\theta)$ of the performance index for a fixed vector θ. The aim is to construct a new vector $\bar{\theta}$ such that $J^0(\bar{\theta}) \ge J^0(\theta)$.

Methods of constructing an optimal vector θ^0 (Bazaraa, Sherali and Shetty, 1993; Gill, Murray and Wright, 1983) are based on calculating a gradient of the performance index of problem (4.1) with respect to θ. It is not difficult to show that

$$\frac{\partial J(\theta,u)}{\partial \theta_k} = c' \frac{\partial x(t^* | \theta, u_{sup}^0(\theta))}{\partial \theta_k} - \nu'(\theta) \left(H_j \frac{\partial x(\theta_j | \theta, u_{sup}^0(\theta))}{\partial \theta_k}, \atop j = \overline{1, j^*} \right), \quad k = \overline{1, j^*-1},$$

where $u^0_{sup}(\theta) = (u^0(t|\theta),\ t \in T^0_{sup}(\theta))$ are support values of the optimal control $u^0(t|\theta)$, $t \in T_h$; $x(t|\theta, u^0_{sup}(\theta))$, $t \in T$, is the corresponding trajectory of system (2.2). Then a step along the direction grad $J = (\frac{\partial J(\theta,u)}{\partial \theta_k},\ k = \overline{1, j^* - 1})$ is chosen (Bazaraa et al., 1993) and a new vector $\bar{\theta}$ is constructed.

5 Optimal Controller for nonlinear dynamical systems

Let us return to problem (1.1). A discrete control $u(t)$, $t \in T$, is said to be an admissible (open-loop) control if $|u(t)| \leq 1$, $t \in T$, and the corresponding trajectory of system (1.1) satisfies the endpoint constraint $x(t^*) \in X^*$. An admissible control $u^0(t)$, $t \in T$, is said to be an open-loop solution (optimal open-loop control) of problem (1.1) if it gives the maximal value to the performance index.

To introduce a concept of positional solution (optimal feedback) of problem (1.1) we assume that the state of control system is known not only at the initial instant $t = 0$ but will be known also at each current instant $\tau \in T_h$ of the control process. Under this assumption, problem (1.1) is imbedded into the family of problems

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = f(x) + bu,\ x(\tau) = z, \quad x(t^*) \in X^*, \quad |u(t)| \leq 1, \quad t \in T^\tau, \quad (5.1)$$

depending on a scalar $\tau \in T_h$ and an n-vector z.

As above, an optimal closed-loop solution to problem (1.1) is defined by expression (3.2) where $u^0(t|\tau, z)$, $t \in T^\tau$, is an optimal open-loop control of problem (5.1), X_τ is a set of all $z \in X$ for which problem (5.1) has open-loop solutions for a fixed τ, the concepts of realization of optimal feedback and Optimal Controller are introduced.

Below an approach to constructing optimal open-loop and closed-loop solutions to problem (1.1) is described. When solving the problem we distinguish local and global optimization of nonlinear system. Details of the algorithms are presented in (Balashevich, Gabasov, Kalinin and Kirillova, 2002).

5.1 Local optimization of nonlinear systems

Let x^* be any point in X, $\hat{f}(x) = A(x - x^*) + d$, $d = f(x^*)$, $A = \partial f(x)/\partial x\,|_{x=x^*}$, δ be accuracy (2.1) of linear approximation of the function $f(x)$ in X. In this section, problem (1.1) is solved on a domain X for which δ is a small value.

Solution of problem (1.1) consists of two procedures: 1) solution to a linearized problem; 2) correction of the solution to the linearized problem.

5.1.1 Solution to a linearized problem

The linearized problem has the form

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = Ax + a + bu,\ x(0) = x_0, \quad x(t^*) \in X^*, \quad |u(t)| \leq 1, \quad t \in T, \quad (5.2)$$

where $a = -Ax^* + f(x^*)$, and is said to be a base problem.

It is solved by the fast algorithm presented in section 2, as a result an optimal support control $\{u^0(\cdot), T^0_{sup}\}$ is constructed, and an auxiliary information is formed to be used for asymptotic correction of the solution.

For small δ the optimal control constructed for linear system (5.2) and used for nonlinear system (1.1) gives satisfactory results. However, to expand the domain X in which the solution to problem (1.1) is constructed with a given accuracy we pass on to the second procedure.

5.1.2 Asymptotic correction of solution of linear problem

Imbed problem (1.1) into the family of problems depending on a small parameter $\mu \to 0$:

$$c'x(t^*) \longrightarrow \max,\ \dot{x} = Ax + a + \mu g(x) + bu,\ x(0) = x_0,\ x(t^*) \in X^*,\ |u(t)| \le 1,\ t \in T, \quad (5.3)$$

called a quasilinear OC problem. Problem (5.3) coincides with problem (1.1) when $\mu = \delta$. Asymptotic method of constructing open-loop solution to problem (5.3) is described in (Gabasov, Kalinin and Kirillova, 1987; Kalinin, 2000). Let us give the main elements of it.

For a fixed parameter μ a piecewise continuous function $u(t,\mu)$, $t \in T$, satisfying the inequality $|u(t,\mu)| \le 1$, $t \in T$, is called an accessible control. An accessible control is called an admissible open-loop control if the generated trajectory $x(t,\mu)$, $t \in T$, of system (5.3) satisfies the constraint $x(t^*,\mu) \in X^*$. An admissible control $u^0(t,\mu)$, $t \in T$, maximizing the performance index of problem (5.3) is said to be an optimal open-loop control. The corresponding optimal trajectory is denoted by $x^0(t,\mu)$, $t \in T$. For a chosen integer s a family $u^s(t) = \{u^s_\mu(t),\ \mu \to 0\}$, $t \in T$, of accessible controls is said to be an asymptotically s-optimal open-loop control of quasilinear problem (5.3) if for trajectories $x^s_\mu(t)$, $t \in T$, $\mu \to 0$, generated by controls $u^s_\mu(t)$, $t \in T$, $\mu \to 0$, the asymptotic equalities $c'x^0(t^*,\mu) - c'x^s_\mu(t^*) = o(\mu^s)$, $Hx^s_\mu(t^*) - g = o(\mu^s)$, $\mu \to 0$, hold.

Theorem 5.1. *Under certain conditions (Kalinin, 2000), problem (5.3) with a small enough μ has a unique OC of the form $u^0(t,\mu) = (-1)^j\gamma$, $t \in [t_j(\mu), t_{j+1}(\mu)[$, $j = \overline{0,k^*}$, ($t_0(\mu) = 0$, $t_{k^*+1}(\mu) = t^*$) where the switching points $t_j(\mu)$, $j = \overline{1,k^*}$, are infinitely differentiable with respect to μ and $t_j(0) = t^0_j$, $j = \overline{1,k^*}$, are optimal switching point in base problem (5.2).*

Thus, the optimal control $u^0(t,\mu)$, $t \in T$, of problem (5.3) is defined by switching points $t_j(\mu)$, $j = \overline{1,k^*}$, and the Lagrange vector $\nu(\mu)$. According to (Gabasov et al., 1987), for constructing an asymptotically s-optimal control we calculate the coefficients $z_k = (t^k_j,\ j = \overline{1,k^*};\ \nu^k_i,\ i = \overline{1,m})$, $k = \overline{0,s}$, of the Taylor polynomials of degree s of functions $t_j(\mu)$, $\nu(\mu)$: $t^s_j(\mu) = \sum_{k=0}^s \mu^k t^k_j$, $j = \overline{1,k^*}$; $\nu^s_i(\mu) = \sum_{k=0}^s \mu^k \nu^k_i$, $i = \overline{1,m}$. Coefficients $z_0 = (t^0_j, = \overline{1,k^*};\ \nu^0_i,\ i = \overline{1,m})$ are calculated as a result of solving base problem (5.2) in the class of piecewise continuous functions. They are constructed by solving a system of equations of the refining procedure (Gabasov et al., 2000):

$$R_0(z) = 0 \quad (5.4)$$

with respect to variables $z = (t_j,\ j = \overline{1,k^*};\ \nu_i,\ i = \overline{1,m})$, where

$$R_0(z) = \begin{bmatrix} \sum_{j=0}^{k^*}(-1)^j\gamma \int_{t_j}^{t_{j+1}} G(t)bdt - \hat{g}, \\ (\psi'_c(t_j) - \nu' G(t_j))b, \quad j = \overline{1,k^*} \end{bmatrix}. \quad (5.5)$$

System (5.4) is solved by the Newton method. Approximation z_0^l of values z_0 is computed as

$$z_0^l = z_0^{l-1} - I_0^{-1}(z_0^{l-1})R_0(z_0^{l-1}), \quad l = \overline{1, l_0},$$

where $I_0(z)$ is the Jacobi matrix of system (5.4):

$$I_0(z) = \begin{pmatrix} 2(-1)^{j-1}\gamma G(t_j)b,\ j = \overline{1,k^*} & 0_{m\times m} \\ \mathrm{diag}(\nu' G(t_j) - \psi_c'(t_j))Ab,\ j = \overline{1,k^*}) & -(G(t_j)b)',\ j = \overline{1,k^*} \end{pmatrix}.$$

Set $z_0 = z_0^{l_0}$ where $z_0^{l_0}$ is an approximation which for a given $\varepsilon \geq 0$ provides $\|R_0(z_0^{l_0})\| \leq \varepsilon$.
As an initial approximation $z_0^0 = (t_j^{0(0)},\ j = \overline{1,k^*};\ \nu_i^{0(0)},\ i = \overline{1,m})$ elements of the set $T_{sn}^0 \backslash \{0, t^*\}$ ($t_0^0 = 0$, $t_{k^*+1}^0 = t^*$) and the optimal Lagrange vector obtained while solving problem (5.2) in the class of discrete controls are taken. On the known values $G(t_j^{0(0)})$, $\psi_c(t_j^{0(0)})$, $j = \overline{1,k^*}$, and vector p vector $R_0(z_0^0)$ and matrix $I_0(z_0^0)$ are formed. On each iteration of the Newton method, the integration of functions $G(t)$, $\psi_c(t)$ is performed on intervals $[t_j^{0(l-1)}, t_j^{0(l)}]$, $j = \overline{1,k^*}$. When these integrations are performed in parallel, the total complexity of the refining procedure equals $\sum_{l=1}^{l_0} \max_{j=\overline{1,k^*}} |t_j^{0(l)} - t_j^{0(l-1)}|/t^*$.
According to (Gabasov et al., 1987), coefficients z_k, $k = \overline{1,s}$, are calculated successively by solving systems of linear equations

$$I_0(z_0)z_1 = -R_1(z_0),$$
$$I_0(z_0)z_2 = -\tfrac{\partial R_1}{\partial z}(z_0)z_1 - \tfrac{1}{2}z_1'\tfrac{\partial^2 R_0}{\partial z^2}(z_0)z_1 - R_2(z_0),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where

$$R_1(z) = \begin{bmatrix} Hx_1(t^*|t_1,\ldots t_{k^*}), \\ \psi_1'(t_j|z)b,\ j = \overline{1,k^*} \end{bmatrix}, \quad R_2(z) = \begin{bmatrix} Hx_2(t^*|t_1,\ldots,t_{k^*}), \\ \psi_2'(t_j|z)b,\ j = \overline{1,k^*} \end{bmatrix},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

functions $x_k(t) = x_k(t|t_1,\ldots t_{k^*})$, $\psi_k(t) = \psi_k(t|z)$, $k = \overline{1,s}$, satisfy equations

$$\dot{x}_1 = Ax_1 + g(x_0(t)), \quad x_1(0) = 0;$$
$$\dot{x}_2 = Ax_2 + \tfrac{\partial g(x_0(t))}{\partial x}x_1(t), \quad x_2(0) = 0;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\dot{\psi}_1 = -A'\psi_1 - \tfrac{\partial H(x_0(t),\psi_0(t))}{\partial x}, \quad \psi_1(t^*) = 0;$$
$$\dot{\psi}_2 = -A'\psi_2 - \tfrac{\partial H(x_0(t),\psi_1(t))}{\partial x} - \tfrac{\partial^2 H(x_0(t),\psi_0(t))}{\partial x^2}x_1(t), \quad \psi_2(t^*) = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where $H(x,\psi) = \psi' g(x)$; $x_0(t)$, $\psi_0(t)$, $t \in T$, are the optimal trajectory and cotrajectory of the base problem.
Calculate coefficients t_j^k, $j = \overline{1,k^*}$; ν_i^k, $i = \overline{1,m}$; $k = \overline{1,s}$, put $\mu = \delta$, and obtain $t_j^s(\delta) = \sum_{k=0}^{s} \delta^k t_j^k$, $j = \overline{1,k^*}$; $\nu_i^s(\delta) = \sum_{k=0}^{s} \delta^k \nu_i^k$, $i = \overline{1,m}$. Control $u^s(t) = (-1)^j\gamma$, $t \in [t_{j-1}^s(\delta), t_j^s(\delta)[$, $j = \overline{1,k^*+1}$, ($t_0^s(\delta) = 0$, $t_{k^*+1}^s(\delta) = t^*$) is taken as an open-loop solution to problem (1.1).
As a rule, there are no strict limitations on the time of constructing the optimal open-loop control so here we are not discussing the ways of calculating required values of functions $x_k(t)$, $\psi_k(t)$, $t \in T$, $k = \overline{1,s}$, and their derivatives. Let us dwell on this question when describing a method of constructing a positional solution.

5.1.3 Realization of optimal feedback

In a particular control process the behavior of a nonlinear system is described by equation

$$\dot{x} = f(x) + bu^*(t) + w(t), \quad x(0) = x_0, \tag{5.6}$$

and the problem is to calculate a realization $u^*(\tau)$ of the optimal feedback at each current instant $\tau \in T_h$. As in the case of constructing an open-loop solution, for constructing a realization of an optimal feedback one can use either a solution to a linearized problem or its asymptotic correction.

Construction of a realization of the optimal feedback for a linearized problem performs by the dual method. In doing so, at each instant $\tau \in T_h$ a problem

$$c'x(t^*) \longrightarrow \max, \ \dot{x} = Ax + a + bu, \ x(\tau) = x^*(\tau), \ x(t^*) \in X^*, \ |u(t)| \leq 1, \ t \in T^\tau, \tag{5.7}$$

is considered. As an initial state $x(\tau)$ a realized state $x^*(\tau)$ of nonlinear system (5.6) constructed under control $u^*(t)$, $t \in [0, \tau[$, and disturbance $w^*(t)$, $t \in [0, \tau[$, is taken.

While constructing a realization $u^*(\tau)$, $\tau \in T$, with the use of asymptotic expansions, family (5.3) is considered as an element of a more general family

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = Ax + a + \mu g(x) + bu, \ x(\tau) = z, \quad x(t^*) \in X^*, \quad |u(t)| \leq 1, \quad t \in T^\tau,$$

depending also on $\tau \in T$, $z \in X$.

Let $u^s(t|\tau, z) = \{u^s_\mu(t|\tau, z), \ \mu \to 0\}$, $t \in T^\tau$, be an asymptotically s-optimal open-loop control for a position (τ, z), Ω_s be a set of positions (τ, z), for which there exist open-loop asymptotically s-optimal controls, $u^s_\mu(\tau, z) = u^s_\mu(\tau|\tau, z)$, $(\tau, z) \in \Omega_s$.

Definition 5.1. A family $u^s(\tau, z) = \{u^s_\mu(\tau, z), \mu \to 0\}$, $(\tau, z) \in \Omega_s$, is said to be an s-optimal feedback control.

Consider the behavior of the system closed by the s-optimal feedback under unknown disturbances

$$\dot{x} = f(x) + bu^s_\delta(t, x) + w(t), \quad x(0) = x_0. \tag{5.8}$$

Denote by $x^*(t)$, $t \in T$, a trajectory of equation (5.8) corresponding to a realizing disturbance $w^*(t)$, $t \in T$.

A function $u^{s*}(t) = u^s_\delta(t, x^*(t))$, $t \in T_h$, is said to be a realization of s-optimal feedback, and any device able to calculate its values in real time is called an s-optimal controller (Gabasov, Kalinin, Kirillova and Naumovich, 1994). An s-optimal controller with a chosen value s is taken as Optimal Controller solving the problem of local optimization of a nonlinear system.

Let us describe an algorithm of operating a 1-optimal controller.

Before starting the control process, on interval T we fix N points s_l, $l = \overline{1, N}$, calculate and store $2N$ matrices $F(s_l)$, $F(t^*)F^{-1}(s_l)$, $l = \overline{1, N}$.

Suppose that the controller has been constructed and operated on interval $[0, \tau[$, and at instant $\tau \in T_h$ system (5.8) is in a state $x^*(\tau)$. We solve problem (5.7) for this state. The switching points and the Lagrange vector $z_0(\tau) = \{t^0_j(\tau), j = \overline{1, k^*(\tau)}; \nu^0(\tau)\}$ are calculated from equation (5.4), (5.5), where now $z = z(\tau)$, $k^* = k^*(\tau)$, $\gamma = \gamma(\tau)$, $\hat{g} = \hat{g}(\tau) = g - G(\tau)x^*(\tau) - \int_\tau^{t^*} G(\vartheta)ad\vartheta$.

A set $S(\tau) = \{\gamma(\tau), k^*(\tau)\}$ is said to be a structure of the optimal control of a base problem. If the structure $S(\tau)$ coincides with the structure $S(\tau - h)$, then we solve (5.4) by the Newton method. As an initial approximation $z_0^0(\tau)$ elements $z_0(\tau - h)$, constructed at instant $\tau - h$ are taken.

In the course of solving the base problem we form and store information which is used further for calculation of the coefficients of the first approximation: $\int_{t_j^0(\tau)}^{t_{j+1}^0(\tau)} F^{-1}(\vartheta)bd\vartheta$, $j = \overline{0, k^*(\tau)}$; $\int_{t_{j(l)}^0(\tau)}^{s_l} F(s_l)F^{-1}(\vartheta)bd\vartheta$, $j(l)$ is such a number that $t_{j(l)}^0(\tau) < s_l \le t_{j(l)+1}^0(\tau)$, $l = \overline{1, N(\tau)}$; $F^{-1}(t_j^0(\tau))$, $j = \overline{0, k^*(\tau)}$; $\int_\tau^{s_l} F(s_l)F^{-1}(\vartheta)ad\vartheta$, $l = \overline{1, N(\tau)}$. If integrations are performed in parallel, the complexity of forming this information is

$$A_\tau = (h + \sum_{l=1}^{l_0(\tau)} \max_{j=\overline{1,k^*(\tau)}} |t_j^{0(l)}(\tau) - t_j^{0(l-1)}(\tau)|)/t^*,$$

where $l_0(\tau)$ is a number of iterations of the Newton method.

To calculate $z_1(\tau) = \{t_j^1(\tau), j = \overline{1, k^*(\tau)}; \nu^1(\tau)\}$ we solve the system

$$I_0(z_0(\tau))z_1(\tau) = -R_1(z_0(\tau)), \tag{5.9}$$

where $I_0(z_0(\tau))$ is the Jacobi matrix of equation (5.4) constructed when solving the base problem. Values $x_1(t^*)$, $\psi_1(t_j^0(\tau))$, $j = \overline{1, k^*(\tau)}$, required for forming the right side of (5.9), are calculated from the equations

$$\dot{x}_1 = Ax_1 + g(x_0(t|\tau)), \quad x_1(\tau) = 0;$$
$$\dot{\psi}_1 = -A'\psi_1 - \frac{\partial g}{\partial x}(x_0(t|\tau))\psi_0(t|\tau), \quad \psi_1(t^*) = 0.$$

Fast calculation of the values $x_1(t^*)$, $\psi_1(t_j^0(\tau))$, $j = \overline{1, k^*(\tau)}$, is based on the use of quadratures

$$x_1(t^*) = \textstyle\sum_{l=1}^{N(\tau)} F(t^*)F^{-1}(s_l)g(x^0(s_l|\tau))h_l,$$
$$\psi_1'(t_j^0(\tau)) = \textstyle\sum_{l:\ s_l \ge t_j^0(\tau)} \psi_0'(s_l|\tau)\frac{\partial g'}{\partial x}(x^0(s_l|\tau))F(s_l)F^{-1}(t_j^0(\tau))h_l$$

where $N(\tau)$ is the number of fixed in advance points s_l, $l = \overline{1, N}$, located on interval T^τ; h_l, $l = \overline{1, N}$, are coefficients defined by the chosen quadrature.

Required values $x^0(s_l|\tau)$, $\psi_0(s_l|\tau)$, $l = \overline{1, N(\tau)}$, are calculated with the use of information formed before starting the controller and after solving the base problem:

$$x^0(s_l|\tau) = F(s_l)F^{-1}(\tau)x^*(\tau) + \textstyle\sum_{j=0}^{j(l)-1} F(s_l) \int_{t_j^0(\tau)}^{t_{j+1}^0(\tau)} F^{-1}(\vartheta)bd\vartheta(-1)^j\gamma(\tau) +$$
$$+ \textstyle\int_{t_{j(l)}^0(\tau)}^{s_l} F(s_l)F^{-1}(\vartheta)bd\vartheta(-1)^{j(l)}\gamma(\tau) + \int_\tau^{s_l} F(s_l)F^{-1}(\vartheta)ad\vartheta;$$
$$\psi_0'(s_l|\tau) = (c' - \nu^{0'}(\tau)H)F(t^*)F^{-1}(s_l), \quad l = \overline{1, N(\tau)}.$$

From equation (5.9) we find $z_1(\tau)$ and calculate $t_j^1(\delta|\tau) = t_j^0(\tau) + \delta t_j^1(\tau)$. If

$$\tau < t_1^1(\delta|\tau) < \cdots < t_{k^*(\tau)}^1(\delta|\tau) < t^*, \tag{5.10}$$

the structure of control $u^1(t, \delta|\tau, x^*(\tau))$, $t \in T^\tau$, coincides with the structure of the optimal control $u^0(t|\tau, x^*(\tau))$, $t \in T^\tau$, of the base problem. In this case, the input of system (5.6) is

fed by the signal $u^{1*}(\tau) = \gamma(\tau)$. The violation of condition (5.10) means that the number δ is greater than such a number $\mu_0 > 0$, (Gabasov et al., 1987; Kalinin, 2000), that for any μ, $|\mu| < \mu_0$ the structure of OC $u^0(t, \mu|\tau, x^*(\tau))$, $t \in T^\tau$, coincides with the structure of the control $u^0(t|\tau, x^*(\tau))$, $t \in T^\tau$. In this case, we put

$$u^{1*}(\tau) = \begin{cases} \gamma(\tau) & \text{if } t_1^1(\delta|\tau) > \tau, \\ -\gamma(\tau) & \text{if } t_1^1(\delta|\tau) < \tau. \end{cases}$$

In doing so, we assume that the length of intervals, where the structures of controls $u^1(t, \delta|\tau, x^*(\tau))$, $t \in T^\tau$, and $u^0(t|\tau, x^*(\tau))$, $t \in T^\tau$, are distinguished, is insignificant in comparison with the duration t^* of the process.

5.2 Global optimization of nonlinear systems

Consider the optimization problem for nonlinear system (1.1) on a domain X for which the accuracy δ of a linear approximation $\hat{f}(x)$, $x \in X$, does not allow to obtain satisfactory results by the method of section 5.1. The idea of extending the method of section 5.1 to more general domains is based on the use of PL approximations of a nonlinear element of the system. As in section 4, we assume that the closure of the domain X can be presented as a unification of polyhedral sets $X_1, X_2, \ldots, X_p$ such that int $X_i \cap$ int $X_j = \emptyset$, $i \neq j$.
As a function $\hat{f}(x)$, $x \in X$, we take a continuous function linear on each domain X_j, $j = \overline{1,p}$. The accuracy of approximation δ is defined again by expression (2.1). The solution to problem (1.1) is constructed with the use of two procedures: 1) solution to a PL problem; 2) asymptotic correction of the solution to a PL problem. The new element with respect to section 5.1 is asymptotic expansion of instants of transition $\theta_1, \ldots, \theta_{j^*-1}$ between domains of linearity of function $\hat{f}(x)$, $x \in X$.

5.3 Example

The suggested approach to optimization of nonlinear systems is illustrated by the problem of optimal damping of a mathematical pendulum

$$\begin{gathered} \int_0^{10} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 + u, \\ x_1(0) = 1.5, \; x_2(0) = 0, \quad x_1(10) = x_2(10) = 0, \quad 0 \leq u(t) \leq 0.5, \; t \in T = [0, 10], \end{gathered} \tag{5.11}$$

in the domain $X = \{(x_1, x_2) : |x_1| < \pi/2\}$.
We use two approximations of the nonlinear element $-\sin x_1$: 1) linear approximation $-x_1$, $x \in X$; 2) PL approximation $(1 - 4/\pi)x_1 + 1 - \pi/2$, $x \in X_1 = \{(x_1, x_2) : \pi/4 < x_1 < \pi/2\}$; $-x_1$, $x \in X_2 = \{(x_1, x_2) : |x_1| < \pi/4\}$. From the linear approximation we obtain $\delta_1 = 0.570796$, using the PL approximation one gets $\delta_2 = 0.110721$.
Thus, in the first case, problem (5.11) is imbedded into the family

$$\begin{gathered} \int_0^{10} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \mu(x_1 - \sin x_1)/\delta_1 + u, \\ x_1(0) = 1.5, \; x_2(0) = 0, \quad x_1(10) = x_2(10) = 0, \quad 0 \leq u(t) \leq 0.5, \; t \in T, \end{gathered} \tag{5.12}$$

in the second case, we use the family

$$\int_0^{10} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_2,$$
$$\dot{x}_2 = \begin{cases} (1-4/\pi)x_1 + 1 - \pi/2 + \mu((4/\pi - 1)x_1 - 1 + \pi/2 - \sin x_1)/\delta_2 + u, & x \in X_1, \\ -x_1 + \mu(x_1 - \sin x_1)/\delta_2 + u, & x \in X_2, \end{cases} \quad (5.13)$$
$$x_1(0) = 1.5,\ x_2(0) = 0, \quad x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5,\ t \in T.$$

The parametrized form of problem (5.13) is

$$\int_0^{10} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_2,$$
$$\dot{x}_2 = (1-4/\pi)x_1 + 1 - \pi/2 + \mu((4/\pi - 1)x_1 - 1 + \pi/2 - \sin x_1)/\delta_2 + u,\ t \in [0, \theta_1[,$$
$$x_1(0) = 1.5,\ x_2(0) = 0, \quad x_1(\theta_1) = \pi/4, \quad (5.14)$$
$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + \mu(x_1 - \sin x_1)/\delta_2 + u,\ t \in [\theta_1, 10],$$
$$x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5,\ t \in T.$$

The base problem for problem (5.14) is as follows

$$\int_0^{10} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_2,$$
$$\dot{x}_2 = (1-4/\pi)x_1 + 1 - \pi/2 + u,\ t \in [0, \theta_1[,\ x_1(0) = 1.5,\ x_2(0) = 0,\ x_1(\theta_1) = \pi/4, \quad (5.15)$$
$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ t \in [\theta_1, 10], \quad x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5,\ t \in T.$$

Table 1: Open-loop solutions

Control	Switching points	Instant θ_1	Performance index	Endpoint state
$u_1^0(t), t \in T$	0.722734, 2.418858 7.005920, 8.702044	—	1.696124	0.017095 -0.516580
$u_1^1(t), t \in T$	1.007916, 2.517443 7.547231, 9.017574	—	1.489935	-0.012199 -0.055845
$u_2^0(t), t \in T$	1.078442, 2.593439 7.371674, 8.876624	1.233553	1.509974	-0.013850 -0.111839
$u_2^1(t), t \in T$	1.064668, 2.573226 7.553250, 9.036841	1.218658	1.496074	-0.001177 -0.008707
$u^0(t,\delta), t \in T$	1.064547, 2.573684 7.566068, 9.049386	—	1.496228	10^{-8} 10^{-8}

Table 1 contains results of open-loop solution to problem (5.11). Trajectories of system (5.11) have been constructed under the following controls: 1) $u_1^0(t)$, $t \in T$, is the optimal control of the linear base problem

$$\int_0^{10} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad x_1(0) = 1.5,\ x_2(0) = 0,$$
$$x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5,\ t \in T;$$

2) $u_1^1(t)$, $t \in T$, is the realization of asymptotically 1-optimal open-loop control of problem (5.12) for the fixed value $\mu = \delta_1$; 3) $u_2^0(t)$, $t \in T$, is the optimal control of PL base problem (5.15); 4) $u_2^1(t)$, $t \in T$, is the realization of asymptotically 1-optimal open-loop control of problem (5.14)

for the fixed value $\mu = \delta_2$; 5) $u^0(t, \delta)$, $t \in T$, is the optimal open-loop control of problem (5.11) constructed by the iterative refining procedure (Gabasov et al., 1998) based on the Pontriagin maximum principle. This procedure requires integration of a nonlinear system on interval T on each iteration, so it is worthy for comparing results at illustrative examples but not for real-time control.

In each case the control has the form

$$u(t) = \begin{cases} 0, & t \in [0, t_1[\cup[t_2, t_3[\cup[t_4, 10[, \\ 0.5, & t \in [t_1, t_2[\cup[t_3, t_4[. \end{cases} \tag{5.16}$$

Table 1 contains switching points of these controls, the values of the transition instant θ_1 when PL approximation is used, the values of the performance index and the endpoint states of system (5.11). Curve 1 in Figure 3 presents the state trajectory of system (5.11) constructed by the control $u_2^1(t)$, $t \in T$.

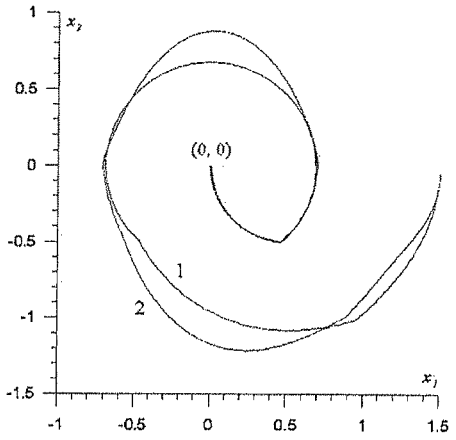

Figure 3: Trajectories of systems (5.11), (5.17)

Let us construct the positional solution to problem (5.11). At first, we analyse the behavior of system (5.11) without disturbances under the realization $u^{1*}(t)$, $t \in T$, constructed by the 1-optimal controller ($h = 0.01$). Necessary values of the functions $x_1(t)$, $\psi_1(t)$, $t \in T^\tau$, $\tau \in T_h$, have been constructed by method (Forsythe, Malcolm and Moler, 1977). The control $u^{1*}(t)$, $t \in T$, has the form (5.16) with switching points 1.06, 2.58, 7.567630, 9.039940 and the transition instant $\theta_1^* = 1.21$. At the instant $t^* = 10$ the trajectory of system (5.11) reaches the state $(-0.000129, -0.000455)$, the value of the performance index is 1.496155.

Table 2: Positional solutions

Control	Switching points	Performance index	Endpoint state
$u^{1*}(t), t \in T$	1.3, 2.7, 7.789350, 9.287743	1.449196	0.000016, -0.001192
$u_{10}^{1*}(t), t \in T$	1.3, 2.7, 7.717981, 9.257927	1.469973	-0.003723, -0.002823
$u_{50}^{1*}(t), t \in T$	1.3, 2.7, 7.778486, 9.278203	1.449858	-0.000074, -0.001229
$u_{100}^{1*}(t), t \in T$	1.3, 2.7, 7.778420, 9.278330	1.449955	10^{-6}, -0.001202
$cu_{300}^{1*}(t), t \in T$	1.3, 2.7, 7.778403, 9.278366	1.449982	0.000014, -0.001133

Now we consider the behavior of the system under the disturbance $w^*(t) = 0.4 \sin 3t$, $t \in [0, 7[$, $w^*(t) \equiv 0$, $t \geq 7$, unknown for the controller:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 + u + w^*(t), \quad x_1(0) = 1.5, \; x_2(0) = 0. \tag{5.17}$$

Trajectories of system (5.17) have been constructed under various realizations of the optimal feedback: 1) $u^{1*}(t)$, $t \in T$, has been constructed by the 1-optimal controller with calculating the values of the functions $x_1(t)$, $\psi_1(t)$, $t \in T^\tau$, $\tau \in T_h$, by the method (Forsythe et al., 1977); 2) $u_N^{1*}(t)$, $t \in T$, has been constructed by the 1-optimal controller with the use of the middle box quadrature with N knots for calculating the values of the functions $x_1(t)$, $\psi_1(t)$, $t \in T^\tau$, $\tau \in T_h$, for $N = 10$, 50, 100, 300. The constructed controls have form (5.16). For each of them the realized transition instant is $\theta_1^* = 1.39$.

Table 2 contains switching points of the realizations, the corresponding values of the performance index and the endpoint states of system (5.17). Curve 2 in Figure 3 presents the state trajectory of system (5.17) under control $u^{1*}(t)$, $t \in T$.

6 Conclusion

In the paper a method of realization of optimal state feedbacks for control of dynamical systems is described. The algorithm of functioning Optimal Controller which generates current values of optimal feedback for systems with nonlinear dynamics is based on fast methods for solving a series of auxiliary OC problems and is supplemented by asymptotic methods of correction of approximate solutions.

The suggested approach provides two tools for increasing the quality of solution: 1) improving the accuracy δ of PL approximation $\hat{f}(x)$, $x \in X$; 2) increasing the degree s of asymptotic expansions of defining elements of OC.

Today, with availability of modern computer facilities one may conclude that the problem of calculating optimal open-loop controls can be solved for the majority of applied problems. The results of the paper show that the combination of mathematical methods of OC theory and computer facilities allows to solve much more complicated and important in applications problem of synthesis of OC of feedback type. In doing so, only quite rough information about the further solution is formed in advance, which does not require any significant amount of computer memory. Then, during actual control process, this solution is being quickly corrected by means of additional calculations by microprocessors according to the incoming data on current states of system being optimized. Thus, one manages to overcome the famous "curse of dimensionality" of dynamic programming. Involvement of up-to-date computer facilities not only at preliminary stages but (which is the most important) also during an actual control process, in addition to modern mathematical methods, is a promising direction for control theory.

The presented approach is currently developed for non-deterministic OC problems with set-membership uncertainty, OC problems with incomplete and inexact measurements of system's state, OC problems for systems governed by time-delayed and partial differential equations.

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Analytical and Numerical Solutions of Linear Singular Integral Equations

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ABSTRACT

In this paper, by a new concept and method we shall give practical and numerical solutions of linear singular integral equations by combining the two theories of the Tikhonov regularization and reproducing kernels.

Keywords: Linear singular integral equation, Hilbert transform, Carleman's equation, approximation of functions, reproducing kernel, Tikhonov regularization, Paley-Wiener space, generalized inverse, approximate inverse, noise, error.

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1 Introduction

Singular integral equations are presently encountered in a wide range of mathematical models, for instance in acoustics, fluid dynamics, elasticity and fracture mechanics. Together with these models, a variety of methods and applications for these integral equations have been developed (see, for example, Estrada and Kanwal, 2000; Mikhlin and Prössdorf, 1986; Muskhelishvili, 1972). As a typical singular integral equation of them, we shall consider the Carleman's equation over a real interval, for any $L_2(-1,1)(:=L_2)$ function g and for real or complex valued L_2 (or bounded integrable) functions a, b

$$(Ly)(t) = a(t)y(t) + \frac{b(t)}{\pi i}\text{p.v.}\int_{-1}^{+1}\frac{y(\zeta)}{\zeta - t}\,\mathrm{d}\zeta = g(t) \quad \text{on} \quad -1 < t < 1 \tag{1.1}$$

in the class of the functions of the Paley-Wiener spaces. We denote by p.v. the Cauchy's principal value of the integral. According to Muskhelishvili 1972, the operator L, satisfying a condition $a^2(t) - b^2(t) \neq 0$ for $-1 < t < 1$, is called a regular type operator. It is well known that

the equation (1.1) always has an explicit solution for a regular type operator. See also (Mikhlin and Prössdorf, 1986). However, when $a^2(t) - b^2(t) = 0$, there exist solutions if and only if g satisfies a special condition (see, Section 5, and also, cf. Estrada and Kanwal, 2000). The analysis of this case is important for the kinked crack problem. Accordingly, we shall introduce a new method which gives simple and natural approximate solutions for linear singular integral equations including the case where the condition of a regular type operator is violated. We can deal with a general linear singular integral equation, however, for simplicity, we shall state the results for this most typical case.

Indeed, in this paper we shall introduce a new approach for some general linear singular integral equations with bounded linear integral operators by transforming the integral equations to integral equations of Fredholm of the second type with sufficiently smooth coefficients and by using the two theories of the Tikhonov regularization and reproducing kernels. See Section 4. We shall use the Paley-Wiener spaces for approximate function spaces and so, in the next Section 2 we shall set their fundamental properties. In Section 3, we shall state our basic idea and method using the Tikhonov regularization and the theory of reproducing kernels. In Section 4, following the method, we shall give our approximate solution for the equation (1.1). In Section 5, we shall give a concrete analytic representation of the approximate solutions for the Carleman's integral equation for the case of the whole space and with constant coefficients. In Section 6, we shall give a constructive representation of the approximate solutions for the general case of (1.1) by using a complete orthonormal system (abbreviated by CONS) expansion. In Section 7, we shall discuss the properties of the approximate solutions when the parameter tends to zero, for the case of Section 6.

2 Paley-Wiener space and reproducing kernels

At first we shall fix notations following the book (Stenger, 1993) and at the same time we shall show the basic relation of the sampling theory and the theory of reproducing kernels.

We shall consider the integral transform, for $L_2(-\pi/h, +\pi/h), (h > 0)$ functions g

$$f(z) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} g(t) e^{-izt} \, \mathrm{d}t. \tag{2.1}$$

In order to identify the image space following the theory of reproducing kernels (Saitoh, 1997), we form the reproducing kernel

$$\begin{aligned} K_h(z, \overline{u}) &= \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-izt} \overline{e^{-iut}} \, \mathrm{d}t \\ &= \frac{1}{\pi(z - \overline{u})} \sin \frac{\pi}{h}(z - \overline{u}) \\ &:= \frac{1}{h} \mathrm{Sinc} \left(\frac{z - \overline{u}}{h} \right) \\ &:= \frac{1}{h} S(k, h)(z - \overline{u} + hk), \end{aligned} \tag{2.2}$$

by the notations in the book (Stenger, 1993). The image space of (2.1) is called the Paley-Wiener space $W\left(\frac{\pi}{h}\right)(:=W_h)$ comprised of all analytic functions of exponential type satisfying, for some constant C and as $z \to \infty$

$$|f(z)| \leq C \exp\left(\frac{\pi|z|}{h}\right)$$

and

$$\int_{\mathrm{R}} |f(x)|^2 \, \mathrm{d}x < \infty.$$

From the identity

$$K_h(jh, j'h) = \frac{1}{h}\delta(j, j')$$

(the Kronecker's δ), since $\delta(j, j')$ is the reproducing kernel for the Hilbert space ℓ^2, from the general theory of integral transforms and the Parseval's identity we have the isometric identities in (2.1)

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} |g(t)|^2 \, \mathrm{d}t &= h\sum_j |f(jh)|^2 \\ &= \int_{\mathrm{R}} |f(x)|^2 \, \mathrm{d}x. \end{aligned}$$

That is, the reproducing kernel Hilbert space H_{K_h} with $K_h(z, \overline{u})$ is characterized as a space comprising the Paley-Wiener space W_h with the norm squares defined above. Here we used the well-known result that $\{jh\}_j$ is a uniqueness set for the Paley-Wiener space W_h; that is, $f(jh) = 0$ for all j implies $f \equiv 0$. Then, the reproducing property of $K_h(z, \overline{u})$ states that

$$\begin{aligned} f(x) &= (f(\cdot), K_h(\cdot, x))_{H_{K_h}} \\ &= h\sum_j f(jh)K_h(jh, x) \\ &= \int_{\mathrm{R}} f(\xi)K_h(\xi, x) \, \mathrm{d}\xi, \end{aligned}$$

in particular, for $x \in \mathbf{R}$. This representation is the sampling theorem which represents the whole data $f(x)$ in terms of the discrete data $\{f(jh)\}_j$. Furthermore, for a general theory for the sampling theory and error estimates for some finite points $\{hj\}_j$, (see Saitoh, 1997).

3 Reproducing kernels and the Tikhonov regularization

The good application of reproducing kernels to the Tikhonov regularization is given by the following propositions:

Proposition 3.1. (Byun and Saitoh, 1994; Saitoh, 2004) *Let H_K be a Hilbert space admitting the reproducing kernel $K(p, q)$ on a set E. Let $L : H_K \to \mathcal{H}$ be a bounded linear operator on H_K into a Hilbert space $\mathcal{H}$. For $\lambda > 0$ introduce the inner product in H_K and call it H_{K_λ} as*

$$(f_1, f_2)_{H_{K_\lambda}} = \lambda(f_1, f_2)_{H_K} + (Lf_1, Lf_2)_{\mathcal{H}}, \tag{3.1}$$

then H_{K_λ} is the Hilbert space with the reproducing kernel $K_\lambda(p,q)$ on E and satisfying the equation

$$K(\cdot,q) = (\lambda I + L^*L)K_\lambda(\cdot,q), \tag{3.2}$$

where L^* is the adjoint of $L : H_K \to \mathcal{H}$.

Proposition 3.2. (Saitoh, 2004) *Let H_K, L, $\mathcal{H}$, E and K_λ be as in* Proposition 3.1. *Then, for any $\lambda > 0$ and for any $g \in \mathcal{H}$, the extremal function in*

$$\inf_{f \in H_K} \left(\lambda \|f\|^2_{H_K} + \|Lf - g\|^2_{\mathcal{H}} \right) \tag{3.3}$$

exists uniquely and the extremal function is represented by

$$f^*_{\lambda,g}(p) = (g, LK_\lambda(.,p))_{\mathcal{H}} \tag{3.4}$$

which is the member of H_K attaining the infimum in (3.3).

In (3.4), when g contains errors or noises, we need its error estimation. For this, we can obtain the general result:

Theorem 3.3. *In* (3.4), *we obtain the estimate*

$$|f^*_{\lambda,g}(p)| \leq \frac{1}{\sqrt{\lambda}} \sqrt{K(p,p)} \|g\|_{\mathcal{H}}.$$

Proof. From (3.4) and (3.2), we have

$$\begin{aligned} |f^*_{\lambda,g}(p)| &= |(\frac{1}{\lambda I + L^*L} L^*g, K(.,p))_{H_K}| \\ &\leq \|\frac{1}{\lambda I + L^*L} L^*g\|_{H_K} \sqrt{K(p,p)}. \end{aligned}$$

As we see from the spectral theory, since

$$\frac{1}{\lambda I + L^*L} L^* = L^* \frac{1}{\lambda I + LL^*},$$

we have the inequality

$$\begin{aligned} \|\frac{1}{\lambda I + L^*L} L^*g\|^2_{H_K} &= \left(L^* \frac{1}{\lambda I + LL^*} g, L^* \frac{1}{\lambda I + LL^*} g \right)_{H_K} \\ &= \left(LL^* \frac{1}{\lambda I + LL^*} g, \frac{1}{\lambda I + LL^*} g \right)_{\mathcal{H}} \\ &\leq \left\| LL^* \frac{1}{\lambda I + LL^*} g \right\|_{\mathcal{H}} \left\| \frac{1}{\lambda I + LL^*} g \right\|_{\mathcal{H}} \\ &\leq \|g\|_{\mathcal{H}} \frac{1}{\lambda} \|g\|_{\mathcal{H}}, \end{aligned}$$

(Engl, Hanke and Neubauer, 2000, pp. 71-73), and so, we obtain the desired result. □

For the properties and error estimates for the limit

$$\lim_{\lambda \to 0} f^*_{\lambda,g}(p),$$

see (Saitoh, 2005b; Saitoh, 2005c). In particular, when there exists the Moore-Penrose generalized solution for the operator equation

$$Lf = g$$

in (3.3), the limit converges uniformly to the Moore-Penrose generalized solution on any subset of E such that $K(p,p)$ is bounded.
For many concrete applications of these general theorems, see, for example, (Asaduzzaman, Matsuura and Saitoh, 2005; Matsuura and Saitoh, 2005; Saitoh, 2004; Saitoh, 2005a).

4 Construction of approximate solutions by solving Fredholm's integral equation

Following the idea and method in Section 3, we shall consider the extremal problem:

$$\inf_{f \in W_h} \left\{ \lambda \|f\|^2_{W_h} + \|Lf - g\|^2_{L_2} \right\}. \tag{4.1}$$

Note that L is a bounded linear operator from W_h into L_2, as we see from the Cauchy-Schwarz inequality and a boundedness

$$\int_{-1}^{1} \left| \frac{1}{\pi} \text{p.v.} \int_{-1}^{1} \frac{F(\xi)}{\xi - \eta} \, \mathrm{d}\xi \right|^2 \mathrm{d}\eta \leq \int_{-1}^{1} |F(\xi)|^2 \, \mathrm{d}\xi$$

(cf. Stenger, 1993) of the finite Hilbert transform (see, also cf. Mikhlin and Prössdorf, 1986). We wish to construct the reproducing kernel for the Hilbert space with the norm square

$$\lambda \|f\|^2_{W_h} + \|Lf\|^2_{L_2}. \tag{4.2}$$

From Proposition 3.1 this reproducing kernel $K_\lambda(t,t')$ is calculated by solving an integral equation of Fredholm of the second kind:

$$\frac{1}{\lambda} K_h(t,t') = K_\lambda(t,t') + \frac{1}{\lambda} (LK_\lambda(\cdot,t')(p), (LK_h(\cdot,t))(p))_{L_2}. \tag{4.3}$$

Then, the extremal function f^*_λ in (4.1) is given by

$$f^*_\lambda(t) = (g, LK_\lambda(\cdot,t))_{L_2}. \tag{4.4}$$

By applying the operator L to (4.3) with respect to functions of t, we have

$$\frac{1}{\lambda}L_tK_h(t,t') = L_tK_\lambda(t,t') + \frac{1}{\lambda}L_t(LK_\lambda(\cdot,t')(p),(LK_h(\cdot,t))(p))_{L_2}. \tag{4.5}$$

Therefore,

$$LK_\lambda(\cdot,t)$$

is given as the solution of the integral equation of Fredholm of the second kind for fixed t. Note that the functions in (4.5)

$$L_tK_h(t,t')$$

and

$$L_t\overline{LK_h(\cdot,t)}$$

are calculated by using Fourier's integral and the formula: For the Fourier transform $\mathcal{F}$,

$$\mathcal{F}[f](\xi) = \int_{-\infty}^{\infty} f(t)e^{-i\xi t}\,\mathrm{d}t, \tag{4.6}$$

and for the Hilbert transform $\mathcal{H}$,

$$[\mathcal{H}y](t) = \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty}\frac{y(\zeta)}{\zeta - t}\,\mathrm{d}\zeta,$$

$$\mathcal{H}F = \mathcal{F}^{-1}\left(-i\text{sgn}\xi(\mathcal{F}F)(\xi)\right). \tag{4.7}$$

In particular, note that the Hilbert transform of the function

$$\frac{\sin(ax)}{x} \quad (a>0)$$

is

$$\frac{\cos(ax)-1}{x}.$$

5 Carleman's equation for the case of the whole line and with complex constant coefficients

In this section as a typical example of applying our method, we shall consider approximate solutions for Carleman's equation for the case of the whole line and with complex constant coefficients a, b:

$$(\tilde{L}y)(t) = ay(t) + \frac{b}{\pi i}\,\text{p.v.}\int_{-\infty}^{\infty}\frac{y(\zeta)}{\zeta - t}\,\mathrm{d}\zeta = g(t) \quad \text{on} \quad -\infty < t < \infty. \tag{5.1}$$

In the same way as (4.5), we obtain

$$\frac{1}{\lambda}\tilde{L}_tK_h(t,t') = \tilde{L}_tK_\lambda(t,t') + \frac{1}{\lambda}\tilde{L}_t(\tilde{L}K_\lambda(\cdot,t'))(p),(\tilde{L}K_h(\cdot,t))(p))_{L_2}. \tag{5.2}$$

By virtue of Propositions in Section 3 if we can find the solution $\tilde{L}K_\lambda(t,t')$ in (5.2), we can obtain the approximate solution of (5.1).

Firstly, we rewrite (5.2) as

$$\frac{1}{\lambda}\tilde{L}_t K_h(t,t') = \tilde{L}_t K_\lambda(t,t') + \frac{1}{\lambda}\int_{\mathbf{R}} \tilde{L}_p K_\lambda(\cdot,t')(p)\tilde{L}_t\overline{\tilde{L}_p K_h(\cdot,t)(p)}\,\mathrm{d}p. \tag{5.3}$$

The left-hand side of (5.3) becomes

$$\frac{1}{\lambda}\tilde{L}_t K_h(t,t') = \frac{1}{\lambda}\left(a\frac{\sin\frac{\pi}{h}(t-t')}{\pi(t-t')} + \frac{b}{i}\frac{\cos\frac{\pi}{h}(t-t')-1}{\pi(t-t')}\right). \tag{5.4}$$

And the kernel of the integral on the right-hand side of (5.3) becomes

$$\begin{aligned}\tilde{L}_{t'}\overline{\tilde{L}_t K_h(t,t')} &= \tilde{L}_{t'}\left(\bar{a}\frac{\sin\frac{\pi}{h}(t-t')}{\pi(t-t')} - \frac{\bar{b}}{i}\frac{\cos\frac{\pi}{h}(t-t')-1}{\pi(t-t')}\right)\\ &= (|a|^2+|b|^2)\frac{\sin\frac{\pi}{h}(t-t')}{\pi(t-t')}\\ &\quad -(a\bar{b}+\bar{a}b)i\frac{\cos\frac{\pi}{h}(t-t')-1}{\pi(t-t')}.\end{aligned} \tag{5.5}$$

Note that

$$\mathcal{F}\left[\frac{\sin\frac{\pi t}{h}}{\pi t}\right](\xi) := p(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| < \frac{\pi}{h},\\ \frac{1}{2} & \text{if} \quad |\xi| = \frac{\pi}{h},\\ 0 & \text{if} \quad |\xi| > \frac{\pi}{h},\end{cases} \tag{5.6}$$

$$\mathcal{F}\left[\frac{1}{\pi t}\right](\xi) := q_1(\xi) = \begin{cases} i & \text{if} \quad \xi < 0,\\ -i & \text{if} \quad \xi \geq 0,\end{cases} \tag{5.7}$$

$$\mathcal{F}\left[\frac{\cos\frac{\pi t}{h}}{\pi t}\right](\xi) := q_2(\xi) = \begin{cases} i & \text{if} \quad \xi < -\frac{\pi}{h},\\ 0 & \text{if} \quad -\frac{\pi}{h} < \xi < \frac{\pi}{h},\\ -i & \text{if} \quad \xi > \frac{\pi}{h},\end{cases} \tag{5.8}$$

$$\mathcal{F}\left[\frac{\cos\frac{\pi t}{h}-1}{\pi t}\right](\xi) := q(\xi) = \begin{cases} 0 & \text{if} \quad |\xi| > \frac{\pi}{h},\\ -i & \text{if} \quad -\frac{\pi}{h} < \xi < 0,\\ i & \text{if} \quad 0 < \xi < \frac{\pi}{h}.\end{cases} \tag{5.9}$$

If we take the Fourier transform of both sides of (5.3) with respect to t and we assume $\tilde{L}_t K_\lambda(t,t') := \varphi(t-t')$, then by using the convolution theorem, we have

$$\begin{aligned}&\frac{e^{-i\xi t'}}{\lambda}\left(ap(\xi) - biq(\xi)\right)\\ &= e^{-i\xi t'}\mathcal{F}[\varphi](\xi)\\ &\quad + \frac{e^{-i\xi t'}}{\lambda}\left(|a|^2+|b|^2)p(\xi) - (a\bar{b}+\bar{a}b)iq(\xi)\right)\mathcal{F}[\varphi](\xi)\end{aligned} \tag{5.10}$$

and so,

$$\mathcal{F}[\varphi](\xi) = \frac{ap(\xi) - biq(\xi)}{\lambda + (|a|^2 + |b|^2)p(\xi) - (a\bar{b} + \bar{a}b)iq(\xi)}. \tag{5.11}$$

From (5.6) and (5.9), (5.11) is rewritten by

$$\mathcal{F}[\varphi](\xi) = \begin{cases} 0 & \text{if} \quad |\xi| > \frac{\pi}{h}, \\ \frac{a-b}{\lambda+|a-b|^2} & \text{if} \quad -\frac{\pi}{h} < \xi < 0, \\ \frac{a+b}{\lambda+|a+b|^2} & \text{if} \quad 0 < \xi < \frac{\pi}{h}. \end{cases} \tag{5.12}$$

Taking the inverse Fourier transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}[f](\xi)e^{i\xi t}\,\mathrm{d}\xi \tag{5.13}$$

of (5.12) we can find the solution of (5.3)

$$\begin{aligned} \tilde{L}_t K_\lambda(t,t') &= \frac{a-b}{\lambda+|a-b|^2}\frac{1}{2\pi i(t-t')}\cdot\left(1 - e^{-\frac{i\pi}{h}(t-t')}\right) \\ &\quad + \frac{a+b}{\lambda+|a+b|^2}\frac{1}{2\pi i(t-t')}\cdot\left(e^{\frac{i\pi}{h}(t-t')} - 1\right). \end{aligned} \tag{5.14}$$

The reproducing kernel $K_\lambda(t,t')$ is represented as follows:

$$\begin{aligned} K_\lambda(t,t') &= \frac{1}{\lambda+|a+b|^2}\frac{1}{2\pi}\int_{-\pi/h}^{0} e^{-i\xi(t-t')}\,\mathrm{d}\xi \\ &\quad + \frac{1}{\lambda+|a-b|^2}\frac{1}{2\pi}\int_{0}^{\pi/h} e^{-i\xi(t-t')}\,\mathrm{d}\xi. \end{aligned}$$

Theorem 5.1. *In* (5.1)*, for any function* $g \in L_2$*, the best approximate solution* $f^*_{\lambda,h,g}$ *is represented by*

$$\begin{aligned} f^*_{\lambda,h,g}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\left[\frac{\overline{(a+b)}}{\lambda+|a+b|^2}\int_{-\pi/h}^{0} e^{i\eta(\xi-t)}\,\mathrm{d}\eta\right. \\ &\quad \left. + \frac{\overline{(a-b)}}{\lambda+|a-b|^2}\int_{0}^{\pi/h} e^{i\eta(\xi-t)}\,\mathrm{d}\eta\right]\,\mathrm{d}\xi. \end{aligned}$$

For a function $f \in W_h$ *if we take* g *as* $\tilde{L}f = g$*, then we obtain the result*

$$\lim_{\lambda\to 0} f^*_{\lambda,h,g}(t) = f(t),$$

uniformly.

Proof. The latter result in this theorem may be derived directly by using the representations (2.1) and $f^*_{\lambda,h,g}(t)$. □

Note that the regular type case, $a^2 - b^2 \neq 0$, we can take $\lambda = 0$; that is, we do not need the Tikhonov regularization in our problem. It is a trivial case.
In Theorem 5.1, we see directly

$$\tilde{L}_t f^*_{\lambda,h,g}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\left[\frac{|a+b|^2}{\lambda+|a+b|^2}\int_{-\pi/h}^{0} e^{i\eta(\xi-t)}\,\mathrm{d}\eta + \frac{|a-b|^2}{\lambda+|a-b|^2}\int_{0}^{\pi/h} e^{i\eta(\xi-t)}\,\mathrm{d}\eta\right]\mathrm{d}\xi.$$

Therefore, for the cases $\lambda = 0$ and $a^2 - b^2 \neq 0$

$$\lim_{h\to 0} \tilde{L}_t f^*_{0,h,g}(t) = g(t)$$

at points t where g is continuous.
In particular, when $a = 0$ and $b = i$; that is, for the Hilbert transform case, we obtain for $\lambda = 0$

Corollary 5.2. *For the extremal problem*

$$\inf_{f\in W_h}\left\{\|\mathcal{H}f - g\|^2_{L_2}\right\}, \tag{5.15}$$

the extremal function $f^*_{H,h,g}$ attaining the infimum exists uniquely and it is given by

$$f^*_{H,h,g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(\xi)\frac{1}{\xi - t}\left[\cos\frac{\pi}{h}(\xi - t) - 1\right]\mathrm{d}\xi$$

and then,

$$\left[\mathcal{H}f^*_{H,h,g}\right](t) = (g, K_h(\cdot, t))_{L_2};$$

that is, $\left[\mathcal{H}f^*_{H,h,g}\right]$ is the orthogonal projection of g onto the Paley-Wiener space W_h.

Corollary 1 also means to give an approximate Hilbert transform for any L_2 function g by an ordinary integral; that is,

$$\lim_{h\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty} g(\xi)\frac{1}{\xi - t}\left[1 - \cos\frac{\pi}{h}(\xi - t)\right]\mathrm{d}\xi = [\mathcal{H}g](t)$$

at the points t where g is continuous.
In particular, note that for the singular cases, $a^2 - b^2 = 0$, the integral equations have the solutions only for very special functions g.
For example, if $a = \pm b$, since $\mathcal{H}(\mathcal{H}y) = -y$, from (5.1),

$$a\mathcal{H}y \mp \frac{a}{i}y = \mathcal{H}g$$

and so, we see that g must satisfy the relation

$$\mathcal{H}g = \pm ig,$$

(see, for the details, Estrada and Kanwal, 2000, pp. 270). The following two Corollaries give the solutions for the singular cases for general L_2 functions g, and coincide with the results for the very special functions g in the book (Estrada and Kanwal, 2000).

Corollary 5.3. *If $a = b = 1$, then we obtain*

$$\begin{aligned}\lim_{\lambda \to 0} f^*_{\lambda,h,g}(t) &:= f^*_{+0,h,g}(t) \\ &= \frac{1}{4}\left[i f^*_{H,h,g}(t) + (g, K_h(\cdot,t))_{L_2}\right].\end{aligned}$$

Furthermore,

$$\lim_{h \to 0} \tilde{L}_t f^*_{+0,h,g}(t) = \frac{i}{2} f^*_{H,h,g}(t) + \frac{1}{2}(g, K_h(\cdot,t))_{L_2}.$$

If the condition $\mathcal{H}g = ig$ for existence of the solutions is satisfied, then

$$\lim_{h \to 0} f^*_{+0,h,g}(t) = \frac{1}{2} g(t)$$

and

$$\lim_{h \to 0} \tilde{L}_t f^*_{+0,h,g}(t) = g(t)$$

at the points t where $g(t)$ is continuous.

Corollary 5.4. *If $a = 1, b = -1$, then we obtain*

$$f^*_{+0,h,g}(t) = \frac{1}{4}\left[-i f^*_{H,h,g}(t) + (g, K_h(\cdot,t))_{L_2}\right].$$

Furthermore,

$$\lim_{h \to 0} \tilde{L}_t f^*_{+0,h,g}(t) = -\frac{i}{2} f^*_{H,h,g}(t) + \frac{1}{2}(g, K_h(\cdot,t))_{L_2}.$$

If the condition $\mathcal{H}g = -ig$ for existence of the solutions is satisfied, then

$$\lim_{h \to 0} f^*_{+0,h,g}(t) = \frac{1}{2} g(t)$$

and

$$\lim_{h \to 0} \tilde{L}_t f^*_{+0,h,g}(t) = g(t)$$

at the points t where $g(t)$ is continuous.

Therefore, in Theorem 5.1, we obtain the explicit representations of the approximate solutions including the singular cases. Surprisingly enough, we can obtain the explicit representations of the "solutions" for any L_2 function g.

6 Construction of approximate solutions by using a CONS expansion

In order to escape the Fredholm's integral equation in Section 4 and in order to derive an effective discretization, we shall consider a new algorithm in the following way:
At first, we introduce a complete orthonormal system $\{\varphi_j\}_{j=1}^{\infty}$ for the Hilbert space L_2. For example, we can take the Legendre functions

$$\varphi_j(t) = \sqrt{\frac{2j+1}{2}} P_j(t);$$

for

$$\begin{aligned} P_j(t) &= \frac{1}{2^j j!}\frac{d^j}{dt^j}(t^2-1)^j \\ &= \sum_{k=0}^{[j/2]} \frac{(-1)^k (2j-2k)!}{2^j k!(j-k)!(j-2k)!} t^{j-2k}. \end{aligned}$$

For a sufficiently large N and for fixed $\{\lambda_j\}_{j=1}^{N}(\lambda_j > 0)$, we consider the extremal problem for (4.1)

$$\inf_{f\in W_h}\left\{\lambda\|f\|_{W_h}^2 + \sum_{j=1}^{N}\lambda_j|((Lf)(t)-g(t),\varphi_j(t))_{L_2}|^2\right\}. \tag{6.1}$$

That is,

$$\|(Lf)(t)-g(t)\|_{L_2}^2$$

is replaced by

$$\sum_{j=1}^{N}\lambda_j|((Lf)(t)-g(t),\varphi_j(t))_{L_2}|^2.$$

Then, we shall give an algorithm constructing the reproducing kernel $K_{\lambda,\lambda_j}^{(N)}(t,t')$ of the Hilbert space $H_{K_{\lambda,\lambda_j}^{(N)}}$ with the norm square

$$\lambda\|f\|_{W_h}^2 + \sum_{j=1}^{N}\lambda_j|((Lf)(p),\varphi_j(p))_{L_2}|^2. \tag{6.2}$$

We shall start with the first step. The reproducing kernel $K^{(1)}(t,t')$ of the Hilbert space with the norm square

$$\lambda\|f\|_{W_h}^2 + \sum_{j=1}^{1}\lambda_j|((Lf)(p),\varphi_j(p))_{L_2}|^2 \tag{6.3}$$

is given by

$$\begin{aligned} K^{(1)}(t,t') &= K^{(0)}(t,t') \\ &\quad - \frac{\lambda_1(\varphi_1(p),(LK^{(0)}(\cdot,t))(p)))_{L_2}((LK^{(0)}(\cdot,t'))(q),\varphi_1(q))_{L_2}}{1+\lambda_1(L(\varphi_1(p),(LK^{(0)}(\cdot,t'))(p))_{L_2}(q),\varphi_1(q))_{L_2}}, \end{aligned} \tag{6.4}$$

for

$$K^{(0)}(t,t') = \frac{1}{\lambda}K_h(t,t')$$

((Saitoh, 1997, pp. 50-52, and 81)). For the second step, the reproducing kernel $K^{(2)}(t,t')$ of the Hilbert space with the norm square

$$\lambda\|f\|^2_{W_h} + \sum_{j=1}^{2}\lambda_j|((Lf)(p),\varphi_j(p))_{L_2}|^2 \tag{6.5}$$

is given by

$$\begin{aligned}K^{(2)}(t,t') &= K^{(1)}(t,t') \\ &\quad - \frac{\lambda_2(\varphi_2(p),(LK^{(1)}(\cdot,t))(p)))_{L_2}((LK^{(1)}(\cdot,t')(q),\varphi_2(q))_{L_2}}{1+\lambda_2(L(\varphi_2(p),(LK^{(1)}(\cdot,t'))(p))_{L_2}(q),\varphi_2(q))_{L_2}},\end{aligned} \tag{6.6}$$

by using the reproducing kernel $K^{(1)}(t,t')$. In this way, we can obtain the desired representation of $K^{(N)}_{\lambda,\lambda_j}(t,t')$. Then, we obtain

Theorem 6.1. *For any $g \in L_2$, the extremal function $f^{(N)}_{\lambda,\lambda_j}$ in the extremal problem* (6.1) *is given by*

$$f^{(N)}_{\lambda,\lambda_j}(t) = \sum_{j=1}^{N}\lambda_j(g,\varphi_j)_{L_2}(\varphi_j(p),(LK^{(N)}_{\lambda,\lambda_j}(\cdot,t))(p))_{L_2}. \tag{6.7}$$

We consider a general extremal problem in (6.1) by considering a general weight $\{\lambda_j\}$. This means that for a larger λ_{j_0}, the speed of the convergence

$$((Lf)(p),\varphi_{j_0}(p))_{L_2} \to (g(p),\varphi_{j_0}(p))_{L_2}$$

is higher. This technique is a very important for practical applications (for example, see Matsuura and Saitoh, 2005).

7 Error estimates

In Theorem 6.1, when the data g contain errors or noises, we need the estimation of our solutions $f^{(N)}_{\lambda,\lambda_j}(t)$ in terms of g. For this, we can obtain a good estimation in the form:

Theorem 7.1. *In* Theorem 6.1, *we obtain the estimate*

$$|f^{(N)}_{\lambda,\lambda_j}(t)|^2 \le \frac{1}{\lambda h}\left(\sum_{j=1}^{N}\lambda_j|(g,\varphi_j)_{L_2}|^2\right). \tag{7.1}$$

Proof. By the Cauchy-Schwarz inequality, we obtain

$$|f_{\lambda,\lambda_j}^{(N)}(t)|^2 \leq \left(\sum_{j=1}^{N}\lambda_j|(g,\varphi_j)_{L_2}|^2\right)\cdot\left(\sum_{j=1}^{N}\lambda_j|(\varphi_j(p),(LK_{\lambda,\lambda_j}^{(N)}(\cdot,t))(p))_{L_2}|^2\right).$$

Here, $K_{\lambda,\lambda_j}^{(N)}(t,t')$ is the reproducing kernel of the Hilbert space $H_{K_{\lambda,\lambda_j}}^{(N)}$ with the norm square (6.2) and so, in particular, we have,

$$\begin{aligned}\frac{1}{\lambda h} &= K^{(0)}(t,t) \geq K_{\lambda,\lambda_j}^{(N)}(t,t)\\ &= \|K_{\lambda,\lambda_j}^{(N)}(\cdot,t)\|^2_{H_{K_{\lambda,\lambda_j}}^{(N)}}\\ &= \lambda\|K_{\lambda,\lambda_j}^{(N)}(\cdot,t)\|^2_{W_h} + \left(\sum_{j=1}^{N}\lambda_j|(\varphi_j(p),(LK_{\lambda,\lambda_j}^{(N)}(\cdot,t))(p))_{L_2}|^2\right)\\ &\geq \left(\sum_{j=1}^{N}\lambda_j|(\varphi_j(p),(LK_{\lambda,\lambda_j}^{(N)}(\cdot,t))(p))_{L_2}|^2\right).\end{aligned}$$

Therefore, we have the desired result. □

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A Hyperchaotic Complex Chen System and its Dynamics

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ABSTRACT

The main goal of this article is to introduce and study the dynamics of a new hyperchaotic complex Chen system. The dynamics of this system which is a 6-dimensional autonomous system is rich and complicated. The new system has chaotic, hyperchaotic attractors, periodic, quasi-periodic solutions and solutions approach fixed points. These solutions are calculated based on Lyapunov exponents. Bifurcation diagram of this system are calculated and in good agreement with Lyapunov exponents. Different hyperchaotic complex Chen systems are suggested. Our results are demonstrated graphically.

Keywords: Hyperchaotic, dynamics, complex, attractors, Lyapunov exponents, bifurcation, dissipative

2000 Mathematics Subject Classification:37, 34c

1 Introduction

Hyperchaotic nonlinear systems have been extensively studied over the last 28 years on a large number of real dynamical systems involving real variables [e.g.1-5 and references therein]. However, there also many interesting systems involving complex variables which have not yet been as actively explored. An example we state here the complex Lorenz, Chen and Lü systems [6-9]. These systems are used to describe the physics of detuned laser and rotating fluids. The electric field and atomic polarization amplitude in these systems are both complex quantities. In communications, systems contain complex variables (or using higher dimensional systems) are used to increase the content and security of the transmitted information [10]. Nonlinear systems with at least two positive Lyapunov exponents are defined as hyperchaotic systems and exhibit complex nonlinear behavior.

In 2007, Mahmoud et al [8] constructed the chaotic complex Chen system expressed by:

$$\begin{aligned} \dot{x} &= \alpha(y-x), \\ \dot{y} &= (\gamma-\alpha)x - xz + \gamma y, \\ \dot{z} &= 1/2\,(\bar{x}y + x\bar{y}) - \beta z, \end{aligned} \tag{1.1}$$

where α, γ and β are positive real parameters, $x = u_1 + iu_2$, $y = u_3 + iu_4$ are complex function, $i = \sqrt{-1}$, u_i and z are real functions. Dots represent derivatives with respect to time and overbar denotes complex conjugate function (variables).

The main goal of this paper is to introduce and study the dynamics of a new hyperchaotic complex Chen system by adding a state feedback controller to system (1.1) as:

$$\begin{aligned} \dot{x} &= \alpha(y-x), \\ \dot{y} &= (\gamma-\alpha)x - xz + \gamma y + w, \\ \dot{z} &= 1/2\,(\bar{x}y + x\bar{y}) - \beta z + w, \\ \dot{w} &= 1/2\,(\bar{x}y + x\bar{y}) - dw, \end{aligned} \tag{1.2}$$

where w is a real function and d is a control parameter. The dynamics of system (1.2) which is a 6-dimensional autonomous hyperchaotic system is very complicated. This dynamics includes symmetries, dissipative, fixed points and stability of trivial fixed point and Lyapunov exponents. This paper is organized as follows: In Section 2, we present different versions (or forms) of the hyperchaotic complex nonlinear Chen system. The classification of solutions of this system is presented. Section 3 contains the dynamics of the hyperchaotic system (1.2). The required condition to ensure that system (1.2) is dissipative is calculated. We obtained, also, the conditions at which our system has unstable trivial fixed points. The values of parameters α, γ, β and d at which hyperchaotic attractors of (1.2) exist are calculated based on Lyapunov exponents. Plotting our results in several figures, we demonstrate graphically Lyapunov exponents of (1.2) and the corresponding hyperchaotic attractors, fixed points, periodic and quasi-periodic solutions. Bifurcations diagrams of (1.2) are calculated in section 4. The results of this section are in good agreement with those of Lyapunov exponents of section 3. Section 5 contains our concluding remarks.

2 Hyperchaotic Complex Chen Systems

In this section we can construct different forms (or versions) of the hyperchaotic complex nonlinear system based on system (1.1) by adding a state feedback controller. One of them is our system (1.2) and its dynamics is studied in more details in section 3. Another system to generate hyperchaotic behavior is:

$$\begin{aligned} \dot{x} &= \alpha(y-x) + w(1+i), \\ \dot{y} &= (\gamma-\alpha)x - xz + \gamma y, \\ \dot{z} &= 1/2\,(\bar{x}y + x\bar{y}) - \beta z, \\ \dot{w} &= 1/2\,(\bar{x}y + x\bar{y}) - dw. \end{aligned} \tag{2.1}$$

The third system has hyperchaotic behavior can be considered as:

$$\begin{aligned} \dot{x} &= \alpha(y-x)+w, \\ \dot{y} &= (\gamma-\alpha)x - xz + \gamma y + w, \\ \dot{z} &= 1/2\,(\bar{x}y + x\bar{y}) - \beta z+, \\ \dot{w} &= 1/2\,(\bar{x}y + x\bar{y}) - dw. \end{aligned} \tag{2.2}$$

A fourth system is described by:

$$\begin{aligned} \dot{x} &= \alpha(y-x)+iw, \\ \dot{y} &= (\gamma-\alpha)x - xz + \gamma y + iw, \\ \dot{z} &= 1/2\,(\bar{x}y + x\bar{y}) - \beta z, \\ \dot{w} &= 1/2\,(\bar{x}y + x\bar{y}) - dw. \end{aligned} \tag{2.3}$$

One constructs the hyperchaotic complex Chen system as:

$$\begin{aligned} \dot{x} &= \alpha(y-x)+w, \\ \dot{y} &= (\gamma-\alpha)x - xz + \gamma y, \\ \dot{z} &= 1/2\,(\bar{x}y + x\bar{y}) - \beta z + w, \\ \dot{w} &= 1/2\,(\bar{x}y + x\bar{y}) - dw. \end{aligned} \tag{2.4}$$

The last system we constructed it is:

$$\begin{aligned} \dot{x} &= \alpha(y-x)+iw, \\ \dot{y} &= (\gamma-\alpha)x - xz + \gamma y + w, \\ \dot{z} &= 1/2\,(\bar{x}y + x\bar{y}) - \beta z, \\ \dot{w} &= 1/2\,(\bar{x}y + x\bar{y}) - dw. \end{aligned} \tag{2.5}$$

All of the above systems have more than one positive Lyapunov exponents and their sum is negative, which means that they are hyperchaotic and dissipative systems. The Lyapunov exponents of these systems are calculated at the end of section 3. The dynamics of systems (2.1), (2.2), (2.3), (2.4) and (2.5) can be similarly studied as we did for system (1.2).

3 Complex Behavior of System (1.2)

This section deals with the basic dynamical properties of system (1.2). The 6D real system of (1.2) is:

$$\begin{aligned} &\dot{u}_1 = \alpha(u_3-u_1),\ \dot{u}_2 = \alpha(u_4-u_2), \\ &\dot{u}_3 = (\gamma-\alpha)u_1 - u_1u_5 + \gamma u_3 + u_6, \\ &\dot{u}_4 = (\gamma-\alpha)u_2 - u_2u_5 + \gamma u_4, \\ &\dot{u}_5 = u_1u_3 + u_2u_4 - \beta u_5 + u_6, \\ &\dot{u}_6 = u_1u_3 + u_2u_4 - du_6. \end{aligned} \tag{3.1}$$

System (3.1) has the following basic dynamical properties:
(1) Symmetry and invariance:

In (3.1), we note that this system is invariant under the transformation
$(u_1, u_2, u_3, u_4, u_5, u_6) \longrightarrow (u_1, -u_2, u_3, -u_4, u_5, u_6)$.
Therefore, if $(u_1, u_2, u_3, u_4, u_5, u_6)$ is a solution of (3.1), then $(u_1, -u_2, u_3, -u_4, u_5, u_6)$ is also a solution of the same system.
(2) Dissipation:
System (3.1) is dissipative under the condition $2\gamma - (2\alpha + \beta + d) < 0$ since:

$$\frac{\partial \dot{u}_1}{\partial u_1} + \frac{\partial \dot{u}_2}{\partial u_2} + ... + \frac{\partial \dot{u}_6}{\partial u_6} = 2\gamma - (2\alpha + \beta + d).$$

(3) Equilibria and their stability:
The equilibria of system (3.1) can be calculated by solving the following system of equations $\dot{u}_1 = 0$, $\dot{u}_2 = 0$, $\dot{u}_3 = 0$, $\dot{u}_4 = 0$, $\dot{u}_5 = 0$ and $\dot{u}_6 = 0$ to get three equilibria $E_0 = (0, 0, 0, 0, 0, 0)$, $E_1 = (u_1^*, 0, u_3^*, 0, u_5^*, u_6^*)$ and $E_2 = (u_1^{**}, 0, u_3^{**}, 0, u_5^{**}, u_6^{**})$ where:

$$\begin{aligned} u_1^* &= u_3^* = \tfrac{\beta+s_1}{2(d+1)},\ u_5^* = \tfrac{2d(1+d)(2\gamma-\alpha)+\beta+s_1}{2d(1+d)},\ u_6^* = \tfrac{\beta[2d(1+d)(2\gamma-\alpha)+\beta+s_1]}{2d(1+d)^2}, \\ u_1^{**} &= u_3^{**} = \tfrac{\beta-s_1}{2(d+1)},\ u_5^{**} = \tfrac{2d(1+d)(2\gamma-\alpha)+\beta-s_1}{2d(1+d)},\ u_6^{**} = \tfrac{\beta[2d(1+d)(2\gamma-\alpha)+\beta-s_1]}{2d(1+d)^2}, \\ s_1 &= \sqrt{\beta[\beta - 4d(d+1)(\alpha - 2\gamma)]}. \end{aligned}$$

To study the stability of E_0 the Jacobian matrix of system (3.1) at E_0 is:

$$J_{E_0} = \begin{pmatrix} -\alpha & 0 & \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & \alpha & 0 & 0 \\ \gamma - \alpha & 0 & \gamma & 0 & 0 & 1 \\ 0 & \gamma - \alpha & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta & 1 \\ 0 & 0 & 0 & 0 & 0 & -d \end{pmatrix}.$$

The characteristic polynomial is:

$$(\lambda + d)(\lambda + \beta)(\lambda^2 + \lambda(\alpha - \gamma) + \alpha^2 - 2\alpha\gamma)^2 = 0.$$

Then the eigenvalues are:

$$\begin{aligned} \lambda_1 &= -d,\ \lambda_2 = -\beta, \\ \lambda_3 &= \lambda_4 = \frac{1}{2}(-\alpha + \gamma - \sqrt{-3\alpha^2 + 6\alpha\gamma + \gamma^2}), \\ \lambda_5 &= \lambda_6 = \frac{1}{2}(-\alpha + \gamma + \sqrt{-3\alpha^2 + 6\alpha\gamma + \gamma^2}). \end{aligned}$$

So this fixed point is **stable** if:

$$2\gamma < \alpha,\ d > 0 \text{ and } \beta > 0.$$

Otherwise it is an unstable fixed point.
(4) Lyapunov exponents and solutions of (1.2):
System (3.1) in vector notation can be written as:

$$\dot{U}(t) = h(U(t); \eta), \tag{3.2}$$

where $U(t) = [u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t)]^t$ is the state space vector, $h = [h_1, h_2, h_3, h_4, h_5, h_6]^t$, η is a set of parameters and $[...]^t$ denoting transpose. The equations for small deviations δU from the trajectory $U(t)$ are:

$$\delta \dot{U}(t) = L_{ij}(U(t); \eta)\delta U, \quad i, j = 1, 2, 3, 4, 5, 6 \tag{3.3}$$

where $L_{i,j} = \dfrac{\partial h_i}{\partial u_j}$ is the Jacobian matrix of the form:

$$L_{i,j} = \begin{pmatrix} -\alpha & 0 & \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & \alpha & 0 & 0 \\ \gamma - \alpha - u_5 & 0 & \gamma & 0 & -u_1 & 1 \\ 0 & \gamma - \alpha - u_5 & 0 & \gamma & -u_2 & 0 \\ u_3 & u_4 & u_1 & u_2 & -\beta & 1 \\ u_3 & u_4 & u_1 & u_2 & 0 & -d \end{pmatrix}.$$

The Lyapunov exponents λ_i of the system is defined by:

$$\lambda_i = \lim_{t \longrightarrow \infty} \frac{1}{t} \log \frac{\|\delta u_i(t)\|}{\|\delta u_i(0)\|}. \tag{3.4}$$

To find λ_i, Eqs. (3.2) and (3.3) must be numerically solved simultaneously. For the case $\alpha = 32$, $\beta = 4$, $\gamma = 25$ and $d = 5$ with the initial conditions $t_0 = 0$, $u_1(0) = 1$, $u_2(0) = 2$, $u_3(0) = 3$, $u_4(0) = 4$, $u_5(0) = 5$ and $u_6(0) = 6$ we calculate the Lyapunov exponents as:
$\lambda_1 = 2.23$, $\lambda_2 = 1.13$, $\lambda_3 = 0$, $\lambda_4 = -8.41$, $\lambda_5 = -11.07$, $\lambda_6 = -17.06$.
This means that our system (3.1) for this choice of α, γ, β and d is a hyperchaotic system since two of the Lyapunov exponents are positive and dissipative system since the sum of Lyapunov exponents is negative.
The solutions of system (1.2) can be classified as:

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	Type of solutions
−	−	−	−	−	−	solutions approach fixed points
0	−	−	−	−	−	periodic solutions (limit cycles)
0	0	−	−	−	−	quasi-periodic solutions (2-torus)
+	0	−	−	−	−	chaotic attractors
+	+	0	−	−	−	hyperchaotic attractors

Based on Lyapunov exponents λ_i eq.(3.4) we calculate the parameters values of system (1.2) at which, chaotic, hyperchaotic, periodic, quasi-periodic attractors and attractors approach fixed points exist.

(4.1) Fix $\gamma = 25$, $d = 5$, $\beta = 4$ and vary α

Using (3.4) we calculate λ_i, $i = 1, 2, ..., 6$ and the values of λ_1, λ_2 and λ_3 versus α are plotted in Figure 1a, while the values of λ_4, λ_5 and λ_6 versus α are shown in Figure 1b. From Figure 1b, it is clear that all the values of λ_4, λ_5 and λ_6 are negative. For other choices of d, γ, β and α we found that λ_4, λ_5 and λ_6 are negative and we omitted their figures.
From figure 1a we can conclude that our system (1.2) has hyperchaotic attractors for α lies in the intervals $(29.81, 33.45]$, $(33.62, 33.95]$, $(33.96, 34.1]$, $(34.35, 34.86]$, $(34.88, 34.93]$, $(34.95, 35.19]$

and $(35.20, 37.07]$, see Figure 1c. The chaotic attractors are exist for $\alpha = 29.81$, 33.62, 33.96, 34.95, 35.20, and $\alpha \in (34.1, 34.35]$, $(34.86, 34.88)$, $(37.07, 37.48]$ and $(37.56, 38.39]$, see Figure 1d. The periodic attractors of (1.2) are exist for α lies in the intervals $(26.3, 29.55]$, $(37.48, 37.56]$ and $(38.39, 41.69]$, see Figure 1e, and quasi-periodic attractors for $\alpha \in (24, 26.3)$ and $(41.69, 41.73]$, see Figure 1f. Finally we found solutions approach fixed points for $\alpha \in (41.73, 78]$, see Figure 1g. It is clear that the dynamics of (1.2) is rich and very complicated.

(4.2) Fix $\alpha = 32$, $\gamma = 25$, $\beta = 4$ and vary d

As we did in (4.1) we plot λ_1, λ_2 and λ_3 versus d in Figure 2a. It is clear that from Figure 2a system (1.2) has hyperchaotic attractors for $d \in (3.36, 16.95]$ and chaotic ones for $d \in (2.74, 3.05]$, $d = 3.36$ and $d \in (16.95, 201.39)$. Periodic solutions are exist for $d \in (0.8, 2.74]$ and $d = 3.25$ and quasi-periodic for $d \in (201.39, 278)$. For d lies in the interval $(0, 0.8]$, solutions of (1.2) approach fixed points.

(4.3) Fix $\alpha = 32$, $d = 5$, $\beta = 4$ and vary γ

The values of λ_1, λ_2 and λ_3 versus γ are plotted in Figure 2b. From this figure one sees that the hyperchaotic attractor are exist for $\gamma \in (21.4, 22.94]$, $(23.09, 23.2]$, $(23.24, 23.56]$ and $(23.7, 27.02]$. System (1.2) has chaotic attractors for small intervals of γ, e.g. $\gamma \in (20.2, 20.64)$ and $(20.73, 21.12]$. Periodic solutions are exist for small intervals of γ, for example, $\gamma \in (19.12, 20.2]$ and $(21.12, 21.4]$ and large values of γ lie in $(27.29, 36]$. For $\gamma \in (0, 19.12]$ our systems (1.2) has solutions approach fixed points.

(4.4) Fix $\alpha = 32$, $d = 5$, $\gamma = 25$ and vary β

Figure 2c depicts λ_1, λ_2 and λ_3 versus β. For the values of these parameters, system (1.2) has, hyperchaotic, chaotic, periodic and quasi-periodic attractors and solutions approach fixed points. For example hyperchaotic attractors exist for β lies in many intervals , e.g. $(2.92, 5.13]$ and $(0.56, 1.06]$. For $\beta \in (18.31, 18.47)$ our system (1.2) has quasi-periodic solutions with $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

(4.5) Lyapunov exponents of systems of section 2

Using similar analysis as we did for system (1.2) in Subsection 4 of this section we calculate the Lyapunov exponents of systems (2.1), (2.2), ..., (2.5) of section 2. For the case $\alpha = 40$, $\beta = 5$, $\gamma = 25$ and $d = 10$ with the initial conditions $t_0 = 0$, $u_1(0) = 1$, $u_2(0) = 2$, $u_3(0) = 3, u_4(0) = 4$, $u_5(0) = 5$ and $u_6(0) = 6$ system (2.1) has Lyapunov exponents as:$\lambda_1 = 1.49$, $\lambda_2 = 0.12$, $\lambda_3 = 0$, $\lambda_4 = -14.4$, $\lambda_5 = -21.98$, $\lambda_6 = -30.15$. This means that our system (2.1) is a hyperchaotic system. System (2.2) for the choice $\alpha = 31$, $\beta = 3$, $\gamma = 25$ and $d = 6$ has two positive Lyapunov exponents where $\lambda_1 = 1.88$, $\lambda_2 = 1.10$, $\lambda_3 = 0$, $\lambda_4 = -7.64$, $\lambda_5 = -10.71$, $\lambda_6 = -14.92$. Lyapunov exponents for system (2.3) are $\lambda_1 = 1.44$, $\lambda_2 = 0.16$, $\lambda_3 = 0$, $\lambda_4 = -11.53$, $\lambda_5 = -21.97$, $\lambda_6 = -30.13$, for $\alpha = 40$, $\beta = 5$, $\gamma = 25$ and $d = 8$, while for system (2.4) they are $\lambda_1 = 1.94$, $\lambda_2 = 0.92$, $\lambda_3 = 0$, $\lambda_4 = -8.19$, $\lambda_5 = -10.82$, $\lambda_6 = -19.93$ for the choice $\alpha = 32$, $\beta = 5$, $\gamma = 25$ and $d = 6$. Finally for the case $\alpha = 31$, $\beta = 3$, $\gamma = 25$ and $d = 5$ our system (2.5) has Lyapunov exponents as: $\lambda_1 = 2.15$, $\lambda_2 = 0.96$, $\lambda_3 = 0$, $\lambda_4 = -7.83$, $\lambda_5 = -9.97$, $\lambda_6 = -14.16$. We used the same initial conditions as those used for system (3.1).

4 Bifurcation Diagrams of System (1.2

In this section we calculate bifurcation diagrams of system (1.2) and are plotted in Figure 3. Figure $3a$ shows (u_3, α) plane. It is clear that the dynamical behavior of system (1.2) is rich, complex and agrees with the regions of chaotic, hyperchaotic, periodic, and fixed points of Subsection (4.1) of Section 3. Figure $3b$ depicts (u_3, d) plane. The results of this figure agree with the regions which are given in (4.2) of Section 3. Figure $3c$ and $3d$ contain (u_3, γ) and (u_3, β) planes respectively. The behavior of these figures are similar to those are given in (4.3) and (4.4) respectively. For example from Figure $3c$ for $\alpha = 32$, $\beta = 4$, $\gamma = 28$ and $d = 5$, system (3.1) has a periodic solution which is shown in Figure 4a. Figure $4b$ depicts periodic-doubling bifurcation for $\gamma = 27.9$ and Figure $4c$ shows, the hyperchaotic attractor of system (3.1) at $\gamma = 27$.

5 Conclusions

In this paper we have introduced a new hyperchaotic complex Chen system (1.2) and studied its dynamics. The values of the parameters α,β,γ and d at which the new system has hyperchaotic and chaotic attractors as well as periodic, quasi-periodic solutions and fixed points are calculated based on Lyapunov exponents . Figures $1a - 1g$ display these attractors. It is clear that from these values of parameters and figure 3, the dynamics of system (1.2) is very complicated. We constructed different versions of hyperchaotic complex Chen system by adding state feedback controllers which are systems $(2.1), (2.2), ..., (2.5)$. All these systems have two positive Lyapunov exponents and the sum of $\lambda_i, i = 1, 2, ..., 6$ is negative. The stability analysis of the trivial fixed points $E_0 = (0, 0, 0, 0, 0, 0)$ is studied. The condition at which system (3.1) is dissipative is calculated. The dynamics of systems (2.1)–(2.5) can be similarly studied as we did for system (3.1).

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Figure 1. For $\beta = 4$, $\gamma = 25$, $d = 5$ and vary α with $t_0 = 0$, $u_1(0) = 1$, $u_2(0) = 2$, $u_3(0) = 3$, $u_4(0) = 4$, $u_5(0) = 5$, $u_6(0) = 6$ (a) Lyapunov exponents of (1.2), λ_1, λ_2 and λ_3 versus α (b) Lyapunov exponents of (1.2), λ_4, λ_5 and λ_6 versus α (c) hyperchaotic attractor of (1.2) when $\alpha = 36$ (d) chaotic attractor of (1.2) when $\alpha = 37.3$, (e) periodic solution of (1.2) when $\alpha = 41$ (f) quasi-periodic solution of (1.2) when $\alpha = 25.8$ (g) fixed point of (1.2) when $\alpha = 44$.

Fig 1

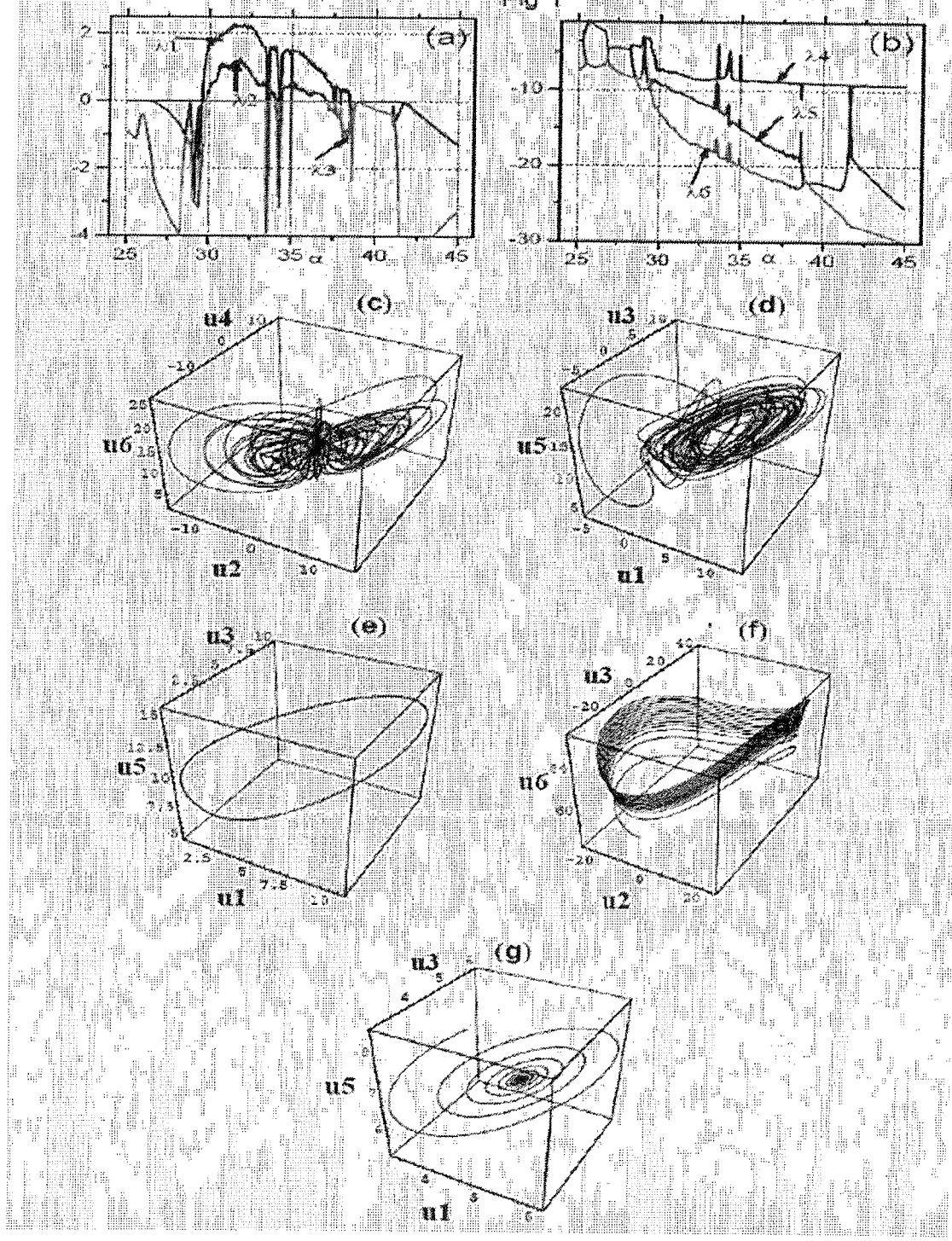


Figure 2: Lyapunov exponents of (1.2) at the same initial conditions as in Figure 1: (a) λ_1, λ_2 and λ_3 versus d (b) λ_1, λ_2 and λ_3 versus γ (c) λ_1, λ_2 and λ_3 versus β.

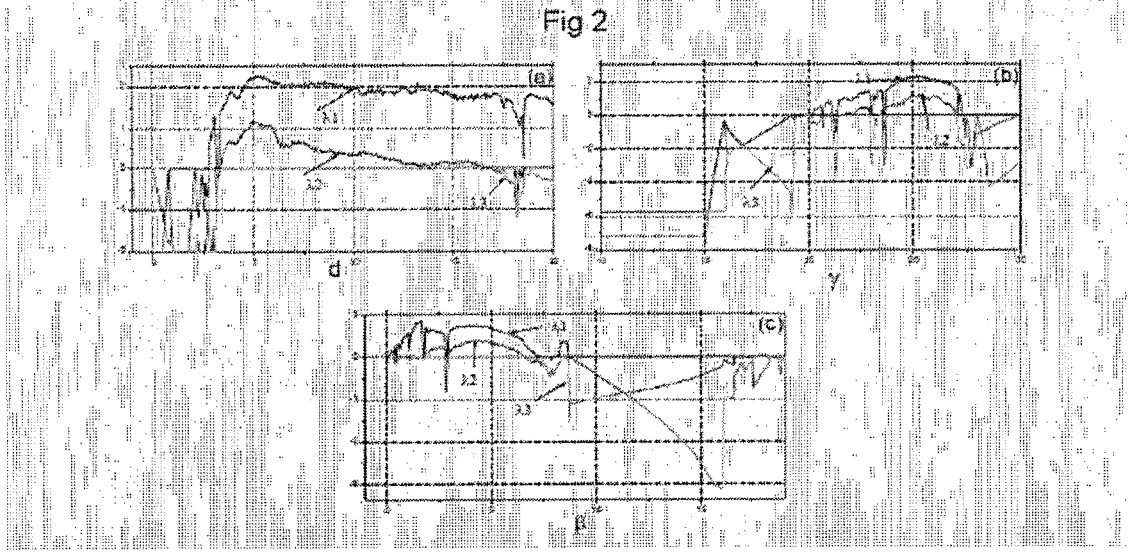


Figure 3:Bifurcation diagrams of system (1.2) (a)(u_3, α) plane, (b)(u_3, d) plane, (c)(u_3, γ) plane, (d) (u_3, β) plane.

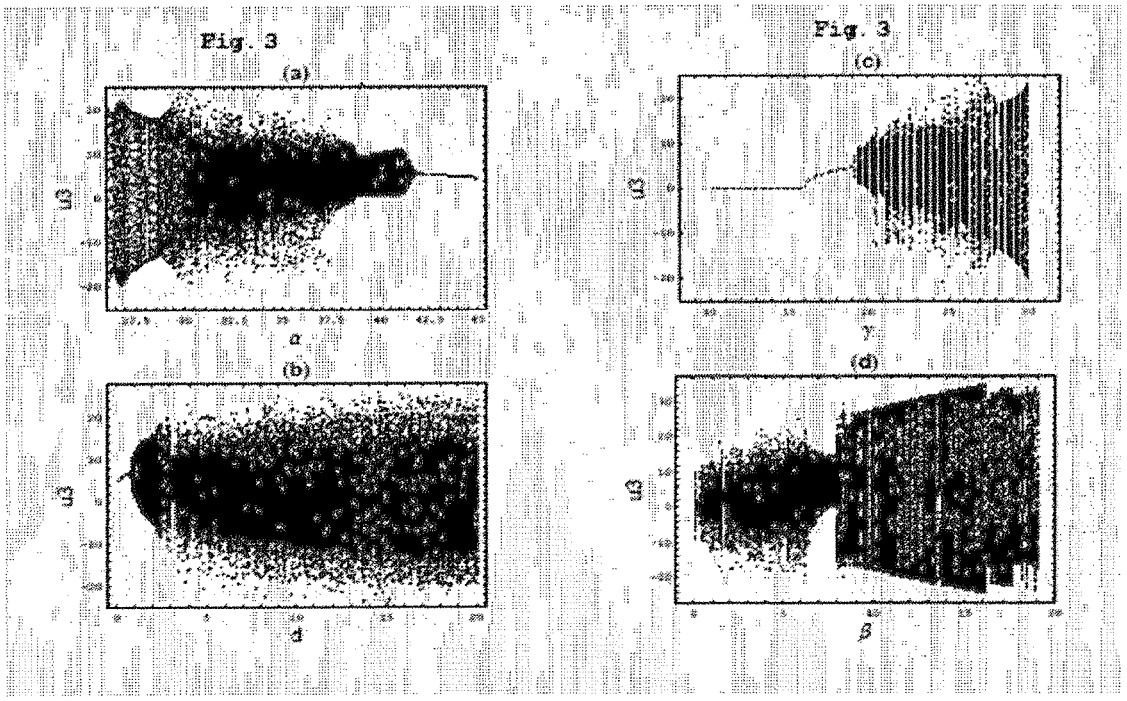
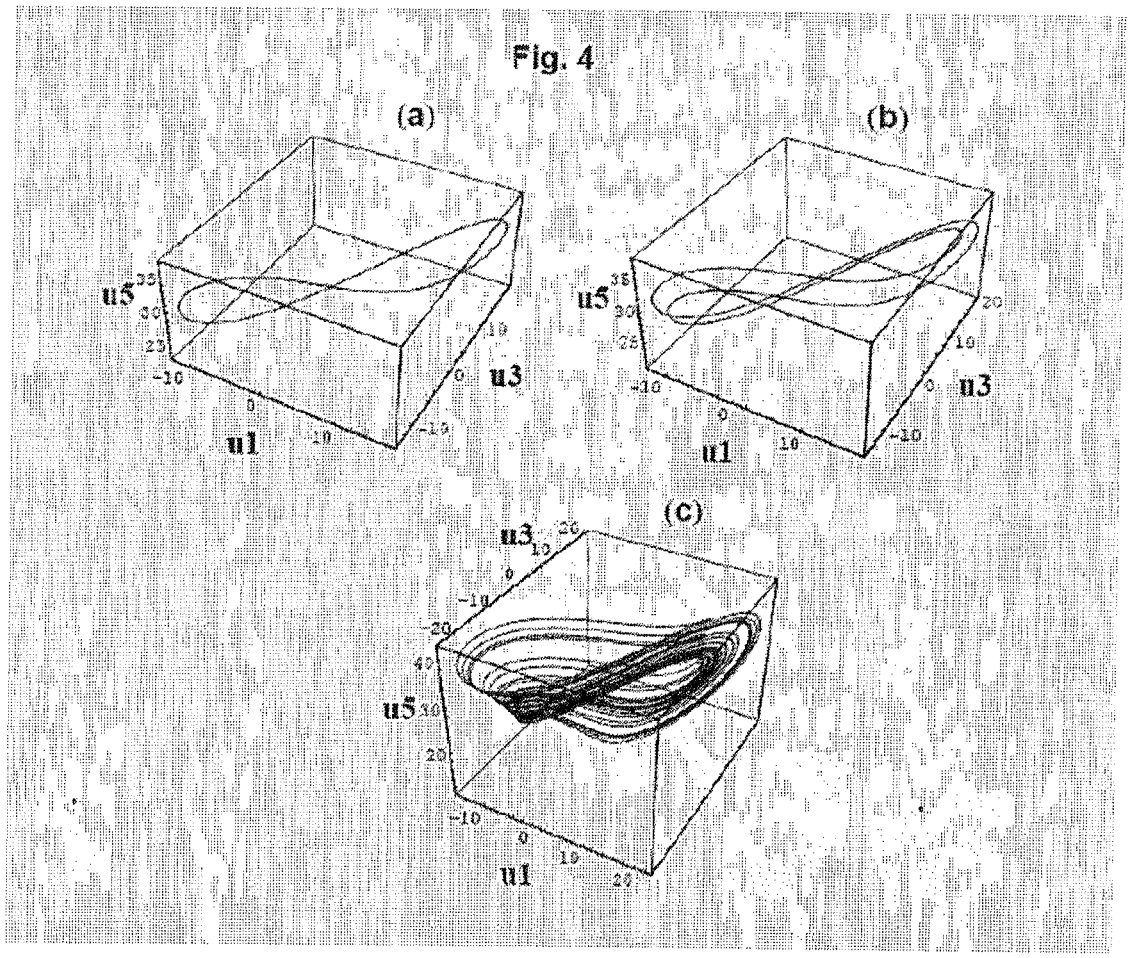


Figure 4: Solutions of system (1.2) for $\alpha = 32$, $\beta = 4$, $\gamma = 28$ and $d = 5$, and at the same initial conditions as in Figure 1 :(a)$\gamma = 28$ periodic solution, (b)$\gamma = 27.9$ period-doubling bifurcation, (c)$\gamma = 27$ hyperchaotic attractor.



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Integro-Differential Equations with Variational Derivatives Generated by Nonlinear Random Partial Integral Equations

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ABSTRACT

Integro-partial differential equations with variational derivatives, that are generated by random partial integral equations of second kind with power-law non-linearity, are considered. Solution existence and uniqueness is investigated and illustrative examples of such equations are given.

Keywords: Random integral equations, variational derivatives, integro-partial differential equation, nonlinear integral equations.

2000 Mathematics Subject Classification: 45G10, 45K05, 45R05.

1 Introduction

Let's consider the random integral equation

$$x(t_1,t_2;\omega)-\int\limits_{T_1}k_1(t_1,t_2,s_1;\omega)x^p(s_1,t_2;\omega)ds_1-\int\limits_{T_2}k_2(t_1,t_2,s_2;\omega)x^q(t_1,s_2;\omega)ds_2-$$

$$\int\limits_{T_3}k_3(t_1,t_2,s_1,s_2;\omega)x^r(s_1,s_2;\omega)ds_1ds_2=y(t_1,t_2;\omega), \tag{1}$$

in unknown function $x(t_1,t_2;\omega)$ where $t_\nu,s_\nu\in T_\nu\subseteq\mathbb{R}$ $(\nu=1,2)$, $T_3=T_1\times T_2$, $\omega\in\Omega$ ($\{\Omega,\Sigma,P\}$ is a probability space); p,q,r are positive integers, $k_\nu\in L_{\infty,1,\infty}(T_3\times T_\nu\times\Omega)$ $(\nu=\overline{1,3})$, $x,y\in X=L_\infty(T_3\times\Omega)$.

The determinate analogue of (1) is considered in (Appel, Kalitvin and Zabrejko, 2000), (Kalitvin, 2002). The solvability of similar linear determinate integral equations of Fredholm and Volterra type is also discussed in (Zabrejko, Kalitvin and Frolova, 2002), (Kalitvin and Frolova, 2001). The work (Romanovski, 1964) is among the first ones, where partial integral equations are considered.

Our problem is to find the characteristic functional of $x(t_1, t_2; \omega)$, supposing that the characteristic functional of vector process $(k_1(t_1, t_2, s_1; \omega), k_2(t_1, t_2, s_2; \omega), k_3(t_1, t_2, s_1, s_2; \omega), y(t_1, t_2; \omega))$ is known. The method we use to solve it is the same as in (Doroshko and Yanovich, 2000): one has to build the corresponding determinate equation with variational derivatives, find its appropriate solution and restore the unknown distribution of the random equation solution.

We'll use the shorter form

$$(I - A)x = y$$

of the equation (1), where

$$(Ax)(t_1, t_2; \omega) = \int\limits_{T_1} k_1(t_1, t_2, s_1; \omega) x^p(s_1, t_2; \omega) ds_1 +$$

$$\int\limits_{T_2} k_2(t_1, t_2, s_2; \omega) x^q(t_1, s_2; \omega) ds_2 + \int\limits_{T_3} k_3(t_1, t_2, s_1, s_2; \omega) x^r(s_1, s_2; \omega) ds_1 ds_2,$$

I is an identity operator. One may prove that A transforms the space X into itself.

2 Integro-partial differential equation. Boundary condition.

Suppose x is a fixed function such that the equality (1) is true. Let's introduce the spaces $V = L_\infty(T_3)$, $U_\nu = L_\infty(T_3 \times T_\nu)$ $(\nu = \overline{1,3})$ and bilinear operators $\langle \cdot, \cdot \rangle : X \times V \to L_\infty(\Omega)$, $\langle \cdot, \cdot \rangle_\nu : K_\nu \times U_\nu \to L_\infty(\Omega)$ $(\nu = \overline{1,3})$ such that

$$\langle k_\nu, u_\nu \rangle_\nu = \int\limits_{T_3 \times T_\nu} k_\nu(t_1, t_2, s_\nu; \omega) u_\nu(t_1, t_2, s_\nu) dt_1 dt_2 ds_\nu \ (\nu = 1, 2),$$

$$\langle k_3, u_3 \rangle_3 = \int\limits_{T_3 \times T_3} k_3(t_1, t_2, s_1, s_2; \omega) u_3(t_1, t_2, s_1, s_2) dt_1 dt_2 ds_1 ds_2,$$

$$\langle x, v \rangle = \int\limits_{T_3} x(t_1, t_2; \omega) v(t_1, t_2) dt_1 dt_2, \ \langle y, w \rangle = \int\limits_{T_3} y(t_1, t_2; \omega) w(t_1, t_2) dt_1 dt_2,$$

where $v, w \in V$, $u_\nu \in U_\nu$ $(\nu = \overline{1,3})$.

We denote the mathematical expectation operator as E such that for all random values $\xi \in L(\Omega)$ we have $E\xi = \int\limits_\Omega \xi(\omega) P(d\omega)$. Let's consider the functional

$$F(u_1, u_2, u_3, v, w) = E\Phi(u_1, u_2, u_3, v, w; \omega),$$

where

$$\Phi(u_1, u_2, u_3, v, w; \omega) = \exp\{i[\sum_{\nu=1}^{3} \langle k_\nu, u_\nu \rangle_\nu + \langle x, v \rangle + \langle y, w \rangle]\} \ (i^2 = -1).$$

This functional is defined on the Cartesian product of the spaces of essentially bounded functions of two, three and four variables, defined in the foregoing regions. Further we'll have to calculate partial differentials and, correspondingly, partial functional derivatives with respect to

functions, taken from the above spaces at arbitrary fixed points.
According to the definition of variational derivative

$$F(u_1,u_2,u_3,v+\lambda h,w)=F(u_1,u_2,u_3,v,w)+$$
$$+\lambda\int\limits_{T_3}\frac{\delta F(u_1,u_2,u_3,v,w)}{\delta v(t_1,t_2)}h(t_1,t_2)dt_1dt_2+o(\lambda)\text{ for all }h\in V\text{if}\lambda\to 0.$$

Analogous equalities are used to find the partial derivatives with respect to the variables $w(t_1,t_2), u_\nu(t_1,t_2,s_\nu)$ $(\nu=1,2)$ and $u_3(t_1,t_2,s_1,s_2)$. It is easy to check that

$$\frac{\delta F(u_1,u_2,u_3,v,w)}{\delta v(t_1,t_2)}=iE[x(t_1,t_2;\omega)\Phi(u_1,u_2,u_3,v,w;\omega)].$$

It is also true that

$$\frac{\delta F(u_1,u_2,u_3,v,w)}{\delta w(t_1,t_2)}=iE[y(t_1,t_2;\omega)\Phi(u_1,u_2,u_3,v,w;\omega)].$$

Variational derivatives of higher order (consequent) are defined according to traditional rules. In particular the following equalities are true for mixed derivatives, that are used below:

$$\frac{\delta^{p+1}F(u_1,u_2,u_3,v,w)}{\delta u_1(t_1,t_2,s_1)\delta v^p(s_1,t_2)}=i^{p+1}E[k_1(t_1,t_2,s_1;\omega)x^p(s_1,t_2;\omega)\Phi(u_1,u_2,u_3,v,w;\omega)],$$

$$\frac{\delta^{q+1}F(u_1,u_2,u_3,v,w)}{\delta u_2(t_1,t_2,s_2)\delta v^q(t_1,s_2)}=i^{q+1}E[k_2(t_1,t_2,s_2;\omega)x^q(t_1,s_2;\omega)\Phi(u_1,u_2,u_3,v,w;\omega)],$$

$$\frac{\delta^{r+1}F(u_1,u_2,u_3,v,w)}{\delta u_3(t_1,t_2,s_1,s_2)\delta v^r(s_1,s_2)}=i^{r+1}E[k_3(t_1,t_2,s_1,s_2;\omega)x^r(s_1,s_2;\omega)\Phi(u_1,u_2,u_3,v,w;\omega)].$$

Let D be such an operator that for every functional $G(u_1,u_2,u_3,v,w,t_1,t_2):U_1\times U_2\times U_3\times V^2\times T_3\to\mathbb{R}$ we have

$$(DG)(u_1,u_2,u_3,v,w,t_1,t_2)=\frac{1}{i^p}\int\limits_{T_1}\frac{\delta^pG(u_1,u_2,u_3,v,w,s_1,t_2)}{\delta u_1(t_1,t_2,s_1)\delta v^{p-1}(s_1,t_2)}ds_1+$$

$$\frac{1}{i^q}\int\limits_{T_2}\frac{\delta^qG(u_1,u_2,u_3,v,w,t_1,s_2)}{\delta u_2(t_1,t_2,s_2)\delta v^{q-1}(t_1,s_2)}ds_2+\frac{1}{i^r}\int\limits_{T_3}\frac{\delta^rG(u_1,u_2,u_3,v,w,s_1,s_2)}{\delta u_3(t_1,t_2,s_1,s_2)\delta v^{r-1}(s_1,s_2)}ds_1ds_2.$$

We obtain

$$D\frac{\delta F(u_1,u_2,u_3,v,w)}{\delta v(t_1,t_2)}=E[(Ax)(t_1,t_2;\omega)\Phi(u_1,u_2,u_3,v,w;\omega)].$$

Then the equality

$$(I-D)\frac{\delta F}{\delta v}=\frac{\delta F}{\delta w},\tag{2}$$

which has the explicit expression

$$\frac{\delta F(u_1,u_2,u_3,v,w)}{\delta v(t_1,t_2)}-\frac{1}{i^p}\int\limits_{T_1}\frac{\delta^{p+1}F(u_1,u_2,u_3,v,w)}{\delta u_1(t_1,t_2,s_1)\delta v^p(s_1,t_2)}ds_1-\frac{1}{i^q}\int\limits_{T_2}\frac{\delta^{q+1}F(u_1,u_2,u_3,v,w)}{\delta u_2(t_1,t_2,s_2)\delta v^q(t_1,s_2)}ds_2-$$

$$\frac{1}{i^r}\int\limits_{T_3}\frac{\delta^{r+1}F(u_1,u_2,u_3,v,w)}{\delta u_3(t_1,t_2,s_1,s_2)\delta v^r(s_1,s_2)}ds_1ds_2=\frac{\delta F(u_1,u_2,u_3,v,w)}{\delta w(t_1,t_2)},\tag{3}$$

is true. Note that

$$F(u_1, u_2, u_3, 0, w) = F_0(u_1, u_2, u_3, w), \tag{4}$$

where $F_0(u_1, u_2, u_3, w) = E[\Phi(u_1, u_2, u_3, 0, w; \omega)] = E[\exp\{i[\sum_{\nu=1}^{3} \langle k_\nu, u_\nu \rangle_\nu + \langle y, w \rangle]\}]$, is known. Thus we've obtained the determinate problem (3), (4) in unknown functional $F : U_1 \times U_2 \times U_3 \times V^2 \to \mathbb{R}$ from the random equation (1) in unknown function $x \in X$. Besides, if $x(t_1, t_2; \omega)$ is the solution of the equation (1) and $F(u_1, u_2, u_3, v, w)$ is the corresponding solution of the problem (3), (4) then

$$F(0, 0, 0, v, 0) = E[\Phi(0, 0, 0, v, 0; \omega)] = E[\exp\{i \langle x, v \rangle\}]$$

is the characteristic functional of $x(t_1, t_2; \omega)$. Consequently, one may find all the higher order moments of the random function $x(t_1, t_2; \omega)$:

$$E[\prod_{j=1}^{n} x^{\nu_j}(\tau_j, \sigma_j; \omega)] = \frac{1}{i^\nu} \frac{\delta^\nu F(0, 0, 0, v, 0)}{\delta^{\nu_1} v(\tau_1, \sigma_1) \delta^{\nu_2} v(\tau_2, \sigma_2) \dots \delta^{\nu_n} v(\tau_n, \sigma_n)} \Big|_{v=0},$$

where $\nu = \sum_{j=1}^{n} \nu_j$, ν_j are integer numbers.

3 Solvability theorem. "Normal" solution.

We know that the problem (3), (4) has a solution if the equation (1) is solvable. But what about the uniqueness of this solution?

Theorem 3.1. *The equation* (3) *under the boundary condition*

$$F(u_1, u_2, u_3, 0, w) = 0 \tag{5}$$

has infinitely dimensional affine solution space.

Proof. Let's notice that if $\widetilde{x} \in X$ and

$$\widetilde{x}(t_1, t_2; \omega) - \int_{T_1} \widetilde{k}_1(t_1, t_2, s_1; \omega) \widetilde{x}^p(s_1, t_2; \omega) ds_1 -$$

$$\int_{T_2} \widetilde{k}_2(t_1, t_2, s_2; \omega) \widetilde{x}^q(t_1, s_2; \omega) ds_2 - \int_{T_3} \widetilde{k}_3(t_1, t_2, s_1, s_2; \omega) \widetilde{x}^r(s_1, s_2; \omega) ds_1 ds_2 = 0,$$

where $\widetilde{k}_\nu \in K_\nu$ $(\nu = 1, 2, 3)$, then the functional

$$\widehat{F}(u_1, u_2, u_3, v) = \widetilde{F}(u_1, u_2, u_3, v) - \widetilde{F}(u_1, u_2, u_3, 0),$$

under the assumption

$$\widetilde{F}(u_1, u_2, u_3, v) = E[\widetilde{\Phi}(u_1, u_2, u_3, v; \omega)], \ \widetilde{\Phi}(u_1, u_2, u_3, v; \omega) = \exp\{i[\sum_{\nu=1}^{3} \left\langle \widetilde{k}_\nu, u_\nu \right\rangle_\nu + \langle \widetilde{x}, v \rangle]\},$$

is the solution of the problem (3), (5). Indeed, let us check whether the equalities (2), (5) are true if $F = \widehat{F}$:

$$(I - D) \frac{\delta \widehat{F}}{\delta v} = (I - D) \frac{\delta \widetilde{F}}{\delta v} = 0, \ \frac{\delta \widehat{F}}{\delta w} = 0,$$

$$\widehat{F}|_{v=0} = 0.$$

Now let us show that (3), (5) has infinitely many linearly independent solutions. Indeed, let n be a positive integer. We'll prove that the dimensionality of the solution space of the problem (3), (5) is not less than n. One may easily find such *determinate* functions $\widetilde{x}_j \in X$ $(j = \overline{1,n})$, that $\sum\limits_{j=1}^{n} \alpha_j \widetilde{x}_j \neq 0$ for every nonzero vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\int\limits_{T_3} \widetilde{x}_j^r(s_1, s_2) ds_1 ds_2 \neq 0$ for all j. Let $\widetilde{k}_{3,j}(t_1, t_2, s_1, s_2) = [\int\limits_{T_3} \widetilde{x}_j^r(s_1, s_2) ds_1 ds_2]^{-1}$ $(j = \overline{1,n})$. Then

$$\widetilde{x}_j(t_1, t_2) - \int\limits_{T_3} \widetilde{k}_{3,j}(t_1, t_2, s_1, s_2) \widetilde{x}_j^r(s_1, s_2) ds_1 ds_2 = 0 \ (j = \overline{1,n}),$$

and

$$\widehat{F}_j(u_1, u_2, u_3, v, w) = \widetilde{F}_j(u_1, u_2, u_3, v, w) - \widetilde{F}_j(u_1, u_2, u_3, 0, w),$$

where

$$\widetilde{F}_j(u_1, u_2, u_3, v, w) = \exp\{i[\left\langle \widetilde{k}_{\nu,3}, u_\nu \right\rangle_\nu + \langle \widetilde{x}_j, v \rangle]\},$$

is the solution of (3), (5).

What is left to prove is that the functionals $\widehat{F}_j(u_1, u_2, u_3, v, w)$ $(j = \overline{1,n})$ are linearly independent. Indeed, imagine there exists such a nonzero vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ that $\sum\limits_{j=1}^{n} \alpha_j \widehat{F}_j = 0$. Then $\sum\limits_{j=1}^{n} \alpha_j \frac{\delta \widehat{F}_j(0,0,0,v,0)}{\delta v}\Big|_{v=0} = 0$. It means that $\sum\limits_{j=1}^{n} \alpha_j \widetilde{x}_j = 0$. A contradiction. □

Thus the problem (3), (4) has infinitely dimensional affine solution space. Therefore let us call its solution "normal" if it is generated by a solution of the corresponding random equation.

4 "Normal" solution obtained from Liouville-Neumann series

Let $\widetilde{X}$ be such a subset of X that

- $A(\widetilde{X}) \subseteq \widetilde{X}$,
- $\exists m < 1$, $\|Ax_1 - Ax_2\| \leq m\|x_1 - x_2\|$ for all $x_1, x_2 \in \widetilde{X}$.

Also let $y \in \widetilde{X}$. Then $x^* = \sum\limits_{\nu=0}^{\infty} A^\nu y$ is the unique solution of the equation (1) (in $\widetilde{X}$) and

$$F = E[\exp\{i\,(x^*, v)\}\Phi_0], \tag{6}$$

where Φ_0 is the denotation for $\Phi(u_1, u_2, u_3, 0, w; \omega)$, is the "normal" solution of the problem (3), (4). Let's express it through F_0.

Lemma 4.1. *Let ξ and η be such random values that $\xi \in L_\infty(\Omega)$, $|\eta| \equiv 1$. Then*

$$E[\exp\{i\xi\}\eta] = E\eta + \sum_{j=1}^{n-1} \frac{i^j}{j!} E[\xi^j \eta] + \frac{\varepsilon}{n!}, \ |\varepsilon| \leq \sqrt{2} E|\xi|^n.$$

Proof. Let's apply Taylor's formula with Lagrange remainder for the real and the imaginative part of $\exp\{i\xi\}$:

$$\cos\xi = 1 + \sum_{j=1}^{n-1} a_j \xi^j + \cos^{(n)}\theta_1 \frac{\xi^n}{n!},\ a_j = \begin{cases} 0,\ j = 2l-1 \\ \frac{(-1)^l}{j!},\ j = 2l \end{cases} \quad (l \in \mathbb{N}),$$

$$\sin\xi = \sum_{j=1}^{n-1} b_j \xi^j + \sin^{(n)}\theta_2 \frac{\xi^n}{n!},\ b_j = \begin{cases} \frac{(-1)^{l-1}}{j!},\ j = 2l-1 \\ 0,\ j = 2l \end{cases} \quad (l \in \mathbb{N}),$$

$\theta_1,\ \theta_2$ are random values. Then

$$\exp\{i\xi\} = 1 + \sum_{j=1}^{n-1}(a_j + ib_j)\xi^j + (\cos^{(n)}\theta_1 + i\sin^{(n)}\theta_2)\frac{\xi^n}{n!} =$$

$$1 + \sum_{j=1}^{n-1} \frac{i^j}{j!}\xi^j + (\cos(\theta_1 + \frac{\pi n}{2}) + i\sin(\theta_2 + \frac{\pi n}{2}))\frac{\xi^n}{n!}.$$

Finally

$$E[\exp\{i\xi\}\eta] = E\eta + \sum_{j=1}^{n-1} \frac{i^j}{j!} E[\xi^j \eta] + \frac{\varepsilon}{n!},\ |\varepsilon| \le \sqrt{2}E|\xi|^n.$$

□

Corollary 4.2. *Suppose* $x \in X,\ v \in V,\ \frac{E|\langle x,v\rangle|^n}{n!} \underset{n\to\infty}{\longrightarrow} 0$. *Then*

$$E[\exp\{i\langle x, v\rangle\}\Phi_0] = E\Phi_0 + \sum_{j=1}^{\infty} \frac{i^j}{j!} E[\langle x, v\rangle^j \Phi_0]. \tag{7}$$

Now let's consider $\langle x^*, v\rangle^j = \left\langle \sum_{\nu=1}^{\infty} A^\nu y, v \right\rangle^j = (\sum_{\nu=1}^{\infty} \langle A^\nu y, v\rangle)^j$. To open the brackets we need the following lemma:

Lemma 4.3. *Let* $a_\nu^{(\kappa)} \in L_\infty(\Omega)$ $(\kappa = \overline{1,j},\ \nu = 1,2,\ldots)$, $\sum_{\nu=1}^{\infty} a_\nu^{(\kappa)}$ $(\kappa = \overline{1,j})$ *converge in* $L_\infty(\Omega)$, $\prod_{\kappa=1}^{j} a_{\nu_\kappa}^{(\kappa)} \in L_\infty(\Omega), (\nu_\kappa = 1,2,\ldots;\kappa = \overline{1,j})$. *Then* $\sum_{\sigma=1}^{\infty} b_\sigma$, *where*

$$b_\sigma = \sum_{\max\{\nu_1,\nu_2,\ldots,\nu_j\}=\sigma} a_{\nu_1}^{(1)} a_{\nu_2}^{(2)} \cdots a_{\nu_j}^{(j)},$$

converges in $L_\infty(\Omega)$ *and*

$$\prod_{\kappa=1}^{j} \sum_{\nu=1}^{\infty} a_\nu^{(\kappa)} = \sum_{\sigma=1}^{\infty} b_\sigma$$

is an equality in $L_\infty(\Omega)$.

Proof. Let's notice that $\prod_{\kappa=1}^{j} \sum_{\nu=1}^{\infty} a_\nu^{(\kappa)} = \prod_{\kappa=1}^{j} (\sum_{\nu=1}^{s} a_\nu^{(\kappa)} + o(1)) = \prod_{\kappa=1}^{j} \sum_{\nu_\kappa=1}^{s} a_{\nu_\kappa}^{(\kappa)} + o(1)$ if $s \to \infty$. Also

$$\prod_{\kappa=1}^{j} \sum_{\nu=1}^{s} a_\nu^{(\kappa)} = \sum_{\nu_1,\nu_2,\ldots,\nu_j \le s} a_{\nu_1}^{(1)} a_{\nu_2}^{(2)} \cdots a_{\nu_j}^{(j)} = \sum_{\sigma=1}^{s} \sum_{\max\{\nu_1,\nu_2,\ldots,\nu_j\}=\sigma} a_{\nu_1}^{(1)} a_{\nu_2}^{(2)} \cdots a_{\nu_j}^{(j)} = \sum_{\sigma=1}^{s} b_\sigma.$$

□

Let's put $a_\nu^{(1)} = a_\nu^{(2)} = \ldots = a_\nu^{(j)} = a_\nu$. Then we may reduce

$$b_\sigma = \sum_{\max\{\nu_1,\nu_2,\ldots,\nu_j\}=\sigma} a_{\nu_1} a_{\nu_2} \ldots a_{\nu_j} \tag{8}$$

to the following form:

$$b_\sigma = \sum_{\nu_1 \le \nu_2 \le \ldots \le \nu_j = \sigma} \frac{j!}{\prod_{\tau=1}^{\sigma} (\sum_{\kappa=1}^{j} \delta_{\nu_k,\tau})!} a_{\nu_1} a_{\nu_2} \ldots a_{\nu_j}. \tag{9},$$

where $\delta_{\nu_k,\tau} = \begin{cases} 1, & \nu_k = \tau \\ 0, & \nu_k \neq \tau \end{cases}$. Indeed, let us say that vectors $\nu = (\nu_1, \nu_2, \ldots, \nu_j)$ and $\nu' = (\nu_1', \nu_2', \ldots, \nu_j')$ are equivalent $(\nu \sim \nu')$ if and only if there exists such a permutation æ : $\{1, 2, \ldots, j\} \leftrightarrow \{1, 2, \ldots, j\}$ that $(\nu_1', \nu_2', \ldots, \nu_j') = (\nu_{æ(1)}, \nu_{æ(2)}, \ldots, \nu_{æ(j)})$. The introduced relationship is symmetrical, reflexive and transitive. Therefore, the whole set $S_j = \{(\nu_1, \nu_2, \ldots, \nu_j) : \nu_\kappa \in \{1, 2, \ldots, s\}, \kappa = \overline{1, j}\}$ may me split into several equivalence classes. Every class has an ascending ordered vector while every ascending ordered vector $\nu \in S_j$ defines a class $C(\nu) = \{\nu' : \nu' \sim \nu\}$. Then

$$b_\sigma = \sum_{\nu_1 \le \nu_2 \le \ldots \le \nu_j = \sigma} \sum_{\nu' \in C(\nu)} a_{\nu_1'} a_{\nu_2'} \ldots a_{\nu_j'} =$$

$$\sum_{\nu_1 \le \nu_2 \le \ldots \le \nu_j = \sigma} |C(\nu)| a_{\nu_1} a_{\nu_2} \ldots a_{\nu_j},$$

where $|C(\nu)|$ is number of elements in $C(\nu)$. Now we'll show that $|C(\nu)| = \frac{j!}{\prod_{\tau=1}^{\sigma} (\sum_{\kappa=1}^{j} \delta_{\nu_k,\tau})!}$. Indeed, the numbers $\nu_1, \nu_2, \ldots, \nu_j$ may be split into g groups, containing $n_1, n_2, \ldots, n_g$ equal numbers correspondingly:

$$\nu_1 = \nu_2 = \ldots = \nu_{n_1} < \nu_{n_1+1} = \nu_{n_1+2} = \ldots = \nu_{n_1+n_2} < \ldots <$$

$$\nu_{n_1+n_2+\ldots+n_{g-1}+1} = \nu_{n_1+n_2+\ldots+n_{g-1}+2} = \ldots = \nu_{n_1+n_2+\ldots+n_{g-1}+n_g}.$$

Then, according to the formula for the number of permutations with repetitions $|C(\nu)| = \frac{j!}{\prod_{\gamma=1}^{g} n_\gamma!}$.

Assuming $\mathrm{N}_g = \{\nu_{n_1}, \nu_{n_1+n_2}, \ldots, \nu_{n_1+n_2+\ldots+n_g}\}$ we get

$$\prod_{\tau=1}^{\sigma} (\sum_{\kappa=1}^{j} \delta_{\nu_k,\tau})! = \prod_{\tau \in \mathrm{N}_g} (\sum_{\kappa=1}^{j} \delta_{\nu_k,\tau})! \prod_{\tau \in \{1,2,\ldots,\sigma\} \setminus \mathrm{N}_g} (\sum_{\kappa=1}^{j} \delta_{\nu_k,\tau})! = \prod_{\gamma=1}^{g} n_\gamma! \prod_{\tau \in \{1,2,\ldots,\sigma\} \setminus \mathrm{N}_g} 0! = \prod_{\gamma=1}^{g} n_\gamma!.$$

Thus, we've proved

Lemma 4.4. *Let $a_\nu \in L_\infty(\Omega)$ $(\nu = 1, 2, \ldots)$, $\sum_{\nu=1}^{\infty} a_\nu$ converge in $L_\infty(\Omega)$. Then $\sum_{\sigma=1}^{\infty} b_\sigma$, where b_σ is defined by equalities (8) and (9), converges in $L_\infty(\Omega)$ and*

$$\left[\sum_{\nu=1}^{\infty} a_\nu\right]^j = \sum_{\sigma=1}^{\infty} b_\sigma$$

is an equality in $L_\infty(\Omega)$.

Applying this lemma to $a_\nu = \langle A^\nu y, v\rangle$ we obtain the following

Corollary 4.5. *Let $v \in V$, $x^* = \sum\limits_{\nu=0}^{\infty} A^\nu y \in X$. Then*

$$\frac{1}{j!}E[\langle x^*, v\rangle^j \Phi_0] = \frac{1}{j!}\sum_{\sigma=1}^{\infty}\sum_{\max\{\nu_1,\nu_2,\ldots,\nu_j\}=\sigma} E[\prod_{\kappa=1}^{j}\langle A^{\nu_\kappa}y, v\rangle \Phi_0] =$$

$$\sum_{\sigma=1}^{\infty}\sum_{\nu_1\le\nu_2\le\ldots\le\nu_j=\sigma}\frac{1}{\prod\limits_{\tau=1}^{\sigma}(\sum\limits_{\kappa=1}^{j}\delta_{\nu_k,\tau})!}E[\prod_{\kappa=1}^{j}\langle A^{\nu_\kappa}y, v\rangle \Phi_0]. \tag{10}$$

Let's widen the definition of the operator D in such a way that it can be also applied to functionals $G(u_1,u_2,u_3,w,t_1,t_2) : U_1\times U_2\times U_3\times V\times T_3 \to \mathbb{R}$, having no v argument:

$$(DG)(u_1,u_2,u_3,w,t_1,t_2) = \frac{1}{i^p}\int_{T_1}\frac{\delta^p G(u_1,u_2,u_3,w,s_1,t_2)}{\delta u_1(t_1,t_2,s_1)\delta w^{p-1}(s_1,t_2)}ds_1 +$$

$$\frac{1}{i^q}\int_{T_2}\frac{\delta^q G(u_1,u_2,u_3,w,t_1,s_2)}{\delta u_2(t_1,t_2,s_2)\delta w^{q-1}(t_1,s_2)}ds_2 + \frac{1}{i^r}\int_{T_3}\frac{\delta^r G(u_1,u_2,u_3,w,s_1,s_2)}{\delta u_3(t_1,t_2,s_1,s_2)\delta w^{r-1}(s_1,s_2)}ds_1ds_2.$$

Lemma 4.6. *Let $v \in V$. Then*

$$E[\prod_{\kappa=1}^{j}\langle A^{\nu_\kappa}y, v\rangle \Phi_0] = \left\langle D^{\nu_j}\left(\frac{\delta}{\delta w}E[\prod_{\kappa=1}^{j-1}\langle A^{\nu_\kappa}y, v\rangle \Phi_0]\right), v\right\rangle.$$

Proof. Indeed, $\frac{\delta}{\delta w}E[\prod\limits_{\kappa=1}^{j-1}\langle A^{\nu_\kappa}y, v\rangle \Phi_0] = E[y\prod\limits_{\kappa=1}^{j-1}\langle A^{\nu_\kappa}y, v\rangle \Phi_0]$. We obtain the formula

$$DE[y\prod_{\kappa=1}^{j-1}\langle A^{\nu_\kappa}y, v\rangle \Phi_0] = E[Ay\prod_{\kappa=1}^{j-1}\langle A^{\nu_\kappa}y, v\rangle \Phi_0],$$

which is generalized by induction: $D^{\nu_j}E[y\prod\limits_{\kappa=1}^{j-1}\langle A^{\nu_\kappa}y, v\rangle \Phi_0] = E[A^{\nu_j}y\prod\limits_{\kappa=1}^{j-1}\langle A^{\nu_\kappa}y, v\rangle \Phi_0]$. Then $\left\langle D^{\nu_j}E[y\prod\limits_{\kappa=1}^{j-1}\langle A^{\nu_\kappa}y, v\rangle \Phi_0], v\right\rangle = E[\langle A^{\nu_j}y, v\rangle\prod\limits_{\kappa=1}^{j-1}\langle A^{\nu_\kappa}y, v\rangle \Phi_0] = E[\prod\limits_{\kappa=1}^{j}\langle A^{\nu_\kappa}y, v\rangle \Phi_0]$. □

Introducing functionals $B_\nu = \left\langle D^\nu\frac{\delta(\cdot)}{\delta w}, v\right\rangle$ $(\nu \in \mathbb{N})$ we obtain

Corollary 4.7. *Let $v \in V$. Then*

$$E[\prod_{\kappa=1}^{j}\langle A^{\nu_\kappa}y, v\rangle \Phi_0] = B_{\nu_j}B_{\nu_{j-1}}\ldots B_{\nu_1}F_0.$$

Putting corollaries 4.2, 4.5, 4.7 and formulas (6), (7), (10) together we come to the following

Theorem 4.8. *Let*

- $\sum\limits_{\nu=0}^{\infty} A^\nu y$ *converge in* X,

- $\frac{1}{n!}E\left|\left\langle\sum\limits_{\nu=0}^{\infty}A^{\nu}y,v\right\rangle\right|^{n}\underset{n\to\infty}{\to}0$ *for all* $v\in V$.

Then the problem (3), (4) *corresponding to the equation* (1) *has the "normal" solution*

$$F(u_1,u_2,u_3,v,w)=F_0(u_1,u_2,u_3,w)+$$

$$\sum_{j=1}^{\infty}\frac{i^j}{j!}\sum_{\sigma=1}^{\infty}\sum_{\max\{\nu_1,\nu_2,\ldots,\nu_j\}=\sigma}(B_{\nu_j}B_{\nu_{j-1}}\ldots B_{\nu_1}F_0)(u_1,u_2,u_3,v,w)=F_0(u_1,u_2,u_3,w)+$$

$$\sum_{j=1}^{\infty}i^j\sum_{\sigma=1}^{\infty}\sum_{\nu_1\le\nu_2\le\ldots\le\nu_j=\sigma}\frac{1}{\prod\limits_{\tau=1}^{\sigma}(\sum\limits_{\kappa=1}^{j}\delta_{\nu_k,\tau})!}(B_{\nu_j}B_{\nu_{j-1}}\ldots B_{\nu_1}F_0)(u_1,u_2,u_3,v,w)\qquad(11)$$

generated by the solution $x^*=\sum\limits_{\nu=0}^{\infty}A^{\nu}y\in X$ *of the equation* (1).

5 Examples

1. Let's consider the following random integral equation:

$$x(t_1,t_2;\omega)-\frac{1}{5}\int_0^1 s_1x^2(s_1,t_2;\omega)ds_1-\frac{1}{5}\int_0^1(s_2+\xi(\omega))x(t_1,s_2;\omega)ds_2=-\frac{3}{10}\xi^2(\omega)-$$

$$\frac{1}{5}[\xi(\omega)(\frac{t_1+t_2}{2}-\frac{47}{12})+\frac{1}{48}(6t_2^2-108t_1-112t_2+11)],$$

where $\xi(\omega)$ is uniformly distributed on $[0,1]$. Let $\|\cdot\|=\operatorname{ess\,sup}\limits_{T_3\times\Omega}|\cdot|$. One may check that for all $x_1,x_2\in X$ it is true that

$$\|Ax_1-Ax_2\|=\|A(x_2+(x_1-x_2))-Ax_2\|\le$$

$$\frac{1}{10}(\|x_1-x_2\|^2+2\|x_1-x_2\|\|x_2\|)+\frac{3}{10}\|x_1-x_2\|.$$

Let $\widetilde{X}=\{x:\|x\|\in[0,2]\}$. Then for all $x\in\widetilde{X}$ we obtain $\|Ax\|=\|Ax-A0\|\le\frac{1}{10}\|x\|^2+\frac{3}{10}\|x\|\le\frac{1}{2}\|x\|$. Therefore $A(\widetilde{X})\subseteq X$. Also for all $x_1,x_2\in\widetilde{X}$ we get $\|Ax_1-Ax_2\|\le m\|x_1-x_2\|$, $m=\frac{9}{10}$. Now let's check if the conditions of theorem 4.8 are met. Since $\|y\|=\frac{271}{240}\in[0,2]$, $y\in\widetilde{X}$ and $\sum\limits_{\nu=0}^{\infty}A^{\nu}y$ converges in X. Let $\|\sum\limits_{\nu=0}^{\infty}A^{\nu}y\|=M$. The second condition is met, too: $\frac{1}{n!}E\left|\left\langle\sum\limits_{\nu=0}^{\infty}A^{\nu}y,v\right\rangle\right|^{n}\le\frac{M^n}{n!}\left(\int\limits_{T_3}|v(s_1,s_2)|ds_1ds_2\right)^{n}\underset{n\to\infty}{\to}0$. Thus the random equation has the solution $x^*(t_1,t_2,\omega)=\sum\limits_{\nu=0}^{\infty}(A^{\nu}y)(t_1,t_2,\omega)$. Substituting $x^{**}(t_1,t_2,\omega)=\frac{t_1+t_2}{2}+\xi(\omega)$ into (1) we obtain that x^{**} is a solution, too. But $\|x^{**}\|=2\in[0,2]$, therefore $x^{**}\in\widetilde{X}$. Because of solution uniqueness (in $\widetilde{X}$) $x^*=x^{**}$.

The corresponding determinate equation under the bounding condition

$$F(u_1,u_2,u_3,0,w)=\exp\left\{\frac{i}{5}\int_0^1\int_0^1\left(\int_0^1 s_1u_1(t_1,t_2,s_1)ds_1+\int_0^1 s_2u_2(t_1,t_2,s_2)ds_2-\right.\right.$$

$$\frac{1}{48}(6t_2^2 - 108t_1 - 112t_2 + 11)w(t_1,t_2)\Big)\,dt_1dt_2\Big\} \times$$

$$\times I\left(\frac{1}{5}\int_0^1\int_0^1\left(\int_0^1 u_2(t_1,t_2,s_2)ds_2 - (\frac{t_1+t_2}{2} - \frac{47}{12})w(t_1,t_2)\right)dt_1dt_2, -\frac{3}{10}\int_0^1 w(t_1,t_2)dt_1dt_2\right),$$

where $I(\alpha,\beta) = \int_0^1 \exp\{i\xi(\alpha+\beta\xi)\}d\xi$, has the "normal" solution determined by (11).

2. Let's consider the equation

$$x(t_1,t_2;\omega) - \int_T \exp\{t_1\xi^2(\omega)\}\cos(t_2\xi(\omega)+s_1)x^p(s_1,t_2;\omega)ds_1 =$$

$$\frac{(-1)^p+1}{p+1}\exp\{t_1\xi^2(\omega)\}\sin^{p+1}(t_2\xi(\omega)) + \sin(t_1+t_2\xi(\omega)),$$

where $(t_1,t_2,s_1)\in T^3$ $(T=[0,\pi])$, $p\in\mathbb{N}$. It has the following solution

$$x(t_1,t_2,\omega) = \sin(t_1+t_2\xi(\omega))$$

The corresponding determinate problem under the obvious additional conditions

$$\frac{\delta F(u_1,u_2,u_3,v,w)}{\delta u_2(t_1,t_2,s_2)} = \frac{\delta F(u_1,u_2,u_3,v,w)}{\delta u_3(t_1,t_2,s_1,s_2)} = 0$$

looks the following way:

$$\frac{\delta F(u_1,v,w)}{\delta v(t_1,t_2)} + \frac{1}{i^p}\int_T \frac{\delta F^{p+1}(u_1,v,w)}{\delta u_1(t_1,t_2,s_1)\delta v^p(s_1,t_2)}ds_1 = \frac{\delta F(u_1,v,w)}{\delta w(t_1,t_2)},$$

$$F(u_1,0,w) = \int_\Omega \Phi_0(u_1,w;\omega)P(d\omega),$$

where

$$\Phi_0(u_1,w;\omega) = \exp\{i\left[\int_{T^2}\left(\int_T(\exp\{t_1\xi^2(\omega)\}\cos(t_2\xi(\omega)+s_1)u_1(t_1,t_2,s_1)ds_1 + \right.\right.$$

$$\left.\left.(\frac{(-1)^p+1}{p+1}\exp\{t_1\xi^2(\omega)\}\sin^{p+1}(t_2\xi(\omega)) + \sin(t_1+t_2\xi(\omega)))w(t_1,t_2)\right)dt_1dt_2\right]\},$$

and has the "normal" solution

$$F(u_1,v,w) = \int_\Omega \exp\{i\int_{T^2}\sin(s_1+s_2\xi(\omega))v(s_1,s_2)ds_1ds_2\}\Phi_0(u_1,w;\omega)P(d\omega).$$

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INEQUALITIES FOR LINEAR POSITIVE OPERATORS

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ABSTRACT

This survey paper grew out of a series of results, achieved at Duisburg in collaboration with mathematicians from Germany and Romania during my visits in the period of last seven years,1999-2005. Direct theorems for approximation of continuous real-valued functions by positive linear operators in terms of the second order modulus of smoothness are presented.Special emphasis is on the magnitude of the absolute constants appearing in the right-hand side of the inequalities. The degree of approximation for Bernstein operator, Schoenberg variation-diminishing spline operator and NURBS (non-uniform rational B-spline operator) is studied. Each of these three linear positive operators is a generalization of the preceding one and has an important applications in broad areas of mathematics such as real and functional analysis, approximation theory, numerical methods, computer-aided geometric design (CAGD), the theory of probabilities and mathematical statistics,geometry.

Keywords: Linear Positive Operators,Degree of Approximation,Moduli of Smoothmess.

2000 Mathematics Subject Classification: 41A15,41A25,41A28,41A36, 41A63,65D07, 65D17

1 INTRODUCTION

With $C[0,1]$ we denote the space of all continuous real-valued functions,defined on the interval $[0,1]$. For a measure of the rate of approximation we use two types of moduli of smoothness. The first one is the classical $s-$order modulus of smoothness $\omega_s(f;h)$ of $f \in C[0,1]$ with argument $h>0$, $s=1,2$ with the following definitions:

$$\omega_1(f;h) := \sup_{0<t\leq h} \max_{x\in[0,1-t]} |f(x+t)-f(x)| \tag{1.1}$$

$$\omega_2(f;h) := \sup_{0<t\leq h} \max_{x\in[t,1-t]} |f(x+t)-2f(x)+f(x-t)|,\ h>0 \tag{1.2}$$

For the properties and applications of these moduli in many approximation problems see [2]. At the beginning of the 80's it had become clear that for characterizing those functions $f \in C[0,1]$ for which the quantities

$$\|f-B_nf\|_\infty := \max_{x\in[0,1]} |f(x)-B_n(f,x)|$$

vanish at a given speed, moduli of smoothness should be used which are based on differences $\Delta_t^2 f(x) := f(x+t) - 2f(x) + f(x-t)$ in which the step t is allowed to depend upon the position of $x \in [0,1]$. Here B_n denotes the famous Bernstein operator given by

$$B_n(f,x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k} \tag{1.3}$$

$n \in N$-the set of natural numbers.
So we arrived at the so-called second order Ditzian-Totik modulus of smoothness $\omega_2^{\varphi}(f;\cdot)$ defined by

$$\omega_2^{\varphi}(f;t) = \sup_{0<h\leq t} \|\Delta_{h\varphi}^2 f\|_{\infty},\ \varphi(x) = \sqrt{x(1-x)} \tag{1.4}$$

where

$$\Delta_{h\varphi}^2 f(x) := f(x - h\varphi(x)) - 2f(x) + f(x + h\varphi(x))$$

if $[x - h\varphi(x), x + h\varphi(x)] \in [0,1]$ and $\Delta_{h\varphi}^2 f(x) := 0$ otherwise.
The main reason to introduce this new constructive characteristic of the function $f(x)$ is the improvement of the approximation near the endpoints of the interval $[0,1]$. For the properties and applications of these moduli see [4]. We only mention that for every $f \in C[0,1]$ the moduli $\omega_s(f,t)$ and $\omega_2^{\varphi}(f,t)$ tends to 0 when $t \to 0$. Further let $L(f;x) : C[0,1] \to C[0,1]$ be a linear positive operator which reproduces constant functions,i.e. $L(1;x) := 1$ for every $x \in [0,1]$.
Our main goal in this paper is to study the error of approximation

$$L(f,x) - f(x)$$

for fixed $x \in [0,1]$ and to establish the bounds for the sup norm of this error. Here we are interested in obtaining the so-called pointwise estimates

$$|L(f,x) - f(x)| < A(f,x) \tag{1.5}$$

and the norm estimates

$$\|Lf - f\|_{\infty} < A(f) \tag{1.6),}$$

where $A(f,x)$ and $A(f)$ are quantities,dependent only on the marked parameters.
In section 2 we consider for L the Bernstein operator B_n defined by (1.3). In sections 3 and 4 we represent the recent results for Schoenberg and NURBS operators respectivelly.

2 BERNSTEIN OPERATOR

For the first time the Bernstein operator was presented in: "D'emonstration du th'eoreme de Weierstrass,fond'ee sur le calcul des probabilit'es,Comm.Soc. Mat.Kharkov (2),13,1912-13,1-2",the paper written by S.N. Bernstein. The main result there is a very elegant and constructive proof of the famous theorem of Weierstrass (formulated in 1885) that each continuous real function f on $[a,b]$ is uniformly approximable by algebraic polynomials. For the proof see also Theorem 1.1,Ch. 1 in [2]. Most likely about 2500 items now exists in the literature dealing with

Bernstein operator and other related operators called "Bernstein-type operators" like Schoenberg variation-diminishing spline operator,NURBS operator and many others. During the last decade remarkable progress has been made in investigating the operator B_n. At the forefront of the development were authors such as Z.Ditzian,K.Ivanov, V.Totik and Xin-long Zhou. We recall the following pioneering result:

Theorem 1. (Knoop and Zhou [11],Totik [18]).
There are constants c_1 and c_2 such that for all $n \in N$ and $f \in C[0,1]$ one has

$$c_1\omega_2^{\varphi}(f;\frac{1}{\sqrt{n}}) \leq \|B_nf - f\|_{\infty} \leq c_2\omega_2^{\varphi}(f;\frac{1}{\sqrt{n}}) \tag{2.1}$$

Remarks:

i) The left inequality is usually called a "strong converse inequality" while the right one can be denoted as a "Jackson inequality" and had been known before. The right inequality was proved simultaneously by X.L.Zhou in [19] and by K.Ivanov in [10].

ii) It was hardly possible to derive magnitude of c_1 and c_2 from the papers [10,11,19,18].

iii) For the classification of different types of "strong converse" results see [3].

iv) Nothing seems to be known at this time -2006 about the possible or the best possible value(s) of c_1 in (2.1)
The first approach by the author and H.Gonska (1998-2000) to establish a concrete value of c_2 was a constructive one, in which a suitable spline g was constructed, see [25,26]. With the help of g we have proved:

Lemma 1. *If $L : C[0,1] \to C[0,1]$ is a positive linear operator which reproduces linear functions, then one has*

$$|L(f;x) - f(x)| \leq 2\|f - g\|_{\infty} + \frac{L((e_1 - x)^2;x)}{x(1-x)} \cdot \|\varphi^2 g''\|_{\infty} \tag{2.2}$$

for each $x \in (0,1)$ and for any $f \in C[0,1]$ and any g for which $\|\varphi^2 g''\|_{\infty}$ is finite.

Here $\varphi^2(x) = x(1-x)$. The construction of the quadratic C^1- spline $g = g(f,m,\varphi)$ leads us to

Lemma 2.

$$\|f - g\|_{\infty} \leq \omega_2^{\varphi}(f;\frac{\sqrt{2}}{m \cdot d(m)}) \tag{2.3}$$

$$\|\varphi^2 g^{(2)}\|_{\infty} \leq 24m^2\omega_2^{\varphi}(f;\frac{\sqrt{2}}{m \cdot d(m)}) \tag{2.4}$$

Here $m \geq 2$ and $d(m)$ is the strictly increasing sequence given by

$$d(m) = \frac{\sqrt{m^4+m^2+1}-1}{\sqrt{m^4+m^2+1}+m^2},\ m \geq 1.$$

This constructive approach was used to prove

Theorem 2.

$$\|B_n f - f\|_\infty \leq 405 \cdot \omega_2^\varphi(f; \frac{1}{\sqrt{n}}),\ n \geq 1. \tag{2.5}$$

In 2000 improvement was made by I.Gavrea [7] using better knots of g:

$$\|B_n f - f\|_\infty \leq 6 \cdot \omega_2^\varphi(f; \frac{1}{\sqrt{n}}),\ n \geq 1. \tag{2.6}$$

The next improvement by Adell and Sangüessa [1] was obtained by probabilistic tools:

$$\|B_n f - f\|_\infty \leq 4 \cdot \omega_2^\varphi(f; \frac{1}{\sqrt{n}}),\ n \geq 1. \tag{2.7}$$

The collaboration of the author with H.Gonska,I.Gavrea and R.Paltanea leads us to:

Theorem 3. (see[27]) *Let $L : C[0,1] \to C[0,1]$ be a positive linear operator reproducing linear functions. Then*

$$|L(f,x) - f(x)| \leq \left[\frac{3}{2} + \frac{3}{2} \cdot \frac{1}{h^2} \cdot \frac{L(|e_1 - xe_0|^2; x)}{x(1-x)}\right] \cdot \omega_2^\varphi(f;h) \tag{2.8}$$

for $x \in (0,1)$, $0 < h \leq \frac{1}{\sqrt{2}}$.

Here e_k, $k = 0,1,2$ are the monomial functions $e_k(x) = x^k$, $x \in [0,1]$. If $L = B_n$ and $h = \frac{1}{\sqrt{n}}$ we get

$$\|B_n f - f\|_\infty \leq 3 \cdot \omega_2^\varphi(f; \frac{1}{\sqrt{n}}),\ n \geq 1. \tag{2.9}$$

The best possible value of c_2 remains unknown.

3 SCHOENBERG VARIATION-DIMINISHING SPLINE OPERATOR

Consider the knot sequence $\Delta_n = \{x_i\}_{-k}^{n+k}$, $n > 0, k > 0$ with

$$x_{-k} = x_{-k+1} = \cdots = x_0 = 0 < x_1 < \cdots < x_n = \ldots x_{n+k} = 1.$$

For a real-valued bounded function f the variation-diminishing spline of degree k w.r.t. Δ_n is given by

$$S_{\Delta_n,k}f(x) := \sum_{j=-k}^{n-1} f(\xi_{j,k}) \cdot N_{j,k}(x)$$

for $0 \leq x < 1$ and

$$S_{\Delta_n,k}f(1) := \lim_{y \to 1, y<1} S_{\Delta_n,k}f(y),$$

with the nodes (Greville abscissas)

$$\xi_{j,k} := \frac{x_{j+1} + \cdots + x_{j+k}}{k}, \quad -k \leq j \leq n-1,$$

and the normalized $B-$ spline as fundamental functions

$$N_{j,k} := (x_{j+k+1} - x_j)[x_j, x_{j+1}, \ldots, x_{j+k+1}](\cdot - x)_+^k.$$

This method of approximation was introduced by Schoenberg [17] in 1965 as a "natural" extension of the classical Bernstein polynomial approximation (set in (3.1) $n = 1$,i.e. no interior knots).

A further key article on the method is one by Marsden and Schoenberg [14] which appeared in Romania, Schoenberg's native country in 1966. We list some of the most important properties of the operator $S_{\Delta_n,k}$:

P_1) $S_{\Delta_n,k}$ is a positive linear operator which reproduces linear functions,i.e.

$$\sum_{j=-k}^{n-1} N_{j,k}(x) = 1, \quad 0 \leq x \leq 1,$$

$$\sum_{j=-k}^{n-1} \xi_{j,k} \cdot N_{j,k}(x) = x, \quad 0 \leq x \leq 1.$$

P_2) $S_{\Delta_n,k}$ is a discretely defined operator,which maps $C[0,1]$ into that subspace of $C^{k-1}[0,1]$ containing all functions which are on each interval $[x_i, x_{i+1}]$ a polynomial of degree at most k.

P_3) Besides of the Bernstein operator, $S_{\Delta_n,k}$ also generalizes piecewise linear interpolation at the knots of Δ_n (set $k = 1$ in (3.1)).

P_4) $S_{\Delta_n,k}$ has the convex hull property and interpolates at the endpoints.

P_5) $S_{\Delta_n,k}$ has the variation-diminishing property,i.e. $V(S_{\Delta_n,k}f - l) \leq V(f - l)$ on $[0,1]$, for all linear functions l, where $V(g)$ denotes the number of sign changes of the function g.Schoenberg spline operator is in much use in CAGD and has become an indispensible tool there. For more details in regard to their use in CAGD see the books Farin [6] and by Hoschek and Lasser [9] where more references can be found.

As for any positive linear operator, the second moments $(S_{\Delta_n,k}(e_1 - x)^2)(x)$, $x \in [0,1]$ play an important role for the quantitative behaviour of $S_{\Delta_n,k}$. It is thus instructive to have an idea of where the graph of the function

$$[0,1] \to (S_{\Delta_n,k}(e_1 - x)^2)(x)$$

is located. In [20] we proved the following crutial inequality

Theorem 4. ([20]).*For $n \geq 1$, $k \geq 1$, $x \in [0,1]$ we have*

$$(S_{n,k}(e_1 - x)^2)(x) \leq 1 \cdot \frac{\min\{2x(1-x); \frac{k}{n}\}}{n+k-1} \tag{3.2}$$

where $S_{n,k}$ denotes the $k-th$ degree Schoenberg splines with equidistant knots $x_j = \frac{j}{n}$, $0 \leq j \leq n$.

The estimate (3.2) is an essential improvement of the previous results of Marsden [12,13]. In papers [21,22,23,24] we have studied various problems concerning approximation by $S_{n,k}$. The following elegant result of Paltanea ([15]) is the key for our subsequent applications to Schoenberg splines:

Theorem 5. ([15]). *If $L : C[0,1] \to C[0,1]$ is a positive linear operator, then for $f \in C[0,1], x \in [0,1]$ and each $0 < h \leq \frac{1}{2}$ the following holds:*

$$\begin{aligned} |(Lf)(x) - f(x)| \leq\ & |(Le_0)(x) - 1| \cdot |f(x)| \\ &+ \frac{1}{h} \cdot |(L(e_1 - x))(x)| \cdot \omega_1(f,h) \\ &+ \left[(Le_0)(x) + \frac{1}{2h^2} \cdot (L(e_1 - x)^2)(x)\right] \cdot \omega_2(f,h). \end{aligned} \tag{3.3}$$

In particular one has:

$$|S_{n,k}f(x) - f(x)| \leq \frac{3}{2} \cdot \omega_2(f; \sqrt{\frac{2x(1-x)}{n+k-1}}) \tag{3.4}$$

$$|S_{n,k}f(x) - f(x)| \leq \frac{3}{2} \cdot \omega_2(f; \sqrt{\frac{k}{n(n+k-1)}}) \tag{3.5}$$

Further from (2.8) it follows

$$|S_{n,k}f(x) - f(x)| \leq \frac{9}{2} \cdot \omega_2^{\varphi}(f; \frac{1}{\sqrt{n+k-1}}) \tag{3.6}$$

To combine the last three inequalities define

$$\Omega_2(f;n,k) := \min\left\{\omega_2^{\varphi}(f; \frac{1}{\sqrt{n+k-1}}), \omega_2(f; \sqrt{\frac{k}{n+k-1}})\right\}. \tag{3.7}$$

The estimates (3.5) and (3.6) lead us to

$$\|S_{n,k}f - f\|_{\infty} \leq \frac{9}{2} \cdot \Omega_2(f;n,k). \tag{3.8}$$

We recall here that Schoenberg's original intention was to introduce a natural "spline extension" of the Bernstein polynomials. This was definitely achieved. Since then impressive progress

was made in the investigation of Bernstein operator. One particular highlight is the result of Knoop and Zhou [11]. They showed that for the second order Ditzian-Totik modulus one has

$$\|B_k f - f\|_\infty \approx \omega_2^\varphi(f, \frac{1}{\sqrt{k}}),\ k \to \infty. \tag{3.9}$$

The author is not aware of any corresponding result for the $S_{\Delta_n,k}$ which generalizes the assertion of Knoop and Zhou. We feel that the proof of a strong converse inequality (see[3] for the theory of converse inequalities) would be a significant and most valuable contribution to both Approximation Theory and CAGD. In this connection we formulate the following

Conjecture 1. *Let $k \geq 2, n \geq 1, n+k \to \infty$. There are constants $c_3, c_4 > 0$, independent on f, n, k such that*

$$c_3 \cdot \Omega_2(f; n, k) \leq \|S_{n,k} f - f\|_\infty \leq c_4 \cdot \Omega_2(f; n, k). \tag{3.10}$$

The right-hand side is clear - (3.8). The left-hand side remains as an open and probably very difficult problem.

4 NURBS OPERATORS

The name NURBS denotes non-uniform rational B-splines.We start with the definition. Let Δ_n be the finite partition of $[0,1]$ given in the previous section, $\xi_{j,k}$-be the Greville abscissae and $N_{j,k}(x)$ - the fundamental B-splines,$-k \leq j \leq n-1$, $k, n \in N$. Then the NURBS operator is given by

$$R_{\Delta_{n,k}}(f;x) := \frac{\sum\limits_{j=-k}^{n-1} \omega_{j,k} \cdot f(\xi_{j,k}) \cdot N_{j,k}(x)}{\sum\limits_{j=-k}^{n-1} \omega_{j,k} \cdot N_{j,k}(x)}, \tag{4.1}$$

for $0 \leq x < 1$ and

$$R_{\Delta_{n,k}}(f;1) := \lim_{y \to 1} R_{\Delta_{n,k}}(f;y).$$

Some properties of $R_{\Delta_n,k}$ are:

P_1) $R_{\Delta_n,k}$ is a positive linear operator reproducing constant functions (compare with B_n and $S_{n,k}$).

P_2) Both the numerator and the denumerator are splines of degree k and in $C^{k-1}[0,1]$.

P_3) $R_{\Delta_n,k}(f;0) = f(0)$; $R_{\Delta_n,k}(f;1) = f(1)$.

P_4) $R_{\Delta_n,k}$ is discretely defined,i.e. it depends only on the $n+k$ values $f(\xi_{j,k})$, $-k \leq j \leq n-1$ and on the weights $\omega_{j,k}$ associated with $\xi_{j,k}$. The numbers $\omega_{j,k}$ are positive for all $-k \leq j \leq n-1$.

Some examples of $R_{\Delta_n,k}$:

1. Suppose that $\omega_{j,k} = \omega > 0$ for $-k \le j \le n-1$. Then

$$R_{\Delta_{n,k}}(f;x) = \sum_{j=-k}^{n-1} f(\xi_{j,k}) \cdot N_{j,k}(x) = S_{\Delta_{n,k}}(f;x),\ x \in [0,1].$$

the latter is the variation-diminishing Schoenberg spline considered in the previous section.
2. $\omega_{j,k} = \omega > 0$, $k = 1, n \in N$. The nodes are $\xi_{j,1} = x_{j+1}$, $-1 \le j \le n-1$. $S_{\Delta_n,1}$ describes piecewise linear interpolation at the points $0 = x_0 < x_1 < \cdots < x_n = 1$.
3. $\omega_{j,k} = \omega > 0$, $n = 1, k \in N$. Now we obtain $B_k(f,\cdot)$-the Bernstein polynomials of degree k, considered in section 2.
4. Not all weights $\omega_{j,k}$ are equal, but again $n = 1$, $k \in N$. In this case we have

$$R_{\Delta_1,k}(f;x) := \frac{\sum_{j=0}^{k} \omega_{j,k} \cdot f(\frac{j}{k}) \cdot p_{k,j}(x)}{\sum_{j=0}^{k} \omega_{j,k} \cdot p_{k,j}(x)}, \tag{4.2}$$

where $p_{k,j}(x)$ are the Bernstein basis polynomials. This is the rational Bernstein function. For the latter H.Gonska,D.Kacso and P.Pitul (2005) proved in [8] the following

Theorem 6. *The rational Bernstein operator $R_{1,k}$ (equidistant knots x_i) reproduces linear functions if and only if $\omega_{j,k} = \omega$ for all j.*

The proof is based upon the representation

$$R_{1,k}(t;x) - x = x(1-x) \cdot \frac{1}{N} \cdot \sum_{j=0}^{k-1} (\omega_{j+1,k} - \omega_{j,k}) \cdot p_{k-1,j}(x), \tag{4.3}$$

with

$$N := \sum_{j=0}^{k} \omega_{j,k} \cdot p_{k,j}(x).$$

The statement in theorem 6 is in contrast to the case of NURBS-curves, where in (4.1) instead of $f(\xi_{j,k})$ we take P_j^--the so-called control points and in this way we obtain the vector-valued variant of (4.1). For the numerous applications, properties and algorithms of NURBS curves and surfaces see the book [16]. In [5] Farin proved that the reproduction of linear functions in the case of NURBS-curves is possible even when weights $\omega_{j,k}$ are different. It would be desirable to have a similar to (4.3) representation for the more general $R_{\Delta_n,k}$ or at least for $R_{n,k}$, $n \ge 2$. Very recently we proved in this direction the following [28]:

Theorem 7. *For all $k \ge 1$, $n \ge 1$ the first moment of $R_{\Delta_n,k}$ is:*

$$R_{\Delta_n,k}(t;x) - x = \frac{1}{N} \cdot \sum_{j=-k}^{n-1} \sum_{i=j+1}^{n-1} (\omega_{j,k} - \omega_{i,k})(\xi_{j,k} - \xi_{i,k}) N_{j,k}(x) N_{i,k}(x), \tag{4.4}$$

with

$$N := \sum_{j=-k}^{n-1} \omega_{j,k} N_{j,k}(x).$$

After simple calculations from (4.4) we get

Theorem 8. *The NURBS operator $R_{\Delta_n,k}(k = 1, 2)$ reproduces linear functions if and only if all weights ω_j are equal.*

The idea in the proof is not working for $k \geq 3$. But we believe that the following holds true:

Conjecture 2. *The statement in theorem 8 is true for $k \geq 3$.*

Further we study the degree of approximation by $R_{n,1}-$equidistant knots. We apply the estimate (3.3) of Paltanea to arrive at

Theorem 9. *For $f \in C[0,1]$, $0 < h \leq \frac{1}{2}$ there holds*

$$\begin{aligned}(4.5)\ |R_{n,1}(f,x) - f(x)| &\leq \\ &\leq \frac{1}{N}\Big|\sum_{j=-1}^{n-1}(\omega_{j,1} - \omega_{j+1,1})N_{j,1}(x)N_{j+1,1}(x)\Big| \cdot \frac{1}{nh} \cdot \omega_1(f,h) \\ &+ \Big(1 + \frac{1}{2} \cdot h^{-2} \cdot \rho_{n,1}\frac{\min\{2x(1-x); \frac{1}{n}\}}{n}\Big) \cdot \omega_2(f,h)\end{aligned}$$

with

$$\rho_{n,1} := \frac{\max \omega_{j,1}}{\min \omega_{j,1}} \geq 1.$$

We set

$$\omega_{j,1} := 1 + \frac{c(j+1)}{n^\alpha},\ -1 \leq j \leq n,\ c \geq 0,\ \alpha > 0.$$

With this choice of the weight numbers it is clear that

$$\frac{\max|\omega_{j,1} - \omega_{j+1,1}|}{\min \omega_{j,1}} \leq \frac{c}{n^\alpha},\ n = 1, 2, \ldots$$

Obviously

$$N_{j,1}(x)N_{j+1,1}(x) \leq \frac{1}{4}.$$

We obtain

$$|R_{n,1}(f,x) - f(x)| \leq \frac{c}{4n^{\alpha+1}h}\omega_1(f,h) + (1 + \frac{\rho_{n,1}}{2} \cdot \frac{1}{h^2n^2})\omega_2(f,h)$$

It is obvious that

$$\rho_{n,1} = 1 + cn^{1-\alpha} \approx 1$$

for $\alpha > 0$ big enough. We set $h = n^{-1}$ and arrive at

$$|R_{n,1}(f,x) - f(x)| \leq \frac{c}{4n^{\alpha}}\omega_1(f,n^{-1}) + (1+\frac{\rho_{n,1}}{2})\omega_2(f,n^{-1}),\ n \to \infty. \tag{4.6}$$

What is missing according to the approximation power of $R_{n,k}$ is a "good" representation of the second moment of $R_{n,k}$,i.e. without using the idea of the factor $\rho_{n,1}$.

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Strong Convergence Theorems for a Finite Family of Nonexpansive Mappings with Their Applications

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ABSTRACT

In this paper, by using some new analysis techniques, we study the approximation problems of common fixed points for a class of finite nonexpansive mappings in uniformly convex Banach spaces by using new control conditions. The main results presented in this paper generalize, extend and improve the corresponding results of Atsushiba and Takahashi (Atsushiba and Takahashi, to appear in Indian J. Math.), Shioji and Takahashi (Shioji and Takahashi, 1997), Takahashi, Tamura and Toyoda (Takahashi et al., 2002), Wittmann (Wittmann, 1992), Xu (Xu, 2002, 2003) and others.

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1 Introduction

Let X be a real Banach space and X^* be the dual space of X. Let $\varphi : [0, \infty) =: R^+ \to R^+$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Such a function φ is called a *gauge*. Associated with a gauge φ is the *daulity mapping* $J_\varphi : X \to X^*$ defined by

$$J_\varphi(x) = \{f \in X^* : \langle x,\ f\rangle = \|x\| \cdot \varphi(\|x\|),\ \|f\| = \varphi(\|x\|)\}, \quad x \in X.$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between X and X^*. In the case that $\varphi(t) = t$, we write J for J_φ and call J the *normalized duality mapping*. It is well known that the following inequality holds in arbitrary normed linear space X: Let X be a real normed linear space. Then, for all x, $y \in X$ and $j(x+y) \in J(x+y)$,

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, J(x+y) > .$$

We shall make use of this inequality in our proofs.

Definition 1.1. (1) A Banach space X is said to be *strictly convex* if

$$\frac{\|x+y\|}{2} < 1$$

for all $x, y \in S_X = \{z \in X : \|z\| = 1\}$.
(2) For any ϵ with $0 \leq \epsilon \leq 2$, we define the *modulus* $\delta(\epsilon)$ of convexity of X by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \ \|x-y\| \geq \epsilon \right\}.$$

A Banach space X is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for any $\epsilon > 0$.
(3) A Banach space X is said to be *smooth* if

$$\lim_{\lambda \to 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda}$$

exists for all $x, y \in S_X$. In this case, the norm of E is said to be *Gâteaux differentiable*
(4) A Banach space X is *uniformly smooth* if the limit

$$\lim_{\lambda \to 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda}$$

exists and is attained uniformly in $x, y \in S_X$.
(5) The norm of X is said to be uniformly *Gâteaux differentiable* if, for any $y \in S_X$,

$$\lim_{\lambda \to 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda}$$

exists uniformly for all $x \in S_X$.

Remark 1.1. (1) Banach space X is strictly convex if and only if $\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$ for all $x, y \in X$ and $0 < \lambda < 1$ implies that $x = y$.
(2) A uniformly convex Banach space X is strictly convex, but the converse is not true.
(3) If a Banach space X is uniformly smooth, then the normalized duality mapping J is single-valued and norm to norm uniformly continuous on any bounded subsets of X. Moreover, if the norm of X is uniformly *Gâteaux differentiable*, then the normalized duality mapping J is norm to weak star uniformly continuous on any bounded subsets of X.

Definition 1.2. Let C be a closed convex subset of a Banach space X and K be a subset of C.
(1) A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x-y\|, \quad x, y \in C.$$

(2) A mapping P of C onto K is said to be *sunny* if $P(Px+t(x-Px))=Px$ for any $x\in C$ and $t\geq 0$ with $Px+t(x-Px)\in C$. P is called a retraction if $Px=x$ for each $x\in K$.
(3) A subset K of C is called a *nonexpansive retract* of C if there exists a nonexpansive retraction of C onto K.

Remark 1.2. (1) Let C be a nonempty closed convex subset of a Hilbert space H. If P is the metric projection of H into C, then P is sunny and nonexpansisve.
(2) Let C be a nonempty convex subset of a smooth Banach space X and $C_0\subset C$, then a retraction P of C onto C_0 is sunny and nonexpansive if and only if

$$< x-Px, J(y-Px) >\leq 0, \quad x\in C,\ y\in C_0.$$

Hence there is at most one sunny nonexpansive retraction of C onto C_0. If there is a sunny nonexpansive retraction of C onto C_0, then C_0 is said to be a sunny nonexpansive retract of C.

The following theorem related to the existence of nonexpansive retractions was proved by Bruck (Bruck, 1973).

Theorem BR. *Let X be a reflexive Banach space, C be a nonempty closed convex subset of X and T be a nonexpansive mapping of C into itself with $F(T)\neq\emptyset$. If T has a fixed point in every nonempty bounded closed convex subset o f X such that T leaves invariant, then $F(T)$ is a nonexpansive retract of C.*

The following theorem related to the existence of sunny nonexpansive retractions was proved by Reich (Reich,1980) (see also Takahashi, 2000, Shioji and W. Takahashi, 1997).

Theorem R. *Let X be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C be a nonempty closed convex subset of X and T be a nonexpansive mapping of C into itself with $F(T)\neq\emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

Let C be a nonempty convex subset of a Banach space X. Let $T_1,T_2,\cdots,T_r$ be finite mappings of C into itself and $\alpha_1,\alpha_2,\cdots,\alpha_r$ be real numbers such that $0\leq\alpha_i\leq 1$ for any $i=1,2,\cdots,r$. Then we define a mapping W of C into itself by

$$W=U_r=\alpha_rT_rU_{r-1}+(1-\alpha_r)I,$$

where

$$U_1=\alpha_1T_1+(1-\alpha_1)I,\ U_2=\alpha_2T_2U_1+(1-\alpha_2)I,\ \cdots,\ U_{r-1}=\alpha_{r-1}T_{r-1}U_{r-2}+(1-\alpha_{r-1})I. \tag{A}$$

Such a mapping W is called the *W−mapping* generated by $T_1,T_2,\cdots,T_r$ and $\alpha_1,\alpha_2,\cdots,\alpha_r$. Let $\alpha_{n1},\alpha_{n2},\cdots,\alpha_{nr}$ $(n=1,2,\cdots,r)$ be real numbers such that $0\leq\alpha_{ni}\leq 1$ for any $i=1,2,\cdots,r$. Let W_n $(n=1,2,\cdots,r)$ be the W−mappings generated by $T_1,T_2,\cdots,T_r$ and $\alpha_{n1},\alpha_{n2},\cdots,\alpha_{nr}$ $(n=1,2,\cdots)$.
Now, we consider the following iteration scheme:

$$u,\ x_1\in C,\ x_{n+1}=\beta_nu+(1-\beta_n)W_nx_n,\quad n\geq 1, \tag{B}$$

where $\{\beta_n\}$ is a sequence of real numbers such that $0\leq\beta_n\leq 1$ for any $n=1,2,\cdots$.

Remark 1.3. From the definition above, we see that, if T_i is nonexpansive mapping for any $i = 1, 2, \cdots, r$, then U_i and W_n are also nonexpansive mappings for any $i = 1, 2, \cdots, r$.

In 1992, Wittmann (Wittmann, 1992) proved the following strong convergence theorem for nonexpansive mapping in a Hilbert space:

Theorem W. *Let H be a Hilbert space, C be a nonempty closed and convex subset of H and $T : C \to C$ be a nonexpansive mapping with a fixed point. Let $\{\beta_n\}$ be a sequence of real numbers with*

$$0 \leq \beta_n \leq 1, \quad \lim_{n\to\infty} \beta_n = 0,$$

$$\sum_{n=1}^{\infty} \beta_n = \infty, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

For arbitrary initial date $x_1 = x \in C$, define a sequence $\{x_n\}$ iteratively as follows:

$$x_{n+1} = \beta_n x + (1 - \beta_n) T x_n, \quad n \geq 1.$$

Then the sequence $\{x_n\}$ converges strongly to the element of $F(T)$ which is nearest to $x \in F(T)$.

In 1997, Shioji and Takanashi (Shioji and Takahashi, 1997) extended Theorem W above to certain Banach space.

On the other hand, in 1996, Bauschke (Bauschke, 1992,) dealt with the iterative process for finding a common fixed point of finite nonexpansive mappings in a Hilbert space (see also Lions, 1977). He proved the following result:

Theorem B. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $\{T_1, T_2, \cdots, T_r\}$ be a finite family of nonexpansive mappings of C into itself with $F = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and*

$$\begin{aligned} \bigcap_{i=1}^{r} F(T_i) &= F(T_r T_{r-1} \cdots T_1) \\ &= F(T_1 T_r \cdots T_2) \\ &= \cdots \\ &= F(T_{r-1} \cdots T_1 T_r). \end{aligned}$$

Let $\{\alpha_n\}$ be a real sequence in [0,1) which satisfies

$$\alpha_n \to 0 \quad (n \to \infty), \tag{C1}$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty \tag{C2}$$

and

$$\sum_{n} |\alpha_n - \alpha_{n+r}| < \infty. \tag{C3}$$

If the sequence $\{x_n\}$ in C is generated by

$$u,\ x_0 \in C,\ x_{n+1} = \alpha_n u + (1 - \alpha_n) T_{n+1} x_n, \quad n \geq 0, \tag{C}$$

where $T_n = T_{n \bmod r}$. *Then the sequence* $\{x_n\}$ *generated by* (C) *converges in norm to* $P_F u$, *where* $P_F : H \to F$ *is the metric projection.*

In 1967, Halpern (Halpern, 1967) introduced initially the sequence $\{x_n\}$ defined by (C) and Xu (Xu, 2002, 2003) and others have studied the control condition $\{\alpha_n\}$ of the sequence $\{x_n\}$. Recently, Takahashi, Tamura and Toyoda extended Theorem B above by showing the strong convergence of the iterative sequence (C) in a uniformly convex Banach space under the control conditions $(C1)$, $(C2)$ and $(C3)$ (Takahashi et al, 2002) and, by using of the $W-$mapping concept, in [1], Atsushiba and Takahashi introduced another kind of iteration (B) which is different from the iterative scheme (C) and showed several strong convergence results (Atsushiba and Takahashi, to appear in Indian J. Math.). Moreover, by virtue of their results, Takahashi et al. (Takahashi et al., 2002) and Atsushiba et al. (Atsushiba and Takahashi, to appear in Indian J. Math.) also considered the problem of image recovery in the setting of Banach spaces.

The purpose of this paper is to introduce the new control conditions and to prove some strong convergence theorems of Atsushiba and Takahashi (Atsushiba and Takahashi, to appear in Indian J. Math.) under these new control conditions.

2 Lemmas

Now, we introduce some lemmas for our main results in this section.

Lemma 2.1. (Weng, 1991) *Let* $\{a_n\}$ *be a real sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - t_n)a_n + o(t_n), \quad n \geq 0,$$

where $t_n \in (0, 1)$ *with* $\sum_{n=0}^{\infty} t_n = \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Note that Lemma 1 is called *Weng's lemma.*

Let μ be a continuous linear functional on l^∞ and $(a_0, a_1, \cdots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \cdots))$. We call μ *Banach's limit* if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \cdots) \in l^\infty$. If μ is Banach's limit, then we have the following:
(i) for all $n \geq 1$ $a_n \leq c_n$ implies $\mu(a_n) \leq \mu(c_n)$,
(ii) $\mu(a_{n+r}) = \mu(a_n)$ for any fixed positive integer r,
(iii) $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, \cdots) \in l^\infty$.
Further, if $a_n \to p$ as $n \to \infty$, then $\mu_n(a_n) = p$.

It also is well-known that, for any $a \in R$ and $(a_1, a_2, \cdots) \in l^\infty$, $\mu_n(a_n) \leq a$ if and only if, for all $\epsilon > 0$, there exists $m \geq 1$ such that

$$\frac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p} \leq a + \epsilon$$

for all $p \geq m$ and all $n \geq 1$. See (Shioji and Takahashi, 1997) and (Takahashi, 2000) for more details.

Lemma 2.2. (Shioji and Takahashi, 1997) *Let* $a \in R$ *be a real number and a sequence* $\{a_n\} \in l^\infty$ *satisfy the condition*

$$\mu_n(a_n) \leq a$$

for all Banach limit μ. If $\limsup_n(a_{n+1}-a_n)\le 0$, then $\limsup_n a_n \le a$.

Lemma 2.3. (Atsushiba and Takahashi, to appear in Indian J. Math.) *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_1,T_2,\cdots,T_r\}$ be a finite family of nonexpansive mappings of C into itself such that $F=\bigcap_{i=1}^r F(T_i)\neq\emptyset$ and $\alpha_1,\alpha_2,\cdots,\alpha_r$ be real numbers such that $0<\alpha_i<1$ for any $i=1,2,\cdots,r-1$ and $0<\alpha_r\le 1$. Let W be the $W-$mapping of C into itself generated by $T_1,T_2,\cdots,T_r$ and $\alpha_1,\alpha_2,\cdots,\alpha_r$. Then $F(W)=F$.*

Lemma 2.4. *Let C be a nonempty bounded closed convex subset of a Banach space X, $\alpha_{n1},\alpha_{n2},\cdots,\alpha_{nr}$ be real numbers such that $0\le\alpha_{ni}\le 1$ for any $i=1,2,\cdots,r$ and $\{T_1,T_2,\cdots,T_r\}$ be a finite family of nonexpansive mappings of C into itself. Let W_n be the $W-$mappings of C into itself generated by $T_1,T_2,\cdots,T_r$ and $\alpha_{n1},\alpha_{n2},\cdots,\alpha_{nr}$ and $\{\beta_n\}$ be a sequence of real numbers such that $0\le\beta_n\le 1$ for any $n=1,2,\cdots$,*

$$|\beta_{n+1}-\beta_n|=o(\beta_{n+1}),\quad \sum_{n=1}^{\infty}\beta_n=\infty.$$

If $\sum_{n=1}^{\infty}|\alpha_{(n+1)i}-\alpha_{ni}|<\infty$ for any $i=1,2,\cdots,r$ and the sequence $\{x_n\}$ is given by (B), then $x_{n+1}-x_n\to 0$ as $n\to\infty$.

Proof. Since $F\neq\emptyset$, one can take a fixed point $z\in F$. Set

$$M=\max\{\|u-z\|,\|x_1-z\|\},\quad M_1=M+\|z\|.$$

Then, by induction, one can prove that $\|x_n-z\|\le M$ for all $n\ge 1$ and hence $\|W_nx_n-z\|\le\|x_n-z\|\le M$ for all $n\ge 1$, which leads to $\|x_n\|,\|W_nx_n\|\le M_1$ for all $n\ge 1$. Observe that

$$\begin{aligned}
&\|U_{nr-1}x_{n-1}-U_{n-1r-1}x_{n-1}\|\\
&=\|\alpha_{nr-1}T_{r-1}U_{nr-2}x_{n-1}+(1-\alpha_{nr-1})x_{n-1}\\
&-\{\alpha_{n-1r-1}T_{r-1}U_{n-1r-2}x_{n-1}+(1-\alpha_{n-1r-1})x_{n-1}\}\|\\
&\le 2|\alpha_{nr-1}-\alpha_{n-1r-1}|M_1+\alpha_{nr-1}\|U_{nr-2}x_{n-1}-U_{n-1r-2}x_{n-1}\|\\
&\le 2M_1\sum_{i=1}^{r-1}|\alpha_{ni}-\alpha_{n-1i}|,
\end{aligned}$$

$$\begin{aligned}
&\|W_nx_{n-1}-W_{n-1}x_{n-1}\|\\
&\le 2M_1|\alpha_{nr}-\alpha_{n-1r}|+\|U_{nr-1}x_{n-1}-U_{n-1r-1}x_{n-1}\|
\end{aligned}$$

and

$$\begin{aligned}
&\|x_{n+1}-x_n\|\\
&=|(\beta_n-\beta_{n-1})u+(1-\beta_n)(W_nx_n-W_nx_{n-1})\\
&+(\beta_{n-1}-\beta_n)W_nx_{n-1}+(1-\beta_{n-1}(W_nx_{n-1}-W_{n-1}x_{n-1})|\\
&\le(1-\beta_n)\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|(\|u\|+M_1)\\
&\quad+\|W_nx_{n-1}-W_{n-1}x_{n-1}\|.
\end{aligned}$$

It follows from the above estimations that

$$\begin{aligned}
&\|x_{n+1}-x_n\|\\
&\le(1-\beta_n)\|x_n-x_{n-1}\|+2M_1\sum_{i=1}^{r}|\alpha_{ni}-\alpha_{n-1i}|\\
&\quad+(\|u\|+M_1)|\beta_n-\beta_{n-1}|.
\end{aligned}$$

Setting $a_n = \|x_n - x_{n-1}\|$, $t_n = \beta_n$ and $b_n = 2M_1 \sum_{i=1}^{r} |\alpha_{ni} - \alpha_{n-1i}|$, then we obtain

$$a_{n+1} \leq (1 - t_n)a_n + b_n + o(t_n),$$

which implies that $a_n \to 0$ as $n \to \infty$ by Lemma 2.1. This completes the proof.

Lemma 2.5. (Reich, 1980) *Let X be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C be a nonempty closed convex subset of X and T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Then there is a unique sunny nonexpansive retraction P from C onto $F(T)$. Furthermore, if $u \in C$ and the sequence $\{z_t\} \subset C$ is given by*

$$z_t = tu + (1 - t)Tz_t, \quad t \in (0, 1),$$

then $\{z_t\}$ converges strongly to $Pu \in F(T)$ as $t \to 0$.

Lemma 2.6. *Let X be uniformly convex Banach space with a uniformly Gâteaux differentiable norm and C be a nonempty closed convex subset of X. Let $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nr}$ $(n = 1, 2, \cdots)$ be real numbers such that $0 < \alpha_{ni} < 1$ for any $i = 1, 2, \cdots, r - 1$ and $0 < \alpha_{nr} \leq 1$ and let $\{T_1, T_2, \cdots, T_r\}$ be a finite family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let W_n $(n = 1, 2, \cdots)$ be the $W-$mappings of C into itself generated by $T_1, T_2, \cdots, T_r$ and $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nr}$ and $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for any $n = 1, 2, \cdots$,*

$$\beta_n \to 0 (n \to \infty), \quad |\beta_{n+1} - \beta_n| = o(\beta_{n+1}), \quad \sum_{n=1}^{\infty} \beta_n = \infty.$$

If $\sum_{n=1}^{\infty} |\alpha_{(n+1)i} - \alpha_n i| < \infty$ for any $i = 1, 2, \cdots, r$ and the sequence $\{x_n\}$ is given by (B), then we have

$$\limsup_{n \to \infty} < u - Pu, J(x_n - Pu) > \leq 0,$$

where P is a unique sunny nonexpansive retraction from C onto F.

Proof. Claim 1: There is a unique sunny nonexpansive retraction P from C onto F. Further, for $t \in (0, 1)$, there is a unique $z_t^n \in C$ such that

$$z_t^n = tu + (1 - t)W_n z_t^n, \quad n \geq 1,$$

and $z_t^n \to Pu$ as $t \to 0$ for all $n \geq 1$. Indeed, by Lemma 2.5, we see that, for any fixed $n \geq 1$, $F(W_n) = F \neq \emptyset$ and so there is a unique sunny nonexpansive retraction P_n from C onto $F(W_n) = F$ by Lemma 2.3.

From Remark 1.2 (2), we know that the existence of sunny nonexpansive retraction of C onto F is unique and so $P_n = P$ for all $n \geq 1$. By Banach's contraction principle, we see that, for any $t \in (0, 1)$, there is a unique $z_t^n \in C$ such that

$$z_t^n = tu + (1 - t)W_n z_t^n, \quad n \geq 1.$$

Set $Pu = \lim_{t \to 0} z_t^n$. Then, in view of Lemma 2.5, we conclude that $z_t^n \to Pu \in F$ as $t \to 0$ for all $n \geq 1$.

Claim 2: $\mu_n < u - Pu, J(x_n - Pu) > \leq 0$.

From Claim 1, we can write

$$z_t^n - x_{n+1} = t(u - x_{n+1}) + (1-t)(W_n z_t^n - x_{n+1}).$$

It follows that

$$\begin{aligned}
&\|x_{n+1} - z_t^n\| \\
&\leq (1-t)^2\|W_n z_t^n - x_{n+1}\|^2 + 2t < u - x_{n+1}, J(z_t^n - x_{n+1}) > \\
&= (1-t)^2\|W_n z_t^n - x_{n+1}\|^2 + 2t < u - z_t^n, J(z_t^n - x_{n+1}) > \\
&\quad + 2t\|z_t^n - x_{n+1}\|^2 \\
&\leq (1-t)^2[\|\|x_n - z_t^n\|^2 + 2\|x_n - z_t^n\|\|x_{n+1} - W_n x_n\| \\
&\quad + \|x_{n+1} - W_n x_n\|^2] + 2t < u - z_t^n, J(z_t^n - x_{n+1}) > \\
&\quad + 2t\|x_{n+1} - z_t^n\|^2,
\end{aligned}$$

which implies that

$$\mu_n < u - z_t^n, J(x_{n+1} - z_t^n) > \leq \frac{t}{2}\mu_n\|x_n - z_t^n\| \to 0 \quad (t \to 0),$$

where we used the facts that $\|z_t^{n+1} - z_t^n\| \to 0$ as $n \to \infty$, $\|x_{n+1} - W_n x_n\| \to 0$ as $n \to \infty$ and $\mu_n\|x_{n+1} - z_t^n\| = \mu_n\|x_n - z_t^n\|$. Since X is smooth, we see that $J : x \to X^*$ is demicontinuous and so

$$< u - Pu, J(x_n - z_t^n) > \to < u - Pu, J(x_n - Pu) > \quad (t \to 0).$$

Observe that

$$\begin{aligned}
&\mu_n < u - z_t^n, J(x_n - z_t^n) > \\
&= \mu_n < u - Pu, J(x_n - z_t^n) > + \mu_n < Pu - z_t^n, J(x_n - z_t^n) > .
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
&\mu_n < u - Pu, J(x_n - Pu) > \\
&= \lim_{t \to 0} \mu_n < u - Pu, J(x_n - z_t^n) > \\
&= \limsup_{t \to 0} < u - z_t^n, J(x_n - z_t^n) > \\
&\leq 0.
\end{aligned}$$

Claim 3: $\limsup_{n \to \infty} < u - Pu, J(x_n - Pu) > \leq 0$.
By Setting $a_n = < u - Pu, J(x_n - Pu) >$; then we have

$$a_{n+1} - a_n = < u - Pu, J(x_{n+1} - Pu) - J(x_n - Pu) > \to 0 \quad (n \to \infty)$$

since J is $\|\cdot\|$ to weak star uniformly continuous on any bounded subsets of X and $x_{n+1} - x_n \to 0$ as $n \to \infty$. By Lemma 2.2, we know that $\limsup_{n \to \infty} a_n \leq 0$. This completes the proof.

3 The Main Results

Now, we give our main results in this paper.

Corollary 3.3. *Let X be uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C be a nonempty closed convex subset of X and T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for any $n = 1, 2, \cdots$,*

$$\beta_n \to 0 \ (n \to \infty), \ |\beta_{n+1} - \beta_n| = o(\beta_{n+1}), \ \sum_{n=1}^{\infty} \beta_n = \infty.$$

If the sequence $\{x_n\}$ is given by

$$u, \ x_1 \in C, x_{n+1} = \beta_n u + (1 - \beta_n) T x_n, \quad n \geq 1,$$

then the sequence $\{x_n\}$ converges strongly to $Pu \in F(T)$, where P is a unique sunny nonexpansive retraction from C onto $F(T)$.

Theorem 3.4. Let X be uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C be a nonempty closed convex subset of X and $C_1, C_2, \cdots, C_r$ be nonexpansive retracts of C such that $\bigcap_{i=1}^{r} C_i \neq \emptyset$. Let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for any $i = 1, 2, \cdots, r-1$ and $0 < \alpha_r \leq 1$ and let $\{P_1, P_2, \cdots, P_r\}$ be a finite family of nonexpansive retractions from C onto C_i. Let W be the $W-$mappings of C into itself generated by $P_1, P_2, \cdots, P_r$ and $\alpha_1, \alpha_2, \cdots, \alpha_r$ and let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for any $n = 1, 2, \cdots$,

$$\beta_n \to 0 \ (n \to \infty), \ |\beta_{n+1} - \beta_n| = o(\beta_{n+1}), \ \sum_{n=1}^{\infty} \beta_n = \infty.$$

If the sequence $\{x_n\}$ is given by

$$u, \quad x_1 \in C, x_{n+1} = \beta_n u + (1 - \beta_n) W x_n, \quad n \geq 1,$$

then the sequence $\{x_n\}$ converges strongly to $Pu \in F(W) = \bigcap_{i=1}^{r} F(P_i) = \bigcap_{i=1}^{r} C_i$, where P is a unique sunny nonexpansive retraction from C onto $F(W) = \bigcap_{i=1}^{r} F(P_i) = \bigcap_{i=1}^{r} C_i$.

Theorem 3.5. *Let X be uniformly convex Banach space with a uniformly Gâteaux differentiable norm and C be a nonempty closed convex subset of X. Let $\{S_1, S_2, \cdots, S_r\}$ be a finite commutive family of nonexpansive mappings of C into itself such that $F(S_i) \neq \emptyset$ for any $i = 1, 2, \cdots, r$. Let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for any $i = 1, 2, \cdots, r-1$ and $0 < \alpha_r \leq 1$ and let $\{P_1, P_2, \cdots, P_r\}$ be a finite family of sunny nonexpansive retractions from C onto $F(S_i)$. Let W be the $W-$mapping of C into itself generated by $P_1, P_2, \cdots, P_r$ and $\alpha_1, \alpha_2, \cdots, \alpha_r$ and let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for any $n = 1, 2, \cdots$,*

$$\beta_n \to 0 \ (n \to \infty), \ |\beta_{n+1} - \beta_n| = o(\beta_{n+1}), \ \sum_{n=1}^{\infty} \beta_n = \infty.$$

If the sequence $\{x_n\}$ is given by

$$u, x_1 \in C, x_{n+1} = \beta_n u + (1 - \beta_n) W x_n, \quad n \geq 1,$$

then the sequence $\{x_n\}$ converges strongly to $Pu \in F(W) = \bigcap_{i=1}^{r} F(P_i) = \bigcap_{i=1}^{r} F(S_i)$, where P is a unique sunny nonexpansive retraction from C onto $F(W) = \bigcap_{i=1}^{r} F(P_i) = \bigcap_{i=1}^{r} F(S_i)$.

Theorem 3.1. *Let X be uniformly convex Banach space with a uniformly Gâteaux differentiable norm and C be a nonempty closed convex subset of X. Let $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nr}$ $(n = 1, 2, \cdots)$ be real numbers such that $0 < \alpha_{ni} < 1$ for any $i = 1, 2, \cdots, r-1$ and $0 < \alpha_{nr} \leq 1$ and let $\{T_1, T_2, \cdots, T_r\}$ be a finite family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let W_n $(n = 1, 2, \cdots)$ be the $W-$mappings of C into itself generated by $T_1, T_2, \cdots, T_r$ and $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nr}$ and $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for any $n = 1, 2, \cdots$,*

$$\beta_n \to 0 (n \to \infty), \quad |\beta_{n+1} - \beta_n| = o(\beta_{n+1}), \quad \sum_{n=1}^{\infty} \beta_n = \infty.$$

If $\sum_{n=1}^{\infty} |\alpha_{(n+1)i} - \alpha_n i| < \infty$ for any $i = 1, 2, \cdots, r$ and the sequence $\{x_n\}$ is given by (B), then the sequence $\{x_n\}$ converges strongly to $Pu \in F$, where P is a unique sunny nonexpansive retraction from C onto F.

Proof. By Lemma 2.6, we have

$$a_n = \limsup_{n \to \infty} < u - Pu, J(x_n - Pu) > \leq 0.$$

Define $\gamma_n = \max\{a_n, 0\}$. Then $\gamma_n \to 0$ as $n \to \infty$. It follows that

$$\begin{aligned} &\|x_{n+1} - Pu\|^2 \\ &\leq (1 - \beta_n)^2 \|x_n - Pu\|^2 + 2\beta_n < u - Pu, J(x_{n+1} - Pu) > \\ &\leq (1 - \beta_n) \|x_n - Pu\|^2 + 2\beta_n \gamma_n \\ &= (1 - \beta_n) \|x_n - Pu\|^2 + o(\beta_n) \end{aligned}$$

and so, by Weng's lemma (Lemma 2.1), we conclude that $x_n \to Pu$ as $n \to \infty$. This completes the proof.

As a direct consequence of Theorem 3.1, we have the following:

Theorem 3.2. *Let X be uniformly convex Banach space with a uniformly Gâteaux differentiable norm and C be a nonempty closed convex subset of X. Let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for any $i = 1, 2, \cdots, r-1$ and $0 < \alpha_r \leq 1$ and let $\{T_1, T_2, \cdots, T_r\}$ be a finite family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let W be the $W-$mappings of C into itself generated by $T_1, T_2, \cdots, T_r$ and $\alpha_1, \alpha_2, \cdots, \alpha_r$ and $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for any $n = 1, 2, \cdots$,*

$$\beta_n \to 0 \ (n \to \infty), \ |\beta_{n+1} - \beta_n| = o(\beta_{n+1}), \ \sum_{n=1}^{\infty} \beta_n = \infty.$$

If the sequence $\{x_n\}$ is given by

$$u, \ \ x_1 \in C, x_{n+1} = \beta_n u + (1 - \beta_n) W x_n, \quad n \geq 1,$$

then the sequence $\{x_n\}$ converges strongly to $Pu \in F(W) = \bigcap_{i=1}^{r} F(T_i)$, where P is a unique sunny nonexpansive retraction from C onto $F(W) = \bigcap_{i=1}^{r} F(T_i)$.

By virtue of Theorem 3.2, we also have the following:

Proof. In view of Theorem R, it follows that there is a sunny nonexpansive retraction P_i from C onto $F(S_i)$ for any $i = 1, 2, \cdots, r$. We also have that, for any $i = 1, 2, \cdots, r$, $F(S_i)$ is closed and convex since X is strictly convex. It follows from the commutative property of S_1 and S_2 that $F(S_1)$ is invariant under S_2. Hence $F(S_1) \bigcap F(S_2) \neq \emptyset$. Similarly, we can show that $F = \bigcap_{i=1}^{r} F(S_i) \neq \emptyset$. It is obvious that $F(S_i) = F(P_i)$ for any $i = 1, 2, \cdots, r$ and hence $\bigcap_{i=1}^{r} F(S_i) = \bigcap_{i=1}^{r} F(P_i)$. By Lemma 2.3, we have $F(W) = \bigcap_{i=1}^{r} F(P_i)$. Therefore, from Theorem 3.2, we conclude that the sequence $\{x_n\}$ converges strongly to $Pu \in F(W) = \bigcap_{i=1}^{r} F(P_i) = \bigcap_{i=1}^{r} F(S_i)$, where P is a unique sunny nonexpansive retraction from C onto $F(W)$. This completes the proof.

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