ISSN 0973-1377 (Print)
ISSN 0973-7545 (Online)

International Journal of Applied Mathematics & Statistics

Special Issue on **Leonhard Paul Euler**'s:
Mathematical Topics and Applications (M. T. A.)

Centre for Environment, Social & Economic Research

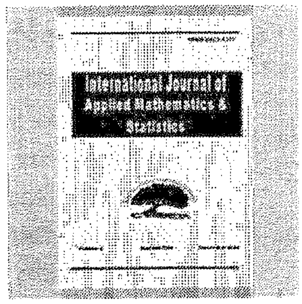
CESER

Volume 11 Number N07 November 2007

International Journal of Applied Mathematics & Statistics

ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Editorial Board



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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

PREFACE

This Euler's commemorating volume entitled :

Functional Equations , Integral Equations, Differential Equations and Applications **(F. I. D. A),**

is a forum for exchanging ideas among eminent mathematicians and physicists, from many parts of the world, as a tribute to the tri-centennial birthday anniversary of Leonhard Paul Euler (April 15,1707 A.D., b. in Basel – September 18, 1783 A.D., d. in St. Petersburg).

This 998 pages long collection is composed of outstanding contributions in mathematical and physical equations and inequalities and other fields of mathematical, physical and life sciences.

In addition, this anniversary volume is unique in its target, as it strives to represent a broad and highly selected participation from across and beyond the scientific and technological country regions. It is intended to boost the cooperation among mathematicians and physicists working on a broad variety of pure and applied mathematical areas.

Moreover, this new volume will provide readers and especially researchers with a detailed overview of many significant insights through advanced developments on Euler's mathematics and physics. This transatlantic collection of mathematical ideas and methods comprises a wide area of applications in which equations, inequalities and computational techniques pertinent to their solutions play a core role.

Euler's influence has been tremendous on our everyday life, because new tools have been developed, and revolutionary research results have been achieved , bringing scientists of exact sciences even closer, by fostering the emergence of new approaches, techniques and perspectives.

The central scope of this commemorating 300birthday anniversary volume is broad, by deeper looking at the impact and the ultimate role of mathematical and physical challenges, both inside and outside research institutes, scientific foundations and organizations.

We have recently observed a more rapid development in the areas of research of Euler worldwide. Leonhard P. Euler (1707-1783) was actually the most influential mathematician and prolific writer of the eighteenth century, by having contributed to almost all the fundamental fields of mathematics and mathematical physics. In calculus of variations, according to C. Caratheodory, Euler's work: *Methodus inveniendi lineas curves…*(1740 A.D.) was one of the most beautiful works ever written. Euler was dubbed *Analysis Incarnate* by his peers for his incredible ability. He was especially great from his writings and that produced more academic work on mathematics than anyone. He could produce an entire new mathematical paper in about thirty minutes and had huge piles of his works lying on his desk. It was not uncommon to find Analysis Incarnate ruminating over a new subject with a child on his lap.

This volume is suitable for graduate students and researchers interested in functional equations, integral equations and differential equations and would make an ideal supplementary reading or independent study research text.

These issues will also be of interest to those working in other areas of mathematics and physics. It is a work of great interest and enjoyable read as well as unique in market.

This **Euler's volume (F. I. D. A.)** consists of six (6) issues containing various parts of contemporary pure and applied mathematics with emphasis to Euler's mathematics and physics.

It contains sixty eight (68) fundamental research papers of one hundred one (101) outstanding research contributors from twenty seven (27) different countries. In particular, these contributors come from:

Algerie (1 contributor); Belgique (2); Bosnia and Herzegovina (2); Brazil (2); Bulgaria (3); China (9); Egypt (1); France (3); Greece (2); India (8); Iran (3); Italy (1); Japan (7); Korea (7); Morocco (3); Oman (2); Poland (3); R. O. Belarus (8); Romania (2); Russia (3); Saudi Arabia (1); Serbia and Montenegro (5); The Netherlands (3); U. A. Emirates (1);U. K. (2); U. S. A. (15); Uzbekistan (2).

First Issue (F. E. I.) contains various parts of ***Functional Equations and Inequalities,*** namely: Euler's Life and Work, Ulam stability, Hyers – Ulam stability and Ulam – Gavruta – Rassias stability of functional equations, Euler – Lagrange type and Euler – Lagrange – Rassias quadratic mappings in Banach and Hilbert spaces, Aleksandrov and isometry Ulam stability problems, stability of Pexider and Drygas functional equations, alternative of fixed point, and Hyers - Ulam stability of differential equations.

Second Issue (MT. PDE) contains various parts of ***Mixed Type Partial Differential Equations,*** namely: Tricomi - Protter problem of nD mixed type partial differential equations, solutions of generalized Rassias' equation, degenerated elliptic equations, mixed type oblique derivative problem, Cauchy problem for Euler – Poisson - Darboux equation, non - local boundary value problems, non-uniqueness of transonic flow past a flattened airfoil, multiplier methods for mixed type equations.

Third Issue (F . D . E.) contains various parts of ***Functional and Differential Equations,*** namely: Iterative method for singular Sturm - Liouville problems, Euler type boundary value problems in quantum mechanics, positive solutions of boundary value problems, controllability of impulsive functional semi-linear differential inclusions in Frechet spaces, asymptotic properties of solutions of the Emden-Fowler equation, comparison theorems for perturbed half-linear Euler differential equations, almost sure asymptotic estimations for solutions of stochastic differential delay equations, , difference equations inspired by Euler's discretization method, extended oligopoly models.

Fourth Issue (D. E. I.) contains various parts of ***Differential Equations and Inequalities,*** namely:

New spaces with wavelets and multi-fractal analysis, mathematical modeling of flow control and wind forces, free convection in conducting fluids, distributions in spaces, strong stability of operator – ifferential equations, slope – bounding procedure, sinc methods and PDE, Fourier type analysis and quantum mechanics.

Fifth Issue (M. T. A.) contains various parts of ***Mathematical Topics and Applications,*** namely:Maximal subgroups and theta pairs in a group, Euler constants on algebraic number fields, characterization of modulated Cox measures on topological spaces, hyper-surfaces with flat r-mean curvature and Ribaucour transformations, Leonhard Euler's methods and ideas live on in the thermodynamic hierarchical theory of biological evolution, zeroes of L-series in characteristic p, Beck's graphs, best co-positive approximation function, Convexity in the theory of the Gamma function, analytical and differential – algebraic properties of Gamma function, Ramanujan's summation formula and related identities, ill – posed problems, zeros of the q-analogues of Euler polynomials, Eulerian and other integral representations for some families of hyper-geometric polynomials, group C*-algebras and their stable rank, complementaries of Greek means to Gini means, class of three- parameter weighted means, research for Bernoulli's inequality.

Sixth Issue (DS. IDE.), contains various parts of ***Dynamical Systems and Integro - Differential Equations,*** namely: Semi-global analysis of dynamical systems, nonlinear functional-differential and integral equations, optimal control of dynamical systems, analytical and numerical solutions of singular integral equations, chaos control of classes of complex dynamical systems, second order integro-differential equation, integro-differential equations with variational derivatives generated by random partial integral equations, inequalities for positive operators, strong convergence for a family of non-expansive mappings.

Deep gratitude is due to all those Guest Editors and Contributors who helped me to carry out this intricate project. My warm thanks to my family:

Matina- Mathematics Ph. D. candidate of the Strathclyde University (Glasgow, United Kingdom), Katia- Senior student of Archaeology and History of Art of the National and Capodistrian University of Athens (Greece), and Vassiliki- M. B. A. of the University of La Verne, Marketing Manager in a FMCG company (Greece).

Finally I express my special appreciation to: The *Executive Editor* of the *International Journal of Applied Mathematics and Statistics* (IJAMAS.) *Dr. Tanuja Srivastava* for her nice cooperation and great patience.

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

On the Number of Maximal Subgroups and Theta Pairs in a Finite Group

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ABSTRACT

Let M be a maximal subgroup of a finite group G. The pair (A, B) of subgroups of G is called a θ-pair of M if the following conditions hold: (a) $B \lhd G, B < A$, (b) $< M, A >= G$ and $B \leq M$ and (c) $\frac{A}{B}$ has no proper normal subgroup of $\frac{G}{B}$. In this paper, we investigate the structure of a finite group G according to the number of its maximal subgroups and $\theta-$pairs.

Keywords: finite group, theta pair, maximal subgroup.

2000 Mathematics Subject Classification: 20D10, 20E28, 20F16.

1 Introduction and Preliminaries

In this paper all groups considered are assumed to be finite groups. For convenience we denote $M < \cdot G$ to indicate that M is a maximal subgroup of a group G. Also, M_G denotes the core of M in G and $\Phi(G)$ is the Frattini subgroup of the group G.

In [7], Mukherjee and Bhattacharya introduced the concept of θ-pairs associated to maximal subgroups of a group, and used this concept to investigate the structure of some groups. Then Beidlemen and Smith [3], generalized the concept to the universe of infinite groups. The investigation on θ-pairs are continued in [2,8] and [11-15]. Let us recall the definition of θ-pair which is introduced by Mukherjee and Bhattacharya.

Definition 1.1. Given a maximal subgroup M of a group G, a θ-pair of M is any pair (A, B) of subgroups satisfying the following conditions:
(a) $B \lhd G, B < A$.
(b) $< M, A >= G$ and $B \leq M$.
(c) $\frac{A}{B}$ has no proper normal subgroup of $\frac{G}{B}$.

In addition, if $A \trianglelefteq G$, then (A, B) is called a normal θ-pair. A θ-pair (A, B) is said to be maximal if there is no θ-pair (C, D) such that $A < C$. The nonempty set of all θ-pairs of M in G is denoted by $\theta(M)$ and $\theta(G) = \bigcup_{M <\cdot G} \theta(M)$.

The aim of this paper is to investigate the problem of existence of finite groups with a given number of theta pairs. From the definition of $\theta(G)$, one can see that the number of $\theta-$pairs in a finite group is related to the number of maximal subgroups of the group under consideration. So it is natural to investigate the same problem for maximal subgroups.

It is well known that if a finite group G has exactly one maximal subgroup, then $|G|$ is divisible by exactly one prime number and G is cyclic. In this connection one might ask about the structure of G, if G has exactly two or three maximal subgroups. A group G has exactly two maximal subgroups then $|G|$ is indeed divisible by two primes and G is cyclic, and if G has exactly three maximal subgroups then neither G needs to be cyclic nor it is required for $|G|$ to be divisible by three primes. In fact, in this case G is a 2-group or a cyclic group with exactly three prime factors, see for details [6].

Definition 1.2. Let n be a natural number. By Ł(n) we denote the set of all non-isomorphic finite groups with exactly n maximal subgroups. We define a binary relation $\preceq$ on Ł(n) as follows:

$$H \preceq G \Longleftrightarrow \exists\, N \trianglelefteq G \;\; s.t. \;\; \tfrac{G}{N} \cong H$$

Lemma 1.1. *The relation $\preceq$ is a partial order on* Ł(n).

Proof. It is obvious that for any $G \in$ Ł(n), $G \preceq G$. Assume that $H \preceq G$ and $G \preceq H$, then there are $N \trianglelefteq G$ and $M \trianglelefteq H$ such that $\frac{G}{N} \cong H$ and $\frac{H}{M} \cong G$. Hence, $|G| \geq |\frac{G}{N}| \geq |\frac{H}{M}| = |G|$ and so $|N| = 1$, i.e. $G \cong H$. Now by definition of Ł(n) we have $H = G$. Finially, suppose that $H \preceq G$ and $G \preceq K$, so there are $N \trianglelefteq G$ and $M \trianglelefteq K$ such that $\frac{G}{N} \cong H$ and $\frac{K}{M} \cong G$. Since $G \cong \frac{K}{M}$, there exists an isomorphism $\eta : G \longrightarrow \frac{K}{M}$. Set $T = \gamma^{-1}(\eta(N)$, where γ is the canonical epimorphism. Now it is easy to see that $T \trianglelefteq K$ and $\frac{K}{T} \cong H$. □

Lemma 1.2. *Suppose $G \in$ Ł(n), then $\frac{G}{\phi(G)}$ is a minimal element of the poset* Ł(n).

Proof. Let $G \in$ Ł(n) and $H \preceq \frac{G}{\phi(G)}$, then there exists $\frac{N}{\phi(G)} \trianglelefteq \frac{G}{\phi(G)}$ such that $H \cong \frac{G}{N}$. Now we show that $N = \phi(G)$, obviously, $\phi(G) \subseteq N$ and since H and G has exactly n maximal subgroups, hence $N \subseteq \phi(G)$. Therefore, $N = \phi(G)$ and so $H \cong \frac{G}{\phi(G)}$, but any distinct two elements of Ł(n) are non-isomorphic, so $H = \frac{G}{\phi(G)}$. □

By the previous lemma, one can see that the problem of classifying all groups with exactly n maximal subgroups is reduced to determining of minimal elements of the poset Ł(n).

Question 1.3. Under what condition, Ł(n) is a lattice?

In this paper, all notations are standard and taken mainly from [1,4,5,10].

2 The Number of $\theta-$Paris in a Finite Group

In this section we obtain the number of θ-pairs of some finite groups and prove that for any positive integer $n \neq 2, 3$, there exists a finite group G such that $|\theta(G)| = n$. We begin with a simple lemma which is useful in counting the number of $\theta-$pair.

Lemma 2.1. *If $(C, D) \in \theta(M)$, then for all $g \in G$, $(C^g, D) \in \theta(M^g)$.*

Proof. Since, $D \lhd G, D < C$ and $C \not\subseteq M$, we have $D < C^g$ and $C^g \not\subseteq M^g$. Assume that $\frac{C^g}{D}$ properly contains a non-trivial normal subgroup $\frac{T}{D}$ of $\frac{G}{D}$. Then we have,

$$\frac{T}{D} = \frac{T^{g^{-1}}}{D} = (\frac{T}{D})^{g^{-1}D} \subset (\frac{C^g}{D})^{g^{-1}D} = \frac{C}{D}.$$

But, $(C, D) \in \theta(M)$, a contradiction. Therefore, $(C^g, D) \in \theta(M^g)$ and the lemma is proved. □

Corollary 2.2. *Let M be a maximal subgroup of the group G. Then, for all $g \in G$, $|\theta(M)| = |\theta(M^g)|$.*

Proof. By Lemma 2.1, the map $\tau : \theta(M) \longrightarrow \theta(M^g)$ that sends (C, D) to (C^g, D) is well-defined. Now, it is easy to see that the map τ is a one-to-one correspondence. □

Let G be a finite group and M be a maximal normal subgroup of G. Then (G, M) is a $\theta-$pair of M in G. So $\theta(M) \neq \emptyset$. In what follows, we investigate the structure of finite groups with exactly 1 and 2 θ-pair.

Lemma 2.3. *A group G has exactly one θ-pair if and only if G is a cyclic group of prime power order.*

Proof. Suppose G has exactly one θ-pair. Then $\frac{G}{\Phi(G)}$ is a simple group and $\theta(G) = \{(G, \Phi(G))\}$. Suppose $m(G) > 1$. Then $\Phi(G)$ is not maximal in G and for any maximal subgroup M of G, $(M, \Phi(G))$ is a θ-pair of L, in which L is a maximal subgroup of G distinct from M, a contradiction. This shows that $m(G) = 1$ and so G is a cyclic group of prime power order. □

Lemma 2.4. *If there exists a maximal subgroup M of G such that $\theta(M) = \theta(G)$, then G has exactly one θ-pair.*

Proof. It is obvious that G has exactly one θ-maximal and so $\frac{G}{\Phi(G)}$ is a simple group. If $m(G) > 1$ then $(M, \Phi(G)) \in \theta(L)$ and $(L, \Phi(G)) \in \theta(M)$, for two distinct maximal subgroups M and L of G, which is a contradiction. Therefore, $m(G) = 1$ and by Lemma 2.3, G has exactly one θ-pair, proving the lemma. □

Lemma 2.5. *There is no $n\theta$-pair cyclic group of order $p_1^{r_1} \cdot p_2^{r_2} \cdots p_n^{r_n}$, $p_1 < p_2 < \cdots < p_n$, in which $n > 1$.*

Proof. Suppose $\{M_1, M_2, \cdots, M_n\}$ is the set of all maximal subgroups of G. Then $(G, M_i), 1 \leq i \leq n$, are n maximal θ-pairs for G and so G has at least n θ-pair. Assume that M is a maximal subgroup of index p_1, A is a maximal subgroup of M of index p_2 and L is a maximal subgroup of G of index p_2. Then $(M, A) \in \theta(L)$, a contradiction. □

We now are ready to state one of our main results. We have:

Theorem 2.6. *There is no finite group with exactly two θ−pairs.*

Proof. Let G has exactly two θ-pairs. By Lemma 2.4, there is no maximal subgroup M of G such that $\theta(M) = \theta(G)$ and so G has exactly two maximal θ-pairs. Thus, $|\{X_G \mid X < \cdot G\}| = 2$. Suppose that (C, L_G) and (G, M_G) are two distinct maximal θ-pairs of G associated to maximal subgroups L and M, respectively. We claim that G has exactly two maximal subgroups. To do this, we assume that T is a maximal subgroup of G different from M and L. If $C \neq G$ then $\Phi(G) = L_G$ and $(L, \Phi(G)) \in \theta(T)$, which is a contradiction. We now assume that $C = G$, then $\frac{G}{M_G}$ and $\frac{G}{L_G}$ are simple groups. Therefore, $T_G = L_G$ or $T_G = M_G$. Suppose $T_G = L_G$ then $(L, L_G) \in \theta(T)$, a contradiction. Also, if $T_G = M_G$ then $(M, M_G) \in \theta(T)$ and so $M_G = L_G$. This implies that $\frac{G}{\Phi(G)}$ is a simple group, which is a contradiction. Therefore, G has exactly two maximal subgroups and so $|G|$ is indeed divisible by two primes. Now by Lemma 2.5, the proof is complete. □

Lemma 2.7. *Let G be a finite group such that all of maximal θ-pairs of G are normal and $\{M_G \mid M < \cdot G\} = \{L_{1G}, \cdots, L_{rG}\}$. Then $\theta_{max}(G) = \theta_{max}(L_1) \cup \cdots \cup \theta_{max}(L_r)$.*

Proof. Suppose (C, D) is an arbitrary maximal θ-pair of G. Then $D = L_{iG}$, for some $1 \leq i \leq r$. If $C \subseteq L_i$ then $C \subseteq D$, a contradiction. Thus $(C, D) \in \theta(L_i)$. Now we assume that (E, F) is a maximal θ-pair of $\theta(L_i)$ such that $(C, D) \leq (E, F)$. Therefore, $C \leq E$, $D = F$, $\frac{C}{D} \leq \frac{E}{D}$ and $\frac{C}{D} \trianglelefteq \frac{G}{D}$. This shows that (C, D) is a maximal θ-pair of $\theta(L_i)$ and the proof is complete. □

In the following theorem, we prove that there is no also finite groups with exactly three θ−pairs.

Theorem 2.8. *There is no finite group with exactly three θ-pairs.*

Proof. Let G be a 3θ-pair group. By Lemma 3, there is no maximal subgroup M of G such that $\theta(M) = \theta(G)$. Our main proof will consider a number of cases.

Case 1. *There are two maximal subgroups M and L of G such that $|\theta(M)| = 2$ and $|\theta(L)| = 1$.* Assume that $(B, M_G), (C, D) \in \theta(M)$ and $(A, L_G) \in \theta(L)$. We can see that $C \trianglelefteq G$ and $C \neq G$. We claim that G has at least three maximal subgroups. By lemma 2.3, G has at least two maximal subgroups. Assume that G has exactly two maximal subgroups, say M and L. Thus, by the mentioned theorem of khazal, G is cyclic and so $(A, L_G) = (G, L)$, $(B, M_G) = (G, M)$. Since $\frac{G}{L}$ is a simple group, we have $(M, \Phi(G)) \in \theta(L)$, a contradiction. Therefore G has at least three maximal subgroups. We now see that $M_G \neq L_G$. Thus, for any maximal subgroup X of G, $X_G = L_G$ or $X_G \leq M_G$. Suppose $A = G$. If L is non-normal and $g \in G - N_G(L)$, then $(L^g, L_G) \in \theta(L)$, which is impossible. So $L \trianglelefteq G$ and we can see that $(M_G, L \cap M_G) \in \theta(L)$, a contradiction. Thus $A \neq G$ and so $A \leq M_G$. Also, $C \leq L_G$ and hence $C \leq L_G \leq A \leq M_G$, which is a contradiction.

Case 2. *G is 3θ-maximal and there are maximal subgroups M, L and K of G such that $(A, L_G) \in \theta(L)$, $(B, K_G) \in \theta(K)$ and $(C, M_G) \in \theta(M)$.* By Lemma 2.7 and Case 1, $|\{M_G \mid M < \cdot G\}| = 3$. We claim that one of the subgroups A, B and C is equal to G and the other two are proper. To do this, suppose $A = C = G$. Then $M, L \triangleleft G$ and $(L, M \cap L) \in \theta(M)$, which is impossible. Therefore, we can assume that $A \neq G, B \neq G$ and $|\theta(\frac{G}{A})| = |\theta(\frac{G}{B})| = 1$. Suppose $\frac{R}{A}$ and $\frac{S}{B}$ are the unique maximal subgroups of $\frac{G}{A}$ and $\frac{G}{B}$, respectively. Thus, $(\frac{G}{A}, \frac{R}{A}) \in \theta(\frac{G}{A})$

and $(\frac{G}{B}, \frac{S}{B}) \in \theta(\frac{G}{B})$. This shows that (G, R) and (G, S) are θ-pairs of G and so $R = S$. We can assume that $M \lhd G$ and $A, B \leq M$. Now $(\frac{A}{L_G}, \frac{L_G}{L_G}), (\frac{G}{L_G}, \frac{M}{L_G}) \in \theta(\frac{G}{L_G})$ and $|\theta_{max}(\frac{G}{L_G})| \leq 3$. Therefore, $|\theta_{max}(\frac{G}{L_G})| = 3$ and there exists another θ-pair $(\frac{R_1}{L_G}, \frac{U_1}{L_G}) \in \theta(\frac{G}{L_G})$. It is easy to see that $L_G \subseteq K_G$. Using similar argument as in above, $K_G \subseteq L_G$ and so $L_G = K_G$, which is a contradiction. □

Theorem 2.9. *There exists a group with exactly n θ-pair, for $n \neq 2, 3$.*

Proof. For $n = 1$, a cyclic group of prime power order has exactly one θ-pair. Suppose $n \geq 4$ and $G = Z_{p^n q}$. Then G has exactly two maximal subgroups M and N of orders p^n and $p^{n-1}q$, respectively. Suppose A_i and B_i, $0 \leq i \leq n$, are subgroups of G of order p^i and $p^i q$. Now it is easy to see that $\theta(M) = \{(B_i, A_i) \mid 0 \leq i \leq n\}$ and $\theta(N) = \{(A_n, A_{n-1}), (B_n, B_{n-1})\}$. Therefore G has exactly $n + 3$, θ-pair, proving the result. □

We end this paper with the following open question:

Question 2.10. What is the number of $\theta-$pair in a finite abelian group?

Acknowledgment

This work was supported by a grant from the Ministry of Science, Research and Technology of Iran and Tarbiat Modarres University.

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Explicit formulas and Euler constants on algebraic number fields

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Abstract

Building upon our work on explicit formulas for the Jorgenson-Lang fundamental class of functions, we deduce expressions for Euler constants in the case of Dedekind zeta function that are analoguous to the Stieltjes results for the Riemann zeta function.

Keywords: Dedekind zeta function, generalized Euler constants, explicit formulas
2000 MSC Classification: 11M36, 11R42, 30B50

1. INTRODUCTION

In 1737, Euler [7] made an observation that if f is a completely multiplicative complex function defined on the positive integers (i.e. $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$) and if the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent, then

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \frac{1}{1-f(p)},$$

where the product is taken over all prime numbers p.

This discovery, made by the Analyst, served as a starting point for probably the most influential ten pages in the history of number theory. Riemann begins his celebrated memoire [12] with a meromorphic continuation of the function initially defined by

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1-p^{-s}}, \text{ for } \operatorname{Re} s > 1,$$

to the whole complex plane and proves the functional equation

$$\varsigma(s)\pi^{-s+\frac{1}{2}} \frac{\Gamma(s/2)}{\Gamma((1-s)/2)} = \varsigma(1-s).$$

He proceeds by explaining how $\log \varsigma$ can be written as an infinite sum involving the zeros of ς.

The constant γ, introduced by Euler in 1734, three years prior to his product formula, reappears as the constant term γ_0 in the Laurent series expansion of the Riemann zeta function around its pole:

$$\varsigma(s)=\frac{1}{s-1}+\sum_{k=0}^{\infty}\gamma_k(s-1)^k .$$

Stieltjes [6, p. 164] has proved that the so-called higher Euler constants γ_k, $k\geq 1$, can be calculated as:

$$\gamma_k=\frac{(-1)^k}{k}\lim_{x\to\infty}\left(\sum_{n<x}\frac{\log^k n}{n}-\frac{\log^{k+1}x}{k+1}\right).$$

Our goal is to deduce analoguous expressions for generalized Euler constants on algebraic number fields by making use of the explicit formula for the corresponding triple in the Jorgenson-Lang fundamental class of functions.

2. DEDEKIND ZETA FUNCTION

Zeta functions happen to encode many deep properties of a number field. The Dedekind zeta function [11, pp. 457 f.] is a Dirichlet series defined for any algebraic number field through

$$\zeta_{\Bbbk}(s)=\sum\frac{1}{(\mathrm{N}a)^s},$$

where the sum is taken over all ideals a of the ring of integers $\mathfrak{O}_{\Bbbk}$ and $\mathrm{N}a=Card\,\mathfrak{O}_{\Bbbk}/a$ is the norm of a.

The series $\zeta_{\Bbbk}(s)$ converges absolutely and uniformly for $\mathrm{Re}\,s\geq 1+\varepsilon$ $(\varepsilon>0)$ and can be represented as an Euler product $\zeta_{\Bbbk}(s)=\prod_{p}\left(1-\mathrm{N}p^{-s}\right)^{-1}$ over all prime ideals of $\Bbbk$. The function $\zeta_{\Bbbk}$ satisfies the functional equation $\zeta_{\Bbbk}(s)\Phi(s)=\zeta_{\Bbbk}(1-s)$, with the factor

$$\Phi(s)=\left|d_{\Bbbk}\right|^{s-\frac{1}{2}}\pi^{\frac{r_1}{2}}\left(\pi^{-s}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\right)^{r_1}(2\pi)^{r_2}\left((2\pi)^{-s}\frac{\Gamma(s)}{\Gamma(1-s)}\right)^{r_2}.$$

Here, $d_{\Bbbk}$ is the discriminant of $\Bbbk$, r_1 and r_2 denote the number of real resp. complex places.

The function $\zeta_{\Bbbk}$ is a meromorphic function of a finite order and $\log\zeta_{\Bbbk}(s)$ has an Euler series representation $\log\zeta_{\Bbbk}(s)=\sum_{p,n}\frac{1}{n\mathrm{N}p^{ns}}$, converging absolutely for $\mathrm{Re}\,s>1$. Therefore, the logarithmic derivative of the Dedekind zeta function is a holomorphic function in the half-plane $\mathrm{Re}\,s>1$, given by the Dirichlet series $\frac{\zeta_{\Bbbk}'}{\zeta_{\Bbbk}}(s)=-\sum_{p,n}\frac{\log\mathrm{N}p}{\mathrm{N}p^{ns}}$.

Thus, the triple (ζ_k, ζ_k, Φ) belongs to the Jorgenson-Lang fundamental class of functions, as defined in [9, pp. 46-47]. The factor Φ is of a regularized product type of reduced order $(0,0)$. (For the definition of a regularized product type and its order, see [9, pp. 36-37]).

In the sequel, we shall assume validity of the generalized Riemann hypothesis, i.e. that all non-trivial zeros of the function ζ_k lie on the critical line $\mathrm{Re}\, s = \frac{1}{2}$. Therefore, all zeros and poles of the function ζ_k in the strip $0 \le \mathrm{Re}\, s \le 1$ consist of the simple zeros $\rho = \frac{1}{2} \pm it$, the zero $s = 0$ of order $r_1 + r_2 - 1$ and the simple pole $s = 1$.

3. EXPLICIT FORMULA FOR THE DEDEKIND ZETA FUNCTION

Explicit formulas in number theory are identities that relate the sum of the Mellin transform of a certain test function over zeros of a zeta (or L) function with the sum of its values over primes (or equivalents of the primes).

The Riemann – von Mangoldt explicit formula for the Riemann zeta function was the first to appear. Later, A. Weil [13] has pointed out that these formulas can be stated more generally. J. Jorgenson and S. Lang [9] introduced the fundamental class of functions, as a class of meromorphic functions of a finite order that satisfy a functional equation and with the logarithm possessing an Euler sum representation. They proved the explicit formula for triples $(Z, \widetilde{Z}, \Phi)$ in this class. In [1], it was shown that the Jorgenson-Lang explicit formula holds for a larger class of test functions, and in [3] a new explicit formula for a more general class $(Z, \widetilde{Z}, \Phi)$ is obtained.

The special case $Z = \widetilde{Z}$, important in applications, deserves a special attention. In [2] and [4] we proved that the explicit formula for triples (Z, Z, Φ) holds for a very large class of test functions (not necessarily differentiable nor even continuous). A benefit was an extension of the Selberg trace formula for a Fuchsian group of the first kind to a larger class of test functions.

Application of [2, Th. 4.1.] to the triple (ζ_k, ζ_k, Φ) yields the following theorem.

Theorem 3.1. *Let F be a regularized function such that*

1. $\left(F(x)+F(-x)\right)e^{\left(\frac{1}{2}+\varepsilon\right)|x|} \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$ *for some* $\varepsilon > 0$

2. $F(x)+F(-x)-2F(0) = O\left(\left|\log|x|\right|^{-\alpha}\right)$ $(\alpha > 2)$ *as* $x \to 0$.

Then, the formula

$$\lim_{T\to\infty} \sum_{\substack{\rho=1/2+it \\ |t|\le T}} M_{1/2}f(\rho) = -\sum_{p,n} \frac{\log \mathrm{N}p}{\mathrm{N}p^{n/2}} \left(f(\mathrm{N}p^{n}) + f(\mathrm{N}p^{-n}) \right) + + M_{1/2}f(0) + M_{1/2}f(1) + W_{\Phi}(F) \qquad (1)$$

holds true.

Here, ϕBV denotes the class of functions of bounded ϕ-variation, where ϕ is a continuous, convex function on $[0,\infty)$, increasing from 0 to ∞ and such that $\sum_n \left(\frac{1}{n}\right)^{1/p} \phi^{-1}\left(\frac{1}{n}\right) < \infty$ for some $p>1$. Functions f and F are related by $f(x)=F(-\log x)$ and $M_{1/2}f$ denotes the translate by $1/2$ of the Mellin transform of f.

Finally,

$$W_\Phi(F)=\frac{1}{\sqrt{2\pi}}\lim_{T\to\infty}\int_{-T}^{T}\widehat{F}(t)\frac{\Phi'}{\Phi}\left(\frac{1}{2}+it\right)dt$$

is the Weil functional.

We omit the proof of Theorem 3.1. here, since it is a direct consequence of [2, Th. 4.1.].

4. LOGARITHMIC DERIVATIVE OF THE DEDEKIND ZETA FUNCTION

Theorem 3.1. implies an interesting representation of the logarithmic derivative of the Dedekind zeta function that we shall use to calculate generalized Euler constants for $\frac{\zeta_{\mathbb{k}}'}{\zeta_{\mathbb{k}}}$.

Theorem 4.1. *The function* $\frac{\zeta_{\mathbb{k}}'}{\zeta_{\mathbb{k}}}$ *has a meromorphic continuation to the half-plane* $\mathrm{Re}\, s>1/2$, *given by the formula:*

$$\frac{\zeta_{\mathbb{k}}'}{\zeta_{\mathbb{k}}}(s)=\frac{(4s-2)x^{s-1/2}}{1+x^{2s-1}}\left(\sum_{t\ge 0}\frac{\cos ty}{(s-1/2)^2+t^2}-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\cos ty dt}{(s-1/2)^2+t^2}\left(2r_2\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right)+r_1\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right)\right)\right)$$
$$+\frac{1}{1+x^{2s-1}}\sum_{\substack{p,n\\ \mathrm{N}p^n<x}}\frac{\log \mathrm{N}p}{\mathrm{N}p^{ns}}\left(\mathrm{N}p^{n(2s-1)}-x^{2s-1}\right)-\frac{\log|d_{\mathbb{k}}|-\log \pi^{r_1}(2\pi)^{r_2}}{1+x^{2s-1}}-\frac{(2s-1)(x^{s-1}+x^s)}{(1+x^{2s-1})s(s-1)}. \quad (2)$$

The sum on the right-hand side of (2) is taken over all $t\ge 0$ such that $\rho=1/2\pm it$ is a zero of the function.

Proof. The test function

$$F_{y,\alpha}(x)=\begin{cases} e^{-\alpha x}, & \text{if } x\ge -y\\ e^{\alpha(x+2y)}, & \text{if } x<-y\end{cases}$$

where $\mathrm{Re}\,\alpha>1/2$ and $y>0$, satisfies conditions posed in Theorem 3.1. Inserting $F_{y,\alpha}$ and the corresponding function $f_{y,\alpha}$ into (1), we get

$$-\sum_{p,n}\frac{\log \mathrm{N}p}{\mathrm{N}p^{n/2}}\left(f_{y,\alpha}(\mathrm{N}p^n)+f_{y,\alpha}(\mathrm{N}p^{-n})\right)=\left(1+e^{2y\alpha}\right)\frac{\zeta_{\mathbb{k}}'}{\zeta_{\mathbb{k}}}\left(\frac{1}{2}+\alpha\right)-\sum_{\substack{p,n\\ \mathrm{N}p^n<e^y}}\frac{\log \mathrm{N}p}{\mathrm{N}p^{n(\alpha+1/2)}}\left(\mathrm{N}p^{2n\alpha}-e^{2y\alpha}\right),$$

$$W_\Phi(F_{y,\alpha})=\log\frac{|d_{\mathbb{k}}|}{\pi^{r_1}(2\pi)^{r_2}}+\frac{4\alpha e^{y\alpha}}{4\pi}\int_{-\infty}^{\infty}\frac{\cos ty dt}{\alpha^2+t^2}\left(2r_2\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right)+r_1\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right)\right) \text{ and}$$

$$M_{1/2}f_{y,\alpha}(\rho)=\frac{2\alpha e^{y\alpha}}{\alpha^2-(\rho-1/2)^2}e^{y(\rho-1/2)}.$$

Putting $x=e^y$ and $\alpha=s-1/2$, we see that the formula (2) is valid for $\mathrm{Re}\,s>1$.

Now, the series $\sum_{t>0}\frac{1}{t^2}$ is convergent. Indeed, let

$$N_{\Bbbk}(T)=Card\left\{\rho=\frac{1}{2}+it\,\middle|\,\varsigma_{\Bbbk}(\rho)=0,\ 0\le t\le T\right\}.$$

Then, $N_{\Bbbk}(T)=O\left(T^{1+\varepsilon}\right)$ ($\varepsilon>0$), see [10, p. 410, with $\sigma=1/2$]. Hence,

$$\sum_{t>0}\frac{1}{t^2}=\int_{\tau}^{\infty}\frac{dN_{\Bbbk}(t)}{t^2}<\infty\text{, for some }\tau>0.$$

Thus, the series $\sum_{t\geq 0}\frac{\cos ty}{(s-1/2)^2+t^2}$ converges absolutely and uniformly in any closed subset of the half-plane $\mathrm{Re}\,s>1/2$.

Since $\frac{\Gamma'}{\Gamma}(a+it)=O(\log|t|)$, when $t\to\infty$, for all $a\geq\frac{1}{4}$, the integral on the right-hand side of (2) is uniformly convergent in any closed subset of the half-plane $\mathrm{Re}\,s>\frac{1}{2}$.

Therefore, the right-hand side of (2) is a meromorphic function of s for $\mathrm{Re}\,s>1/2$. This completes the proof of the theorem.

5. GENERALIZED EULER CONSTANTS

The Dedekind zeta function can be expanded around $s=1$ as

$$\zeta_{\Bbbk}(s)=\frac{\gamma_{-1}(\Bbbk)}{s-1}+\sum_{l=0}^{\infty}\gamma_l(\Bbbk)(s-1)^l,$$

where $\gamma_0(\Bbbk)$ is called the Euler constant and $\gamma_l(\Bbbk)$ are higher Euler constants for a number field $\Bbbk$. The residue $\gamma_{-1}(\Bbbk)$ has been calculated by Dedekind and is known as a class field formula [11, p. 467].

We shall consider the Laurent expansion of $\frac{\zeta'_{\Bbbk}}{\zeta_{\Bbbk}}$ around $s=1$:

$$\frac{\zeta'_{\Bbbk}}{\zeta_{\Bbbk}}=\frac{\gamma^{(-1)}(\Bbbk)}{s-1}+\sum_{l=0}^{\infty}\gamma^{(l)}(\Bbbk)(s-1)^l,$$

and call the constants $\gamma^{(l)}(\Bbbk)$ generalized Euler constants.

It is known that $\gamma^{(-1)}(\Bbbk)=-1$ and $\gamma^{(0)}(\Bbbk)=\frac{\gamma_0(\Bbbk)}{\gamma_{-1}(\Bbbk)}$. The relations between higher constants $\gamma^{(l)}(\Bbbk)$ and $\gamma_l(\Bbbk)$ can also be deduced.

The following theorem gives us formulas for evaluation of constants $\gamma^{(l)}(\Bbbk)$ $(l \geq 0)$ that are analoguous to the Stieltjes formulas above.

Theorem 5.1. *For any* $l \geq 0$ *we have*

$$\gamma^{(l)}(\Bbbk) = \frac{(-1)^l}{l!} \lim_{x\to\infty}\left(\frac{\log^{l+1} x}{l+1} - \sum_{\mathrm{N}a<x} \frac{\Lambda a}{\mathrm{N}a} \log^l \mathrm{N}a \right), \tag{3}$$

where $\Lambda a = \begin{cases} \log \mathrm{N}\mathfrak{p}, \text{ if } a = \mathfrak{p}^n \\ 0, \quad\quad \text{otherwise} \end{cases}$.

Proof. We shall prove that the function $\frac{\zeta_\Bbbk'}{\zeta_\Bbbk}$ can be expressed in a vicinity of $s=1$ as

$$\frac{\zeta_\Bbbk'}{\zeta_\Bbbk} = \frac{f_{-1}(x,s)}{s-1} + \sum_{l=0}^{\infty} f_l(x,s)(s-1)^l,$$

where functions $f_l(x,s)$ $(l \geq -1)$ are such that, for a fixed $0 < \delta < \frac{1}{4}$ (depending on a number field $\Bbbk$ only) we have $\lim_{x\to\infty} f_l(x,s) = f_l < \infty$ in the disc $|s-1| < \delta$. Then, it will easily follow (see [5]) that $\gamma^{(l)}(\Bbbk) = f_l$ $(l \geq -1)$. This will prove the theorem once we show that f_l, $l \geq 0$, is equal to the right-hand side of (3).

According to Theorem 4.1., we have

$$\frac{\zeta_\Bbbk'}{\zeta_\Bbbk}(s) = \frac{(4s-2)x^{s-1/2}}{1+x^{2s-1}}\left(\sum_{t\geq 0} \frac{\cos ty}{(s-1/2)^2+t^2} - \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\cos tydt}{(s-1/2)^2+t^2}\left(2r_2 \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right) + r_1 \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right)\right)\right)$$
$$-\frac{\log|d_\Bbbk| - \log \pi^{r_1}(2\pi)^{r_2}}{1+x^{2s-1}} - \frac{1}{s}\frac{x^{s-1}}{(1+x^{2s-1})} - \frac{1}{s-1}\frac{x^{s-1}}{(1+x^{2s-1})} + \frac{1}{1+x^{2s-1}}\sum_{\mathrm{N}\mathfrak{a}<x} \Lambda\mathfrak{a}\mathrm{N}\mathfrak{a}^{s-1}$$
$$-\frac{x^{2s-1}}{1+x^{2s-1}}\left[\frac{x^{1-s}}{s} + \frac{1}{s-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\log^n x(s-1)^n + \sum_{\mathrm{N}\mathfrak{a}<x}\frac{\Lambda\mathfrak{a}}{\mathrm{N}\mathfrak{a}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\log^n \mathrm{N}\mathfrak{a}(s-1)^n \right].$$

Now, $f_{-1}(x,s) = -\frac{x^{s-1}}{(1+x^{2s-1})} - \frac{x^{2s-1}}{(1+x^{2s-1})}$,

$$f_0(x,s) = \frac{(4s-2)x^{s-1/2}}{1+x^{2s-1}}\left(\sum_{t\geq 0}\frac{\cos ty}{(s-1/2)^2+t^2} - \int_{-\infty}^{\infty}\frac{\cos tydt}{(s-1/2)^2+t^2}\left(\frac{r_2}{2\pi}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right) + \frac{r_1}{4\pi}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right)\right)\right)$$
$$-\frac{\log|d_\Bbbk| - \log\pi^{r_1}(2\pi)^{r_2}}{1+x^{2s-1}} - \frac{1}{s}\frac{x^{s-1}}{1+x^{2s-1}} + \frac{1}{1+x^{2s-1}}\left(\sum_{\mathrm{N}\mathfrak{a}<x}\Lambda\mathfrak{a}\mathrm{N}\mathfrak{a}^{s-1} - \frac{x^s}{s}\right) + \frac{x^{2s-1}}{1+x^{2s-1}}\left(\log x - \sum_{\mathrm{N}\mathfrak{a}<x}\frac{\Lambda\mathfrak{a}}{\mathrm{N}\mathfrak{a}}\right)$$
$$= A(x,s) + \frac{1}{1+x^{2s-1}}\left(\sum_{\mathrm{N}\mathfrak{a}<x}\Lambda\mathfrak{a}\mathrm{N}\mathfrak{a}^{s-1} - \frac{x^s}{s}\right) + \frac{x^{2s-1}}{1+x^{2s-1}}\left(\log x - \sum_{\mathrm{N}\mathfrak{a}<x}\frac{\Lambda\mathfrak{a}}{\mathrm{N}\mathfrak{a}}\right)$$

and, for $l \geq 1$

$$f_l(x,s) = \frac{(-1)^l}{l!}\frac{x^{2s-1}}{1+x^{2s-1}}\left(\frac{\log^{l+1} x}{l+1} - \sum_{\mathrm{N}\mathfrak{a}<x}\frac{\Lambda\mathfrak{a}}{\mathrm{N}\mathfrak{a}}\log^l \mathrm{N}\mathfrak{a}\right).$$

Obviously, $\lim_{x\to\infty} f_{-1}(x,s) = -1$ and $\lim_{x\to\infty} A(x,s) = 0$ in the disc $|s-1| < \delta$.

Let $\psi(x) = \sum_{\mathrm{N}\mathfrak{a}\le x} \Lambda\mathfrak{a}$. Then,

$$\sum_{\mathrm{N}\mathfrak{a}<x} \Lambda\mathfrak{a}\mathrm{N}\mathfrak{a}^{s-1} = \int_1^x t^{s-1} d\psi(t) = \psi(x)x^{s-1} - (s-1)\int_1^x t^{s-2}\psi(t)\,dt + C.$$

Furthermore, $\psi(x) = x + g(x)$, for $x \ge 2$, where the function g is such that $g(x) = O\left(x^{1-\eta}\right)$, as $x \to \infty$, for some $\eta > 0$ depending on the number field $\Bbbk$ only. (This can be seen using [10, Cor. (ii) on p.369] and the relationship between the functions $\pi_{\Bbbk}(x)$, $\theta(x)$ and $\psi(x)$.)

In the disc $|s-1| < \frac{\eta}{2}$ we have

$$\lim_{x\to\infty} \frac{1}{1+x^{2s-1}}\left(\sum_{\mathrm{N}\mathfrak{a}<x} \Lambda\mathfrak{a}\mathrm{N}\mathfrak{a}^{s-1} - \frac{x^s}{s}\right) = 0.$$

Therefore,

$$\lim_{x\to\infty} f_0(x,s) = \lim_{x\to\infty}\left[\frac{x^{2s-1}}{1+x^{2s-1}}\left(\log x - \sum_{\mathrm{N}\mathfrak{a}<x}\frac{\Lambda\mathfrak{a}}{\mathrm{N}\mathfrak{a}}\right)\right] = \lim_{x\to\infty}\left(\log x - \sum_{\mathrm{N}\mathfrak{a}<x}\frac{\Lambda\mathfrak{a}}{\mathrm{N}\mathfrak{a}}\right)$$

in the disc $|s-1| < \delta_0 = \min\left\{\frac{\eta}{2}, \frac{1}{4}\right\}$.

Similarly, for $l \ge 1$ we have

$$\sum_{\mathrm{N}\mathfrak{a}<x} \frac{\Lambda\mathfrak{a}}{\mathrm{N}\mathfrak{a}} \log^l \mathrm{N}\mathfrak{a} = \int_1^x \frac{\log^l t}{t} d\psi(t) = \frac{\log^l x}{x}\psi(x) - \int_1^x \psi(t)\left(l\frac{\log^{l-1} t}{t^2} - \frac{\log^l t}{t^2}\right)dt,$$

and

$$\int_1^x \psi(t)\left(l\frac{\log^{l-1} t}{t^2} - \frac{\log^l t}{t^2}\right)dt = \log^l x - \frac{1}{l+1}\log^{l+1} x + l\int_1^x \frac{g(t)}{t}\log^{l-1} t\,dt - \int_1^x \frac{g(t)}{t^2}\log^l t\,dt.$$

Therefore, for $l \ge 1$

$$\lim_{x\to\infty} f_l(x,s) = \lim_{x\to\infty} \frac{(-1)^l}{l!}\left(\frac{\log^{l+1} x}{l+1} - \sum_{\mathrm{N}\mathfrak{a}<x}\frac{\Lambda\mathfrak{a}}{\mathrm{N}\mathfrak{a}}\log^l \mathrm{N}\mathfrak{a}\right) =$$

$$= \lim_{x\to\infty} \frac{(-1)^l}{l!}\left(\log^l x\left(1 - \frac{\psi(x)}{x}\right) + l\int_1^x \frac{g(t)}{t}\log^{l-1} t\,dt - \int_1^x \frac{g(t)}{t^2}\log^l t\,dt\right) < \infty,$$

since $g(x) = O\left(x^{1-\eta}\right)$, as $x \to \infty$, and $\eta > 0$.

Repeating the above steps, we obtain that $\lim_{x\to\infty}\left(\log x - \sum_{\mathrm{N}\mathfrak{a}<x}\frac{\Lambda\mathfrak{a}}{\mathrm{N}\mathfrak{a}}\right) < \infty$.

Thus, we get that $\lim_{x\to\infty} f_l(x,s) = f_l < \infty$ in the disc $|s-1| < \delta_0$, for all $l \ge -1$. This finishes the proof of the theorem.

Corollary 5.2. The Euler constant of a number field $\Bbbk$ can be expressed as

$$\gamma_0(\mathbb{k}) = \gamma_{-1}(\mathbb{k}) \lim_{x\to\infty}\left(\log x - \sum_{\mathrm{N}\mathfrak{a}<x} \frac{\Lambda\mathfrak{a}}{\mathrm{N}\mathfrak{a}}\right).$$

Remark 5.3. Y. Hashimoto, Y. Iijima, N. Kurokawa, and M. Wakayama in [8, Th. B. (1)] have proved a result similar to the one obtained in Corollary 5.2. by using completely different techniques.

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

CHARACTERIZATION OF MODULATED COX MEASURES ON TOPOLOGICAL SPACES *

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ABSTRACT

This article introduces the notion of a modulated stochastic process that is a family of stochastic processes $\{\xi_y; y \in Y\}$ connected through a leading jump process η valued in Y. So, once η takes on value y, the *modulated* process ξ_η assumes the identity of ξ_y and evolves in accordance with ξ_y up until η enters another state (or subset). Such processes arise in control theory, and in this article more emphasis is made on modulated Cox processes defined on a locally compact and σ-compact Hausdorff space. The paper also studies potential functions for modulated Cox processes and intensity rates for which some explicit formulas are derived.

Key words: Random measure, Poisson measure, Cox measure, modulated measure, modulated Cox measure, potential, potential intensity rate.
AMS subject classifications: Primary 60G57, 60K15, 60K99; Secondary 60K10, 60K25.

1 INTRODUCTION

In this article we examine a class of Cox processes under modulation. A counting process (random measure) $N = \sum_{i=1}^{\nu} \varepsilon_{\tau_i}$ (ε_a is a point mass) is Poisson if the random variables (r.v.'s) $\tau_1, \tau_2, ...$ are independent and belong to the equivalence class $[\tau]$ (valued in a locally compact Hausdorff space $\mathfrak{X}$) and ν is a Poisson r.v. N is said to be with mean measure $\mu = E\nu\, P_\tau$ if its moment generating function $Ee^{\theta N(\cdot)}$ satisfies the formula $Ee^{\theta N(\cdot)} = e^{\mu(\cdot)(e^\theta - 1)}$. Poisson processes on arbitrary spaces are quite common in physical sciences and technology, but they have their limitations. To make them fit more real-world situations we can attach "marks" $X_1, X_2, ...$ to the points $\tau_1, \tau_2, ...$ and also modify the mean measure μ of N to a random measure. If μ is continuous w.r.t. a locally finite Radon measure, say σ, then there is a stochastic process Λ (from the class $\left[\frac{d\mu}{d\sigma}\right]$ of Radon-Nikodym random density functions) known

*This article is devoted to the Threecentenial Leonard Euler's birthday.
†This research is supported by NSA grant No. H98230-05-1-0250

as the "intensity rate" of the Poisson process. Such an embellishment of N turns it to a *marked Cox measure* and it makes N much more versatile. If the mean measure μ is nonrandom, then N is a called a *marked Poisson measure.*

The utility of a marked Cox measure N can be significantly enhanced if we allow N to be controlled by a third party stochastic process η. For example, if η assumes one of its countably many states $\{y_1, y_2, ...\}$, say y_i, we will assign to N the mean measure μ_i (which can be random) and thus, N will change its profile in accordance with the evolution of η. We will say that N is *modulated by* η and consider the couple (N, η) or rather the modulated process in notation N_η.

One needs very little imagination to recognize an abundance of real-world scenarios to which N_η can be well adapted. Among many other things, we would like to mention the stock market constantly perturbed by economic and political news. In reality, some of the stocks and indexes will change its course forever or up until they get hit by another major political cataclysm.

The term "modulation" probably originates from electrical circuits where network processes are "excited" from time to time by surges that can change their parameters and even routing patterns. The same principle was applied in the early celebrated work by Neuts [20,21] that generalized Poisson processes by allowing their rates to change in accordance with the evolution of a Markov process. Not only did they emulate rather general classes of renewal processes (which have been notoriously untamed analytically and computationally), but they also provided a very handy algorithms from within classes of stochastic models, most predominantly, queues. This work inspired hundreds of his followers in thousands of papers to explore these ideas in computational probability.

Along with Neuts' seminal work on *Markov modulated Poisson processes* (as they were called) and later on *batch Markov modulated Poisson processes* (derived from a marked Poisson process), there were some other work related to the reward rates of marked Poisson processes modulated by semi-Markov and semi-regenerative processes all related to stochastic control of queueing models (cf. [6,7] for pertinent references). Calculation of these reward rates required some formalism of modulation attempted in [6,7] by the author and then further refined in [1] and [9]. In particular, in [9], the author allowed a formalism of modulated Poisson measures on locally compact Hausdorff spaces, and in the present article, we continue this trend for modulated Cox measures. This article is an extension and refinement to [9] (to appear in the recent Proceedings of a Conference on Differential Equations). For consistency, we provide the same background as in [9] and even some results, but we further embellish them for Cox processes and bring more development to the potentials of modulated random measures and their intensity rates also by generalizing and refining the results in [1] for Euclidean spaces. We managed to obtain some explicit formulas for the intensity rates and demonstrate it in an example with a semi-Markov modulated Cox measure.

This work topically falls into the area of random measures best presented in a classic monograph [14] by Kallenberg. Many other works and articles are related to random measures but titled *point processes* and they range from less rigorous [3,19,24] to more theoretical in various degrees [4,5,12-18,22,25]. There are also many interesting monographs and papers on point processes in queueing theory [2,11] and simulation [19], to name a few.

2 BACKGROUND

Remark 1. We will use the following notation for a function from [8]. We will often write $[X, Y, f]$ to mean that f is a function from X into Y. If Y' is a subset of Y, by $f^*(Y')$ we denote the inverse image of Y'. If $\mathcal{S}(Y)$ is a system of subsets of Y, then the set of all inverse images of $\mathcal{S}(Y)$ under f will be denoted by $f^{**}(\mathcal{S}(Y))$. An image of a subset $A \subseteq X$ under f will be denoted by $f_*(A)$. □

The rest of the notation will be as follows. Suppose $(\mathfrak{X}, \tau_\mathfrak{X})$ (where $\tau_\mathfrak{X}$ is a topology in $\mathfrak{X}$) is a LCHS (locally compact Hausdorff space) often abbreviated as $\mathfrak{X}$ and

$$\mathcal{B}_\mathfrak{X} = \mathcal{B}(\mathfrak{X}) = \mathcal{B}(\tau_\mathfrak{X})$$

be the corresponding Borel σ-algebra. Denote by $\mathcal{R}_\mathfrak{X}$ the set of all relatively compact Borel subsets of $\mathfrak{X}$ (which is a ring) and by $\mathcal{K}_\mathfrak{X}$ - the set of all compact subsets. A Borel measure μ on $\mathcal{B}_\mathfrak{X}$ is called *locally finite* if μ is finite on $\mathcal{R}_\mathfrak{X}$. Notice that μ is a *Borel-Lebesgue-Stieltjes* measure on $\mathcal{B}_{\mathbb{R}^d}$ (cf. [8]) if and only if μ is locally finite. If the space is metrizable, then $\mathcal{R}_\mathfrak{X}$ is commonly replaced by the family of all bounded measurable sets.

Let $\mathfrak{M}_\mathfrak{X}$ denote the set of all locally finite (Borel) measures on $\mathcal{B}_\mathfrak{X}$. Unless specified otherwise, all measures we will deal with will be positive.

Definition 1. (Regularity, cf. [8]) Let A be a Borel set and $\mu \in \mathfrak{M}_\mathfrak{X}$.

μ is *inner regular* at A if A can be μ-approximated from below by compact subsets, or more formally,

$$\mu(A) = \sup\{\mu(K) : K \subseteq A, A \in \mathcal{K}_\mathfrak{X}\}.$$

μ is *outer regular* at A if A can be μ-approximated from above by open supersets, or more formally,

$$\mu(A) = \inf\{\mu(U) : A \subseteq U, U \in \tau_\mathfrak{X}\}.$$

Measure μ is *weakly regular* or *Radon* if it is inner regular on $\tau_\mathfrak{X}$ (i.e. at each open set), and it is outer regular on $\mathcal{B}_{\mathfrak{X}}$ (i.e. at each Borel set). Measure μ is *regular* if it is inner and outer regular on $\mathcal{B}_{\mathfrak{X}}$. □

Regularity of locally finite Borel measures is a special feature and it is to be either assumed or it follows from some assumptions imposed on the topology $\tau_{\mathfrak{X}}$. One of them is as follows:

Theorem 1 (cf. [8]). If $(\mathfrak{X}, \tau_{\mathfrak{X}})$ is σ-compact, then any locally finite measure on $\mathcal{B}_{\mathfrak{X}}$ is regular (in particular, Radon) and σ-finite.

Thus assuming $(\mathfrak{X}, \tau_{\mathfrak{X}})$ to be σ-compact and LCHS, $\mathfrak{M}_{\mathfrak{X}}$ turns out to be the set of all **regular** measures, in particular, Radon measures.

Remark 2. In the literature they often assume $\mathfrak{X}$ to be second countable. Then, $\mathfrak{X}$ is metrizable, separable, and σ-compact. We observe that metrization is not always necessary; and σ-compactness of $\mathfrak{X}$, while often sufficient, is a much weaker assumption. The Euclidean space with its natural topology is LCHS, σ-compact and Lindelöf compact. Thus, any locally finite measure on $\mathcal{B}_{\mathbb{R}^d}$ is Borel-Lebesgue-Stieltjes, Radon, and regular. Of course, Euclidean space is also second countable and complete and, therefore, it is Polish. Notice that in spite of some claims, not every LCHS $\mathfrak{X}$, which is second countable, is Polish. For instance, if $\mathfrak{X} = (0,1)$ with the relative topology $\tau_e \cap (0,1)$, then it is a second countable LCHS, but not complete.

□

Thus, under the assumption that $(\mathfrak{X}, \tau_{\mathfrak{X}})$ is σ-compact (and LCHS), we have that $\mathfrak{M}_{\mathfrak{X}}$ is the set of all locally finite regular (in particular, Radon) measures. We will continue to assume this throughout the rest of the paper.

Now we define a σ-algebra in $\mathfrak{M}_{\mathfrak{X}}$.

Definition 2. Given a Borel set B, denote the map $\mu \mapsto \mu(B)$ by $\psi_B : \mathfrak{M}_{\mathfrak{X}} \to \bar{\mathbb{R}}_+$. The family $\{\psi_B : B \in \mathcal{B}_{\mathfrak{X}}\}$ of all such maps indexed by elements of $\mathcal{B}_{\mathfrak{X}}$ induces the smallest σ-algebra $\mathcal{M}_{\mathfrak{X}}$ in $\mathfrak{M}_{\mathfrak{X}}$ relative to which every such map ψ_B is measurable, i.e.

$$\mathcal{M}_{\mathfrak{X}} = \sigma\left(\bigcup_{B \in \mathcal{B}_{\mathfrak{X}}} \psi_B^{**}(\mathcal{B}(\bar{\mathbb{R}}_+)\right). \tag{1}$$

(See Remark 1 relative to equation (1).) □

Definition 3. Let $(\Omega, \mathfrak{A}(\Omega), P)$ be a probability space. A *random measure* ξ is any measurable mapping from $(\Omega, \mathfrak{A}(\Omega), P)$ to $(\mathfrak{M}_{\mathfrak{X}}, \mathcal{M}_{\mathfrak{X}})$. [It is a parametric family of measures in $\mathfrak{M}_{\mathfrak{X}}$ indexed by $\omega \in \Omega$ such that $\xi^{**}(\mathcal{M}_{\mathfrak{X}}) \subseteq \mathfrak{A}(\Omega)$. It is also a random variable (r.v.) generating a family $P\xi^*$ of distributions on $\mathcal{M}_{\mathfrak{X}}$.] The integral measure

$$E\xi = \int \xi dP \tag{2}$$

is called the *intensity* of ξ. Observe that while $E\xi$ is a Borel measure on $\mathcal{B}_{\mathfrak{X}}$, in general, $E\xi \notin \mathfrak{M}_{\mathfrak{X}}$. □

3 MODULATED RANDOM MEASURES

Example 1. We begin with a special construction of a random measure to be used for modulation. Let $(\Omega, \mathfrak{A}(\Omega), P)$ be a probability space, let $\mu \in \mathfrak{M}_{\mathfrak{X}}$ and let G be a measurable subset of the product space $\Omega \times \mathfrak{X}$, i.e. $G \in \mathfrak{A}(\Omega) \otimes \mathcal{B}_{\mathfrak{X}}$ and, for any $\omega \in \Omega$, let G_ω denote the ω-section of G.

Since μ is σ-finite, due to σ-compactness of $\mathfrak{X}$ (see Theorem 1), the map $\omega \mapsto \mu(G_\omega)$ is $\mathfrak{A}(\Omega)$-$\mathcal{B}(\bar{\mathbb{R}}_+)$-measurable and thus it can be regarded as a random variable on $(\Omega, \mathfrak{A}(\Omega), P)$. Hence, given a fixed G, the map

$$(\omega, B) \mapsto \nu_G(\omega, B) := \mu(G_\omega \cap B),\ B \in \mathcal{B}_{\mathfrak{X}}, \tag{3}$$

is a random measure from $(\Omega, \mathfrak{A}(\Omega), P)$ to $(\mathfrak{M}_{\mathfrak{X}}, \mathcal{M}_{\mathfrak{X}})$. The intensity $E\nu_G$, as for any random measure, is a measure itself, but in this case, it is also locally finite and thus regular. Indeed, it is readily seen that

$$E\nu_G \le E\nu_{\Omega\times\mathfrak{X}} = \mu \tag{4}$$

and thus, for every $G \in \mathfrak{A}(\Omega) \otimes \mathcal{B}_{\mathfrak{X}}$, $E\nu_G \in \mathfrak{M}_{\mathfrak{X}}$. In particular, it implies that the measure $E\nu_G$ is absolutely continuous w.r.t. μ, i.e. if $[g]_\mu$ is the corresponding Radon-Nikodym derivative, then

$$E\nu_G \left(= \int \mu(G_\omega \cap \cdot) P(d\omega)\right) = \mu[g]_\mu \left(= \int [g]_\mu d\mu\right) \tag{5}$$

□

Now modulation is based on the following concept.

Definition 4. We will say that a sequence of locally finite regular measures $\{\mu_1, \mu_2, ...\}$ is *locally bounded* if for any relatively compact Borel set $R \in \mathcal{R}_{\mathfrak{X}}$, there is a locally finite regular Borel measure δ_R such that

$$\mu_i(B) \le \delta_R(B),\ \forall B \in R \cap \mathcal{B}_{\mathfrak{X}}, i = 1, 2, ... \tag{6}$$

Example 2 (A random measure induced by a locally bounded sequence from $\mathfrak{M}_{\mathfrak{X}}$ and a sequence of measurable sets.) Let $\{\mu_1, \mu_2, ...\}$ be a locally bounded sequence from $\mathfrak{M}_{\mathfrak{X}}$, let

$$\{G_1, G_2, ...\} \in \mathfrak{A}(\Omega) \otimes \mathcal{B}_{\mathfrak{X}}$$

be a measurable partition of $\Omega \times \mathfrak{X}$, and let $\{a_1, a_2, ...\}$ be a bounded sequence of nonnegative real numbers. Define

$$\xi = \xi(\omega, \cdot) = \sum_{i=1}^{\infty} a_i \mu_i((G_i)_\omega) \cap \cdot). \tag{7}$$

We show that ξ is indeed a random measure. □

Proposition 1. Given a locally bounded sequence of regular measures $\{\mu_1, \mu_2, ...\}$, a measurable partition $\{G_1, G_2, ...\}$ of $\Omega \times \mathfrak{X}$, and a bounded sequence $\{a_1, a_2, ...\}$, ξ is a random measure and its intensity $E\xi$ is a locally finite regular measure.

Proof. Since each μ_i is σ-finite, as in Example 1, $\mu_i((G_i)_\omega) \cap B)$ is measurable for each i and for each $B \in \mathcal{B}_\mathfrak{X}$ and so is $\xi(\cdot, B)$. Now, the condition of local boundedness of the sequence $\{\mu_1, \mu_2, ...\}$ is equivalent to the existence of a family $\{\delta_R : R \in \mathcal{R}_\mathfrak{X}\} \subseteq \mathfrak{M}_\mathfrak{X}$ such that for each relatively compact Borel set R, there is a locally finite Borel measure δ_R such that for each $B \in R \cap \mathcal{B}_\mathfrak{X}$,

$\mu_i(B) \leq \delta_R(B)$, for all $i = 1, 2, ...$.(8)

Then, if a is an upper bound for $\{a_i\}$, we have that

$$\xi(\omega, R) \leq a \sum_i \delta_R((G_i)_\omega) \cap R) = a\delta_R(\sum_i (G_i)_\omega \cap R) = a\delta_R(R) < \infty, \tag{9}$$

good for all ω. It means that for every ω, ξ is locally finite and thus regular and, consequently, is a random measure. Finally, $E\xi(R) \leq a\delta_R(R) < \infty$ and hence $E\xi \in \mathfrak{M}_\mathfrak{X}$. □

Notice that local boundedness of the sequence $\{\mu_1, \mu_2, ...\}$ is a relatively weak constrain on the sequence (resembling pointwise boundedness of a sequence of functions) applied to only the ring of relatively compact Borel sets, and this is obviously a weaker condition than $\mu_i \leq \delta$ for all i.

Example 3. Let η be a stochastic process from $(\Omega, \mathfrak{A}(\Omega), P)$ to a measurable space $(Y, \mathcal{B}(\tau(Y)))$ parametrized by $t \in \mathfrak{X}$ (which, as before, is LCHS and σ-compact) and let $\{Y_1, Y_2, ...,\}$ be a measurable countable partition of Y. Then, $\{\eta^*(Y_i), i = 1, 2, ...\}$ is a measurable partition of $\Omega \times \mathfrak{X}$. Under the condition of Example 2 (as regards μ_i's and a), denote

$$\xi_\eta = \sum_i a_i \mu_i \left((\eta^*(Y_i))_\omega \cap \cdot \right). \tag{10}$$

Then, by Proposition 1, ξ_η is a random measure. We will call ξ_η the *random measure modulated by stochastic process η with respect to the sequence* $\{\mu_1, \mu_2, ...\}$. Obviously, the intensity

$$E\xi_\eta = \sum_i a_i E\mu_i \left((\eta^*(Y_i))_\omega \cap \cdot \right) \tag{11}$$

is a locally finite regular measure as per Proposition 1. □

Example 4. Under the condition of Example 3, let μ_i's be random measures. To distinguish them from nonrandom measures, for convenience we will replace μ_i's with characters $\{M_1, M_2, ...\}$. We also assume that this sequence is *locally bounded*, i.e. a sequence $\{M_1, M_2, ...\}$ of random measures is said to be *locally bounded* if for any relatively compact Borel set $R \in \mathcal{R}_{\mathfrak{X}}$, there is a locally finite regular Borel measure δ_R such that

$$M_i(\omega, B) \leq \delta_R(B),\ \forall B \in R \cap \mathcal{B}_{\mathfrak{X}}, \omega \in \Omega,\ i = 1, 2, \dots\ . \tag{12}$$

Then, the random measure ξ_η of (10) will be modified to

$$\Upsilon_\eta = \sum_i a_i M_i \left(\omega, (\eta^*(Y_i))_\omega \cap \cdot\right). \tag{13}$$

We easily conclude that Υ_η is again a random measure by the same arguments as those for (10). However, Υ_η is "doubly stochastic," although this term is being reserved exclusively for Cox processes, and surely its intensity

$$E\Upsilon_\eta = \sum_i a_i E M_i \left(\omega, (\eta^*(Y_i))_\omega \cap \cdot\right) \tag{14}$$

is a locally finite regular measure. Furthermore, assume that for each i, $M_i \ll \sigma \in \mathfrak{M}_{\mathfrak{X}}$, and let Λ_i be a Radon-Nikodym (random) density from the class $\frac{dM_i}{d\sigma}$. Then,

$$M_i\eta^*(Y_i) = \int \mathbf{1}_{Y_i} \circ \eta \, dM_i = \int \mathbf{1}_{Y_i} \circ \eta \, \Lambda_i \, d\sigma. \tag{15}$$

Proposition 2. Assuming Λ_i and η to be uncorrelated we have

$$E M_i \eta^*(Y_i) = \int E[\mathbf{1}_{Y_i} \circ \eta]\, E[\Lambda_i]\, d\sigma$$

$$= \int_{t \in \mathfrak{X}} P\{\eta(t) \in Y_i\} E[\Lambda_i(t)]\, d\sigma(t)\ \ i = 1, 2, \dots\ , \tag{16}$$

and thus the intensity $E\Upsilon_\eta$ is

$$E\Upsilon_\eta = \sum_i a_i \int_{t \in \mathfrak{X}} P\{\eta(t) \in Y_i\} E[\Lambda_i(t)]\, d\sigma(t). \tag{17}$$

4 MODULATED POISSON AND COX PROCESSES

The notion of modulation stems from electrical engineering and queueing, when a stochastic process, most prominently, Poisson, is governed by a Markov chain with finitely many states. That is the intensity of the Poisson process from being constant turns to random subject to the evolution of the underlying Markov chain. We will explore this notion more rigorously. We start with

Definition 14 (cf. Kallenberg, [14]). Let

$$\xi = \sum_{i=1}^{\nu} X_i \varepsilon_{\tau_i} \tag{18}$$

where $X_1, X_2, ...$ are i.i.d. nonnegative r.v.'s, $\tau_1, \tau_2, ...$ be i.i.d. r.v.'s valued in $\mathfrak{X}$, and ν be a Poisson r.v. with finite mean $E\nu$. Then ξ is a random measure and it is called a *marked Poisson random measure* (MPRM). We assume position independent marking, i.e. X_i's and τ_i are independent. Alternatively, ξ is referred to as a *compound Poisson process* or *marked Poisson process*, in this case, with position independent marking. The associated random measure $N = \sum_{i=1}^{\nu} \varepsilon_{\tau_i}$ is called the *support counting measure* of ξ. □

The below properties of ξ and N are widely known.

Proposition 3. Random measures ξ and N are known to have the properties:

(i) For disjoint $B_1, ..., B_k \in \mathcal{B}_{\mathfrak{X}}$, the r.v.'s $\xi(B_1), ..., \xi(B_k)$ are independent.

(ii) There is a regular locally finite measure $\mu \in \mathfrak{M}_{\mathfrak{X}}$ such that

$$Ee^{\theta\xi(\cdot)} = e^{\mu(\cdot)[m(\theta)-1]}, \tag{19}$$

where $m(\theta) = Ee^{\theta X_1}$ and, in particular,

$$Ee^{\theta N(\cdot)} = e^{\mu(\cdot)(e^{\theta}-1)}, \tag{20}$$

with μ being called the *mean measure* of N. ξ is said to be *directed by measure* μ. Notice that

$$\mu = E\nu P_\tau \tag{21}$$

where $\tau \sim \tau_1$. The intensity of ξ is

$$E\xi = a\mu, \tag{22}$$

where $a = EX$, and $X \sim X_1$. □

Definition 6. Now, a MPRM ξ can get *mixed* by replacing the mean measure μ with a random measure, say M. The process of mixture allows one to modify a random measure like MPRM from within its property (ii) of Proposition 3. It is a standard procedure in the theory of random measures, cf. Kallenberg [14], p. 16. In other words, due to (22), aM is the "conditional intensity" $E[\xi|M]$. Now, ξ is *directed by random measure* M and the intensity of ξ becomes

$$E\xi = aEM. \tag{23}$$

Equation (19) of property (ii) from Proposition 3 is modified as

$$E[e^{\theta\xi(\cdot)}|M] = e^{M(\cdot)[m(\theta)-1]} \tag{24}$$

This way, the mixed random measure ξ is called a *marked Cox measure directed by* M, whereas with $X = 1$ a.s., ξ is simply a *Cox measure directed by* M. □

Definition 7.

(i) Assume that $\xi_1, \xi_2, ...$ be a locally bounded sequence of MPRM's such that for a fixed k,

$$\xi_k = \sum_{i=1}^{\nu_k} X_i^{(k)} \varepsilon_{\tau_i^{(k)}} \tag{25}$$

directed by a mean measure μ_k and with intensity $E\xi_k = a_k\mu_k$. Then, under the condition in Example 4,

$$\Pi_\eta = \sum_i \xi_i\left(\omega, (\eta^*(Y_i))_\omega \cap \cdot\right) \tag{26}$$

is a random measure modulated by η, which we will call a *marked Poisson random measure modulated by* η.

(ii) From formula (22), the intensity of Π_η is

$$E\Pi_\eta = \sum_i a_i E\mu_i\left((\eta^*(Y_i))_\cdot \cap \cdot\right). \tag{27}$$

Now, if each respective mean measure μ_i is continuous w.r.t. some $\sigma \in \mathfrak{M}_{\mathfrak{X}}$ and if $\lambda_i \in \frac{d\mu_i}{d\sigma}$, then

$$E\mu_i\left((\eta^*(Y_i))_\cdot \cap \cdot\right) = E\int 1_{Y_i} \circ \eta \, d\mu_i = E\int 1_{Y_i} \circ \eta \, \lambda_i \, d\sigma$$

$$= \int_{t\in\mathfrak{X}} P\{\eta(t) \in Y_i\} \, \lambda_i(t) \, d\sigma(t) \tag{28}$$

If the above sequence $\xi_1, \xi_2, ...$ defined by (25) is a locally bounded sequence of marked Cox measures, each directed by its respective random measure M_k and $E\xi_k = a_k EM_k$, then (26) will be modified as

$$\mathfrak{C}_\eta = \sum_i \xi_i\left(\omega, (\eta^*(Y_i))_\omega \cap \cdot\right) \tag{29}$$

which looks exactly as (26), and we call it a *marked Cox measure modulated by* η. Of course, (29) is in fact not the same as (26), because ξ_i is directed by a random measure M_i for each i, compared to the marked Poisson measure of (26). □

The analog of formula (27) for the intensity of $\mathfrak{C}_\eta$ is

$$E\mathfrak{C}_\eta = \sum_i a_i EM_i\left((\eta^*(Y_i))_\cdot \cap \cdot\right). \tag{30}$$

Now, if each respective directed measure M_i is continuous w.r.t. some $\sigma \in \mathfrak{M}_{\mathfrak{X}}$ and if $\Lambda_i \in \frac{dM_i}{d\sigma}$ (where the Radon-Nikodym stochastic density Λ_i is a stochastic process), then

$$EM_i\left((\eta^*(Y_i)).\cap\cdot\right) = E\int 1_{Y_i}\circ\eta\, dM_i = E\int 1_{Y_i}\circ\eta\,\Lambda_i\, d\sigma. \tag{31}$$

If furthermore, η and Λ_i are independent, then

$$EM_i\left((\eta^*(Y_i)).\cap\cdot\right) = \int_{t\in\mathfrak{X}} P\{\eta(t)\in Y_i\}\, E[\Lambda_i(\cdot,t)]\, d\sigma(t). \tag{32}$$

Formula (32) is the representation of the intensity EM_i of M_i modulated by η.

5 SEMI-MARKOV MODULATED COX RANDOM MEASURES ON EUCLIDEAN SPACE

Suppose $\mathfrak{X} = \mathbb{R}_+$, let $(\Omega, \mathfrak{A}(\Omega), P^i, S(t), t\in\mathbb{R}_+)$ be a semi-regenerative process valued in at most countable subset $Y := \{y_1, y_2, ...\}$ of $\mathbb{R}$, $\{T_n\}$ be a monotone increasing sequence of stopping times of S, such that S has a locally strong Markov property at T_n (cf. [1]). Let $\eta = \eta(t)$ be the minimal semi-Markov process w.r.t. $\{T_n\}$ and suppose that the Markov chain $\{S(T_n)\}$ is ergodic with the limiting distribution

$$P_{S(T_\infty)} = \sum_{k\geq 0} p_k \varepsilon_{y_k}, \tag{33}$$

The (row) vector **p**$= (p_k)$ is known as the *invariant probability measure* of the Markov chain $\{S(T_n)\}$.

Let us denote **b**$= (b_k)^{\mathsf{T}}$, $b_k := E^{y_k}[T_1]$, and let **pb** stand for the dot product of **p** and **b**. Then the limiting distribution $P_\eta = \sum_{k\geq 0}\pi_k\varepsilon_{\{y_k\}}$ of η is known to satisfy the formula

$$\pi_k = \lim_{t\to\infty}\frac{1}{t}E\int_{[0,t]} 1_{\{y_k\}}(\eta(\cdot,t))dl(t) = \frac{1}{\mathbf{pb}}p_k b_k, \tag{34}$$

where the integral in brackets is w.r.t. the Borel-Lebesgue measure l. Now, let $\xi_1, \xi_2, ...$ be a locally bounded sequence of marked Cox measures. We examine the Cox measure $\mathfrak{C}_\eta$ modulated by the semi-Markov process η. Here is the interpretation of the evolution of the modulated Cox process $\mathfrak{C}_\eta$. At time T_n, when $\eta(T_n)$ assumes value, say y_k, $\mathfrak{C}_\eta$ assumes the identity of ξ_k and all the way up until T_{n+1}.

Definition 8. Consider

$$r = \lim_{t\to\infty}\frac{1}{t}E\mathfrak{C}_\eta[0,t]. \tag{35}$$

and call it the *intensity rate of* $\mathfrak{C}_\eta$ *w.r.t. the Borel-Lebesgue measure.* □

Using formula (30), we calculate

$$\frac{1}{t}E\mathfrak{C}_\eta[0,t] = \frac{1}{t}\sum_i a_i EM_i\left((\eta^*(\{y_i\}))\cap[0,t]\right) = \frac{1}{t}\sum_i a_i E\int_0^t \mathbf{1}_{\{y_i\}}\circ\eta\, dM_i$$

$$= \frac{1}{t}\sum_i a_i E\int_0^t \mathbf{1}_{\{y_i\}}\circ\eta\, \Lambda_i dl, \tag{36}$$

where $\Lambda_i \in \frac{dM_i}{dl}$ is the Radon-Nikodym intensity function w.r.t. the Borel-Lebesgue measure l. Now, assuming Λ_i independent of η and furthermore, letting Λ_i be a r.v. (i.e. a stationary process) we arrive at

$$\frac{1}{t}E\mathfrak{C}_\eta[0,t] = \frac{1}{t}\sum_i a_i E\Lambda_i \int_{y=0}^t P\{\eta(y) = y_i\}dl(y) \tag{37}$$

Combining (37) with (34) we have

$$r = \sum_i a_i E\Lambda_i \frac{1}{\mathbf{pb}} p_i b_i = Eg\left(\eta(\infty)\right), \tag{38}$$

where

$$g = \sum_i a_i E\Lambda_i \mathbf{1}_{\{y_i\}}. \tag{38a}$$

Formula (38) can be interpreted as the reward rate of a Cox stochastic process that is perturbed by a semi-Markov process in such a way that the Cox process is piecewise stationary conditioned on the random interval $[T_n, T_{n+1})$ in accordance with the stationary intensity Λ_i when the semi-Markov process assumes value y_i and keeps on in this state from T_n to T_{n+1}. The value a_i is the corresponding reward for the modulated process, which we can also allow to be real-valued.

Notice that a special case of the semi-Markov modulated piecewise stationary Cox measure is a more general version of a so-called *Markov modulated Poisson process* very well known in stochastic modeling. It was primarily developed by M. Neuts [20] and then embellished by his followers [10,23] in connection with various applications to queueing theory and stochastic networks [26]. In this process their inventors (most predominantly, Neuts) allowed the Cox process intensity functions evolve in accordance with the evolution of a Markov chain with finitely many states. While the applications of this process are entirely different from ours, it should be noted that Neuts' construction is very sophisticated and it allowed one to form a large class of versatile input processes to a stochastic servicing system that can be evaluated numerically and gave rise to various generalizations described in hundreds of articles and books.

6 POTENTIALS OF MODULATED COX MEASURES

Recall from (23) that the intensity of a marked Cox measure ξ directed by a random measure M is $E\xi = aEM$. We will present an analog of this equation for a class of functionals of stochastic processes w.r.t. random measures and then extend it to modulated Cox measures.

Definition 9. Let η be a stochastic process from $(\Omega, \mathfrak{A}(\Omega), P)$ to $(Y, \mathcal{B}(\tau(Y)))$ parametrized by $t \in \mathfrak{X}$ (LCHS and σ-compact), let $[Y, \mathbb{R}, f]$ be Borel measurable function, and let ξ be a random measure. Then, we will call $E\int f(\eta)d\xi$ the *potential of* η *w.r.t.* f *and random measure* ξ (provided that the integral exists). □

Theorem 2. In the condition of Definition 6, the potential of stochastic process η w.r.t. f and a marked Cox measure ξ directed by a random measure M allows the following explicit representation:

$$E\int f(\eta)d\xi = aE\int f(\eta)dM, \tag{39}$$

provided that the integrals in (39) exist.

Proof. Let Y' be a measurable subset of Y. By (23),

$$E\xi\eta^*(Y') = aEM\eta^*(Y'). \tag{40}$$

On the other hand, the left- and right-hand-side of (40) yield

$$E\int \mathbf{1}_{Y'} \circ \eta \, d\xi = aE\int \mathbf{1}_{Y'} \circ \eta \, dM. \tag{41}$$

Thus, for a simple function $h = \sum_{i=1}^{k} b_i \mathbf{1}_{Y_i}$, (41) yields

$$E\int \sum_{i=1}^{k} b_i \mathbf{1}_{Y_i} \, d\xi\eta^* = aE\int \sum_{i=1}^{k} b_i \mathbf{1}_{Y_i} dM\eta^* \tag{42}$$

and for any measurable nonnegative $f = \sup h_n$

$E\int f d\xi\eta^* = \sup E\int h_n d\xi\eta^* = a\sup E\int h_n dM\eta^*$.(43)

Using the change of variables and extending (43) to real-valued function, we are done with the proof. □

Definition 10. Let $B \in \mathcal{R}_{\mathfrak{X}}$ (relatively compact Borel subsets of $\mathfrak{X}$) and $\rho \in \mathfrak{M}_{\mathfrak{X}}$. Under the condition of Definition 6 and Theorem 2, define

$$r(\mathfrak{C}_\eta f(\eta)(B)) = \frac{E\left[\mathfrak{C}_\eta f(\eta)(B)\right]}{\rho(B)} \tag{44}$$

assuming that $\rho(B) \neq 0$ and call it the *potential intensity rate of* $\mathfrak{C}_\eta$ *on set* B *w.r.t. measure* ρ. Let $\{B_\alpha; \alpha \in \mathcal{N}\}$ be a monotone increasing family of relatively compact Borel sets along a net $\mathcal{N}$ such that $B_\alpha \uparrow \mathfrak{X}$ and $\mu(B_\alpha) \neq 0$ for all α. Now, if the limit

$$r(\mathfrak{C}_\eta f(\eta)) = \lim_{\alpha \in \mathcal{N}} r(\mathfrak{C}_\eta f(\eta)(B_\alpha)) \tag{45}$$

exists, we will call its value $r(\mathfrak{C}_\eta f(\eta))$ *the potential intensity rate of* $\mathfrak{C}_\eta$ *w.r.t. measure* ρ. We are not going to investigate whether or not such a number exists and if it does, whether or not $r(F)$ is invariant of the choice of $\{B_\alpha; \alpha \in \mathcal{N}\} \uparrow \mathfrak{X}$. □

Suppose that the process η is valued in a countable subset $\{y_1, y_2, ...\} \subseteq Y$, it is ergodic and its stationary distribution w.r.t. measure ρ is

$$P_{\eta(\infty)} = \sum_{k \geq 0} x_k \varepsilon_{y_k}, \tag{46}$$

i.e.,

$$\lim_{B_\alpha \uparrow \mathfrak{X}} \frac{1}{\rho(B_\alpha)} E\left[\int_{B_\alpha} \mathbf{1}_{\{y_k\}}(\eta(\cdot, y))\rho(dy)\right] = x_k \tag{47}$$

Then,

$$\begin{aligned} &\frac{1}{\rho(B_\alpha)} \int_{B_\alpha} P\{\eta(\cdot, y) = y_k\} d\rho(y) \\ &= \frac{1}{\rho(B_\alpha)} E\left[\int_{B_\alpha} \mathbf{1}_{\{y_k\}}(\eta(\cdot, y))\rho(dy)\right] \to x_k \text{ as } B_\alpha \uparrow \mathfrak{X}. \end{aligned} \tag{48}$$

By Proposition 2,

$$\begin{aligned} &\lim_{B_\alpha \uparrow \mathfrak{X}} \frac{1}{\rho(B_\alpha)} E\Upsilon_\eta B_\alpha \\ &= \lim_{B_\alpha \uparrow \mathfrak{X}} \frac{1}{\rho(B_\alpha)} \sum_i a_i \, E\, A_i \int_{[0,t]} E[\mathbf{1}_{\{y_i\}} \circ \eta] \, d\rho \\ &= \sum_i a_i \, E\, A_i \, x_i = \int g dP_{\eta(\infty)} = Eg(\eta(\infty)) \end{aligned} \tag{49}$$

where

$$g = \sum_i a_i E A_i \mathbf{1}_{\{y_i\}} \tag{50}$$

Definition 11. Let η be a stochastic process from $(\Omega, \mathfrak{A}(\Omega), P)$ to $(Y, \mathcal{B}(\tau(Y)))$ parametrized by $t \in \mathfrak{X}$ and let $F = \{f_i\}$ be a sequence of measurable functions defined on Y. The *potential of* η *w.r.t.* F *and an* η*-modulated Cox measure*

$$\mathfrak{C}_\eta = \sum_i \xi_i \left(\omega, \left(\eta^*(Y_i)\right)_\omega \cap \cdot\right)$$

is defined as

$$E\mathfrak{C}_\eta F(\eta) = E \int \sum_i f_i(\eta) d\xi_i, \tag{51}$$

provided that the integral on the right of (51) exists. In particular, for a Borel subset $B \subseteq \mathfrak{X}$, let

$$E\mathfrak{C}_\eta F(\eta)(B) = E$$

B$\int \sum_i f_i(\eta) d\xi_i.(52)$

□

From Theorem 2, we have:

Theorem 3. Under the condition of Definition 10, the potential of η w.r.t. F and an η-modulated Cox measure satisfies the formula:

$$E\mathfrak{C}_\eta F(\eta) = \sum_i a_i E \int f_i(\eta) dM_i. \tag{53}$$

□

Let $B \in \mathcal{R}_\mathfrak{X}$ (relatively compact Borel subsets of $\mathfrak{X}$) and $\mu \in \mathfrak{M}_\mathfrak{X}$. Under the condition of (52), define

$$r(F, B) = \frac{E[\mathfrak{C}_\eta F(\eta)](B)}{\rho(B)} \tag{54}$$

assuming that $\rho(B) \neq 0$ and call it the *potential intensity rate of* $\mathfrak{C}_\eta$ *on set* B *w.r.t. measure* ρ. Let $\{B_\alpha; \alpha \in \mathcal{N}\}$ be a monotone increasing family of relatively compact Borel sets along a net $\mathcal{N}$ such that $B_\alpha \uparrow \mathfrak{X}$ and $\mu(B_\alpha) \neq 0$ for all α. Now, if the limit

$$r(F) = \lim_{\alpha \in \mathcal{N}} r(F, B_\alpha) \tag{55}$$

exists, we will call its value $r(F)$ the *potential intensity rate of* $\mathfrak{C}_\eta$ *w.r.t. measure* ρ. We are not going to investigate whether or not such a number exists and if it does, whether or not $r(F)$ is invariant of the choice of $\{B_\alpha; \alpha \in \mathcal{N}\} \uparrow \mathfrak{X}$.

From (52) we have

$$E\mathfrak{C}_\eta F(\eta)(B) = \sum_i a_i E \int_B f_i(\eta) dM_i \tag{56}$$

Assuming $dM_i = \Lambda_i d\rho$ and Λ_i being independent of η, as many times earlier above, we have

$$E\mathfrak{C}_\eta F(\eta)(B) = \sum_i a_i \int_B Ef_i(\eta) E\Lambda_i d\rho = \sum_i a_i E\Lambda_i \int_B Ef_i(\eta) d\rho. \tag{57}$$

The latter will due to the assumption that A_i's are stationary (i.e. r.v.'s).

Our next effort is to calculate

$$\frac{1}{\rho(B_\alpha)}\int B_\alpha Ef_i(\eta)d\rho \ as \ B_\alpha \uparrow \mathfrak{X}. \tag{58}$$

If f_i is a simple function, say $\sum_{s=1}^{N_i} b_{si}\mathbf{1}_{Y_{si}}$, where $\{Y_{1i},...,Y_{N_i i}\}$ is a finite decomposition of $\{y_1, y_2, ...\}$ such that b_{si}'s differ on distinct Y_{si}'s.

Assuming η to be ergodic, from

$$\frac{1}{\rho(B_\alpha)}E\int_{B_\alpha}\mathbf{1}_{\{y_k\}}(\eta(\cdot,y))\rho(dy) \ \rightarrow \ x_k \ as \ B_\alpha \uparrow \mathfrak{X}$$

we have

$$\frac{1}{\rho(B_\alpha)}\int B_\alpha Ef_i(\eta)d\rho = \sum_s \hat{b}_s x_s = Ef_i(\eta(\infty)), \tag{59}$$

where $\hat{b}_s$, $s = 1, 2, ...$, are collected from b_{si}'s and some can be identical and there are only finitely many of them distinct. Now, the same result holds for a real-valued measurable function f_i by using the monotone convergence theorem. From (54), (55), and (59), using the Lebesgue dominated convergence theorem, we get

$$r(F) = \lim_{B_\alpha \uparrow \mathfrak{X}}\frac{E[\mathfrak{C}_\eta F(\eta)](B_\alpha)}{\rho(B_\alpha)} = \sum_i a_i EA_i Ef_i(\eta(\infty)). \tag{60}$$

The above can be summarized in the following theorem.

Theorem 4. Let $\mathfrak{C}_\eta$ be a marked Cox measure modulated by an ergodic process η valued in a countable subset $\{y_1, y_2, ...\}$ of Y and let $F = \{f_1, f_2, ...\}$ be a sequence of Borel measurable real-valued functions with their common domain Y. Let $\eta(\infty)$ denote η in its stationary mode (in probability) w.r.t. a measure $\rho \in \mathfrak{M}_{\mathfrak{X}}$, as per (48). Then, the potential intensity rate $r(F)$ w.r.t. ρ satisfies formula (60). □

The results of Theorem 4 and its various modifications can be useful in control theory. A very special case of this was discussed in section 5.

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Hypersurfaces with flat r-mean curvature and Ribaucour transformations

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ABSTRACT

We consider Ribaucour transformations as a method of obtaining hypersurfaces with flat r-mean curvature. Solutions of an integrable system of equations provide hypersurfaces with flat Guass-Kronecker from a given such hypersurface. Nonlinear differential equations are exhibited, whose solutions provide n-dimensional hypersurfaces of the euclidean space with flat r-mean curvature. These differential equations are obtained by Ribaucour transformations applied to a totally geodesic hypersurface. Applications of the theory produce families of hypersurfaces with flat r-mean curvature.

Keywords: flat r-mean curvature, Ribaucour transformations.

2000 Mathematics Subject Classification: 53A07, 53C42, 37K35.

1 Introduction

Euler, in 1760, introduced one of the fundamental concepts in differential geometry, namely the concept of principal curvatures of a surface contained in the euclidean space R^3. Later, in 1827, Gauss defined what is now called the Gaussian curvature of a surface and he showed, that in fact, his curvature corresponds exactly to the product of the principal curvatures introduced by Euler.

Nowadays, for each integer, $1 \leq r \leq n$, one defines the r-mean curvature of a hypersurface M^n, contained in a space form $\bar{M}(\bar{K})^{n+1}$, of constant curvature $\bar{K}$, by

$$H_r = \frac{1}{\binom{n}{r}} \sum_{1 \leq i_1 < \ldots < i_r \leq n} \lambda^{i_1} \ldots \lambda^{i_r}$$

where $\lambda^1, ..., \lambda^n$ are the principal curvatures of M. H_1 is called simply the mean curvature, H_2 the scalar curvature and H_n the Gauss-Kronecker curvature respectively. A hypersurface whose mean curvature vanishes identically is said to be minimal. Reilly, 1973, showed that, in a variational context, the flat r-mean curvature hypersurfaces are natural generalizations of the minimal hypersurfaces.

The theory of n-dimensional hypersurfaces of the Euclidean space with vanishing r-mean curvature, H_r, has been a very active topic of research, in the last few years. In view of the theoretical advances, there has been a growing interest in obtaining nontrivial, explicit examples of such hypersurfaces, specially for $r < n$. The main difficulty lies in the complexity of the differential equation $H_r = 0$, for hypersurfaces given as graphs. Hounie and Leite, 1995, showed that the linearization of the differential equation $H_r = 0$, $1 \leq r \leq n-1$, is elliptic if $H_{r+1} \neq 0$.

On the other hand, Ribaucour transformations for hypersurfaces were classically studied by Bianchi (Bianchi 1918, 1919, 1927). They can be used to provide families of surfaces of constant Gaussian curvature from a given such surface. Similarly, by using Ribaucour transformations one may obtain minimal surfaces from a given such surface. The first applications of this method for minimal surfaces were obtained recently, when Ribaucour transformations were shown to be useful to provide new complete minimal surfaces (Corro, Ferreira and Tenenblat, 2003(1)), (Lemes and Tenenblat, 2005), constant mean curvature and also linear Weingarten surfaces (Corro, Ferreira and Tenenblat, 2003(2)), (Tenenblat, 2005), (Tenenblat and Wang, 2006).

In this article, we present a survey on recent results, contained in (Corro, Ferreira and Tenenblat, 2005) and (Ferreira and Tenenblat), based on the use of Ribaucour transformations for producing flat r-mean curvature hypersurfaces. For details and proofs of the results mentioned in this paper we refer the reader to these articles. We provide an alternative differential equation whose solutions produce the hypersurfaces, contained in a space form, obtained by this procedure. Moreover, using Ribaucour transformations, we provide a method of obtaining families of hypersurfaces with flat Gauss-Kronecker curvature, from a given such hypersurfaces. By applying the theory we construct several families of such hypersurfaces.

In the euclidean space R^3, the only complete surfaces with flat Gaussian curvature are the planes and the cylinders over plane curves (Hartman and Niremberg, 1959). In general, complete hypersurfaces with flat Gauss-Kronecker curvature are not necessarily cylinders. Such an example was given by Sacksteder, 1960. Results on complete minimal hypersurfaces in S^4 with vanishing Gauss-Kronecker curvature were obtained in (Almeida and Brito, 1987), (Lusala, 2004) and (Ramanathan, 1990).

The classification of rotational hypersurfaces with flat r-mean curvature can be found in (Leite, 1990) and (Palmas, 1999). Minimal hypersurfaces of the Euclidean R^{2m} space that are invariant under the action of the group $O(m) \times O(m)$ were considered in (Alencar, 1993). A classification of the hypersurfaces invariant by the action of this group which have zero scalar curvature can be found in (Sato, 2000) and (Palmas, 2000).

In section 2, we consider the basic facts of the theory of Ribaucour transformations that will be used in the following sections. In section 3, by considering an integrable system of differential equations on M, we provide a method of construction of families of hypersurfaces $\tilde{M}^n$, locally associated by Ribaucour transformations to M, such that $\tilde{M}$ has also flat Gauss-Kronecker curvature.

In section 4, we present the general differential equation whose solutions provide hypersurfaces with vanishing r-mean curvature in a space form, obtained by a Ribaucour transformation applied to a totally geodesic hypersurface. In section 5, we present several applications of the preceding sections.

2 Ribaucour transformations

The classical definition of a Ribaucour transformation considers two surfaces M^2, $\tilde{M}^2 \subset R^3$ and a diffeomorphism $\psi : M \to \tilde{M}$, such that: the normal lines at corresponding points intersect at an equidistant point, the set of intersection points is a surface and ψ preserves lines of curvature. The classical theory includes the case of hypersurfaces M^n and $\tilde{M}^n \subset \bar{M}^{n+1}(\bar{K})$ parametrized by lines of curvature, where the principal curvatures have multiplicity one (Bianchi, 1918,1919). When the principal curvatures of M have multiplicity > 1, one needs a more precise definition, introduced in (Corro and Tenenblat, 2004).

In what follows $\bar{M}(\bar{K})^{n+1}$ will be a space form of constant sectional curvature $\bar{K} \in \{-1, 0, 1\}$, i.e.

$$\bar{M}(\bar{K})^{n+1} = \begin{cases} S^{n+1} \subset R^{n+2} & \text{if } \bar{K} = 1, \\ R^{n+1} \text{ se } \bar{K} = 0, \\ H^{n+1} \subset L^{n+2} & \text{if } \bar{K} = -1, \end{cases}$$

where L^{n+2} is the Lorentzian space and H^{n+1} is given by $x = (x_1, ..., x_{n+2}) \in L^{n+2}$ such that $-x_1^2 + \sum_{j=2}^{n+2} x_j^2 = -1$ and $x_1 > 0$.

Definition 2.1. Let M^n and $\tilde{M}^n$ be an orientable hypersurface of a space form $\bar{M}^{n+1}(\bar{K})$, of curvature $\bar{K}$, with unit normal vector fields N and $\tilde{N}$. Assume there exist, n orthonormal principal vector fields e_i on M. We say that $\tilde{M}$ *is associated to* M *by a Ribaucour transformation with respect to* $\{e_i\}$ if $\exists$ a diffeomorphism $\psi : M \to \tilde{M}$, a differentiable function $l : M \to R$ such that:

a) $exp_q l(q)N(q) = exp_{\psi(q)} l(q)\tilde{N}(\psi(q))$, $\forall q \in M$;

b) the subset $S = \{exp_q l(q)N(q),\ q \in M\}$ is n-dimensional;

c) $d\psi(e_i)$ are orthogonal principal directions on $\tilde{M}$.

This transformation is invertible in the sense that there exist principal direction vector fields $\tilde{e}_i$ on $\tilde{M}$, such that M is associated to $\tilde{M}$ by a Ribaucour tramsformation w.r.t $\{\tilde{e}_i\}$. One can also consider $\tilde{M}^n$ locally associated to M by Ribaucour transformation with respect to orthonormal principal vector fields globally defined on M. The definition can be extended to immersions. In general one has an immersion locally associated by Ribaucour transformation to a given one, even when both manifolds are complete.

Let M be a hypersurface of $\bar{M}^{n+1}(\bar{K})$. Let e_i, $1 \leq i \leq n$, be an orthonormal frame of principal directions on M and let N a unit vector filed normal to M. We denote by ω_i the dual forms and by ω_{ij}, $1 \leq i, j \leq n$, the connection forms determined by

$$d\omega_i = \sum_{j \neq i} \omega_j \wedge \omega_{ji}, \qquad \omega_{ij} + \omega_{ji} = 0.$$

Let $\omega_{in+1} =< de_i, N >$ the normal connection forms. Since e_i, $1 \leq i \leq n$, is a set of principal directions we have $dN(e_i) = \lambda^i e_i$, and $\omega_{in+1} = -\lambda^i \omega_i$.

One can show that a characterization, for the hypersurfaces which are locally associated to a given hypersurface by Ribaucour transformations, is determined by means of a system of nonlinear differential equations for a function $h : M \to R$, where

$$h = \begin{cases} \tan \ell & \text{if} \quad \bar{K} = 1, \\ \ell & \text{if} \quad \bar{K} = 0, \\ \tanh \ell & \text{if} \quad \bar{K} = -1, \end{cases}$$

and ℓ is the function of the definition of a Ribaucour transformation. We observe that condition a) of the definition is equivalent to saying that

$$\tilde{X} = X + h(N - \tilde{N}),$$

where X and $\tilde{X}$ are local parametrizations of M and $\tilde{M}$.

The nonlinear system of equations for h can be linearized by defining $h = \frac{\Omega}{W}$ and $\Omega^i = d\Omega(e_i)$. Then one can prove the following result (see (Corro, Ferreira and Tenenblat, 1999), (Corro and Tenenblat, 2004), (Tenenblat and Wang, 2006).

Theorem 2.1. *Let M^n be an orientable hypersurface of $\bar{M}^{n+1}(\bar{K})$ and let e_i, $1 \leq i \leq n$ be orthonormal principal vector fields on M, N a unit normal vector field and $dN(e_i) = \lambda^i e_i$. Then $\tilde{M}^n$ is locally associated to M by a Ribaucour transformation w.r.t $\{e_i\}$ if, and only if, $\forall \tilde{q} \in \tilde{M}$, $\exists$ differentiable functions W, Ω, Ω^i, which satisfy*

$$d\Omega^i(e_j) = \sum_{k=1}^{n} \Omega^k \omega_{ik}(e_j), \quad \text{for } i \neq j, \tag{2.1}$$

$$d\Omega = \sum_{i=1}^{n} \Omega^i \omega_i, \tag{2.2}$$

$$dW = -\sum_{i=1}^{n} \Omega^i \lambda^i \omega_i. \tag{2.3}$$

such that

$$WS(W + \lambda^i \Omega)(S - \Omega T_i) \neq 0, \quad 1 \leq i \leq n,$$

where

$$S = \sum_{j=1}^{n} (\Omega^j)^2 + W^2 + \bar{K}(\Omega)^2$$

and

$$T_i = 2\left(d\Omega^i(e_i) + \sum_k \Omega^k \omega_{ki}(e_i) - W\lambda^i + \bar{K}\Omega\right). \tag{2.4}$$

If M is parametrized by $X : U \subset R^n \to M$, then a parametrization of a neighborhood of $\tilde{q} \in \tilde{M}$ is given by

$$\tilde{X} = (1 - \frac{2\bar{K}(\Omega)^2}{S})X - \frac{2\Omega}{S}\left(\sum_{i=1}^{n} \Omega^i e_i - WN\right) \tag{2.5}$$

and the principal curvatures for $\tilde{M}$ are given by

$$\tilde{\lambda}^i = \frac{WT_i + \lambda^i S}{S - \Omega T_i}. \tag{2.6}$$

If M admits a parametrization $X(u_1, u_2, ..., u_n)$ by orthogonal lines of curvature, then $e_i = X_{u_i}/a_i$, where $a_i = |X_{u_i}|$, are principal directions. The dual forms are $\omega_i = a_i du_i$ and the connection forms are $\omega_{ij} = \frac{1}{a_i}\frac{\partial a_j}{\partial u_i}du_j - \frac{1}{a_j}\frac{\partial a_i}{\partial u_j}du_i$. Therefore, in this case, the system (2.1)-(2.3) reduces, in the classical notation, to

$$\frac{\partial \Omega^i}{\partial u_j} = \frac{1}{a_i}\frac{\partial a_j}{\partial u_i}\Omega^j, \quad i \neq j, \tag{2.7}$$

$$\frac{\partial \Omega}{\partial u_i} = a_i \Omega^i, \tag{2.8}$$

$$\frac{\partial W}{\partial u_i} = -a_i \lambda^i \Omega^i. \tag{2.9}$$

In principle, Ribaucour transformations are local transformations determined by the solutions of the system of equations (2.1)-(2.3) (or (2.7)-(2.9)) defined on a simply connected domain. Even if the solution is globally defined on the universal covering of M, a point where $h = \Omega/W$ tends to infinity may not correspond to any point on the associated hypersurface. For example this is how planar ends are produced on minimal surfaces by Ribaucour transformations (see Corro, Ferreira and Tenenblat, 2003(1)). One can also show that the parametrization $\tilde{X}$ given by (2.5) may extend regularly to points where $W(W + \lambda^i\Omega) = 0$, whenever $S(S - \Omega T_i) \neq 0$, since $|d\tilde{X}(e_i)|^2 = (S - \Omega T_i)^2/S^2$.

The following result asserts that for any orientable hypersurface M^n of the space form $\bar{M}^{n+1}(\bar{K})$, there is an open subset of a totally umbilic submanifold of $\bar{M}$ which is locally associated to M by a Ribaucour transformation.

Theorem 2.2. *(Corro and Tenenblat, 2004; Tenenblat and Wang, 2006) Let $M^n \subset \bar{M}^{n+1}(\bar{K})$ be an orientable hypersurface that admits n orthonormal principal vector fields e_i. For any real constants $b \neq 0$ and b_0, the system of equations*

$$d\Omega^i = \sum_k \Omega^k \omega_{ik} + (b_0 - \bar{K}\Omega)\omega_i + (b - W)\omega_{in+1}, \tag{2.10}$$

$$d\Omega = \sum_{i=1}^n \Omega^i \omega_i, \tag{2.11}$$

$$dW = \sum_{i=1}^n \Omega^i \omega_{in+1} \tag{2.12}$$

is integrable. For any such solution, the function $S - 2(b_0\Omega + bW) = c$ is a constant determined by the initial conditions. Considering $c = 0$, the associated hypersurface $\tilde{M}$ is an open subset of a totally umbilic submanifold of $\bar{M}$. In particular, if $b_0 = 0$ and $c = 0$ then $\tilde{M}$ is totally geodesic.

Since Ribaucour transformations are in a sense invertible, as a consequence of the above theorem we have the following corollary, that will be used in Sections 3 and 4.

Corollary 2.3. *Any orientable hypersurface of $\bar{M}^{n+1}(\bar{K})$ is locally associated to a totally geodesic hypersurface M by a Ribaucour transformation, with respect to some frame of orthonormal principal vector fields of M.*

If $M^n \subset \bar{M}^{n+1}(\bar{K})$ is an orientable hypersurface that admit a parametrization $X(u_1, u_2, ..., u_n)$ by orthogonal lines of curvature then the system of equations (2.10)-(2.12) reduces to

$$\begin{aligned}
\frac{\partial \Omega^i}{\partial u_j} &= \frac{1}{a_i}\frac{\partial a_j}{\partial u_i}\Omega^j, \quad i \neq j, \\
\frac{\partial \Omega}{\partial u_i} &= a_i \Omega^i, \\
\frac{\partial W}{\partial u_i} &= -a_i \lambda^i \Omega^i, \\
\frac{\partial \Omega^i}{\partial u_i} &= -\sum_{j\neq i}^{n} \frac{1}{a_j}\frac{\partial a_i}{\partial u_j}\Omega^j + a_i\lambda_i(W-b) + a_i(b_0 - \bar{K}\Omega).
\end{aligned}$$

3 Ribaucour Transformations for hypersurfaces with flat Gauss-Kronecker curvature

For detailed proofs of the results stated in this section, we refer to (Corro, Ferreira and Tenenblat, 2005). We begin by observing that, as an immediate consequence of Theorem 2.1, if M is a hypersurface of $\bar{M}^{n+1}(\bar{K})$ with flat Gauss-Kronecker curvature and $\{e_i\}$, $1 \leq i \leq n$, is an orthonormal frame of principal directions on M, such that $\lambda^{i_0} = 0$, then for any solution of the system (2.1)-(2.3), satisfying $T^{i_0} = 0$ (T^i defined by (2.4)), the hypersurface $\tilde{M}$, locally associated to M as in Theorem 2.1, has also flat Gauss-Kronecker curvature. See Propositions 5.9 - 5.10 for families of such hypersurfaces, obtained by this procedure.

In this section we consider solutions of (2.1)-(2.3), such that T^i is a multiple of λ^i, say $T^i = -2b\lambda^i$, $b \in R$. We observe that this condition together with (2.1) is equivalent to requiring

$$d\Omega_i = \sum_k \Omega_k \omega_{ik} + (b - W)\omega_{in+1} - \bar{K}\Omega\omega_i.$$

Hence any such solution satisfies $S - 2bW = 2c \in R$ and it determines a hypersurface $\tilde{M}$ locally associated to M by a Ribaucour transformation with respect to $\{e_i\}$. The regularity condition requires that $S(S - \Omega T^i) \neq 0$. From (2.6) the principal curvatures of the associated hypersurfaces are given by

$$\tilde{\lambda}^i = \frac{c\lambda^i}{b(W + \lambda^i\Omega) + c}.$$

In our next result, we obtain all hypersurfaces $\tilde{M}$ associated to a given hypersurface $M^n \subset \bar{M}^{n+1}(\bar{K})$ as in Theorem 2.2 with $b_0 = 0$. We observe that for $\bar{K} = \pm 1$, we consider the unit sphere as a subset of R^{n+2} and the hyperbolic space as a subset of the Lorentzian space. Hence, $<,>$ will denote the usual metric on R^{n+1} or R^{n+2} if $\bar{K} = 0$ or 1 and it will denote the Lorentzian metric on R^{n+2} if $\bar{K} = -1$. Moreover, we will denote $||Y||^2 = <Y, Y>$.

Theorem 3.1. *(Corro, Ferreira and Tenenblat, 2005) Let $X : M^n \to \bar{M}^{n+1}(\bar{K})$ be a parametrized hypersurface. Then any hypersurface $\tilde{M}$, locally associated to M by a Ribaucour transformation, with respect to any orthonormal frame of principal directions $\{e_i\}$ on M, as in Theorem 2.2, with $b_0 = 0$, is given by*

$$\tilde{X}_{brV} = X - \frac{2(<V, X> + r)}{||V - bN||^2}(V - bN), \tag{3.1}$$

where N is a unit vector field normal to M, $b, r \in R$, $\bar{K}r = 0$ and V is a vector of R^{n+1} (resp. $V \in R^{n+2}$) if $\bar{K} = 0$, (resp. $\bar{K} = \pm 1$), are such that $(||V||^2 - b^2)(||V - bN||) \neq 0$.

We conclude this section by providing a geometric interpretation of the family of hypersurfaces described by (3.1). For fixed b, r, $\in R$ such that $\bar{K}r = 0$ and V_1 a unit vector, consider the set of hypersurfaces in $\bar{M}(\bar{K})$ given by

$$Y^t_{brV_1} = X - \frac{2(t < V_1, X > +(1-t)r)}{||tV_1 - (1-t)bN||^2}[tV_1 - (1-t)bN]$$

where $t \in R$. The family $Y^t_{brV_1}$ contains the parallel surface ($t = 0$) and the reflection of X with respect to a hyperplane orthogonal to V_1 passing through the origin ($t = 1$). This family is associated to the solution of (2.1)-(2.3) given by

$$\begin{aligned} \Omega^t_i &= t < V_1, e_i >, \\ \Omega^t &= t < V_1, X > +(1-t)r, \\ W^t &= -t < V_1, N > +(1-t)b. \end{aligned}$$

It is easy to see that the family $\tilde{X}_{brV}$ given by (3.1) coincides with $Y^t_{brV_1}$.

4 Hypersurfaces with flat r-mean curvature obtained by Ribaucour transformations

In this section, we present differential equations whose solutions provide hypersurfaces with flat r-mean curvature in space forms. Corollary 2.3 asserts that any orientable hypersurface of $\bar{M}^{n+1}(\bar{K})$ is locally associated to a totally geodesic hypersurface of $\bar{M}$, by a Ribaucour transformation. In particular, flat r-mean curvature hypersurfaces are locally associated to a totally geodesic hypersurface, with respect to an apropriate orthonormal frame. Complete proofs of the results stated in this section will appear in (Ferreira and Tenenblat).

Let V be an open subset of a totally geodesic hypersurface of $\bar{M}$. Then we may consider V contained on the hypersurface $x_{n+1} = 0$ when $\bar{K} = 0$ and $x_{n+2} = 0$ when $\bar{K} = \pm 1$. Therefore, we may consider $N = (0, ..., 0, 1)$ to be normal to V and the totally geodesic submanifold V will be parametrized by $X : U \subset R^n \to V \subset \bar{M}^{n+1}(\bar{K})$. Moreover, we will consider functions $\Omega : V \to R$ and we will denote by $\nabla_g \Omega$ and $Hess_g \Omega$ the gradient and the Hessian of Ω on the metric g induced on the totally geodesic hypersurface V, by the standard metric of $\bar{M}^{n+1}(\bar{K})$.

Proposition 4.1. *Let $X : U \subset R^n \to \bar{M}^{n+1}(\bar{K})$ be a parametrization of a totally geodesic submanifold, orthogonal to $N = (0, ..., 0, 1)$. Let $e_1, \ldots, e_n$ be an orthonormal frame tangent to $X(U)$. Consider a differentiable function $\Omega : X(U) \to R$, $\Omega^i = d\Omega(e_i)$, $\alpha \neq 0$ a real constant and*

$$S = |\nabla \Omega|^2 + \bar{K}\Omega^2 + \alpha^2. \tag{4.1}$$

Then $\tilde{X}$ given by

$$\tilde{X} = (1 - \frac{2\bar{K}(\Omega)^2}{S})X - \frac{2\Omega}{S}\left(\sum_{i=1}^{n} \Omega^i e_i - \alpha N\right) \tag{4.2}$$

is locally associated to X by a Ribaucour transformation, with respect to e_i if, and only if, e_i diagonalize the matrix $Hess_g \Omega$ and

$$S \prod_{j=1}^{n} (S - 2\Omega(L_j + \bar{K}\Omega)) \neq 0, \quad \text{on } X(U), \tag{4.3}$$

where

$$L_i = d\Omega^i(e_i) + \sum_s \Omega^s \omega_{si}(e_i).$$

are the eigenvalues of $Hess_g\Omega$.

Given an $n \times n$ matrix $A = [a_{ij}]$ and a set I of indices, $1 \leq i_1 < ... < i_k \leq n$, we denote by A_I the $k \times k$ submatrix obtained from A by considering the indices in I. Moreover, we denote by $|I|$ the number k of indices. If A is a symmetric matrix, let $k_1, ..., k_n$ be the eigenvalues of A. We denote by

$$Q_0(A) = 1 \quad \text{and } Q_r(A) = \sum_{1 \leq i_1 < ... < i_r \leq n} k_{i_1} \dots k_{i_r} = \sum_{|I|=r} \det A_I,$$

the r-symmetric functions of the roots of the characteristic polynomial of A.

Our next result provides differential equations whose solutions characterize the hypersurfaces with flat r-mean curvature.

Theorem 4.2. *Let $X : U \subset R^n \to \bar{M}^{n+1}(\bar{K})$ be a totally geodesic immersion contained in the hyperplane orthogonal to $N = (0, ..., 0, 1)$. Let $\Omega : X(U) \to R$ be a differentiable function and let $e_i, 1 \leq i \leq n,$ be orthonormal vector fields tangent to $X(U)$, which diagonalize $Hess_g\Omega$. Consider $\Omega^i = d\Omega(e_i)$, $\alpha \neq 0$ a real constant and S defined by (4.1). Assume $\tilde{X}$, given by (4.2), is a hypersurface of $\bar{M}$ with flat r-mean curvature, then Ω satisfies*

$$\sum_{j=0}^{n-r} (-1)^j \, 2^j \begin{pmatrix} r+j \\ r \end{pmatrix} S^{n-r-j} \; (\Omega)^j \; Q_{j+r}(Hess_g\Omega + \bar{K}\Omega \, I) = 0, \tag{4.4}$$

where I is the identity matrix.

The following result shows that the converse of Theorem 4.2 holds.

Theorem 4.3. *For each fixed integer r, $1 \leq r \leq n$, and real constant $\alpha \neq 0$, consider the differential equation (4.4) for a function Ω defined on an open subset of a totally geodesic submanifold of $\bar{M}^{n+1}(\bar{K})$, orthogonal to $N = (0, ..., 0, 1)$, given by $X : U \subset R^n \to \bar{M}^{n+1}(\bar{K})$. Let Ω be a solution of this equation and let $L_j$and e_j be the eigenvalues and orthonormal eigenvectors of $Hess_g\Omega$, respectively. Then Ω determines a hypersurface of $\bar{M}^{n+1}(\bar{K})$, with flat r-mean curvature, given by (4.2) and defined on the subset where (4.3) holds.*

From Theorems 4.2 and 4.3 we have the following.

Corollary 4.4. *A hypersurface of $\bar{M}^{n+1}(\bar{K})$ has flat r-mean curvature if, and only if, it is locally determined by a solution Ω of the differential equation (4.4) defined on an open subset of a totally geodesic hypersurface of $\bar{M}^{n+1}(\bar{K})$, where (4.3) holds.*

In particular, minimal surfaces contained in R^3 are determined by solutions Ω of the differential equation

$$(|\nabla\Omega|^2 + \alpha^2)\Delta\Omega - 4\Omega \det(\text{Hess } \Omega) = 0,$$

while minimal hypersurfaces of R^4, are determined by the solutions of the differential equation

$$(|\nabla\Omega|^2 + \alpha^2)^2\Delta\Omega - 4S\Omega Q_2 + 12\Omega^2 \det(\text{Hess } \Omega) = 0.$$

One can also also see that the hypersurfaces of R^4 , with flat scalar curvature are determined by the solutions of the equation

$$(|\nabla\Omega|^2+\alpha^2)Q_2 - 6\Omega\det(\text{Hess }\Omega) = 0$$

where

$$Q_2 = \Omega_{u_1u_1}\Omega_{u_2u_2}+\Omega_{u_1u_1}\Omega_{u_3u_3}+\Omega_{u_2u_2}\Omega_{u_3u_3}-\Omega^2_{u_1u_2}-\Omega^2_{u_1u_3}-\Omega^2_{u_2u_3}.$$

5 Applications

In this section, by applying the theory of sections 3 and 4, we will produce families of flat r-mean curvature hypersurfaces. These families will be obtained by considering special solutions of the differential equation (4.4), restricted to the case in which $\bar{K}=0$, or by applications of Theorem 3.1 and the theory of section 3. For detailed proofs of the results in this section we refer to (Corro, Ferreira and Tenenblat, 2005) and (Ferreira and Tenenblat).

We start by rewriting Theorem 4.3 for the Euclidean case, in an equivalent form. From now on, for a differentiable function $\Omega: U\subset R^n \to R$, we will denote by

$$S = |\nabla\Omega|^2+\alpha^2; \quad \alpha\neq 0,\ \alpha\in R$$

and by L_j, $1\leq j\leq n$, the eigenvalues of $Hess\ \Omega$. Moreover, we will also need to consider domains for Ω, where

$$S-2\Omega L_j\neq 0, \quad \forall j. \tag{5.1}$$

We observe that if Ω is a generic solution of (4.4) defined on an open set U, then there exists an open subset $U_0\subset U$, where (5.1) holds at each point of U_0, since the relation $\Pi^n_{j=1}(S-2\Omega L_j)=0$, defines generically a hypersurface of U.

Theorem 5.1. *Let $\Omega(u_1,...,u_n)$ be a real differentiable function such that (5.1) holds on $U\subset R^n$. For a fixed integer r, $1\leq r\leq n$, and a real number $\alpha\neq 0$, assume Ω is a solution of the differentiable equation*

$$\sum_{1\leq i_1<...<i_r\leq n}\frac{L_{i_1}}{(S-2\Omega L_{i_1})}\cdots\frac{L_{i_r}}{(S-2\Omega L_{i_r})}=0. \tag{5.2}$$

Then Ω determines a hypersurface in R^{n+1}, with flat r-mean curvature, given by

$$\tilde{X}(u_1,...,u_n) = (u_1,...,u_n,\frac{2\alpha\Omega}{S}) - \frac{2\Omega}{S}\nabla\Omega, \tag{5.3}$$

where $\{e_i\}$ are orthonormal eigenvectors of $Hess\ \Omega$, considered in R^{n+1} and $\Omega^i = d\Omega(e_i)$.

We remark that we have replaced equation (4.4), with the equivalent equation (5.2). This equation is more convenient when there are eigenvalues of $Hess\ \Omega$, with multiplicity bigger than one.

A simple application of Theorem 5.1 gives the following result.

Theorem 5.2. *Let $\Omega = f(u_1,...u_s)$, with $1\leq s<n$, be a differentiable function defined on a domain where (5.1) holds. Then Ω is a solution of (5.2) for each r, such that $r\geq s+1$. The hypersurface $\tilde{X}$ locally defined by (5.3) is a cylinder given by*

$$\tilde{X}(u_1,...,u_n) = \left(u_1-\frac{2ff_{u_1}}{|\nabla f|^2+\alpha^2}, u_s-\frac{2ff_{u_s}}{|\nabla f|^2+\alpha^2}, u_{s+1},...,u_n,\frac{2\alpha f}{|\nabla f|^2+\alpha^2}\right)$$

which has flat r-mean curvature and $(n-s)$ flat principal curvatures.

Our next result shows that solutions of (5.2), with $r = n$ (resp. $r = n - 1$), such that Ω is a sum of functions of separated variables, provide hypersurfaces with flat Gauss-Kronecker curvature (resp. flat $(n-1)$-mean curvature) and one flat principal curvature (resp. two flat principal curvatures).

Proposition 5.3. *Let $\Omega = \sum_{i=1}^{n} f_i(u_i)$, be a differentiable function, such that $f_i' \neq 0$ and (5.1) holds on an open subset of R^n. Then Ω is a solution of (5.2), with $r = n$ (resp. $r = n - 1$), $n \geq 2$ (resp. $n \geq 3$) if, and only if, at least one function f_i is linear (resp. two functions f_i are linear). In this case, w.l.o.g. considering $f_1(u_1)$ (resp. $f_1(u_1)$ and $f_2(u_2)$) to be linear and $f_i(u_i)$, $i \geq 2$(resp.$i > 2$) to be arbitrary differentiable functions, then the hypersurface defined by (5.3) is given by*

$$\tilde{X} = \left(u_1 - \frac{2\sum f_i}{\sum (f_i')^2 + \alpha^2} f_1', \dots, u_n - \frac{2\sum f_i}{\sum (f_i')^2 + \alpha^2} f_n', \frac{2\alpha \sum f_i}{\sum (f_i')^2 + \alpha^2} \right)$$

The hypersurface $\tilde{X}$ has flat Gauss-Kronecker curvature (resp. flat $(n-1)$-mean curvature) and at least one flat principal curvature (resp. two flat principal curvatures).

As an application of Theorem 3.1, we obtain a 6-parameter family of hypersurfaces with flat Gauss-Kronecker curvature contained in R^4, which are associated to a hypersurface given by Sacksteder, 1960, which has flat Gauss-Kronecker curvature. Generically, a hypersurface of this family will have singularities. However, we can show that the family contains a 1-parameter family of complete, non-cylindrical hypersurfaces (see Corro, Ferreira and Tenenblat, 2005).

Proposition 5.4. *Consider the hypersurface of R^4 defined by $X(x,y,z) = (x,\, y,\, z,\, f(x,y,z))$, where $f = x \cos z + y \sin z$, and its Guass map $N = (\cos z, \sin z, f_z, -1)/\sqrt{2 + f_z^2}$.*

i) *For any vector V of R^4 and real numbers b, r such that $|V|^2 - b^2 \neq 0$,*

$$\tilde{X}_{brV} = X - \frac{2(< V, X > + r)}{|V - bN|^2} (V - bN), \tag{5.4}$$

is a hypersurface with flat Gauss-Kronecker curvature, which is locally associated to X by a Ribaucour transformation.

ii) *If $r = 0$ and $V = (0,0,0,\varepsilon)$, where $\varepsilon = \pm 1$, then for any constant b such that $\varepsilon b < 0$ and $b^2 + 2\sqrt{2}\varepsilon b + 1 > 0$, $\tilde{X}_b$, defined by (5.4), is a complete, non-cylindrical hypersurface, not congruent to X, with flat Gauss-Kronecker curvature.*

In our next result, by considering solutions of (5.2) of the form $\Omega = f(y,z)$, where $y = u_1^2 + \dots + u_s^2$, and $z = u_{s+1}^2 + \dots + u_n^2$, we obtain hypersurfaces with flat r-mean curvature that are foliated by product of spheres $S^{s-1}(r_1(y,z)) \times S^{n-s-1}(r_2(y,z))$ and invariant by $O(s) \times O(n-s)$ (see Ferreira and Tenenblat).

Proposition 5.5. *Let $\Omega = f(y,z)$, where $y = u_1^2 + \dots + u_s^2$, $z = u_{s+1}^2 + \dots + u_n^2$, $n \geq 4$, $2 \leq s \leq n-2$ be a function such that (5.1) holds on an open subset $U \subset R_+^2$. For a fixed integer r, $1 \leq r \leq n$, Ω is a solution of (5.2) if, and only if, f satisfies*

$$Q_r\left(\frac{2f_y}{S - 4ff_y}, \frac{2f_z}{S - 4ff_z}, \frac{L_+}{S - 2fL_+}, \frac{L_-}{S - 2fL_-}\right) = 0,$$

where $S = 4(yf_y^2 + zf_z^2) + \alpha^2$, $L_\pm$ *are the solutions of the equation*

$$L^2 - (A_1 + A_2)L + P = 0,$$

where

$$A_1 = 2(f_y + 2yf_{yy}) \quad A_2 = 2(f_z + 2zf_{zz}) \text{ and } P = A_1A_2 - 16yzf_{yz}^2.$$

For such a function f, *there is a hypersurface in* R^{n+1} *with flat* r*-mean curvature given by*

$$\tilde{X}(u_1, ..., u_n) = (u_1, ..., u_n, \frac{2\alpha f}{S}) - \frac{2f}{S}\nabla\Omega$$

Moreover, $\tilde{X}$ *is foliated by product of spheres* $S^{s-1}(r_1(y,z)) \times S^{n-s-1}(r_2(y,z))$.

In particular for $n = 4$ and $s = 2$ the hypersurfaces with $H_3 = 0$ are determined by solutions $\Omega = f(y,z)$ of the differential equation

$$S(P(f_y + f_z) + 2(A_1 + A_2)f_yf_z) - 16ff_yf_zP = 0$$

An application of Theorems 2.1, 2.2 and 3.1 provides a 5-parameter family of compact Dupin hypersurfaces in the unit sphere S^4, whose Gauss-Kronecker curvature vanishes. Whenever $\lambda^i, \forall i, 1 \leq i \leq n$, are constant along the integral curves of e_i, i.e. $d\lambda^i(e_i) = 0$, M is said to be a *Dupin hypersurface*.

We start by considering the Veronese surface described by $X : S^2_{\sqrt{3}} \to S^4 \subset R^5$,

$$X(x,y,z) = \frac{1}{\sqrt{3}}(xy,\ xz,\ yz,\ \frac{x^2 - y^2}{2},\ \frac{\sqrt{3}}{2}(1 - z^2)),$$

where $S^2_{\sqrt{3}} \subset R^3$ is the sphere of radius $\sqrt{3}$. We denote by $T_1X^\perp$ the unit normal bundle of X, i.e.

$$T_1X^\perp = \{(p,\xi);\ p \in S^2_{\sqrt{3}},\ \xi \in (TpX)^\perp \subset T_pS^4 \text{ and } |\xi| = 1\}.$$

The tube of geodesic ray $R = \pi/2$, around X is the hypersurface $Y : T_1X^\perp \to S^4$ given by

$$Y(p,\xi) = exp_{X(p)}(\frac{\pi}{2}\xi) = \xi.$$

A vector field normal to Y (tangent to S^4 along Y) $N : T_1X^\perp \to S^4 \subset R^5$ is given by $N(p,\xi) = X(p)$. One can show (see Almeida and Brito 1987) that Y is an isoparametric minimal hypersurface in S^4 whose principal curvatures are $\lambda^1 = 0$, $\lambda^2 = \sqrt{3}$, $\lambda^3 = -\sqrt{3}$.

Proposition 5.6. *Let* Y *be the tube of geodesic ray* $\pi/2$ *around the Veronese surface* X. *Then the map* $\tilde{Y}_{bV} : T_1X^\perp \to S^4 \subset R^5$ *given by*

$$\tilde{Y}_{bV} = Y - \frac{2 < V, Y >}{|V - bX|^2}(V - bX)$$

is a regular, compact, Dupin hypersurface of S^4, *with flat Gauss-Kronecker curvature, locally associated to* Y *by a Ribaucour transformation,* $\forall b \in R$ *and any unit vector* $V \in R^5$ *such that*

$$b^2 + 1 - 2|b|(1 + \sqrt{3}) > 0.$$

We observe that each hypersurface $\tilde{Y}_{bV}$ is a tube of geodesic ray $\pi/2$ over the image of the its Gaussian normal map $\tilde{N}_{bV} : T_1X^{\perp} \rightarrow S^4$ given by

$$\tilde{N}_{bV} = X + \frac{2(- < V, X > +b)}{|V - bX|^2}(V - bX).$$

By considering radial solutions of (5.2), we obtain rotational hypersurfaces with flat r-mean curvature.

Proposition 5.7. *Let $\Omega = f(t)$, $t = \sum_{i=1}^{n} u_i^2$, be a radial function which is not linear. For a fixed integer r, $1 \leq r \leq n-1$, the function Ω is a solution of (5.2) if, and only if, f satisfies*

$$2t(\frac{r}{n} - 2C(t)f')f'' + (1 - 2C(t)f')f' = 0 \tag{5.5}$$

where

$$C(t) = \frac{2f}{4t(f')^2 + \alpha^2}, \qquad \alpha \neq 0, \alpha \in R.$$

If f is a solution of (5.5), then the immersion $\tilde{X}$, defined by (5.3), is a rotational hypersurface of R^{n+1}, with flat r-mean curvature, given by

$$\tilde{X}(u_1, ..., u_n) = (1 - 2C(t)f')(u_1, ..., u_n, 0) + (0, ..., 0, \alpha C(t)).$$

If we consider solutions of (5.2) of the form $\Omega = f(x, y)$, where $x = u_1$, and $y = u_2^2 + ... + u_n^2$, $n \geq 3$, we obtain hypersurfaces with flat r-mean curvature that are foliated by $n-2$ dimensional spheres, centered on the x_{n+1} axis.

Proposition 5.8. *Let $\Omega = f(x, y)$, where $x = u_1$ and $y = u_2^2 + ... + u_n^2$, $n \geq 3$, be a differentiable function such that (5.1) holds on an open subset $U \subset \{(x, y) \in R^2 \; ; \; y > 0\}$. For a fixed integer r, $1 \leq r \leq n-1$, and a real number $\alpha \neq 0$, Ω is a solution of (5.2) if, and only if, f satisfies*

$$2(n-r)f_y[2(n-r-1)(S^2 - 2fSA + 4f^2P)f_y + r(S - 4ff_y)(SA - 4fP)] + r(r-1)P(S - 4ff_y)^2 = 0,$$

where

$$S = f_x^2 + 4yf_y^2 + \alpha^2$$
$$A = 2f_y + f_{xx} + 4yf_{yy}, \qquad P = 4y(f_{xx}f_{yy} - f_{xy}^2) + 2f_yf_{xx}.$$

For such a function f, there is a hypersurface in R^{n+1} with flat r-mean curvature given by

$$\tilde{X}(u_1, ..., u_n) = \left(u_1, ..., u_n, \frac{2\alpha f}{S}\right) - \frac{2f}{S}\nabla\Omega.$$

Moreover, $\tilde{X}$ is foliated by $(n-2)$ dimensional spheres, centered on the x_{n+1} axis.

As a consequence of the basic theory on Ribaucour transformations (Theorem 2.1), we get the following results.

Proposition 5.9. *Let $X = (\cos u_1, \sin u_1, u_2, ... u_n)$ be a parametrized cylinder in R^{n+1} and let $e_i = X_{u_i}$, $1 \leq i \leq n$. Consider arbitrary differentiable functions $f_i(u_i)$ of u_i such that for some $i_0 \geq 2$, $f_{i_0} = au_{i_0} + b$, $a, b \in R$ and $\gamma, \alpha \in R$. Then*

$$\tilde{X} = X - \frac{2(\sum_{i=1}^{n} f_i + \gamma)}{\sum_{i=1}^{n}(f_i')^2 + (f_1 - \alpha)^2}(-f_1' \sin x_1 - (\alpha - f_1)\cos x_1, f_1' \cos x_1 - (\alpha - f_1)\sin x_1, f_2', ..., f_n')$$

provides a family of hypersurfaces with flat Gauss-Kronecker curvature in R^{n+1}, locally associated to the cylinder by a Ribaucour transformation with respect to $\{e_i\}$.

Proposition 5.10. *Consider a parametrization of the hyperbolic space H^3, as a hypersurface of H^4, contained in the Lorentzian space L^5, given by*

$$X = (\cosh u_3, 0, 0, 0, 0) + \sinh u_3 (0, \cos u_2 \cos u_1, \cos u_2 \sin u_1, \sin u_2, 0),$$

where $-\pi/2 < u_2 < \pi/2$ and $u_3 > 0$. Let $e_i = X_{u_i}/|X_{u_i}|$, $i = 1, 2, 3$ and let $N = (0, 0, 0, 0, 1)$ be the normal map. Then the hypersurfaces of H^4, locally associated to X by a Ribaucour transformation with respect to e_i, are given by

$$\tilde{X} = X - \frac{2\Omega}{S}\left(\sum_i \Omega_i e_i - WN - \Omega X\right),$$

where

$$S = \sum_i \Omega_i^2 + W^2 - \Omega^2, \quad \Omega_1 = f_1', \quad \Omega_2 = -f_1 \sin u_2 + f_2', \quad W = b \neq 0, \ b \in R,$$

$$\Omega_3 = (f_1 \cos u_2 + f_2) \cosh u_3 + f_3', \qquad \Omega = (f_1 \cos u_2 + f_2) \sinh u_3 + f_3,$$

and f_i is an arbitrary differentiable real function of u_i. Moreover, if

$$f_3 = c_1 \sinh u_3 + c_2 \cosh u_3, \quad c_1, c_2 \in R,$$

then $\tilde{X}$ has flat Gauss-Kronecker curvature.

Acknowledgment

This work was supported by CNPq and CNPq/PADCT.

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Leonhard Euler's methods and ideas live in the thermodynamic hierarchical theory of biological evolution

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Abstract

It is well known that Leonhard Euler's works were of outstanding importance for the creation of modern science. In this article, the role of the brilliant Euler in the development of classical (equilibrium) thermodynamics and the quasi-equilibrium thermodynamics of quasi-closed systems is considered. Some of Euler's important methods and ideas are a significant part of the mathematical basis of not only equilibrium thermodynamics but also modern hierarchical thermodynamics. To a certain approximation, Clausius and Gibbs' thermodynamics is applied to describing the evolution of living systems. This is possible due to the law of temporal hierarchies and the premise that the functions of state of living systems have real physical meaning at practically all hierarchical levels and at every moment of time. It is shown that the principle of substance stability – the thermodynamic feedback principle – is applicable to all biological systems. It boils down for different temporal hierarchies to the following: during the formation (self-assembly) of the most thermodynamically stable structures at the highest hierarchical level (j), e.g., the supramolecular level, in accordance with the second law, Nature spontaneously uses predominantly the (available for the given local part of the biological system) least thermodynamically stable structures belonging to a lower level, for example, the molecular level (j-1). The principle can be also applied to understructure hierarchical levels of any temporal hierarchy. There are also facts that corroborate the application of the principle to social temporal and structural hierarchies. The application of the principle to problems of sociology and politics is discussed.

Keywords. Euler's methods, hierarchical thermodynamic, biological evolution, law of temporal hierarchies, principle of substance stability

AMS Mathematical Subject Classification. 74A15, 80-06, 80A50

PACS: 05.70. -a, 64, 82.60. -s, 95.30.Tg

1. Introduction

Leonhard Euler, undoubtedly, is one of the greatest mathematicians of all time. His outstanding works have enriched not only mathematics but also many sciences: mechanics, elasticity theory, mathematical physics, optics, machine theory, ballistics, the marine sciences, music theory, economics, and many other branches of science [1-7].

Euler was not only an unexcelled genius of "calculations" he was also a pioneer of ideas in numerous fields of knowledge [1 -3, 7]; for example, he was the first to use the term "function" to describe an expression involving various arguments, i.e. $y = F(x)$.

In this article it is impossible to mention all of Euler's theorems and methods that are found in current use and application. Only some of these theorems and methods are used with his name in the literature. This leads to the fact that Euler's services have undeservedly receive little attention. Furthermore, in the new interdisciplinary sciences, his name is very often practically forgotten.

2. Leonhard Euler and thermodynamics

In this article I would like to pay attention only to the role of the ingenious Euler in the creation of classical (equilibrium) thermodynamics and the thermodynamic methods and models that are applied in the evolutionary thermodynamics of near-equilibrium processes [8-16].

On the basis of classical thermodynamics, the modern hierarchical thermodynamic theory of the biological evolution and aging of living beings has appeared and is being developed [16]. This theory is based upon the thermodynamic models of quasi-closed quasi-equilibrium systems. Although the above theory is an approximate (general) theory, it rests, as does classical thermodynamics, on the ideas and methods of Euler. These ideas and methods of Euler enter into the hierarchical thermodynamics of the evolution of living matter, as well as into the evolutionary thermodynamic model of the development of our universe, via the scientific works of J.L. Lagrange, J.W. Gibbs, and other creators of classical thermodynamics [14, 15].

In 1744, Euler first correctly formulated the mechanical principle of least action. He presented for the first time examples of the application of this principle. Hereinafter, this principle found development in the Hamiltonian principle, the Le Chatelier–Braun principle, the principles of quantum theory, and many other principles and rules. It is important to note that the Le Chatelier–Braun principle is a particular case of the general formulation of Gibbs, which was found earlier.

Developing the theory of ships, Euler made an invaluable contribution to the theory of stability. His work in this field laid the groundwork for the development by Lagrange of the system of analytical mechanics, on which Gibbs relied in creating his thermodynamic theory.

The deduction of fundamental integral equations of classical thermodynamics by means of integration of differential equations, as a rule, is carried out with reference to Euler's theorem of homogeneous functions. This strictly establishes that the additive thermodynamic value be a homogeneous function of the first degree with respect to the additive variables. For example, in this way, one can derive equations for the internal energy U, the enthalpy H, the Helmholtz function F, the Gibbs function G, and other functions of state.

The equation

$$G = \sum_{i=1}^{k} \mu_i \, n_i \ , \tag{2. 1}$$

where μ_i is the chemical potential of the ith substance (or ith component) and n_i is the mass of the ith component expressed in the number of moles, is of special interest since it shows that the Gibbs function (the Gibbs free energy, or the Gibbs energy) is the sum of the contributions of the different *i*th components [8, 10, 11, 13].

Euler is considered by rights as the creator of variational calculus. In 1744 he devoted one of his works to its fundamentals. After the enhancement of his method by Lagrange, Euler returned again to his ideas and explicated in an original way variational calculus [1]. These and subsequent works of Euler and Lagrange showed the ways of application of variational principles in the Gibbs thermodynamics and in works of other great scientists.

For thermodynamics, the approach referred to as Euler's criterion (test) is very important. This criterion allows one to determine whether a differential of a certain function Z is full (exact). The thermodynamic functions that have a full (exact) differential are the functions of state, i.e., the functions that have concrete numerical meanings when a system exists in a definite state.

A differential

$$dZ = M(x)dx,$$

a function of only one variable, is always full.

In order that a differential

$$dZ = M(x, y)dx + N(x, y)dy, \tag{2.2}$$

a function of two variables, be a full differential in a region R of the xy plane, it is necessary and sufficient that between M and N there exists the relation

$$(\partial M / \partial y)_x = (\partial N / \partial x)_y. \tag{2.3}$$

If condition (2.3) is not fulfilled, then the form in the right part of relation (2.2) is not a full differential. In this case, the function Z is a function of the process.

Euler's criterion (2.3) is an important instrument in the mathematical apparatus of thermodynamics.

It is interesting to note the likeness of Euler's relationship, also known as "Euler's formula", which relates the number of edges, vertices, and faces of a simply connected polyhedron, and a simple form of the Gibbs phase rule. This likeness demonstrates a certain commonality of the mathematical approaches to obtaining the above relationships.

I focused on only some aspects of Euler's works that, in my opinion, have an incontestable significance for phenomenological equilibrium and quasi-equilibrium thermodynamics. We can become acquainted with other results of the work of this ingenious creator that have a relation to thermodynamics by studying his works [1-7].

In concluding this section, I would like to remark that Euler's works were the starting point of various investigations of P.S. Laplace, Lagrange, G. Monge, A.M. Legendre, and C.F. Gauss, and subsequently A. Cauchy, J.W. Gibbs, M.V. Ostrogradsky, P.L. Chebyshev, and many other mathematicians, theoretical physicists, physical chemists, and scientists of other specialties [1]. Euler's ideas and methods have influenced practically all areas of modern science.

3. The second law of thermodynamics is one of general laws of nature

Classical Formulations: During the first half of the 19th century, the second law of thermodynamics, one of general laws of nature, was enunciated by some of the greatest names in science. Well-known formulations of this law are associated with the names of Sadi Carnot (1824), Rudolf Clausius (1850), and William Thomson (Lord Kelvin) (1851). Although the formulations themselves are different, mainly because of differences in phrasing, they may be considered equivalent. Many authors have attempted to change or improve the formulations as regards to their physical meaning, yet none have succeeded. The meaning and essence of these formulations has not been disproved to date [8, 10, 14]. New concepts, however, have extended the possible applications of the second law of thermodynamics to different sciences, especially chemistry and, as it turned out later, biology. This became possible mainly due to J.W. Gibbs' works performed in 1873–1878. To a certain approximation, the methodologies of Gibbs' thermodynamics [8, 10, 14] have been extended to date to all hierarchies of natural systems, which are generally open ones [14, 17-19]

The discovery of the law of temporal hierarchies, which may be considered a new general law of nature, has determined the extension of Gibbs's theory to living systems [14, 17-21]. This law [16, 17] makes it possible to apply thermodynamics, or more precisely the hierarchical thermodynamics of quasi-closed systems, to all hierarchies of the real world, particularly, living objects and biological systems, to quite a good approximation. The discovery of this law confirms the universality of classical thermodynamic methods, and the name of Josiah Willard Gibbs even more vividly symbolizes the future of science that confirms the validity of general laws of nature as applied to the evolution of all material systems at all organizational levels of our world.

Advances in classical thermodynamics as well as approximate thermodynamics, i.e., the quasi-equilibrium thermodynamics of quasi-closed systems, are described in a number of textbooks and monographs. They are certainly numerous; the reader is encouraged to refer to the works [1–17], which will be very useful for all beginning researchers.

Clausius' formulation of the second law of thermodynamics, also known as the Clausius principle, states that: *"a process that involves no changes except for the transfer of heat from a warmer body to a colder body is irreversible, i.e., heat cannot spontaneously pass from a colder body to a warmer one"* [10, 11]. Building on this principle, in 1865, Clausius introduced the concept of entropy (*S*), a function of state of a system (a function that has a full differential) according to the Clausius inequality:

$$dS \geq \delta Q / T, \tag{3.1}$$

where the equality sign pertains to reversible processes and the inequality (greater-than) sign, to irreversible ones. Expression (3.1) is suitable for a simple isolated system, which can exchange neither substance nor energy with the environment and whose internal energy (*U*) and volume (*V*) are constant. In such systems only the work of expansion or no work at all is performed. In this case, the second law of thermodynamics may be written as:

$$dS_{U,V} \geq 0 \tag{3.2}$$

Thus, the entropy of this system increases when irreversible processes occur, and it is a maximum in the state of thermodynamic equilibrium.

The second law of thermodynamics according to Thomson, i.e. Thomson's principle, states that: *"the process during which work is transformed into heat without any other changes in the system's state is irreversible."* This means that all heat withdrawn from a body cannot be entirely transformed into work unless the system is changed in other respects. This formulation is equivalent to the statement that a perpetual mobile of the second kind is impossible [10, 11, 13].

Similarly, Carnot's theorem is also equivalent to the impossibility of the perpetual mobile of the second kind. According to this theorem, no heat engine can have a higher efficiency than that of the Carnot cycle,

$$\eta = (T_1 - T_2)/T_1,$$

which is determined only by the temperatures of the heat reservoir and the cooling condenser; T_1 and T_2, respectively. Carnot's theorem lays the basis for the absolute temperature scale. Sometimes, the second law of thermodynamics is formulated as the well-known Caratheodory' s principle (1909).

In the kinetic theory of gases, the second law of thermodynamics is substantiated by Boltzmann's H theorem. Here, H is the Boltzmann H function (to be precise, functional) determined from the mean logarithm of the particle distribution function. The Boltzmann H function is proportional to the entropy of a perfect gas; hence, the physical meaning of entropy is revealed in statistical physics. Boltzmann demonstrated that entropy is related to the logarithm of thermodynamic probability (W):

$$S = k \ln W, \tag{3.3}$$

where k is the Boltzmann constant. Note that Boltzmann's substantiating the statistical basis of the second law of thermodynamics, as well as the statistical substantiation of phenomenological thermodynamics suggested by Gibbs, involves ideal models, e.g., a perfect gas. In the case of more complex systems [10, 11, 13, 14], where pronounced (especially, strong) interactions between particles (molecules) are observed, it is difficult to perform the calculations. Therefore, it is obvious that these models are unlikely to be effective when studying most natural systems (e.g., biological), i.e., systems that are far from corresponding to ideal or simple models.

The thermodynamics of nonequilibrium processes deals with the rate of increase in or, as it is sometimes called, production of entropy. Therefore, it is sometimes asserted that nonequilibrium thermodynamics provides "the quantitative characteristic of the second law of thermodynamics" [10]. In the given case, however, this statement is reasonable only when applied to transformations in simple isolated systems where all processes are close to equilibrium. Only in a system that is close to equilibrium can the differential of this function of state of the system (entropy) be considered to be a full one, to an acceptable approximation. However, all the aforesaid is usually underestimated; therefore, many works on nonequilibrium thermodynamics, especially the thermodynamics of systems that are far from equilibrium, remain a faint "future hope." Some of these works, I daresay, are mere "mathematically trimmed" fantasies useless for real life [14].

Comparison of classical and non-classical formulations of the second law: Historically, the formulations of the second law of thermodynamics were closely associated with the study of heat engines. This approach has been developed by physicists, mainly thermal physicists, and heat engineers. Another trend in the use of the second law of thermodynamics is related to the attempts of some mathematicians and physicists constructing ideal and simple models to explain many natural phenomena in statistical terms. However, since all interactions in real systems are near to impossible to take into account, there is but little hope that calculations in the framework of these models will successfully solve the problem. Hence, only the phenomenological thermodynamics of systems close to equilibrium, i.e. equilibrium or quasi-equilibrium thermodynamics, will likely ensure insight into many natural phenomena and subsequently make reliable quantitative predictions

The above formulations of the second law of thermodynamics are, in a sense, somewhat outside the realm of the chemistry of molecular and supramolecular systems. These formulations may seem to be even farther from biology, sociology, and other sciences that are mainly based on chemistry, both molecular chemistry per se and the chemistry of supramolecular structures, which we perceive as "chemistry around us". Therefore, it is not unexpected that a purely physical, rather than a physicochemical, approach to the origin of life, biological evolution, and aging of living organisms has lead to numerous misunderstandings—one might say, even to tragic errors—in life science. It should suffice to mention L. Boltzmann's, E. Schrödinger's, I. Prigogine's, and other researchers' fallacies accounted for by neglecting to some or another extent, Gibbs's works and underestimating the possibilities offered by thermodynamics [14].

Prigogine [14, 16, 17], for example, supposed that the phenomenon of life is hardly consistent with the second law of thermodynamics. He noted, *"During the last decades, an opinion has widely spread that there is the apparent contradiction between biological order and laws of physics—particularly the second law of thermodynamics"* (1980). Prigogine also emphasized that *"this contradiction cannot be removed as long as one tries to understand living systems by the methods of equilibrium thermodynamics"*.

In order to solve these "contradiction", Prigogine [14] developed the theory of dissipative structures, i.e., structures appearing in systems that are **far from equilibrium.** Later, it turned out that the theory did not allow overcoming the aforementioned "contradictions." In fact, it made the imbroglio even more intricate. It later became obvious that Prigogine's views do not agree with the second law of thermodynamics [19, 20]. This is so in many respects. Suffice it to say that, in the general case, the Prigogine entropy (S' or S_i) has no full differential. Therefore, his theory cannot be regarded as thermodynamic. This is a kinetic theory based on an "entropy" (Prigogine's entropy, S') which can be neither calculated nor measured.

As noted above, only in recent decades were the principles of hierarchical thermodynamics (macrothermodynamics) formulated. I have managed to extend Gibbs's methodology so that it might be used for creating the physical (physicochemical) theories of the origin of life, biological evolution, and aging of living organisms [16 -20].

As noted, the physical substantiation of the second law of thermodynamics deals with ideal processes and is based on the concept of statistical entropy. The nonequilibrium thermodynamics of systems that are far from equilibrium tries to study the changes in "kinetic entropy" (e.g., Prigogine's entropy S' or (S' or S_i), which, as mentioned above, has no full differential (even an approximate one) and cannot be calculated in principle! In addition, the approaches used in the nonequilibrium thermodynamics of systems far from equilibrium create difficulties related, e.g., to the notions on the thermodynamics of processes and the thermodynamics of systems.

One of the greatest merits of Gibbs and other renowned founders of classical thermodynamics is that they used the works by L. Euler, J.L. Lagrange and other outstanding mathematicians (specifically, the variation principles developed by them) as a basis for the concepts on the functions of state of the system other than entropy (which, like entropy, have full differentials). The functions of state permit determining the directions of spontaneous processes and estimating the extent of their advancement in individual thermodynamic systems identified in the real world. In other words, the evolution of systems themselves can now be studied, to a certain approximation, if certain natural (independent) variables are constant. The Gibbs function G can be used for studying equilibrium (quasi-equilibrium) processes and closed systems (the quasi-closed systems in which quasi-equilibrium transformations occur) at constant temperature and pressure. The Helmholtz function F (A) is applicable to studying these processes and systems at constant temperature and volume. Certainly, this is only true on the assumption that the functions of state (of the systems studied) have actual physical sense at any moment of time. This is true for systems close to equilibrium but not for those far from equilibrium. I would like to emphasize once more that the law of temporal hierarchies gives grounds for the use of the functions of state when the direction and the extent of advancement of the evolutionary processes that occur in quasi-closed systems are estimated at different hierarchical levels of living matter [16-17].

Misunderstandings of the second law of thermodynamics: It is impossible in this short article to list all important conditions for the use of all types of state functions for different system. Moreover, I do not think I have noted all of the main "delicate" points that beginners should take into account. Besides, I refer to just a few publications, those that are most important for me. It should also be noted that my paper, as well as most publications on thermodynamics, may contain some inaccuracies of wording resulting from the ambiguity of translation. For example, most professional scientists know about inexcusable confusions with the terms *isolated system* and *closed system* (originally English). Both terms are sometimes translated into Russian as *замкнутая система* (literally, *closed system*). So the terms are often regarded as equivalent or identical. Other errors result from semantic coincidence of some terms. For example, the Gibbs (or Helmholtz) free energy is often confused with energy in the ordinary sense. This is why many researchers have attempted to replace this term with the term *the Gibbs function* [14, 19]. Another example is the term *complex system*. Here, the word *complex* has a double meaning. In thermodynamics, a complex system (as opposed to a simple one) usually means a system in which (or on which) a work other than the work of expansion is done [14-16]. Sometimes, however, the word *complex* is used to emphasize a structural or some other heterogeneity of the system itself or the diversity of its elements. This also applies to the term *simple system,* and so on. Certainly, these and other such confusions may lead to blunders that escape a nonprofessional's notice. These and other similar errors creep into some textbooks, reference books, and then into the Internet. I presume the possible inaccuracy of my English in this paper is insignificant, and I hope that the above remarks will warn beginners about the erroneous views that may exist in thermodynamics. I think that all physicists, chemists, biologists, and other specialists that deal with thermodynamics should study the Gibbs phenomenological thermodynamics first of all. As noted previously, this authentic (in a certain sense, true) thermodynamics is based on the notion of full differentials. To point out here once again, Euler's criteria (2.3) characterizes the nature of the full differential. Note that this approach to understanding the world surrounding us is intrinsically irrefutable. We may only discuss the accuracy of Gibbs thermodynamics as applied to, e.g., quasi-closed systems the processes in which are close to equilibrium. In accord with the very essence of the full differential, i.e. its mathematical meaning, as well as the first law of thermodynamics, the change in the function of state of the system accompanying the transition from one equilibrium state to another is independent of the path or mechanisms of its transition. Probably, the lack of our knowledge on actual complex systems may be partly attributed to the changes in entropy during this transition, because the entropy cannot, in principle, be measured directly. The changes in phenomenological entropy accompanying transformations in both simple and complex systems may be calculated only if one has studied the corresponding thermal processes. In statistical terms, the entropy is calculated only for ideal systems (or systems close to ideal). It is impossible to perform any precise calculations of this function of state for systems with strong interactions between particles (molecules and supramolecular structures) on a statistical basis. I would like to emphasize that this applies to complex thermodynamic systems, i.e., the systems in which strong interactions occur.

Thermodynamics, owing to its impeccably reliable mathematical basis of L. Euler, J.L Lagrange and other classics, may be regarded as a "machine" that always yields the right result if the premises are correct. Physical chemistry has repeatedly confirmed this [8–10, 14, 19]. Unfortunately, some physicists, biophysicists, biologists, and, especially, modern "philosophers" are still unaware of this experience of chemists and chemical technologists.

I would like to repeat that the aforementioned ambiguities, which are mainly related to the disregard of the correct use of many terms that are semantically similar but differ in physical meaning, result in confusion and misunderstandings. These misunderstandings discredit, at least in nonprofessionals' opinion, thermodynamics itself and science as a whole. Hence the numerous incorrect interpretations of the second law of thermodynamics, various dubious "views" on entropy [14, 19], and many far-fetched "functions of state of systems" in the literature are apparent. Many authors, ignoring classical works in this field, apply different formulations of the second law of thermodynamics to systems where they are inherently inapplicable. Some of these authors suggest their own interpretations of this general law of nature. This debases science and education. Moreover, it can be said that several "second laws of thermodynamics" have appeared, none of which having anything to do with reality. A good example is the aforementioned Prigogine's [14] interpretation of the second law of thermodynamics. This interpretation "extends" the well-known incorrect and indemonstrable statement by the great Boltzmann [14, 19], who neglected the important concepts put forward by L. Euler, J.L Lagrange, R. Clausius and J.W. Gibbs.

4. The hierarchical thermodynamic theory of aging in action

For decades, the opinion was widespread that natural open biological systems are far from an equilibrium state. It was also believed that far from equilibrium processes take place in these systems. Indeed, if this is true, then thermodynamics (i.e. thermostatics), or the thermodynamics of quasi-equilibrium systems and processes, cannot be applied.

However, recently, the law of temporal hierarchies was formulated [16-20]. This law substantiates the possibility of identifying, or discerning, quasi-closed monohierarchical systems or subsystems within open polyhierarchical biological systems. It was also established, as a rule, that the processes of evolution in living natural systems are quasi-equilibrium processes. It was shown that models of living systems are analogues of models of equilibrium or quasi-equilibrium chromatographic columns.

These facts facilitated the development of quasi-equilibrium thermodynamics of near to equilibrium quasi-closed systems. This variation of thermodynamics is based on the statement that the functions of state, to within a good approximation, at any moment of time in quasi-closed monohierarchical systems have a real physical meaning, i.e. a quantitative sense. Thus, classical thermodynamics using a linear approximation, i.e. the thermodynamics of near to equilibrium systems, at the phenomenological level can be used for the investigation of the origin of life, biological evolution, and the development and aging of organisms. Such investigations are carried out in terms of kinetic, or dynamical, linear thermodynamics.

It has been shown that the variation of the chemical composition of living beings in the course of ontogenesis and phylogenesis is a consequence of change in the mean specific value of the Gibbs function of formation for supramolecular and intermolecular interactions operating during the formation of supramolecular structures of an organism's tissues, which in quantitative value tends to a minimum. Strictly speaking, this variation is connected with the trend for mean specific values of the Gibbs function related to a unit of volume or mass, at all hierarchical levels, to seek a minimum.

The principle of the substance stability and feedback has been formulated. It is applicable to any biological system belonging to different hierarchies. For instance, this principle explains the accumulation of a substance with chemically high energy capacity by biological systems in the course of evolution and aging of living beings. This energetic accumulation of substance forces water out of these systems.

The arguments presented here, being well-substantiated, indicates that practically all concrete, i.e. detailed, recommendations relating to nutrition and lifestyle are individual. Such recommendations should be formulated on the basis of general and anti-aging medicine, from gerontology, and should take into account the findings of physicochemical dietetics.

Nevertheless, the thermodynamic theory of biological evolution and aging of living organisms, as built on the foundation of classical science, provides an opportunity to formulate general concepts pertaining to nutrition. These formulations and concepts will encourage and stimulate behavioral and dietary changes thermodynamically-favored towards the development of long and healthy human lives.

Diets promoting a healthy life style should, of course, comprise only ecologically clean foods. They should be balanced as to composition and caloric value. It is extremely important for a diet to include foods from cold, i.e. deep, regions of the sea and foods made from plants and animals inhabiting cold and Alpine regions. It is also desirable that the biomass used should be that of young plants and animals; being that this biomass has a higher anti-aging value. Moreover, food stuffs should be prepared from the biomass of ancient species, i.e., living organisms with a low phylogenetic development level, being those situated at the early stages of phylogenesis. Food for which young, ontogenetically and phylogenetically, plants and animals are used is not only gerontologically valuable but also, for obvious reasons, has a low caloric value. These steps are known to prolong life in a healthy manner and certain to increase general longevity. Pure, practically salt-free drinking water, as unadulterated glacial water, should be used in the maximal quantity acceptable for every patient. Medicinal mineral waters should be specifically indicated.

Generally, it is advisable to use foods and water that meet general up-to-date standards developed on a strictly scientific basis. It is also desirable that these foods and water intake recommendations, according to well-known patents, should be recognized to have high anti-aging value and that the water should be "gerontologically pure." Specific recommendations, which are an object of current research, are also available.

Lastly, it is important to take into account, from the viewpoint of hierarchical thermodynamics, that anti-aging diets and many drugs can be used for the prophylaxis and treatment of cardiovascular diseases, cancer, and for numerous other illnesses.

5. The law of temporal hierarchies

The law of temporal hierarchies, which some researchers have begun to call Gladyshev's law [19-20], can be presented as a series of strong inequalities. The direction of this series is towards increasing average life-spans of structures on going from lower to higher structures. In the simplest case, this law can be presented as:

$$\ldots << t^{m} << t^{im} << t^{organism} << t^{pop} << \ldots, \tag{5.1}$$

where t^{m} is the average life-span, or duration of existence, of the organism's molecules (or chemical compounds) taking part in metabolism; t^{im} is the average life-span of any intermolecular (or supramolecular) structures of the organism's tissues renovated in the process of its growth and development; $t^{organism}$ is the average life-span of organisms in the population; and t^{pop} is the average life-span of the populations. For the sake of simplicity and clarity, in the series of strong inequalities (5.1) the life-span of cells and some other complex supramolecular structures are omitted. Needless to say, this series, as determined by the presence of metabolism in the world of living matter, accords well with reality, and reflects the existence of temporal hierarchies in living systems. This rigorously substantiates the possibility of identifying, i.e. separating, monohierarchical quasi-closed systems (or subsystems) belonging to different temporal or structural polyhierarchies in open biological systems. Note that each type or species of organism is characterized by its own average life-span value for each respective hierarchy. However, series (5.1) is observed for each species of organism.

The series of times of imagined relaxation of different-hierarchy structures postulated by the author in 1976 had a reverse direction as compared to series (5.1), as in its originally form it was based relaxation times of dissociation and thermalisation rather than component life-spans in its current form. Nevertheless, both of these series give reason to make a conclusion on the possibility of identifying quasi-closed systems in open biological spheres and structures. There is a profound link between the direction of these series of the times of imagined relaxation and the life-spans of different-hierarchy structures. The sources of this link can be identified on a statistical basis for an ideal structural hierarchical model. In any case, there is a simple route towards comprehending the existence of this link.

Thus, the law of temporal hierarchies *makes it possible to identify, in open biological systems, quasi-closed thermodynamic systems*, i.e. subsystems, and to study their development (ontogenesis) and evolution (phylogenesis) by measuring the *change in the specific (per unit of volume or mass) value of the Gibbs function of formation of the given higher hierarchical structure from structures of a lower level.*

It was established, in the process of ontogenesis as well as phylogenesis and evolution generally, that the specific value of the Gibbs function of formation of supramolecular structures of the tissues of an ith organism $\bar{\tilde{G}}_i^{im}$ tends toward a minimum. First, this tendency is defined by the variation of chemical composition of the system during its evolutionary development. This tendency towards a minimum can be pressed in the form:

$$\bar{\tilde{G}}_i^{im} = \frac{1}{V}\int_0^V \frac{\partial \tilde{G}_i^{im}}{\partial m}(x,y,z)dxdydz \to \min_2$$

(5.2)

For the phase of supramolecular structures of varying composition
(for the times of ontogenesis, phylogenies, etc.)

Here, V is the volume of the investigated system; m is the mass of the identified microvolumes; x, y, and z are coordinates; the symbol "–" means that value $\bar{\tilde{G}}_i^{im}$ is specific, i.e. relating to the macrovolume; and the symbol "~" stresses the heterogeneous character of the system. The subscript " i " of G is related to the systems of different of chemical composition.

Note, once again, that the value $\bar{\tilde{G}}_i^{im}$ tends toward a minimum as a result of the system's tendency to seek the equilibrium state; i.e., the tendency of the "the investigated system within the constant temperature constant pressure environment" to evolve towards a stable potential energy well, as defined by a minimum of free energy. In this case, the environment is the physical thermostat, i.e. the medium of higher hierarchical structures (e.g. the hierarchy of organisms within a population) which evolves gradually.

From another perspective, as a consequence of the quick attainment of supramolecular equilibrium at each moment of time (t = 1, 2, 3, ...), in the investigated system, $\bar{\tilde{G}}_i^{im}$, where subscript " i " corresponds a system of constant chemical composition, attaches or reconfigures molecularly to a minimum that corresponds to the resultant stable supramolecular structure .

Thus, we have:

$$\bar{\tilde{G}}_i^{im} = \frac{1}{V}\int_0^V \frac{\partial \tilde{G}_i^{im}}{\partial m}(x,y,z)dxdydz \to \min_1 \qquad (5.3)$$

For the phase of supramolecular structures with constant composition
(at times of relaxation to local equilibrium)

It is important to note that correlations, i.e. the schematic tendencies, (5.2) and (5.3) imply taking into account the intermolecular interactions in all supramolecular structures of the biological tissue, both intracellular and extracellular. This is fully justified since structural hierarchy does not always coincide with temporal hierarchy. Thus, some types of cells do not divide (according to current views) and, like organs, age along with the organism. However, for any supramolecular hierarchy (j-1) there exists some higher ($j + x$) hierarchy, such that:

$$t^{j-1} << t^{j+x},$$

where t^{j-1} and t^{j+x} are the average life-spans (lifetimes) of elementary structures of the corresponding structural hierarchies in a living system, $x = 0, 1, 2, \ldots$, etc.

The presented detailed explanations given here characterize the "principle of substance stability".

6. The principle of substance stability

The principle of substance stability describes the tendency or trend of natural systems to seek local and general equilibria at all temporal and structural levels of the organization of matter [16, 17, 19-23, 23]. These tendencies derive from the second law thermodynamics (the Clausius–Gibbs variation) in coordination with the Le Chatelier–Braun principle.

The principle of substance stability is determined by the limited energetic potential, i.e. the Gibbs potential energy, of associated interacting elementary structures of every hierarchy. This principle appears at all hierarchical levels, temporal and structural, of living matter. It is connected with the fact that we can observe stabilizing tendencies and actions at time scales corresponding to our capabilities.

This principle was formulated by the author, whose focus is on an understanding of the greatest simplicity for the creation of thermodynamic models of our world. Such an approach was applied by classics of science, including J. Willard Gibbs [8, 16, 17] and Henri Poincaré [9].

Earlier, the author proposed different formulations of the principle of substance stability, which do not contradict each other [16, 17, 19].

The principle applied to molecular and supramolecular structures was named "the principle of the stability of a chemical substance". Subsequently this principle was applied by the author to various hierarchies as a part of the theory of the evolution of life. This principle is also known as: the principle of stability of matter, the principle of substance stability, the feedback principle, and Gladyshev's principle.

It boils down to the following: during the formation or self-assembly of the *most thermodynamically stable structures at the highest hierarchical level* (j), e.g., the supramolecular level, Nature, in accordance with the second law, spontaneously uses predominantly the *least thermodynamically stable structures* available from a given local part of the biological system, belonging to a lower level, i.e. molecular level (j-1), and incorporates these unstable structures into next higher level, i.e. supramolecular level (j). The justice of the principle is proved on a quantitative basis as applied to the molecular and supramolecular structural levels of biological tissues.

As an illustration, in the course of ontogenesis and phylogenesis the supramolecular structures of tissues, i.e., a higher level of structure j, as compared to the molecular level j-1, accumulate relatively unstable molecules or substances with a relatively high chemical energy capacity; for example, triglycerides, which force water out of these tissues. Similar phenomena occur in molecular chromatographic columns; specifically, hydrophobic cells and columns [16, 21-25]. All chemists know about it. These columns accumulate substance with a high energy capacity. These facts do not surprise us, although open heterogeneous *adsorbent (absorbent) – adsorbate* systems, **approaching supramolecular equilibria**, on the whole, **move away from chemical equilibrium with the environment.**

In this environment there are precisely those chemical substances that penetrate the column. The removal from chemical equilibrium with the environment is the consequence of the trend toward a minimum of the specific supramolecular component of the Gibbs function – the Gibbs free energy of formation for biological tissue $\overline{\widetilde{G}}^{im}$.

The author applied the principle of substance stability to the structural hierarchies that function inside any temporal hierarchy. These structural hierarchies have been named "understructure hierarchies." An illustration of the principle is provided by the selection of a sequence of nucleic acids including AU pairs in evolution, although these pairs are less stable from the standpoint of supramolecular thermodynamics than GC pairs. Hence, the selection of natural (AUGC) sequences takes into account not only the stability of the lower understructure supramolecular hierarchy, as was sometimes previously believed, but also the stability of the highest understructure supramolecular hierarchy, as well as tertiary, quaternary, and the highest supramolecular structures – nucleic acid–protein complexes.

There are some facts that call for application of the principle of substance stability to the hierarchy of cells. Thus, tumor cells have a lower ability for aggregation. As a result, they easily move in the body, which leads to the appearance of metastases. The cell membranes of tumor cells are, apparently, formed from supramolecular structures of increased stability. Hence, the supramolecular stability of cell aggregates formed with the participation of tumor cells should be lowered according to the principle in question. In order to increase the adhesive ability of the cells, the structure of membranes should be "diluted" and made less thermodynamically stable. Hence, it is clear why experimental anticancer diets propose the use of plant oils, fats of animals from cold seas, and other products containing residues of unsaturated low-melting-point fatty acids. The anti-tumor effect of aspirin can also be explained on the basis of such statements. These ideas agree with the recommendations made using the thermodynamic theory of aging [16, 19, 24-25] (in [23] there are many inadmissible errors in English, e.g. p. 437; in the Russian version of this journal there are no such errors).

The principle of substance stability allows us to understand the effect of the influence of some chemical substances on the supramolecular structures of nucleic acids [16]. As a result of the action of such substances, "sleeping" ancient genes, accumulated during the evolution of living beings, can awaken. These genes can stimulate some types of cancer.

A well-known fact in the sphere of sociology concerning family ties illustrates the relationship between the principle of substance stability and a social hierarchy. Here, we have in mind the substance, i.e. elemental structures, of any "inside" social hierarchy being here defined as an understructure hierarchy; e.g., a hierarchy of organisms, groups of organisms, etc. For example, the stronger the love and mutual understanding between a couple, i.e. the understructure hierarchy, the less time they spend "outside" the family, i.e. the "overstructure hierarchy". Such spouses do not have the desire, power, or time for this. Here we see, surprisingly, that hierarchical thermodynamics applies aptly to human life situations.

Furthermore, the principle of substance stability corresponds with the well-known rules of maintenance of stability of parties, unions, states, and nations. One can comprehend in this way the age-old social management methods as "divide and rule." Moreover, recently, the principle has found successful correspondence with the human thermodynamics of L. Thims [14, 15]

Professor L. Gumilev's and Professor A. Akhiezer's dynamic models of the development of communities and nations can be also confirmed using the quantitative basis of hierarchical thermodynamics [16]. Here it is useful to make an apposite remark concerning the possibility of prediction of the history of mankind. This history, in theory, can be predicted on the basis of the principles of hierarchical thermodynamics. In these examples, the quantitative thermodynamics of social hierarchy and the concept of sociological potential can be used [18].

Here, we note that the author's conception of the evolution and life, which were first put forth in 1977–1978, corresponds to James Lovelock's Gaia theory of life on the Earth as proposed in 1979. The basis of his well-known ecological theory is that the Earth is a self-regulating organism that adjusts to changes in order to maintain suitable conditions for life.

These facts confirm the author's point of view that feedback between all hierarchical levels of the biological world is based on hierarchical thermodynamics. These feedbacks can be schematically presented as a sequence [16, 19, 20]:

Noosphere→ biosphere → ecosystems → populations' → organisms' → cells → supramolecular structures → macromolecules (proteins) → DNA (RNA).

Hence, the principle in question is applicable to all hierarchies, understructure and overstructure, including the molecular and supramolecular structures for which it was first formulated.

Now some remarks on thermodynamics and politics. From the viewpoint of the principle of substance stability, it is clear why people who, as a rule, are distinguished by their independence and audacity seek power - and often achieve it. These individuals use techniques and methods known only to them and unavailable, due to moral considerations, to the average cultured person. However, having achieved a high position, these members of society begin, under favorable conditions, to come into confrontation with similar members of society. To achieve great power requires not only a combination of favorable factors but also a person's intelligence, which, however, usually shows itself in various peculiar and particular variations. To create an algorithm of the coming of a particular person into power is without a doubt a difficult task. In such cases, we deal with a varying algorithm, which is constantly transforming, i.e. changing, under the influence of change in the environment. Here the situation calls to mind the hopelessness of the creation of artificial intelligence [5], as well as the search for a rigorous genetic program of aging used to predict the life span of a person, being perhaps synonymous with his or her fate. In such cases, we can speak of averaged thermodynamic tendencies determined by the blurred "fan of thermodynamics."

The possibility of applying the principle to reveal the evolutionary trend of human society should especially be noted.

The rapid development of humanity in our time is associated with the preferential selection of energy-consuming systems and devices. These systems and devices, while making life easier for the people who own them, and for society as a whole, invariably result to increase humanities' thermodynamic or sociological potential. However, in accordance with the principle of substance stability, humanity together with its technogenic environment, as a single system, will, with time, become unstable. This, sooner or later, will lead to the partial degradation of the system and, in the end, to its complete destruction. Hierarchical thermodynamics, however, does not deny the possibility of the rebirth of humanity anew in a less than perfect, from the point of view of modern morality, form. This is all that we, the inhabitants of the planet, can count on in the future. The laws of thermodynamics are relentless. They are in effect everywhere in our universe.

In sum, the principle of substance stability, in various forms, can be extended to all hierarchies of matter [14, 16, 17].

7. The supramolecular thermodynamics and the anti-aging quality of foodstuff

It follows from the thermodynamic theory that the changes in the Gibbs specific function when supramolecular structures of a foodstuff (a substance) are formed, as well as the value of gerontological (anti-aging) quality of foodstuff, i.e. the GPG_i index, connected with it can be easily assessed from the approximated Gibbs – Helmholtz – Gladyshev equation [16, 20], which is an **analogue** of the classical Gibbs – Helmholtz approximated equation. As applied to natural fats and oils, it can be written down as:

$$\Delta\overline{\widetilde{G}}_i^{im} = (\Delta\overline{\widetilde{H}}_{mi}^{im} / T_{mi})(T_{mi} - T_0) = \Delta\overline{\widetilde{S}}_{mi}^{im} \Delta T, \quad (7.3)$$

where $\Delta\overline{\widetilde{G}}_i^{im}$ is the Gibbs specific function or Gibbs specific free energy of the supramolecular or intermolecular formation of the condensed phase i, $\Delta\overline{\widetilde{H}}_{mi}^{im}$; $\Delta\overline{\widetilde{S}}_{mi}^{im}$ is the change of specific enthalpy and entropy during the solidification of natural fat (oil); T_{mi} is the pour or melting point; and T_0 is the standard temperature (e.g., 25, 0, - 25, - 50 °C) at which values $\Delta\overline{\widetilde{G}}_i^{im}$, and consequently GPG_i, are compared. Value T_0 must be lower than value T_{mi}. When the gerontological value of a food is assessed, the choice of T_0 is determined by the melting point of the lowest melting-point substance in the series of compared products. It is assumed that the low melting-point substances take part in the formation of corresponding low melting-point supramolecular structures in an organism's tissues.

Let me note that the Gibbs – Helmholtz equation is correct for an individual substance in a closed system in which chemical, phase or other transformations may take place. The **analogue** of this equation, often with a good approximation, can be applied to various substances of the same type and for variable composition systems. The Gibbs – Helmholtz equation and its **analogue** (7.3) were used, with good results, by the author when determining the thermodynamic direction of evolutionary processes [16-17, 20, 24-27]. Such relationships, by default, are widely used in the study of synthetic copolymers, biological polymers, and other variable composition systems [16, 19, 20].

It follows from equation (7.3) that a correlation between $\Delta\overline{\widetilde{G}}_i^{im}$ calculated for standard temperature and the pour or melting point of fats or oils, T_{mi} should often be observed, with acceptable approximation [16]. Such a correlation should of course, be also observed between the indicator of the anti-aging (gerontological) value of the food in question, GPG_i and T_{mi} [16, 20]. Indeed, such a correlation does exist [16, 20-27].

All conclusions of the theory are fully conformed to the experience of medicine and dietetics. Hear we can only delighted by the efficiency of thermodynamic methods.

ACKNOWLEDGMENT. I am grateful to Prof. G. Arrhenius, Prof. N. N. Bogolyubov, Jr., Prof. K. G. Denbigh, Prof. V. P. Kazakov, Prof. Yu .S. Lipatov, Prof. A.A. Logunov, Prof. Yu. B. Monakov, Prof. John M. Rassias, Prof. V.V. Sychev, Dr. Libb Thims, and Prof. V. M. Frolov for the support, and valuable advice.

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Pierre-Simon, Marquis de Laplace

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Zeroes of L-series in characteristic p

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ABSTRACT

In the classical theory of L-series, the exact order (of zero) at a trivial zero is easily computed via the functional equation. In the characteristic p theory, it has long been known that a functional equation of classical $s \mapsto 1-s$ type could not exist. In fact, there exist trivial zeroes whose order of zero is "too high;" we call such trivial zeroes "non-classical." This class of trivial zeroes was originally studied by Dinesh Thakur (Thakur 1995) and quite recently, Javier Diaz-Vargas (Diaz-Vargas 2006). In the examples computed it was found that these non-classical trivial zeroes were correlated with integers having *bounded* sum of p-adic coefficients. In this paper we present a general conjecture along these lines and explain how this conjecture fits in with previous work on the zeroes of such characteristic p functions. In particular, a solution to this conjecture might entail finding the "correct" functional equations in finite characteristic.

Keywords: Drinfeld modules, T-modules, characteristic p L-series, L-zeroes, multi-valued operators

2000 Mathematics Subject Classification: Primary 11G09

1 Introduction

The debt all mathematicians owe to Euler is obvious and universally known. Euler's instincts and mathematical taste have had the most profound effect on all subsequent generations of researchers. Nowhere is this more evident than in Euler's fantastic contributions to number theory and, in his work on number theory (see, for instance, (Dunham 1999)), nothing is more fabulous than Euler's investigation into what we now call the Riemann zeta function $\zeta(s) = \zeta_{\mathbb{Q}}(s) := \sum_{n=1}^{\infty} n^{-s}$. In fact, Euler was the first to appreciate that $\zeta(s)$ had a functional equation, (see (Ayoub 1974) for a wonderful discussion of Euler's insights).

The fact that there are infinitely many primes goes back to Euclid. Euler gave an elegant refinement of this result by establishing that

$$\sum_{p \text{ prime}} \frac{1}{p}$$

*This work is in honor of my mother Barbara Goss Alter.

diverges thereby giving some indication of how well spaced the primes actually are. Euler's proof of this fact uses the "Euler product" associated to $\zeta(s)$ as well as the divergence of the harmonic series $\zeta(1)=\sum_{n=1}^{\infty}\frac{1}{n}$.

Euler was also fascinated by the special values of $\zeta(s)$ at all the other integers, both positive and negative, and he devoted much energy to their computation. In fact, it was by studying these values that Euler discovered the above mentioned functional equation. A modern number theorist cannot read these discoveries of Euler without wonder at the sheer beauty and audacity of Euler's methods. While the rigorous deduction of the functional equation of $\zeta(s)$ had to wait until Riemann and the methods of complex analysis, Euler's argument makes thrilling use of both convergent *and* divergent series (see Section 2 below).

In the process of evaluating $\zeta(s)$ where s is an integer, Euler discovered the "trivial zeroes" of $\zeta(s)$ at the negative even integers. These zeroes play a crucial role in Euler's calculations as they form the numerator of a rational quantity Euler needs to compute *and* where his methods (and those of everybody else since) fail to compute the denominator in closed form (see Remark 1 and Equation 2.6). Moreover, the functional equation of $\zeta(s)$ then allows one to then show that these trivial zeroes are simple. As is well-known in modern number theory, the functional equation of $\zeta(s)$ is merely the beginning of a vast undertaking whose goal is to show that *all* arithmetically interesting Dirichlet series ultimately behave in similar ways.

Euler's work on $\zeta(s)$ has always been an inspiration in our work on L-series in characteristic p. We will briefly review the definition of such characteristic p functions in Section 3 where the base ring A of the theory of Drinfeld modules plays the role of the integers $\mathbb{Z}$ in classical theory. In particular, in the prototypical case of $A=\mathbb{F}_r[T]$, $r=p^m$, p prime, (which, like $\mathbb{Z}$, is also Euclidean) one is able to describe a function $\zeta_A(s):=\sum_{n\in A \text{ monic}} n^{-s}$ (see Chapter 8 of (Goss 1996)) which is a very close cousin of $\zeta(s)$. Indeed, using the period of the Carlitz module instead of $2\pi i$, one could readily establish an analog of Euler's calculation of $\zeta(2n)$ $n=1,2,\cdots$ for those positive j which are "A-even" (i.e. j divisible by $r-1=\#A^*$). At the negative integers, $-j$ $(j\geq 0)$ one obtains divergent sums of the form $\sum_{n\in A \text{ monic}} n^j$ which, upon regrouping according to the degree of n, become finite. When j is again A-even, this sum is 0 giving a "trivial zero."

Thus, on the surface, the special values of $\zeta_A(s)$ behave very similarly to those of $\zeta_{\mathbb{Q}}(s)$, and so a first hope would be to follow Euler and guess at "the functional equation" for $\zeta_{\mathbb{F}_r[T]}(s)$. This fails to work for two basic reasons: 1. Obviously the mapping $s\mapsto 1-s$ is a bijection between even and odd integers; this fails for A-odd and A-even numbers when $r\neq 3$; 2. Even with $r=3$ there are *two* distinct analogs of Bernoulli numbers in the characteristic p theory; this results in considerably more complicated quotients than in classical theory.

This state of affairs persisted until the mid 1990's when two seemingly independent developments occurred. In the first (Wan 1996) (see also (Diaz-Vargas 1996)) Daqing Wan computed the Newton polygons associated to $\zeta_A(s)$ when $r=p$ (later extended to all r by B. Poonen and J. Sheats (Sheats 1998)) thereby establishing that the absolute value of a zero *uniquely* determines it (including the multiplicity of the zero). Therefore the zeroes of $\zeta_{\mathbb{F}_r[T]}$ lie "on the line" $\mathbb{F}_r((1/T))$ and are simple. Wan's calculations were prompted by the observation by the present author that, in some cases at least, the coefficients of $\zeta_A(s)$ go to 0 exponentially. Such a rate of decay is far faster than is necessitated to simply establish the basic analyticity properties of such a function and is implied by having the degrees of certain "special polynomials" (see Section 3) grow logarithmically. Such logarithmic growth is now known to be a completely general phenomenon (Böckle 2002), (Goss 2005).

In the second development, D. Thakur (Thakur 1995) looked into the possibility that, for general A, the trivial zeroes had higher order multiplicities; such a phenomenon *never* happens classically. In other words, the construction of trivial zeroes comes equipped with a "classical" lower-bound on the order of zero. However as Thakur found, there are many instances where this lower-bound is *not* the exact order; such a trivial zero is called "non-classical." It is totally remarkable, and very important for us, that Thakur's results on non-classical trivial zeroes involve having the sum of the p-adic digits of the trivial zero be *bounded*. Thakur's computations have recently been extended by Javier Diaz-Vargas (Diaz-Vargas 2006). These basic results will be recalled in Section 4.

The results of Wan, Sheats, etc., are clearly a type of "Riemann hypothesis" (see (Goss 2000), (Goss 2004)) and one wants to be able to put them into a general conjecture about the zeroes in *full* generality for all motives (τ-sheaves, etc.) and interpolations at all places of the quotient field k of A. Our first attempt to do so (Goss 2000) simply ignored the trivial zeroes (as they are ignored classically in the Riemann hypothesis). As was reported in (Goss 2004), this conjecture was wrong *precisely* because of the impact of higher order trivial zeroes! More precisely, using the topology on the domain space of our L-series, one is able to use higher order trivial zeroes to inductively construct p-adic integers where the conjecture is false. The construction produces such p-adic integers by building up their canonical p-adic expansion out of the expansions of trivial zeroes with higher orders.

It is precisely here that the computations of Thakur and Diaz-Vargas now fit. Indeed, their computations lead naturally to the conjecture (Conjecture 1) that those integers j for which the trivial zero at $-j$ is non-classical have *bounded* sum of their p-adic digits. Presenting this conjecture is the goal of this work and we discuss the conjecture both at ∞ (i.e., the analog of the complex field) *and* at the interpolations of our functions at finite primes. If this conjecture is true, then it places a limit on how one can construct counter-examples to our original conjecture. Indeed, it implies that we need not worry about non-classical trivial zeroes *alone* leading to counter-examples as they can have no effect on our construction once the sum of the p-adic digits of the integers being used becomes sufficiently large (see the discussion in Section 5; in particular, Example 4). We view this as positive evidence for the conjecture.

As the reader may see, the trivial zeroes play a special role in both the classical and characteristic p theory. But what is the right general conjecture on the zeroes in characteristic p? As of now, we do not know. However, since the functional equation classically is what allows one to compute the order of a trivial zero, it seems to us quite reasonable that a proof of the above conjecture in our case will generate the correct ideas and techniques.

It is clear that this work owes a great deal to Euler. It should also be clear that it owes a great deal to the mathematical taste and insight of Dinesh Thakur and Javier Diaz-Vargas. It is moreover my pleasure to thank Thakur and J.-P. Serre for helpful comments.

2 Euler's discovery of the functional equation for $\zeta_{\mathbb{Q}}(s)$

Our treatment here follows that of (Ayoub 1974); the reader is referred there for references and any elided details. Let $\zeta(s)$ be the Riemann zeta function.

After many years of work, Euler succeeded in computing the values $\zeta(2n)$, $n = 1, 2 \cdots$ in terms of Bernoulli numbers. Euler then turned his attention to the values $\zeta(s)$ at negative integers. Of course, Euler did not have analytic continuation to work with and relied on his instincts for beauty; nevertheless, he got it right! Euler begins with the very well known expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots . \tag{2.1}$$

Clearly this expansion is only valid when $|x| < 1$, but that does not stop Euler. Upon putting $x = -1$, he deduces

$$1/2 = 1 - 1 + 1 - 1 + 1 \cdots . \tag{2.2}$$

To the modern eye, this is a nonsensical statement about divergent series; however following in Euler's bold steps, we won't let that stop us! Indeed, upon applying $x(d/dx)$ to Equation 2.1 and plugging in $x = -1$, we obtain

$$1/4 = 1 - 2 + 3 - 4 + 5 \cdots . \tag{2.3}$$

Applying the process again, Euler finds the "trivial zero"

$$0 = 1 - 2^2 + 3^2 - \cdots , \tag{2.4}$$

and so on. Obviously, Euler is not working with the values at the negative integers of $\zeta(s)$ but rather the function

$$\zeta^*(s) := (1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1}/n^s , \tag{2.5}$$

however this is of little consequence and the zeta-values Euler obtains are exactly those rigorously obtained much later by Riemann. (In (Ayoub 1974), our $\zeta^*(s)$ is denoted $\phi(s)$.)

Nine years later, in (Euler 1768) (N.B.: while (Euler 1768) was published in 1768, it was written in 1749) Euler notices, at least for small $n \geq 2$, that his calculations imply

$$\frac{\zeta^*(1-n)}{\zeta^*(n)} = \begin{cases} \frac{(-1)^{(n/2)+1}(2^n-1)(n-1)!}{(2^{n-1}-1)\pi^n} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{2.6}$$

Upon rewriting Equation 2.6 using his gamma function $\Gamma(s)$ and the cosine, Euler then "hazards" to conjecture

$$\frac{\zeta^*(1-s)}{\zeta^*(s)} = \frac{-\Gamma(s)(2^s - 1)\cos(\pi s/2)}{(2^{s-1} - 1)\pi^s} , \tag{2.7}$$

which translates easily into the functional equation of $\zeta(s)$!

Remark 1. Note the important role played by the trivial zeroes in Equation 2.6 in that they render harmless Euler's inability to calculate explicitly $\zeta^*(n)$, or $\zeta(n)$, at odd integers > 1.

But there is still more! The value $\zeta^*(1)$ is precisely the alternating harmonic series

$$1 - 1/2 + 1/3 - 1/4 \cdots$$

which Euler knows converges to $\log 2$; so his calculations tell him that evaluating the left hand side of Equation 2.7 at $s = 1$ gives $\frac{1}{2\log 2}$. Euler then takes the limit on the right hand side and obtains the same value! To Euler, this is "strong justification" for his conjecture which Riemann much later proved. (This quote is from Euler's paper (Euler 1768), the translation is found at the bottom of page 1083 of (Ayoub 1974).)

3 A quick introduction to L-series in characteristic p

We now briefly go over the basic definitions of characteristic p L-series. We will present the general definitions but the reader will lose very little by always assuming $A = \mathbb{F}_q[T]$ in what follows.

Let k be an arbitrary global function field of transcendency degree 1 and full field of constants $\mathbb{F}_r$. Let ∞ be a fixed place of k of degree d_∞ over $\mathbb{F}_r$ and let $|?|_\infty$ be the associated absolute value. Let A be the Dedekind domain of those functions regular outside ∞. It is easy to see that the unit group of A is the set of non-zero constants and that one has

$$h_A = d_\infty \cdot h_k\,, \tag{3.1}$$

where d_∞ is the degree of ∞ and $h_?$ is the respective class number.

We let K be the completion of k at ∞ and $\mathbb{F}_\infty \simeq \mathbb{F}_{r^{d_\infty}} \subset K$ be the associated finite field. We let $\pi \in K$ be a uniformizing element so that every non-zero element α of K may be written

$$\alpha = \zeta_\alpha \cdot \pi^{n_\alpha} \cdot \langle \alpha \rangle \tag{3.2}$$

where $\zeta_\alpha \in \mathbb{F}_\infty^*$, $n_\alpha \in \mathbb{Z}$ and $\langle \alpha \rangle \in U_1(K) = \{x \in K \mid |x-1|_\infty < 1\}$ has absolute value 1. The elements ζ_α and $\langle \alpha \rangle$ depend on our choice of π. The element ζ_α is called the "sign of α" and denoted sgn(α).

Example 1. When $k = \mathbb{F}_r(T)$ and $A = \mathbb{F}_r[T]$, the simplest choice is $\pi = 1/T$ so that for $n \in A$ monic of degree d, one has

$$n = \pi^{-d}\langle n \rangle = T^d \langle n \rangle\,, \tag{3.3}$$

with $\langle n \rangle = n/T^d \equiv 1 \pmod{\pi}$.

In general, an element $\alpha \in K^*$ is said to be *monic* or *positive* if and only if sgn$(\alpha) = \zeta_\alpha = 1$, which clearly depends on the choice of π. Notice that the positive elements clearly form a subgroup of K^*.

Let X be the smooth projective curve associated to k. For any fractional ideal I of A, we let $\deg_k(I)$ be the degree over $\mathbb{F}_r$ of the divisor associated to I on the affine curve $X - \infty$. For $\alpha \in k^*$, one sets $\deg_k(\alpha) = \deg_k((\alpha))$ where (α) is the associated fractional ideal; this clearly gives the correct degree of a polynomial in $\mathbb{F}_r[T]$.

Let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure $\bar{K}$ of K equipped with the canonical extension of the normalized absolute value on K.

Definition 1. Set $S_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p$.

The space S_∞ plays the role of the complex numbers in our theory in that it is the domain of "n^s." Indeed, let $s = (x, y) \in S_\infty$ and let $\alpha \in k$ be positive. The element $u = \langle\alpha\rangle - 1$ has absolute value < 1; thus $\langle\alpha\rangle^y = (1+u)^y$ is easily defined and computed via the binomial theorem.

Definition 2. We set

$$\alpha^s := x^{\deg_k(\alpha)}\langle\alpha\rangle^y\,. \tag{3.4}$$

Clearly S_∞ is a group whose operation is written additively. Suppose that $j \in \mathbb{Z}$ and α^j is defined in the usual sense of the canonical $\mathbb{Z}$-action on the multiplicative group. Let $\pi_* \in \mathbb{C}_\infty^*$ be a fixed d_∞-th root of π. Set $s_j := (\pi_*^{-j}, j) \in S_\infty$. One checks easily that Definition 2 gives $\alpha^{s_j} = \alpha^j$. When there is no chance of confusion, we denote s_j simply by "j."
In the basic case $A = \mathbb{F}_r[T]$ one can now proceed to define L-series in complete generality. However, in general there are non-principal ideals. Fortunately there is a canonical and simple procedure to extend Definition 2 to them as follows. Let $\mathcal{I}$ be the group of fractional ideals of the Dedekind domain A and let $\mathcal{P} \subseteq \mathcal{I}$ be the subgroup of principal ideals. Let $\mathcal{P}^+ \subseteq \mathcal{P}$ be the subgroup of principal ideals which have positive generators. It is a standard fact that $\mathcal{I}/\mathcal{P}^+$ is a finite abelian group. The association

$$\mathfrak{h} \in \mathcal{P}^+ \mapsto \langle\mathfrak{h}\rangle := \langle\lambda\rangle\,, \tag{3.5}$$

where λ is the unique positive generator of $\mathfrak{h}$, is obviously a homomorphism from $\mathcal{P}^+$ to $U_1(K) \subset \mathbb{C}_\infty^*$.
Let $U_1(\mathbb{C}_\infty) \subset \mathbb{C}_\infty^*$ be the group of 1-units defined in the obvious fashion. The binomial theorem again shows that $U_1(\mathbb{C}_\infty)$ is a $\mathbb{Z}_p$-module. However, it is also closed under the unique operation of taking p-th roots; as such $U_1(\mathbb{C}_\infty)$ is a $\mathbb{Q}_p$-vector space.

Lemma 3.1. *The mapping $\mathcal{P}^+ \to U_1(\mathbb{C}_\infty)$ given by $\mathfrak{h} \mapsto \langle\mathfrak{h}\rangle$ has a unique extension to $\mathfrak{I}$ (which we also denote by $\langle ? \rangle$).*

Proof. As $U_1(\mathbb{C}_\infty)$ is a $\mathbb{Q}_p$-vector space, it is a divisible group; thus the extension follows by general theory. The uniqueness then follows by the finitude of $\mathcal{I}/\mathcal{P}^+$. □

If $s \in S_\infty$ and I as above, we now set

$$I^s := x^{\deg_k(I)}\langle I\rangle^y\,. \tag{3.6}$$

Thus if $\alpha \in k$ is positive one sees that $(\alpha)^s$ agrees with α^s as in Equation 3.4.

3.1 Definition of L-series

Let $G := \mathrm{Gal}(k^{\mathrm{sep}}/k)$ be the absolute Galois group of k where k^{sep} is a fixed separable closure of k. Let $\bar{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and let $\chi\colon G \to \bar{\mathbb{Q}}_p^*$ be a homomorphism of Galois type; i.e., χ factors through the Galois group G_1 of a *finite* abelian extension k_1 of k. Obviously the image of χ consists of roots of unity and viewing these as sitting in $\mathbb{C}$ (via some injection) allows us to think of χ as also being complex valued. In particular, for each place w of k (including ∞) one attaches a local factor as follows: 1. The place w is ramified for χ; in which case the factor is simply 1. The place w is unramified; in which case the factor is $(1 - \chi(F_w)t)$ where F_w is the (arithmetic) Frobenius element at w.

Let $R_p \subset \bar{\mathbb{Q}}_p$ be the ring of integers with maximal ideal M_p. We fix an injection of R_p/M_p into $\mathbb{C}_\infty$ and so we now obtain local factors in $\mathbb{C}_\infty[t]$ for which we will use the same notation $(1 - \chi(F_w)t)$ etc.

Remark 2. The reader may be wondering why we simply did not use the obvious reduction $\bar{\chi}\colon G \to (R_p/M_p)^*$ to begin with. The answer is that there are no non-trivial p-power roots of unity in characteristic p and so one is hard pressed to get the local factors right. For instance, in the case G_1 is a p-group, the reduced homomorphism $\bar{\chi}$ is the trivial character. If however, we would simply use the trivial character to obtain local factors we would be off at the ramified primes. Thus is is far better to use the characteristic 0 factors in the above fashion.

Let $s \in S_\infty$ and χ as above.

Definition 3. We set

$$L(\chi, s) := \prod_{\substack{v \in \mathrm{Spec}(A) \\ v \text{ unramified}}} (1 - \chi(F_v)v^{-s})^{-1}. \tag{3.7}$$

As in Section 8.9 of (Goss 1996), it is known that $L(\chi, s)$ converges on a "half-plane" of S_∞ and can be analytically extended to an "entire" function on S_∞. Thus, one can view $L(\chi, s)$ as a continuous 1-parameter, where $y \in \mathbb{Z}_p$ is the parameter, family of entire power series in x^{-1} etc.

While we have only discussed abelian χ here for simplicity of exposition, it is clear how to proceed in the non-abelian case.

3.2 Special polynomials

Let j now be a non-negative integer with χ, as above and let $s = (x, y) \in S_\infty$.

Definition 4. We set

$$z_L(\chi, x, -j) := L(\chi, \pi_*^j x, -j). \tag{3.8}$$

It is known that $L(\chi, x, -j)$ is a polynomial in x^{-1} and all such polynomials are called the *special polynomials* of $L(\chi, s)$. By unraveling the definition of $z_L(\chi, x, -j)$, one sees that the coefficients of this polynomial lie in

$$\mathcal{O} := \mathcal{O}_{\mathrm{V}}[\zeta] \tag{3.9}$$

where ζ is a primitive n-th root of unity and n is the order of the reduction $\bar{\chi}$ of χ (as a finite character) and $\mathcal{O}_{\mathrm{V}} \subset \mathbb{C}_\infty$ is the ring of integers in the *value field* generated by the elements $\{I^{s_1}\}$ (see Subsection 8.2 of (Goss 1996)). As mentioned at the end of (Goss 2005), elementary estimates imply that the degree (in x^{-1}) of $L(\chi, x, -j)$ grows logarithmically in j.

Remark 3. For L-series associated to "τ-sheaves" etc., the logarithmic growth of the special polynomials is due to Böckle (Böckle 2002). For arbitrary L-series associated to representations of Galois type (not necessarily abelian) one can use Böckle's results and the fact that the Artin Conjecture is true (Goss 1985) for these functions to deduce logarithmic growth.

3.3 Trivial zeroes

The classical, characteristic 0 valued, L-series associated to χ also attaches a local factor to the prime ∞ if it is unramified for χ. In the case that χ is non-principal, one knows that this classical L-series is entire; i.e., is a polynomial in $u = r^{-s}$. These infinite factors are missing in the definition of our L-series and thereby equip them with *trivial zeroes* as we shall explain here.

Let ψ be a Hayes-module (i.e., a sign normalized rank one Drinfeld module) (Hayes 1992), Section 7 of (Goss 1996) associated to a twisting of sgn. The module ψ analytically arises from a rank one lattice generated by an element $\xi \in \mathbb{C}_\infty$; one knows that $\xi^{r^{d_\infty}-1} \in K^*$. The extension $K_1 := K(\xi)/K$ is a totally ramified abelian extension with Galois group g_∞ isomorphic to $\mathbb{F}_\infty^*$ via the action on ξ. This local extension is also obtained by adjoining to K *any* non-trivial division point for ψ.

Let $W \subset \bar{\mathbb{Q}}_p$ be the Witt ring of $\mathbb{F}_\infty$ and let $t\colon g_\infty \to \mathbb{F}_\infty^*$ be the homomorphism given by the action of g_∞ on ξ and let $T_\infty\colon g_\infty \to W^*$ be the composition of t and the *Teichmüller character* of $\mathbb{F}_\infty^*$.

We view T_∞ as being extended in the obvious way to a character of the absolute Galois group G_∞ of K^{sep}/K where $K^{\mathrm{sep}} \subset \mathbb{C}_\infty$ is the separable closure.

Let χ_∞ be the local factor at ∞ associated to χ which we also view as a character on G_∞. Assume that for some non-negative j the character $\chi_\infty \cdot T_\infty^j$ is *unramified*. Then, as in Theorem 8.12.5 of (Goss 1996), a double congruence implies that

$$z_L(\chi, x, -j)/(1-(\chi_\infty \cdot T_\infty^j)(F_\infty)x^{-d_\infty}) \in \mathcal{O}[x^{-1}]\,, \tag{3.10}$$

where $\mathcal{O}$ is given in Equation 3.9. Thus zeroes of $(1-(\chi_\infty \cdot T_\infty^j)(F_\infty)x^{-d_\infty})$ clearly give rise to zeroes of the original L-series $L(\chi, s)$.

Definition 5. The zeroes of $L(\chi, s)$ arising from the factor $(1-(\chi_\infty \cdot T_\infty^j)(F_\infty)x^{-d_\infty})$ are called *trivial zeroes for χ at $-j$.*

Remark 4. It is clear how to generalize the above construction of trivial zeroes to arbitrary representations of Galois type. For general L-series arising from Drinfeld modules, t-modules, τ-sheaves etc., one proceeds cohomologically as in (Böckle 2002).

Remark 5. If $\chi_\infty \cdot T_\infty^j$ is ramified, then the local factor is 1 and so no non-trivial information is deduced. In this case, it is reasonable to expect that $z_L(\chi, x, -j)$ has no zeroes of absolute value 1.

Definition 6. Let t be a trivial zero for $L(\chi, s)$ at $-j$; so that $\pi_*^j t$ is a root of

$$(1-(\chi_\infty \cdot T_\infty^j)(F_\infty)x^{-d_\infty})$$

of order $v_0(t)$. Let $v_1(t)$ be the order of t as a zero of $z_L(\chi, x, -j)$. By (3.10) we know that $v_0(t) \le v_1(t)$. If this inequality is strict, then we say that t is *non-classical*. The set of all non-negative j such that $L(\chi, s)$ has a non-classical trivial zero at $-j$ will be called *the non-classical set for $L(\chi, s)$*.

Let $\bar{\chi}$ be the reduction of χ considered as a homomorphism into $\mathbb{C}_\infty^*$ via our fixed embedding of R_p/M_p into $\mathbb{C}_\infty$. Let $\mathbb{F}_\chi$ be the finite field generated by the values of $\bar{\chi}$ over the base field $\mathbb{F}_p$; obviously $\mathbb{F}_\chi$ is finite and we let $q_\chi := p^{e(\chi)}$ be its order.
The next proposition is implicit in (Thakur 1995) and (Diaz-Vargas 2006)

Proposition 3.2. *The non-classical set for $L(\chi, s)$ is closed under multiplication by q_χ.*

Proof. This follow upon applying the q_χ-th power mapping to the coefficients. □

3.4 v-adic theory and v-adic trivial zeroes

Let v be a finite prime of A of degree d_v over $\mathbb{F}_r$. Let k_v be the local field at v with fixed algebraic closure $\bar{k}_v$, equipped with the canonical topology, and let $\mathbb{C}_v$ be the associated complete field. As before let $\mathbf{V}$ be the value field and n the order of the reduction of χ. Fix an embedding $\sigma\colon \mathbf{V}[\zeta] \to \mathbb{C}_v$ where ζ is a primitive n-th root.
Via σ, the functions $\{z_L(\chi, x, -j)\}_{j=0}^{\infty}$ can be thought of as lying in $\mathbb{C}_v[x^{-d_\infty}]$. Upon setting $x = x_\sigma \in \mathbb{C}_v^*$, they interpolate to a continuous function $L_{\sigma,v}(\chi, x_\sigma, s_\sigma)$ where $(x_\sigma, s_\sigma) \in \mathbb{C}_v^* \times S_{\sigma,v}$; here $S_{\sigma,v} = \mathbb{Z}_p \times \mathbb{Z}/(r^t-1) = \varprojlim \mathbb{Z}/p^n(r^t-1)$ is the inductive limit over n of $\mathbb{Z}/p^n(r^t-1)$ and where r^t-1 is the number of roots of unity in the extension of k_v generated by the image of σ.
As in the previous subsection, let j be chosen so that $\chi_\infty \cdot T_\infty^j$ is unramified at ∞. The local factor $(1-(\chi_\infty \cdot T_\infty^j)(F_\infty)x^{-d_\infty})$ obviously has roots of unity for its zeroes. Thus, these roots have bounded v-adic absolute value (as of course their absolute value is 1). Their effect on the Newton Polygons for $L_{\sigma,v}(\chi, x_\sigma, s_\sigma)$ is thus very limited and so can essentially be ignored.
However, the process of interpolating v-adically precisely *kills* the Euler factor at v in the following manner. Let σ be extended to the natural action on polynomials via its action on the coefficients.

Proposition 3.3. *One has*

$$L_{\sigma,v}(\chi, x_\sigma, -j) = \sigma\left((1-\chi(F_v)v^j x_\sigma^{-d_v})z_L(\chi, x_\sigma, -j)\right). \tag{3.11}$$

Proof. This follows immediately as $\lim_{i\to\infty} v^i = 0$ in $\mathbb{C}_v$. □

We are thus led to a very interesting class of zeroes for $L_{\sigma,v}(\chi, x_\sigma, s_\sigma)$.

Definition 7. The zeroes of $1-\sigma(\chi(F_v)v^j)x_\sigma^{-d_v}$ in $\mathbb{C}_v$ are called the *v-adic trivial zeroes of* $L_{\sigma,v}(\chi, x_\sigma, s_\sigma)$ *at* $-j \in S_{\sigma,v}$.

The v-adic trivial zeroes are remarkably similar to their counterparts in S_∞. The definition of *non-classical v-adic trivial zeroes* is now obvious as is the definition of the *non-classical set for* $L_{\sigma,v}(\chi, x_\sigma, s_\sigma)$.

Remark 6. Actually, our whole construction of v-adic trivial zeroes *is* non-classical; indeed we know of no analog of our construction of v-adic trivial zeroes in the theory of p-adic L-series. However, we will continue to use "non-classical" to refer to those trivial zeroes whose order is higher than expected.

The next result is the obvious analog of Proposition 3.2 and has the same proof.

Proposition 3.4. *The non-classical set for $L_{\sigma,v}(\chi, x_\sigma, s_\sigma)$ is closed under multiplication by q_χ.*

4 The calculations of Thakur and Diaz-Vargas

Let $\chi = \chi_0$ be the trivial character with constant value 1.

Definition 8. We call the function $L(\chi, s)$ the *zeta function of* A and denote it $\zeta_A(s)$. The v-adic interpolations of $\zeta_A(s)$ are denoted $\zeta_{\sigma,v}(x_\sigma, s_\sigma)$.

Clearly one has $\zeta_A(s) = \sum_I I^{-s}$ where I runs over the ideals of A.
The trivial zeroes of $\zeta_A(s)$ then occur at the negative integers $-j \in S_\infty$ where $j \equiv 0 \pmod{r^{d_\infty} - 1}$; indeed the local factor at ∞ in this case is simply $1 - x^{-d_\infty}$. In the case $A = \mathbb{F}_r[T]$, one can show that these zeroes are simple; thus the non-classical set for $\zeta_{\mathbb{F}_r[T]}(s)$ is empty.

Remark 7. Let $A = \mathbb{F}_r[T]$ and let $s = (x, y) \in S_\infty$. In (Wan 1996), (Sheats 1998) it is shown that for fixed y, a zero of $\zeta_A(x, y)$, has multiplicity 1 and is *uniquely* determined by its absolute value; thus *all* zeroes are simple and must lie in K. Suppose now that θ is a classical Dirichlet character with classical (complex) L-series, $L(\theta, s)$. Let

$$l(\theta, t) := L(\theta, 1/2 + it)\,.$$

At the end of (Goss 2004), it is shown how the classical functional equation combined with the action of complex conjugation imply that the expansion about $t = 0$ of $l(\theta, t)$ is, up to possible multiplication by a non-trivial constant, a *real* power series. Of course the Riemann hypothesis is equivalent to having the zeroes of $l(\theta, t)$ be real; so that classical theory looks quite similar to what was established for $\zeta_A(s)$.

Now let $r = p = 2$.

Example 2. Let $A := \mathbb{F}_2[T_1, T_2]/(T_1^2 + T_1 + T_2^3 + T_2 + 1)$. In this case the quotient field k has genus 1.

Example 3. Let $A := \mathbb{F}_2[T_1, T_2]/(T_1^2 + T_1 + T_2^5 + T_2^3 + 1)$. Here the quotient field k has genus 2.

In both cases, one finds that A has class number 1 which implies that $d_\infty = 1$ also. In both cases, $\zeta_A(s)$ will have trivial zeroes at all negative integers (as $r - 1 = 1$).
Let j be an integer and let $l_p(j)$ be the sum of its p-adic digits. After some hand and machine calculations, Thakur (Thakur 1995) established the following result.

Theorem 4.1. *Let A be as in Example 2 or Example 3. Then the order of vanishing of $\zeta_A(s)$ at $s = -j$ is 2 if $l_2(j) \le g$ where g is the genus of the quotient field k.*

Thus, in particular, the non-classical set for $\zeta_A(s)$ is non-empty.
For very small j, Thakur also shows the converse to his result. Thus, for instance, in the case of Example 3, the trivial zero at -7 is simple. His paper contains other such examples.
In Theorem 5.4.9 of (Thakur 2004), Thakur establishes a partial converse to Theorem 4.1 in the case of Example 2. More precisely he shows in this case that the trivial zero at $s = -j$ is simple if $l_2(j) = 2$ or $j \equiv 0, 3, 5$ or $6 \pmod 7$.
In (Diaz-Vargas 2006), Javier Diaz-Vargas extends these calculations to more general A where the degree of ∞ is 1 but where A has non-trivial class number and so our exponentiation of non-trivial ideals is used. The same general phenomenon appears to hold.

5 A general conjecture

Let w be any place of k, where k is now completely general, and consider the non-classical set N_w for the interpolation of $L(\chi, s)$ at w (so that if $w = \infty$, this interpolation is $L(\chi, s)$ itself). We know from Propositions 3.2 and 3.4 that N_w is closed under multiplication by q_χ. Extrapolating vastly from the calculations presented in the previous section we are led to the following conjecture.

Conjecture 1. The non-classical set N_w consists of elements with *bounded* sum of p-adic digits.

Let $C \geq 0$ be some bound. Then, of course, $\{j \mid l_p(j) \leq C\}$ is a particularly simple set of positive integers which is closed under multiplication by any power of p.

Suppose that one can find infinitely many $-j$ so that the factor $1 - (\chi_\infty \cdot T_\infty^j)(F_\infty)x^{-d_\infty}$ has many zeroes of the same absolute value (which will clearly happen if $d_\infty > 1$). Then, in (Goss 2004), we showed how to construct elements $\alpha \in \mathbb{Z}_p$ so that the power series arising from $L(\chi, x, \alpha)$ had the strange property that there were infinitely many slopes (of the associated Newton Polygon) of length greater than 1. One does this inductively by building up α via its p-adic digits; so that in particular α is built inductively of integers $\{\alpha_i\}$ with $l_p(\alpha_i)$ increasing.

If Conjecture 1 is true, then such counter-examples **cannot** be created out of non-classical trivial zeroes alone. This is illustrated by the next example.

Example 4. Let A be as in Example 2 or 3 and suppose that Conjecture 1 is true in the sense that Thakur's result Theorem 4.1 is both necessary and sufficient. Then one cannot use the construction in (Goss 2004) to obtain strange α as above. Indeed, once our approximations have sufficiently large sum of p-adic digits the effect of the non-classical trivial zeroes is negligible. In fact, let α_i be an approximation to α with $l_p(\alpha_i) > 2$. Then the trivial zero at $-\alpha_i$ must now be simple and so *cannot* contribute a slope to α of length bigger than 1.

We view Example 4 as being some "justification" for our conjecture in the Eulerian sense of Section 2.

Conjecture 1 is very general but clearly not the final word. One would like to know the exact structure of the non-classical sets as well as the exact orders of the associated trivial zeroes. Moreover, one would like to know how the bounds change as the place w varies, etc. Still Conjecture 1 is a precise statement that indicates a much deeper theory of the zeroes.

Finally, in this paper we have worked with representations of Galois type. It is reasonable to ask for a similar conjecture for arbitrary motives etc. As of this writing, we do not know how to formulate such a conjecture. However, the following simple example indicates some possible structure.

Example 5. Let A be as in Example 2 or 3 so that $p = 2$. Let ψ be the Hayes module associated to A and let $L(\psi, s)$ be its L-series. Then $L(\psi, s) = \zeta_A(s-1)$. Thus j is non-classical for $L(\psi, s)$ if and only if $j+1$ is non-classical for $\zeta_A(s)$. Thus Conjecture 1 implies that $l_p(j+1)$ is bounded. Note that clearly $l_p(j+1)$ can be bounded while $l_p(j)$ goes to infinity (e.g., $j = 2^t - 1, t = 1, 2, \ldots$).

It might be that having $l_p(j+1)$ be bounded instead of $l_p(j)$ is somehow analogous to having a functional equation of the form $s \mapsto k - s$ classically for $k \neq 1$.

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Beck's graphs associated with $\mathbb{Z}_n$ and their characteristic polynomials

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ABSTRACT

In this paper, the graphs $G(\mathbf{Z}_n)$ associated with the residue class rings $\mathbf{Z}_n$ defined by I. Beck (?) where the vertices set $\mathbf{Z}_n$ and the edges set such that an edge $[a,b]$ if and only if $ab = 0$ for two distinct vertices a, b in $\mathbf{Z}_n$. Let A_n be an adjacent matrix of a graph $G(\mathbf{Z}_n)$. When the chromatic number of $G(\mathbf{Z}_n)$ is three we consider the characteristic polynomial of A_n and the number of distinct 4-cycles of $G(\mathbf{Z}_n)$. Also, we give some examples of Beck's graphs associated with $R_{n,m} = \mathbf{Z}_n[x] = \mathbf{Z}_n[X]/(X^m)$.

Keywords: Beck's graphs, characteristic polynomials, Adjacent matrix, ring, 4-cycle.

2000 Mathematics Subject Classification: 05C15, 05C50, 11C08, 13F20.

1 Definitions and Examples

Let $\mathbf{Z}_n = \{0, 1, 2, \cdots, n-1\}$ be a residue class ring of the integers ring $\mathbf{Z}$ by an ideal (n) generated by a positive integer n. Let $G(\mathbf{Z}_n)$ be a graph such that $\mathbf{Z}_n$ is the vertex set and an edge $[a,b]$ is a property that $ab = 0$ if and only if a and b are adjacent for two distinct vertices a and b of a graph $G(\mathbf{Z}_n)$. Also, the (vertex) chromatic number of $G(\mathbf{Z}_n)$ denote by $\chi(\mathbf{Z}_n)$ (cf. (?)). Also, let $\mathbf{Z}_n[X]$ be a polynomial ring over $\mathbf{Z}_n$ and let $R_{n,m} = \mathbf{Z}_n[x] = \mathbf{Z}_n[X]/(X^m)$.

Definition 1.1. (?) An alternated sequence $C = v_0e_1v_1e_2v_2\cdots e_kv_0$ is called a **k-cycle** of a graph G if $v_0, v_1, v_2, \cdots, v_{k-1}$ are all distinct vertices of G and $e_1, e_2, \cdots, e_k$ are all distinct edges and the initial vertex coincide with the end vertex. In particular, 3-cycle is called a **triangle**.

Example 1.1. Let $R_{2,3} = \mathbf{Z}_2/(X^3) = \mathbf{Z}_2[x]\ (x^3 = 0)$. Beck's graph $G(R_{2,3})$ is not Euler graph and $G(R_{2,3}) = (V(G(R_{2,3})), E(G(R_{2,3})))$ where $E(G(R_{2,3})) = \{[0,1], [0,x], [0,x^2], [0,1+x], [0,1+x^2], [0,1+x+x^2], [x,x^2], [x^2,x+x^2]\}$. $T_1 = 0[0,x]x[x,x^2]x^2[x^2,0]0$ and $T_2 = 0[0,x^2]x^2[x^2,(x+x^2)](x+x^2)[x+x^2,0]0$ are two triangles in this graph. $C = 0[0,x]x[x,x^2]x^2[x^2,x+x^2](x+x^2)[x+x^2,0]0$ is only one 4-cycle.

Example 1.2. Let $R_{2,4} = \mathbf{Z}_2/(X^4) = \mathbf{Z}_2[x]\ (x^4 = 0)$. Beck's graph $G(R_{2,4})$ is not Euler graph and $G(R_{2,4}) = (V(G(R_{2,4})), E(G(R_{2,4})))$ where
$E(G(R_{2,4})) = \{[0,1], [0,x], [0,x^2], [0,x^3], [0,x+x^2], [0,x+x^3], [0,x^2+x^3], [0,x+x^2+x^3], [0,1+x], [0,1+x^2], [0,1+x+x^2], [0,1+x^3], [0,1+x+x^3], [0,1+x^2+x^3], [0,1+x+x^2+x^3], [x,x^3], [x^2,x^3], [x^2,x^2+x^3], [x^3,x+x^2], [x^3,x+x^3], [x^3,x^2+x^3], [x^3,x+x^2+x^3]\}$.
Also, we have that triangles are the following statements.

$T_1 = 0[0,x]x[x,x^3]0,$

$T_2 = 0[0,x^2]x^2[x^2,x^3]x^3[x^3,0]0,$

$T_3 = 0[0,x^2]x^2[x^2,x^2+x^3](x^2+x^3)[x^2+x^3,0]0,$

$T_4 = 0[0,x^3]x^3[x^3,x^2+x^3](x^2+x^3)[x^2+x^3,0]0,$

$T_5 = 0[0,x^3]x^3[x^3,x+x^2+x^3](x+x^2+x^3)[x+x^2+x^3,0]0,$

$T_6 = 0[0,x^3]x^3[x+x^3,0]0,$

$T_7 = 0[0,x^3]x^3[x^3,x+x^2](x+x^2)[x+x^2,0]0,$

$T_8 = x^2[x^2,x^3]x^3[x^3,x^2+x^3].$

Moreover, we have that distinct 4-cycles are the following statements.

$C_1 = 0[0,x]x[x,x^3]x^3[x^3,x+x^2](x+x^2)[x+x^2,0]0,$

$C_2 = 0[0,x]x[x,x^3]x^3[x^3,x+x^3](x+x^3)[x+x^3,0]0,$

$C_3 = 0[0,x]x[x,x^3]x^3[x^3,x^2+x^3](x^2+x^3)[x^2+x^3,0]0,$

$C_4 = 0[0,x]x[x,x^3]x^3[x^3,x+x^2+x^3](x+x^2+x^3)[x+x^2+x^3,0]0,$

$C_5 = 0[0,x^2]x^2[x^2,x^3]x^3[x^3,x+x^2](x+x^2)[x+x^2,0]0,$

$C_6 = 0[0,x^2]x^2[x^2,x^3]x^3[x^3,x+x^3](x+x^3)[x+x^3,0]0,$

$C_7 = 0[0,x^2]x^2[x^2,x^3]x^3[x^3,x^2+x^3](x^2+x^3)[x^2+x^3,0]0,$

$C_8 = 0[0,x^2]x^2[x^2,x^3]x^3[x^3,x+x^2+x^3](x+x^2+x^3)[x+x^2+x^3,0]0.$

Definition 1.2 ((?), p. 43). Let $G = (V(G), E(G))$ be a graph where $V(G)$ ia the vertices set and $E(G)$ is the edges set. For $v_i,\ v_j \in V(G)$, let a_{ij} be a number of edges $[v_i, v_j]$. A matrix $A = (a_{ij})$ is called an **adjacent matrix** of G.

Definition 1.3. (?) Let R be a commutative ring with identity and let $Z(R)$ be its set of zero-divisors of R. By the **zero-divisor graph** $\Gamma(R)$ of R we mean the graph with vertices $Z(R) - \{0\}$ such that there is an (undirected) edge between vertices a and b if and only if $a \neq b$ and $ab = 0$.

Example 1.3. Since $\mathbf{Z}_{21} = \{0, 1, 2, \cdots, 20\}$, we have that the Euler function $\varphi(21) = 21(1 - \frac{1}{3})(1 - \frac{1}{7}) = 12$ and $|Z(\mathbf{Z}_{21})| = 21 - \varphi(21) - 1 = 8$. Therefore $\Gamma(\mathbf{Z}_{21}) = (V(\Gamma(\mathbf{Z}_{21})), E(\Gamma(\mathbf{Z}_{21})))$ where $V(\Gamma(\mathbf{Z}_{21})) = \{\mathbf{Z}_{21} - \{0\}\} = \{3, 6, 7, 9, 12, 14, 15, 18\}$ and $E(\Gamma(\mathbf{Z}_{15})) = \{[3,7], [3,14], [6,7], [6,14], [7,9], [7,12], [7,15], [7,18], [9,14], [12,14], [14,15], [14,18]\}$. Thus the number of edges of $\Gamma(\mathbf{Z}_{21})$ is 12.

Example 1.4. Let $A_{2,3}$ be a adjacent matrix of the graph $G(R_{2,3})$.

$$A_{2,3} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 1.5. Let B_{15} be a adjacent matrix of the zero-divisor graph $\Gamma(\mathbf{Z}_{15})$.

$$B_{15} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Since $G(\mathbf{Z}_n)$ is a simple (undirected) graph, the adjacent matrix A_n is a Boolean (that is, $a_{ij} = 0$ or 1) symmetric matrix by Definition 1.2.

Definition 1.4. (**?**) For an adjacent matrix A of a graph G,

$$\Phi(G;\lambda) = |\lambda E - A| = \lambda^n + C_1\lambda^{n-1} + C_2\lambda^{n-2} + \cdots + C_n \tag{1.1}$$

is called a **characteristic polynomial of** G where E is an identity matrix. Also, the solutions of $\Phi(G;\lambda) = 0$ are called **eigenvalues** of A.

Moreover, by [(**?**), p. 87], we have that

$$\Phi(G;\lambda) = |\lambda E - A| = \lambda^n + C_1\lambda^{n-1} + C_2\lambda^{n-2} + \cdots + C_n \tag{1.2}$$

be a characteristic polynomial $\Phi(G;\lambda)$ of G. Then we have that

$$C_k = (-1)^k \sum_{i_1<i_2<\cdots<i_k} \begin{vmatrix} a_{i_1i_1} & a_{i_1i_2} & \cdots & a_{i_1i_k} \\ a_{i_2i_1} & a_{i_2i_2} & \cdots & a_{i_2i_k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_ki_1} & a_{i_ki_2} & \cdots & a_{i_ki_k} \end{vmatrix} \tag{1.3}$$

Example 1.6. $\Phi(G(R_{2,3}),\lambda) = |\lambda E - A_{2,3}| = \lambda^8 - 9\lambda^6 - 4\lambda^5 + 8\lambda^4$.
$\Phi(\Gamma(\mathbf{Z}_{21}),\lambda) = \lambda^8 - 12\lambda^6$.

Proposition 1.1 ((**?**), p. 8). *If G is an undirected graph having no loops, then we have that the following statements.*

1. $C_1 = 0$;
2. $-C_2$ *equals the size* q *of* G;
3. $-C_3$ *equals twice number of triangles.*

Example 1.7. Let $\Phi(G(R_{2,3})) = \lambda^8 + C_1\lambda^7 + C_2\lambda^6 + C_3\lambda^5 + C_4\lambda^4 + C_5\lambda^3 + C_6\lambda^2 + C_7\lambda + C_8$ be a characteristic polynomial of $G(R_{2,3})$. By Example 1.6, we have that
$C_1 = 0, C_2 = -9, C_3 = -4 = -2 \times 2, C_5 = C_6 = C_7 = C_8 = 0$.

Definition 1.5. (**?**) Let M be a subset of the edges set $E(G)$ of G. Then M is called a **2-matching** if M consists two elements e, f and e, f is a property that e and f are independent, that is, e and f are not incident each other.

Example 1.8. We consider the 2-matchings in $G(R_{2,3})$. Each M_i is a 2-matching in $G(R_{2,3})$.
$M_1 = \{[0,1],[x^2,x+x^2]\}, M_2 = \{[0,x],[x^2,x+x^2]\}, M_3 = \{[x,x^2],[0,1+x]\}$,
$M_4 = \{[x,x^2],[0,1+x+x^2]\}, M_5 = \{[x,x^2],[0,x+x^2]\}, M_6 = \{[x,x^2],[1+x^2]\}$,
$M_7 = \{[0,1+x],[x^2,x+x^2]\}, M_8 = \{[0,1+x+x^2],[x^2,x+x^2]\}, M_9 = \{[0,1+x^2],[x+x^2]\}$
$M_{10} = \{[0,1],[x,x^2]\}$.

Proposition 1.2 ((?), P. 46). *If an undirected graph G has no loops and n_M, n_C are the number of 2-matchings, 4-cycles, respectively. Then we have that*

$$C_4 = n_M - 2n_C \tag{1.4}$$

Example 1.9. $\Phi(G(R_{2,3})) = \lambda^8+C_1\lambda^7+C_2\lambda^6+C_3\lambda^5+C_4\lambda^4+C_5\lambda^3+C_6\lambda^2+C_7\lambda+C_8$. We obtain that $n_M = 10$ and $n_C = 1$ in the graph $G(R_{2,3})$. We have that $C_4 = 8 = 10-2\times 1$. Also, $\chi(G(R_{2,3})) = 3$ and $\chi'(R_{2,3}) = 7$.

2 The Main Theorems

By Yinglie Jin and Mitsuo Kanemitsu [(?), Theorem 1], it is proved that $\chi(\mathbf{Z}_n) = 2$ is equivalent to $\Phi(G(\mathbf{Z}_n);\lambda) = \lambda^n - (n-1)\lambda^{n-2}$.
In this paper, when $\chi(\mathbf{Z}_n) = 3$, we study the characteristic polynomial $\Phi(G(\mathbf{Z}_n);\lambda)$ of a graph $G(\mathbf{Z}_n)$ and the number of 4-cycles n_C.

Proposition 2.1 ((?), Theorem 2). *For a graph $G(\mathbf{Z}_n)$, the following statements are equivalent.*
1. $\chi(\mathbf{Z}_n) = 3$;
2. *There exists a triangle and each triangle has a vertex 0;*
3. $n = pq$ (p, q *are distinct prime numbers*), $n = 2^2p$ (p *is a prime number*) *or* $n = 9$.

Remark 2.1. For any non-zero distinct vertices a, b of a graph $G(\mathbf{Z}_n)$, we have that a and b are adjacent if and only if (a,n) and (b,n) are adjacent, where (x,y) ia the greatest common divisor of x and y. In fact, $(a,0) = (a,n) = a$ in $\mathbf{Z}_n$.

Theorem 2.2. *Assume that $\chi(\mathbf{Z}_n) = 3$. Then the following statements hold.*
1. *If $n = pq$ (p, q are distinct prime numbers), then*

$$\begin{aligned}\Phi(G(\mathbf{Z}_n);\lambda) &= \lambda^n - (2pq-p-q)\lambda^{n-2} - 2(p-1)(q-1)\lambda^{n-3} \\ &\quad +(p-1)^2(q-1)^2\lambda^{n-4}\end{aligned} \tag{2.1}$$

2. *If* $n = 2^2 p$ (p *is a prime number,* $p \neq 2$), *then*

$$\begin{aligned}\Phi(G(\mathbf{Z}_n);\lambda) &= \lambda^n - (8p-5)\lambda^{n-2} - 8(p-1)\lambda^{n-3} + 2(p-1)(6p-5)\lambda^{n-4} \\ &\quad +4(p-1)^2\lambda^{n-5} - 4(p-1)^3\lambda^{n-6} \qquad (2.2)\end{aligned}$$

3. *If* $n = 2^2 \times 2 = 8$, *then*

$$\Phi(G(\mathbf{Z}_n);\lambda) = \lambda^8 - 9\lambda^6 - 4\lambda^5 + 8\lambda^4 \qquad (2.3)$$

4. *If* $n = 9$, *then*

$$\Phi(G(\mathbf{Z}_n);\lambda) = \lambda^9 - 9\lambda^7 - 2\lambda^6 + 6\lambda^5 \qquad (2.4)$$

Proof. By Proposition 2.1, we consider $\Phi(G(\mathbf{Z}_n);\lambda)$ where $n = pq$ (p, q are distinct prime numbers), $n = 2^2p$ (p is a prime number) and $n = 9$.
When $\forall a_1 \in M_1, \forall a_2 \in M_2, \cdots, \forall a_n \in M_n$, we denote $\forall(a_1, a_2, \cdots, a_n) \in (M_1, M_2, \cdots, M_n)$.
Also, we denote an adjacent matrices of a graph $G(\mathbf{Z}_n)$ by A_n.

1. $n = pq$ (p, q are distinct prime numbers). Since all divisors of n are $1,\ p,\ q,\ pq$, for $\forall m \in Z_n$, (m, n) is an element among $1,\ p,\ q,\ pq$.

   (1) In case $(m, n) = pq$. m is an element of $M_1 = \{0\}$, $|M_1| = 1$.

   (2) In case $(m, n) = p$. m is an element of $M_2 = \{p, 2p, \cdots, (q-1)p\}$, $|M_2| = q-1$.

   (3) In case $(m, n) = q$. m is an element of $M_3 = \{q, 2q, \cdots, (p-1)q\}$, $|M_3| = p-1$.

   (4) In case $(m, n) = 1$. m is an element of $M_4 = Z_n - M_1 - M_2 - M_3, |M_4| = (p-1)(q-1)$.

   Then, $\forall a, b \in M_i (i = 2, 3, 4)$ such that $a \neq b$, $n \not| ab$ and so $ab \neq 0$. Therefore a and b are not adjacent by Remark. Also, $\forall a, b \in M_i$ ($i = 2, 3, 4$), since the correspondence rows, columns, respectively of a and b in a minor determinant of A_n containing vertices a, b equal, those minor determinants are 0.
   Therefore $C_i = 0$ ($i \geq 5$). By Proposition 1.1,

$$\Phi(G(\mathbf{Z}_n);\lambda) = \lambda^n + C_2\lambda^{n-2} + C_3\lambda^{n-3} + C_4\lambda^{n-4} \qquad (2.5)$$

Next, any element of M_1 and any element of M_2, M_3, M_4 are adjacent. Similarly, any element of M_2 and any element of M_3 are also adjacent. Those are only adjacent elements. Since the number of edges equals $2pq - p - q$, we have that $C_2 = -(2pq - p - q)$ by Proposition 1.1. By Proposition 2.1, since the triangles of a graph $G(\mathbf{Z}_n)$ consists of each element of M_1, M_2, M_3 respectively, we have that the number of triangles equals $(p-1)(q-1)$. By Proposition 1.1, we have that $C_3 = -2(p-1)(q-1)$.

Also, the minor determinants of $\Phi(G(\mathbf{Z}_n); \lambda)$ correspondent to

$$\forall(a, b, c, d) \in (M_1, M_2, M_3, M_4),$$

equals 1, we have that the number of such the number of determinant having 1 equals $(p-1)^2(q-1)^2$. Then it follows from (1.3) that $C_4 = (p-1)^2(q-1)^2$. Consequently,

$$\begin{aligned} \Phi(G(\mathbf{Z}_n); \lambda) &= \lambda^n - (2pq - p - q)\lambda^{n-2} - 2(p-1)(q-1)\lambda^{n-3} \\ &\quad + (p-1)^2(q-1)^2\lambda^{n-4} \end{aligned} \tag{2.6}$$

2. In case $n = 2^2p$ ($p(\neq 2)$ is a prime number). Since all divisors of n are $1, 2, 2^2, p, 2p, 2^2p$, we have that, for $m \in \mathbf{Z}_n$, (m, n) is an element among $1, 2, 2^2, p, 2p, 2^2p$.
For $\forall m \in \mathbf{Z}_n$,

   (1) In case $(m, n) = 2^2p$. m is an element of $M_1 = \{0\}$, $|M_1| = 1$.

   (2) In case $(m, n) = 2p$. m is an element of $M_2 = \{2p\}$, $|M_2| = 1$.

   (3) In case $(m, n) = p$. m is an element of $M_3 = \{p, 3p\}$, $|M_3| = 2$.

   (4) In case $(m, n) = 2^2$. m is an element of $M_4 = \{2^2, 2^2 \cdot 2, \cdots, 2^2(p-1)\}$, $|M_4| = p - 1$.

   (5) In case $(m, n) = 2$. m is an element of $M_5 = \{2, 2 \cdot 3, \cdots, 2(p-2), 2(p+2), \cdots, 2(2p-1)\}$, $|M_5| = p - 1$.

   (6) In case $(m, n) = 1$. m is an element of $M_6 = Z_n - M_1 - M_2 - M_3 - M_4 - M_5$, $|M_6| = 2(p-1)$.

Also, by the statements in the proof of 1, for $\forall a, b \in M_i (i = 3, 4, 5, 6)$, the minor determinant containing vertices a, b of A_n equals 0, and so $C_i = 0$ $(i \geq 7)$. Hence,

$$\Phi(G(\mathbf{Z}_n); \lambda) = \lambda^n + C_2\lambda^{n-2} + C_3\lambda^{n-3} + C_4\lambda^{n-4} + C_5\lambda^{n-5} + C_6\lambda^{n-6} \tag{2.7}$$

Next, only the adjacent elements are the followings: any element of M_1 and any element of M_2, M_3, M_4, M_5, M_6; any element of M_2 and any element of M_4, M_5; any element of M_3 and any element of M_4, so the number of the all edges equals $8p-5$. By Proposition 1.1, we have that $C_2 = -(8p-5)$. By Proposition 2.1, a triangle of $G(\mathbf{Z}_n)$ are only the following: any three elements of M_1, M_2, M_4; M_1, M_2, M_5; M_1, M_3, M_4. The number of all triangles of $G(\mathbf{Z}_n)$ equals $4(p-1)$. By Proposition 1.1, we have that $C_3 = -8(p-1)$.

Also, the non-zero minor determinants of A_n correspondent $\forall(a,b,c,d) \in (M_{i_1}, M_{i_2}, M_{i_3}, M_{i_4})$, the following

$$\begin{aligned}\forall(a,b,c,d) \in \{&(M_1,M_2,M_3,M_5),\ (M_1,M_2,M_4,M_6),\ (M_1,M_2,M_5,M_6),\\ &(M_1,M_3,M_4,M_5),\ (M_1,M_3,M_4,M_6),\ (M_2,M_3,M_4,M_5)\},\end{aligned}$$

their determinants equal 1 and the their number equals $2(p-1)(6p-5)$. Then it follows from (1.3) that $C_4 = 2(p-1)(6p-5)$.

Moreover, the non-zero minor determinants of A_n correspondent $\forall(a,b,c,d,e) \in (M_{i_1}, M_{i_2}, M_{i_3}, M_{i_4}, M_{i_5})$ are only the following case:

$\forall(a,b,c,d,e) \in (M_1,M_2,M_3,M_4,M_5)$. Their correspondent minor determinant equals -2, and their number equals $2(p-1)^2$. Then it follows from (1.3) that $C_5 = 4(p-1)^2$.

Finally, the minor determinants of A_n correspondent

$\forall(a,b,c,d,e,f) \in (M_1,M_2,M_3,M_4,M_5,M_6)$ all equal -1, the number of their minor determinants equal $4(p-1)^3$. Then it follows from (1.3) that $C_6 = -4(p-1)^3$.

Hence,

$$\begin{aligned}\Phi(G(\mathbf{Z}_n);\lambda) &= \lambda^n - (8p-5)\lambda^{n-2} - 8(p-1)\lambda^{n-3} + 2(p-1)(6p-5)\lambda^{n-4}\\ &\quad +4(p-1)^2\lambda^{n-5} - 4(p-1)^3\lambda^{n-6} \qquad (2.8)\end{aligned}$$

3. $n = 8$. Since all divisors of n are $1,2,4,8$, for $\forall m \in \mathbf{Z}_n$, (m,n) is an element among $1,2,4,8$.

   (1) $(m,n) = 8$. m is an element of $M_1 = \{0\}$, $|M_1| = 1$.

   (2) $(m,n) = 4$. m is an element of $M_2 = \{4\}$, $|M_2| = 1$.

   (3) $(m,n) = 2$. m is an element of $M_3 = \{2,6\}$, $|M_3| = 2$.

(4) $(m, n) = 1$. m is an element of $M_4 = \{1, 3, 5, 7\}$, $|M_4| = 4$.

Then, in the same manner of the proof of 1, we have that

$$\Phi(G(\mathbf{Z}_n); \lambda) = \lambda^8 - 9\lambda^6 - 4\lambda^5 + 8\lambda^4 \tag{2.9}$$

4. $n = 9$. Since the divisors of n are $1, 3, 9$, for $\forall m \in \mathbf{Z}_n$, (m, n) is any one among $1, 3, 9$.

   (1) $(m, n) = 9$. m is an element of $M_1 = \{0\}$, $|M_1| = 1$.

   (2) $(m, n) = 3$. m is an element of $M_2 = \{3, 6\}$, $|M_2| = 2$.

   (3) $(m, n) = 1$. m is an element of $M_3 = \{1, 2, 4, 5, 7, 8\}$, $|M_3| = 6$.

   Then any two distinct elements of M_2 are adjacent, and in the same manner of the proof in 1,

$$\Phi(G(\mathbf{Z}_n); \lambda) = \lambda^9 - 9\lambda^7 - 2\lambda^6 + 6\lambda^5 \tag{2.10}$$

□

Theorem 2.3. *Let G be a simple graph having the order n and let $(d_1, d_2, \cdots, d_n)$ be the degree sequence of G. Then the number of 2-matching of G*

$$\frac{1}{8}\left(\sum_{i=1}^{n} d_i\right)^2 - \frac{1}{2}\sum_{i=1}^{n} {d_i}^2 + \frac{1}{4}\sum_{i=1}^{n} d_i \tag{2.11}$$

where d_i is a degree of a vertex v_i $(i = 1, 2, \cdots, n)$ of G.

Proof. Let $A = (a_{ij})$ be an adjacent matrix of G and let $||A|| = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$. Then $||A|| = \sum_{i=1}^{n} d_i$. Also, two edges e_{ij}, e_{kl} is a 2-matching if and only if i, j, k, l are each their distinct numbers. The number of a 2-matching $\{e_{ij}, e_{kl}\}$ equals $a_{ij}a_{kl}$ (where, $a_{ij} = a_{kl} = 1$). Therefore, by the definitions of adjacent matrix and degree of vertex, the number of all 2-matchings containing an edge e_{ij}

$$\frac{1}{2}a_{ij}(||A|| - 2d_i - 2d_j + 2) = a_{ij}\left(\frac{1}{2}||A|| - d_i - d_j + 1\right) \tag{2.12}$$

Hence, since $\sum_{j=1}^{n} a_{ij} = d_i$ and $\sum_{i=1}^{n} a_{ij} = d_j$, the number of all 2-matching of a graph G

$$\begin{aligned}
& \frac{1}{2}\sum_{i=1}^{n}\frac{1}{2}\left[\sum_{j=1}^{n}\frac{1}{2}a_{ij}\left(\frac{1}{2}||A|| - d_i - d_j + 1\right)\right] \\
= & \frac{1}{8}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}||A|| - \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}d_i - \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}d_j + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij} \\
= & \frac{1}{8}||A||\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij} - \frac{1}{4}\sum_{i=1}^{n}d_i\sum_{j=1}^{n}a_{ij} - \frac{1}{4}\sum_{j=1}^{n}d_j\sum_{i=1}^{n}a_{ij} + \frac{1}{4}||A|| \\
= & \frac{1}{8}\left(\sum_{i=1}^{n}d_i\right)^2 - \frac{1}{2}\sum_{i=1}^{n}{d_i}^2 + \frac{1}{4}\sum_{i=1}^{n}d_i \qquad (2.13)
\end{aligned}$$

□

Example 2.1. Beck's graph $G(R_{2,4})$ associated with $R_{2,4} = \mathbf{Z}_2[X]/(X^4) = \mathbf{Z}_2[x]$ $(x^4 = 0)$ has eight vertices. This graph is not a Euler graph. The degree sequence of this graph is $(15, 7, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1)$. We have that $n_M = 95$ by Theorem 2.3.

Theorem 2.4. *Assume that $\chi(\mathbf{Z}_n) = 3$. Then the following statements hold.*

1. $n = pq$ *(p, q are two distinct prime numbers), the number of all 4-cycles in a graph $G(\mathbf{Z}_n)$ equals $\frac{1}{4}(p-1)(q-1)(pq-4)$.*

2. $n = 2^2p$ *($p(\neq 2)$ is a prime number), the number of all 4-cycles of a graph $G(\mathbf{Z}_n)$ equals $\frac{1}{2}(p-1)(9p-10)$.*

3. $n = 2^2 \times 2 = 8$*, the number of all 4-cycles in a graph $G(\mathbf{Z}_n)$ equals* 1.

4. $n = 9$*, there is no 4-cycle of a graph $G(\mathbf{Z}_n)$.*

Proof. We consider the cases $n = pq$ (p, q are distinct prime numbers), $n = 2^2p$ (p is a prime number) and $n = 9$. In a graph $G(\mathbf{Z}_n)$, by Proposition 2.1, we consider the number of all 4-cycles in the above cases.

1. $n = pq$ (p, q are distinct prime numbers). By the proof in Theorem 2.2, let $M_1 = \{0\}$; $M_2 = \{p, 2p, \cdots, (q-1)p\}$; $M_3 = \{q, 2q, \cdots, (p-1)q\}$; $M_4 = Z_n - M_1 - M_2 - M_3$, we have that $Z_n = M_1 \bigcup M_2 \bigcup M_3 \bigcup M_4$, $|M_1| = 1$; $|M_2| = q - 1$; $|M_3| = p - 1$; $|M_4| = (p-1)(q-1)$. Also, any distinct two elements a, b of $M_i (i = 2, 3, 4)$ are not adjacent. The adjacent elements are the following

case: any element of M_1 and any element of M_2, M_3, M_4; any element of M_2 and any element of M_3 are adjacent.

Therefore, the degree sequence of a graph $G(\mathbf{Z}_n)$ is a following:

$$(pq-1, \underbrace{p, \cdots, p}_{q-1}, \underbrace{q, \cdots, q}_{p-1}, \underbrace{1, \cdots, 1}_{(p-1)(q-1)}).$$

Also, $\sum_{j=1}^{n} d_i = 4pq - 2p - 2q$.

Hence, by Theorem 2.3, the number of 2-matching of a graph $G(\mathbf{Z}_n)$ is

$$\begin{aligned} n_M &= \frac{1}{8}\left(\sum_{i=1}^{n} d_i\right)^2 - \frac{1}{2}\sum_{i=1}^{n} d_i{}^2 + \frac{1}{4}\sum_{i=1}^{n} d_i \\ &= \frac{3}{2}p^2q^2 - \frac{5}{2}p^2q - \frac{5}{2}pq^2 + \frac{5}{2}pq + p^2 + q^2 - 1 \\ &= \frac{1}{2}(p-1)(q-1)(3pq - 2p - 2q - 2). \end{aligned} \qquad (2.14)$$

By Proposition 1.2 and Theorem 2.2, we have that

$$n_C = \frac{1}{2}(n_M - C_4) = \frac{1}{4}(p-1)(q-1)(pq-4). \qquad (2.15)$$

Therefore, in the case $n = pq$ (p, q are distinct prime numbers), the number of 4-cycles in a graph $G(\mathbf{Z}_n)$ equals $\frac{1}{4}(p-1)(q-1)(pq-4)$.

2. $n = 2^2p$ ($p(\neq 2)$ is a prime number). By the proof of Theorem 2.2, let $M_1 = \{0\}$; $M_2 = \{2p\}$; $M_3 = \{p, 3p\}$; $M_4 = \{2^2, 2^2\cdot 2, \cdots, 2^2(p-1)\}$; $M_5 = \{2, 2\cdot 3, \cdots, 2(p-2), 2(p+2), \cdots, 2(2p-1)\}$; $M_6 = Z_n - \sum_{i=1}^{5} M_i$, we have that $Z_n = \bigcup_{i=1}^{6} M_i$, $|M_1| = |M_2| = 1$; $|M_3| = 2$; $|M_4| = |M_5| = p-1$; $|M_6| = 2(p-1)$. Also, any distinct two elements a, b of $M_i (i = 3, 4, 5, 6)$ are not adjacent. Moreover, any element of M_1 and any element of M_2, M_3, M_4, M_5, M_6; any element of M_2 and any element of M_4, M_5; any element of M_3 and any element of M_4 only adjacent elements.
Therefore, the degree sequence of $G(\mathbf{Z}_n)$ is

$$(4p-1, 2p-1, p, p, \underbrace{4, \cdots, 4}_{p-1}, \underbrace{2, \cdots, 2}_{p-1}, \underbrace{1, \cdots, 1}_{2(p-1)})$$

and $\sum_{j=1}^{n} d_i = 16p - 10$.

Hence, the number of all 2-matchings equals

$$\begin{aligned} n_M &= \frac{1}{8}\left(\sum_{i=1}^{n} d_i\right)^2 - \frac{1}{2}\sum_{i=1}^{n} {d_i}^2 + \frac{1}{4}\sum_{i=1}^{n} d_i \\ &= 21p^2 - 41p - 20 \\ &= (p-1)(21p-20) \end{aligned} \quad (2.16)$$

by Theorem 2.3. By Proposition 1.2 and Theorem 2.2,

$$n_C = \frac{1}{2}(n_M - C_4) = \frac{1}{2}(p-1)(9p-10) \quad (2.17)$$

Therefore, in the case $n = 2^2p$ ($p(\neq 2)$ ia a prime number), the number of 4-cycles in a graph $G(\mathbf{Z}_n)$ equals $\frac{1}{2}(p-1)(9p-10)$.

3. $n = 8$. By the proof in Theorem 2.2, let $M_1 = \{0\}$; $M_2 = \{4\}$; $M_3 = \{2,6\}$; $M_4 = \{1,3,5,7\}$, we have that $\mathbf{Z}_8 = M_1 \bigcup M_2 \bigcup M_3 \bigcup M_4$, $|M_1| = |M_2| = 1$; $|M_3| = 2$; $|M_4| = 4$. Also, any distinct two elements of $M_i (i = 3,4)$ are not adjacent. Moreover, all adjacent elements of this graph are the following: any element of M_1 and any element of M_2, M_3, M_4; any element of M_2 and any element of M_3.
Hence, the degree sequence of a graph $\mathbf{Z}_n$ is $(7,3,2,2,1,1,1,1)$, and $\sum_{j=1}^{n} d_i = 18$.
Therefore, by Theorem 2.3, the number of all 2-matchings in a graph $G(\mathbf{Z}_n)$ equals

$$n_M = \frac{1}{8} \times 18^2 - \frac{1}{2} \times (7^2 + 3^2 + 2^2 + 2^2 + 4) + \frac{1}{4} \times 18 = 10 \quad (2.18)$$

By Proposition 1.2 and Theorem 2.2,

$$n_C = \frac{1}{2}(n_M - C_4) = \frac{1}{2} \times (10-8) = 1 \quad (2.19)$$

Hence, in the case $n = 8$, the number of 4-cycles in a graph $G(\mathbf{Z}_n)$ equals 1.

4. $n = 9$. By Theorem 2.2, let $M_1 = \{0\}$; $M_2 = \{3,6\}$; $M_3 = \{1,2,4,5,7,8\}$, $\mathbf{Z}_8 = M_1 \bigcup M_2 \bigcup M_3$, $|M_1| = 1$; $|M_2| = 2$; $|M_3| = 6$. Also, any distinct two elements a, b of M_4 are not adjacent. Moreover, only adjacent elements are the following: any element of M_1 and any element of M_2, M_3; any two elements of M_2.
Therefore, the degree sequence of a graph $G(\mathbf{Z}_n)$ is $(8,2,2,1,1,1,1,1,1)$ and

$\sum_{j=1}^{n} d_i = 18.$

Hence, by Theorem 2.3, the number of all 2-matchings in a graph $G(\mathbf{Z}_n)$

$$n_M = \frac{1}{8} \times 18^2 - \frac{1}{2} \times (8^2 + 2^2 + 2^2 + 6) + \frac{1}{4} \times 18 = 6 \quad (2.20)$$

and so, by Proposition 1.2 and Theorem 2.2,

$$n_C = \frac{1}{2}(n_M - C_4) = \frac{1}{2} \times (6 - 6) = 0. \quad (2.21)$$

Therefore, in the case $n = 9$, a graph $G(\mathbf{Z}_n)$ has no 4-cycle.

□

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

On the Continuity of the Best Copositive Approximation Function

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ABSTRACT

In this paper the author studies the continuity of the best copositive approximation function that maps C(Q) onto any of its finite dimensional Haar subspaces, when Q is any compact subset of the real numbers. In the case when M is a Z-subspace of C(Q), the author characterizes those $f \in C(Q)$ at which the copositive metric projection is continuous. He also proves that the copositive metric projection as a function, is always discontinuous.

Key words: Chebyshev spaces, Haar Subspaces, Z-subspace, best copositive approximation, copositive metric projection.

AMS Subject Classification: 41A65.

1. INTRODUCTION

If A is a subset of the normed linear space X, and $x \in X$, then the *distance* d(x, A) from x to A is defined to be; $d(x, A) = \inf\{\| x-y \|; y \in A\}$. The element $y_0 \in A$ is called a *best approximation* for x from A if $d(x, A) = \|x-y_0\|$. The subset A is said to be *proximinal* in X iff the best approximation for each $x \in X$ from A is attained. The best approximation for the element x from A need not be unique. If for each $x \in X$, the best approximation for x from A is unique then A is called a *Chebyshev subset* of X. If M is proximinal in X, then the set valued function P_M defined on X by $P_M(f) = \{g \in M;$ g is a best approximation for f from M$\}$ is called the *metric projection* from X onto M. If Q is a compact Hausdorff space then C(Q) denotes the Banach space of all continuous real valued functions on Q together with the uniform norm, that is, $\|f\| = \max\{|f(x)|; x \in Q\}$. If Q is a compact totally ordered space then the n-dimensional subspace M of C(Q) is a Chebyshev subspace of C(Q) iff each $g \neq 0$ in M has at most n-1 zeros, [See singer [9] Theorem 2.2, page 215). In the case when Q = [a,b], a closed and bounded interval of real numbers, the n-dimensional Chebyshev subspace M of C[a,b] has the property that each $g \neq 0$ in M, has no more than n-1 changes of sign, that is, for each $g \neq 0$ in M, there do not exist n+1 points, $x_1 < x_2 < \dots < x_{n+1}$ in Q such that $g(x_i)g(x_{i+1}) < 0$, for all i =1, 2, . . ., n. This property turn out to be essential to guarantee the alternation of the error in approximation. (See Kamal [4]). Unfortunately this property fail to exist when Q is not connected. To overcome this problem some authors add it implicitly to the definition of the n- dimensional Chebyshev subspace of C(Q) (See Taylor [10, page 241]). As in Zielke [12, page 7], the n-dimensional Chebyshev subspace M of C(Q) is called a Haar subspace of C(Q) iff each $g\neq 0$ in M has no more than n-1 changes of sign. If M is a subspace of C(Q), and $f\in C(Q)$, then $g\in M$ is said to be *copositive* with f on Q iff $f(x)g(x)\geq 0$ for all $x\in Q$. The element $g_0\in M$ is called a *best copositive approximation* for f from M iff g_0 is copositive with f on Q and $\|f- g_0\| = \inf\{\| f- g \|;\ g\in M$, and g is copositive with f on Q}. The set $\{g\in M$; g is copositive with f on Q} is closed, so if the dimension of M is finite, then the best copositive approximation for f from M is attained. In this case the set valued function C_M defined on X by $C_M(f) = \{g\in M$; g is a best copositive approximation for f from M} will be called the *copositive metric projection* from C(Q) onto M. Copositive approximation is as important as ordinary approximation. In some fields of science (for example electricity), one want to approximate a complicated function by a simpler one without loosing the sign of the original function. Schumaker and Taylor [8] were the first to study the uniqueness of best copositive approximation as a part of their study on polynomial having restricted ranges. Zhong [11], showed that if Q is a closed and bounded interval [a, b] of the real numbers, and M is a finite dimensional Haar subspace of C[a,b] then the best copositive approximation is unique for any element $f\in C[a,b]$ that does not vanish on any infinite subinterval of [a,b]. Kamal [4], obtained the same result when Q is any compact subset of real numbers, and M is any finite dimensional Haar subspace of C(Q). So if Q is any compact subset of the real numbers, G is the set of all elements $f\in C(Q)$ that does not vanish on any subinterval of Q, and M is any finite dimensional Haar subspace of C(Q) then the copositive metric projection C_M from G onto M is single valued function.

The continuity of the metric projection P_N, and the copositive metric projection C_M are very important in the applications of Approximation Theory in the other branches of science. It helps in writing an algorithm to calculate the best approximation, and the best copositive approximation. From the Corollary of Lemma 1, page 580 of Brown [1] it is clear that if Q is any compact subset of real numbers, and M is a finite dimensional Chebyshev subspace of C(Q) then the metric projection $P_M : C(Q)\to M$ is continuous as a single valued function. It is easy to show that generally the copositive metric projection $C_M : C(Q)\to M$ is not continuous for any finite dimensional Haar subspace M of C(Q), and any compact subset Q of the real numbers, that have more than two elements. (See Theorem 2.8 in this paper). The more interesting question is " For the finite dimensional Haar subspace M of C(Q), what are the elements $f_0 \in C(Q)$ at which C_M is continuous". The answer for these question is included in solving the following conjecture .

1.1 Problem: *Let n be any natural number, and assume that Q is a compact subset of real numbers containing at least n+1 points. Let M be an n- dimensional Haar subspace of C(Q), and let f_0 be a non zero element in C(Q). The copositive metric projection C_M is continuous at f_0 iff "$f_0(x) \neq 0$ for all $x \in Q$ or the number of changes of f_0 is at least n".*

#

The subspace M of C(Q) is called a Z-subspace of C(Q) if no $g \neq 0$ in M vanishes on some non empty open interval in Q. When M is an n-dimensional Haar subspace of C(Q), then M is a Z-subspace of C(Q) iff $g(x) \neq 0$ for any $g \neq 0$ in M, and any isolated point x in Q. The Z-subspaces played an important role in the study of continuous selection for the metric projection. It was introduced for the first time in 1969, by A.J. Lazar, D.E. Wulbert, and P.D. Morris [6]. For the importance of the Z-subspaces in the theory of continuous selection, on can see also Al. Brown[2], and A. Kamal [5].

In this paper the author studies problem 1.1 in details, and gives a positive answer when M is a Z-subspace of C(Q). He shows that this condition is essential. When Q has no isolated points, (for example Q = [a,b], a closed and bounded interval, or when Q is a compact union of intervals), then every n-dimensional Haar subspace of C(Q) is a Z-subspace. So in this case one can ignore this condition.

In the rest of this section, some notations and results will be mentioned to be used later in section 2. If Q is a compact subset of real numbers, and $x<y$ in Q then the intervals [x,y], (x,y), and (x,y] are defined in Q as usual, that is; $[x,y] = \{q \in Q;\ x \leq q \leq y\}$. The function $f \in C(Q)$ is said to be "*admissible*" on Q, if there do not exist $x < y$ in Q such that the interval [x,y] in Q contains an infinite number of points and $f(z) = 0$ for all $z \in [x,y]$. The definition of changing sigh is clear when Q is a connected interval. But in the general case it needs to be clarified. The function $f \in C(Q)$ is said to have at "*least k changes of sign in Q*" if there are k+1 point $t_1<t_2< \dots ,t_{k+1}$ in Q, so that $f(t_i)f(t_{i+1})< 0$ for all i = 1, 2, ... ,k. The "*number of change of sign of f*" is defined to be *sup{k; f has at least k changes of sign}*.

1.2. Proposition : *Assume that Q is a compact subset of real numbers containing at least n+1 points, M is an n-dimensional Haar subspace of C(Q), and f is an element in C(Q)\M, having h changes of sign. If g is a best copositive approximation for f from M, then $g \neq 0$ iff $h \leq n-1$.*

1.3. Theorem : *Assume that Q is a compact subset of real numbers having at least n+1 points, and that M is an n-dimensional Haar subspace of C(Q). If f is an admissible function in C(Q)\M, then the best copositive approximation for f from M is unique.*

Proposition 1.2, and Theorem 1.3 were proved first by Zhong [11] for Q = [a,b], a closed and bounded interval of real numbers, and by Kamal [3] for any compact subset of real numbers.

#

The following proposition concerns Haar subspaces, its proof can be found in Zielke [12].

1.4. Proposition : *Assume that Q is a compact subset of real numbers containing at least n+1 points, and that M is an n-dimensional Haar subspace of C(Q). The following facts hold;.*

i- If $x_1 < x_2 < \ldots < x_{n-1}$ are n-1 points in Q, then there is $g \in M$, satisfying that g(x)=0 for all $x \in \{x_1, x_2, \ldots, x_{n-1}\}$, and

1- g(x)>0 if $x < x_1$,

2- $(-1)^n g(x) > 0$ if $x > x_{n-1}$, and

3- $(-1)^i g(x) > 0$ if $x \in (x_i, x_{i+1})$, and i=1,2, ... ,n-2.

ii- No $g \neq 0$ in M alternates weakly at n+1 points in Q, that is, there do not exist $x_1 < x_2 < \ldots < x_{n+1}$ in Q, and $g \neq 0$ in M such that $(-1)^i g(x_i) \geq 0$ for each i= 1, 2, ... ,n+1.

Part i in Proposition 1.4, can be obtained from Lemma 6.5.a, page 25 of Zielke [12]. Part ii, is Lemma 3.1.b, page 8 of Zielke [12].

#

2. THE CONTINUITY OF BEST COPOSITIVE APPROXIMATION

We start this section by writing the proof of one direction of the conjecture in Problem 1.1.

2.1. Theorem: *Let n be any positive integer, Q be any compact subset of the real numbers containing at least n+1 points, and let M be any n-dimensional Haar subspace of C(Q). If f_0 is an element in C(Q) satisfying that $f_0(x) \neq 0$ for all $x \in Q$, or if f_0 has at least n changes of sign, then the copositive metric projection C_M is continuous at f_0.*

Proof: Assume first that f_0 has at least n changes of sign. Let $\{f_m\}$ be any sequence in C(Q) converging to f_0, then one can assume without loss of generality that each f_m has at least n changes of sign. By proposition 1.2, $C_M(f_m) = 0$ for all m, and $C_M(f_0) = 0$. Thus C_M is continuous at f_0. Second assume that $f_0(x) \neq 0$ for all $x \in Q$. Let $\{f_m\}$ be any sequence in C(Q) converging to f_0. Since $f_0(x) \neq 0$ for all $x \in Q$, and Q is compact, then one can assume without loss of generality that for each m=1, 2, ..., $f_m(x) f_0(x) > 0$ for all $x \in Q$. Thus if $g \in M$ is copositive with f_0 then it is copositive with f_m for each m. Let g_0 be a best copositive approximation for f_0 from M, and for each m, let g_m be a best copositive approximation for f_m from M. It is clear that f_0 is admissible. Thus by Theorem 1.3, g_0 is unique. Since $\{g_m\}$ is a bounded sequence in the finite dimensional space M, then to show that $\{g_m\}$ converges to g_0, it is enough to show that for each subsequence $\{g_{m_i}\}$ of $\{g_m\}$, if $\{g_{m_i}\}$ converges, then it converges to g_0. Assume not, and let $\{g_{m_i}\}$ be a subsequence of $\{g_m\}$ that converges to $h_0 \neq g_0$. It is obvious that h_0 is copositive with f_0, and since g_0 is the unique best copositive approximation for f_0 from M, it follows that there is a positive number $\varepsilon>0$, satisfying that $\|f_0-g_0\| < \|f_0-h_0\|-4\varepsilon$. But $\{f_{m_i}\}$ converges to f_0, and $\{g_{m_i}\}$ converges to h_0. So there is a natural number N such that if $m_i>N$, then $\|f_{m_i}-f_0\|<\varepsilon$, and $\|g_{m_i}-h_0\|<\varepsilon$. Now for each $m_i>N$, $\|f_{m_i}-g_0\| \leq \|f_0-g_0\| + \|f_{m_i}-f_0\| < \|f_0-g_0\|+\varepsilon < \|f_0-h_0\|-3\varepsilon \leq \|f_0-f_{m_i}\| + \|f_{m_i}-g_{m_i}\| + \|g_{m_i}-h_0\|-3\varepsilon < \varepsilon + \|f_{m_i}-g_{m_i}\|+\varepsilon-3\varepsilon = \|f_{m_i}-g_{m_i}\|-\varepsilon$. Since g_0 is copositive with f_{m_i}, it follows that g_{m_i} is not a best copositive approximation for f_{m_i} from M, which is a contradiction.

#

The next Theorem gives a partial solution for the other direction of the conjecture in Problem 1.1.

2.2. Theorem: *Assume that Q is any infinite compact subset of the real number, M is an n-dimensional Haar subspace of C(Q), and let f_0 be a non zero element in C(Q). If the copositive metric projection C_M is continuous at f_0. then either $f_0(x) \neq 0$ for all limit points $x \in Q$,, or the number of changes of f_0 is at least n,*

Proof: By contradiction assume that $f_0(x_0) = 0$ for some limit points $x_0 \in Q$, and that the number of changes of f_0 is at most n-1. It will be shown that the metric projection C_M is not continuous at f_0. Let h be the number of changes of sign of f_0. Since x_0 is a limit point in Q, and $f_0(x_0) = 0$, then there is a nested family of neighborhoods $\{U_m\}$ for x_0 in Q, satisfying that $U_{m+1} \subseteq U_m$ for each m, $\bigcap_{m=1}^{\infty} U_m = \{x_0\}$, and $|f_0(x)| \leq \frac{1}{m}$ for each $x \in U_m$. For each m, define f_m in C(Q) so that $f_m(x) = f_0(x)$ for all $x \notin U_m$, $|f_m(x)| \leq \frac{1}{m}$ for all $x \in U_m$, and f_m has at least n-h+1 changes of sign in U_m. It is clear that f_m has more than n-1 changes of sign in Q. By Proposition 1.2, if g_m is a best copositive approximation for f_m from M, then $g_m = 0$, and if g_0 is a best copositive approximation for f_0 from M then $g_0 \neq 0$. It is obvious that the sequence $\{f_m\}$

in C(Q) converges to f_0, meanwhile the sequence $\{C_M(f_m)\}$ does not converges to $C_M(f_0)$. So the copositive metric projection C_M is not continuous at f_0.

#

When Q is a compact union of closed and bounded real interval, then every point is a limit point. In this case, and other more general cases, Theorem 2.1 and Theorem 2.2 give the proof of Corollary 2.3

2.3 Corollary: *Assume that Q is a compact subset of real numbers that has no isolated points, and that M is an n-dimensional Haar subspace of C(Q). If f_0 is a non zero element in C(Q), then the copositive metric projection C_M is continuous at f_0 if and only if $f_0(x) \neq 0$ for all $x \in Q$ or the number of changes of f_0 is at least n.*

#

Corollary 2.3 gives a positive answer for Problem 1.1 when Q is a compact subset of real numbers that has no isolated points. Next Theorem discusses the case when Q has an isolated point.

2.4. Theorem: *Let Q be a compact subset of real numbers containing at least n+1 points, M be any n-dimensional Haar Z-subspace of C(Q), f_0 be any non zero element in C(Q). If the copositive metric projection C_M is continuous at f_0, then either $f_0(x) \neq 0$ for all $x \in Q$,; or the number of changes of f_0 is at least n,*

Proof: By contradiction assume that the number of changes of f_0 is at most n-1 If $f_0(x_0) = 0$ for some limit points $x_0 \in Q$ then by Theorem 2.2, the copositive metric projection C_M is not continuous at f_0. So one may without loss of generality that $f(x) \neq 0$ for each limit point x in Q. Since Q is compact, then in this case, f will never vanish on any infinite interval. So f is admissible function. By Theorem 1.3, its best copositive approximation from M is unique. Assume there is isolated point x_0 in Q satisfying that $f(x_0) = 0$. Let g_0 be the unique best copositive approximation for f from M. Since the number of changes of f_0 is at most n-1. then by Proposition 1.2, $g_0 \neq 0$, and since M is a Z-subspace, then $g_0(x_0) \neq 0$. Define $f_m \in C(Q)$ by $f_m(x) = f_0(x)$ for each $x \neq x_0$, and $f_m(x_0) = \frac{\delta}{m}$, and choose $\delta = \pm 1$, so that $\delta g_0(x_0) < 0$. It is clear that the sequence $\{f_m\}$ converges to f_0. But if g_m is the best copositive approximation for f_m from M then since $g_m(x_0) f_m(x_0) \geq 0$. it follows that $\delta g_m(x_0) \geq 0$. Thus the sequence $\{g_m(x_0)\}$ does not converges to $g_0(x_0)$. So $\{g_m\}$ does not converges to g_0 in M. Therefore the copositive metric projection C_M is not continuous at f_0

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2.5. Corollary: *Assume that Q is a compact subset of real numbers that has at least n+1 points, and that M is an n-dimensional Haar Z-subspace of C(Q). If f_0 is a non zero element* *in C(Q), then the copositive metric projection C_M is continuous at f_0 if and only if $f_0(x) \neq 0$ for all $x \in Q$ or the number of changes of f_0 is at least n.*

#

In Theorem 2.4, the assumption that M is a Z-subspace is essential. If M is not Z-subspace of C(Q), then it is possible to find a non zero element f_0 in C(Q) that has no more than n-1 changes of sign, and that has a zero at some isolated point x_0 in Q, so that the copositive metric projection C_M is continuous at f_0. To show that we need the following proposition that can be obtained from Theorem 3 of Passow and Taylor [7] page 133.

2.6. Proposition : *Assume that Q is a finite subset of the real numbers containing at least n+1 points, and that M is an n-dimensional Haar subspace of C(Q). Let $f \in C(Q) \setminus M$ be a function that has no more than n-1 changes of sign, and let $g \in M$ be copositive with f. Define $X(f, g) = \{q \in Q;\ |f(q) - g(q)| = \|f - g\| \text{ or } g(q) = 0 \text{ and } f(q) \neq 0\}$, and define $\sigma : X(f,g) \Rightarrow \{1, -1\}$ by $\sigma(q) = sign[f(q) - g(q)]$. Then g is the unique best copositive approximation for f from M if and only if there are $t_1 < t_2 < \ldots < t_{n+1}$ in X(f,g) and $\delta = \pm 1$ so that $(-1)^i \delta \sigma(t_i) = 1$ for each i.*

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2.7. Theorem: *Assume that $n \geq 2$, and that Q is a finite subset of real number having at least n+2 elements. Let M be an n-dimensional Haar subspace of C(Q). If M is not Z-subspace then there is a non zero function f_0 in C(Q) with the properties that f_0 has no more than n-1 changes of sign, and that f_0 has a zero at some isolated point x_0 in Q, but the copositive metric projection C_M is continuous at f_0.*

Proof: Since M is not Z-subspace of M then there is a non zero element g_0 in M, and an isolated point x_0 in Q so that $g_0(x_0) = 0$. Choose g_0 so that $g_0(x) \neq 0$ for at least n+1 points in Q. Let $x_1 < x_2 < \ldots < x_{n+1}$ be points in Q so that $g_0(x_i) \neq 0$ for all i = 1, 2, ... ,n+1. Let $\delta = \min\{|g_0(x_1)|, |g_0(x_2)|, \ldots, |g_0(x_{n+1})|\}$, $A = \{x \in Q;\ g_0(x) = 0\}$, and $B = Q \setminus (A \cup \{x_1, x_2, \ldots, x_{n+1}\})$. Define $f_0 \in C(Q)$ by $f_0(x_i) = g_0(x_i) + (-1)^i \frac{\delta}{2}$ for i = 1, 2, ... ,n+1, $f_0(x) = 0$ for $x \in A$. For $x \in B$, choose the value of $f_0(x)$ so that $f_0(x)g_0(x) > 0$, and $|f_0(x) - g_0(x)| = \frac{\delta}{4}$. Since g_0 is copositive with f_0 on Q, $\|f_0 - g_0\| = \frac{\delta}{2}$, and $[f_0(x_i) - g_0(x_i)] = (-1)^i \frac{\delta}{2}$ for each i =1, 2, ... ,n+1, then by Proposition 2.6, g_0 is the unique best copositive approximation for f_0 from M. Let $\{f_m\}$ be any sequence in C(Q) converging to f_m. There is a natural number N, so that if m>N then $f_m(x)f_0(x) > 0$ for all $x \notin A$. So f_m is copositive with g_0 for all m>N. Without loss of generality assume that for all m, f_m is copositive with g_0. Let h_m be the best copositive approximation for f_m from M. The sequence $\{h_m\}$ is bounded in the finite dimensional space M. So to complete the proof it is enough to show that every convergent subsequence of $\{h_m\}$ converges to g_0. Without loss of generality assume that $\{h_m\}$ converges to h_0. Since $f_m(x)f_0(x)>0$, for all $x \notin A$, and $h_m(x)f_m(x) \geq 0$ for all $x \in Q$, it follows that $h_m(x)f_0(x) \geq 0$ for all x in Q. So $h_0(x)f_0(x) \geq 0$ for all x in Q. Thus h_0 is copositive with f_0 on Q. On the other hand $\| f_0 - h_0 \| = \lim_{m\to\infty} \| f_m - h_m \| \leq \lim_{m\to\infty} \| f_m - g_0 \| = \| f_0 - g_0 \|$. Thus h_0 is a best copositive approximation for f_0 from M. But the best copositive approximation is unique. So $h_0 = g_0$.

#

The next Theorem shows that the copositive metric projection is always discontinuous.

2.8. Theorem: *Let Q be any compact subset of the real number containing more than n+1 points, and let M be any n-dimensional Haar subspace of C(Q). There is a function $f_0 \in C(Q)$ so that the copositive metric projection C_M is not continuous at f_0.*

Proof: Since Q is a compact subset of real numbers, then either Q has a limit point, or Q has an isolated point. If Q has a limit point x_0, then choose $f_0 \in C(Q)$ so that f_0 has at most n-1 changes of sign, and that $f_0(x_0) = 0$. By Theorem 2.2, the metric projection C_M is not continuous at f_0. In the second case, that is, when Q has an isolated point x_0, choose f_0 in C(Q), so that f_0 is admissible function having exactly n-1 changes of sign, $f_0(x_0) = 0$, and f_0 does not change sign at x_0. It will be shown that the copositive metric projection C_M is not continuous at f_0. Let g_0 be the unique best copositive approximation for f_0 from M. Since f_0 has n-1 changes of sign then by Proposition 1.4, $g_0 \neq 0$. For each natural number m, define $f_m \in C(Q)$ by $f_m(x) = f_0(x)$ for $x \neq x_0$, and $f_m(x_0) = \frac{\delta}{m}$, and choose $\delta = \pm 1$, so that f_m changes sign at x_0. Since f_0 has n-1 changes of sign then f_m has at least n changes of sign. It is clear that the sequence $\{f_m\}$ converges to f_0. For each m = 1, 2, ..., let g_m be the unique best copositive approximation for f_m from M. Since each f_m has at least n changes of sign, it follows by Proposition 1.2, that $g_m = 0$ for all m. But $g_0 \neq 0$. Therefore $\{g_m\}$ does not converge to g_0. So the copositive metric projection C_M is not continuous at f_0.

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Convexity in the Theory of the Gamma Function

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ABSTRACT

Convexity is a fundamental property of the Gamma function, as shown by pioneering work of Emil Artin, Wolfgang Krull and others. We start with revisiting Krull's work about the functional equation $f(x+1) - f(x) = g(x)$, in a more modern presentation and a slightly more general setup. We present applications of these results to deriving classical and new expansions, respresentations and characterizations of the Gamma function. **Keywords:** Functional equation, Gamma function, convexity. **2000 Mathematics Subject Classification:** 26A51, 33B22, 33B15.

1 Introduction

Leonhard Euler described what we call today the Gamma function, in two letters to German mathematician Christian Goldbach in 1729–1730. He published his discovery in (Euler, 1738), which can be found in Internet, translated from Latin to English by Stacy G. Langton in 1999. In fact, Euler discovered two representations of a function $x \mapsto f(x)$ that for $x = n \in \mathbb{N}$ takes value of $n!$. One is infinite product, and the other is integral. Euler notices that the infinite product

$$\prod_{k=1}^{+\infty} \frac{k^{1-x}(k+1)^x}{x+k} \tag{1.1}$$

takes value $x!$ when $x \in \mathbb{N}$. He states that he found a general expression to describe a "progression"

$$1,\ 1\cdot 2,\ 1\cdot 2\cdot 3,\ 1\cdot 2\cdot 3\cdot 4,\ \ldots$$

and he observes that the formula (1.1) is suitable "for interpolating terms whose indices are fractional numbers". In the same paper, Euler derives the integral form of (1.1):

$$\int_0^1 (-\log x)^n \, \mathrm{d}x.$$

Today's familiar form

$$\Gamma(x)=\int_0^{+\infty}e^{-t}t^{x-1}\,\mathrm{d}t,\qquad \Gamma(n)=(n-1)!,\ n\in\mathbb{N}, \tag{1.2}$$

is due to Legendre. Many celebrated mathematicians contributed to the theory of the Gamma function; a more detailed historical account can be found in (Davis, 1959). This paper is concentrated on convexity: it turns out that this simple tool can be used to produce many interesting properties, inequalities and expansions related to the Gamma function of a positive argument. Immediatelly from (1.2), we can conclude that $x\mapsto\Gamma(x)$ is convex on $x>0$, because the function $x\mapsto t^{x-1}$ is convex for each $t>0$, and $e^{-t}>0$. By a more subtle argument, that can be found in (Artin, 1964), it can be shown that the sum of log-convex functions (i.e., functions whose logarithm is convex) is also log-convex; hence the Gamma function is also log-convex, because the function $x\mapsto\log t^{x-1}$ is convex for each $t>0$. In the first half of 20th century, convexity was a contemporary topic in mathematics. The interest in the subject arose with Jensen's papers (Jensen, 1905) and (Jensen, 1906), which promted mathematical community to search various applications of convexity. Bohr and Mollerup (Bohr and Mollerup, 1922) in 1922 gave a characterization of the Gamma function via convexity. Emil Artin in his tiny monograph (Artin, 1964) of 1931 and Krull in the paper (Krull, 1948) of 1948 extended and ramified the original ideas. While Artin's work is well known, Krull's results are rarely cited in the literature, with an exception of a brief note in Kuczma's monograph (Kuczma, 1968). We start with revisiting Krull's work in Sections 2 and 3, in a more modern presentation and slightly more general setup. In Section 4, we present applications to the Gamma function, and Sections 5 and 6 are based on further applications of convexity, along the lines of papers (Merkle, 2005)-(Merkle, 1996).

2 Krull's work revisited: some auxilliary results

In this section we will repeatedly make use of the following characterization of convexity (Marshall and Olkin, 1979, 16B.3.a): A function f is convex on an interval I if and only if

$$\frac{f(y_1)-f(x_1)}{y_1-x_1}\le\frac{f(y_2)-f(x_2)}{y_2-x_2}, \tag{2.1}$$

whenever $x_1<y_1\le y_2$ and $x_1\le x_2<y_2$, for all $x_1,x_2,y_1,y_2\in I$.

Lemma 2.1. *Let f be a convex function on $(a,+\infty)$ and suppose that $\lim_{x\to+\infty}(f(x+1)-f(x))=0$. Then*

$$\lim_{x\to+\infty}(f(x+y)-f(x))=0\quad\text{for every } y\in\mathbb{R}. \tag{2.2}$$

Proof. We first prove the result for $y=h\in(0,1]$. Let $\varepsilon>0$ be given. Then there is an $x_0>a$ so that $-\varepsilon<f(x+1)-f(x)<\varepsilon$ for every $x\ge x_0$. Let now $x\ge x_0+1$. Then by convexity of f we have

$$-\varepsilon<f(x)-f(x-1)\le\frac{f(x+h)-f(x)}{h}\le f(x+1)-f(x)<\varepsilon,\quad\text{for } h\in(0,1],$$

wherefrom it follows that $|f(x+h)-f(x)|<\varepsilon h$ for any $x \geq x_0+1$. Therefore,

$$\lim_{x\to+\infty}(f(x+h)-f(x))=0 \qquad \text{for any } h\in[0,1]. \tag{2.3}$$

Let now $y>1$ be fixed, and let $[y]=m$, so that $y=m+h$, $h\in[0,1)$. Then

$$\begin{aligned} f(x+y)-f(x) &= (f(x+m+h)-f(x+m)) \\ &\quad +(f(x+m)-f(x+m-1))+\cdots+(f(x+1)-f(x)) \end{aligned}$$

and by (2.3) and the assumption we get (2.2) for $y \geq 0$. The statement for $y<0$ follows from the symmetry. □

Lemma 2.2. *Let f be a convex function on $(a,+\infty)$ and suppose that $\lim_{x\to+\infty}(f(x+1)-f(x))=0$. Then f is non-increasing on $(a,+\infty)$.*

Proof. By convexity, for each $x>a$, $h>0$ and $n>0$ we have

$$f(x+h)-f(x)\leq f(x+n+h)-f(x+n).$$

Letting here $n\to+\infty$ and using Lemma 2.1, we get that $f(x+h)-f(x)\leq 0$ for any $x>a$, $h>0$. □

Lemma 2.3. *Let f be a non-increasing convex function on $(a,+\infty)$. Then for every $x_0 \geq a$ and $x\in(x_0,x_0+1]$, the series*

$$\sum_{k=0}^{+\infty}\left(\frac{f(x+k)-f(x_0+k)}{x-x_0}-(f(x_0+k+1)-f(x_0+k))\right) \tag{2.4}$$

is convergent.

Proof. Let S_n be a partial sum of the series in (2.4). Then, by convexity criterion (2.1),

$$S_{n+1}-S_n=\frac{f(x+n+1)-f(x_0+n+1)}{x-x_0}-(f(x_0+n+2)-f(x_0+n+1))\leq 0,$$

and, further,

$$\begin{aligned} S_n &\geq \frac{f(x)-f(x_0)}{x-x_0}-(f(x_0+1)-f(x_0)) \\ &\quad +\sum_{k=1}^{n}((f(x_0+k)-f(x_0+k-1))-(f(x_0+k+1)-f(x_0+k))) \\ &= \frac{f(x)-f(x_0)}{x-x_0}+f(x_0+n)-f(x_0+n+1) \\ &\geq \frac{f(x)-f(x_0)}{x-x_0}. \end{aligned}$$

Hence, the sequence $\{S_n\}$ is convergent, being non-increasing and bounded from below. □

Lemma 2.4. *The series (2.4) is convergent for all $x\neq x_0$.*

Proof. Let $S_n(x)$ be the n-th partial sum of the series (2.4) and let $\varphi_n(x) = (x - x_0)S_n(x)$ for $x \neq x_0$ and $\varphi_n(x_0) = 0$. Then

$$\varphi_n(x+1) = \varphi_n(x) + f(x+n+1) - f(x_0+n+1) + f(x_0) - f(x). \tag{2.5}$$

By Lemma 2.1, $f(x+n+1) - f(x_0+n+1) \to 0$ as $n \to +\infty$ and so $\varphi_n(x+1)$ is convergent if and only if $\varphi_n(x)$ is. Therefore, if $\varphi_n(x)$ converges in an interval of a length one, then it converges for all $x \in \mathbb{R}$. Note that (2.5) reveals that $\varphi_n(x_0+1) = \varphi_n(x_0) = 0$. □

Lemma 2.5. *Let f be a convex function on $(a, +\infty)$ and suppose that $\lim_{x\to+\infty}(f(x+1) - f(x)) = 0$. Then for every $x > a$, $h_1, h_2 \in (0,1)$ such that $x - h_1 > a$, the series*

$$\sum_{k=0}^{+\infty}\left(\frac{f(x+k+h_2) - f(x+k)}{h_2} - \frac{f(x+k) - f(x+k-h_1)}{h_1}\right) \tag{2.6}$$

converges and for its sum $S(x)$ we have

$$\lim_{x\to+\infty} S(x) = 0. \tag{2.7}$$

Proof. By convexity, all terms of the series in (2.6) are non-negative. Also by convexity,

$$\frac{f(x+k+h_2) - f(x+k)}{h_2} \leq \frac{f(x+k+1) - f(x+k+1-h_1)}{h_1}.$$

Denoting by D_{k+1} the right hand side above and by S_n the n-th partial sum of the series (2.6), we have that

$$\begin{aligned} 0 \leq S_n &\leq \sum_{k=0}^{n}(D_{k+1} - D_k) = D_{n+1} - D_0 \\ &= \frac{f(x+n+1) - f(x+n+1-h_1)}{h_1} - \frac{f(x) - f(x-h_1)}{h_1}. \end{aligned}$$

Now, by Lemma 2.2, $f(x+n+1) \leq f(x+n+1-h_1)$ and $f(x) \leq f(x-h_1)$ for $x - h_1 > a$, so

$$0 \leq S_n \leq \frac{f(x-h_1) - f(x)}{h_1} \to 0 \qquad \text{as } x \to +\infty. \tag{2.8}$$

This shows that the series in (2.6) is convergent and that its sum converges to 0 as $x \to +\infty$. □

Lemma 2.6. *Suppose that $\rho(x+1) = \rho(x)$ for every $x > a$ and suppose that*

$$\frac{\rho(x+h_2) - \rho(x)}{h_2} - \frac{\rho(x) - \rho(x-h_1)}{h_1} \to 0 \quad \text{as } x \to +\infty, \tag{2.9}$$

for all $h_1, h_2 < \delta$, for some $\delta > 0$. Then the function $x \mapsto \rho(x)$, $x > a$, is identically equal to a constant.

Proof. For any fixed $x_0 > a$, choose $h_1 < \delta$ so that $x_0 - h_1 > a$ and take arbitrary $h_2 < \delta$. Then by the assumptions we have that

$$\begin{aligned} &\frac{\rho(x_0+h_2) - \rho(x_0)}{h_2} - \frac{\rho(x_0) - \rho(x_0-h_1)}{h_1} \\ = \; &\frac{\rho(x_0+n+h_2) - \rho(x_0+n)}{h_2} - \frac{\rho(x_0+n) - \rho(x_0+n-h_1)}{h_1} \to 0 \end{aligned}$$

as $n \to +\infty$, and we conclude that

$$\frac{\rho(x_0+h_2)-\rho(x_0)}{h_2} - \frac{\rho(x_0)-\rho(x_0-h_1)}{h_1} = 0$$

for any h_1, h_2 as specified above. Then ρ is both convex and concave, hence $\rho(x) = bx + c$ for $x \in [x_0 - h_1, x_0 + h_2]$. Since x_0 was arbitrary we must have $\rho(x) = bx + c$ for some constants b, c and all $x > a$; since $\rho(x+1) = \rho(x)$, we conclude that $b = 0$ and so, ρ is a constant. □

Lemma 2.7. *Lemmas 2.1,2.3,2.4 and 2.5 hold true if conditions on f are replaced by $f(x) = f_1(x)+f_2(x)$, where f_1 is convex and f_2 is concave on $(a, +\infty)$ and $\lim_{x\to+\infty}\big(f_i(x+1)-f_i(x)\big) = 0$, $i = 1, 2$.*

Proof. It is easy to see that the domain of validity of mentioned lemmas is linear, i.e., if the stated results hold for some functions f and g, they hold for any linear combination $\alpha f + \beta g$. In particular, since the results are proved for convex functions f, they must hold for concave functions $-f$ and then for the sum of a convex and a concave function. □

3 Krull's work revisited: main results

Let us observe the functional equation

$$g(x+1) - g(x) = f(x), \qquad x \geq a, \tag{3.1}$$

where g is an unknown function and f is given. Note that the case $f(x) = \log x$ is the recurrence relation for the logarithm of the Gamma function. In what follows we will make use of the following two conditions. **Condition A.** We say that f satisfies condition A if $f(x) = f_1(x) + f_2(x)$, where f_1 is convex and f_2 is concave on $x > a$, and also

$$\lim_{x\to+\infty}(f_i(x+1) - f_i(x)) = 0, \qquad i = 1, 2.$$

Condition B. A function g satisfies condition B if for any h_1, h_2 such that $0 < h_i < \delta$ for some $\delta > 0$, we have that

$$\lim_{x\to+\infty}\left(\frac{g(x+h_2)-g(x)}{h_2} - \frac{g(x)-g(x-h_1)}{h_1}\right) = 0.$$

Remark 3.1. Let $\varphi(x,t)$ be a function of two real variables, such that $x \mapsto \varphi(x,t)$ is convex on $x > a$ for each $t \in \mathbb{R}$, and let μ be a finite signed measure on $\mathbb{R}$, such that the function $t \mapsto \varphi(x,t)$ is μ-integrable for each x. Assume, further, that

$$\lim_{x\to+\infty}(\varphi(x+1,t) - \varphi(x,t)) = 0 \quad \text{uniformly in } t \in \mathbb{R}. \tag{3.2}$$

Then the function f defined by

$$f(x) = \int_{\mathbb{R}} \varphi(x,t)\, \mathrm{d}\mu(t) \tag{3.3}$$

satisfies condition A. Indeed, by Jordan decomposition, $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are positive finite measures, and we may define

$$f_1(x) = \int_{\mathbb{R}} \varphi(x,t)\,\mathrm{d}\mu^+(t), \quad f_2(x) = -\int_{\mathbb{R}} \varphi(x,t)\,\mathrm{d}\mu^-(t).$$

Then $f(x) = f_1(x) + f_2(x)$, where f_1 is convex and f_2 is concave; the rest follows from (3.2). In particular, condition A is satisfied by every function of the form

$$f(x) = \int_{-\infty}^{+\infty} K(t)\varphi(x,t)\,\mathrm{d}t,$$

where K is an integrable function, and $x \mapsto \varphi(x,t)$ is convex and satisfies (3.2).

Remark 3.2. Note that the condition B is satisfied with any twice differentiable function g such that $g''(x) \to 0$ as $x \to +\infty$.

Lemma 3.1. *Suppose that the equation (3.1) has a solution g that satisfies condition B. Then all solutions that satisfy condition B are of the form $g + C$, where C is an arbitrary constant.*

Proof. Suppose that g_1 is another solution of (3.1) that satisfies condition B and let $\rho(x) = g_1(x) - g(x)$. Then by Lemma 2.6, ρ is a constant, which had to be proved. □

Lemma 3.2. *Suppose that f satisfies the condition A, and let g be a convex solution of (3.1). Then g satisfies the condition B. Moreover, any solution of (3.1) that satisfies the condition B is of the form $g + C$, where C is an arbitrary constant. In particular, any convex solution of (3.1) is of the form $g + C$ and any twice differentiable solution with a second derivative converging to zero as $x \to +\infty$ is also of the form $g + C$.*

Proof. Let f satisfies the condition A, and let g be a convex solution of (3.1). Then for any $h_1, h_2 \in (0,1)$ we have that

$$\begin{aligned} f(x+1) - f(x) &= (g(x+2) - g(x+1)) - (g(x+1) - g(x)) \\ &\geq \frac{g(x+1+h_2) - g(x+1)}{h_2} - \frac{g(x+1) - g(x+1-h_1)}{h_1} \geq 0 \end{aligned}$$

By condition A, $f(x+1) - f(x) \to 0$ as $x \to +\infty$ and so we conclude that g satisfies the condition B. Then by Lemma 3.1, all solutions of (3.1) that satisfy the condition B are of the form $g + C$. As we just proved, any convex solution of (3.1) satisfies the condition B, and so, any convex soluton is of the form $g + C$. Finally, as we noticed in Remark 3.2, any function with the second derivative converging to zero as $x \to +\infty$ satisfies condition B and hence any such a solution is of the form $g + C$. □

Theorem 3.3. *Suppose that f satisfies condition A. For $x_0 \geq a$ being fixed, define a function g by*

$$\begin{aligned} g(x) &= \int_{x_0}^{x} f(u)\,\mathrm{d}u - \frac{1}{2}f(x) \\ &+ \sum_{k=0}^{+\infty}\left(\int_{x+k}^{x+k+1} f(u)\,\mathrm{d}u - \frac{1}{2}(f(x+k+1) + f(x+k))\right), \end{aligned} \quad (3.4)$$

where $x > a$. Then (i) The function g is a solution of equation (3.1) on $x > a$. (ii) The function g satisfies condition B. (iii) The following relation holds:

$$g(x) = \int_{x_0}^{x} f(u)\,\mathrm{d}u - \frac{1}{2}f(x) + o(1) \qquad (x \to +\infty). \tag{3.5}$$

(iv) In addition, if f is either convex or concave, then

$$|S(x)| \le \frac{1}{2}\,|f(x+1/2) - f(x)|\,, \tag{3.6}$$

where $S(x)$ is the sum of the series in (3.4).

Proof. It is clear that if the theorem holds in the case of convex f satisfying condition A, then it must hold for any function that satisfies condition A. So, it suffices to assume that f is a convex function that satisfies condition A. We firstly have to prove that the series in (3.4) converges. Let us introduce the notation

$$D(x) = \int_{x}^{x+1} f(u)\,\mathrm{d}u - \frac{1}{2}(f(x+1) + f(x))\,.$$

Then by Hadamard's inequalities,

$$f(x+1/2) \le \int_{x}^{x+1} f(u)\,\mathrm{d}u \le \frac{1}{2}(f(x) + f(x+1))$$

and therefore

$$\begin{aligned} 0 &\ge D(x+k) \ge f(x+k+1/2) - \frac{1}{2}(f(x+k+1) + f(x+k)) \\ &= -\frac{1}{4}\left(\frac{f(x+k+1) - f(x+k+1/2)}{1/2} - \frac{f(x+k+1/2) - f(x+k)}{1/2}\right). \end{aligned} \tag{3.7}$$

Denote by $h(x+k)$ the expression on the right hand side of (3.7). By Lemma 2.5, applied with $h_1 = h_2 = 1/2$ and with x replaced by $x + 1/2$, the series $\sum h(x+k)$ converges for any $x > a$ and its sum converges to zero as $x \to +\infty$. By (3.7), the same holds true for the series $\sum D(x+k)$. This proves (iii) in the case of convex f. By linearity, (iii) holds for any function f that satisfies condition A. If f is convex, (iv) follows from (3.7) and (2.8); in a similar way one can obtain (iv) for a concave f. Note that (iv) need not hold under general setup of condition A, as it was incorrectly deduced in Krull's original paper. Let us now prove that g satisfies the condition B. By (iii) and Lemma 2.5, it suffices to show that the condition B is satisfied by the function ψ defined by $\psi(x) = \int_{x_0}^{x} f(u)\,\mathrm{d}u$. For any $h_1, h_2 > 0$ we have

$$\begin{aligned} \frac{\psi(x+h_2) - \psi(x)}{h_2} - \frac{\psi(x) - \psi(x-h_1)}{h_1} &= \frac{1}{h_2}\int_{x}^{x+h_2} f(u)\,\mathrm{d}u - \frac{1}{h_1}\int_{x-h_1}^{x} f(u)\,\mathrm{d}u \\ &= f(x_2) - f(x_1), \end{aligned}$$

where $x_1 \in (x - h_1, x)$ and $x_2 \in (x, x + h_2)$ (by the mean value theorem and continuity of a convex function in an open interval). By Lemma 2.2, f is non-increasing and therefore,

$$0 \ge f(x_2) - f(x_1) \ge f(x+h_2) - f(x-h_1) \to 0 \quad \text{as } x \to +\infty \qquad \text{(by Lemma 2.1).}$$

Therefore, the condition B holds for g. Finally, let us verify that g is a solution of (3.1). By (3.4) we have that $g(x) = \lim_{n\to+\infty} g_n(x)$, where

$$g_n(x) = \int_{x_0}^{x+n+1} f(u)\,\mathrm{d}u - \sum_{k=0}^{n+1} f(x+k) + \frac{1}{2}f(x+n+1),$$

and

$$g_n(x+1) - g_n(x) = \int_{x+n+1}^{x+n+2} f(u)\,\mathrm{d}u - f(x+n+2) + f(x) + \frac{1}{2}(f(x+n+2) - f(x+n+1))$$

Now by Lemma 2.1,

$$\lim_{n\to+\infty}(f(x+n+2) - f(x+n+1)) = 0.$$

If f is convex then by Hadamard's inequalities,

$$f(x+n+1/2) - f(x+n+2) \le \int_{x+n+1}^{x+n+2} f(u)\,\mathrm{d}u - f(x+n+2) \le \frac{1}{2}(f(x+n+1) - f(x+n+2))$$

and so

$$\lim_{n\to+\infty}\left(\int_{x+n+1}^{x+n+2} f(u)\,\mathrm{d}u - f(x+n+2)\right) = 0.$$

By linearity, the last relation holds also if f is the sum of a convex and a concave function. Therefore, we have shown that

$$g(x+1) - g(x) = \lim_{n\to+\infty}(g_n(x+1) - g_n(x)) = f(x),$$

i.e., the equation (3.1) is satisfied. □

Theorem 3.4. *Suppose that f satisfies condition A. For $x_0 \ge a$ and $y_0 \in \mathbb{R}$ being fixed, define a function g^* by*

$$\begin{aligned} g^*(x) &= y_0 + (x - x_0)f(x_0) \\ &\quad - \sum_{k=0}^{+\infty}\left(f(x+k) - f(x_0+k)\right) - (x-x_0)(f(x_0+k+1) - f(x_0+k)) . \end{aligned} \tag{3.8}$$

Then (i) The function g^* satisfies (3.1) on $x > a$ and $g^*(x_0) = y_0$. (ii) The function g^* satisfies the condition B.

Proof. The series in (3.8) converges by Lemma 2.4. Now it is easy to verify (i). Condition B follows from Lemma 2.5. □

Corollary 3.5. *Let g and g^* be as defined in Theorems 3.3 and 3.4 respectively. Then for $x > a$, $g(x) = g^*(x) + C$, where C is a constant.*

Proof. By Lemma 3.1 and Theorems 3.3 and 3.4. □

Corollary 3.6. *If f is a concave function satisfying the condition A, then the function g^* is a unique convex solution of (3.1) under the initial condition $g(x_0) = y_0$.*

Proof. We only have to show that g^* is convex. Then the rest follows from Lemma 3.2. However, from (3.8) it follows easily that

$$\begin{aligned} & g(\lambda x + (1-\lambda)y) - \lambda g(x) - (1-\lambda)g(y) \\ = & -\sum_{k=0}^{+\infty} f(\lambda(x+k) + (1-\lambda)(y+k)) - \lambda f(x+k) - (1-\lambda)f(y+k), \end{aligned}$$

and therefore, concavity of f implies the convexity of g. □

Theorem 3.7. *Suppose that f satisfies the condition A and assume also that f is r times differentiable, with $f^{(r)}(x)$ monotone for x large enough. Then the solution g of (3.1), introduced in Theorems 3.3 and 3.4 is also r times differentiable and we have*

$$g^{(j)}(x) = -\sum_{k=0}^{+\infty} f^{(j)}(x+k) \qquad (j \geq 2) \tag{3.9}$$

$$g'(x) = \lim_{n\to+\infty} \left(f(x+n) - \sum_{k=0}^{n} f'(x+k) \right). \tag{3.10}$$

Proof. By repeated formal differentiation in (3.4) we obtain

$$\begin{aligned} g^{(j)}(x) = & f^{(j-1)}(x) - \frac{1}{2} f^{(j)}(x) \\ & + \sum_{k=0}^{+\infty} \left(\int_{x+k}^{x+k+1} f^{(j)}(u)\,\mathrm{d}u - \frac{1}{2}(f^{(j)}(x+k+1) + f^{(j)}(x+k)) \right) \end{aligned} \tag{3.11}$$

Let us show that the term-wise differentiation is allowed. Firstly, note that if f is not monotone in any interval $[b, +\infty)$, then its derivative has infinitely many changes of sign, hence it can not be monotone. Therefore, if f' is monotone for x large enough, then so is f, and hence the monotonicity of $f^{(r)}$ for x large enough implies the same property for $f, f', \ldots, f^{(r-1)}$. Next, observe that if $x_2 = x_1 + 1$ and if φ is any integrable and monotone function, say monotonically nondecreasing, then

$$\varphi(x_1) \leq \int_{x_1}^{x_2} \varphi(u)\,\mathrm{d}u \leq \varphi(x_2),$$

and opposite inequalities hold if φ is nonincreasing; in a general case we have

$$\left| \int_{x_1}^{x_2} \varphi(u)\,\mathrm{d}u - \frac{1}{2}(\varphi(x_1) + \varphi(x_2)) \right| \leq \frac{1}{2} |\varphi(x_2) - \varphi(x_1)|.$$

An application to the sum in (3.11) yields

$$\begin{aligned} & \left| \sum_{k=n+1}^{n+m} \int_{x+k}^{x+k+1} f^{(j)}(u)\,\mathrm{d}u - \frac{1}{2}(f^{(j)}(x+k+1) + f^{(j)}(x+k)) \right| \\ \leq & \sum_{k=n+1}^{n+m} \frac{1}{2} |f^{(j)}(x+k+1) - f^{(j)}(x+k)| \\ = & \frac{1}{2} |f^{(j)}(x+n+m+1) - f^{(j)}(x+n+1)| \quad \text{(by monotonicity)}, \end{aligned} \tag{3.12}$$

where we assumed (without a loss of generality) that $f^{(r)}(x)$ is monotone for $x > a$. Now the monotonicity of the first derivative implies

$$|f(x+1) - f(x)| = |f'(x+\theta)| \geq |f'(x)| \qquad \text{or}$$

$$|f(x+1)-f(x)| = |f'(x+\theta)| \geq |f'(x+1)|$$

for all x large enough. Then by the condition A we conclude that $f'(x) \to 0$ as $x \to +\infty$ and further $\lim_{x\to+\infty} (f'(x+1) - f'(x) = 0)$. Repeating the procedure for higher derivatives, we conclude that $\lim_{x\to+\infty} f^{(j)}(x) = 0$ for all $j = 1, \ldots, r$. This, together with (3.12) shows that the series in (3.11) is uniformly convergent and the term-wise differentiation in (3.4) is allowed. Retaining the first n terms in (3.11) we obtain

$$g^{(j)}(x) = \lim_{n\to+\infty} \left(f^{(j-1)}(x+n+1) - \frac{1}{2} f^{(j)}(x+n+1) - \sum_{k=0}^{n} f^{(j)}(x+k) \right).$$

By the above discussion, the second term above vanishes as $n \to +\infty$ for all $j \geq 1$ and so does the first term for $j > 1$. For $j = 1$, the first term may not vanish and we get (3.10). □

Corollary 3.8. *If f satisfies the condition A and if f'' is monotone for large enough x, then*

$$g(x)'' = -\sum_{k=0}^{+\infty} f''(x+k) \quad \text{and} \quad \lim_{x\to+\infty} g''(x) = 0.$$

Proof. The expression for g'' in terms of the series follows directly from Theorem 3.7. Since f'' is monotone it follows that it does not change sign for x large enough and so, f is either concave or convex. Then by Lemma 2.5 we conclude that $g''(x) \to 0$ as $x \to +\infty$. □

4 Applications to the Gamma function

In this section we present some classical results related to the Gamma function as straightforward consequences of Krull's theory. It seems that the result stated in Theorem 4.2 has not been observed in the literature.

Theorem 4.1 (Bohr-Mollerup theorem). *If G is a logarithmically convex solution of the functional equation*

$$xG(x) = G(x+1) \qquad (x > 0)$$

and if $G(1) = 1$, then $G(x) \equiv \Gamma(x)$ on $x > 0$.

Proof. By Lemma 3.2, with $g(x) = \log G(x)$ and $f(x) = \log x$. □

Theorem 4.2. *If g is a twice differentiable function that satisfies*

$$g(x+1) - g(x) = \log x, \quad g(1) = 0, \quad \text{and} \quad \lim_{x\to+\infty} g''(x) = 0,$$

then $g(x) = \log \Gamma(x)$.

Proof. By Lemma 3.1, with $f(x) = \log x$. □

Theorem 4.3. *The following representation holds for $x > 0$:*

$$\Gamma(x) = \lim_{n\to+\infty} \frac{(x+n+1)^{x+n+1/2} e^{-(x+n+1)} \sqrt{2\pi}}{x(x+1)\cdots(x+n)}.$$

Proof. Any solution of equation (3.1) can be expressed, by Theorem 1 and Lemma 8, as

$$g(x) = \lim_{n\to+\infty}\left(F(x+n+1) - \sum_{k=0}^{n} f(x+k) - \frac{1}{2}f(x+n+1)\right) + C,$$

where C is a constant, F is a primitive function for f. In the case of the Gamma function, we have that $f(x) = \log x, F(x) = x\log x - x$, $x > 0$ and so

$$\begin{aligned} g(x) = \quad & \lim_{n\to+\infty}\Big((x+n+1)\log(x+n+1) - (x+n+1) \\ & -\sum_{k=0}^{n}\log(x+k) - \frac{1}{2}\log(x+n+1)\Big) + C. \end{aligned}$$

To determine C, let $x = 1$, then $g(1) = \log\Gamma(1) = 0$ and so

$$\lim_{n\to+\infty}\left((n+2)\log(n+2) - (n+2) - \sum_{k=1}^{n+1}\log k - \frac{1}{2}\log(n+2)\right) + C = 0.$$

Using Stirling's formula for the factorial, the expression under limit can be transformed as

$$\log\frac{(n+2)^{n+3/2}e^{-(n+2)}}{(n+1)!} \sim \log\frac{(n+2)^{n+3/2}e^{-(n+2)}}{\sqrt{2\pi}(n+1)^{n+3/2}e^{-(n+1)}} \sim -\frac{1}{2}\log 2\pi,$$

and so, $C = \frac{1}{2}\log 2\pi$. This yields the desired expansion. □

Theorem 4.4 (Euler's product). *For $x > 0$, the Gamma function can be expressed as*

$$\Gamma(x) = \frac{1}{x}\prod_{k=1}^{+\infty}\left(1+\frac{1}{k}\right)^{x}\left(1+\frac{x}{k}\right)^{-1}.$$

Proof. A straightforward application of (3.8), with $f(x) = \log x$, $x_0 = 1$, $y_0 = 0$ and with x replaced by $x+1$ gives

$$\log\Gamma(x+1) = \log\prod_{k=1}^{+\infty}\frac{k^{1-x}(k+1)^{x}}{(x+k)},$$

which yields the desired result. □

Theorem 4.5. *The derivative of the function $x \mapsto \log\Gamma(x)$ can be expressed as*

$$\Psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x) = \lim_{n\to+\infty}\left(\log(x+n) - \sum_{k=0}^{n}\frac{1}{x+k}\right) \tag{4.1}$$

$$= -\gamma - \frac{1}{x} + \sum_{k=1}^{+\infty}\frac{x}{k(x+k)}, \tag{4.2}$$

where γ is Euler's constant. Further, for higher derivatives we have

$$\Psi^{(j-1)}(x) = \frac{\mathrm{d}^j}{\mathrm{d}x^j}\log\Gamma(x) = \sum_{k=0}^{+\infty}\frac{(-1)^j(j-1)!}{(x+k)^j}, \qquad j = 2, 3\ldots \tag{4.3}$$

Proof. Expression (4.1) follows from (3.10) of Theorem 3.7, by letting $f(x) = \log x$. Its equivalent form (4.2) is easy to obtain using the definition of Euler's constant,

$$\gamma = \lim_{n\to+\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right).$$

The expression (4.3) is obtained from (3.9) with $f(x) = \log x$. □

5 Asymptotic expansions via convexity

We start this section with a result from (Merkle, 1999):

Theorem 5.1. *Let g be a twice continuously differentiable function on an interval $I = (a, +\infty)$. For $\lambda \in [0, 1]$, $x, y \in I$, let*

$$R(\lambda, x, y) = \lambda g(x) + (1 - \lambda)g(y) - g(\lambda x + (1 - \lambda)y).$$

If $\lim_{x\to+\infty} g''(x) = 0$, then $R(\lambda, x, y) \to 0$ as $\min(x, y) \to +\infty$.

Using Theorem 5.1, we will now derive a more informative version of expansion (3.8). In this and next section, we use the notation $a_n \lesssim b_n$ for sequences a_n and b_n with

$$a_n \le b_n \quad \text{for all } n \quad \text{and} \quad \lim_{n\to+\infty}(a_n - b_n) = 0.$$

The notation $a_n \gtrsim b_n$ is equivalent to $b_n \lesssim a_n$

Theorem 5.2. *For a given function f, let g be a twice differentiable solution of (3.1) on $x > a$, with the property that $\lim_{x\to+\infty} g''(x) = 0$. Then, for every $x, y \in (a, +\infty)$, we have*

$$\begin{aligned} g(y) - g(x) &= \lim_{n\to+\infty}\left((y - x)f(x + n) + \sum_{k=0}^{n-1}\left(f(x + k) - f(y + k)\right)\right), \\ g(y) - g(x) &= \lim_{n\to+\infty}\left((y - x)f(y + n - 1) + \sum_{k=0}^{n-1}\left(f(x + k) - f(y + k)\right)\right). \end{aligned}$$

If g is a convex function on $(a, +\infty)$, then for $0 < y - x < 1$, we have

$$\begin{aligned} g(y) - g(x) &\lesssim (y - x)f(x + n) + \sum_{k=0}^{n-1}\left(f(x + k) - f(y + k)\right), \\ g(y) - g(x) &\gtrsim (y - x)f(y + n - 1) + \sum_{k=0}^{n-1}\left(f(x + k) - f(y + k)\right), \end{aligned}$$

and for $y - x < 0$ or $y - x > 1$:

$$\begin{aligned} g(y) - g(x) &\gtrsim (y - x)f(x + n) + \sum_{k=0}^{n-1}\left(f(x + k) - f(y + k)\right), \\ g(y) - g(x) &\lesssim (y - x)f(y + n - 1) + \sum_{k=0}^{n-1}\left(f(x + k) - f(y + k)\right). \end{aligned}$$

If g is a concave function, the last four expansions remain valid with $\lesssim$ and $\gtrsim$ interchanged.

Proof. For $0 < y - x < 1$, and an integer $n \ge 1$, we can express $y + n$ as a convex combination of $x + n$ and $x + n + 1$ as follows:

$$y + n = (1 - (y - x))(x + n) + (y - x)(x + n + 1).$$

Then, by Theorem 4.5, $R_n := R(1 - (y - x), x + n, x + n + 1)) \to 0$ as $n \to +\infty$, where

$$R(1 - (y - x), x + n, x + n + 1) = (1 - (y - x))g(x + n) + (y - x)g(x + n + 1) - g(y + n).$$

Since g satisfies (3.1), we have that

$$g(x+n) = g(x) + \sum_{k=0}^{n-1} f(x+k),$$

and so

$$R_n = g(x) - g(y) + \sum_{k=0}^{n-1} \big(f(x+k) - f(y+k)\big) + (y-x)f(x+n+1),$$

and the first expansion is proved. To prove the second one, we start with a convex combination of $y+n-1$ and $y+n$

$$x+n = (y-x)(y+n-1) + (1-(y-x))(y+n),$$

and proceed in the same way to find that

$$R(y-x, y+n-1, y+n) = g(y) - g(x) + \sum_{k=0}^{n-1} \big(f(y+k) - f(x+k)\big) - (y-x)f(y+n-1)$$

converges to zero as $n \to +\infty$. To prove third and fourth relation, note that for a convex gunction g we have

$$R(1-(y-x), x+n, x+n+1) \geq 0 \quad \text{and} \quad R(y-x, y+n-1, y+n) \geq 0.$$

To get fifth and sixth relation if $y-x>1$, we start with

$$\begin{aligned} x+n+1 &= \frac{y-x+1}{y-x}(x+n) + \frac{1}{y-x}(y+n) \quad \text{and} \\ y+n-1 &= \frac{1}{y-x}(x+n) + \frac{y-x-1}{y-x}(y+n), \end{aligned}$$

and if $y-x<0$ with

$$\begin{aligned} x+n &= \frac{x-y}{1+x-y}(x+n+1) + \frac{1}{1+x-y}(y+n) \quad \text{and} \\ y+n &= \frac{1}{1+x-y}(x+n) + \frac{x-y}{1+x-y}(y+n-1), \end{aligned}$$

and proceed in a same way as above. Finally, the statement for a concave function g follows from the fact that $R \leq 0$ in that case. □

Theorem 5.2 yields various two sided expansions for the Gamma function. For example, letting $x = m$ and $y = m+\beta$, where m is a positive integer, and $\beta \in [0,1]$, and applying results of Theorem 10 with $f(x) = \log x$, we get

$$(m-1)!(m+n-1+\beta)^{\beta} \prod_{k=0}^{n-1} \frac{m+k}{m+\beta+k} \lesssim \Gamma(m+\beta)$$

$$\lesssim (m-1)!(m+n)^{\beta} \prod_{k=0}^{n-1} \frac{m+k}{m+\beta+k}, \quad \text{as } n \to +\infty.$$

In the next theorem, we give a refinement of the representation of Theorem 4.3.

Theorem 5.3. *For $x > 0$, we have*

$$\frac{(x+n)^{x+n-1/2}e^{-(x+n)}\sqrt{2\pi}}{x(x+1)\cdots(x+n-1)} \lesssim \Gamma(x) \lesssim \frac{(x+n)^{x+n-1}\sqrt{x+n+\frac{1}{2}}e^{-(x+n)}\sqrt{2\pi}}{x(x+1)\cdots(x+n-1)}.$$

Proof. Using Theorem 6, we can evaluate the sum of the series in (3.4), with $f(x) = \log x$ and $g(x) = \log\Gamma(x)$, as follows:

$$S(x) = \log\Gamma(x) + x - (x - \frac{1}{2})\log x - \frac{1}{2}\log 2\pi. \tag{5.1}$$

By inequality (3.6) for a concave function $f(x) = \log x$, we have that

$$1 \le e^{S(x)} \le \sqrt{\frac{x+\frac{1}{2}}{x}}. \tag{5.2}$$

Replacing x with $x + n$ and using the fact that

$$\Gamma(x+n) = x(x+1)\cdots(x+n-1),$$

we get the desired inequalities from (5.2). The asymptotics as $n \to +\infty$ follows from comparison with the expansion of Theorem 4.3. □

6 Gautchi's and Gurland's ratio

Motivated by various applications, the inequalities related to the Gamma function have been a subject of an intensive research. Apart from inequalities for the Gamma function alone, there is a considerable number of results about two ratios of Gamma functions. Gautchi's ratio (Gautschi, 1959) is defined by

$$Q(x,\beta) = \frac{\Gamma(x+\beta)}{\Gamma(x)}, \tag{6.1}$$

and has been usually studied with $\beta \in (0,1)$, see (Merkle, 1996) and references therein. Gurland's ratio (Gurland, 1956) is defined as

$$T(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma^2((x+y)/2)}, \qquad x, y > 0. \tag{6.2}$$

A survey of results about Gurland's ratio can be found in (Merkle, 2005). Logarithms of both ratios satisfy Krull's functional equation (3.1), with conditions A and B being satisfied:

$$\log Q(x+1,\beta) - \log Q(x,\beta) = \log(x+\beta) - \log x,$$

$$\log T(x+1, x+1+2\beta) - \log T(x, x+2\beta) = \log(x+2\beta) + \log x - 2\log(x+\beta).$$

Therefore, the results and methods presented in previous sections can be applied to produce inequalities and expansions for both ratios. A more detailed account of convexity techniques for inequalities can be found in (Merkle, 2008), as well as in papers (Merkle, 1997), (Merkle, 1998a), (Merkle, 1998b), (Merkle, 2001) and (Merkle, 2004).

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

ANALYTICAL AND DIFFERENTIAL-ALGEBRAIC PROPERTIES OF GAMMA FUNCTION

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ABSTRACT

In this paper we consider some analytical relations between gamma function $\Gamma(z)$ and related functions such as the Kurepa's function $K(z)$ and alternating Kurepa's function $A(z)$. It is well-known in the physics that the Casimir energy is defined by the principal part of the Riemann function $\zeta(z)$ (Blau, Visser, Wipf; Elizalde). Analogously, we consider the principal parts for functions $\Gamma(z)$, $K(z)$, $A(z)$ and we also define and consider the principal part for arbitrary meromorphic functions. Next, in this paper we consider some differential-algebraic (d.a.) properties of functions $\Gamma(z)$, $\zeta(z)$, $K(z)$, $A(z)$. As it is well-known (Hölder; Ostrowski) $\Gamma(z)$ is not a solution of any d.a. equation. It appears that this property of $\Gamma(z)$ is universal. Namely, a large class of solutions of functional differential equations also has that property. Proof of these facts is reduced, by the use of the theory of differential algebraic fields (Ritt; Kaplansky; Kolchin), to the d.a. transcendency of $\Gamma(z)$.

Keywords: principal part at a point, Gamma function, Kurepa's function, Zeta function, Casimir energy, model-theoretic algebra, differential algebra, differential transcendency.

2000 Mathematics Subject Classification: 03C60, 11R42, 11J91, 12H05, 30E20, 34M15.

1 Analytical properties

In this section we consider analytical properties of the gamma and related functions which pertain to the principal part of a function at a point.

1.1 The principal part of the gamma function

Gamma function is defined by the integral:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \tag{1.1}$$

which converges for $Re(z) > 0$. It is possible to form analytical continuation of this function over the whole set of the complex numbers $\mathbb{C}$ except at $z = -k$, where $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. One approach to analytical continuation is given by:

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)} + \int_{1}^{\infty} e^{-t} t^{z-1}\, dt, \tag{1.2}$$

see (Markushevich, 1965), (Leibbrandt, 1975). Residue at $z = -k$, $k \in \mathbb{N}_0$, is:

$$\operatorname*{res}_{z=-k} \Gamma(z) = \frac{(-1)^k}{k!}. \tag{1.3}$$

It is possible to extend the domain of the gamma function to the set of all complex numbers $\mathbb{C}$ in the sense of the principal part at a point as follows. For a meromorphic function $f(z)$, on the basis of Cauchy's integral formula, we define *the principal part at point* a (see: (Slavić, 1970), (Malešević, 2003)):

$$\operatorname*{p.p.}_{z=a} f(z) = \lim_{\rho \to 0_+} \frac{1}{2\pi i} \oint_{|z-a|=\rho} \frac{f(z)}{z-a}\, dz. \tag{1.4}$$

If the point a is regular for the function $f(z)$ then $\operatorname*{p.p.}_{z=a} f(z) = f(a)$; otherwise the principal part at the pole $z = a$ exists as a finite complex number:

$$\operatorname*{p.p.}_{z=a} f(z) = \operatorname*{res}_{z=a} \left(\frac{f(z)}{z-a} \right), \tag{1.5}$$

as cited in (Malešević, 2006). Let us determine basic properties of the principal part at the point. For two meromorphic functions $f_1(z)$ and $f_2(z)$ additivity holds (Slavić, 1970):

$$\operatorname*{p.p.}_{z=a} \Big(f_1(z) + f_2(z) \Big) = \operatorname*{p.p.}_{z=a} f_1(z) + \operatorname*{p.p.}_{z=a} f_2(z). \tag{1.6}$$

In the paper (Slavić, 1970) it is proved that multiplicativity of the principal part does not hold. Namely for the principal part the following statement is true.

Theorem 1.1. *Let $f_1(z)$ be a holomorphic function at the point a and let $f_2(z)$ be a meromorphic function with pole of the m-th order at the same point a. Then:*

$$\operatorname*{p.p.}_{z=a} \Big(f_1(z) \cdot f_2(z) \Big) = \sum_{k=0}^{m} \frac{f_1^{(k)}(a)}{k!} \operatorname*{p.p.}_{z=a} \Big((z-a)^k \cdot f_2(z) \Big). \tag{1.7}$$

Proof Let $f_1(z)$ and let $f_2(z)$ be represented by the series:

$$f_1(z) = \sum_{i=0}^{\infty} \frac{f_1^{(i)}(a)}{i!} (z-a)^i \quad \text{and} \quad f_2(z) = \sum_{j=-m}^{\infty} c_j (z-a)^j, \tag{1.8}$$

for some $c_j \in \mathbb{C}$ $(j \geq -m)$, $c_{-m} \neq 0$ and $z \neq a$. Let us notice that $\operatorname*{p.p.}_{z=a} f_2(z) = c_0$. Multiplying the following series:

$$\frac{f_1(z) - f_1(a)}{z-a} \cdot f_2(z) = \sum_{i=1}^{\infty} \frac{f_1^{(i)}(a)}{i!} (z-a)^{i-1} \cdot \sum_{j=-m}^{\infty} c_j (z-a)^j \tag{1.9}$$

we obtain:

$$\operatorname*{res}_{z=a}\left(\frac{f_1(z)-f_1(a)}{z-a}\cdot f_2(z)\right)=\sum_{k=1}^{m}\frac{f_1^{(k)}(a)}{k!}\cdot c_{-k}\,. \tag{1.10}$$

Hence:

$$\begin{aligned}
\operatorname*{p.p.}_{z=a}\Big(f_1(z)\cdot f_2(z)\Big) &= \operatorname*{res}_{z=a}\left(\frac{f_1(z)\cdot f_2(z)}{z-a}\right)\\
&= \operatorname*{res}_{z=a}\left(\frac{f_2(z)}{z-a}\right)\cdot f_1(a)+\operatorname*{res}_{z=a}\left(\frac{f_1(z)-f_1(a)}{z-a}\cdot f_2(z)\right)\\
&= f_1(a)\cdot c_0+\sum_{k=1}^{m}\frac{f_1^{(k)}(a)}{k!}\cdot c_{-k}\\
&= \sum_{k=0}^{m}\frac{f_1^{(k)}(a)}{k!}\operatorname*{p.p.}_{z=a}\Big((z-a)^k\cdot f_2(z)\Big).\ \nabla
\end{aligned} \tag{1.11}$$

Remark 1.1. The phrase "function is holomorphic at the point a" means not just function is differentiable at a, but differentiable everywhere within some open disk centered at a in the complex plane.

Corollary 1.2. *Let $f_1(z)$ be a holomorphic function at the point a and let $f_2(z)$ be a meromorphic function with simple pole at the same point a. Then:*

$$\operatorname*{p.p.}_{z=a}\Big(f_1(z)\cdot f_2(z)\Big)=f_1(a)\cdot\operatorname*{p.p.}_{z=a}f_2(z)+f_1'(a)\cdot\operatorname*{res}_{z=a}f_2(z). \tag{1.12}$$

The previous formula, in the case of the zeta function $f_2(z)=\zeta(z)$, is also given in (Blau, Visser and Wipf, 1988a), (Blau, Visser and Wipf, 1988b).

For meromorphic function $f(z)$ with simple pole at the point $z=a$ the following formula is true (Malešević, 2003):

$$\operatorname*{p.p.}_{z=a}f(z)=\lim_{\varepsilon\to 0}\frac{f(a-\varepsilon)+f(a+\varepsilon)}{2}. \tag{1.13}$$

Especially for gamma function $\Gamma(z)$ it is true (Slavić, 1970), (Malešević, 2003):

$$\operatorname*{p.p.}_{z=-n}\Gamma(z)=(-1)^n\frac{\Gamma'(n+1)}{\Gamma(n+1)^2}=\frac{-\gamma+\sum\limits_{k=1}^{n}\frac{1}{k}}{n!}, \tag{1.14}$$

where γ is Euler's constant and $n\in\mathbf{N}_0$.

1.2 The principal part of the Kurepa's functions

Đ. Kurepa introduced in paper (Kurepa, 1971) function $K(z)$ by integral:

$$K(z)=\int_0^\infty e^{-t}\frac{t^z-1}{t-1}\,dt, \tag{1.15}$$

which converges for $Re(z)>0$, and it represents one analytical extension of the sum of factorials:

$$K(n)=\sum_{i=0}^{n-1}i!\,. \tag{1.16}$$

For the function $K(z)$ we use the term *Kurepa's function* and it is one solution of the functional equation:

$$K(z) - K(z-1) = \Gamma(z). \tag{1.17}$$

Let us observe that it is possible to make analytical continuation of Kurepa's function $K(z)$ for $Re(z) \leq 0$. In that way, the Kurepa's function $K(z)$ is a meromorphic function with simple poles at $z = -1$ and $z = -n$ $(n \geq 3)$. At point $z = -2$ Kurepa's function has a removable singularity and $K(-2) \stackrel{\text{def}}{=} \lim_{z \to -2} K(z) = 1$. Kurepa's function has the following residues:

$$\operatorname*{res}_{z=-1} K(z) = -1 \quad \text{and} \quad \operatorname*{res}_{z=-n} K(z) = \sum_{k=2}^{n-1} \frac{(-1)^{k-1}}{k!} \quad (n \geq 3). \tag{1.18}$$

Previous results for Kurepa's function are given according to (Kurepa, 1973) and (Slavić, 1973). The functional equation (1.17), besides Kurepa's function $K(z)$, has another solution by series:

$$K_1(z) = \sum_{n=0}^{\infty} \Gamma(z-n), \tag{1.19}$$

which converges over the set $\mathbf{C} \backslash \mathbf{Z}$ (Malešević, 2003).

Extension of domain of functions $K(z)$ and $K_1(z)$ in the sense of the principal part at the point is given by the following statements (Slavić, 1973), (Malešević, 2003).

Lemma 1.3. *Let us define* $L_1 = -\sum_{n=0}^{\infty} \operatorname*{p.p.}_{z=-n} \Gamma(z)$. *Then:*

$$L_1 = \frac{1}{e}\Big(\gamma + \sum_{n=1}^{\infty} \frac{1}{n!n}\Big) = \frac{\mathrm{Ei}(1)}{e} \approx 0.697\,174\,883\,, \tag{1.20}$$

where Ei *is function of exponential integral.*

Theorem 1.4. *For the functions* $K(z)$ *and* $K_1(z)$ *are true:*

$$\operatorname*{p.p.}_{z=-n} K(z) = -\sum_{i=0}^{n-1} \operatorname*{p.p.}_{z=-i} \Gamma(z) = \sum_{i=0}^{n-1} (-1)^{i+1} \frac{\Gamma'(i+1)}{\Gamma(i+1)^2} \quad (n \in \mathbf{N}) \tag{1.21}$$

and

$$\operatorname*{p.p.}_{z=n} K_1(z) = \operatorname*{p.p.}_{z=n} K(z) - L_1 \quad (n \in \mathbf{Z}). \tag{1.22}$$

The connection between functions $K(z)$ and $K_1(z)$ is given by Slavić's formula which is presented in the following statement (Slavić, 1973), (Marichev, 1983), (Malešević, 2003).

Theorem 1.5. *It is true:*

$$K(z) = \frac{1}{e}\Big(\gamma + \sum_{n=1}^{\infty} \frac{1}{n!n}\Big) - \frac{\pi}{e} \operatorname{ctg} \pi z + \sum_{n=0}^{\infty} \Gamma(z-n), \tag{1.23}$$

where the values in the previous formula, in integer points z, *are determined in the sense of the principal part.*

Analogously to Kurepa's function we consider the function $A(z)$ given by the integral:

$$A(z) = \int_0^\infty e^{-t} \frac{t^{z+1} - (-1)^z t}{t+1} \, dt, \tag{1.24}$$

which converges for $Re(z) > 0$ (Petojević, 2002), and it represents one analytical extension of the alternating sum of factorials:

$$A(n) = \sum_{i=1}^{n} (-1)^{n-i} i! \, . \tag{1.25}$$

For the function $A(z)$ we use term *alternating Kurepa's function* and it is one solution of the functional equation:

$$A(z) + A(z-1) = \Gamma(z+1). \tag{1.26}$$

Let us observe that it is possible to make analytical continuation of alternating Kurepa's function $A(z)$ for $Re(z) \leq 0$. In that way, the alternating Kurepa's function $A(z)$ is a meromorphic function with simple poles at $z = -n$ $(n \geq 2)$. Alternating Kurepa's function has the following residues:

$$\operatorname*{res}_{z=-n} A(z) = (-1)^n \sum_{k=0}^{n-2} \frac{1}{k!} \quad (n \geq 2). \tag{1.27}$$

Previous results for alternating Kurepa's function are given according to (Petojević, 2002). The functional equation (1.26), besides alternating Kurepa's function $A(z)$, has another solution by series:

$$A_1(z) = \sum_{n=0}^{\infty} (-1)^n \Gamma(z+1-n), \tag{1.28}$$

which converges over the set $\mathbf{C} \backslash \mathbf{Z}$ (Malešević, 2006).

Extension of domain of functions $A(z)$ and $A_1(z)$ in the sense of the principal part at the point is given by following statements (Malešević, 2006).

Lemma 1.6. *Let us define* $L_2 = \sum_{n=0}^{\infty} (-1)^n \operatorname*{p.p.}_{z=-(n-1)} \Gamma(z)$, *then:*

$$L_2 = 1 + e\gamma - e\Big(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n! n} \Big) = 1 + e\text{Ei}(-1) \approx 0.403\,652\,337 \, . \tag{1.29}$$

Theorem 1.7. *For the functions* $A(z)$ *and* $A_1(z)$ *we have:*

$$\operatorname*{p.p.}_{z=-n} A(z) = \sum_{i=0}^{n-1} (-1)^{n+1-i} \operatorname*{p.p.}_{z=-(i-1)} \Gamma(z) = (-1)^{n+1} \Big(1 - \sum_{i=1}^{n-1} \frac{\Gamma'(i)}{\Gamma(i)^2} \Big) \quad (n \in \mathbf{N}) \tag{1.30}$$

and

$$\operatorname*{p.p.}_{z=n} A_1(z) = (-1)^n L_2 + \operatorname*{p.p.}_{z=n} A(z) \quad (n \in \mathbf{Z}). \tag{1.31}$$

The connection between functions $A(z)$ and $A_1(z)$ is given by a formula of the Slavić's type in the following statement (Malešević, 2006).

Theorem 1.8. *It is true that:*

$$A(z) = \Big(e\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!\,n} - 1 - e\gamma\Big)(-1)^z + \frac{\pi e}{\sin \pi z} + \sum_{n=0}^{\infty}(-1)^n\Gamma(z+1-n), \tag{1.32}$$

where the values in the previous formula, in integer points z, are determined in the sense of the principal part.

1.3 Principal part of the zeta function and Casimir energy

Riemann zeta function $\zeta(s)$ is a meromorphic function and it has only simple pole at $s=1$ with the principal part:

$$\underset{s=1}{\text{p.p.}}\,\zeta(s) = \gamma, \tag{1.33}$$

as cited in (Slavić, 1970). We consider the principal part of the global spectral zeta function $\zeta_L(s)$ which is a direct extension of the Riemann zeta function $\zeta(s)$ (Nesterenko, Lambiase and Scarpetta, 2004). Namely, let L be an elliptic differential operator of the second order acting only on the variable x and let $\varphi(t,x) = e^{\pm i\omega t}\varphi_n(x)$ be a solution of the following equation:

$$\Big(L + \frac{\partial^2}{c^2\partial t^2}\Big)\varphi(t,x) = 0, \tag{1.34}$$

where ω and c are constants. Let scalars λ_n fulfill $L\varphi_n(x) = \lambda_n\varphi_n(x)$. Then we define *global spectral zeta function* by (Hawking, 1977), (Elizalde, Odintsov, Romeo, Bytsenko and Zerbini, 1994), (Nesterenko et al., 2004):

$$\zeta_L(s) = \sum_n \lambda_n^{-s}. \tag{1.35}$$

Casimir energy of the field $\varphi(t,x)$ is defined by (Blau et al., 1988b), (Nesterenko et al., 2004):

$$E_0 = \frac{1}{2}\lim_{\varepsilon\to 0}\frac{\zeta_L(-\frac{1}{2}+\varepsilon)+\zeta_L(-\frac{1}{2}-\varepsilon)}{2} = \frac{1}{2}\underset{s=-\frac{1}{2}}{\text{p.p.}}\ \zeta_L(s). \tag{1.36}$$

In paper (Nesterenko et al., 2004) some values of Casimir energy have been given, dependent of the fields which are considered. All computations in (Nesterenko et al., 2004), based on paper (Blau et al., 1988b), are related to the global spectral zeta functions with, as a rule, simple poles.

2 Differential - algebraic properties

In this section we present a method for proving that certain analytic functions are not solutions of algebraic differential equations. The method is based on model-theoretic properties of differential fields and that $\Gamma(x)$ is a transcendental differential function.

2.1 Differential fields

The theory DF_0 of differential fields of characteristic 0 is the theory of fields with following axioms that relate to the derivative D:

$$D(x+y) = Dx + Dy, \quad D(xy) = xDy + yDx. \tag{2.1}$$

Thus, a model of DF_0 is a differential field $\mathbf{K} = (K, +, \cdot, D, 0, 1)$ where $(K, +, \cdot, 0, 1)$ is a field and D is a differential operator satisfying the above axioms. Abraham Robinson proved that DF_0 has a model completion, and then defined DCF_0 to be the model completion of DF_0. Afterwards Leonore Blum found simple axioms of DFC_0 not mentioning of differential polynomials in more than one variable (Sacks, 1972). In the following, if not otherwise stated, $\mathbf{F}, \mathbf{K}, \mathbf{L}, \ldots$ will denote differential fields, $F, L, K, \ldots$ their domains while $\mathbf{F}^*, \mathbf{K}^*, \mathbf{L}^*, \ldots$ will denote their field parts, i.e. $\mathbf{F}^*{=}(F, +, \cdot, 0, 1)$. It is customary to denote by $\mathbf{L}\{X\}$ the ring of differential polynomials over $\mathbf{L}$ in the variable X, see (Marker, Messmer and Pillay, 1996). Thus, if $f \in L\{X\}$ then for some natural number n, is true $f = f(X, DX, D^2X, \ldots, D^nX)$, where $f(x, y_1, y_2, \ldots, y_n)$ is the ordinary algebraic polynomial over $\mathbf{L}^*$. Then the order of f, denoted by $\operatorname{ord} f$, is the largest n such that D^nX occurs in f.

If $b \in K$, then $\mathbf{L}(b)$ will denote the simple differential extension of $\mathbf{L}$ in $\mathbf{K}$, i.e. $\mathbf{L}(b)$ is the smallest differential subfield of $\mathbf{K}$ containing both L and b. Also, we shall use the following abbreviations:

d.p. is standing for *differential polynomial.* Thus, f is a d.p. over $\mathbf{L}$ in the variable X if and only if $f \in L\{X\}$.

d.a. is standing for *differential algebraic.* Hence, if $\mathbf{L} \subseteq \mathbf{K}$ then $b \in K$ is d.a. over $\mathbf{L}$ if and only if there is a non-zero d.p. f such that $f(b, Db, D^2b, \ldots, D^nb) = 0$, otherwise b is transcendental. The field $\mathbf{K}$ is a d.a. extension of $\mathbf{L}$ if every $b \in K$ is d.a. over $\mathbf{L}$.

d.e. is standing for *differential equation,*

a.d.e. is standing for *algebraic differential equation.* Hence, $f = 0$ is a.d.e. if $f \in L\{X\}$.

Models of DCF_0 are differentially closed fields. A differential field $\mathbf{K}$ is *differentially closed* if, whenever $f, g \in \mathbf{K}\{X\}$, g is non-zero and $\operatorname{ord} f > \operatorname{ord} g$, there is $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$. The theory DCF_0 admits elimination of quantifiers and it is submodel complete (A. Robinson): if $\mathbf{F} \subseteq \mathbf{L}, \mathbf{K}$ then $\mathbf{K}_F \equiv \mathbf{L}_F$, i.e. $\mathbf{K}$ and $\mathbf{L}$ are elementary equivalent over $\mathbf{L}$. In the following we shall use the next theorem, see (Mijajlović and Malešević, 2007):

Theorem 2.1. *Suppose* $\mathbf{F} \subseteq \mathbf{K}$ *and let* $L = \{b \in K\colon\ b$ *is d.a. over* $\mathbf{F}\}$. *Then*

a. L *is a differential subfield of* $\mathbf{K}$ *extending* $\mathbf{F}$.

b. *If* $\mathbf{K}$ *is d.a. closed then* L *is d.a. closed.*

Other notations, notions and results concerning differential fields that will be used corresponds to those in (Sacks, 1972) or (Marker et al., 1996).

2.2 Transcendental differential functions

Suppose $\mathbf{L} \subseteq \mathbf{K}$. Let $\mathcal{R} = \mathbf{R}(x)$ be the differential field of real rational functions and $\mathcal{C} = \mathbf{C}(z)$ the differential field of complex rational functions. The following Hölder's famous theorem asserts the differential transcendentality of Gamma function.

Theorem 2.2. **a.** $\Gamma(x)$ *is not d.a. over* $\mathbf{R}(x)$. **b.** $\Gamma(z)$ *is not d.a. over* $\mathbf{C}(z)$. ∇

Now we shall use the transcendentality of $\Gamma(z)$ and properties of differential fields to prove differential transcendentality over $\mathcal{C}$ of some analytic functions. Let us denote by $\mathcal{M}_{\mathrm{D}}$ the class of complex functions meromorphic on a complex domain D (a connected open set in the complex z-plane C). If $\mathrm{D} = \mathrm{C}$ then we shall write $\mathcal{M}$ instead of $\mathcal{M}_{\mathrm{D}}$. Then $\mathcal{M}_{\mathrm{D}}$ is differential field and $\mathcal{C} \subseteq \mathcal{M}$. Further, let $\mathcal{L} = \{f \in \mathcal{M} : f \text{ d.a. over } \mathcal{C}\}$. By Theorem 2.1 $\mathcal{L}$ is a differential subfield of $\mathcal{M}$ extending $\mathcal{C}$. The function $\Gamma(z)$ is meromorphic and by Hölder's theorem $\Gamma(z) \notin \mathcal{L}$.

Example 2.3. As we have seen in the first part, Kurepa's function $K(z)$ can be continued meromorphically to whole complex plane. Therefore, $K(z-1)$ is meromorphic either. Also, as we have seen in the first part, Kurepa's function satisfies the recurrence relation

$$K(z) - K(z-1) = \Gamma(z), \tag{2.2}$$

Now, suppose that $K(z)$ belongs to $\mathcal{L}$. Then $K(z)$ satisfies an a.d.e.

$$f(z, y, Dy, D^2y, \ldots, D^n y) = 0 \tag{2.3}$$

where $f(z, y_1, y_2, \ldots, y_n) \in \mathcal{C}[x, y_1, y_2, \ldots, y_n]$. Then $K(z+1)$ satisfies the a.d.e.

$$f(z+1, y, Dy, D^2y, \ldots, D^n y) = 0 \tag{2.4}$$

so $K(z+1)$ belongs to $\mathcal{L}$. As $\mathcal{L}$ is a field, by (2.2) it follows that $\Gamma(z)$ belongs to $\mathcal{L}$, what yields a contradiction. Hence, $K(z)$ is a transcendental differential function.
Using previous method we can conclude that each meromorphic solution of a functional equation (2.2) is transcendental differential function over the field $\mathcal{C}$. For example, another solution of this functional equation is series (1.19). Therefore, $K_1(z)$ is a transcendental differential function too. ∇

In a similar way, one can prove that alternating functions $A(z)$ and $A_1(z)$ are differentially transcendental too over $\mathcal{C}$.

The following general proposition concerning transcendental differential functions holds.

Theorem 2.4. *Let $a(z)$ be a meromorphic differentially transcendental function over $\mathcal{C}$ and $f(z, u_0, u_1, \ldots, u_m, y_1, \ldots, y_n)$ be a polynomial over $\mathcal{C}$. If b is meromorphic and $f(z, b, E_1 b, \ldots, E_m b, Db, \ldots, D^n b) = a(z)$, where $E_i f(z) = \alpha_i z + \beta_i$, $\alpha_i, \beta_i \in \mathrm{C}$, then b is differentially transcendental over $\mathcal{C}$.*

Proof Suppose that b is d.a. over $\mathcal{C}$, i.e. that $b \in \mathcal{L}$. Then $Db, \ldots, D^n b$ belong to $\mathcal{L}$. Further, there is a.d.e. $f(z, y, Dy, \ldots, D^k y) = 0$ satisfied by b, so $E_i b$ satisfies

$$f(\alpha_i z + \beta_i, y, \alpha_1^{-1} Dy, \ldots, \alpha_k^{-1} D^k y) = 0, \tag{2.5}$$

i.e. $E_i b \in \mathcal{L}$, too. Therefore, $g(z, b, E_1 b, \ldots, E_m b, Db, \ldots, D^n b) \in \mathcal{L}$, so $a(z)$ belongs to $\mathcal{L}$, a contradiction. ∇

Example 2.5. The Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p\text{-prime}} (1 - p^{-s})^{-1}, \quad \mathrm{Re}(s) > 1, \tag{2.6}$$

is differentially transcendental over $\mathcal{C}$ (Hilbert). First we observe that $\zeta(s)$ can be continued meromorphically to whole complex plane with a simple pole at $s=1$ and that $\zeta(s)$ satisfies the well-known functional equation (Ivić, 1985):

$$\zeta(s)=\chi(s)\zeta(1-s), \quad \text{where} \quad \chi(s)=\frac{(2\pi)^s}{2\Gamma(s)\cos(\frac{\pi s}{2})}. \tag{2.7}$$

Now, suppose that $\zeta(s)$ is d.a. over $\mathcal{C}$, i.e. that $\zeta(s)\in\mathcal{L}$. Then $\zeta(1-s)$ and $\zeta(s)/\zeta(1-s)$ belong to $\mathcal{L}$, too, so $\chi(s)$ belongs to $\mathcal{L}$. The elementary functions $(2\pi)^s$, and $\cos(\frac{\pi s}{2})$ obviously are d.a. over $\mathcal{C}$ i.e. they belong to $\mathcal{L}$. As $\mathcal{L}$ is a field, it follows that $\Gamma(z)$ belong to $\mathcal{L}$, too. But this yield a contradiction, therefore $\zeta(s)$ is differentially transcendental function over $\mathcal{C}$. Generally, Dirichlet L-series

$$L_k(s)=\sum_{n=1}^{\infty}\kappa_k(n)\frac{1}{n^s} \quad (k\in Z), \tag{2.8}$$

where $\kappa_k(n)$ is Dirichlet character (Ireland and Rosen, 1982), is differentially transcendental function over $\mathcal{C}$. This follows from a well-known functional equations

$$L_{-k}(s)=2^s\pi^{s-1}k^{-s+\frac{1}{2}}\Gamma(1-s)\cos(\frac{\pi s}{2})L_{-k}(1-s) \tag{2.9}$$

and

$$L_{+k}(s)=2^s\pi^{s-1}k^{-s+\frac{1}{2}}\Gamma(1-s)\sin(\frac{\pi s}{2})L_{+k}(1-s)\,. \tag{2.10}$$

Besides Riemann zeta function $\zeta(s)=L_{+1}(s)$, Dirichlet eta function

$$\eta(s)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^s}=(1-2^{1-s})L_{+1}(s) \tag{2.11}$$

and Dirichlet beta function

$$\beta(s)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)^s}=L_{-4}(s) \tag{2.12}$$

are transcendental differential functions as examples of Dirichlet series (Borwein and Borwein, 1987). ▽

Example 2.6. The meromorphic function

$$H_1(z)=\sum_{n=0}^{\infty}\frac{1}{(n+z)^2} \tag{2.13}$$

is differentially transcendental over $\mathcal{C}$. Really, $D^2\ln(\Gamma(z))=H_1(z)$, i.e. $\Gamma(z)$ satisfies the a.d.e. $(D^2\Gamma)\Gamma-(D\Gamma)^2-H_1\Gamma^2=0$ over $\mathcal{C}(H_1)$. Thus, if H_1 would be d.a. over $\mathcal{C}$, then by Theorem 2.4, Γ would be too, what yields a contradiction. Hence, $H_1(z)$ is differentially transcendental function over $\mathcal{C}$. Let us notice the well-known fact that gamma function is the solution of the following non-algebraic differential equation: $df(z)/dz=\psi(z)f(z)$ for $f(z)=\Gamma(z)$. ▽

Remark 2.1. We see that in Example 2.6 functions $\Gamma(z)$ and $H_1(z)$ are differentially algebraically dependent, i.e. $g(\Gamma,H_1)=0$, where $g(x,y)=x''x-(x')^2-yx^2$. We do not know if similar dependencies exist for pairs (K,Γ) and (ζ,Γ). It is very likely that these pairs are in fact differentially transcendental.

2.3 Operator δ

Let $\mathbf{F}_D = (F, +, \cdot, D, 0, 1)$ be a differential field and $\theta \in F$. We can introduce a new differential operator $\delta = \theta \cdot D$, i.e. by $\delta(x) = \theta \cdot D(x)$, $x \in F$. Then $\mathbf{F}_\delta = (F, +, \cdot, \delta, 0, 1)$ becomes a new differential filed. Let $\overline{\mathbf{F}}_D$ and $\overline{\mathbf{F}}_\delta$ denote differential closures of fields $\mathbf{F}_D$ and $\mathbf{F}_\delta$ respectively.

Proposition 2.7. *Domains of fields $\overline{\mathbf{F}}_D$ and $\overline{\mathbf{F}}_\delta$ are same, i.e. $\bar{F}_D = \bar{F}_\delta$.*

Proof If $a \in \bar{F}_\delta$ then a is a solution of an a.d.e. $\mathcal{E}(\delta)$ in respect to the operator δ. We can substitute in this equation operator δ with $\theta \cdot D$, and we shall obtain again an a.d.e. $\mathcal{E}'(\mathcal{D})$ but now in respect to D. Then a is a solution of this equation, hence $a \in \bar{F}_D$. So we proved that $\bar{F}_\delta \subseteq \bar{F}_D$. On the other hand, $D = \theta^{-1}\delta$, so we may apply a symmetrical argument, hence $a \in \bar{F}_D$ implies $a \in \bar{F}_\delta$, i.e. $\bar{F}_D \subseteq \bar{F}_\delta$. Therefore, we proved $\bar{F}_D = \bar{F}_\delta$. ∇

We can ask the natural question if fields $\overline{\mathbf{F}}_D$, $\overline{\mathbf{F}}_\delta$ are isomorphic. We observe that it is not necessary $\mathbf{F}_D \cong \mathbf{F}_\delta$. For example, if $\mathbf{F} = \mathbf{R}(x)$, D is the ordinary differentiation operator and $\delta = xD$, then the equation $\delta y = y$ has a solution in $\mathbf{F}_\delta$, $y = x$, while the equation $Dy = y$ has no solution in $\mathbf{F}_D$. Hence $\mathbf{F}_D \ncong \mathbf{F}_\delta$. Let us remind that $\tau\colon F_\delta \to F_D$ is an isomorphism if τ satisfies:

$$\tau(x+y) = \tau x + \tau y,\ \ \tau(xy) = \tau x\, \tau y,\ \ \tau(\delta x) = D\tau(x),\ \ \tau(0) = 0, \tau(1) = 1. \tag{2.14}$$

Under some circumstances the isomorphism exists between fields $\overline{\mathbf{F}}_D$, $\overline{\mathbf{F}}_\delta$, or between certain intermediate fields. For example, the conditions will be fulfilled if these fields have functional representation and a particular differential equation has a solution. Let $\mathcal{L}_D = \{f \in \mathcal{M} : f$ d.a. over $\mathcal{C}$ in respect to the operator $D\}$ and $\mathcal{L}_\delta = \{f \in \mathcal{M} : f$ d.a. over $\mathcal{C}$ in respect to the operator $\delta\}$.

Theorem 2.8. *If $\theta \in \mathcal{L}_D$ is non-constant and g is a non-constant solution of a.d.e. $Dx = \theta \cdot x$ then $\mathcal{L}_D \cong \mathcal{L}_\delta$.*

Proof First, we observe, using the argument as in the proof of the above proposition, that domains of $\mathcal{L}_D$ and $\mathcal{L}_\delta$ are same. Let $\tau\colon \mathcal{L}_\delta \to \mathcal{L}_D$ be defined by $\tau(x) = x \circ g$, where $\circ$ is the composition operator. We see that g is meromorphic and is a.d. over $\mathcal{C}$, therefore $g \in \mathcal{L}_D$. τ is well defined since $\mathcal{L}_D$ is closed under composition. Obviously it satisfies $\tau(x+y) = \tau x + \tau y$, $\tau(xy) = \tau x\, \tau y$. Further, as $Dg = \theta \circ g$,

$$\tau(\delta x) = (\theta D x) \circ g = (\theta \circ g)((Dx) \circ g) = Dg((Dx) \circ g) = D(x \circ g) = D(\tau x). \tag{2.15}$$

τ is 1–1 function, since g takes infinitely many values over a bounded region. Therefore, $\tau\colon \mathcal{L}_D \cong \mathcal{L}_\delta$. ∇

In the case of $\theta = x$, x here denotes a variable (i.e. the polynomial of the degree one), we can produce an explicit isomorphism $\tau\colon \mathcal{L}_D \cong \mathcal{L}_\delta$. We can define τ by $\tau : f \to f \circ g$, $f \in \mathcal{L}_D$, where $g(x) = e^x$. Observe that this isomorphism corresponds to the transformation $x = e^z$ in the algorithm of solving of Euler linear differential equations. This observation give us a new, the algebraic insight into the classical method of solving Euler and similar types (e.g. Legendre linear equation) of differential equations. We shall give an illustration by example:

Example 2.9. Solve $x^3y''' + 3x^2y'' - 2xy' + 2y = 0$.

Solution This equation is equivalent to

$$(\delta(\delta-1)(\delta-2) + 3\delta(\delta-1) - 2\delta + 2)y = 0, \tag{2.16}$$

i.e. to the equation $(\delta^3 - 3\delta + 2)y = 0$ in $\mathcal{L}_\delta$. The corresponding equation $(D^3 - 3D + 2)y = 0$ in $\mathcal{L}_D$ has general solution $c_1h_1 + c_2h_2 + c_3h_3$, where $h_1(x) = e^x, h_2(x) = xe^x, h_3(x) = e^{-2x}$. As $\tau\colon \mathcal{L}_D \cong \mathcal{L}_\delta$, and τ^{-1} is given by $\tau^{-1}\colon f \to f \circ g^{-1}$ (here $g^{-1}(x) = \ln x$), it follows that

$$\tau^{-1}(c_1h_1 + c_2h_2 + c_3h_3) = c_1h_1 \circ g^{-1} + c_2h_2 \circ g^{-1} + c_3h_3 \circ g^{-1} \tag{2.17}$$

is the general solution of $(\delta^3 - 3\delta + 2)y = 0$ in $\mathcal{L}_\delta$, and so the solution of the starting equation is $y = c_1x + c_2x\ln x + c_3x^{-2}$.

Hence, one should expect that standard methods of solving differential equations which are done by "properly chosen transformations of the independent variable" correspond in fact to constructions of an isomorphism between $\mathcal{L}_D$ and $\mathcal{L}_\delta$, or some other intermediate fields, for properly chosen differential operators δ.

Acknowledgment

Second author supported in part by the project MNTRS, Grant No. ON144020.

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

RAMANUJAN'S $_1\Psi_1$ - SUMMATION FORMULA AND RELATED IDENTITIES

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ABSTRACT

Ramanujan recorded [7] several identities involving Lambert series and product or quotient of theta-functions which are equivalent to some important classical results in Number theory. Recently M. D. Hirschhorn [5], M. S. Mahadeva Naika and H. S. Madhusudhan [6] have derived some of these identities. In this paper we generalise these identities using $_1\psi_1$ summation formula of Ramanujan.

Key Words: Jocobi's two square theorem, Lambert series, $_1\psi_1$ - summation formula, Theta-function.

2000 AMS Mathematics subject classification: 11E25, 14K25

1. INTRODUCTION

In Chapter 16 of his second notebook [1], [2], Ramanujan develops the theory of theta-functions. His general theta-function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, |ab|<1.$$

Let

$$\varphi(q) := f(q,q) = \frac{(-q;-q)_\infty}{(q;-q)_\infty},$$

$$\psi(q) := f(q,q^3) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := f(-q;-q^2) = (q;q)_\infty.$$

The product representations of these theta functions can be derived by using the Jacobi triple product identity:

$$f(a,b) := (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab|<1.$$

Ramanujan recorded [7, pp.353-355], several identities involving Lambert series and product or quotient of theta-functions. Hirschhorn [5] has derived the following identities of Ramanujan:

$$\prod_{n\ge1}(1-q^n)^3$$

$$=\prod_{n\ge1}(1+q^{4n-3})(1+q^{4n-1})(1-q^{4n})\left[1-4\sum_{n\ge1}\left(\frac{q^{4n-3}}{1+q^{4n-3}}-\frac{q^{4n-1}}{1+q^{4n-1}}\right)\right], \tag{1.1}$$

$$\frac{\varphi^3(q)}{\varphi(q^3)}=1-6\sum_{n\ge1}(-1)^n\left(\frac{q^{3n-2}}{1+(-1)^nq^{3n-2}}+\frac{q^{3n-1}}{1-(-1)^nq^{3n-1}}\right). \tag{1.2}$$

$$\frac{\psi^3(q)}{\psi(q^3)}=1+3\sum_{n\ge1}\left(\frac{q^{6n-5}}{1-q^{6n-5}}-\frac{q^{6n-1}}{1-q^{6n-1}}\right). \tag{1.3}$$

He has also drawn combinatorial conclusions from these identities. He has shown that (1.1) can be transformed into

$$\varphi^2(q)=1+4\sum_{n\ge1}\left(\frac{q^{4n-3}}{1-q^{4n-3}}-\frac{q^{4n-1}}{1-q^{4n-1}}\right), \tag{1.4}$$

which is equivalent to Jacobi's two square theorem, which states that the number of representations of n as a sum of two squares is four times the differences between the number of divisors of n congruent to 1 modulo 4 and the number of divisors of n congruent to 3 modulo 4. Motivated by these works of Hirschhorn [5], Mahadeva Naika and Madhusudhan [6] gave a simple and a unified approach to proving many identities of Ramanujan found in the scattered literature.

The main purpose of this paper is to generalise some of the identities of Ramanujan involving Lambert series and product or quotients of theta-functions using ${}_1\psi_1$ - summation formula of Ramanujan.

2. SOME PRELIMINARY RESULTS

In the following theorem, we collect the various results needed for proving our main results.

Theorem 2.1. If $|q|<1$, then the following theta-function identities hold:

(i) $$\prod_{n\ge1}(1+aq^{2n-1})(1+a^{-1}q^{2n-1})(1-q^{2n})=\sum_{n=-\infty}^{\infty}a^nq^{n^2}, \tag{2.1}$$

(ii) $$\prod_{n\ge1}(1-q^n)^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}, \tag{2.2}$$

(iii) $\varphi^2(q)-\varphi^2(-q)=8q\psi^2(q^4)$, (2.3)

(iv) $\varphi^2(q)f(-q)=\sum_{n=-\infty}^{\infty}(6n+1)q^{(3n^2+n)/2}$, (2.4)

(v) $\psi(q^2)f^2(-q)=\sum_{n=-\infty}^{\infty}(3n+1)q^{3n^2+2n}$, (2.5)

(vi) $4q\psi(q^2)\psi(q^6)=\varphi(q)\varphi(q^3)-\varphi(-q)\varphi(-q^3)$, (2.6)

(vii) $\dfrac{\varphi^3(q)}{\varphi(q^3)}+2\dfrac{\varphi^3(-q^2)}{\varphi(-q^6)}=3\varphi(q)\varphi(q^3)$, (2.7)

(viii) $a(q)=\varphi(q)\varphi(q^3)+4q\psi(q^2)\psi(q^6)$, (2.8)

(ix) $a(q)=\dfrac{\varphi^3(-q^3)}{\varphi(-q)}+4q\dfrac{\psi^3(q^3)}{\psi(q)}$ (2.9)

and

(x) $a(q)=\dfrac{\psi^3(q)}{\psi(q^3)}+3q\dfrac{\psi^3(q^3)}{\psi(q)}$. (2.10)

Proof. For a proof of (2.1) see [4, Theorem 352, p.282]; for a proof of (2.2) see [4, Theorem 357, p.285]; for a proof of (2.3) see [2,Entry 25(v),ch.16 p.40]; for a proof of (2.4) see [2, Entry 8(ix), ch.17,p.114]; for a proof of (2.5) see [2,Entry 8(x), ch.17,p.115]; for proofs of (2.6) and (2.7) see [2, p.232]; for a proof of (2.8) see [3, p.93]; for a proof of (2.9) see [3, p.110]; for a proof of (2.10) see [3, p.111].

3. MAIN THEOREMS

Theorem 3.1. If $|\alpha\beta q^k|<|\alpha q^{(k+m)/2}|<1$, then

$$\sum_{n=-\infty}^{\infty}\frac{(pn+x)(1/\alpha;q^k)_n(-\alpha q^{(k+m)/2})^n}{(\beta q^k;q^k)_n}=f(q^{(k-m)/2},q^{(k+m)/2})$$
$$\times\prod_{n=1}^{\infty}\frac{(1-\alpha\beta q^{kn})}{(1-\alpha q^{kn})(1-\beta q^{kn})(1+\alpha q^{kn-((k-m)/2)})(1+\beta q^{kn-((k+m)/2)})}$$
$$\times\left[x+p(1-\alpha)\sum_{n=1}^{\infty}\frac{q^{kn-((k-m)/2)}}{(1+q^{kn-((k-m)/2)})(1+\alpha q^{kn-((k-m)/2)})}\right.$$
$$\left.-p(1-\beta)\sum_{n=1}^{\infty}\frac{q^{kn-((k+m)/2)}}{(1+q^{kn-((k+m)/2)})(1+\beta q^{kn-((k+m)/2)})}\right],\quad (3.1)$$

where p and m are non-negative integers and k is a positive integer.

Proof. By Entry 17 of Chapter 16 of Ramanujan's second notebook [7, p.196], we have

$$\sum_{n=-\infty}^{\infty}\frac{(1/\alpha;q^2)_n(-\alpha q)^n z^n}{(\beta q^2;q^2)_n}$$

$$=f(qz,q/z)\prod_{n=1}^{\infty}\frac{(1-\alpha\beta q^{2n})}{(1-\alpha q^{2n})(1-\beta q^{2n})(1+\alpha q^{2n-1}z)(1+\beta q^{2n-1}z^{-1})},$$

where

$$|\beta q|<|z|<\frac{1}{|\alpha q|},\ |q|<1. \tag{3.2}$$

Replacing q by q^k and z by $z^p q^m$ in (6.2.2) we find that

$$\sum_{n=-\infty}^{\infty}\frac{(1/\alpha;q^{2k})_n(-\alpha q^{k+m})^n z^{pn}}{(\beta q^{2k};q^{2k})_n}=f(z^p q^{k+m},z^{-p}q^{k-m})$$

$$\times\prod_{n=1}^{\infty}\frac{(1-\alpha\beta q^{2kn})}{(1-\alpha q^{2kn})(1-\beta q^{2kn})(1+\alpha q^{2kn-(k-m)}z^p)(1+\beta q^{2kn-(k+m)}z^{-p})}. \tag{3.3}$$

Replacing q by $q^{1/2}$ and then multiplying both sides of (3.3) by z^x, we deduce that

$$\sum_{n=-\infty}^{\infty}\frac{(1/\alpha;q^{k})_n(-\alpha q^{(k+m)/2})^n z^{pn+x}}{(\beta q^{k};q^{k})_n}=z^x f(z^p q^{(k+m)/2},z^{-p}q^{(k-m)/2})$$

$$\times\prod_{n=1}^{\infty}\frac{(1-\alpha\beta q^{kn})}{(1-\alpha q^{kn})(1-\beta q^{kn})(1+\alpha q^{kn-((k-m)/2)}z^p)(1+\beta q^{kn-((k+m)/2)}z^{-p})}. \tag{3.4}$$

Differentiating both sides of (3.4) with respect to z and then putting $z = 1$, we obtain (3.1).

Theorem 3.2. If $\left|\alpha\beta q^k\right|<\left|\alpha q^{(k+m)/2}\right|<1$, then

$$\sum_{n=-\infty}^{\infty}\frac{(pn+x)(1/\alpha;\ q^k)_n(\alpha q^{(k+m)/2})^n}{(\beta q^k;\ q^k)_n}=f(-q^{(k-m)/2},-q^{(k+m)/2})$$

$$\times\prod_{n=1}^{\infty}\frac{(1-\alpha\beta q^{kn})}{(1-\alpha q^{kn})\ (1-\beta q^{kn})(1-\alpha q^{kn-((k-m)/2)})(1-\beta q^{kn-((k+m)/2)})}$$

$$\times\left[x-p(1-\alpha)\sum_{n=1}^{\infty}\frac{q^{kn-((k-m)/2)}}{(1-q^{kn-((k-m)/2)})(1-\alpha q^{kn-((k-m)/2)})}\right.$$

$$\left.+p(1-\beta)\sum_{n=1}^{\infty}\frac{q^{kn-((k+m)/2)}}{(1-q^{kn-((k+m)/2)})(1-\beta q^{kn-((k+m)/2)})}\right], \tag{3.5}$$

where p and m are non-negative integers and k is a positive integer.

Proof. Replacing q by q^k and then z by $-z^p q^m$ in (3.2) we find that

$$\sum_{n=-\infty}^{\infty}\frac{(1/\alpha;q^{2k})_n(\alpha q^{k+m})^n z^{pn}}{(\beta q^{2k};q^{2k})_n}=f(-z^p q^{k+m},-z^{-p}q^{k-m})$$

$$\times\prod_{n=1}^{\infty}\frac{(1-\alpha\beta q^{2kn})}{(1-\alpha q^{2kn})(1-\beta q^{2kn})(1-\alpha q^{2kn-(k-m)}z^p)(1-\beta q^{2kn-(k+m)}z^{-p})}. \tag{3.6}$$

Again replacing q by $q^{1/2}$ and then multiplying both sides of the above identity (3.6) by z^x, we deduce that

$$\sum_{n=-\infty}^{\infty}\frac{(1/\alpha;q^{k})_n(\alpha q^{(k+m)/2})^n z^{pn+x}}{(\beta q^{k};q^{k})_n}=z^x f(-z^p q^{(k+m)/2},-z^{-p}q^{(k-m)/2})$$

$$\times\prod_{n=1}^{\infty}\frac{(1-\alpha\beta q^{kn})}{(1-\alpha q^{kn})(1-\beta q^{kn})(1-\alpha q^{kn-((k-m)/2)}z^p)(1-\beta q^{kn-((k+m)/2)}z^{-p})}.$$

Differentiating both sides of the above identity with respect to z and then putting $z = 1$, we obtain (3.5).

Now we not only establish two results in [6] using Theorems 3.1 and 3.2 but also deduce other interesting special cases, namely (3.9) and (3.10). Several Lambert series of Ramanujan found in his second and 'lost' notebooks can also be deduced from Theorems 3.1 and 3.2.

Corollary 3.1. [6, Theorem 3.1] We have

$$\frac{\sum_{n=-\infty}^{\infty}(pn+x)q^{(kn^2+mn)/2}}{f(q^{(k-m)/2},q^{(k+m)/2})}=x+p\sum_{n=1}^{\infty}\left(\frac{q^{kn-(k-m)/2}}{1+q^{kn-(k-m)/2}}-\frac{q^{kn-(k+m)/2}}{1+q^{kn-(k+m)/2}}\right). \tag{3.7}$$

Proof. Putting $\alpha=\beta=0$ in (3.1), we obtain (3.7).

Corollary 3.2. [6, Theorem 3.2] We have

$$\frac{\sum_{n=-\infty}^{\infty}(-1)^n(pn+x)q^{(kn^2+mn)/2}}{f(-q^{(k-m)/2},-q^{(k+m)/2})}$$

$$=x-p\sum_{n=1}^{\infty}\left(\frac{q^{kn-(k-m)/2}}{1-q^{kn-(k-m)/2}}-\frac{q^{kn-(k+m)/2}}{1-q^{kn-(k+m)/2}}\right). \tag{3.8}$$

Proof. Putting $\alpha=\beta=0$ in (3.5), we obtain (3.8).

Corollary 3.3. We have

$$\sum_{n=0}^{\infty}\frac{(pn+x)q^{(kn^2+mn)/2}}{(q^k;q^k)_n}=\frac{f(q^{(k-m)/2},q^{(k+m)/2})}{\prod_{n=1}^{\infty}(1-q^{kn})(1+q^{kn-((k+m)/2)})}$$

$$\times\left[x+p\sum_{n=1}^{\infty}\frac{q^{kn-(k-m)/2}}{1+q^{kn-(k-m)/2}}\right]. \tag{3.9}$$

Proof. Putting $\alpha=0$ and $\beta=1$ in (3.1), we obtain (3.9).

Corollary 3.4. We have

$$\sum_{n=0}^{\infty}\frac{(-1)^n(pn+x)q^{(kn^2+mn)/2}}{(q^k;q^k)_n}=\frac{f(-q^{(k-m)/2},-q^{(k+m)/2})}{\prod_{n=1}^{\infty}(1-q^{kn})(1-q^{kn-((k+m)/2)})}$$

$$\times\left[x-p\sum_{n=1}^{\infty}\frac{q^{kn-(k-m)/2}}{1-q^{kn-(k-m)/2}}\right]. \tag{3.10}$$

Proof. Putting $\alpha=0$ and $\beta=1$ in (3.5), we obtain (3.10).

Acknowledgement

*The authors wish to thank Professor **C. Adiga** for his suggestions and encouragement during the preparation of this paper. The Authors are thankful to the referee for suggestions.*

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Two results on ill-posed problems

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ABSTRACT

Let $A = A^*$ be a linear operator in a Hilbert space H. Assume that equation $Au = f$ (1) is solvable, not necessarily uniquely, and y is its minimal-norm solution. Assume that problem (1) is ill-posed. Let f_δ, $||f - f_\delta|| \leq \delta$, be noisy data, which are given, while f is not known. Variational regularization of problem (1) leads to an equation $A^*Au + \alpha u = A^* f_\delta$. Operation count for solving this equation is much higher, than for solving the equation $(A + ia)u = f_\delta$ (2). The first result is the theorem which says that if $a = a(\delta)$, $\lim_{\delta\to 0} a(\delta) = 0$ and $\lim_{\delta\to 0} \frac{\delta}{a(\delta)} = 0$, then the unique solution u_δ to equation (2), with $a = a(\delta)$, has the property $\lim_{\delta\to 0} ||u_\delta - y|| = 0$. The second result is an iterative method for stable calculation of the values of unbounded operator on elements given with an error.

Keywords:linear operators,ill-posed problems, regularization, discrepancy principle.

Math Subject Classification: 47A05, 45A50,35R30

1 Introduction

The results of this paper are formulated as Theorems 1 and 2 and proved in Sections 1 and 2 respectively. For the notions related to ill-posed problems one may consult, for example, (Bakushinsky and Goncharsky, 1989), (Morozov, 1984), (Ramm, 2005a), (Vainikko and Veretennikov, 1986), and the references cited there.
Let $A = A^*$ be a linear operator in a Hilbert space H. Assume that equation

$$Au = f. \tag{1.1}$$

is solvable, not necessarily uniquely, and y is its minimal-norm solution, $y \perp N := \{u : Au = 0\}$. Assume that problem (1.1) is ill-posed. In this case small perturbations of f may cause large perturbations of the solution to (1.1) or may throw f out of the range $R(A)$ of A. Let f_δ, $||f - f_\delta|| \leq \delta$, be noisy data, which are given, while f is not known. Variational regularization of problem (1.1) for bounded operators A leads to an equation $A^*Au + \alpha u = A^* f_\delta$, where $A^* = A$ since we assume A to be selfadjoint. Operation count for solving this equation is much higher than that for solving the equation

$$(A + ia)u = f_\delta. \tag{1.2}$$

The emphasis in this paper is on unbounded operators A. If A is unbounded, then f_δ may not belong to $D(A^*)$, and in this case the usual equation $A^*Au + \alpha u = A^* f_\delta$ is not well-defined. See a discussion of this point in (Ramm, To appear 2006)-(Ramm, Submitted 2005b). A regularization method for unbounded operators is proposed in (Ramm, 2002). In (Bakushinsky

and Goncharsky, 1989) and (Vainikko and Veretennikov, 1986) one can find a presentation of iterative methods for regularization of linear ill-posed problems.
The results of this paper are theorems 1 and 2, proved in Sections 1 and 2 respectively.

Theorem 1.1. *Let $A = A^*$ be a linear bounded, or densely defined, unbounded, selfadjoint operator in a Hilbert space. Assume that $a = a(\delta) > 0$, $\lim_{\delta\to 0} a(\delta) = 0$ and $\lim_{\delta\to 0} \frac{\delta}{a(\delta)} = 0$, then the unique solution u_δ to equation (1.2) with $a = a(\delta)$ has the property*

$$\lim_{\delta\to 0} ||u_\delta - y|| = 0. \tag{1.3}$$

Why should one be interested in the above theorem? Because the numerical implementation of the solution to equation (1.2) requires less computer memory than the solution of the equation $(A^*A + \alpha I)u = A^*f_\delta$, used in the variational regularization method for stable solution of equation (1.1). Here I is the identity operator. Also, a discretized version of (1.2) leads to matrices whose condition number is of the order of square root of the condition number of the matrix corresponding to the operator $A^*A + \alpha(\delta)I$.
Moreover, if A is unbounded, then f may not belong to $D(A^*)$ and then the equation $A^*Au = A^*f$, which is equivalent to $Au = f$ if A is bounded and $f \in R(A)$, may have no sense. In (Ramm, To appear 2006) this issue is discussed in detail. See also (Ramm, Submitted 2005b) and (Ramm, 2002), where the ill-posed problems with unbounded operators are discussed.

Proof of Theorem 1. One has

$$||u_{a,\delta} - y|| \leq ||(A + ia)^{-1}(f_\delta - f)|| + ||(A + ia)^{-1}Ay - y|| \leq \frac{\delta}{a} + a||(A + ia)^{-1}y||. \tag{1.4}$$

Moreover,

$$\lim_{a\to 0} a^2||(A + ia)^{-1}Ay - y||^2 = \lim_{a\to 0} a^2 \int_{-\infty}^{\infty} \frac{d(E_s y, y)}{s^2 + a^2} = 0, \tag{1.5}$$

where the spectral theorem was used, E_s is the resolution of the identity corresponding to the selfadjoint operator A, and we have taken into account that

$$\lim_{a\to 0} a^2 \int_{-0}^{0} \frac{d(E_s y, y)}{s^2 + a^2} = 0$$

because $y \perp N$. From formulas (1.4) and (1.5) one concludes that if $a = a(\delta) > 0$, $\lim_{\delta\to 0} a(\delta) = 0$ and $\lim_{\delta\to 0} \frac{\delta}{a(\delta)} = 0$, then the unique solution u_δ to equation (1.2) with $a = a(\delta)$ satisfies equation (1.3). Theorem 1 is proved. □

Remark 1.1. In Theorem 1 we approximate the minimal-norm solution to equation (1.1), where A is a selfadjoint operator, by the element $(A + ia(\delta))^{-1}f_\delta$, where $a(\delta)$ is suitably chosen. In the literature ((Bakushinsky and Goncharsky, 1989), (Vainikko and Veretennikov, 1986)) more general approximations were studied. For example, see (Bakushinsky and Goncharsky, 1989), p.28, one may use an approximation of the form $g(T, a(\delta))A^*f_\delta$, where $T := A^*A$ and g is a function, defined on $[0, \infty)$, such that $\lim_{a\to 0} sg(s, a) = 1$, $s|g(s, a)| \leq c(1 + s)^{1/2}$, $c = const > 0$ does not depend on a, and $ess \sup_{s\geq 0} s^{1/2}|g(s, a)| \leq c(a)$, where $c(a) < \infty$ for all $a > 0$ and the $ess \sup$ is taken with respect to all the measures $(F_s h, h)$ $h \in N(T)$, where F_s is the resolution of the identity corresponding to the operator T. An example of such an approximation is

$T^{-1}_{a(\delta)}A^* f_\delta$, where $T_a := T + aI$, and I is the identity operator. This approximation appears in the variational regularization method. Such approximations require in numerical implementations higher operation counts and more computer memory.

2 Calculation of values of unbounded operators

Assume that A is a densely defined closed linear operator in H. We do not assume in this Section that A is selfadjoint. If $f \in D(A)$, then we want to compute Af given noisy data f_δ, $||f_\delta - f|| \le \delta$. Note that f_δ may not belong to $D(A)$. The problem of stable calculation of Af given the data $\{f_\delta, \delta, A\}$ is ill-posed. It was studied in the literature (see, e.g., (Morozov, 1984)) by a variational regularization method. Our aim is to reduce this problem to a standard equation with a selfadjoint bounded operator $0 \le B \le I$, and solve this equation stably by an iterative method.

Let $v = Af$. This relation is equivalent to

$$Bv = Ff, \tag{2.1}$$

where $B := (I+Q)^{-1}$, $F := BA = (I+Q)^{-1}A$, $Q := AA^*$ is a densely defined, non-negative, selfadjoint operator, the range of $I+Q$ is the whole space H, and B is a selfadjoint operator, $0 \le B \le I$, where the inequalities are understood in the sense of quadratic forms, e.g., $B \ge 0$ means $(Bg, g) \ge 0$ for all $g \in H$.

Lemma 2.1. *(see (Ramm, Submitted 2005b)) The operator $(I+Q)^{-1}A$, originally defined on $D(A)$, is closable. Its closure is a bounded, defined on all of H linear operator with the norm $\le \frac{1}{2}$. One has $(I+Q)^{-1}A = A(I+T)^{-1}$, where $T = A^*A$ is a non-negative, densely defined selfadjoint operator, and $||A(I+T)^{-1}|| \le \frac{1}{2}$.*

If f_δ is given in place of f, then we stably solve equation (2.1) for v using the following iterative process:

$$v_{n+1} = (I-B)v_n + Ff_\delta, \tag{2.2}$$

where v_0 is arbitrary. Let y be the unique minimal-norm solution to equation (6), $By = Ff$. Note that $y = Hy + Ff$, where $H := I - B$.

Theorem 2.2. *If $n = n(\delta)$ is an integer, $\lim_{\delta\to 0} n(\delta) = \infty$ and $\lim_{\delta\to 0}[\delta n(\delta)] = 0$, then*

$$\lim_{\delta\to 0} ||v_\delta - y|| = 0, \tag{2.3}$$

where $v_\delta := v_{n(\delta)}$, and v_n is defined in (2.2).

Proof of Theorem 2. From (2.2) one gets $v_{n+1} = \sum_{j=0}^{n} H^j F f_\delta + H^{n+1}u_0$, where $H := I - B$. One has $y = Hy + Ff$. Let $w_n := v_n - y$. Then $w_n = \sum_{j=0}^{n-1} H^j F g_\delta + H^n w_0$, where $g_\delta := f_\delta - f$, and w_0 is an arbitrary element such that $w_0 \perp N^*$. Since $0 \le H \le I$, $||F|| \le \frac{1}{2}$, and $||g_\delta|| \le \delta$, one gets

$$||w_n|| \le \frac{n\delta}{2} + [\int_0^1 (1-s)^{2n} d(E_s w_0, w_0)]^{1/2}, \quad (2.4)$$

where E_s is the resolution of the identity corresponding to the selfadjoint operator B. Note that $N(B) = 0$. Therefore

$$\lim_{n\to\infty} \int_0^1 (1-s)^{2n} d(E_s w_0, w_0) = ||Pw_0||^2 = 0, \quad (2.5)$$

where P is the orthoprojector onto the subspace $N(B)$. The conclusion of Theorem 2 can now be derived. Given an arbitrary small $\epsilon > 0$, find $n = n(\delta)$ sufficiently large such that $\int_0^1 (1-s)^{2n} d(E_s, w_0, w_0) < \epsilon$ and $\delta n(\delta) < \epsilon$. This is possible if δ is sufficiently small, because $\lim_{\delta\to 0} n(\delta) = \infty$ and $\lim_{\delta\to 0}[\delta n(\delta)] = 0$. Then inequality (2.4) shows that (2.3) holds. Theorem 2 is proved. □

Remark 2.1. It is not possible to estimate the rate of convergence in (2.3) without making additional assumptions on y or on f, as is well-known. In (Ramm, 2005a) one can find examples illustrating similar statements concerning various methods for solving ill-posed problems.

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Numerical computations of the zeros of the q-analogues of ordinary Euler polynomials

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ABSTRACT

In this paper we observe the shapes of the q-analogues of ordinary Euler numbers $K_{n,q}$ and the q-analogues of ordinary Euler polynomials $K_{n,q}(x)$. Finally, we describe the structure of the roots of the q-analogues of ordinary Euler polynomials $K_{n,q}(x)$ for values of the index n by using computer. By numerical experiments, we demonstrate a remarkably regular structure of the complex roots of $K_{n,q}(x)$ for $q = -1/2, -1/3$..

Keywords: q-Euler numbers, q-Euler polynomials, q-Volkenborn Integration, Non-Archimedean Analysis.

2000 Mathematics Subject Classification: 11B68, 11S40.

1 Introduction

The Euler numbers E_n are usually defined by means of the following generating function:

$$F(t) := e^{Et} = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, (|t| < 2\pi), \tag{1}$$

where the symbol E_n is interpreted to mean that E^n must be replaced by E_n when we expand the one on the left. These numbers are classical and important in mathematics and in various places like analysis, number theory. Frobenius extended such numbers as E_n to the so-called Frobenius-Euler numbers $H_n(u)$ belonging to an algebraic number u with $|u| > 1$. Let u be an algebraic number. For $u \in \mathbb{C}$ with $|u| > 1$, the Frobenius-Euler numbers $H_n(u)$ belonging to u are defined by the generating function

$$e^{H(u)t} = \frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, (|t| < 2\pi)$$

with the usual convention of symbolically replacing H^n by H_n. The Euler polynomials $E_n(x)$ are defined by

$$F(x,t) := e^{E(x)t} = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{2}$$

For $u \in \mathbb{C}$ with $|u| > 1$, the Frobenius-Euler polynomials $H_n(u,x)$ belonging to u are defined by

$$e^{H(u,x)t} = \frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} H_n(u,x)\frac{t^n}{n!}.$$

In this paper, we introduce the q-analogues of ordinary Euler polynomials $K_{n,q}(x)$ of the Euler polynomials $E_n(x)$. we explore the shapes of the q-analogues of ordinary Euler numbers $K_{n,q}$ and polynomials $K_{n,q}(x)$ and observe the structure of the real roots of our q-analogues of ordinary Euler polynomials, $K_{n,q}(x)$, using numerical investigation. By computer experiments, we demonstrate a remarkably regular structure of the complex roots of $K_{n,q}(x)$. Finally, we give a table for the zeros of our q-analogues of ordinary Euler polynomials. This numerical investigation is especially exciting because these steps are essential for most students to truly understand even basic concept of q-analogues of ordinary Euler numbers $K_{n,q}$ and q-analogues of ordinary Euler polynomials $K_{n,q}(x)$. Throughout this paper we use the following notations. By $\mathbb{Z}$ we denote the ring of rational integers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{R}$ denotes field of real numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. Throughout this paper we use the below notation:

$$[x]_q = [x:q] = \frac{1-q^x}{1-q}.$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$X = X_d = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$, set

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$$

and this is known to be a distribution on X (see [1,2,3]). This distribution admits an integral for each non-negative integer m:

$$\int_{\mathbb{Z}_p}[x]_q^n d\mu_q(x) = \int_X [x]_q^n d\mu_q(x) = \frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\frac{i+1}{[i+1]_q}.$$

For the variable x in $\mathbb{C}_p$ with $|x|_p \leq 1$, we obtain

$$\int_{\mathbb{Z}_p} [x+t]_q^n d\mu_q(t) = \frac{1}{(1-q)^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{xj} \frac{j+1}{[j+1]_q}.$$

2 q-analogues of ordinary Euler numbers and polynomials

In this section, our primary goal is to give generating functions of the q-analogues of ordinary Euler numbers and polynomials. These numbers will be used to prove the analytic continuation of the q-L-series. In complex plane, we assume that $q \in \mathbb{C}$ with $|q| < 1$.

We define a q-version of each of the functions $F(t)$ and $F(x,t)$ occurring in (1) and (2), respectively. Now, we introduce the generating function $F_q(t)$ and $F_q(x,t)$. Let $F_q(t) = \sum_{n=0}^{\infty} K_{n,q} \frac{t^n}{n!}$. Then we have

$$F_q(t) = e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{1+q}{1+q^{j+1}} (-1)^j (\frac{1}{1-q})^j \frac{t^j}{j!}. \tag{3}$$

By (3), we can find the nice generating function of $K_{n,q}$ as follows:

$$F_q(t) = \sum_{n=0}^{\infty} K_{n,q} \frac{t^n}{n!} = (1+q) \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t}. \tag{4}$$

Note that $\lim_{q\to 1} F_q(t) = \frac{2}{e^t+1}$. By (4), we see that $\lim_{q\to 1} K_{n,q} = E_n$. Hence, $K_{n,q}$ seems to be the nice q-analogue of E_n.

Similarly, the generating function of $F_q(x,t)$ of the q-analogues of ordinary polynomials $K_{n,q}(x)$ is defined analogously as follows:

$$F_q(x,t) = \sum_{n=0}^{\infty} K_{n,q}(x) \frac{t^n}{n!} = (1+q) \sum_{n=0}^{\infty} (-1)^n q^{n+x} e^{[n+x]_q t}. \tag{5}$$

We have the following theorem.

Theorem 2.1. *For $n \in \mathbb{N}$, we have*

$$\lim_{q\to 1} K_{n,q}(x) = E_n(x), \quad \lim_{q\to 1} F_q(x,t) = \frac{2}{e^t+1} e^{xt},$$

$$\lim_{q\to 1} K_{n,q} = E_n, \text{ and } \quad \lim_{q\to 1} F_q(t) = \frac{2}{e^t+1}.$$

Next, we consider the q-analogues of ordinary Euler numbers by using q-Volkenborn integrals as follows:

$$K_{k,q} = \int_{X_m} [t]_q^k d\mu_{-q}(t) = \int_{\mathbb{Z}_p} [t]_q^k d\mu_{-q}(t), \text{ for } k, m \in \mathbb{N}.$$

Thus, we obtain

$$K_{k,q} = (1+q) \left(\frac{1}{1-q} \right)^k \sum_{l=0}^{k} \binom{k}{l} (-1)^l \frac{1}{1+q^{l+1}},$$

where $\binom{k}{i}$ is the binomial coefficient.

For $x \in \mathbb{Z}_p$, we introduce the q-analogues of ordinary Euler polynomials as follows:

$$K_{k,q}(x) = \int_{\mathbb{Z}_p} [x+t]_q^k d\mu_{-q}(t). \tag{6}$$

By (6), we obtain

$$K_{k,q}(x) = \sum_{n=0}^{k} \binom{k}{n} [x]_q^{k-n} q^{nx} K_{n,q}. \tag{7}$$

Using q-Volkenborn integrals, we obtain the following formula: If m is odd number, then

$$K_{k,q}(x) = \frac{(1+q)[m]_q^n}{1-(-q)^m} \sum_{a=0}^{m-1} (-q)^a \int_{\mathbb{Z}_p} [\frac{a+x}{m} + t]_{q^m}^n d\mu_{-q^m}(t). \tag{8}$$

By (6) and (8), we obtain the following proposition.

Proposition 2.2. *Let m be the odd integer. Then we have*

$$K_{n,q}(x) = \frac{(1+q)[m]_q^n}{1-(-q)^m} \sum_{a=0}^{m-1} (-q)^a K_{n,q^m}(\frac{a+x}{m}).$$

Finally, we now observe that

$$\frac{1}{[p^N]_{-q}} \sum_{n=0}^{p^N-1} [x+y]_q^n (-q)^y = (1+q) \left(\frac{1}{1-q}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{xk} \frac{1}{1+q^{p^N}} \frac{1+q^{p^N(k+1)}}{1+q^{k+1}}.$$

Thus, we have the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}$, we obtain*

$$K_{n,q}(x) = (1+q) \left(\frac{1}{1-q}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{xk} \frac{1}{1+q^{k+1}}.$$

Hence we obtain the following corollary.

Corollary 2.4.

$$K_{n,q} = (1+q) \left(\frac{1}{1-q}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{1+q^{k+1}}.$$

3 Beautiful zeros of the q-analogues of ordinary Euler numbers and polynomials

In this section, we display the shapes of the q-analogues of ordinary Euler numbers $K_{n,q}$. For $n = 1, 2, \cdots, 10$, we can draw plot of $K_{n,q}$, respectively. This shows the ten plots combined into one. For $n = 1, \cdots, 10, q$, we display the shapes of the q-analogues of ordinary Euler numbers $K_{n,q}$.

Our numerical results for the q-analogues of ordinary Euler numbers $K_{n,q}$ are displayed in Table 1.

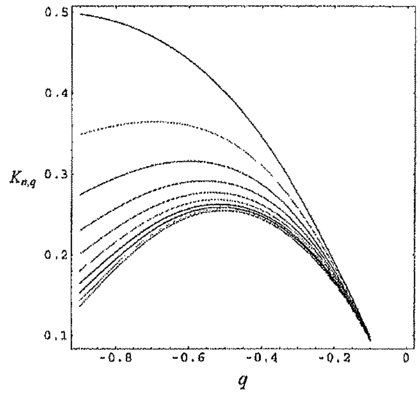

Figure 1: $-9/10 \leq q \leq -1/10$

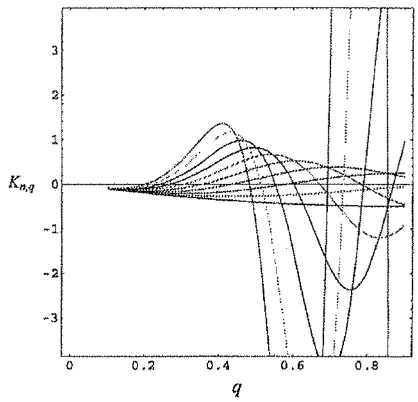

Figure 2: $1/10 \leq q \leq 9/10$

Table 1. q-analogues of ordinary Euler numbers

degree n	$q = -1/2$	$q = 1/2$
1	$2/5$	$-2/5$
2	$12/35$	$-4/15$
3	$184/595$	$-8/85$
4	$5328/18445$	$112/935$
5	$66208/239785$	$13408/36465$
6	$8158656/30452695$	$325184/522665$
7	$293482112/1118048945$	$109743488/134324905$
8	$29575594752/114264602179$	$3747514112/4593911751$
9	$30019734752768/117121217233475$	$188707174912/523195504975$
10	$12205533463962624/47949426335384665$	$-71929487408128/71468505979585$

We describe the shapes of the q-analogues of ordinary Euler polynomials $K_{n,q}(x)$ for $n = 1, \cdots, 10, -1 \leq x \leq 1$. We display the plots of $K_{n,q}(x)$ for $1/10 \leq q \leq 9/10, -2 \leq x \leq 2$.

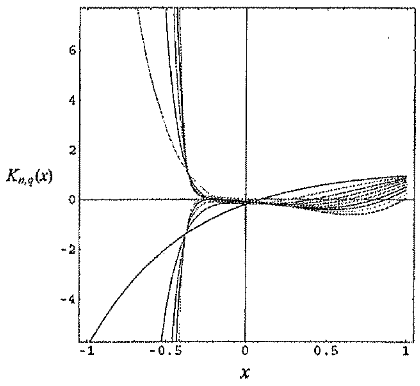

Figure 3: shape of $K_{n,1/5}(x)$

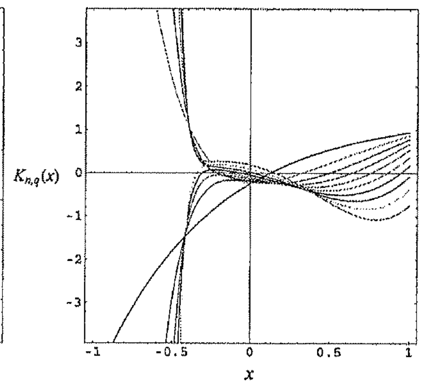

Figure 4: shape of $K_{n,1/4}(x)$

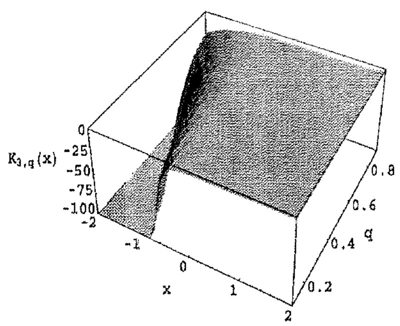

Figure 5: shape of $K_{3,q}(x)$

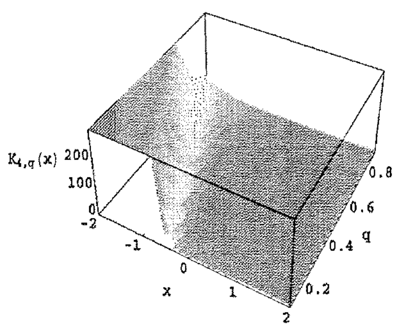

Figure 6: shape of $K_{4,q}(x)$

We plot the zeros of the q-analogues of ordinary Euler polynomials $K_{n,q}(x), x \in \mathbb{C}$.
In these figures, $K_{n,q}(x), q > 0, x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry. This translates to the following open problem: Prove that $\beta_{n,q}(x), K_{n,q}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry. However, $K_{n,q}(x), q < 0, x \in \mathbb{C}$, has not reflection symmetry. For $q < 0$, plot the zeros of q-analogues of ordinary Euler polynomials $K_{n,q}(x), x \in \mathbb{C}$.
Our numerical results for approximate solutions of real zeros of the $K_{n,q}(x)$, $q = 1/2$ are displayed in Table 4. The result is obtained by Mathematica software.

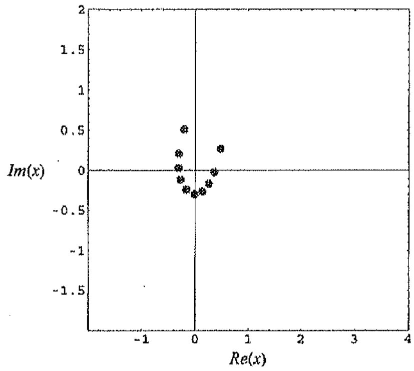

Figure 7: zeros of $K_{10,-1/3}(x)$

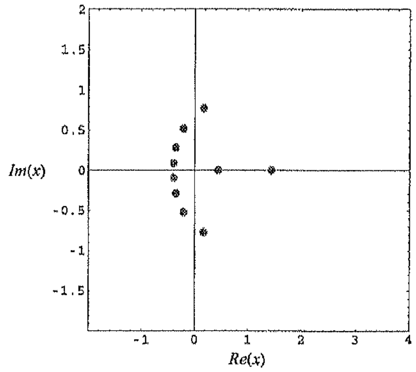

Figure 8: zeros of $K_{10,1/3}(x)$

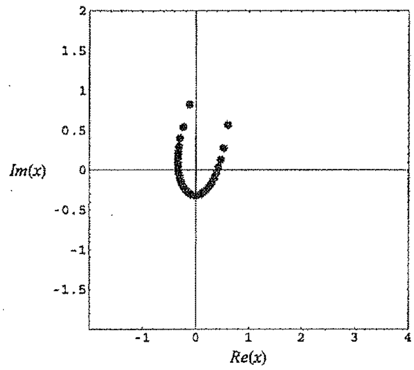

Figure 9: zeros of $K_{30,-1/3}(x)$

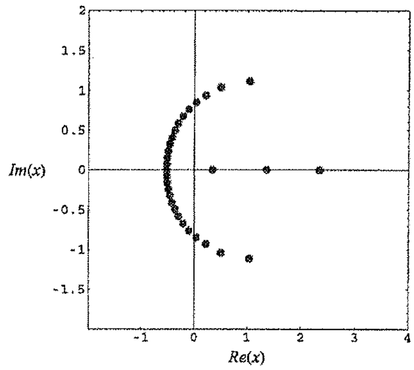

Figure 10: zeros of $K_{30,1/3}(x)$

Table 2. Numbers of real and complex zeros

	$-1/3$		$-1/5$	
degree n	real zeros	complex zeros	real zeros	complex zeros
1	0	1	0	1
3	0	3	0	3
5	0	5	0	5
7	0	7	0	7
9	0	9	0	9
11	0	11	0	11
13	0	13	0	13
15	0	15	0	15
17	0	17	0	17
⋮	⋮	⋮	⋮	⋮
51	0	51	0	51
53	0	53	0	53

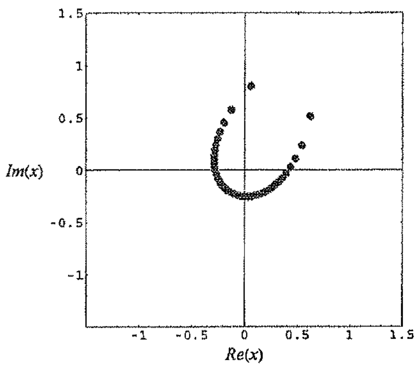

Figure 11: zeros of $K_{40,-1/5}(x)$

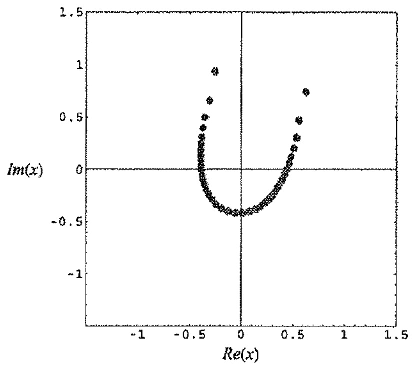

Figure 12: zeros of $K_{40,-1/2}(x)$

By numerical experiments, we demonstrate a remarkably regular structure of the complex roots of $K_{n,q}(x)$ for $q < 0$.

Table 3. Numbers of real and complex zeros

	$-1/2$		$1/2$	
degree n	real zeros	complex zeros	real zeros	complex zeros
1	0	1	1	0
3	0	3	3	0
5	0	5	3	2
7	0	7	3	4
9	0	9	3	6
11	0	11	3	8
13	0	13	3	9
15	0	15	5	10
17	0	17	5	12
19	0	19	5	14
21	0	21	5	16
⋮	⋮	⋮	⋮	⋮
31	0	31	5	26
33	0	33	5	28
⋮	⋮	⋮	⋮	⋮
41	0	41	5	36
43	0	43	5	38
⋮	⋮	⋮	⋮	⋮
51	0	51	7	44
53	0	53	7	46

Table 4. Approximate solutions of $K_{n,1/2}(x) = 0, x \in \mathbb{R}$

degree n	real zeros
1	0.263034
2	-0.195283, 0.610321
3	-0.268477, -0.122484, 0.888461
4	0.119298, 1.11848
5	-0.426196, 0.322836, 1.31475
6	0.505176, 1.48613
7	-0.325429, 0.66923, 1.63838
8	-0.557938, -0.182186, 0.817824, 1.77545
9	-0.0466248, 0.953375, 1.90015
10	-0.605574, 0.0778538, 1.07785, 2.01459
11	0.192853, 1.19285, 2.12035

Finally, we shall consider the more general problems. Since n is the degree of the polynomial $K_{n,q}(x)$, the number of real zeros $r_{K_{n,q}(x)}$ lying on the real plane $Im(x) = 0$ is then $r_{K_{n,q}(x)} = n - c_{K_{n,q}(x)}$, where $c_{K_{n,q}(x)}$ denotes complex zeros. See Table 3 for tabulated values of $r_{K_{n,q}(x)}$ and $c_{K_{n,q}(x)}$. Find the numbers of complex zeros $c_{K_{n,q}(x)}$ of the $K_{n,q}(x)$, the equation of envelope curves boundary the real zeros lying on the plane, and the equation of a trajectory curve running through the complex zeros on any one of the arcs. It would be very interesting to find a mathematical explanation for this. In this paper we observed the regular behaviour of the real roots of the q-analogues of ordinary Euler polynomials $K_{n,q}(x)$. It is surprising that these results seem to have not been noted before. Using computer-generated graphs, we should mention that our interest also started from a graphical analysis of the q-analogues of ordinary Euler polynomials $K_{n,q}(x)$. In any case, these calculations are too complicated to compute by hand, we have to use computer. We hope to make progress in this direction in the future (cf.[1,3]).

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Eulerian and Other Integral Representations for Some Families of Hypergeometric Polynomials

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ABSTRACT

In the present contribution to the Special Volume dedicated to the Tricentennial Birthday Anniversary of Leonhard Euler (1707-1783), the author systematically investigates several general families of hypergeometric polynomials and their associated single-, double-, and triple-integral representations of the Eulerian and other types. Some interesting consequences of the general results presented here, involving such classical orthogonal polynomials as the Jacobi, Laguerre, Hermite, and Bessel polynomials, and also various other relatively less familiar hypergeometric polynomials such as (for example) the Konhauser-Toscano and Gould-Hopper polynomials, are also considered. Each of the many classes of integral representations, which are presented in this article, may be viewed also as a potentially useful linearization relationship for the product of two different members of the associated family of hypergeometric polynomials.

Keywords: Eulerian Beta integral, Integral representations, linearization relationships, Jacobi polynomials, Laguerre polynomials, Bessel polynomials, Hermite polynomials, hypergeometric functions and polynomials, Laplace and inverse Laplace transforms, Kampé de Fériet's function, multivariable hypergeometric polynomials, Konhauser-Toscano polynomials, Gould-Hopper polynomials.

2000 Mathematics Subject Classification: Primary 33C20, 33C45; Secondary 33C65.

1 Introduction, Definitions and Preliminaries

We begin by letting $(\lambda)_\nu$ denote the Pochhammer symbol (or the *shifted factorial*, since $(1)_n = n!$ for $n \in \mathbb{N}_0$) defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the Euler Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\};\ \lambda \in \mathbb{C}), \end{cases} \tag{1.1}$$

where $\mathbb{N}_0$ is the set of *nonnegative* integers. Also, as usual, we denote by ${}_pF_q$ a generalized hypergeometric function with p numerator and q denominator parameters. The classical Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x),$$

of order (or indices) (α, β) and degree n in x, defined (in terms of the Gauss hypergeometric ${}_2F_1$ function) by

$$P_n^{(\alpha,\beta)}(x) := \binom{\alpha+n}{n} {}_2F_1\left(-n, \alpha+\beta+n+1; \alpha+1; \frac{1-x}{2}\right) \tag{1.2}$$

or, equivalently, by the following Rodrigues formula:

$$P_n^{(\alpha,\beta)}(x) := \frac{(-1)^n (1-x)^{-\alpha} (1+x)^{-\beta}}{2^n\, n!} \cdot D_x^n \left\{(1-x)^{\alpha+n} (1+x)^{\beta+n}\right\} \qquad \left(D_x := \frac{d}{dx}\right), \tag{1.3}$$

are orthogonal over the interval $(-1, 1)$ with respect to the following weight function:

$$w(x) := (1-x)^{\alpha} (1+x)^{\beta}; \tag{1.4}$$

in fact, we have (*cf.*, *e.g.*, Szegö [27, p. 68, Equation (3.3.3)])

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta}\; P_m^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(x)\, dx = \frac{2^{\alpha+\beta+1}\, \Gamma(\alpha+n+1)\, \Gamma(\beta+n+1)}{n!\, (\alpha+\beta+2n+1)\, \Gamma(\alpha+\beta+n+1)}\, \delta_{m,n} \tag{1.5}$$

$$\left(\min\{\Re(\alpha), \Re(\beta)\} > -1;\; m, n \in \mathbb{N}_0\right),$$

where $\delta_{m,n}$ denotes the Kronecker delta. In many recent investigations, a remarkably great deal of attention seems to have been paid to an *obvious* variant of the classical Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x).$$

These so-called *extended* Jacobi polynomials

$$F_n^{(\alpha,\beta)}(x; a, b, c),$$

studied (among others) by Izuru Fujiwara (1928-1985) [8] in an attempt to give a unified presentation of the classical orthogonal polynomials (especially Jacobi, Laguerre, and Hermite polynomials), are defined by the following Rodrigues formula:

$$F_n^{(\alpha,\beta)}(x; a, b, c) := \frac{(-c)^n}{n!} (x-a)^{-\alpha} (b-x)^{-\beta} \cdot D_x^n \left\{(x-a)^{\alpha+n} (b-x)^{\beta+n}\right\} \qquad \left(c := \frac{\lambda}{b-a} > 0\right) \tag{1.6}$$

and are orthogonal over the interval (a, b) with respect to the weight function [*cf.* Equation (1.4)]:

$$w(x; a, b) := (x-a)^{\alpha} (b-x)^{\beta}. \tag{1.7}$$

The polynomials

$$F_n^{(\alpha,\beta)}(x; a, b, c)$$

are essentially those that were considered already by Szegö [27, p. 58], who showed (by means of a *simple* linear transformation) that these polynomials are just a constant multiple of the classical Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x)$$

themselves. As a matter of fact, by merely comparing the Rodrigues formulas (1.3) and (1.6), it is not difficult to rewrite Szegö's observation [27, p. 58, Equation (4.1.2)] in the following *explicit* form (*cf.*, *e.g.*, Srivastava and Manocha [24, p. 388, Problem 11]):

$$\begin{aligned} F_n^{(\alpha,\beta)}(x;a,b,c) &= \{c(a-b)\}^n \; P_n^{(\alpha,\beta)}\left(\frac{2(x-a)}{a-b}+1\right) \\ &= \{-c(a-b)\}^n \; P_n^{(\beta,\alpha)}\left(1-\frac{2(x-b)}{a-b}\right) \end{aligned} \tag{1.8}$$

or, equivalently,

$$P_n^{(\alpha,\beta)}(x) = \{c(a-b)\}^{-n} \; F_n^{(\alpha,\beta)}\left(\frac{1}{2}\{a+b+(a-b)x\};a,b,c\right). \tag{1.9}$$

Thus, as already pointed out by Srivastava and Manocha [*loc. cit.*], the polynomials

$$F_n^{(\alpha,\beta)}(x;a,b,c)$$

may be looked upon as being equivalent to (and *not* as a generalization of) the classical Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x).$$

Furthermore, by recourse to certain limiting processes, it is easily seen that the polynomials

$$F_n^{(\alpha,\beta)}(x;a,b,c)$$

would give rise to the Laguerre and Hermite polynomials (and indeed also to the Bessel polynomials) just as the classical Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x)$$

are known to do. Consequently, the *main* purpose of Fujiwara's investigation [8] is already served *adequately* by the classical Jacobi polynomials *themselves*. Even after the aforementioned observation by Szegö [27] and by others (*cf.*, *e.g.*, Srivastava and Manocha [24]), the polynomials

$$F_n^{(\alpha,\beta)}(x;a,b,c)$$

have been (and are still being) made, in recent years, a tool for the purpose of *generalizing* what is already known in the context of the classical Jacobi polynomials (see also Pittaluga *et al.* [17] and González *et al.* [9]). For example, in a paper which appeared very recently, Chongdar and Majumdar [6] derived a triple-integral representation for the product:

$$F_m^{(\alpha,\beta)}(x;a,b,c)\,F_n^{(\alpha',\beta')}(y;a,b,c)$$

with a view to "*generalizing*" the corresponding known result of Chatterjea [4] for the product:

$$P_m^{(\alpha,\beta)}(x)\,P_n^{(\alpha',\beta')}(y)$$

of two Jacobi polynomials. Since x and y are *not* the variables of integration in each of these triple-integral representations, the equivalence of the results of Chatterjea [4] and Chongdar-Majumdar [6] can be demonstrated fairly readily by means of the relationships (1.8) and (1.9). In a sequel to several earlier works on the subject (including, for example, [4] and [6], Srivastava [20] presented several double- and triple-integral representations for two general families of generalized hypergeometric polynomials and also indicated the relevant connections of his general results with those obtained in a number of earlier works. The main object of this article is to briefly revisit Srivastava's work [6] and to systematically investigate several general families of hypergeometric polynomials and their associated single-, double-, and triple-integral representations of the Eulerian and other types. We also consider some interesting consequences of the general results presented here, involving such classical orthogonal polynomials as the Jacobi, Laguerre, Hermite, and Bessel polynomials, and various other relatively less familiar hypergeometric polynomials such as (for example) the Konhauser-Toscano and Gould-Hopper polynomials. Each of the many classes of integral representations, which are presented in this article, may be viewed also as a potentially useful *linearization relationship* for the product of two different members of the associated family of hypergeometric polynomials involved in it (see also [25]).

2 Integral Representations for a Family of Kampé de Fériet Polynomials

For the the following product of two Laguerre polynomials of the same order α and the same argument x:

$$L_m^{(\alpha)}(x)\,L_n^{(\alpha)}(x),$$

where

$$L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \tag{2.1}$$

or, equivalently,

$$L_n^{(\alpha)}(x) := \binom{\alpha+n}{n} {}_1F_1(-n;\alpha+1;x) \tag{2.2}$$

in terms of the confluent (Kummer) hypergeometric ${}_1F_1$ function, Watson [29, p. 207] gave an elegant integral representation which was subsequently generalized by Carlitz [2] in the following form:

$$\begin{aligned} L_m^{(\alpha)}(x)\,L_n^{(\beta)}(y) = {} & \frac{2^{\alpha+\beta+m+n}}{\pi^2} \frac{\Gamma(\alpha+m+1)\,\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+m+n+1)} \\ & \cdot \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} f_{m,n}^{(\alpha,\beta)}(\vartheta,\varphi)\, L_{m+n}^{(\alpha+\beta)}(\Omega[x,y;\vartheta,\varphi])\, d\varphi\, d\vartheta \end{aligned} \tag{2.3}$$

$$\big(m,n \in \mathbb{N}_0;\ \Re(\alpha+\beta) > -1\big),$$

where, for convenience,

$$f_{m,n}^{(\alpha,\beta)}(\vartheta,\varphi) := e^{\{(m-n)\varphi i+(\alpha-\beta)\vartheta i\}} \cos^{m+n}\varphi \cos^{\alpha+\beta}\vartheta \tag{2.4}$$

and

$$\Omega[x, y; \vartheta, \varphi] := \frac{\cos\vartheta}{\cos\varphi}\left[xe^{(\vartheta-\varphi)i} + ye^{-(\vartheta-\varphi)i}\right]. \tag{2.5}$$

Integral representations for similar products of many other polynomials have since then appeared in the mathematical literature. A detailed account of these results may be found in a paper by Srivastava and Joshi (*cf.* [22, p. 920]; see also [19]), who also gave a general $(p+q+2)$-dimensional integral representation for the product:

$$\Phi_m\left[\begin{array}{c}(a_p), \alpha; \\ \qquad\qquad x \\ (b_q)+1;\end{array}\right]\Phi_n\left[\begin{array}{c}(a'_p), \alpha'; \\ \qquad\qquad y \\ (b'_q)+1;\end{array}\right],$$

where, *and throughout this article*, (a_p) abbreviates the array of p parameters

$$a_1, \ldots, a_p,$$

with similar interpretations for (b_q), *et cetera*, and

$$\Phi_n\left[\begin{array}{c}(a_p), \alpha; \\ \qquad\qquad x \\ (b_q);\end{array}\right] := \frac{(\alpha)_n}{n!}\,{}_{p+2}F_q\left[\begin{array}{c}-n, \alpha+n, a_1, \ldots, a_p; \\ \qquad\qquad x \\ b_1, \ldots, b_q;\end{array}\right] \tag{2.6}$$

in terms of an obviously terminating generalized hypergeometric series. The family of hypergeometric polynomials defined by (2.6) possesses a generating function in the form:

$$\sum_{n=0}^{\infty}\Phi_n\left[\begin{array}{c}(a_p), \alpha; \\ \qquad\qquad x \\ (b_q);\end{array}\right]t^n = (1-t)^{-\alpha}\,{}_{p+2}F_q\left[\begin{array}{c}\frac{1}{2}\alpha, \frac{1}{2}\alpha+\frac{1}{2}, a_1, \ldots, a_p; \\ \qquad\qquad -\frac{4xt}{(1-t)^2} \\ b_1, \ldots, b_q;\end{array}\right] \tag{2.7}$$

$$(|t| < 1)$$

and includes, as a special or limit case, several known polynomial systems such as the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined by (1.2), the Bessel polynomials of Krall and Frink [12] (see also [11]):

$$y_n(x; \alpha, \beta) := {}_2F_0\left(-n, \alpha+n-1; —; -\frac{x}{\beta}\right), \tag{2.8}$$

the general Rice polynomials (*cf.*, *e.g.*, [24, p. 140, Equation 2.6 (13)]):

$$H_n^{(\alpha,\beta)}(\zeta, u, v) := \binom{\alpha+n}{n}\,{}_3F_2(-n, \alpha+\beta+n+1; \zeta; \alpha+1, u; v), \tag{2.9}$$

and the Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by (2.2), since

$$\lim_{|\alpha|\to\infty}\,{}_{p+2}F_q\left[\begin{array}{c}-n, \alpha+n, (a_p); \\ \qquad\qquad \frac{x}{\alpha} \\ (b_q);\end{array}\right] = {}_{p+1}F_q\left[\begin{array}{c}-n, (a_p); \\ \qquad\qquad x \\ (b_q);\end{array}\right]. \tag{2.10}$$

Now, starting from Carlitz's formula (2.3) and making repeated use of the relationship:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{\Gamma(\alpha+\beta+n+1)}\int_0^1\left(\log\frac{1}{t}\right)^{\alpha+\beta+n}L_n^{(\alpha)}\left(\frac{1}{2}(1-x)\log\frac{1}{t}\right)dt \tag{2.11}$$

$$\left(\Re(\alpha+\beta) > -1;\ n \in \mathbb{N}_0\right),$$

which is, in fact, equivalent to Feldheim's formula [24, p. 94, Problem 24]:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{\Gamma(\alpha+\beta+n+1)} \int_0^\infty t^{\alpha+\beta+n}\, e^{-t}\, L_n^{(\alpha)}\left(\frac{1}{2}(1-x)\,t\right) dt \tag{2.12}$$

$$\left(\Re(\alpha+\beta) > -1;\ n \in \mathbb{N}_0\right),$$

it is fairly straightforward to observe, by the principle of multidimensional mathematical induction based upon the Laplace and the inverse Laplace transforms, that (*cf.* [26, p. 424, Equation (30)])

$$\begin{aligned} &F^{u:p+2;r+2}_{v:q+1;s+1}\left[\begin{array}{l} (\gamma_u) : -m, \lambda+m, (a_p) ; -n, \mu+n, (c_r) ; \\ \\ (\delta_v) : \qquad \alpha+1, (b_q) ; \qquad \beta+1, (d_s) ; \end{array} x, y\right] \\ &= \Delta^{(\alpha,\beta)}_{m,n} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} f^{(\alpha,\beta)}_{m,n}(\vartheta,\varphi) \\ &\cdot F^{u+1:p+1;r+1}_{v+1:\ q;\ s}\left[\begin{array}{l} -m-n, (\gamma_u) : \lambda+m, (a_p) ; \mu+n, (c_r) ; \\ \\ \alpha+\beta+1, (\delta_v) : \qquad (b_q) ; \qquad (d_s) ; \end{array} \xi(x;\vartheta,\varphi), \eta(y;\vartheta,\varphi)\right] d\varphi\, d\vartheta \end{aligned} \tag{2.13}$$

$$\left(\Re(\alpha+\beta) > -1;\ m,n,p,q,r,s,u,v \in \mathbb{N}_0\right),$$

where $f^{(\alpha,\beta)}_{m,n}(\vartheta,\varphi)$ is given by (2.4),

$$\Delta^{(\alpha,\beta)}_{m,n} := \frac{2^{\alpha+\beta+m+n}}{\pi^2} \frac{m!\, n!}{(m+n)!} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}, \tag{2.14}$$

$$\xi(x;\vartheta,\varphi) := x\, \frac{\cos\vartheta}{\cos\varphi}\, e^{(\vartheta-\varphi)i} \quad \text{and} \quad \eta(y;\vartheta,\varphi) := y\, \frac{\cos\vartheta}{\cos\varphi}\, e^{-(\vartheta-\varphi)i}, \tag{2.15}$$

and

$$F^{p:r;u}_{q:s;v} \qquad (p,q,r,s,u,v \in \mathbb{N}_0)$$

denotes a general (Kampé de Fériet's) double hypergeometric function defined by (*cf., e.g.*, [23, p. 27, Equation 1.3 (28)]; see also [1, p. 150])

$$\begin{aligned} &F^{p:r;u}_{q:s;v}\left[\begin{array}{l} \alpha_1,\ldots,\alpha_p : a_1,\ldots,a_r;\ c_1,\ldots,c_u; \\ \\ \beta_1,\ldots,\beta_q : b_1,\ldots,b_s;\ d_1,\ldots,d_v; \end{array} x, y\right] \\ &:= \sum_{l,m=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_{l+m} \prod_{j=1}^{r} (a_j)_l \prod_{j=1}^{u} (c_j)_m}{\prod_{j=1}^{q} (\beta_j)_{l+m} \prod_{j=1}^{s} (b_j)_l \prod_{j=1}^{v} (d_j)_m} \frac{x^l}{l!} \frac{y^m}{m!}, \end{aligned} \tag{2.16}$$

where, for convergence of the double hypergeometric series,

$$p+r \leqq q+s+1 \quad \text{and} \quad p+u \leqq q+v+1,$$

with equality only when

$$\begin{cases} |x|^{1/(p-q)} + |y|^{1/(p-q)} < 1 & (p > q) \\ \max\{|x|, |y|\} < 1 & (p \leqq q). \end{cases}$$

In light of the limit formula (2.10), the double-integral representation (2.13) *with*

$$x \longmapsto \frac{x}{\lambda}, \quad y \longmapsto \frac{y}{\mu}, \quad \text{and} \quad \min\{|\lambda|, |\mu|\} \to \infty$$

would readily yield

$$F^{u:p+1;r+1}_{v:q+1;s+1}\left[\begin{matrix} (\gamma_u): \; -m, (a_p); \; -n, (c_r); \\ \\ (\delta_v): \alpha+1, (b_q); \beta+1, (d_s); \end{matrix} \; x, y\right]$$
$$= \Delta^{(\alpha,\beta)}_{m,n} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} f^{(\alpha,\beta)}_{m,n}(\vartheta, \varphi)$$
$$\cdot F^{u+1:p;r}_{v+1:q;s}\left[\begin{matrix} -m-n, (\gamma_u): (a_p); (c_r); \\ \\ \alpha+\beta+1, (\delta_v): (b_q); (d_s); \end{matrix} \; \xi(x; \vartheta, \varphi), \eta(y; \vartheta, \varphi)\right] d\varphi \, d\vartheta \quad (2.17)$$

$$\left(\Re(\alpha+\beta) > -1; \; m, n, p, q, r, s, u, v \in \mathbb{N}_0\right).$$

Furthermore, since [23, p. 28, Equation 1.3 (30)]

$$F^{p:0;0}_{q:0;0}\left[\begin{matrix} \alpha_1, \ldots, \alpha_p : \text{—};\text{—}; \\ \\ \beta_1, \ldots, \beta_q : \text{—};\text{—}; \end{matrix} \; x, y\right]$$

$$= {}_pF_q\left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \\ \beta_1, \ldots, \beta_q; \end{matrix} \; x+y\right], \quad (2.18)$$

a special case of (2.17) when

$$p = q = r = s = u = v = 0$$

is precisely Carlitz's formula (2.3) by virtue of the definition (2.2). Next, by means of the definition (2.16) in conjunction with the Eulerian Beta-function formula:

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\alpha, \beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt \quad (2.19)$$

$$\left(\Re(\alpha) > 0; \; \Re(\beta) > 0\right),$$

we find that

$$F^{u+1:1;1}_{v+1:0;0}\left[\begin{array}{c} -m-n,(\gamma_u):\lambda+m;\mu+n; \\ \\ \alpha+\beta+1,(\delta_v):\text{———};\text{———}; \end{array} x,y\right]$$

$$=\sum_{k.l=0}^{k+l\leqq m+n}\frac{(-m-n)_{k+l}\prod_{j=1}^{u}(\gamma_j)_{k+l}}{(\alpha+\beta+1)_{k+l}\prod_{j=1}^{v}(\delta_j)_{k+l}}\frac{\Gamma(\lambda+m+k)\Gamma(\mu+n+l)}{\Gamma(\lambda+m)\Gamma(\mu+n)}\frac{x^k}{k!}\frac{y^l}{l!}$$

$$=\frac{\Gamma(\lambda+\mu+m+n)}{\Gamma(\lambda+m)\Gamma(\mu+n)}\int_0^1 t^{\lambda+m-1}(1-t)^{\mu+n-1}$$

$$\cdot F^{u+2:0;0}_{v+1:0;0}\left[\begin{array}{c} -m-n,\lambda+\mu+m+n,(\gamma_u):\text{—};\text{—}; \\ \\ \alpha+\beta+1,(\delta_v):\text{—};\text{—}; \end{array} xt,y(1-t)\right]dt$$

$$=\frac{\Gamma(\lambda+\mu+m+n)}{\Gamma(\lambda+m)\Gamma(\mu+n)}\int_0^1 t^{\lambda+m-1}(1-t)^{\mu+n-1}$$

$$\cdot\ {}_{u+2}F_{v+1}\left[\begin{array}{c} -m-n,\lambda+\mu+m+n,(\gamma_u); \\ \\ \alpha+\beta+1,(\delta_v); \end{array} xt+y(1-t)\right]dt \tag{2.20}$$

$$\left(\Re(\lambda)>0;\ \Re(\mu)>0;\ m,n,u,v\in\mathbb{N}_0\right),$$

where we have also used the reduction formula (2.18). Upon substituting from (2.20) into the integrand of (2.13) with, of course,

$$p=q=r=s=0,$$

we thus obtain the following general family of triple-integral representations:

$$F^{u:2;2}_{v:1;1}\left[\begin{matrix}(\gamma_u): -m, \lambda+m; -n, \mu+n; \\ (\delta_v): \quad \alpha+1; \quad \beta+1;\end{matrix}\; x, y\right]$$

$$= \frac{\Gamma(\lambda+\mu+m+n)}{\Gamma(\lambda+m)\Gamma(\mu+n)} \Delta^{(\alpha,\beta)}_{m,n}$$

$$\cdot \int_0^1 \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} t^{\lambda+m-1}(1-t)^{\mu+n-1} f^{(\alpha,\beta)}_{m,n}(\vartheta,\varphi)$$

$$\cdot {}_{u+2}F_{v+1}\left[\begin{matrix}-m-n, \lambda+\mu+m+n, (\gamma_u); \\ \alpha+\beta+1, (\delta_v);\end{matrix}\; \Omega[xt, y(1-t); \vartheta, \varphi]\right] d\varphi\, d\vartheta\, dt \tag{2.21}$$

$$(\Re(\lambda) > 0;\ \Re(\mu) > 0;\ \Re(\alpha+\beta) > -1;\ m, n, u, v \in \mathbb{N}_0),$$

where $f^{(\alpha,\beta)}_{m,n}(\vartheta,\varphi)$ and $\Delta^{(\alpha,\beta)}_{m,n}$ are given by (2.4) and (2.14), respectively, and

$$\Omega[x, y; \vartheta, \varphi] = \xi(x; \vartheta, \varphi) + \eta(y; \vartheta, \varphi) \tag{2.22}$$

is defined, as before, by (2.5).

3 Applications to the Classical Jacobi and Related Polynomials

For $u = v = 0$, each of the *two-variable* hypergeometric polynomials occurring on the left-hand sides of (2.13), (2.17), and (2.21) would obviously reduce to the product of two *one-variable* hypergeometric polynomials of the types involved in (2.6) and (2.10). Thus, by specializing in this manner, we can deduce the corresponding double- and triple-integral representations for the products of two such *one-variable* hypergeometric polynomials. In particular, upon putting $u = v = 0$ in (2.21), we obtain

$${}_2F_1(-m, \lambda+m; \alpha+1; x)\ {}_2F_1(-n, \mu+n; \beta+1; y)$$

$$= \frac{\Gamma(\lambda+\mu+m+n)}{\Gamma(\lambda+m)\Gamma(\mu+n)} \Delta^{(\alpha,\beta)}_{m,n} \int_0^1 \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} t^{\lambda+m-1}(1-t)^{\mu+n-1} f^{(\alpha,\beta)}_{m,n}(\vartheta,\varphi)$$

$$\cdot {}_2F_1\big(-m-n, \lambda+\mu+m+n; \alpha+\beta+1; \Omega[xt, y(1-t); \vartheta, \varphi]\big) d\varphi\, d\vartheta\, dt \tag{3.1}$$

$$(\Re(\lambda) > 0;\ \Re(\mu) > 0;\ \Re(\alpha+\beta) > -1;\ m, n \in \mathbb{N}_0),$$

where $f_{m,n}^{(\alpha,\beta)}(\vartheta,\varphi)$, $\Omega[x,y;\vartheta,\varphi]$, and $\Delta_{m,n}^{(\alpha,\beta)}$ are given by (2.4), (2.5), and (2.14), respectively. Finally, upon setting

$$\beta \longmapsto \gamma, \quad \lambda = \alpha+\beta+1, \quad \mu = \gamma+\delta+1, \quad x \longmapsto \frac{1-x}{2}, \quad \text{and} \quad y \longmapsto \frac{1-y}{2},$$

if we apply the definition (1.2), it is easily observed from (3.1) that

$$\begin{aligned} P_m^{(\alpha,\beta)}(x)\,P_n^{(\gamma,\delta)}(y) &= \frac{2^{\alpha+\gamma+m+n}}{\pi^2} \\ &\cdot \frac{\Gamma(\alpha+m+1)\,\Gamma(\gamma+n+1)\,\Gamma(\alpha+\beta+\gamma+\delta+m+n+2)}{\Gamma(\alpha+\beta+m+1)\,\Gamma(\gamma+\delta+n+1)\,\Gamma(\alpha+\gamma+m+n+1)} \\ &\cdot \int_0^1 \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} t^{\alpha+\beta+m}(1-t)^{\gamma+\delta+n}\; f_{m,n}^{(\alpha,\gamma)}(\vartheta,\varphi) \\ &\cdot P_{m+n}^{(\alpha+\gamma,\beta+\delta+1)}\big(1-\Omega[(1-x)\,t,(1-y)(1-t);\vartheta,\varphi]\big)\,d\varphi\, d\vartheta\, dt \end{aligned} \tag{3.2}$$

$$\big(\Re(\alpha+\beta)>-1;\ \Re(\gamma+\delta)>-1;\ \Re(\alpha+\gamma)>-1;\ m,n\in\mathbb{N}_0\big),$$

where $f_{m,n}^{(\alpha,\gamma)}(\vartheta,\varphi)$ and $\Omega[x,y;\vartheta,\varphi]$ are defined, as before, by (2.4) and (2.5), respectively. The triple-integral representation (3.2) provides the corrected (and notationally slightly modified) version of the main result of Chatterjea [4, p. 756, Equation (2.13)], just as it was observed also by Chongdar and Majumdar [6, p. 63, Equation (2.9)]. More importantly, in view of the relationships in (1.8), the *main* result of Chongdar and Majumdar [6, p. 62, Equation (2.8)] can be deduced from Chatterjea's result (3.2) *itself* by first setting

$$x \longmapsto \frac{2(x-a)}{a-b}+1 \quad \text{and} \quad y \longmapsto \frac{2(y-a)}{a-b}+1$$

or, alternatively,

$$x \longmapsto 1-\frac{2(x-b)}{a-b} \quad \text{and} \quad y \longmapsto 1-\frac{2(y-b)}{a-b},$$

and then making some rather obvious notational adjustments. Thus, in the latter case, the argument z of the Jacobi polynomial occurring in the integrand of the triple-integral representation (3.2), in the F-notation of (1.8) and (1.9), is given by

$$\begin{aligned} a+b-z &= \frac{1}{2}\left\{a+b+(a-b)\left(1-\frac{2}{a-b}\Omega[(x-b)\,t,(y-b)(1-t);\vartheta,\varphi]\right)\right\} \\ &= a-\Omega[(x-b)\,t,(y-b)(1-t);\vartheta,\varphi], \end{aligned} \tag{3.3}$$

that is, by

$$\begin{aligned} z &= b+\Omega[(x-b)\,t,(y-b)(1-t);\vartheta,\varphi] \\ &= b+\left\{(x-b)\,te^{(\vartheta-\varphi)i}+(y-b)(1-t)\,e^{-(\vartheta-\varphi)i}\right\}\frac{\cos\vartheta}{\cos\varphi}, \end{aligned} \tag{3.4}$$

in view of the definition (2.5) and the relationship given below:

$$F_n^{(\alpha,\beta)}(a+b-x;a,b,c) = (-1)^n\; F_n^{(\beta,\alpha)}(x;a,b,c), \tag{3.5}$$

which does indeed follow readily from the well-known relationship [27, p. 59, Equation (4.1.3)]:

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n \; P_n^{(\beta,\alpha)}(x). \tag{3.6}$$

Obviously, Equation (3.4) corresponds to the argument of the Jacobi polynomial occurring in the integrand of the *main* result of Chongdar and Majumdar [6, p. 62, Equation (2.8)], which is essentially the same as Chatterjea's result (3.2) simply rewritten (or *translated*) in the *seemingly inconsequential* F-notation.

4 Further Classes of Single-, Double-, and Triple-Integral Representations

Let $\{A_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of essentially arbitrary (real or complex) parameters. Over three decades ago, Srivastava [18] considered the following general family of polynomials:

$$S_n^N(z) := \sum_{k=0}^{[n/N]} \frac{(-n)_{Nk}}{k!} A_{n,k} \, z^k \qquad (n \in \mathbb{N}_0;\; N \in \mathbb{N}), \tag{4.1}$$

where (*and throughout this investigation*) $[\kappa]$ denotes the *greatest* integer in $\kappa \in \mathbb{R}$ and $(\lambda)_\nu$ denotes the Pochhammer symbol (or the *shifted factorial*, since $(1)_n = n!$ for $n \in \mathbb{N}_0$) defined already by (1.1). The polynomials $S_n^N(z)$ and a number of their variants have been considered, in recent years, by *numerous* other workers on the subject (see, for details, González *et al.* [18, p. 145 *et seq.*] and Lin *et al.* [15, p. 448 *et seq.*]). Here, in this section, we consider the following four members of the polynomial family defined by (4.1):

$$\mathcal{S}_{n,N}^{L,M}(z;\lambda,\alpha) := \sum_{k=0}^{[n/N]} \frac{(-n)_{Nk}(\lambda+n)_{Lk}}{(\alpha+1)_{Mk}} \frac{z^k}{k!} \tag{4.2}$$

$$(L, M, N \in \mathbb{N};\; n \in \mathbb{N}_0),$$

$$\mathcal{P}_{n,N}^{L}(z;\lambda) := \sum_{k=0}^{[n/N]} (-n)_{Nk}(\lambda+n)_{Lk} \frac{z^k}{k!} \tag{4.3}$$

$$(L, N \in \mathbb{N};\; n \in \mathbb{N}_0),$$

$$\mathcal{Q}_{n,N}^{M}(z;\alpha) := \sum_{k=0}^{[n/N]} \frac{(-n)_{Nk}}{(\alpha+1)_{Mk}} \frac{z^k}{k!} \tag{4.4}$$

$$(M, N \in \mathbb{N};\; n \in \mathbb{N}_0),$$

and

$$\mathcal{R}_{n,N}(z) := \sum_{k=0}^{[n/N]} (-n)_{Nk} \frac{z^k}{k!} \tag{4.5}$$

$$(N \in \mathbb{N};\; n \in \mathbb{N}_0).$$

Thus, in view of the following limit relationship:

$$\lim_{|\lambda|\to\infty}\left\{(\lambda)_n\left(\frac{z}{\lambda}\right)^n\right\}=z^\lambda=\lim_{|\mu|\to\infty}\left\{\frac{(\mu z)^n}{(\mu)_n}\right\}\qquad(n\in\mathbb{N}_0),\tag{4.6}$$

it is easily seen from the definitions (4.2) to (4.5) that

$$\mathcal{P}_{n,N}^{L}(z;\lambda)=\lim_{|\alpha|\to\infty}\left\{\mathcal{S}_{n,N}^{L,M}\left(z\alpha^M;\lambda,\alpha\right)\right\},\tag{4.7}$$

$$\mathcal{Q}_{n,N}^{M}(z;\alpha)=\lim_{|\lambda|\to\infty}\left\{\mathcal{S}_{n,N}^{L,M}\left(\frac{z}{\lambda^L};\lambda,\alpha\right)\right\},\tag{4.8}$$

and

$$\mathcal{R}_{n,N}(z)=\lim_{|\alpha|\to\infty}\left\{\mathcal{Q}_{n,N}^{M}\left(z\alpha^M;\alpha\right)\right\}\tag{4.9}$$

or, equivalently,

$$\mathcal{R}_{n,N}(z)=\lim_{\min\{|\lambda|,|\alpha|\}\to\infty}\left\{\mathcal{S}_{n,N}^{L,M}\left(\frac{z\alpha^M}{\lambda^L};\lambda,\alpha\right)\right\},\tag{4.10}$$

which exhibit the fact that each of the polynomials $\mathcal{P}_{n,N}^{L}(z;\lambda)$, $\mathcal{Q}_{n,N}^{M}(z;\alpha)$, and $\mathcal{R}_{n,N}(z)$ is a limit case of the polynomials $\mathcal{S}_{n,N}^{L,M}(z;\lambda,\alpha)$ which, in turn, are a special case of the polynomial system $S_n^N(z)$ defined by (4.1) when

$$A_{n,k}=\frac{(\lambda+n)_{Lk}}{(\alpha+1)_{Mk}}\qquad(L,M\in\mathbb{N};\ n,k\in\mathbb{N}_0).\tag{4.11}$$

In this section, we first derive some single-, double-, and triple-integral representations associated with the polynomials defined by (4.2) to (4.5). As we shall briefly investigate in Section 5 below, these classes of integral representations are motivated essentially by several known or new special cases and consequences of the main results presented here, involving such classical orthogonal polynomials as the Jacobi, Laguerre, Hermite, and Bessel polynomials, and various other related hypergeometric polynomials (see also Lin *et al.* [14]). First of all, for the polynomials $\mathcal{S}_{n,N}^{L,M}(z;\lambda,\alpha)$ defined by (4.2), we consider the following product:

$$\begin{aligned}
&\mathcal{S}_{m,N}^{L,M}(x;\lambda,\alpha)\ \mathcal{S}_{n,N}^{L,M}(y;\mu,\beta)\\
&=\sum_{k=0}^{[m/N]}\sum_{l=0}^{[n/N]}\frac{(-m)_{Nk}(-n)_{Nl}(\lambda+m)_{Lk}(\mu+n)_{Ll}}{(\alpha+1)_{Mk}(\beta+1)_{Ml}}\frac{x^k}{k!}\frac{y^l}{l!}\\
&=\frac{m!\,n!}{(m+n)!}\frac{\Gamma(\lambda+\mu+m+n)}{\Gamma(\lambda+m)\Gamma(\mu+n)}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\\
&\quad\cdot\sum_{k,l=0}^{k+l\leqq[(m+n)/N]}\frac{(-m-n)_{(k+l)N}(\lambda+\mu+m+n)_{(k+l)L}}{(\alpha+\beta+1)_{(k+l)M}}\frac{x^k}{k!}\frac{y^l}{l!}\\
&\quad\cdot\frac{\Gamma(m+n-(k+l)N+1)}{\Gamma(m-Nk+1)\Gamma(n-Nl+1)}\cdot\frac{\Gamma(\lambda+m+Lk)\Gamma(\mu+n+Ll)}{\Gamma(\lambda+\mu+m+n+(k+l)L)}\\
&\quad\cdot\frac{\Gamma(\alpha+\beta+(k+l)M+1)}{\Gamma(\alpha+Mk+1)\Gamma(\beta+Ml+1)},
\end{aligned}\tag{4.12}$$

where we have made repeated use of the elementary identity given below:

$$(-n)_{Nk}=(-1)^{Nk}\frac{n!}{(n-Nk)!}\qquad(0\leqq k\leqq[n/N];\ N\in\mathbb{N};\ n\in\mathbb{N}_0).\tag{4.13}$$

Now we recall the following known integral formula (*cf.*, *e.g.*, [30, p. 263, Example 39]; see also [16, p. 9]):

$$\int_0^{\pi/2} \cos^{\alpha+\beta} \vartheta \cos(\alpha-\beta)\vartheta \, d\vartheta = \frac{\pi}{2^{\alpha+\beta+1}} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \tag{4.14}$$

$$\left(\Re(\alpha+\beta) > -1\right),$$

which can easily be rewritten in a more convenient form as follows:

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} = \frac{2^{\alpha+\beta}}{\pi} \int_{-\pi/2}^{\pi/2} e^{i(\alpha-\beta)\vartheta} \cos^{\alpha+\beta} \vartheta \, d\vartheta \tag{4.15}$$

$$\left(\Re(\alpha+\beta) > -1; \; i := \sqrt{-1}\right).$$

By appropriately applying (4.15) *as well as* the familiar Eulerian Beta integral in the form (see, for example, [21, p. 10, Equation 1.1 (69)]; see also Equation (2.19) above):

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} dt = (b-a)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{4.16}$$

$$\left(\Re(\alpha) > 0; \; \Re(\beta) > 0; \; a \neq b\right),$$

we find from (4.12) that

$$\begin{aligned} &\mathcal{S}_{m,N}^{L,M}(x;\lambda,\alpha)\, \mathcal{S}_{n,N}^{L,M}(y;\mu,\beta) \\ &= \frac{2^{\alpha+\beta+m+n}}{\pi^2 (b-a)^{\lambda+\mu+m+n-1}} \frac{m!\, n!}{(m+n)!} \frac{\Gamma(\lambda+\mu+m+n)\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\lambda+m)\Gamma(\mu+n)\Gamma(\alpha+\beta+1)} \\ &\quad \cdot \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_a^b (t-a)^{\lambda+m-1} (b-t)^{\mu+n-1} \Theta_{m,n}^{(\alpha,\beta)}(\vartheta,\varphi) \\ &\quad \cdot \sum_{k,l=0}^{k+l \leqq [(m+n)/N]} \frac{(-m-n)_{(k+l)N} (\lambda+\mu+m+n)_{(k+l)L}}{(\alpha+\beta+1)_{(k+l)M}} \\ &\quad \cdot \frac{\{\xi(x;t,\vartheta,\varphi)\}^k}{k!} \frac{\{\eta(y;t,\vartheta,\varphi)\}^l}{l!} \, dt\, d\vartheta\, d\varphi \end{aligned} \tag{4.17}$$

$$\left(\Re(\alpha+\beta) > -1; \; \min\{\Re(\lambda), \Re(\mu)\} > 0; \; a \neq b; \; L, M, N \in \mathbb{N}; \; m, n \in \mathbb{N}_0\right),$$

where, for convenience,

$$\Theta_{m,n}^{(\alpha,\beta)}(\vartheta,\varphi) := e^{i(m-n)\vartheta + i(\alpha-\beta)\varphi} \cos^{m+n}\vartheta \cos^{\alpha+\beta}\varphi, \tag{4.18}$$

$$\xi(x;t,\vartheta,\varphi) := \frac{(2\cos\varphi)^M}{(2\cos\vartheta)^N} \left(\frac{t-a}{b-a}\right)^L x e^{i(M\varphi - N\vartheta)}, \tag{4.19}$$

and

$$\eta(y;t,\vartheta,\varphi) := \frac{(2\cos\varphi)^M}{(2\cos\vartheta)^N} \left(\frac{b-t}{b-a}\right)^L y e^{-i(M\varphi - N\vartheta)}. \tag{4.20}$$

Finally, in view of the familiar series identity:

$$\sum_{m,n=0}^{\infty} f(m+n) \frac{x^m}{m!} \frac{y^n}{n!} = \sum_{n=0}^{\infty} f(n) \frac{(x+y)^n}{n!}, \tag{4.21}$$

which holds true whenever each of the series involved is absolutely convergent, the double sum in (4.17) can be reduced to a single sum which, in turn, is interpretable by the definition (4.2). We are thus led to the following triple-integral representation for the product of two polynomials of the class defined by (4.2):

$$\begin{aligned}
&\mathcal{S}_{m,N}^{L,M}(x;\lambda,\alpha)\,\mathcal{S}_{n,N}^{L,M}(y;\mu,\beta) \\
&= \frac{2^{\alpha+\beta+m+n}}{\pi^2(b-a)^{\lambda+\mu+m+n-1}}\,\frac{m!\,n!}{(m+n)!}\,\frac{\Gamma(\lambda+\mu+m+n)\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\lambda+m)\,\Gamma(\mu+n)\,\Gamma(\alpha+\beta+1)} \\
&\cdot\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_a^b (t-a)^{\lambda+m-1}(b-t)^{\mu+n-1}\,\Theta_{m,n}^{(\alpha,\beta)}(\vartheta,\varphi) \\
&\cdot\mathcal{S}_{m+n,N}^{L,M}\left(\Phi_{M,N}\left[x\left(\frac{t-a}{b-a}\right)^L, y\left(\frac{b-t}{b-a}\right)^L;\vartheta,\varphi\right];\lambda+\mu,\alpha+\beta\right)dt\,d\vartheta\,d\varphi
\end{aligned} \tag{4.22}$$

$$\left(\Re(\alpha+\beta)>-1;\ \min\{\Re(\lambda),\Re(\mu)\}>0;\ a\neq b;\ L,M,N\in\mathbb{N};\ m,n\in\mathbb{N}_0\right),$$

where $\Theta_{m,n}^{(\alpha,\beta)}(\vartheta,\varphi)$ is given by (4.18) and

$$\Phi_{M,N}[x,y;\vartheta,\varphi] := \frac{(2\cos\varphi)^M}{(2\cos\vartheta)^N}\left\{xe^{i(M\varphi-N\vartheta)}+ye^{-i(M\varphi-N\vartheta)}\right\}. \tag{4.23}$$

In a similar manner, we can derive the following single- and double-integral representations for the product of two polynomials of the classes defined by (4.3), (4.4), and (4.5):

$$\begin{aligned}
\mathcal{P}_{m,N}^{L}(x;\lambda)\,\mathcal{P}_{n,N}^{L}(y;\mu) &= \frac{2^{m+n}}{\pi(b-a)^{\lambda+\mu+m+n-1}}\,\frac{m!\,n!}{(m+n)!}\,\frac{\Gamma(\lambda+\mu+m+n)}{\Gamma(\lambda+m)\,\Gamma(\mu+n)} \\
&\cdot\int_{-\pi/2}^{\pi/2}\int_a^b (t-a)^{\lambda+m-1}(b-t)^{\mu+n-1}e^{i(m-n)\vartheta}\cos^{m+n}\vartheta \\
&\cdot\mathcal{P}_{m+n,N}^{L}\left(\Psi_N\left[x\left(\frac{t-a}{b-a}\right)^L, y\left(\frac{b-t}{b-a}\right)^L;\vartheta\right];\lambda+\mu\right)dt\,d\vartheta
\end{aligned} \tag{4.24}$$

$$\left(\min\{\Re(\lambda),\Re(\mu)\}>0;\ a\neq b;\ L,N\in\mathbb{N};\ m,n\in\mathbb{N}_0\right),$$

where [*cf.* Equation (4.23)]

$$\Psi_N[x,y;\vartheta] = (2\cos\vartheta)^{-N}\left(xe^{-iN\vartheta}+ye^{iN\vartheta}\right); \tag{4.25}$$

$$\begin{aligned}
\mathcal{Q}_{m,N}^{M}(x;\alpha)\,\mathcal{Q}_{n,N}^{M}(y;\beta) &= \frac{2^{\alpha+\beta+m+n}}{\pi^2}\,\frac{m!\,n!}{(m+n)!}\,\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \\
&\cdot\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\Theta_{m,n}^{(\alpha,\beta)}(\vartheta,\varphi)\,\mathcal{Q}_{m+n,N}^{M}\left(\Phi_{M,N}[x,y;\vartheta,\varphi];\alpha+\beta\right)d\vartheta\,d\varphi
\end{aligned} \tag{4.26}$$

$$\left(\Re(\alpha+\beta)>-1;\ M,N\in\mathbb{N};\ m,n\in\mathbb{N}_0\right),$$

where $\Theta_{m,n}^{(\alpha,\beta)}(\vartheta,\varphi)$ and $\Phi_{M,N}[x,y;\vartheta,\varphi]$ are given by (4.18) and (4.23), respectively; and

$$\begin{aligned}
\mathcal{R}_{m,N}(x)\,\mathcal{R}_{n,N}(y) &= \frac{2^{m+n}m!\,n!}{\pi\cdot(m+n)!}\int_{-\pi/2}^{\pi/2}e^{i(m-n)\vartheta}\cos^{m+n}\vartheta \\
&\cdot\mathcal{R}_{m+n,N}\left(\Psi_N[x,y;\vartheta]\right)d\vartheta
\end{aligned} \tag{4.27}$$

$$\left(N\in\mathbb{N};\ m,n\in\mathbb{N}_0\right),$$

where $\Psi_N[x,y;\vartheta]$ is given by (4.25).

5 Applications to the Associated Systems of Hypergeometric Polynomials

By means of the following identity involving the Pochhammer symbol defined by (1.1) above (*cf.*, *e.g.*, [21, p. 8, Equation 1.1 (50)]):

$$(\lambda)_{Nk} = N^{Nk} \prod_{j=1}^{N} \left(\frac{\lambda+j-1}{N}\right)_k \qquad (N \in \mathbb{N};\ k \in \mathbb{N}_0;\ \lambda \in \mathbb{C}), \tag{5.1}$$

each of the polynomials defined by (4.2) to (4.5) can easily be rewritten as a generalized hypergeometric polynomial as follows:

$$\mathcal{S}_{n,N}^{L,M}(z;\lambda,\alpha) = {}_{N+L}F_M \left[\begin{array}{c} \Delta(N;-n), \Delta(L;\lambda+n); \\ \\ \Delta(M;\alpha+1); \end{array} \left(\frac{N^N L^L}{M^M}\right) z\right], \tag{5.2}$$

$$\mathcal{P}_{n,N}^{L}(z;\lambda) = {}_{N+L}F_0 \left[\begin{array}{c} \Delta(N;-n), \Delta(L;\lambda+n); \\ \\ \text{———————}; \end{array} N^N\, L^L\, z\right], \tag{5.3}$$

$$\mathcal{Q}_{n,N}^{M}(z;\alpha) = {}_{N}F_M \left[\begin{array}{c} \Delta(N;-n); \\ \\ \Delta(M;\alpha+1); \end{array} \left(\frac{N^N}{M^M}\right) z\right], \tag{5.4}$$

and

$$\mathcal{R}_{n,N}(z) = {}_{N}F_0 \left[\begin{array}{c} \Delta(N;-n); \\ \\ \text{————}; \end{array} N^N\, z\right], \tag{5.5}$$

where $\Delta(N;\lambda)$ abbreviates the array of N parameters

$$\frac{\lambda}{N}, \frac{\lambda+1}{N}, \ldots, \frac{\lambda+N-1}{N} \qquad (N \in \mathbb{N})$$

and (as before) ${}_pF_q$ denotes a generalized hypergeometric function with p numerator and q denominator parameters, defined by [7, Chapter 4]

$$\begin{aligned} {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) &= {}_pF_q \left[\begin{array}{c} \alpha_1,\ldots,\alpha_p; \\ \\ \beta_1,\ldots,\beta_q; \end{array} z\right] \\ &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \end{aligned} \tag{5.6}$$

$$\big(p,q \in \mathbb{N}_0;\ p \leqq q+1;\ p \leqq q \ \text{ and } \ |z| < \infty;$$

$$p = q+1 \ \text{ and } \ |z| < 1;\ p = q+1,\ |z| = 1, \ \text{ and } \ \Re(\omega) > 0\big),$$

where

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \qquad (\alpha_j \in \mathbb{C}\ (j = 1, \ldots, p);\ \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-\ (j = 1, \ldots, q)). \tag{5.7}$$

In terms of these families of generalized hypergeometric polynomials, the integral representations (4.22), (4.24), (4.26), and (4.27) can be rewritten as follows:

$$\begin{aligned}
&{}_{N+L}F_M \left[\begin{array}{r} \Delta(N; -m), \Delta(L; \lambda + m); \\ \\ \Delta(M; \alpha + 1); \end{array} x \right] {}_{N+L}F_M \left[\begin{array}{r} \Delta(N; -n), \Delta(L; \mu + n); \\ \\ \Delta(M; \beta + 1); \end{array} y \right] \\
&= \frac{2^{\alpha+\beta+m+n}}{\pi^2 (b-a)^{\lambda+\mu+m+n-1}} \, \frac{m!\, n!}{(m+n)!} \, \frac{\Gamma(\lambda+\mu+m+n)\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\lambda+m)\,\Gamma(\mu+n)\,\Gamma(\alpha+\beta+1)} \\
&\quad \cdot \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_a^b (t-a)^{\lambda+m-1} (b-t)^{\mu+n-1}\, \Theta_{m,n}^{(\alpha,\beta)}(\vartheta, \varphi) \\
&\quad \cdot {}_{N+L}F_M \left[\begin{array}{r} \Delta(N; -m-n), \Delta(L; \lambda+\mu+m+n); \\ \\ \Delta(M; \alpha+\beta+1); \end{array} \right. \\
&\qquad \left. \Phi_{M,N} \left[x \left(\frac{t-a}{b-a} \right)^L, y \left(\frac{b-t}{b-a} \right)^L; \vartheta, \varphi \right] \right] dt\, d\vartheta\, d\varphi
\end{aligned} \tag{5.8}$$

$$\left(\Re(\alpha+\beta) > -1;\ \min\{\Re(\lambda), \Re(\mu)\} > 0;\ a \neq b;\ L, M, N \in \mathbb{N};\ m, n \in \mathbb{N}_0\right),$$

where $\Theta_{m,n}^{(\alpha,\beta)}(\vartheta, \varphi)$ and $\Phi_{M,N}[x, y; \vartheta, \varphi]$ are given by (4.18) and (4.23), respectively;

$$\begin{aligned}
&{}_{N+L}F_0 \left[\begin{array}{r} \Delta(N; -m), \Delta(L; \lambda + m); \\ \\ \text{———}; \end{array} x \right] {}_{N+L}F_0 \left[\begin{array}{r} \Delta(N; -n), \Delta(L; \mu + n); \\ \\ \text{———}; \end{array} y \right] \\
&= \frac{2^{m+n}}{\pi (b-a)^{\lambda+\mu+m+n-1}} \, \frac{m!\, n!}{(m+n)!} \, \frac{\Gamma(\lambda+\mu+m+n)}{\Gamma(\lambda+m)\,\Gamma(\mu+n)} \\
&\quad \cdot \int_{-\pi/2}^{\pi/2} \int_a^b (t-a)^{\lambda+m-1} (b-t)^{\mu+n-1}\, e^{i(m-n)\vartheta} \cos^{m+n} \vartheta \\
&\quad \cdot {}_{N+L}F_0 \left[\begin{array}{r} \Delta(N; -m-n), \Delta(L; \lambda+\mu+m+n); \\ \\ \text{———}; \end{array} \right. \\
&\qquad \left. \Psi_N \left[x \left(\frac{t-a}{b-a} \right)^L, y \left(\frac{b-t}{b-a} \right)^L; \vartheta \right] \right] dt\, d\vartheta
\end{aligned} \tag{5.9}$$

$$\left(\min\left\{\Re(\lambda),\Re(\mu)\right\}>0;\ a\neq b;\ L,N\in\mathbb{N};\ m,n\in\mathbb{N}_0\right),$$

where $\Psi_N[x,y;\vartheta]$ is given by (4.25);

$$ {}_NF_M\left[\begin{array}{c}\Delta(N;-m);\\ \\ \Delta(M;\alpha+1);\end{array} x\right] {}_NF_M\left[\begin{array}{c}\Delta(N;-n);\\ \\ \Delta(M;\beta+1);\end{array} y\right]$$

$$=\frac{2^{\alpha+\beta+m+n}}{\pi^2}\,\frac{m!\,n!}{(m+n)!}\,\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}$$

$$\cdot\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\Theta_{m,n}^{(\alpha,\beta)}(\vartheta,\varphi)\ {}_NF_M\left[\begin{array}{c}\Delta(N;-m-n);\\ \\ \Delta(M;\alpha+\beta+1);\end{array}\Phi_{M,N}[x,y;\vartheta,\varphi]\right]d\vartheta\,d\varphi \tag{5.10}$$

$$(\Re(\alpha+\beta)>-1;\ M,N\in\mathbb{N};\ m,n\in\mathbb{N}_0),$$

where $\Theta_{m,n}^{(\alpha,\beta)}(\vartheta,\varphi)$ and $\Phi_{M,N}[x,y;\vartheta,\varphi]$ are given by (5.18) and (5.23), respectively;

$$ {}_NF_0\left[\begin{array}{c}\Delta(N;-m);\\ \\ \text{———};\end{array} x\right] {}_NF_0\left[\begin{array}{c}\Delta(N;-n);\\ \\ \text{———};\end{array} y\right]=\frac{2^{m+n}}{\pi}\,\frac{m!\,n!}{(m+n)!}$$

$$\cdot\int_{-\pi/2}^{\pi/2}e^{i(m-n)\vartheta}\cos^{m+n}\vartheta\ {}_NF_0\left[\begin{array}{c}\Delta(N;-m-n);\\ \\ \text{———};\end{array}\Psi_N[x,y;\vartheta]\right]d\vartheta \tag{5.11}$$

$$(N\in\mathbb{N};\ m,n\in\mathbb{N}_0),$$

where $\Psi_N[x,y;\vartheta]$ is given by (4.25). Next we proceed to apply the integral representations (5.8) to (5.11) to a number of widely-investigated specific hypergeometric polynomials (see, for example, Chen and Srivastava [5]).

I. Jacobi Polynomials. The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ of degree n in x (and with parameters or indices α and β) may be defined by (*cf.*, *e.g.*, [27, Chapter 4])

$$P_n^{(\alpha,\beta)}(x):=\sum_{k=0}^{n}\binom{\alpha+n}{n-k}\binom{\beta+n}{k}\left(\frac{x+1}{2}\right)^{n-k}\left(\frac{x-1}{2}\right)^{k} \tag{5.12}$$

or, equivalently, by its hypergeometric representation given already by (1.2). Thus, by letting

$L = M = N = 1$ in (5.8), we obtain

$$
\begin{aligned}
&{}_2F_1\left(-m, \lambda + m; \alpha + 1; x\right) \; {}_2F_1\left(-n, \mu + n; \beta + 1; y\right) \\
&= \frac{2^{\alpha+\beta+m+n}}{\pi^2 (b-a)^{\lambda+\mu+m+n-1}} \, \frac{m! \, n!}{(m+n)!} \, \frac{\Gamma(\lambda+\mu+m+n)\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\lambda+m)\,\Gamma(\mu+n)\,\Gamma(\alpha+\beta+1)} \\
&\quad \cdot \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_a^b (t-a)^{\lambda+m-1} (b-t)^{\mu+n-1} \, \Theta_{m,n}^{(\alpha,\beta)}(\vartheta, \varphi) \\
&\quad \cdot {}_2F_1\Bigg(-m-n, \lambda+\mu+m+n; \alpha+\beta+1; \\
&\qquad \Phi_{1,1}\left[x\left(\frac{t-a}{b-a}\right), y\left(\frac{b-t}{b-a}\right); \vartheta, \varphi\right]\Bigg) \, dt \, d\vartheta \, d\varphi \qquad (5.13)
\end{aligned}
$$

$$
\left(\Re(\alpha+\beta) > -1; \; \min\{\Re(\lambda), \Re(\mu)\} > 0; \; a \neq b; \; m, n \in \mathbb{N}_0\right),
$$

which, by setting

$$
\beta \longmapsto \gamma, \quad x \longmapsto \frac{1-x}{2}, \quad y \longmapsto \frac{1-y}{2}, \quad \lambda = \alpha + \beta + 1, \quad \text{and} \quad \mu = \gamma + \delta + 1,
$$

yields the following integral representation:

$$
\begin{aligned}
P_m^{(\alpha,\beta)}(x) \, P_n^{(\gamma,\delta)}(y) &= \frac{2^{\alpha+\gamma+m+n}}{\pi^2 (b-a)^{\alpha+\beta+\gamma+\delta+m+n+1}} \\
&\quad \cdot \frac{\Gamma(\alpha+\beta+\gamma+\delta+m+n+2)}{\Gamma(\alpha+\beta+m+1)\,\Gamma(\gamma+\delta+n+1)} \cdot \frac{\Gamma(\alpha+m+1)\,\Gamma(\gamma+n+1)}{\Gamma(\alpha+\gamma+m+n+1)} \\
&\quad \cdot \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_a^b (t-a)^{\alpha+\beta+m} (b-t)^{\gamma+\delta+n} \, \Theta_{m,n}^{(\alpha,\beta)}(\vartheta, \varphi) \\
&\quad \cdot P_{m+n}^{(\alpha+\beta,\gamma+\delta+1)}\left(1 - \Phi_{1,1}\left[(1-x)\left(\frac{t-a}{b-a}\right), (1-y)\left(\frac{b-t}{b-a}\right); \vartheta, \varphi\right]\right) dt \, d\vartheta \, d\varphi \qquad (5.14)
\end{aligned}
$$

$$
\left(\min\{\Re(\alpha+\beta), \Re(\alpha+\gamma), \Re(\gamma+\delta)\} > -1; \; a \neq b; \; m, n \in \mathbb{N}_0\right),
$$

where $\Theta_{m,n}^{(\alpha,\beta)}(\vartheta, \varphi)$ and $\Phi_{M,N}[x, y; \vartheta, \varphi]$ are given by (4.18) and (4.23), respectively. The case $a = b - 1 = 0$ of (5.14) would correspond to the aforecited *corrected* (and notationally slightly modified) version (3.2) of the *main* result of an earlier paper by Chatterjea [4, p. 756, Equation (2.13)]. As a matter of fact, Formula (5.14) can also be *further* specialized appropriately in order to deduce the corresponding integral representations associated with such particular cases of the classical Jacobi polynomials

$$
P_n^{(\alpha,\beta)}(x)
$$

as (for example) the Gegenbauer (or ultraspherical) polynomials for which $\alpha = \beta$, the Chebyshev polynomials of the first and second kind for which

$$
\alpha = \beta = -\frac{1}{2} \qquad \text{and} \qquad \alpha = \beta = \frac{1}{2},
$$

respectively, and the Legendre (or spherical) polynomials for which $\alpha = \beta = 0$.

II. Bessel Polynomials. For the Bessel polynomials $y_n(x;\alpha,\beta)$ defined by (*cf.* [12]; see also [24, p. 75, Equation 1.9 (1)])

$$y_n(x;\alpha,\beta) := \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+n+k-2}{k} k! \left(\frac{x}{\beta}\right)^k \tag{5.15}$$

or, in its hypergeometric form, by (2.8), we find from (5.9) with $L = N = 1$ that (*cf.* [3])

$$y_m(x;\alpha,\beta)\ y_n(y;\gamma,\delta) = \frac{2^{m+n}}{\pi(b-a)^{\alpha+\gamma+m+n-3}}$$

$$\cdot \frac{m!\,n!}{(m+n)!} \frac{\Gamma(\alpha+\gamma+m+n-2)}{\Gamma(\alpha+m-1)\Gamma(\gamma+n-1)}$$

$$\cdot \int_{-\pi/2}^{\pi/2} \int_a^b (t-a)^{\alpha+m-2} (b-t)^{\gamma+n-2}\ e^{i(m-n)\vartheta}\ \cos^{m+n}\vartheta$$

$$\cdot\, y_{m+n}\left(\Psi_1\left[\delta x\left(\frac{t-a}{b-a}\right), \beta y\left(\frac{b-t}{b-a}\right); \vartheta\right]; \alpha+\gamma-1, \beta\delta\right) dt\, d\vartheta \tag{5.16}$$

$$\left(\min\{\Re(\alpha), \Re(\gamma)\} > 1;\ a \neq b;\ m,n \in \mathbb{N}_0\right),$$

where $\Psi_N[x,y;\vartheta]$ is given by (4.25).

III. Laguerre Polynomials. The classical Laguerre polynomials $L_n^{(\alpha)}(x)$ of degree n in x (and with parameter or index α) are defined already by (2.1) and (in hypergeometric form) by (2.2). Thus, in its special case when $M = N = 1$, our general result (5.10) would obviously correspond to the aforecited (Carlitz's) formula (2.3). Several markedly different generalizations of the above-mentioned integral representations of (for example) Carlitz [2] and Chatterjea [3], [4]) were investigated systematically by Srivastava *et al.* [19], [22], and [26]).

IV. Hermite Polynomials. For the classical Hermite polynomials $H_n(x)$ defined by (*cf.*, *e.g.*, [27, Chapter 5])

$$H_n(x) := \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \frac{(2k)!}{k!} (2x)^{n-2k} \tag{5.17}$$

or, equivalently, by

$$H_n(x) = (2x)^n\ {}_2F_0\left(\Delta(2;-n); \text{——}; -\frac{1}{x^2}\right), \tag{5.18}$$

we find from (5.11) with $N = 2$ that

$$H_m(x)\, H_n(y) = \frac{1}{\pi} \left(\frac{2}{y}\right)^m \left(\frac{2}{x}\right)^n$$

$$\cdot \frac{m!\, n!}{(m+n)!} \int_{-\pi/2}^{\pi/2} e^{i(m-n)\vartheta} \left(\cos\vartheta \sqrt{\Psi_2\left[y^2, x^2; \vartheta\right]}\right)^{m+n}$$

$$\cdot H_{m+n}\left(\frac{xy}{\sqrt{\Psi_2\left[y^2, x^2; \vartheta\right]}}\right) d\vartheta \qquad (m, n \in \mathbb{N}_0), \tag{5.19}$$

where $\Psi_N[x, y; \vartheta]$ is given by (4.25).

V. Konhauser-Toscano Polynomials. The *biorthogonal* polynomials

$$Z_n^{\alpha}(x; M) \qquad (M \in \mathbb{N})$$

of the *second* kind, defined by (*cf.* [13, p. 304, Equation (5)]; see also [24, p. 197, Problem 65])

$$Z_n^{\alpha}(x; M) := \sum_{k=0}^{n} \binom{\alpha + Mn}{Mn - Mk} \frac{(Mn - Mk)!}{(n-k)!} \frac{\left(-x^M\right)^k}{k!} \tag{5.20}$$

or, equivalently, by

$$Z_n^{\alpha}(x; M) := \binom{\alpha + Mn}{Mn} \frac{(Mn)!}{n!}\, {}_1F_M \left[\begin{array}{c} -n; \\ \Delta(M; \alpha + 1); \end{array} \left(\frac{x}{M}\right)^M\right], \tag{5.21}$$

were considered by Toscano [28] *without* their *biorthogonality* property (which was subsequently emphasized upon by Konhauser [13]). For these Konhauser-Toscano polynomials, it is not difficult to deduce from (5.10) with $N = 1$ that

$$Z_m^{\alpha}(x; M) Z_n^{\beta}(y; M) = \frac{2^{\alpha+\beta+m+n}}{\pi^2} \frac{\Gamma(\alpha + Mm + 1)\, \Gamma(\beta + Mn + 1)}{\Gamma(\alpha + \beta + M(m+n) + 1)}$$

$$\cdot \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \Theta_{m,n}^{(\alpha,\beta)}(\vartheta, \varphi)\, Z_{m+n}^{\alpha+\beta}\left(\left\{\Phi_{M,1}\left[x^M, y^M; \vartheta, \varphi\right]\right\}^{1/M}; M\right) d\vartheta\, d\varphi \tag{5.22}$$

$$\left(\Re(\alpha + \beta) > -1;\ M \in \mathbb{N};\ m, n \in \mathbb{N}_0\right),$$

where $\Theta_{m,n}^{(\alpha,\beta)}(\vartheta, \varphi)$ and $\Phi_{M,N}[x, y; \vartheta, \varphi]$ are given by (4.18) and (4.23), respectively. Since

$$Z_n^{\alpha}(x; 1) = L_n^{(\alpha)}(x) \qquad (n \in \mathbb{N}_0), \tag{5.23}$$

which follows from the definitions (2.2) and (5.21), Carlitz's formula (2.3) is a special case of (5.23) when $M = 1$.

VI. Gould-Hopper Polynomials. For the Gould-Hopper generalization of the classical Hermite polynomials $H_n(x)$, which is defined by (*cf.* [10, p. 58, Equation (6.2)]; see also [24, p. 76, Equation 1.9 (6)])

$$g_n^N(x, \rho) := \sum_{k=0}^{[n/N]} \binom{n}{Nk} \frac{(Nk)!}{k!} \rho^k\, x^{n-Nk} \tag{5.24}$$

or, equivalently, by

$$g_n^N(x,\rho) := x^n \; {}_N F_0 \left[\begin{array}{c} \Delta(N;-n); \\ \overline{}; \end{array} \rho \left(-\frac{N}{x} \right)^N \right], \tag{5.25}$$

so that, obviously,

$$g_n^2(2x,-1) = H_n(x), \tag{5.26}$$

we find from the integral representation (5.11) that

$$g_m^N(x,-\rho) g_n^N(y,-\sigma) = \frac{1}{\pi} \left(\frac{2}{y}\right)^m \left(\frac{2}{x}\right)^n \frac{m!\, n!}{(m+n)!}$$

$$\cdot \int_{-\pi/2}^{\pi/2} e^{i(m-n)\vartheta} \left(\cos\vartheta \left\{ \Psi_N \left[y^N/\sigma, x^N/\rho; \vartheta \right] \right\}^{1/N} \right)^{m+n}$$

$$\cdot g_{m+n}^N \left(\frac{xy}{\left\{ \Psi_N \left[y^N/\sigma, x^N/\rho; \vartheta \right] \right\}^{1/N}}; -\rho\sigma \right) d\vartheta \qquad (m,n \in \mathbb{N}_0), \tag{5.27}$$

where $\Psi_N[x,y;\vartheta]$ is given by (4.25). By virtue of the relationship (5.26) with the classical Hermite polynomials, this last result (5.27) would reduce at once to (5.19) when

$$N = 2, \quad \rho = \sigma = 1, \quad x \longmapsto 2x, \quad \text{and} \quad y \longmapsto 2y.$$

Many more consequences and applications of the general single-, double-, and triple-integral representations of the preceding sections can be deduced for other classes of hypergeometric polynomials as well. We do, however, choose to omit the details involved in these types of applications of the general results presented here. In conclusion, it may be of interest to remark that each of the many classes of integral representations (which we have presented in this article) may be looked upon as a potentially useful *linearization relationship* for the product of two different members of the associated family of hypergeometric polynomials.

Acknowledgements

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

GROUP C^*-ALGEBRAS AND THEIR STABLE RANK

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ABSTRACT

We study the stable rank of group C^*-algebras of connected or disconnected Lie groups and discrete groups. The main purpose is to collect and reveal our results on this topic obtained so far for further research. Beyond group C^*-algebras we also consider the stable rank for some classes of C^*-algebras.

2000 Mathematics Subject Classification: Primary 46L05.
Keywords: Group C^*-algebras, Stable rank, Lie groups.

Introduction

Group C^*-algebras have been very important objects in the theory of C^*-algebras in some senses such as the representation theory, structure theory and K-theory. In particular, it has been known (see Dixmier [3]) that the group (unitary) representation theory is contained in (or equivalent to) the representation theory of group C^*-algebras in the sense that any unitary irreducible representation of a locally compact group and its unitary equivalence class correspond to those of an irreducible representation of its group C^*-algebra, and the unitary dual (or the spectrum) of a locally compact group consisting of equivalence classes of its unitary irreducible representations is identified with the spectrum of its group C^*-algebra. However, it is difficult to determine the spectrums in general. Actually, for solvable Lie groups of non type I, this problem is hard and unsolved. But for type I solvable Lie groups such as nilpotent or exponential groups, this problem has been settled by the orbit method in the unitary representation theory of solvable Lie groups. More fortunately, simple quotients of group C^*-algebras of connected solvable Lie groups are determined by Poguntke [12] and Green [6].

On the other hand, the stable rank of C^*-algebras has been introduced by Rieffel [13]. The question by Rieffel [13] for the stable rank of Lie group C^*-algebras has been solved by Sheu [15] in the case of certain simply connected nilpotent Lie groups and solved by Takai and the author [33] in the case of all simply connected nilpotent Lie groups. Extensively, the case of simply connected solvable Lie groups of type I, the case of simply connected amenable Lie groups of type I, and that of connected nilpotent Lie groups are considered in [34], [17] and [19] respectively. Morever, the case of certain simply connected solvable Lie groups (of non type I) containing the real 5-dimensional Mautner group is considered in [20], and its generalizations are also obtained by [21]. Also, the case of certain simply connected solvable Lie groups (of non type I) containing the real 7-dimensional Dixmier group is considered in [22].

Furthermore, the cases of certain disconnected solvable Lie groups are considered ([23], [24] and [25]). For non-amenable connected Lie groups such as semi-simple and reductive ones, we have considered the stable rank of their reduced group C^*-algebras in [16] (see also [18]). See also [26] for the (full or reduced) group C^*-algebras of some connected Lie groups with stable rank one.

For discrete groups we have considered the stable rank of their (full or reduced) group C^*-algebras. For the nilpotent discrete case containing the (generalized) discrete Heisenberg groups, see [7] and see [8] for its generalization to the case of two-step nilpotent discrete groups. For the solvable discrete case of certain semi-direct products, see [27]. Moreover, see [28] for the stable rank of residually finite dimensional C^*-algebras and its applications for the reduced group C^*-algebras of residually finite discrete groups with the property (T).

Now, what is the stable rank for ? Roughly speaking, the stable rank of C^*-algebras measures (complex) dimension of their spectrums. In particular, the stable rank of group C^*-algebras measures (complex) dimension of the spaces of their 1-dimensional representations. As we mentioned spectrums above, it is difficult to know about them in general. However, we can know their dimension from computing stable rank in such difficult situations. Then we can know size of spectrums from stable rank, which should be a benefit. Also, the stable rank plays an important role in K-theory of C^*-algebras. For example, stable rank one implies the cancellation property of projections. Furthermore, from stable rank we can know about K_1-groups of C^*-algebras (see [14]). See also Blackadar [1].

The contents of this paper after Preliminaries below:

1 The stable rank of Lie group C^*-algebras

2 The stable rank of disconnected Lie group C^*-algebras

3 The stable rank of discrete group C^*-algebras

4 Beyond group C^*-algebras

Preliminaries

Group C*-algebras [3].

Let G be a locally compact group, $C^*(G)$ the (full) group C^*-algebra of G and $C_r^*(G)$ the reduced group C^*-algebra of G. The group C^*-algebra $C^*(G)$ is defined to be the universal C^*-algebra generated by the image of $L^1(G)$ the Banach $*$-algebras of integrable measurable functions on G with convolution and involution under the universal representation on the universal Hilbert space, and it is also the C^*-completion of $L^1(G)$ with the universal C^*-norm. On the other hand, the reduced group C^*-algebra $C_r^*(G)$ is defined to be the C^*-algebra generated by the image of $L^1(G)$ under the left regular representation on the Hilbert space $L^2(G)$ of square integrable measurable functions on G.

The spectrum $G^\wedge$ of a locally compact group G is the space of unitary equivalence classes of irreducible unitary representations of G, and the spectrum $C^*(G)^\wedge$ of $C^*(G)$ is the space of equivalence classes of irreducible (non-degenerate) representations of $C^*(G)$ with the hull kernel topology. Denote by $G_1^\wedge$, $C^*(G)_1^\wedge$ the spaces of all 1-dimensional representations of G, $C^*(G)$ respectively.

Stable rank [13].

Let $\mathfrak{A}$ be a unital C^*-algebra. The stable rank $\mathrm{sr}(\mathfrak{A})$ of $\mathfrak{A}$ is defined to be the smallest positive integer n such that for any $\varepsilon > 0$ and for any element (a_j) of $\mathfrak{A}^n$ the n-direct sum of $\mathfrak{A}$, there exists an element (b_j) of $\mathfrak{A}^n$ such that $\|a_j - b_j\| < \varepsilon$ for $1 \le j \le n$ and $\sum_{j=1}^n \mathfrak{A} b_j = \mathfrak{A}$, that is, there exists $(c_j) \in \mathfrak{A}^n$ such that $\sum_{j=1}^n c_j b_j = 1$ (or $\sum_{j=1}^n b_j^* b_j$ is invertible in $\mathfrak{A}$). The connected stable rank $\mathrm{csr}(\mathfrak{A})$ of $\mathfrak{A}$ is defined to be the smallest positive integer n such that the set of all elements $(b_j) \in \mathfrak{A}^n$ with $\sum_{j=1}^n \mathfrak{A} b_j = \mathfrak{A}$ is connected. For a nonunital C^*-algebra $\mathfrak{A}$, its stable rank and connected stable rank are defined to be those of the unitization $\mathfrak{A}^+$ of $\mathfrak{A}$ by $\mathbb{C}$.

Let X be a locally compact Hausdorff space and $C_0(X)$ the C^*-algebra of continuous functions on X vanishing at infinity. Set $C(X) = C_0(X)$ when X is compact. Then (F1) ([13] and [9]):

$$\begin{cases} \mathrm{sr}(C_0(X)) = [\dim X^+/2] + 1, \\ \mathrm{csr}(C_0(X)) \le [(1 + \dim X^+)/2] + 1, \end{cases}$$

where X^+ means the one point compactification of X, and $C_0(X)^+ \cong C(X^+)$, and $[x]$ means the maximum integer $\le x$. Set $\dim_{\mathbb{C}} X = [\dim X^+/2] + 1$.

For a short exact sequence: $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{J} \to 0$ of C^*-algebras,

$$(\mathrm{F2}): \quad \begin{cases} \max\{\mathrm{sr}(\mathfrak{J}), \mathrm{sr}(\mathfrak{A}/\mathfrak{J})\} \le \mathrm{sr}(\mathfrak{A}) \le \max\{\mathrm{sr}(\mathfrak{J}), \mathrm{sr}(\mathfrak{A}/\mathfrak{J}), \mathrm{csr}(\mathfrak{A}/\mathfrak{J})\}, \\ \mathrm{csr}(\mathfrak{A}) \le \max\{\mathrm{csr}(\mathfrak{J}), \mathrm{csr}(\mathfrak{A}/\mathfrak{J})\} \quad \text{([13] and [15]).} \end{cases}$$

Let $\mathfrak{A} \otimes \mathbb{K}$ be the C^*-tensor product of a C^*-algebra $\mathfrak{A}$ with $\mathbb{K}$ the C^*-algebra of compact operators on an infinite dimensional Hilbert space. Then we have (F3) ([13] and [9], [15]):

$$\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) = \min\{2, \mathrm{sr}(\mathfrak{A})\}, \quad \mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \le \min\{2, \mathrm{csr}(\mathfrak{A})\}.$$

In what follows, denote by $\vee$, $\wedge$ the maximum and minimum respectively.

1 The stable rank of Lie group C^*-algebras

Table 1: **Classes and examples of (connected) Lie groups**

Classes	Examples
Commutative	$\mathbb{R}^n$, $\mathbb{T}^s$, $\mathbb{R}^n \times \mathbb{T}^s$
Nilpotent	Heisenberg Lie group
Exponential	$ax + b$ group
Type I solvable	Real algebraic linear groups
Type I amenable	$SO(n)$, Motion groups
Non type I solvable	Mautner and Dixmier groups
Non type I amenable	Ext of Compact by Non type I solvable
Type I non-amenable	Semi-simple $SL_n(\mathbb{R})$, Reductive $GL_n(\mathbb{R})$
Non type I non-amenable	Ext of Noncompact by Non type I solvable

It is known that all nilpotent Lie groups are exponential, and all exponential solvable Lie groups are of type I, and all solvable Lie groups are amenable. Also, extensions of compact groups by solvable

Lie groups are amenable. "Ext of Compact by Non type I solvable" means extenstions of compact Lie groups by non type I solvable Lie groups, and "Ext of Noncompact by Non type I solvable" means extensions of non compact semi-simple Lie groups by non type I solvable Lie groups. Note that those extensions are not necessarily of non type I.

Amenable Lie group C^*-algebras of type I

Theorem 1.1. Let G be a connected commutative Lie group. Then

$$\mathrm{sr}(C^*(G)) = [\dim G^\wedge/2] + 1 = \dim_{\mathbb{C}} G^\wedge, \quad \mathrm{csr}(C^*(G)) \leq [(\dim G^\wedge + 1)/2] + 1.$$

Proof. Since G is a connected commutative Lie group, it is isomorphic to $\mathbb{R}^n \times \mathbb{T}^k$ for some $n, k \geq 0$. Thus, $C^*(G) \cong C_0(\mathbb{R}^n \times \mathbb{Z}^k)$ by the Fourier transform, and $G^\wedge = \mathbb{R}^n \times \mathbb{Z}^k$. Then use (F1). In fact, we obtain $\mathrm{csr}(C^*(G)) = \mathrm{csr}(C_0(\mathbb{R}^n)) = \mathrm{csr}(C(S^n)) = 2$ if $n = 1$, 1 if $n = 2$, and $[(n+1)/2] + 1$ if $n \geq 3$, where $G \cong \mathbb{R}^n \times \mathbb{T}^k$ and S^n is the n-dimensional sphere (see [15]). □

As the first step Sheu has obtained the following:

Theorem 1.2. [15] Let G be a simply connected nilpotent Lie group which is a semi-direct product $\mathbb{R}^n \rtimes \mathbb{R}$. Then

$$\mathrm{sr}(C^*(G)) = \mathrm{sr}(C_0(\mathbb{R}^{r(G)})) = [r(G)/2] + 1,$$
$$\mathrm{csr}(C^*(G)) = \max\{2, \mathrm{csr}(C_0(\mathbb{R}^{r(G)}))\} \leq [(r(G)+1)/2] + 1,$$

where $r(G) = \dim(\mathfrak{G}^*)^G$ for $\mathfrak{G}$ the Lie algebra of G and $\mathfrak{G}^*$ its real dual space and $(\mathfrak{G}^*)^G$ the fixed point subspace of $\mathfrak{G}^*$ under the coadjoint action of G.

Generalizing the theorem above Takai and the author have obtained:

Theorem 1.3. [33] Let G be a simply connected nilpotent Lie group. Then

$$\mathrm{sr}(C^*(G)) = \mathrm{sr}(C_0(\mathbb{R}^{r(G)})) = [r(G)/2] + 1 = \dim_{\mathbb{C}}(\mathfrak{G}^*)^G,$$

where $r(G) = \dim(\mathfrak{G}^*)^G$ defined as above. Moreover, $\mathrm{sr}(C^*(G)) = 1$ if and only if $G \cong \mathbb{R}$ if and only if $\dim(\mathfrak{G}^*)^G = 1$.

Example 1.4. Let H_{2n+1} be the real $(2n+1)$-dimensional generalized Heisenberg Lie group defined by the semi-direct product $\mathbb{R}^{n+1} \rtimes_\alpha \mathbb{R}^n$ such that $\alpha_a(c, b) = (c + \sum_{j=1}^n a_j b_j, b)$ for $a = (a_j), b = (b_j) \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then we have the short exact sequence: $0 \to C_0(\mathbb{R}\setminus\{0\}) \otimes \mathbb{K} \to C^*(H_{2n+1}) \to C_0(\mathbb{R}^{2n}) \to 0$, where $\mathbb{K}$ is the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space, and $\mathrm{sr}(C^*(H_{2n+1})) = n + 1 = \mathrm{sr}(C_0(\mathbb{R}^{2n}))$.

Remark. It is known that there exists a bjiction between the spectrum $G^\wedge$ of a simply connected nilpotent Lie group G and its coadjoint orbit space $\mathfrak{G}^*/G$ (in fact, this bijection is a homeomorphism under the hull-kernel topology and the quotient topology).

Furthermore, Takai and the author have obtained the following:

Theorem 1.5. [34] For G a simply connected solvable Lie group of type I,

$$\mathrm{sr}(C^*(G)) = (2 \vee \dim_{\mathbb{C}}(\mathfrak{G}^*)^G) \wedge \dim G.$$

Moreover, for G a simply connected solvable Lie group, we have $\mathrm{sr}(C^*(G)) = 1$ if and only if $G \cong \mathbb{R}$.

Remark. Note that a C^*-algebra is of type I if and only if it has a composition series of closed ideals such that their subquotients have continuous trace. To show the second statement, we used K-theory for C^*-algebras and the (generalized) index theory for K-groups of C^*-algebras.

Example 1.6. Let A_{n+1} be the real $(n+1)$-dimensional generalized $ax+b$ Lie group defined by the semi-direct product $\mathbb{R}^n \rtimes_\alpha \mathbb{R}$ such that $\alpha_t(a) = (e^t a_j)$ for $a = (a_j) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then $C^*(A_{n+1})$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^{n+1}$ such that $\mathfrak{I}_{n+1}/\mathfrak{I}_n \cong C_0(\mathbb{R})$ and

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \oplus^{{}_n\mathrm{C}_{n+1-j}} \oplus^{2^{n+1-j}} C_0(\mathbb{R}_+^{n-j}) \otimes \mathbb{K}, \quad (1 \le j \le n)$$

where ${}_n\mathrm{C}_{n+1-j}$ means combination and $\mathbb{R}_+$ is the space of all positive real numbers, and $\oplus^k$ is the k-direct sum. Then we have

$$\mathrm{sr}(C^*(A_{n+1})) = 2 = 2 \vee \mathrm{sr}(C_0(\mathbb{R}))$$

but $\dim(A_{n+1})_1^\wedge = 1$. Note that $(\mathfrak{G}^*)^G$ is homeomorphic to $G_1^\wedge$ for G a simply connected Lie group.

Extensively, the author has obtained the following:

Theorem 1.7. [17] Let G be a connected amenable Lie group of type I. Then

$$\dim_{\mathbb{C}} G_1^\wedge \le \mathrm{sr}(C^*(G)) \le \max\{2, \dim_{\mathbb{C}} G_1^\wedge\}.$$

Moreover, let R be the radical of G. Then

$$\dim_{\mathbb{C}} G_1^\wedge \vee (2 \wedge \dim_{\mathbb{C}}(R_1^\wedge/G)) \le \mathrm{sr}(C^*(G)) \le (2 \vee \dim_{\mathbb{C}} G_1^\wedge) \wedge (\dim R \vee 1),$$

where $R_1^\wedge/G$ means the adjoint orbit space of $R_1^\wedge$ by G. Furethermore, if R is commutative, then $\mathrm{sr}(C^*(G)) = \dim_{\mathbb{C}} G_1^\wedge \vee (2 \wedge \dim_{\mathbb{C}}(R_1^\wedge/G))$.

Remark. Note that we have the following short exact sequence:

$$0 \to \mathfrak{I}_G \to C^*(G) \to C_0(G_1^\wedge) \to 0$$

for G an amenable Lie group, where $\mathfrak{I}_G$ is the closed ideal corresponding to the open subspace $G^\wedge \setminus G_1^\wedge$.

Example 1.8. Let $G = \mathbb{R}^n \rtimes_\alpha \mathrm{Spin}(n)$ be the simply connected motion groups, where $\mathrm{Spin}(n)$ is the universal covering group of $SO(n)$. Then we have the following short exact sequence:

$$0 \to C_0(\mathbb{R}_+) \otimes C^*(\mathrm{Spin}(n)_\chi) \otimes \mathbb{K} \to C^*(G) \to C^*(\mathrm{Spin}(n)) \to 0,$$

where $\mathrm{Spin}(n)_\chi$ is the stabilizer of a non trivial representation χ of $\mathbb{R}^n$, and we obtain $\mathrm{sr}(C^*(G)) = 1 = \dim_{\mathbb{C}} G_1^\wedge$, where $G_1^\wedge$ consists of only the trivial representation. See [26] for more other variations.

Non-amenable Lie group C^*-algebras of type I

Theorem 1.9. [16] Let G be a noncompact connected real semi-simple Lie group. Then

$$\mathrm{sr}(C_r^*(G)) = \min\{2, \mathrm{rr}(G)\},$$

where $\mathrm{rr}(G)$ means the real rank of G defined by the real dimension of A for $G = KAN$ the Iwasawa decomposition.

Example 1.10. In particular, $\mathrm{sr}(C_r^*(SL_2(\mathbb{R}))) = 1$, and $\mathrm{sr}(C_r^*(SL_n(\mathbb{R}))) = 2$ $(n \geq 3)$ since we have $\mathrm{rr}(SL_n(\mathbb{R})) = n - 1$.

Moreover,

Theorem 1.11. [17] (and [18]) Let G be a non-amenable connected real reductive Lie group. Then

$$\mathrm{sr}(C_r^*(G)) = 2 \wedge (\mathrm{rr}([G,G]) \vee (\dim Z^\wedge + 1)),$$

where $[G,G]$ is the commutator group of G, and Z is the center of G.

Example 1.12. In particular, $\mathrm{sr}(C_r^*(GL_n(\mathbb{R})_0)) = 2$ $(n \geq 2)$, where $GL_n(\mathbb{R})_0$ is the connected component of $GL_n(\mathbb{R})$ containing the identity matrix.

Furthermore,

Theorem 1.13. [17] Let G be a non-amenable connected Lie group of type I and R its radical. Then

$$\min\{2, \mathrm{rr}(G/R)\} \leq \mathrm{sr}(C_r^*(G)) \leq 2.$$

Theorem 1.14. [19] Let G be a connected Lie group of type I. Then

$$\dim_{\mathbb{C}}(G_1^\wedge \cap G_r^\wedge) \leq \mathrm{sr}(C_r^*(G)) \leq 2 \vee \dim_{\mathbb{C}}(G_1^\wedge \cap G_r^\wedge),$$

where $G_r^\wedge$ is the reduced dual of G.

Remark. Note that if G is amenable, then $G^\wedge = G_r^\wedge$, and if G is non-amenable, then $G_1^\wedge \cap G_r^\wedge = \emptyset$ and $\dim_{\mathbb{C}}(\emptyset) = 0$ since $\dim \emptyset = -1$ by definition.

Solvable Lie group C^*-algebras of non type I

Moreover, for certain semi-direct products containing some solvable Lie groups of non type I such as the Mautner group, we have

Theorem 1.15. [20] Let G be a Lie semi-direct product $\mathbb{C}^n \rtimes \mathbb{R}$. Then

$$\begin{cases} \mathrm{sr}(C^*(G)) = \dim_{\mathbb{C}} G_1^\wedge & \text{if } \dim G_1^\wedge \text{ even,} \\ 2 \vee \dim_{\mathbb{C}} G_1^\wedge \leq \mathrm{sr}(C^*(G)) \leq 1 + \dim_{\mathbb{C}} G_1^\wedge, & \text{if } \dim G_1^\wedge \text{ odd,} \end{cases}$$
$$2 \leq \mathrm{csr}(C^*(G)) \leq 1 + \dim_{\mathbb{C}} G_1^\wedge.$$

Example 1.16. Let M_{2n+1} be the real $(2n+1)$-dimensional generalized Mautner group defined by the semi-direct product $\mathbb{C}^n \rtimes_\alpha \mathbb{R}$ such that $\alpha_t(z) = (e^{2\pi i t\theta_j} z_j)$ for $z = (z_j) \in \mathbb{C}^n$ and $t \in \mathbb{R}$. Then $C^*(M_{2n+1})$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^K$ with $K = 1 + \sum_{k=1}^n {}_nC_{n-k+1}$ such that $\mathfrak{I}_K/\mathfrak{I}_{K-1} \cong C_0(\mathbb{R})$ and

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong C_0(\mathbb{R}_+^{n-l+1}) \otimes \mathfrak{A}_{\Theta_{n-l+1}} \otimes \mathbb{K}$$

for $1 + \sum_{k=1}^{l-1} {}_nC_{n-k+1} \leq j \leq \sum_{k=1}^{l} {}_nC_{n-k+1}$ $(1 \leq l \leq n)$ and $\mathfrak{A}_{\Theta_k}$ means the noncommutative k-torus, and

$$\mathrm{sr}(C^*(M_{2n+1})) = 2 = 1 + \dim_{\mathbb{C}} C^*(M_{2n+1})_1^\wedge,$$
$$\mathrm{csr}(C^*(M_{2n+1})) = 2.$$

Corollary 1.17. [20] Let G be a Lie semi-direct product $\mathbb{R}^n \rtimes \mathbb{R}$. Then the stable and connected stable ranks of $C^*(G)$ are estimated as in Theorem 1.15.

Moreover, for another series of simply connected solvable Lie groups,

Theorem 1.18. [22] Let D_{6n+1} be the real $(6n+1)$-dimensional generalized Dixmier group defined by the semi-direct product $\mathbb{C}^{2n} \rtimes_\beta H_{2n+1}$ by the generalized Heisenberg Lie group H_{2n+1} such that $\beta_g(z,w) = (e^{ia_j}z_j, e^{ib_j}w_j)$ for $z = (z_j), w = (w_j) \in \mathbb{C}^n$ and $g = (c,(b_j),(a_j)) \in H_{2n+1}$. Then

$$\mathrm{sr}(C^*(D_{6n+1})) = n+1 = \dim_{\mathbb{C}}(D_{6n+1})_1^\wedge,$$
$$2 \leq \mathrm{csr}(C^*(D_{6n+1})) \leq n+1.$$

Remark. More other variations of the theorem above are obtained in [22]. As a note, let Γ be the discrete central subgroup of H_{2n+1} (and D_{6n+1}) consisting of $(2\pi k,(0),(0)) \in H_{2n+1}$ for $k \in \mathbb{Z}$. Then $C^*(D_{6n+1}/\Gamma)$ is of type I while $C^*(D_{6n+1})$ is of non type I, which generalizes the fact by Dixmier that D_7/Γ is of type I but its universal cover D_7 is of non type I.

2 The stable rank of disconnected Lie group C^*-algebras

Let $\lceil x \rceil$ means the least integer $\geq x$, and $M_n(\mathfrak{A})$ be the $n \times n$ matrix algebra over a C^*-algebra $\mathfrak{A}$. Then (F4) ([13] and [14]):

$$\begin{cases} \mathrm{sr}(M_n(\mathfrak{A})) = \lceil (\mathrm{sr}(\mathfrak{A})-1)/n \rceil + 1, \\ \mathrm{csr}(M_n(\mathfrak{A})) \leq \lceil (\mathrm{csr}(\mathfrak{A})-1)/n \rceil + 1. \end{cases}$$

Table 2: **Classes and examples of disconnected Lie groups**

Classes	Examples
Commutative	$\mathbb{R}^n \times \mathbb{T}^m \times \mathbb{Z}^k \times \mathbb{Z}_l$
Nilpotent	Connected nilpotent by $\mathbb{Z}^k$ or $\mathbb{Z}_l$
Solvable	Connected solvable by $\mathbb{Z}^k$ or $\mathbb{Z}_l$
Amenable	Connected amenable by $\mathbb{Z}^k$ or $\mathbb{Z}_l$
Non-amenable	Connected non-amenable by $\mathbb{Z}^k$ or $\mathbb{Z}_l$
More general	Ext of Discrete by Connected Lie

where "Connected $\cdots$ by $\mathbb{Z}^k$ or $\mathbb{Z}_l$" mean semi-direct products of connected $\cdots$ groups by actions of $\mathbb{Z}^k$ or $\mathbb{Z}_l$, and "Ext of Discrete by Connected Lie" means extensions of discrete groups by connected Lie groups.

Disconnected solvable Lie group C^*-algebras

Theorem 2.1. [23] Let $G = \mathbb{C}^n \rtimes_\alpha \mathbb{Z}$ be the semi-direct product of $\mathbb{C}^n$ by an action α of $\mathbb{Z}$. Then there exists a finite composition series $\{\mathfrak{I}_j\}_{j=1}^r$ of $C^*(G)$ such that

$$\mathfrak{I}_r/\mathfrak{I}_{r-1} \cong C_0(G_1^\wedge) = C_0(\mathbb{C}^g \times \mathbb{T})$$

for $g \geq 0$ and

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong C_0(X_j/\mathbb{Z}) \otimes \mathbb{K} \text{ or } C_0(\mathbb{R}^{2g_0+u_j}) \otimes (C(\mathbb{T}^{u_j}) \rtimes_{\Theta_j} \mathbb{Z})$$

for $1 \leq j \leq r-1$, where $u_{j-1} \geq u_j$, $\dim X_{j-1} \geq \dim X_j$, and $X_j/\mathbb{Z}$ means the orbit space for a certain locally compact Hausdorff space X_j by $\mathbb{Z}$, and the action Θ_j of $\mathbb{Z}$ is a multi-rotation. Furthermore, by using this structure

$$2 \vee \dim_{\mathbb{C}} G_1^{\wedge} \vee \max_j\{\lceil (g_0 + u_j)/p_j \rceil + 1\} \leq$$
$$\mathrm{sr}(C^*(G)) \leq (1 + \dim_{\mathbb{C}} G_1^{\wedge}) \vee \max_j\{\lceil (g_0 + u_j + 1)/p_j \rceil + 1\},$$
$$\mathrm{csr}(C^*(G)) \leq (1 + \dim_{\mathbb{C}} G_1^{\wedge}) \vee \max_j\{\lceil (g_0 + u_j + 1)/p_j \rceil + 1\},$$

where p_j means the period of Θ_j when it is a rational rotation. If all the periods p_j are large enough, the same estimates as in Theorem 1.15 hold.

Theorem 2.2. [24] Let D_{4n}^d be the (real $4n$-dimensional) generalized disconnected Dixmier group defined by the semi-direct product $\mathbb{C}^{2n} \rtimes_\beta H_{2n+1}^{\mathbb{Z}}$ such that $\beta_g(z, w) = (e^{iu_j} z_j, e^{iv_j} w_j)$ for $z = (z_j), w = (w_j) \in \mathbb{C}^n$ and $g = (w, (v_j), (u_j)) \in H_{2n+1}^{\mathbb{Z}}$. Then

$$\mathrm{sr}(C^*(D_{4n}^d)) = n + 1 = \dim_{\mathbb{C}}(D_{4n}^d)_1^{\wedge},$$
$$2 \leq \mathrm{csr}(C^*(D_{4n}^d)) \leq n + 1.$$

Furthermore,

Theorem 2.3. [25] Let $M_{2n,m}^d$ be the (real $2n$-dimensional) generalized disconnected Mautner group defined by the semi-direct product $\mathbb{C}^n \rtimes_\alpha \mathbb{Z}^m$ such that $\alpha_t(z) = (e^{i\sum_{k=1}^m c_{jk} t_k} z_j)$ for $z = (z_j) \in \mathbb{C}^n$, $t = (t_j) \in \mathbb{Z}^m$ and $c_{jk} \in \mathbb{R}$. Then there exists a finite composition series $\{\mathfrak{I}_j\}_{j=1}^N$ of $C^*(M_{2n,m}^d)$ such that

$$\mathfrak{I}_N/\mathfrak{I}_{N-1} \cong C_0(\mathbb{C}^l \times \mathbb{T}^m),$$
$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong C_0(\mathbb{C}^l \times \mathbb{R}^{k_j} \times \mathbb{T}^{m_j}) \otimes (C(\mathbb{T}^{k_j'}) \rtimes_{\hat{\alpha}} \mathbb{Z}^{m-m_j'})$$

for $1 \leq j \leq N-1$, and for some $0 \leq l \leq n$ and $1 \leq k_j \leq k_{j-1} \leq n$ and $0 \leq m_j' \leq m_j \leq m-1$ and $m_j = k_j - k_j' + m_j'$.
Furthermore, by this structure

$$2 \vee \dim_{\mathbb{C}}(M_{2n,m}^d)_1^{\wedge} \vee \sup_{s\in M}\lceil [(2l + k_s + i_s + m_s + k_s')/2]/P_s \rceil + 1 \leq$$
$$\mathrm{sr}(C^*(M_{2n,m}^d)) \leq \begin{cases} 2 \vee \dim_{\mathbb{C}}(M_{2n,m}^d)_1^{\wedge} \vee W & \text{if } \dim_{\mathbb{C}}(M_{2n,m}^d)_1^{\wedge} \text{ even,} \\ (\dim_{\mathbb{C}}(M_{2n,m}^d)_1^{\wedge} + 1) \vee W & \text{if } \dim_{\mathbb{C}}(M_{2n,m}^d)_1^{\wedge} \text{ odd,} \end{cases}$$
$$2 \leq \mathrm{csr}(C^*(M_{2n,m}^d)) \leq ([(1 + \dim_{\mathbb{C}}(M_{2n,m}^d)_1^{\wedge})/2] + 1) \vee W,$$

where

$$W = \sup_{s\in M}\lceil [(2l + k_s + i_s + m_s + k_s' + 1)/2]/P_s \rceil + 1$$

for $s \in M$ corresponds to a homogeneous $\mathfrak{I}_s/\mathfrak{I}_{s-1}$; $0 \leq i_s \leq m - m_s'$; P_s is the product of the periods associated with $\mathfrak{I}_s/\mathfrak{I}_{s-1}$ $(s \in M)$.
If $C^*(M_{2n,m}^d)$ has no homogeneous subquotients $\mathfrak{I}_s/\mathfrak{I}_{s-1}$, the same estimates as in Theorem 1.15 hold.

Remark. Theorems 2.1 and 2.3 suggests that it is much more complicated to estimate the stable rank of disconnected Lie group C^*-algebras than connected ones since homogeneous subquotients are involved in general.

3 The stable rank of discrete group C^*-algebras

First recall that

Table 3: **Classes and examples of discrete groups**

Classes	Examples
Commutative	$\mathbb{Z}^n$, $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, $\mathbb{Z}^n \times \mathbb{Z}_m$
Nilpotent	Heisenberg discrete group
Solvable	Discrete $ax+b$ group
	Discrete Mautner and Dixmier groups
Amenable	Ext of Finite by Solvable
Non-amenable and non (T)	Free groups F_n, An amalgam $SL_2(\mathbb{Z})$
Non-amenable and (T)	Non amalgams $SL_n(\mathbb{Z})$ $(n \geq 3)$
More general	Ext of Non-amenable by Solvable

where "Ext of Finite by Solvable" means extensions of finite groups by solvable discrete groups, and "Ext of non-amenable by Solvable" means extensions of non-amenable discrete groups by solvable ones, and "(T)" means Kazhdan's property T.

Nilpotent discrete group C^*-algebras

Theorem 3.1. Let Γ be a finitely generated commutative discrete group. Then

$$\mathrm{sr}(C^*(\Gamma)) = [n/2] + 1 = \dim_{\mathbb{C}} \Gamma_1^\wedge,$$

where n is the free rank of Γ, that is, $\Gamma \cong \mathbb{Z}^n \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ for some $n_1, \cdots n_k \geq 2$ for $k \geq 0$.

Proof. Note that $C^*(\Gamma) \cong C(\mathbb{T}^n) \otimes C^*(\mathbb{Z}_{n_1}) \otimes \cdots \otimes C^*(\mathbb{Z}_{n_k})$, and $C^*(\mathbb{Z}_{n_j}) \cong \mathbb{C}^{n_j}$ for $1 \leq j \leq k$. □

Theorem 3.2. [7] Let $H_{2n+1}^{\mathbb{Z}}$ be the discrete Heisenberg group of rank $2n+1$ defined by $\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$, where $\alpha_u(w, v) = (w + \sum_{j=1}^n u_j v_j, v)$ for $w \in \mathbb{Z}$, $v = (v_j), u = (u_j) \in \mathbb{Z}^n$. Then

$$\mathrm{sr}(C^*(H_{2n+1}^{\mathbb{Z}})) = n + 1 = \dim_{\mathbb{C}} \Gamma_1^\wedge.$$

Moreover, Ng and the author have obtained

Theorem 3.3. [8] Let Γ be a finitely generated, torsion free, two-step nilpotent group. Then we obtain

$$\mathrm{sr}(C^*(\Gamma)) = \mathrm{sr}(C(\Gamma_1^\wedge)) = \dim_{\mathbb{C}} \Gamma_1^\wedge.$$

For the proofs of the theorems above, we used (a version of) the following:

Theorem 3.4. [29] Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ be a continuous field C^*-algebra on a locally compact, paracompact Hausdorff space X with $\mathfrak{A}_t$ fiebers. Then

$$\mathrm{sr}(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})) \leq \sup_{t\in X} \mathrm{sr}(C_0(X, \mathfrak{A}_t))$$
$$\mathrm{csr}(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})) \leq \sup_{t\in X} \max\{\mathrm{csr}(C_0(X, \mathfrak{A}_t)), \mathrm{sr}(C_0(X, \mathfrak{A}_t))\},$$

where $C_0(X, \mathfrak{A}_t)$ means the C^*-algebra of continuous $\mathfrak{A}_t$-valued functions on X vanishing at infinity.

Solvable discrete group C^*-algebras

Furthermore, some (non-nilpotent) solvable discrete groups of semi-direct products are considered as follows:

Theorem 3.5. [27] Let Γ_{n+1} be the generalized discrete $ax+b$ group of rank $n+1$ defined by $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$, where $\alpha_t(s) = ((e^{\pi i t} s_j))$ for $s = (s_j) \in \mathbb{Z}^n$, $t \in \mathbb{Z}$. Then we have $2 \leq \mathrm{csr}(C^*(\Gamma_{n+1})) \leq \lceil[(n+1)/2]/2\rceil + 1$, and

$$\mathrm{sr}(C_0(\mathbb{R}^n, M_2(\mathbb{C}))) = \lceil[n/2]/2\rceil + 1 \leq \mathrm{sr}(C^*(\Gamma_{n+1})) \leq \lceil[(n+1)/2]/2\rceil + 1.$$

Moreover, let $M_{2n}^{\mathbb{Z}}$ be the generalized discrete Mautner group of rank $2n$ defined by $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}^n$, where $\alpha_t(s) = ((e^{\pi i t_j} s_j))$ for $s = (s_j), t = (t_j) \in \mathbb{Z}^n$. Then we have $2 \leq \mathrm{csr}(C^*(M_{2n}^{\mathbb{Z}})) \leq [(n+1)/2] + 1$, and

$$\mathrm{sr}(C(\mathbb{T}^n)) = [n/2] + 1 \leq \mathrm{sr}(C^*(M_{2n}^{\mathbb{Z}})) \leq [(n+1)/2] + 1.$$

Non-amenable discrete group C^*-algebras

On the other hand, for non-amenable discrete groups such as free groups,

Theorem 3.6. [5] (Dykema-Haagerup-Rørdam) Let Γ be a free product of discrete groups Γ_1, Γ_2 with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$. Then the reduced group $C_r^*(\Gamma)$ has stable rank one.

Theorem 3.7. [4] (Dykema-De la Harpe) Let Γ be a hyperbolic group such that it is either torsion free and non-elementary or is a cocompact lattice in a real noncompact simple connected Lie group of real rank one and with trivial center. Then $C_r^*(\Gamma)$ has stable rank one.
Also, let Γ be an amalgam over a finite subgroup H such that $g^{-1}Hg \cap H = \{1\}$ (the identity) for some $g \in \Gamma$. Then $C_r^*(\Gamma)$ has stable rank one.

Furthermore, for discrete groups with the property T, we obtain

Theorem 3.8. [28] Let Γ be a residually finite, countable discrete group with Kazhdan's property T. Then $\mathrm{sr}(C_r^*(\Gamma)) = 1$ and $\mathrm{csr}(C_r^*(\Gamma)) \leq 2$.

Corollary 3.9. [28] Let Γ be either $SL_n(\mathbb{Z})$, $PSL_n(\mathbb{Z})$, $\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})$ for $n \geq 3$, or $Sp(n,1)_{\mathbb{Z}}$ for $n \geq 2$. Then $\mathrm{sr}(C_r^*(\Gamma)) = 1$ and $\mathrm{csr}(C_r^*(\Gamma)) \leq 2$.

Table 4: **Examples of groups with or without (T)**

Classes	Examples
Groups with (T)	Compact groups, $Sp(n,1)$, $Sp(n,1)_{\mathbb{Z}}$ $(n \geq 2)$ $SL_n(\mathbb{R})$, $SL_n(\mathbb{Z})$, $PSL_n(\mathbb{Z})$ $(n \geq 3)$ $\mathbb{R}^n \rtimes SL_n(\mathbb{R})$, $\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})$ $(n \geq 3)$
Groups without (T)	Noncompact amenable groups $SO_0(n,1)$, $SU(n,1)$ $(n \geq 2)$ $SL_2(\mathbb{R})$, $SL_2(\mathbb{Z})$, $PSL_2(\mathbb{Z})$, F_n

where the actions of $\mathbb{R}^n \rtimes SL_n(\mathbb{R})$, $\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})$ $(n \geq 3)$ are defined by matrix multiplication on $\mathbb{R}^n$, $\mathbb{Z}^n$ by $SL_n(\mathbb{R})$, $SL_n(\mathbb{Z})$ respectively, and $SO_0(n,1)$ is the connected component of $SO(n,1)$ containing the identity matrix.

4 Beyond group C^*-algebras

Table 5: **Classes and examples of C^*-algebras**

Classes	Examples
Nuclear	Connected or amenable group C^*-algebras Inductive limits of type I C^*-algebras Noncommutative tori $\mathfrak{A}_\Theta$, Cuntz algebras O_n
Non nuclear	$C_r^*(F_n)$, $C_r^*(SL_n(\mathbb{Z}))$ $(n \geq 2)$, $\mathbb{B}(H)$
Quasidiagonal	Commutative, AF, $\mathfrak{A}_\Theta$, RFD, $C^*(F_n)$

where note that the reduced group C^*-algebras of non-amenable discrete groups are non-nuclear (see [1, Section 15.8]), and $\mathbb{B}(H)$ is the C^*-algebra of bounded operators on a separable infinite dimensional Hilbert space H, and a C^*-algebra $\mathfrak{A}$ is quasidiagonal if it has a faithful representation whose image is a quasidiagonal set of operators, that is, there exists an increasing sequence $\{P_j\}$ of finite rank projections such that $\|[a, P_j]\| \to 0$ $(n \to \infty)$ for any $a \in \mathfrak{A}$, and $\{P_j\}$ converges to the identify operator in strong operator topology, and "AF" means inductive limits of finite dimensional C^*-algebras, and "RFD" means residually finite dimensional C^*-algebras.

Type I C^*-algebras

Theorem 4.1. [30] Let $\mathfrak{A}$ be a C^*-algebra of type I. Then

$$\sup_{1 \leq n < \infty} \mathrm{sr}(C_0(\mathfrak{A}_n^\wedge, M_n(\mathbb{C}))) \leq$$
$$\mathrm{sr}(\mathfrak{A}) \leq 2 \vee \sup_{1 \leq n < \infty} \{\mathrm{sr}(C_0(\mathfrak{A}_n^\wedge, M_n(\mathbb{C}))) \vee \mathrm{csr}(C_0(\mathfrak{A}_n^\wedge, M_n(\mathbb{C})))\},$$

and

$$\mathrm{csr}(\mathfrak{A}) \leq 2 \vee \sup_{1 \leq n < \infty} \{\mathrm{sr}(C_0(\mathfrak{A}_n^\wedge, M_n(\mathbb{C}))) \vee \mathrm{csr}(C_0(\mathfrak{A}_n^\wedge, M_n(\mathbb{C})))\},$$

where $\mathfrak{A}_n^\wedge$ means the subspace of n-dimensional irreducible representations in the spectrum of $\mathfrak{A}$.

Remark. The first estimate from below is true for any C^*-algebra.

Corollary 4.2. [30] Let $\mathfrak{A}$ be a C^*-algebra of type I. Then

$$\sup_{1 \leq n < \infty} \lceil [\dim \mathfrak{A}_n^\wedge / 2] / n \rceil + 1 \leq \mathrm{sr}(\mathfrak{A}) \leq 2 \vee \sup_{1 \leq n < \infty} \lceil [(1 + \dim \mathfrak{A}_n^\wedge)/2]/n \rceil + 1,$$

and

$$\mathrm{csr}(\mathfrak{A}) \leq 2 \vee \sup_{1 \leq n < \infty} \lceil [(1 + \dim \mathfrak{A}_n^\wedge)/2]/n \rceil + 1.$$

Theorem 4.3. [30] Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras of type I. Then

$$\mathrm{sr}(\mathfrak{A} \otimes \mathfrak{B}) \leq \mathrm{sr}(\mathfrak{A}) + \mathrm{sr}(\mathfrak{B}).$$

We also have

$$\mathrm{csr}(\mathfrak{A} \otimes \mathfrak{B}) \leq \mathrm{sr}(\mathfrak{A}) + \mathrm{sr}(\mathfrak{B}).$$

Remark. The first estimate says that the product formula for the stable rank holds in the case of type I C^*-algebras. As a key step for the proof, it is shown that the same estimate holds for liminal C^*-algebras.

Theorem 4.4. [30] Let $\mathfrak{A}, \mathfrak{B}$ be inductive limits of type I C^*-algebras $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$, $\{\mathfrak{B}_\mu\}_{\mu\in M}$ respectively. Then

$$\mathrm{sr}(\mathfrak{A}\otimes\mathfrak{B}) \le \sup_{\lambda\in\Lambda}\mathrm{sr}(\mathfrak{A}_\lambda) + \sup_{\mu\in M}\mathrm{sr}(\mathfrak{B}_\mu),$$

and

$$\mathrm{csr}(\mathfrak{A}\otimes\mathfrak{B}) \le \sup_{\lambda\in\Lambda}\mathrm{sr}(\mathfrak{A}_\lambda) + \sup_{\mu\in M}\mathrm{sr}(\mathfrak{B}_\mu).$$

Remark. Unfortunately, we just know $\sup_{\lambda\in\Lambda}\mathrm{sr}(\mathfrak{A}_\lambda) \ge \mathrm{sr}(\mathfrak{A})$ in general ([13]).

RFD C^*-algebras

Theorem 4.5. [28] Let $\mathfrak{A}$ be a redidually finite dimensional (RFD) C^*-algebra with a continuous separating family $\{\pi_t\}_{t\in X}$ of finite dimensional irreducible representations of $\mathfrak{A}$ for X a locally compact Hausdorff space. Then

$$\mathrm{sr}(\mathfrak{A}) = \sup_{t\in X}\mathrm{sr}(C_0(\mathfrak{A}^\wedge_{n_t}, \pi_t(\mathfrak{A}))) = \sup_{t\in X}\lceil[\dim\mathfrak{A}^\wedge_{n_t}/2]/n_t\rceil + 1,$$

where $\pi_t(\mathfrak{A}) = M_{n_t}(\mathbb{C})$ and $\mathfrak{A}^\wedge_{n_t}$ is the subspace of n_t-dimensional irreducible representations in the spectrum of $\mathfrak{A}$.

Corollary 4.6. [28] Let $\mathfrak{A}, \mathfrak{B}$ be RFD C^*-algebras. Suppose that the C^*-tensor product $\mathfrak{A}\otimes\mathfrak{B}$ with a suitable C^*-norm is RFD by a continuous separating family of its finite dimensional irreducible representations on a locally compact Hausdorff space. Then

$$\mathrm{sr}(\mathfrak{A}\otimes\mathfrak{B}) \le \mathrm{sr}(\mathfrak{A}) + \mathrm{sr}(\mathfrak{B}).$$

Quasidiagonal C^*-algebras

It is shown by [2, Proposition 9.8] that

Proposition 4.7. A C^*-algebra $\mathfrak{A}$ is nuclear and quasidiagonal if and only if it is isomorphic to an inductive limit of nuclear RFD C^*-algebras.

Using the proposition above and Theorem 4.5 we have

Theorem 4.8. Let $\mathfrak{A} = \varinjlim \mathfrak{B}_j$ be a quasidiagonal nuclear C^*-algebra that is isomorphic to an inductive limit of nuclear RFD C^*-algebras $\mathfrak{B}_j$. Then

$$\mathrm{sr}(\mathfrak{A}) \le \underline{\lim}\,\mathrm{sr}(\mathfrak{B}_j) = \underline{\lim}\sup_{t\in X_j}\lceil[\dim\mathfrak{B}^\wedge_{j,n_t}/2]/n_t\rceil + 1,$$

where each X_j is the base space of a continuous separating family of irreducible finite dimensional representations of $\mathfrak{B}_j$, and $\mathfrak{B}^\wedge_{j,n_t}$ is the subspace of the spectrum of $\mathfrak{B}_j$ consisting of n_t-dimensional irreducible representations.

Morevoer, it is shown by [2, Proposition 9.3] that

Proposition 4.9. A C^*-algebra $\mathfrak{A}$ is MF if and only if it is isomorphic to an inductive limit of RFD C^*-algebras, where MF $\mathfrak{A}$ is isomorphic to a C^*-subalgebra of the quotient $\Pi M_{n_j}(\mathbb{C})/\sum M_{n_j}(\mathbb{C})$ for some sequence $\{n_j\}_{j=1}^\infty$.

Using the proposition above and Theorem 4.5 we have

Theorem 4.10. Let $\mathfrak{A} = \varinjlim \mathfrak{B}_j$ be a MF C^*-algebra that is isomorphic to an inductive limit of RFD C^*-algebras $\mathfrak{B}_j$. Then

$$\mathrm{sr}(\mathfrak{A}) \leq \underline{\lim}\, \mathrm{sr}(\mathfrak{B}_j) = \underline{\lim} \sup_{t \in X_j} \lceil [\dim \mathfrak{B}_{j,n_t}^{\wedge}/2]/n_t \rceil + 1,$$

where X_j and $\mathfrak{B}_{j,n_t}^{\wedge}$ mean the same as in Theorem 4.8.

Crossed products of C^*-algebras

Theorem 4.11. [13] Let $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ be the crossed product of a C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{Z}$. Then

$$\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \leq \mathrm{sr}(\mathfrak{A}) + 1.$$

If $\mathfrak{A}$ is unital, then

$$\mathrm{csr}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \leq \mathrm{sr}(\mathfrak{A}) + 1.$$

Remark. A noncommutative n-torus $\mathfrak{A}_\Theta$ generated by n unitaries U_j with the relations $U_jU_i = e^{2\pi\theta_{ij}}U_iU_j$ for some $\theta_{ij} \in \mathbb{R}$ $(1 \leq i \leq j \leq n)$ can be written as a successive crossed product by $\mathbb{Z}$: $C(\mathbb{T}) \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$ ($n-1$ times).

Theorem 4.12. [32] Let $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ be the crossed product of a C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{R}$. Then

$$\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{R}) \leq \mathrm{sr}(\mathfrak{A}) + 1.$$

We also have

$$\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{T}) \leq \mathrm{sr}(\mathfrak{A}) + 1.$$

Corollary 4.13. [32] Let G be a simply connected solvable Lie group. Then $\mathrm{sr}(C^*(G)) \leq \dim G$.

Remark. It is known that G can be written as a successive semi-direct product by $\mathbb{R}$, that is, $G \cong \mathbb{R} \rtimes \mathbb{R} \cdots \rtimes \mathbb{R}$ ($\dim G - 1$ times). Thus, $C^*(G) \cong C_0(\mathbb{R}) \rtimes \mathbb{R} \cdots \rtimes \mathbb{R}$ a successive crossed product by $\mathbb{R}$ ($\dim G - 1$ times).

C^*-algebras with stable rank infinite

Proposition 4.14. [13] Let $\mathfrak{A}$ be a unital C^*-algebra. If $\mathfrak{A}$ contains two isometries with their ranges orthogonal, then $\mathrm{sr}(\mathfrak{A}) = \infty$.

Example 4.15. [13] The Cuntz algebras O_n generated by orthogonal n isometries $(2 \leq n \leq \infty)$, $\mathbb{B}(H)$ with $\dim H = \infty$, and $C^*(F_n)$ have stable rank ∞.

Furthermore,

Theorem 4.16. [31] Let $\mathfrak{A}$ be a σ-unital C^*-algebra and $M(\mathfrak{A})$ its multiplier algebra. If $\mathfrak{A}$ has a stable quotient, then $\mathrm{sr}(M(\mathfrak{A})) = \infty$ and $\mathrm{csr}(M(\mathfrak{A})) = \infty$.
In particular, $\mathrm{sr}(M(\mathfrak{A} \otimes \mathbb{K})) = \infty$ and $\mathrm{csr}(M(\mathfrak{A} \otimes \mathbb{K})) = \infty$.

Remark. For this theorem we can take $\mathfrak{A}$ as the group C^*-algebra of a connected nilpotent Lie group or a connected semi-simple Lie group. Furthermore, we can take $\mathfrak{A}$ as $C^*(M_{2n+1})$ or $C^*(D_{6n+1})$ as in Section 1.

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Complementaries of Greek Means with Respect to Gini Means

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Abstract

A mean P is called invariant with respect to a pair of means (M,N) if it is a solution of the generalized Gauss' functional equation

$$f(M(a,b),N(a,b)) = f(a,b), \quad a,b>0.$$

Equivalently N is called complementary of M with respect to P. Determining the complementary of a mean with respect to another mean gives the possibility of definition of a double sequence with known limit. We study the complementary of Greek means with respect to weighted Gini means in the family of Greek means.

Keywords: Greek means, complementary means, Gini means.

Mathematics Subject Classification: 26E60.

1. GREEK MEANS

Using the method of proportions, the Pythagorean school defined ten means. They are the arithmetic mean *A*, the geometric mean *G*, the harmonic mean *H*, the contraharmonic mean *C*, and six unnamed means F_i, $i = 5,\ldots,10$. For $a > b > 0$, they are given, in order, by the following expressions:

$$A(a,b) = \frac{a+b}{2}; \quad G(a,b) = \sqrt{ab}; \quad H(a,b) = \frac{2ab}{a+b};$$

$$C(a,b) = \frac{a^2+b^2}{a+b}; \quad F_5(a,b) = \frac{a-b+\sqrt{(a-b)^2+4b^2}}{2};$$

$$F_6(a,b) = \frac{b-a+\sqrt{(a-b)^2+4a^2}}{2}; \quad F_7(a,b) = \frac{a^2-ab+b^2}{a};$$

$$F_8(a,b) = \frac{a^2}{2a-b}; \quad F_9(a,b) = \frac{b(2a-b)}{a}; \quad F_{10}(a,b) = \frac{b+\sqrt{b(4a-3b)}}{2}.$$

We have to replace *a* with *b* to define the means on $0 < a < b$. We can denote also

$$A = F_1, \quad G = F_2, \quad H = F_3 \quad \text{and} \quad C = F_4.$$

More details on the Greek means can be found in [Toader, G., Toader, S., 2005].

As an abstract definition of means (on $\mathbf{R}_+$), usually is given the following

Definition 1. *A* ***mean*** *is a function* $M : \mathbf{R}_+^2 \to \mathbf{R}_+$, *which has the property*

$$\min(a,b) \le M(a,b) \le \max(a,b), \ \forall\ a,b > 0.$$

A mean can have also some additional properties.

Definition 2. *The mean M is called:*

a) ***symmetric*** *if*

$$M(a,b) = M(b,a),\ \forall\ a,b > 0;$$

b) ***homogeneous*** *(of degree one) if*

$$M(ta,tb) = t \cdot M(a,b),\ \forall\ t,a,b > 0.$$

Each Greek mean is a homogeneous symmetric mean. In what follows we use weighted Gini means defined by

$$B_{r,s;\lambda}(a,b) = \left[\frac{\lambda \cdot a^r + (1-\lambda)\cdot b^r}{\lambda \cdot a^s + (1-\lambda)\cdot b^s}\right]^{\frac{1}{r-s}},\ r \neq s,$$

with $\lambda \in [0,1]$ fixed. We can remark that $G_\lambda = B_{r,-r;\lambda}$ is the weighted geometric mean. For $\lambda = 1/2$ we get symmetric means which we denote by $B_{r,s}$.

2. COMPLEMENTARY MEANS

Given three means *M, N* and *P*, their **composition**

$$P(M,N)(a,b) = P(M(a,b),N(a,b)),\ \forall\ a,b > 0,$$

defines also a mean $P(M,N)$.

Definition 3. *A mean P is called* (M,N)***-invariant*** *if it verifies*

$$P(M,N) = P.$$

In this case the mean N is called ***complementary to M with respect to P*** *(or* ***P-complementary*** *to M). If a given mean M has a unique P-complementary mean N, denote it by* $N = M^P$.

Of course, $M^M = M$ for every mean *M*. We call it as **trivial case** of complementariness.

The famous arithmetic-geometric mean of Gauss is (A,G)-invariant. It is well known (see [Borwein, J.M., Borwein, P.B., 1986]) that it was very difficult to be determined. That is why we prefer to study the (equivalent notion of) complementary of a mean with respect to another.

In [Toader, S., Toader, G., 2006] we have proved the following results.

Theorem 4. *If the means M and N are symmetric and P is* (M,N)*-invariant, then P is also symmetric.*

Theorem 5. *If the symmetric mean P is* (M,N)*-invariant, then it is also* (N,M)*-invariant.*

For instance, if we look after the complementary of a Greek mean with respect to a weighted Gini mean in the set of Greek means, we deduce that the Gini mean must to be symmetric, thus $\lambda = 1/2$.

More comments on this notion and its importance in the determination of the limit of a double sequence can be found in [Borwein, J.M., Borwein, P.B., 1986] or [Toader, G., Toader, S., 2005].

3. SERIES EXPANSION OF MEANS

For the study of some problems related to means in [Lehmer, D.H., 1971] is used the power series expansions of means. Let *M* be a symmetric and homogeneous mean. Without loss of generality we may assume that *M* acts on the positive numbers $a \ge b$ and

$$M(a,b) = aM(1,b/a) = aM(1,1-t),$$

where

$$0 \le t = 1 - b/a < 1.$$

For many problems it suffices to consider only the normalized function $M(1,1-t)$ even if the mean *M* is not symmetric nor homogeneous. In the case of Greek means we get in order (see [Toader, S., Toader, G., 2003]):

$$A(1,1-t) = 1 - \frac{t}{2};\quad G(1,1-t) = 1 - \frac{t}{2} - \frac{t^2}{8} - \frac{t^3}{16} - \frac{5t^4}{128} + \dots;$$

$$H(1,1-t) = 1 - \frac{t}{2} - \frac{t^2}{4} - \frac{t^3}{8} - \frac{t^4}{16} + \dots;$$

$$C(1,1-t) = 1 - \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} + \frac{t^4}{16} + \dots;$$

$$F_5(1,1-t) = 1 - \frac{t}{2} + \frac{t^2}{8} + \frac{t^3}{8} + \frac{95t^4}{128} + \dots;$$

$$F_6(1,1-t) = 1 - \frac{t}{2} + \frac{t^2}{8} - \frac{t^4}{128} + \dots;$$

$$F_7(1,1-t) = 1 - t + t^2;\quad F_8(1,1-t) = 1 - t + t^2 - t^3 + t^4 + \dots;$$

$$F_9(1,1-t) = 1 - t^2;\quad F_{10}(1,1-t) = 1 - t^2 + t^3 - \frac{5t^4}{4} + \dots$$

Remark 6. For some means it is very difficult, or even impossible to determine all the coefficients. In these cases, a recurrence relation for the coefficients will be very useful. It gives a way to calculate as many coefficients as desired. To obtain it, we can apply an old **formula of Euler** presented, together with his history, in [Gould, H.W., 1974].

Theorem 7. If the function *f* has the Taylor series

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot x^n,$$

p is a real number and

$$[f(x)]^p = \sum_{n=0}^{\infty} b_n \cdot x^n,$$

then we have the recurrence relation

$$\sum_{k=0}^{n} [k(p+1) - n] \cdot a_k \cdot b_{n-k} = 0,\ n \ge 0.$$

4. $B_{p,q}$-COMPLEMENTARY OF GREEK MEANS

We study the complementariness with respect to the Gini mean $B_{p,q}$. Denote the $B_{p,q}$-complementary of *M* by $M^{B(p,q)}$. Using Euler's formula, in [Costin, I., 2003] is given the following

Lemma 8. *If the mean M has the series expansion*

$$M(1,1-x)=1+\sum_{n=0}^{\infty}a_n x^n,$$

then the first terms of the series expansion of $M^{B(q,q-r)}$, *for* $r\neq 0$ *are*

$$M^{B(q,q-r)}(1,1-x)=1-(1+a_1)x-[a_1(2q-r-1)(1+a_1)+a_2]x^2$$
$$-[2a_3+2(2a_1+1)(2q-1-r)a_2+2(4q^2+r^2-4qr-4q+2r+1)a_1^3$$
$$+(12q^2-12qr-10q+3r^2+5r+2)a_1^2-(4q^2-4qr-2q+r+r^2)a_1]\frac{x^3}{2}$$
$$+\{-12a_4+12a_3(r-2q+1)(2a_1+1)+12a_2^2(r-2q+1)$$
$$+[12a_1^2(12qr-12q^2+12q-6r-3r^2-3)$$
$$-12a_1(12q^2-12qr-10q+3r^2+5r+2)+24q(r-q)-6(r^2+r-2q)]a_2$$
$$+a_1^4(35r-79q-144rq-82r^2q+12+168rq^2+36r^2-112q^3+144q^2+13r^3)$$
$$+2a_1^3(6+23r-46q-120rq-82r^2q+168rq^2+30r^2+120q^2+13r^3-112q^3)$$
$$+a_1^2(9r-18q-96rq-96r^2q+198rq^2+24r^2+15r^3+96q^2+132q^3)$$
$$+2a_1(r-2q)(r^2-1+5q^2-5rq)\}\frac{x^4}{12}+\ldots$$

Using it we get necessary conditions for $M^{B(q,q-r)}$ to be a given mean *N*, or to belong to a given family of means.

Theorem 9. *We have*

$$F_i^{B(q,q-r)}=F_j,\quad \text{with}\quad i,j\in\{1,2,\ldots,10\},$$

in and only in the following non-trivial cases

$$B_{q,q-r}=A\quad \text{and}\quad (i,j)\in\{(3,4),(7,9)\},$$
$$B_{q,q-r}=G\quad \text{and}\quad (i,j)\in\{(1,3),(8,9)\},$$

or in the equivalent cases given by Theorem 5.

Proof. The first terms of the series expansion of the $B_{q,q-r}$-complementary of the Greek means are:

$$A^{B(q,q-r)}(1,1-x)=1-\frac{x}{2}+\frac{2q-1-r}{4}x^2+\frac{2q-1-r}{4}x^3$$
$$+\frac{x^4}{192}(5r^3+12r^2-5r-32q^2-26qr^2+10q-12+48q^2-48rq+48rq^2)+\ldots;$$

$$G^{B(q,q-r)}(1,1-x)=1-\frac{x}{2}+\frac{4q-1-2r}{8}x^2+\frac{4q-1-2r}{16}x^3$$
$$+\frac{x^4}{192}(5r^3+6r^2-14r-32q^3-26qr^2+28q-10+24q^2-24rq+48rq^2)+\ldots;$$

$$H^{B(q,q-r)}(1,1-x)=1-\frac{x}{2}+\frac{2q-r}{4}x^2+\frac{2q-r}{8}x^3$$

$$+\frac{x^4}{192}(r-2q)(5r^2+16q^2-17-16rq)+\ldots;$$

$$C^{B(q,q-r)}(1,1-x)=1-\frac{x}{2}+\frac{2q-2-r}{4}x^2+\frac{2q-2-r}{8}x^3$$

$$+\frac{x^4}{192}(r-2q)(5r^2+16q^2-16rq+24r-48q+31)+\ldots;$$

$$F_5^{B(q,q-r)}(1,1-x)=1-\frac{x}{2}+\frac{4q-3-2r}{8}x^2+\frac{2q-2-r}{8}x^3$$

$$+\frac{x^4}{192}(5r^3+18r^2+10r-32q^3-26qr^2-20q+86+72q^2-72rq+48rq^2)+\ldots;$$

$$F_6^{B(q,q-r)}(1,1-x)=1-\frac{x}{2}+\frac{4q-3-2r}{8}x^2+\frac{2q-1-r}{8}x^3$$

$$+\frac{x^4}{192}(5r^3+18r^2+10r-32q^3-26qr^2-20q-2+72q^2-72rq+48rq^2)+\ldots;$$

$$F_7^{B(q,q-r)}(1,1-x)=1-x^2+(2q-1-r)x^3-\frac{x^4}{2}(2q-r)(2q-r-1)+\ldots;$$

$$F_8^{B(q,q-r)}(1,1-x)=1-x^2+(2q-r)x^3-\frac{x^4}{2}(2q-r)(2q-r+1)+\ldots;$$

$$F_9^{B(q,q-r)}(1,1-x)=1-x+x^2+(2q-1-r)x^3$$

$$+\frac{x^4}{2}(2q-r-2)(2q-r-1)+\ldots$$

respectively

$$F_{10}^{B(q,q-r)}(1,1-x)=1-x+x^2+(2q-2-r)x^3$$

$$+\frac{x^4}{2}(r^2+5r-10q+8+4q^2-4rq)+\ldots$$

Comparing their first coefficients with those of the series expansion of Greek means, we get the following solutions with the corresponding conclusions.

i	j	q	r	Conclusion
1	1	1	1	$A^A=A$ (trivial case)
1	2	3/4	1	No, for $a=1$, $b=4$
1	3	q	$2q$	$A^G=H$
1	4	3/2	1	No, for $a=1$, $b=9$
2	2	q	$2q$	$G^G=G$ (trivial case)
2	3	1/4	1	No, for $a=1$, $b=120$
2	4	5/4	1	No, for $a=1$, $b=16$
3	3	0	1	$H^H=H$ (trivial case)
3	4	1	1	$H^A=C$
4	4	2	1	$C^C=C$ (trivial case)
7	9	1	1	$F_7^A=F_9$
8	9	q	$2q$	$F_8^G=F_9$

Remark 10. So, using Gini means we get no new case of complementaries than those obtained in [Toader, S., Toader, G., 2004] or [Toader, S., Toader, G., 2006] where power means respectively Lehmer means have been considered.

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

A CLASS OF NEW THREE-PARAMETER GENERALIZED WEIGHTED MEANS

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ABSTRACT

In this paper, a class of new three-parameter generalized weighted mean in n variables is defined. Some elementary properties are listed. Using the generalized Van der Monde determinant, an explicit form of this mean is given.

Keywords: Weighted mean, three-parameter, functional, inequality, Van der Monde determinant.

2000 Mathematics Subject Classification: 26D15.

1 Introduction

The generalized weighted means of the function f with weight p and two parameters r and s are defined in [2] by

$$M_{r,s}(f;p;u,v) = \begin{cases} \left[\dfrac{\int_u^v p(x)f^r(x)dx}{\int_u^v p(x)f^s(x)dx}\right]^{\frac{1}{r-s}}, & (r-s)(u-v)\neq 0; \\ \exp\left\{\dfrac{\int_u^v p(x)f^r(x)\ln f(x)dx}{\int_u^v p(x)f^r(x)dx}\right\}, & r=s, u-v\neq 0; \\ f(x), & r=s, u=v; \end{cases} \tag{1.1}$$

where $u, v, r, s \in \mathbb{R}$, $p \geq 0, f > 0$ integrable functions on the interval $[u, v] \subset \mathbb{R}$.
A generalization of $M_{r,s}(f; p; u, v)$ is stated in [3]:

$$M_{r,s}^{[1]}(f;p) = \begin{cases} \left[\dfrac{\int_E p(x) f^r(x)dx}{\int_E p(x) f^s(x)dx}\right]^{\frac{1}{r-s}}, & r \neq s; \\ \exp\left(\dfrac{\int_E p(x) f^r(x)\ln f(x)dx}{\int_E p(x) f^r(x)dx}\right), & r = s. \end{cases} \tag{1.2}$$

where $x = (x_0, x_1, x_2, \cdots, x_n), x_0 = 1 - \sum_{i=1}^{n} x_i, dx = dx_1 dx_2 \cdots dx_n, f$ be a positive real function and p a nonnegative integrable function on E:

$$E = \{(x_1, x_2, \cdots, x_n) : \textstyle\sum_{i=1}^{n} x_i \leqslant 1, x_i \geqslant 0, i = 1, 2, \cdots, n\}.$$

As an application, authors [3] obtained the so-called first three-parameter mean $E_{r,s,t}^{[1]}(a)$ in n variables:

$$E_{r,s,t}^{[1]}(a) = \begin{cases} \left[\dfrac{\int_E M_t^r(a;x)dx}{\int_E M_t^s(a;x)dx}\right]^{\frac{1}{r-s}}, & t(r-s) \neq 0; \\ \exp\left\{\dfrac{\int_E M_t^r(a;x)\ln M_t(a;x)dx}{\int_E M_t^r(a;x)dx}\right\}, & t \neq 0, r = s; \\ \left[\dfrac{\int_E G^r(a;x)dx}{\int_E G^s(a;x)dx}\right]^{\frac{1}{r-s}}, & t = 0, r \neq s; \\ \exp\left\{\dfrac{\int_E G^r(a;x)\ln G(a;x)dx}{\int_E G^r(a;x)dx}\right\}, & t = 0, r = s \neq 0; \\ \exp\left\{n! \int_E \ln G(a;x)dx\right\}, & t = r = s = 0; \end{cases} \tag{1.3}$$

where $r, s, t \in \mathbb{R}$, $a_k > 0$ for $0 \leqslant k \leqslant 0$, $M_r(a;x) = [a_0^r + \sum_{i=1}^{n}(a_i^r - a_0^r)x_i]^{1/r} = (\sum_{i=0}^{n} a_i^r x_i)^{1/r}$, and $M_0(a;x) = G(a;x) = \prod_{i=0}^{n} a_i^{x_i}$.
In this paper, we will define a new generalized weighted means in n variables for three parameters, and prove its monotonicity. By using the generalized Van der Monde determinant, an explicit form of this mean is given.

2 Definition and Properties

Definition 2.1. Let r, s, t, E, x_0, a_k are defined above. if

$$M_t(a^r; x) = \left[a_0^{rt} + \sum_{i=1}^{n}(a_i^{rt} - a_0^{rt})x_i\right]^{1/t} = \left(\sum_{i=0}^{n} a_i^{rt} x_i\right)^{1/t},$$

and $M_0(a^r; x) = G(a^r; x) = \prod_{i=0}^{n} a_i^{rx_i}$, then the second three-parameter generalized weighted mean values $E_{r,s,t}^{[2]}(a)$ as follows

$$E_{r,s,t}^{[2]}(a) = \begin{cases} \left[\dfrac{\int_E M_t(a^r;x)dx}{\int_E M_t(a^s;x)dx}\right]^{1/(r-s)}, & (r-s)t \neq 0; \\ \exp\left\{\dfrac{\int_E M_t^{1-t}(a^r;x)\left(\sum_{k=1}^{n} a_k^{rt} x_k \ln a_k\right)dx}{\int_E M_t(a^r;x)dx}\right\}, & r = s, t \neq 0; \\ \left[\dfrac{\int_E G(a^r;x)dx}{\int_E G(a^s;x)dx}\right]^{1/(r-s)}, & r \neq s, t = 0; \\ \exp\left\{\dfrac{\int_E G(a^r;x)\ln G(a;x)dt}{\int_E G(a^r;x)dx}\right\}, & r = s, t = 0. \end{cases} \tag{2.1}$$

According to Definition 2.1, we easily obtain the following characteristic properties of $E^{[2]}_{r,s,t}(a)$.

Theorem 2.1. *If $E^{[2]}_{r,s,t}(a)$ be the second three-parameter mean, then*

(a) $E^{[2]}_{r,s,t}(a) = E^{[2]}_{s,r,t}(a)$;

(b) $\lim_{r\to\infty} E^{[2]}_{r,s,t}(a) = \lim_{s\to\infty} E^{[2]}_{r,s,t}(a) = \lim_{t\to\infty} E^{[2]}_{r,s,t}(a) = a_{\max}$;

(c) $\lim_{r\to-\infty} E^{[2]}_{r,s,t}(a) = \lim_{s\to-\infty} E^{[2]}_{s,r,t}(a) = \lim_{t\to-\infty} E^{[2]}_{r,s,t}(a) = a_{\min}$;

(d) $\lim_{r\to s} E^{[2]}_{r,s,t}(a) = E^{[2]}_{r,r,t}(a)$;

(e) $a_{\min} \leqslant E^{[2]}_{r,s,t}(a) \leqslant a_{\max}$;

(f) $E^{[2]}_{r,s,t}(a) = a_0$ *if and only if* $a_0 = a_1 = \cdots = a_n$;

(g) $E^{[2]}_{r,s,t}(ta) = tE^{[2]}_{r,s,t}(a), t > 0,$ *when* $ta = (ta_0, ta_1, \cdots, ta_n)$;

(h) $(E^{[2]}_{r,s,t}(a))^{r-s} = (E^{[2]}_{r,u,t}(a))^{r-u} \cdot (E^{[2]}_{u,s,t}(a))^{u-s}$.

Theorem 2.2. *The second three-parameter generalized weighted mean values $E^{[2]}_{r,s,t}(a)$ are monotone increasing functions with both r and s if $t > 0$.*

Proof. By taking $T(a;r,t) = \int_E M_t(a^r;x)dx$, if $t \neq 0$, then

$$T'_r(a;r,t) = \int_E \left(\sum_{k=0}^n a_k^{rt} x_k\right)^{1/t-1} \left(\sum_{k=0}^n a_k^{rt} x_k \ln a_k\right) dx, \tag{2.2}$$

$$\begin{aligned} T''_r(a;r,t) = \int_E \Bigg[&(1-t)\left(\sum_{k=0}^n x_k^{rt} x_k\right)^{1/t-2} \left(\sum_{k=0}^n x_k^{rt} x_k \ln x_k\right)^2 \\ &+t\left(\sum_{k=0}^n x_k^{rt} x_k\right)^{1/t-1} \left(\sum_{k=0}^n x_k^{rt} x_k \ln^2 x_k\right)\Bigg] dt, \end{aligned} \tag{2.3}$$

by using Cauchy integral inequality, from (2.2) and (2.3), yields

$$\begin{aligned} &T(a;r,t)T''_{rr}(a;r,t) - [T'_r(a;r,t)]^2 \\ &= \int_E \left(\sum_{k=0}^n a_k^{rt} x_k\right)^{1/t} dx \cdot \int_E \left(\sum_{k=0}^n a_k^{rt} x_k\right)^{1/t-2} \left(\sum_{k=0}^n a_k^{rt} x_k \ln x_k\right)^2 dx \\ &- \left[\int_E \left(\sum_{k=0}^n a_k^{rt} x_k\right)^{1/t-1} \left(\sum_{k=0}^n a_k^{rt} t_k \ln a_k\right) dt\right]^2 \\ &+ t\int_E M_t(x^r;x)dx \cdot \int_E \left(\sum_{k=0}^n a_k^{rt} x_k\right)^{1/t-2} \\ &\left[\left(\sum_{k=0}^n a_k^{rt} x_k\right)\left(\sum_{k=0}^n a_k^{rt} x_k \ln^2 a_k\right) - \left(\sum_{k=0}^n a_k^{rt} x_k \ln a_k\right)^2\right] dx > 0. \end{aligned} \tag{2.4}$$

For $r = s$, then

$$\frac{\partial}{\partial r}\left(\ln E^{[2]}_{r,r,t}(a)\right) = \frac{\partial}{\partial r}\left(\frac{\int_E M_t^{1-t}(a^r;x)\left(\sum_{k=1}^n a_k^r x_k \ln a_k\right)dx}{\int_E M_t(a^r;x)dx}\right) \tag{2.5}$$
$$= \frac{\partial}{\partial r}\left(\frac{T'_r(a;r,t)}{T(a;r,t)}\right)$$
$$= \frac{T(a;r,t)T''_{rr}(a;r,t) - [T'_r(a;r,t)]^2}{[T(a;r,t)]^2} > 0.$$

For $r \neq s$, using mean values theorem, we have

$$\frac{\partial}{\partial r}\left(\ln E^{[2]}_{r,s,t}(a)\right) = \frac{\partial}{\partial r}\left(\frac{\ln T(a;r,t) - \ln T(a;s,t)}{r-s}\right) \tag{2.6}$$
$$= \frac{1}{r-s}\left(\frac{T'_r(a;r,t)}{T(a;r,t)} - \frac{\ln T(a;r,t) - \ln T(a;s,t)}{r-s}\right)$$
$$= \frac{1}{r-s}\left(\frac{T'_r(a;r,t)}{T(a;r,t)} - \frac{T'_r(a;u,t)}{T(a;u,t)}\right)$$
$$= \frac{r-v}{r-s}\cdot\frac{T(a;v,t)T''_{rr}(a;v,t) - [T'_r(a;v,t)]^2}{[T(a;v,t)]^2} > 0,$$

where u is between r and s, and v is between r and u. These imply Theorem 2.2. We can obtain similarly Theorem 2.2 if $t = 0$. Then proof is completed. □

3 The Generalized Van der Monde Determinant and Explicit Form of $E^{[2]}_{r,s,t}(a)$

In this section, let φ be a continuous function on an interval $\mathbb{I} \subseteq \mathbb{R}$, for

$$V(a;\varphi) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & \varphi(a_0) \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & \varphi(a_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & \varphi(a_n) \end{vmatrix}. \tag{3.1}$$

Let $\varphi(x) = x^{n+r}\ln^k x$ in (3.1), we have

$$V(a;r,k) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & a_0^{n+r}\ln^k a_0 \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^{n+r}\ln^k a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & a_n^{n+r}\ln^k a_n \end{vmatrix}. \tag{3.2}$$

$V(a;r,k)$ is called the generalized determinant of Van Der Monde's matrix because the case $r = 0$ and $k = 0$ is just the determinant of Van Der Monde's matrix of the n-th order:

$$V(a;0,0) = \sum_{i=0}^{n}(-1)^{n+i}a_i^n V_i(a) = \prod_{0\leqslant i<j\leqslant n}(a_j - a_i), \tag{3.3}$$

where for $0 \leqslant i \leqslant n$,

$$V_i(a) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{i-1} & a_{i-1}^2 & \cdots & a_{i-1}^{n-1} \\ 1 & a_{i+1} & a_{i+1}^2 & \cdots & a_{i+1}^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{vmatrix}. \tag{3.4}$$

Let $\ln a = (\ln a_0, \ln a_1, \cdots, \ln a_n)$, we denote

$$V(\ln a; r, k) = \begin{vmatrix} 1 & \ln a_0 & \ln^2 a_0 & \cdots & \ln^{n-1} a_0 & a_0^r \ln^k a_0 \\ 1 & \ln a_1 & \ln^2 a_1 & \cdots & \ln^{n-1} a_1 & a_1^r \ln^k a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \ln a_n & \ln^2 a_n & \cdots & \ln^{n-1} a_n & a_n^r \ln^k a_n \end{vmatrix}. \tag{3.5}$$

Also let $0 \leqslant i \leqslant n$, we set

$$V(a, i; \varphi) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & \varphi(a_0) \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & \varphi(a_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_i & a_i^2 & \cdots & a_i^{n-1} & \varphi(a_i) \\ 0 & 1 & 2a_i & \cdots & (n-1)a_i^{n-2} & \varphi'(a_i) \\ 1 & a_{i+1} & a_{i+1}^2 & \cdots & a_{i+1}^{n-1} & \varphi(a_{i+1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & \varphi(a_n) \end{vmatrix}, \tag{3.6}$$

and taking $\varphi(x) = x^{n+r+1}$ in (3.6), we have

$$V(a, i; r) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n} & a_0^{n+r+1} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n} & a_1^{n+r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_i & a_i^2 & \cdots & a_i^{n} & a_i^{n+r+1} \\ 0 & 1 & 2a_i & \cdots & na_i^{n-1} & (n+r+1)a_i^{n+r} \\ 1 & a_{i+1} & a_{i+1}^2 & \cdots & a_{i+1}^{n} & a_{i+1}^{n+r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n} & a_n^{n+r+1} \end{vmatrix} \tag{3.7}$$

for $r \neq -1, -2, \cdots, -(n+1)$, and

$$V(a, i; 0) = (-1)^{i+1} V(a; 0, 0) \prod_{j=0, j\neq i}^{n} (a_j - a_i) = (-1)^{i+1} V^2(a; 0, 0)/V_i(a), \tag{3.8}$$

$$V_{\ln}(a,i;r)=\begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^n & a_0^{n+r+1}\ln a_0 \\ 1 & a_1 & a_1^2 & \cdots & a_1^n & a_1^{n+r+1}\ln a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_i & a_i^2 & \cdots & a_i^n & a_i^{n+r+1}\ln a_i \\ 0 & 1 & 2a_i & \cdots & na_i^{n-1} & a_i^{n+r}\left[(n+r+1)\ln a_i-1\right] \\ 1 & a_{i+1} & a_{i+1}^2 & \cdots & a_{i+1}^n & a_{i+1}^{n+r+1}\ln a_{i+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^n & a_n^{n+r+1}\ln a_n \end{vmatrix}. \tag{3.9}$$

Lemma 3.1. *If $n\in\mathbb{N}$, φ be a $(n+1)$-order differentiable function on interval $\mathbb{I}\subset\mathbb{R}_+$, then we have*

$$V(a;\varphi)=V(a;0,0)\int_E\varphi^{(n)}(A(a,x))dx_1dx_2\cdots dx_n, \tag{3.10}$$

and

$$\sum_{i=0}^{n}(-1)^{i+1}\lambda_iV(a,i;\varphi)V_i(a)=V^2(a;0,0)\int_E A(\lambda,x)\varphi^{(n+1)}(A(a,x))dx, \tag{3.11}$$

where $a_i,\in I,\ A(a,x)=a_0+\sum_{i=1}^n(a_i-a_0)x_i=\sum_{i=0}^n a_ix_i,\ x_0=1-\sum_{i=1}^n x_i$.

Proof. Identity (3.10) is proved in [3]. It is easy to know

$$\begin{aligned} &V(a,i;r) \qquad\qquad (3.12)\\ &=\sum_{k=0}^{i-1}(-1)^{n+k+i}\varphi(a_k)\cdot V_k(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)\\ &+\varphi(a_i)\cdot\left[\sum_{k=0}^{i-1}(-1)^{n+k+i+1}V_k(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)+\sum_{k=i+1}^{n}(-1)^{n+k+i}V_k(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)\right]\\ &+\sum_{k=i+1}^{n}(-1)^{n+k+i+1}\varphi(a_k)\cdot V_k(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)+(-1)^{n+i}\varphi'(a_i)\cdot V(a;0,0). \end{aligned}$$

Let $A_0=\int_E\left(1-\sum_{k=1}^n x_k\right)\varphi^{(n+1)}(A(a,x))dx,\ A_i=\int_E x_i\varphi^{(n)}(A(a,x))dx\ (1\leqslant i\leqslant n)$. Then

$$\begin{aligned} &\int_E A(\lambda,x)\varphi^{(n+1)}(A(a,x))dx \qquad\qquad (3.13)\\ &=\int_E\left(\sum_{i=0}^n\lambda_ix_i\right)\varphi^{(n+1)}(A(a,x))dx\\ &=\lambda_0\int_E\left(1-\sum_{k=1}^n x_k\right)\varphi^{(n+1)}(A(a,x))dx+\sum_{i=1}^n\lambda_i\int_E x_i\varphi^{(n+1)}(A(a,x))dx\\ &=\lambda_0A_0+\sum_{i=1}^n\lambda_iA_i. \end{aligned}$$

If $x_0 = 1 - \sum_{i=1}^{n} x_i$, then $x_n = 1 - \sum_{i=0}^{n-1} x_i$, and we have

$$E = \left\{ (x_1, x_2, \cdots, x_n) : \sum_{i=1}^{n} x_i \leqslant 1, x_i \geqslant 0, i = 1, 2, \cdots, n \right\},$$

$$F = \left\{ (x_0, x_1, \cdots, x_{n-1}) : \sum_{i=0}^{n-1} x_i \leqslant 1, x_i \geqslant 0, i = 0, 1, \cdots, n-1 \right\},$$

$$\varphi^{(n+1)}(A(a,x)) = \varphi^{(n+1)} \left(a_0 + \sum_{i=1}^{n} (a_i - a_0) x_i \right) = \varphi^{(n+1)} \left(a_n + \sum_{i=0}^{n-1} (a_i - a_n) x_i \right).$$

Therefore

$$A_0 = \int_E \left(1 - \sum_{k=1}^{n} x_k \right) \varphi^{(n+1)}(A(a,x)) dx = \int_F x_0 \varphi^{(n+1)}(A(a,x)) dx^*,$$

where $dx^* = dx_0 dx_1 \cdots dx_{n-1}$, that is still the form of $A_i = \int_E x_i \varphi^{(n+1)}(x) dx$ for $1 \leqslant i \leqslant n$. Let $\bar{a}_i = (a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$, then

$$\begin{aligned} V(a, i; 0) &= (-1)^{i+1} V(a; 0, 0) \prod_{j=0, j \neq i}^{n} (a_j - a_i) \\ &= V(\bar{a}_i; 0, 0) \prod_{j=0, j \neq i}^{n} (a_j - a_i)^2 V_k(a) \\ &= \begin{cases} (-1)^i V_k(\bar{a}_i) \prod_{j=0, j \neq i, k}^{n} (a_j - a_i), & (0 \leqslant k < i), \\ (-1)^{i+1} V_{k-1}(\bar{a}_i) \prod_{j=0, j \neq i, k}^{n} (a_j - a_i), & (i < k \leqslant n); \end{cases} \end{aligned} \tag{3.14}$$

$$\sum_{k=0, k \neq i}^{n} (-1)^{n+k} V_k(a) = (-1)^{n+i-1} V_i(a). \tag{3.15}$$

From expression (3.10), we obtain

$$\begin{aligned} & V(\bar{a}; 0, 0) \int_0^{1-x_i} \int_0^{1-x_i-x_1} \cdots \int_0^{1-\sum_{i=1}^{n-1} x_i} \varphi^{(n+1)}(A(a,x)) dx_1 dx_2 \cdots dx_n \\ &= \sum_{k=0}^{i-1} (-1)^{n+k} V_k(\bar{a}_k) \varphi''(a_k + (a_i - a_k) x_i) + \sum_{k=i+1}^{n} (-1)^{n+k-1} V_{k-1}(\bar{a}_k) \varphi''(a_k + (a_i - a_k) x_i), \end{aligned} \tag{3.16}$$

and

$$(a_k - a_i)^2 \int_0^1 x_i \varphi''(a_k + (a_i - a_k) x_i) dx_i = \varphi(a_k) - \varphi(a_i) - (a_k - a_i) \varphi'(a_i). \tag{3.17}$$

Therefore

$$
\begin{aligned}
&V(a,i;0)A_i \\
&= (-1)^{i+1}V(a;0,0)\prod_{j=0,j\neq i}^{n}(a_j-a_i)\int_E x_i\varphi^{(n+1)}(A(a,x))dx \\
&= V(\bar{a}_i;0,0)\prod_{j=0,j\neq i}^{n}(a_j-a_i)^2\int_0^1\int_0^{1-x_1}\int_0^{1-\sum_{i=1}^{n-1}x_i} x_i\varphi^{(n+1)}(A(a,x))dx_1dx_2\cdots dx_n \\
&= \prod_{j=0,j\neq i}^{n}(a_j-a_i)^2\int_0^1 x_i\left[V(\bar{a}_i;0,0)\int_0^{1-x_i}\int_0^{1-x_i-x_1}\cdots\int_0^{1-\sum_{i=1}^{n-1}x_i}\varphi^{(n+1)}(A(a,x))dx_1\cdots dx_n\right]dx_i \\
&= \sum_{k=0}^{i-1}(-1)^{n+k}V_k(\bar{a}_i)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)^2[\varphi(a_k)-\varphi(a_i)-(a_k-a_i)\varphi'(a_i)] \\
&+ \sum_{k=i+1}^{n}(-1)^{n+k-1}V_{k-1}(\bar{a}_i)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)^2[\varphi(a_k)-\varphi(a_i)-(a_k-a_i)\varphi'(a_i)] \\
&= \sum_{k=1}^{i-1}(-1)^{n+k+i}\varphi(a_k)\cdot V_k(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)+\varphi(a_i)\cdot\left[\sum_{k=0}^{i-1}(-1)^{n+k+i+1}V_k(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)\right. \\
&\left.+\sum_{k=i+1}^{n}(-1)^{n+k+i}V_k(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)\right]+\sum_{k=i+1}^{n}(-1)^{n+k+i+1}\varphi(a_k)\cdot V_k(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i) \\
&+ (-1)^{n+i}V(a;0,0)\cdot\varphi'(a_i) \\
&= V(a,i;\varphi),
\end{aligned}
$$

i.e.

$$
A_i=\int_E x_i\varphi^{(n+1)}(A(a,x))dx=\frac{V(a,i;\varphi)}{V(a,i;0)}=(-1)^{i+1}\frac{V(a,i;\varphi)\cdot V_i(a)}{V^2(a;0,0)}. \tag{3.18}
$$

Combining (3.13) and (3.18), we find expression (3.11). The proof of Lemma 3.1 is completed. □

Theorem 3.2. *If $E^{[2]}_{r,s,t}(a)$ is the second three-parameter generalized weighted mean values, then*

$$
E^{[2]}_{r,s,t}(a)=\begin{cases}
\left[\dfrac{V(a^{st};0,0)}{V(a^{rt};0,0)}\cdot\dfrac{V(a^{rt};1/t,0)}{V(a^{st};1/t,0)}\right]^{\frac{1}{r-s}}, & r\neq s, t\neq 0,-1,-\frac{1}{2},\cdots,-\frac{1}{n};\\
\left[\dfrac{V(a^{st};0,0)}{V(a^{rt};0,0)}\cdot\dfrac{V(a^{rt};1/t,1)}{V(a^{st};1/t,1)}\right]^{\frac{1}{r-s}}, & r\neq s, t=-1,-\frac{1}{2},\cdots,-\frac{1}{n};\\
\exp\left\{\dfrac{\sum_{i=0}^{n}(-1)^{i+1}a_i^{rt}\ln a_i\cdot V(a^{rt},i;1/t)\cdot V_i(a^{rt})}{tV(a^{rt};0,0)\cdot V(a^{rt};1/t,0)}\right\}, & r=s, t\neq 0,-1,-\frac{1}{2},\cdots,-\frac{1}{n};\\
\exp\left\{\dfrac{\sum_{i=0}^{n}(-1)^{i+1}a_i^{rt}\ln a_i\cdot V_{\ln}(a^{rt},i;1/t)\cdot V_i(a^{rt})}{tV(a^{rt};0,0)\cdot V(a^{rt};1/t,0)}\right\}, & r=s, t=-1,-\frac{1}{2},\cdots,-\frac{1}{n};\\
\left[\dfrac{s^n}{r^n}\cdot\dfrac{V(\ln a;r,0)}{V(\ln a;s,0)}\right]^{\frac{1}{r-s}}, & r\neq s, t=0;\\
\exp\left\{\dfrac{V(\ln a;r,1)}{V(\ln a;r,0)}-\dfrac{n}{r}\right\}, & r=s\neq 0, t=0;\\
\prod_{k=0}^{n}a_k^{\frac{1}{n+1}}, & r=s=t=0;
\end{cases}
\tag{3.19}
$$

where $a = (a_0, a_1, \cdots, a_n) \in \mathbb{R}_+^{n+1}$, $a^t = (a_0^t, a_1^t, \cdots, a_n^t)$, $\ln a = (\ln a_0, \ln a_1, \cdots, \ln a_n)$ *and* $a_i \neq a_j$ *for* $1 \leqslant i \neq j \leqslant n$.

Proof. For $t \neq 0$, setting a function

$$\varphi_1(t;x) = \begin{cases} \prod_{k=1}^{n}(k+1/t)^{-1}x^{n+1/t}, & t \neq -1, -\frac{1}{2}, \cdots, -\frac{1}{n}; \\ \left[(-1)^{k+1}(k-1)!(n-k)!\right]^{-1} x^{n+1/t}\ln x, & t = -1, -\frac{1}{2}, \cdots, -\frac{1}{n}; \end{cases}$$

then $\varphi_1^{(n)}(t;x) = x^{1/t}$.

From Definition 2.1, Lemma 3.1 and straightforward computation, we immediately obtain Theorem 3.2.

For $t = 0$, it is proved in [4]. The proof of Theorem 3.2 is completed. □

4 Applications

When r, s, t are given some special values, we can obtain many interesting mean values in n variables.

Theorem 4.1. $E_{r,s,t}^{[2]}(a)$ *include the power mean* $M_r(a)$, *one-parameter Gini type mean* $K_r(a)$ *and Gini mean* $D_{r,s}(a)$ *with positive numbers* $a_0, a_1, \cdots, a_n$ *in [1]:*

$$M_r(a) = E_{r,0,1}^{[2]}(a) = \begin{cases} \left(\frac{1}{n+1}\sum_{k=0}^{n} a_k^r\right)^{\frac{1}{r}}, & r \neq 0; \\ \prod_{k=0}^{n} a_k^{\frac{1}{n+1}}, & r = 0; \end{cases} \tag{4.1}$$

$$K_r(a) = E_{r,r-1,1}^{[2]}(a) = \frac{\sum_{k=0}^{n} a_k^r}{\sum_{k=0}^{n} a_k^{r-1}}, \tag{4.2}$$

and

$$D_{r,s}(a) = E_{r,s,1}^{[2]}(a) = \left(\frac{\sum_{k=0}^{n} a_k^r}{\sum_{k=0}^{n} a_k^s}\right)^{\frac{1}{r-s}}. \tag{4.3}$$

Proof. This follows utilizing the fact [3] that $V(a^r;1,0) = V(a^r;0,0) \cdot \sum_{k=0}^{n} a_k^r$, and Theorem 3.2. □

Theorem 4.2. $E_{r,s,t}^{[2]}(a)$ *include also Alzer type mean* $J_r^{[2]}(a)$ *and Stolarsky type mean* $S_r^{[2]}(a)$ *in* $n+1$ *variables* $a_0, a_1, \cdots, a_n$*:*

$$J_{r,t}^{[2]}(x) = E_{r,r-1,t}^{[2]}(a) = \begin{cases} \dfrac{\int_E M_t(a^r;x)dx}{\int_{E'} M_t(a^{r-1};x)dx}, & t \neq 0; \\ \dfrac{\int_E G(a^r;x)dx}{\int_E G(a^{r-1};x)dx}, & t = 0. \end{cases} \tag{4.4}$$

and

$$S_{r,t}^{[2]}(a) = E_{r,0,t}^{[2]}(a) = \begin{cases} \left[n! \cdot \int_E M_t(a^r;x)dx\right]^{\frac{1}{r}}, & tr \neq 0; \\ \left[n! \int_E G(a^r;x)dx\right]^{\frac{1}{r}}, & t = 0, r \neq 0; \\ \exp\left\{n! \cdot \int_E \ln G(a;x)dx\right\}, & r = 0; \end{cases} \tag{4.5}$$

respectively, where the notions of $E, M_t(a;x), G(a;x)$ are the same as the Definition 2.1. We have also

$$J^{[2]}_{r,t} = \begin{cases} \dfrac{V(a^{(r-1)t};0,0)}{V(a^{rt};0,0)} \cdot \dfrac{V(a^{rt};1/t,0)}{V(a^{(r-1)t};1/t,0)}, & t \neq 0, -1, -\frac{1}{2}, \cdots, -\frac{1}{n}; \\ \dfrac{V(a^{(r-1)t};0,0)}{V(a^{rt};0,0)} \cdot \dfrac{V(a^{rt};1/t,1)}{V(a^{(r-1)t};1/t,1)}, & t = -1, -\frac{1}{2}, \cdots, -\frac{1}{n}; \\ \left(\dfrac{r-1}{r}\right)^n \cdot \dfrac{V(\ln a; r, 0)}{V(\ln a; r-1, 0)}, & t = 0; \end{cases} \tag{4.6}$$

and

$$S^{[2]}_{r,t}(a) = \begin{cases} \left[\dfrac{n!}{\prod_{k=1}^{n}(k+1/t)} \cdot \dfrac{V(a^{rt};1/t,0)}{V(a^{rt};0,0)}\right]^{\frac{1}{r}}, & r \neq 0, t \neq 0, -1, -\frac{1}{2}, \cdots, -\frac{1}{n}; \\ \left[\dfrac{n!}{(-1)^{1+1/t}(-1-1/t)!(n+1/t)!} \cdot \dfrac{V(a^{rt};1/t,1)}{V(a^{rt};0,0)}\right]^{\frac{1}{r}}, & r \neq 0, t = -1, -\frac{1}{2}, \cdots, -\frac{1}{n}; \\ \exp\left\{\dfrac{V(\ln a; r, 1)}{V(\ln a; r, 0)} - \dfrac{n}{r}\right\}, & r \neq 0, t = 0; \\ \prod_{k=0}^{n} a_k^{\frac{1}{n+1}}, & r = t = 0; \end{cases} \tag{4.7}$$

where $a = (a_0, a_1, \cdots, a_n) \in \mathbb{R}_+^{n+1}$, $a^t = (a_0^t, a_1^t, \cdots, a_n^t)$, $\ln a = (\ln a_0, \ln a_1, \cdots, \ln a_n)$ and $a_i \neq a_j$ for $1 \leqslant i \neq j \leqslant n$.

Proof. These are special cases of Definition 2.1 and Theorem 3.2. □

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International Journal of Applied Mathematics & Statistics,
Int. J. Appl. Math. Stat.; Vol. 11; No. V07; November 2007
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Some Research for Bernoulli's Inequality

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ABSTRACT

Bernoulli's Inequality "if $h > -1$, then $(1+h)^n \geq 1+nh$, " is important and almost can be found in any books on mathematics, sometimes we found $h > 0$, but we notice that the condition of the Bernoulli's Inequality can be extended to $h \geq -2$, and the condition is not only sufficient but also necessary. We show that $h \geq -2$ if and only if $(1+h)^n \geq 1+nh$.

Key words and phrases: Bernoulli's Inequality,Binomial Theorem,Mathematical Induction

AMS(MOS) Subject Classification: 00, 01.

1 Introduction

In this paper, we will discuss inequality in Mathematics Analysis, Bernoulli's Inequality. We use mathematical induction to prove it.

2 Main Result

Theorem 2.1. $(1+h)^n \geq 1+nh$ *(for any $n \in N$) if and only if $h \geq -2$.*

Proof. Sufficiency: Suppose that $h \geq -2$.
(1).We will prove that if n is an odd number, $(1+h)^n \geq 1+nh$.
As $n = 1$, it is obvious.
As $n = 2k-1$, assume that $(1+h)^{2k-1} \geq 1+(2k-1)h$
Then,

$$\begin{aligned}(1+h)^{2k+1} = (1+h)^{2k-1}(1+h)^2 &\geq (1+(2k-1)h)(1+2h+h^2)\\ &= 1+(2k+1)h+h^2(4k-2+(2k-1)h)\end{aligned}$$

Since $h \geq -2$, then $4k-2+(2k-1)h \geq 4k-2+(2k-1)(-2) = 0$. So we show that

$$(1+h)^{2k+1} \geq 1+(2k+1)h,$$

thus n is an odd number with $(1+h)^n \geq 1+nh$.
(2). We will prove that for n is an even number, $(1+h)^n \geq 1+nh$.
Firstly, we can deduce $(1+h)^n \geq 1+nh$ from the original condition of Bernoulli's Inequality $h > -1$, and as $h \leq -1$, now since n is an even number, then

$$(1+h)^n \geq 0 > 1-n \geq 1+nh.$$

thus

$$(1+h)^n \geq 1+nh$$

is always true for any real h while n is an even number.
Now, we have proved that $(1+h)^n \geq 1+nh$ as $h \geq -2$.

Conversely, suppose that $(1+h)^n \geq 1+nh$ for any $n \in N$, we will obtain $h \geq -2$.
Otherwise, if we have some $h_0 < -2$ such that $(1+h_0)^n \geq 1+nh_0$. Let

$$h_0 = -2-\alpha, \alpha > 0.$$

If n is odd number,

$$\begin{aligned}(1+h_0)^n-(1+nh_0) &= (-1-\alpha)^n-(1-2n-2\alpha)\\ &= -(1+\alpha)^n-1+2n+\alpha n.\end{aligned}$$

Since $\alpha > 0$, by binomial theorem,

$$(1+\alpha)^n \geq 1+n\alpha+\frac{n(n-1)}{2}\alpha^2.$$

So,

$$\begin{aligned}(1+h_0)^n-(1+nh_0) &\leq -1-n\alpha-\frac{n(n-1)}{2}\alpha^2-1+2n+\alpha n\\ &= -2-\frac{n(n-1)}{2}\alpha^2+2n\end{aligned}$$

Let $n > [\frac{4}{\alpha^2}+1]$, then $\frac{n(n-1)}{2}\alpha^2 > 2n$, hence that

$$(1+h_0)^n-(1+nh_0) \leq -2-2n+2n < 0.$$

This is contradiction. □

Acknowledgment

This work is supported by SF of Dalian Nationalities University.

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Submission of Manuscripts

International Journal of Applied Mathematics & Statistics (IJAMAS)

ISSN 0973-1377 (Print); ISSN 0973-7545 (Online)

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