ISSN 0973-1377 (Print)
ISSN 0973-7545 (Online)

International Journal of Applied Mathematics & Statistics

Special Issue on **Leonhard Paul Euler**'s:
Differential Equations and Inequalities (D. E. I.)

Centre for Environment, Social & Economic Research
CESER

Volume 10 Number S07 September 2007

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CESER

International Journal of Applied Mathematics & Statistics

ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

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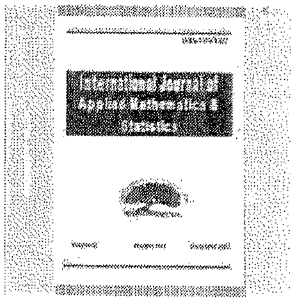

Special Issue on **Leonhard Paul Euler's**: Differential Equations and Inequalities (D. E. I.)

International Journal of Applied Mathematics & Statistics

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Volume 10 | Number S07 | September 2007

International Journal of Applied Mathematics & Statistics
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

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PREFACE

This Euler's commemorating volume entitled :

Functional Equations , Integral Equations, Differential Equations and Applications **(F. I. D. A)**,

is a forum for exchanging ideas among eminent mathematicians and physicists, from many parts of the world, as a tribute to the tri-centennial birthday anniversary of Leonhard Paul Euler (April 15,1707 A.D., b. in Basel – September 18, 1783 A.D., d. in St. Petersburg).

This 998 pages long collection is composed of outstanding contributions in mathematical and physical equations and inequalities and other fields of mathematical, physical and life sciences.

In addition, this anniversary volume is unique in its target, as it strives to represent a broad and highly selected participation from across and beyond the scientific and technological country regions. It is intended to boost the cooperation among mathematicians and physicists working on a broad variety of pure and applied mathematical areas.

Moreover, this new volume will provide readers and especially researchers with a detailed overview of many significant insights through advanced developments on Euler's mathematics and physics. This transatlantic collection of mathematical ideas and methods comprises a wide area of applications in which equations, inequalities and computational techniques pertinent to their solutions play a core role.

Euler's influence has been tremendous on our everyday life, because new tools have been developed, and revolutionary research results have been achieved , bringing scientists of exact sciences even closer, by fostering the emergence of new approaches, techniques and perspectives.

The central scope of this commemorating 300birthday anniversary volume is broad, by deeper looking at the impact and the ultimate role of mathematical and physical challenges, both inside and outside research institutes, scientific foundations and organizations.

We have recently observed a more rapid development in the areas of research of Euler worldwide. Leonhard P. Euler (1707-1783) was actually the most influential mathematician and prolific writer of the eighteenth century, by having contributed to almost all the fundamental fields of mathematics and mathematical physics. In calculus of variations, according to C. Caratheodory, Euler's work: *Methodus inveniendi lineas curves…*(1740 A.D.) was one of the most beautiful works ever written. Euler was dubbed *Analysis Incarnate* by his peers for his incredible ability. He was especially great from his writings and that produced more academic work on mathematics than anyone. He could produce an entire new mathematical paper in about thirty minutes and had huge piles of his works lying on his desk. It was not uncommon to find Analysis Incarnate ruminating over a new subject with a child on his lap.

This volume is suitable for graduate students and researchers interested in functional equations, integral equations and differential equations and would make an ideal supplementary reading or independent study research text.

These issues will also be of interest to those working in other areas of mathematics and physics. It is a work of great interest and enjoyable read as well as unique in market.

This **Euler's volume (F. I. D. A.)** consists of six (6) issues containing various parts of contemporary pure and applied mathematics with emphasis to Euler's mathematics and physics.

It contains sixty eight (68) fundamental research papers of one hundred one (101) outstanding research contributors from twenty seven (27) different countries. In particular, these contributors come from:

Algerie (1 contributor); Belgique (2); Bosnia and Herzegovina (2); Brazil (2); Bulgaria (3); China (9); Egypt (1); France (3); Greece (2); India (8); Iran (3); Italy (1); Japan (7); Korea (7); Morocco (3); Oman (2); Poland (3); R. O. Belarus (8); Romania (2); Russia (3); Saudi Arabia (1); Serbia and Montenegro (5); The Netherlands (3); U. A. Emirates (1);U. K. (2); U. S. A. (15); Uzbekistan (2).

First Issue (F. E. I.) contains various parts of ***Functional Equations and Inequalities***, namely: Euler's Life and Work, Ulam stability, Hyers – Ulam stability and Ulam – Gavruta – Rassias stability of functional equations, Euler – Lagrange type and Euler – Lagrange – Rassias quadratic mappings in Banach and Hilbert spaces, Aleksandrov and isometry Ulam stability problems, stability of Pexider and Drygas functional equations, alternative of fixed point, and Hyers - Ulam stability of differential equations.

Second Issue (MT. PDE) contains various parts of ***Mixed Type Partial Differential Equations***, namely: Tricomi - Protter problem of nD mixed type partial differential equations, solutions of generalized Rassias' equation, degenerated elliptic equations, mixed type oblique derivative problem, Cauchy problem for Euler – Poisson - Darboux equation, non - local boundary value problems, non-uniqueness of transonic flow past a flattened airfoil, multiplier methods for mixed type equations.

Third Issue (F . D . E.) contains various parts of ***Functional and Differential Equations***, namely: Iterative method for singular Sturm - Liouville problems, Euler type boundary value problems in quantum mechanics, positive solutions of boundary value problems, controllability of impulsive functional semi-linear differential inclusions in Frechet spaces, asymptotic properties of solutions of the Emden-Fowler equation, comparison theorems for perturbed half-linear Euler differential equations, almost sure asymptotic estimations for solutions of stochastic differential delay equations, , difference equations inspired by Euler's discretization method, extended oligopoly models.

Fourth Issue (D. E. I.) contains various parts of ***Differential Equations and Inequalities***, namely:

New spaces with wavelets and multi-fractal analysis, mathematical modeling of flow control and wind forces, free convection in conducting fluids, distributions in spaces, strong stability of operator – ifferential equations, slope – bounding procedure, sinc methods and PDE, Fourier type analysis and quantum mechanics.

Fifth Issue (M. T. A.) contains various parts of ***Mathematical Topics and Applications***, namely:Maximal subgroups and theta pairs in a group, Euler constants on algebraic number fields, characterization of modulated Cox measures on topological spaces, hyper-surfaces with flat r-mean curvature and Ribaucour transformations, Leonhard Euler's methods and ideas live on in the thermodynamic hierarchical theory of biological evolution, zeroes of L-series in characteristic p, Beck's graphs, best co-positive approximation function, Convexity in the theory of the Gamma function, analytical and differential – algebraic properties of Gamma function, Ramanujan's summation formula and related identities, ill – posed problems, zeros of the q-analogues of Euler polynomials, Eulerian and other integral representations for some families of hyper-geometric polynomials, group C*-algebras and their stable rank, complementaries of Greek means to Gini means, class of three- parameter weighted means, research for Bernoulli's inequality.

Sixth Issue (DS. IDE.), contains various parts of ***Dynamical Systems and Integro - Differential Equations***, namely: Semi-global analysis of dynamical systems, nonlinear functional-differential and integral equations, optimal control of dynamical systems, analytical and numerical solutions of singular integral equations, chaos control of classes of complex dynamical systems, second order integro-differential equation, integro-differential equations with variational derivatives generated by random partial integral equations, inequalities for positive operators, strong convergence for a family of non-expansive mappings.

Deep gratitude is due to all those Guest Editors and Contributors who helped me to carry out this intricate project. My warm thanks to my family:

Matina- Mathematics Ph. D. candidate of the Strathclyde University (Glasgow, United Kingdom), Katia- Senior student of Archaeology and History of Art of the National and Capodistrian University of Athens (Greece), and Vassiliki- M. B. A. of the University of La Verne, Marketing Manager in a FMCG company (Greece).

Finally I express my special appreciation to: The *Executive Editor* of the *International Journal of Applied Mathematics and Statistics* (IJAMAS.) *Dr. Tanuja Srivastava* for her nice cooperation and great patience.

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International Journal of Applied Mathematics & Statistics
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

IJAMAS: September 2007, Vol. 10, No. S07, 8-24

Some aspects of free convection in electrically conducting fluids: Flow near vertical plates

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ABSTRACT

Free convection in incompressible, electrically conducting fluids in the presence of applied magnetic field has been a subject of active research due to numerous applications. Both numerical and analytical methods have been widely used in the literature to solve the equations governing such convective flows. Among the physically important hydromagnetic flows, there are special types of flows for which the nonlinear magnetofluiddynamical equations simplify to linear ones whose solutions can be obtained in closed form. In this work, we have analysed a specific facet of the transient free convection when the flow of an electrically conducting viscous fluid takes place near a moving vertical surface. It is assumed that this bounding surface is subjected to uniform heat flux. Analytical solutions of the energy and momentum equations, under Boussinesq approximation and an applied magnetic field – either fixed relative to the fluid or to the bounding surface – have been presented for two types of boundary motion. There arises a number of nondimensional parameters characterising various magneto-thermo-fluiddynamical features. The composite effects of these parameters, especially on the developing velocity profiles, have been discussed.

Keywords: hydromagnetic flow, heat flux, free convection, impulsive motion, accelerated motion.

2000 Mathematics Subject Classification: 76R10, 76D99, 76W05.

1 Introduction

The flow of electrically conducting fluids near permeable and impermeable surfaces, under the influence of externally applied magnetic field, has been a subject of extensive study during the last several decades. Such flows find applications in several industrial and technological fields such as aerospace, biomedical, chemical and nuclear engineering. As is known, study of the flow in the presence of magnetic and electric fields can be divided into two separate areas, namely, magnetohydrodynamics and electrohydrodynamics. In the former, one studies the fluid flow without free electric charges and electric field, under the influence of a magnetic field. On the other hand, the latter area encompasses the study of fluids containing electric charges under the influence of an electric field without the consideration of magnetic field. Historically, this division has been necessitated to simplify the mathematical intractability of the coupled system of Navier-Stokes, Maxwell and the related constitutive equations describing the electro-magneto-hydrodynamic flow. Despite this division, the mathematical analysis of magnetohydrodynamic flow — the subject matter of this paper — is a highly complex area, primarily due to the non-linear nature of the governing partial differential equations. For instance, the flow in the presence of viscous, buoyancy and magnetic forces renders the governing equations highly non-linear and coupled. Although the advances in numerical techniques and computing power witnessed in the last couple of decades have enabled the scientific community to analyse such nonlinear systems for specific flow, heat and mass transfer problems, it must, however, be admitted that the numerical solutions of non-linear problems have great limitations as they are mostly ad-hoc in nature. Major efforts in the early literature have therefore been directed to obtaining analytical or semi-analytical solutions of a variety of hydrodynamic and magnetohydrodynamic flow problems, albeit under certain idealised flow situations as, for instance, flow near moving infinite horizontal plates (Rossow, 1958; Kakutani, 1958; Ong and Nicholls, 1959; Gupta, 1960; Kelleher, 1971).

Free convection in electrically conducting fluids under the influence of an external magnetic field has been a subject of considerable research interest for a long time due to a number of applications in varied fields such as nuclear reactors, geothermal engineering, liquid metals and plasma flows, among others. Moreover, when the fluid motion takes place near a moving boundary, subject also to buoyancy forces — as are encountered in free convection problems — the resulting boundary layer flow analysis assumes importance in the above application areas. Such boundary layer flows are significant in the flow control and design of machines. Also, fluid flow control through magnetic forces is known to find applications in magnetohydrodynamic generators and a host of magnetic devices used in industries. It is therefore important to study the features of transport phenomena in hydromagnetic flows under different physical conditions for both transient and steady state cases.

It is known that the fluid dynamical equations of hydromagnetic free convection are, in general, coupled and nonlinear. However, there are special classes of flows such as unidirectional flows near horizontal or vertical flat surfaces for which these equations can be simplified considerably. Of the hydromagnetic fluid flow models for which exact analytical treatment can be carried out, the modified Stokes and Rayleigh problems , for instance, have helped in displaying the basic interaction features between magnetic and viscous forces. In these problems, the quadratic convection terms in the governing equations drop out rendering the flow equations linear, and one can obtain closed form solutions of the linearized problem in a number of important boundary layer flows such as the flow taking place near a stationary or a non-stationary bounding surface subject to cooling, heating or uniform heat flux. However, some practically important boundary conditions can still pose difficulties in solving the flow equations. Gebhart *et al.* (1988) have given a detailed account of the implications of different types of boundary conditions for convection problems.

The solutions of hydromagnetic free convection boundary layer flow past horizontal or vertical flat surfaces have been reported extensively in the literature (Gupta, 1961; Pera and Gebhart, 1973; Soundalgekar, 1977; Soundalgekar and Patil, 1980; Georgantopoulous, 1979; Raptis and Singh, 1983, 1985, Singh and Singh, 1983; Singh, 1984). The present authors have carried out analytical investigations dealing with certain facets of free convection in magneto-hydrodynamical flows for viscous incompressible fluids (Sacheti *et al.*, 1994; Chandran *et al.* 1998, 2001, 2002). In this work, we wish to report certain aspects of magnetohydrodynamic free convection broadly based on some of our earlier studies. We have considered unidirectional developing flow near a vertical impermeable plate which is subjected to uniform heat flux, assuming that an external applied magnetic field is applied transverse to the bounding plate. Two cases of boundary motion have been considered: (i) impulsive motion and (ii) accelerated motion. In each case, analytical solutions have been presented for the temperature and velocity variables. The composite effects of a host of non-dimensional parameters representing different physical features have been examined on the flow.

2 General formulation – Governing equations

We consider the developing free convection in an electrically conducting incompressible fluid near a non-stationary infinite vertical plate. With respect to the rectangular Cartesian coordinate system $Oxyz$, the $z-$axis is taken along the wall in the upward direction, and the axis of y is taken perpendicular to it, directed into the fluid. The flow is assumed to be in the $z-$direction, and under the influence of a transversely applied magnetic field.

For the flow problem under consideration, the governing equations will be the standard conservation equations – mass, momentum and energy. When the magnetic field is also present, the momentum and energy equations are modified due to the magnetic forces. In the present study, we neglect the dissipation arising due to Ohmic heating and viscous forces. Furthermore, the thermophysical properties of the fluid have been assumed to be constant. However, in the buoyancy term, the fluid density is assumed to vary linearly with the temperature, in accordance with the widely used Boussinesq approximation. The governing equations for this type of flow can be written as

$$\nabla \cdot \mathbf{V} = 0 \tag{2.1}$$

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{V} + g\beta(T - T_\infty)\hat{z} + \mathbf{J} \times \mathbf{B} \tag{2.2}$$

$$\rho c_p \frac{dT}{dt} = k\nabla^2 T \tag{2.3}$$

where $\mathbf{V} = (V_x, V_y, V_z)$ is the fluid velocity, p the pressure, T the temperature, ρ the density, $\mathbf{B}$ the applied magnetic field, $\mathbf{J}$ the current density, μ the fluid viscosity, g the acceleration due to gravity, β the volumetric coefficient of thermal expansion, k the thermal conductivity of the fluid, T_∞ a reference temperature and c_p is the specific heat of the fluid at constant pressure. In Eqs (2.2) and (2.3), d/dt is the convective derivative operator.

The term $\mathbf{J} \times \mathbf{B}$ occurring in Eq (2.2) is the Lorentz force, which is the body force due to the applied magnetic field. Moreover, the current density $\mathbf{J}$ and the magnetic field $\mathbf{B}$ are related through the well-known Ohm's law, adjusted to the moving plate velocity:

$$\mathbf{J} = \sigma(\mathbf{V} - \mathbf{U}) \times \mathbf{B} \tag{2.4}$$

Here, $\mathbf{U}$ is the velocity of the bounding plate and σ is the electrical conductivity of the fluid. In Eqs (2.2)–(2.4), we have neglected the effects due to the electric field and the charge density. Furthermore, we also assume that the magnetic Reynolds number is very small so that the induced magnetic field produced by the motion of the electrically conducting fluid is negligible in comparison with the applied one.

Let us consider the two-dimensional free convection in an incompressible fluid near an infinite vertical plate. At times $t \leq 0$, the plate and the fluid medium are assumed to be at rest and maintained at uniform temperature T_∞. At time $t > 0$, the vertical boundary starts moving in its own plane with a velocity proportional to t^n. Simultaneously, the boundary plate is subjected to heat flux at a constant rate. A uniform magnetic field of strength B_y is applied in the y direction. We shall investigate the resulting flow under two situations with respect to the magnetic field: (i) the magnetic lines of force are fixed relative to the fluid, and (ii) the magnetic lines of force are fixed relative to the bounding plate. It is worth mentioning here that for the shear flow being investigated, in the absence of applied pressure gradients, the only non-zero component of velocity, V_z, and the temperature T will be functions of the time variable t and the space variable y only. As a consequence, the convective terms in the momentum and energy equations (2.2) and (2.3) will drop out. For notational convenience, we shall hereafter denote the non-zero velocity component by $u(y,t)$. Under the above assumptions, the momentum equation (2.2) can be written in the form (Cramer and Pai, 1973)

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty) - \frac{\sigma B_y^2}{\rho}(u - K\lambda t^n) \tag{2.5}$$

where ν is the kinematic viscosity, λ a proportionality constant and

$$K = \begin{cases} 0, & \text{if } B_y \text{ is fixed relative to the fluid} \\ 1, & \text{if } B_y \text{ is fixed relative to the plate.} \end{cases} \tag{2.6}$$

The energy equation becomes

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} \tag{2.7}$$

The initial and boundary conditions corresponding to the physical situation described earlier, and subject to a power-law velocity of the bounding plate, can be written as

$$\begin{aligned} &u = 0, \quad T = T_\infty, \quad \text{for} \quad y \geq 0 \quad \text{and} \quad t \leq 0 \\ &u = \lambda t^n, \quad \frac{\partial T}{\partial y} = -\frac{q}{k} \quad \text{at} \quad y = 0 \quad \text{for} \quad t > 0 \\ &u \to 0, \quad T \to T_\infty \quad \text{as} \quad y \to \infty \quad \text{for} \quad t > 0 \end{aligned} \tag{2.8}$$

where q is the heat flux per unit area at the plate.

Having introduced the governing momentum and energy equations subject to a set of initial and general boundary conditions, we next set out to obtain analytical solutions of these initial-boundary-value problems (IBVP) in some situations of practical interest. We shall consider two specific boundary conditions – corresponding to (i) impulsive motion and (ii) accelerated motion – of the bounding plate. In each flow situation, our endeavour will be to obtain unified solutions incorporating the two cases of the magnetic field [cf. Eq (2.6)]. The solution in either flow situation will be obtained after transforming the initial-boundary-value problems into appropriate dimensionless forms. This non-dimensionalisation will enable us to analyze the flow in terms of some well-known parameters. As the non-dimensionalisation process follows slightly different procedures for the cases of impulsive and accelerated motion of the boundary, we shall present them in two separate subsections.

3 Solution of the governing equations

3.1 Impulsive motion of the bounding plate

This situation corresponds to imparting a sudden uniform velocity to the bounding plate in its own plane so that $u(0,t) = \lambda,\ t > 0$. Simultaneously, the plate is also subjected to a uniform heat flux, as described before. We now introduce the following non-dimensional variables and parameters:

$$\begin{aligned} \bar{y} &= y/L_1, \quad \bar{t} = \lambda t/L_1, \quad \bar{u} = u/\lambda, \quad \bar{T} = k(T - T_\infty)/(qL_1) \\ \mathrm{Pr} &= \rho\nu c_p/k, \quad G = qg\beta L_1^2/(k\lambda^2), \quad m = \sigma L_1 B_y^2/(\rho\lambda) \end{aligned} \tag{3.1}$$

In Eq (3.1), $L_1(=\nu/\lambda)$ is a characteristic length scale while the dimensionless parameters Pr, G, and $\sqrt{m}$ denote the Prandtl number of the fluid, Grashof number and the Hartmann number, respectively. Using Eq (3.1), we can recast the equations (2.5), (2.7) and (2.8) as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} - m(u - K) + GT \tag{3.2}$$

$$\frac{\partial T}{\partial t} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 T}{\partial y^2} \tag{3.3}$$

$$\begin{aligned} u &= 0, \quad T = 0 \quad \text{for} \quad y \geq 0 \quad \text{and} \quad t \leq 0 \\ u &= 1, \quad \partial T/\partial y = -1 \quad \text{at} \quad y = 0 \quad \text{for} \quad t > 0 \\ u &\to 0, \quad T \to 0 \quad \text{as} \quad y \to \infty \quad \text{for} \quad t > 0 \end{aligned} \tag{3.4}$$

In the above equations, we have dropped the 'bar' occurring in the non-dimensional variables, for convenience.

Since the temperature equation (3.3) is uncoupled from the momentum equation, we first solve for $T(y,t)$. This, in turn, will be used in Eq (3.2) to obtain the velocity $u(y,t)$. We use Laplace transforms to solve the IBVP for both T and u.

Let $\tilde{u}(y,s)$ and $\tilde{T}(y,s)$ denote the Laplace transforms of $u(y,t)$ and $T(y,t)$, respectively, given by

$$\tilde{u}(y,s) = \int_0^\infty u(y,t)\,\exp(-st)\,\mathrm{d}t; \quad \tilde{T}(y,s) = \int_0^\infty T(y,t)\,\exp(-st)\,\mathrm{d}t \tag{3.5}$$

Using Eq (3.5) in Eqs (3.2)–(3.4) will lead to two differential equations with the transformed boundary conditions. These can be solved to yield

$$\tilde{T}(y,s) = \frac{1}{\sqrt{\Pr s^3}} \exp\left(-y\sqrt{\Pr s}\right) \tag{3.6}$$

$$\begin{aligned} \tilde{u}(y,s) &= \frac{Km}{s(s+m)}\left[1-\exp\left(-y\sqrt{s+m}\right)\right] + \frac{1}{s}\exp\left(-y\sqrt{s+m}\right) \\ &+ \frac{a}{(s+b)\sqrt{s^3}}\left[\exp\left(-y\sqrt{\Pr s}\right) - \exp\left(-y\sqrt{s+m}\right)\right] \end{aligned} \tag{3.7}$$

where

$$a = \frac{G}{(1-\Pr)\sqrt{\Pr}}, \quad b = \frac{m}{1-\Pr}, \quad (\Pr \neq 1)$$

On inverting Eqs (3.6) and (3.7) (Erdelyi, 1954; Abramowitz and Stegun, 1965), we can obtain the physical variables $T(y,t)$ and $u(y,t)$ as

$$T(y,t) = 2\sqrt{\frac{t}{\pi \Pr}} \exp\left(-\frac{\Pr y^2}{4t}\right) - y \,\mathrm{erfc}\left(\frac{\sqrt{\Pr}\, y}{2\sqrt{t}}\right) \tag{3.8}$$

$$\begin{aligned} u(y,t) &= K\left[1-\exp(-mt)+\exp(-mt)\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right] \\ &+ \frac{1-K}{2}\left[\exp(-y\sqrt{m})\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}-\sqrt{mt}\right)\right. \\ &\left. + \exp(y\sqrt{m})\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{mt}\right)\right] \\ &- \frac{2a}{\sqrt{\pi}}\left[\int_0^t \{\varphi_1(y,\xi)+\varphi_2(y,\xi)\}\sqrt{t-\xi}\,\mathrm{d}\xi\right. \\ &\left. - \int_0^t \{\varphi_3(y,\xi)+\varphi_4(y,\xi)\}\sqrt{t-\xi}\,\mathrm{d}\xi\right] \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \varphi_{1,2}(y,t) &= \frac{1}{2}\exp(-bt \mp iy\sqrt{b\Pr})\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}} \mp i\sqrt{b\Pr t}\right) \\ \varphi_{3,4}(y,t) &= \frac{1}{2}\exp(-bt \mp iy\sqrt{b\Pr})\,\mathrm{erfc}\left(\frac{y\sqrt{\Pr}}{2\sqrt{t}} \mp i\sqrt{bt}\right) \end{aligned}$$

and erfc(x) is the complementary error function defined by

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x), \quad \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-\xi^2\right)\mathrm{d}\xi$$

In the functions $\varphi_{i,j}(y,t)$, $(i=1,3;\ j=2,4)$ above, the upper sign goes with i and the lower sign with j. It may be noted that Eq (3.9) gives a unified solution incorporating both cases of modes of application of the magnetic field. The individual case of the magnetic field being fixed relative to the fluid or to the plate can be obtained by choosing $K=0$ or $K=1$.

3.1.1 Particular cases (Impulsive)

(a) Fluids of Prandtl number unity

The temperature distribution of the fluids for which the Prandtl number Pr is unity, can be obtained directly from Eq (3.8). However, the solution for the velocity, $u(y,t)$, given by Eq (3.9) is not valid for $\mathrm{Pr}=1$. Since the Prandtl number of a fluid is a measure of the relative importance of its viscosity vis-à-vis the thermal conductivity, the fluids whose Prandtl numbers are close to unity, are of practical interest in several applications. The solution for velocity in this special case can be obtained by solving Eqs (3.2) and (3.3) afresh by setting $\mathrm{Pr}=1$. Following the same procedure as before, it can be shown that the velocity distribution is given by

$$\begin{aligned} u(y,t) = {} & K\left[1-\exp(-mt)+\exp(-mt)\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right] \\ & +\frac{1-K}{2}\left[\exp(-y\sqrt{m})\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}-\sqrt{mt}\right)\right. \\ & \left.+\exp(y\sqrt{m})\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{mt}\right)\right] \\ & +\frac{G}{m}\left[2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{y^2}{4t}\right)-y\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right] \\ & -\frac{G}{2\sqrt{\pi}m}\int_0^t\left[\exp(-y\sqrt{m})\,\mathrm{erfc}\left(\frac{y}{2\sqrt{\xi}}-\sqrt{m\xi}\right)\right. \\ & \left.+\exp(y\sqrt{m})\,\mathrm{erfc}\left(\frac{y}{2\sqrt{\xi}}+\sqrt{m\xi}\right)\right](t-\xi)^{-1/2}\,\mathrm{d}\xi \end{aligned} \tag{3.10}$$

(b) Non-magnetic case ($\mathrm{Pr}\neq 1$)

Another case of interest is when the applied magnetic field is absent. By setting $m=0$ in Eq (3.9), and after detailed simplifications, it can be shown that the velocity distribution is given by

$$\begin{aligned} u(y,t) = {} & \mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right)+\frac{a}{6\sqrt{\pi}}\left[\sqrt{\pi}y(y^2+6t)\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right. \\ & -\sqrt{\pi\,\mathrm{Pr}}y(\mathrm{Pr}\,y^2+6t)\mathrm{erfc}\left(\frac{y}{2}\sqrt{\frac{\mathrm{Pr}}{t}}\right) \\ & -\sqrt{t}(y^2+8t)\exp\left(-\frac{y^2}{4t}\right) \\ & \left.+\sqrt{t}(\mathrm{Pr}\,y^2+8t)\exp\left(-\frac{\mathrm{Pr}\,y^2}{4t}\right)\right] \end{aligned} \tag{3.11}$$

3.1.2 Discussion – Impulsive case

We shall now discuss the variation of boundary layer temperature and velocity profiles with the non-dimensional parameters – Grashoff number G, Prandtl number Pr, magnetic parameter m – and the nondimensional time variable t. In the Fig 1, we have shown temperature variation with the Prandtl number and time variable. Clearly, for a fixed t, temperature decreases with Prandtl number, while the opposite happens with time.

As stated before, the free convection velocity is dependent on the temperature besides other nondimensional parameters. The variation of velocity for the impulsive motion of the boundary is shown in Fig 2 for both modes of application of the magnetic field. When the magnetic field is applied relative to the fluid ($K = 0$ case), the velocity decreases monotonically with the magnetic field, thus inhibiting the fluid movement in the boundary layer. On the other hand, when the magnetic field is applied relative to the moving boundary ($K = 1$ case), the fluid velocity increases with the magnetic field. Moreover, close to the boundary, the velocity changes with m are apparently more pronounced for this case as compared to the former ($K = 0$). In Fig 3, we have illustrated the dependence of velocity on the parameter G. This parameter is a measure of the buoyancy effects on the flow, and can assume positive or negative values depending upon whether the heat flux at the plate is negative or positive. It is seen from this figure that relative to the zero-flux curve ($G = 0$), the increase in G causes enhancement of velocity, whereas the reverse occurs for negative values of G. Furthermore, the buoyancy force parameter has been seen to influence the velocity in a thinner layer as compared to other parameters. The temporal evolution of the velocity profiles is shown in Fig 4 (for $K = 0$). The transient velocity profiles show that the velocity increases gradually with time at any given cross-section. In the very early stages of motion, the velocity curves were found to merge with their free stream value rather rapidly.

In several industrial applications, the role of the Prandtl number on the flow variables assumes great significance. In fact, Prandtl numbers of industrially important fluids can vary from a very small value (e.g., mercury, with $\mathrm{Pr} = 0.044$) to much higher values (e.g., water, with $\mathrm{Pr} = 7.0$). In our study, a single solution valid for all possible Pr values could not be obtained. The singularity of the solution for the unit Prandtl number case has already been pointed out earlier in the text. Furthermore, we observe that the analytical solution, Eq (3.9), will appear in different real forms depending upon whether Pr is greater than or less than unity, since b is positive when $\mathrm{Pr} < 1$ and negative when $\mathrm{Pr} > 1$. When $\mathrm{Pr} < 1$, the arguments of the complementary error functions and exponential functions occurring in Eq (3.9) are complex, while for $\mathrm{Pr} > 1$, they are real. Thus the solution (3.9) is the real form of the velocity for fluids of Prandtl number greater than unity only. When $\mathrm{Pr} < 1$, one could obtain the real form of the velocity by first writing down the complex form of erfc$(x + iy)$ (Strand, 1965)

$$\mathrm{erfc}(x + iy) = f(x, y) + ih(x, y) \tag{3.12}$$

where

$$\begin{aligned}
f(x,y) &= \sum_{n=0}^{\infty} \left[(xy)^{2n} g_n(x) \cos 2xy - (n+1)(xy)^{2n+1} g_{n+1}(x) \sin 2xy\right] \\
h(x,y) &= -\sum_{n=0}^{\infty} \left[(xy)^{2n} g_n(x) \sin 2xy + (n+1)(xy)^{2n+1} g_{n+1}(x) \cos 2xy\right] \\
g_{n+1}(x) &= \frac{2}{2n+1}\left[\frac{\exp\left(-x^2\right)}{\sqrt{\pi}(n+1)!\, x^{2n+1}} - \frac{g_n(x)}{n+1}\right] \\
g_0(x) &= \mathrm{erfc}(x)
\end{aligned}$$

We have computed velocity profiles for a host of Pr values. The profiles have been shown in Fig 5, for fixed values of G, m and t. It can be readily seen that the fluid velocity in the boundary layer is more sensitive to low Prandtl number regimes. For such low values, we observe overshooting phenomenon for the fluid velocity near the bounding surface – a feature akin to boundary layer flows of viscoelastic fluids (Beard and Walters, 1964; Sarpkaya and Rainey, 1971; Gorla 1978; Sacheti and Chandran, 1998). For moderate to high values of Prandtl number, velocity rapidly approaches its free stream value. Incidentally, it may be remarked that the velocity tends to overshoot its value at the boundary with respect to increasing t also, as can be seen from Fig 4.

3.2 Accelerated motion of the bounding plate

This case corresponds to the velocity of the vertical boundary being $u = \lambda t$. The solution procedure for this accelerated plate motion follows the same steps as in Section 3.1. However, we need to use different length and velocity scales in the non-dimensionalisation process because the dimension of λ in this case is different from that of the impulsive one. We write

$$\begin{aligned} \bar{y} &= y/L_2, \quad \bar{t} = \nu t/L_2^2, \quad \bar{u} = L_2 u/\nu, \quad \bar{T} = k(T - T_\infty)/(qL_2) \\ P &= \rho\nu c_p/k, \quad G = qg\beta L_2^4/(k\nu^2), \quad m = \sigma L_2^2 B_y^2/(\rho\nu) \end{aligned} \tag{3.13}$$

where the new length scale is, $L_2 = (\nu^2/\lambda)^{1/3}$. It may be remarked that the Grashoff number G and the Hartmann number $\sqrt{m}$, for this case have been defined differently. In view of Eq (3.13), the dimensionless governing equations and the associated conditions, dropping the 'bar', as before, will now be

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} - m(u - Kt) + GT \tag{3.14}$$

$$\frac{\partial T}{\partial t} = \frac{1}{P}\frac{\partial^2 T}{\partial y^2} \tag{3.15}$$

$$\begin{aligned} &u = 0, \quad T = 0 \quad \text{for} \quad y \geq 0 \quad \text{and} \quad t \leq 0 \\ &u = t, \quad \partial T/\partial y = -1 \quad \text{at} \quad y = 0 \quad \text{for} \quad t > 0 \\ &u \to 0, \quad T \to 0 \quad \text{as} \quad y \to \infty \quad \text{for} \quad t > 0 \end{aligned} \tag{3.16}$$

The solution of Eq (3.15) is given by Eq (3.8). To obtain the solution of Eq (3.14), we first obtain the Laplace transform $\tilde{u}(y, s)$ in the form

$$\begin{aligned} \tilde{u}(y,s) = {} & \frac{Km}{(s+m)s^2}\left[1 - \exp\left(-y\sqrt{s+m}\right)\right] + \frac{1}{s^2}\exp\left(-y\sqrt{s+m}\right) \\ & + \frac{a}{(s+b)\sqrt{s^3}}\left[\exp\left(-y\sqrt{\Pr s}\right) - \exp\left(-y\sqrt{s+m}\right)\right] \end{aligned} \tag{3.17}$$

where a and b are as defined in Section 3.1. On inverting the function in Eq (3.17), we can express the velocity as

$$\begin{aligned}
u(y,t) \;=\;& \frac{K}{m}\left[mt - 1 + 0.5\exp(-y\sqrt{m})\,\text{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{mt}\right)\right.\\
&\left. + 0.5\exp(y\sqrt{m})\,\text{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{mt}\right) + \exp(-mt)\,\text{erf}\left(\frac{y}{2\sqrt{t}}\right)\right]\\
&+ (1-K)\left[\left(0.5t - \frac{y}{4\sqrt{m}}\right)\exp(-y\sqrt{m})\,\text{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{mt}\right)\right.\\
&\left. + \left(0.5t + \frac{y}{4\sqrt{m}}\right)\exp(y\sqrt{m})\,\text{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{mt}\right)\right]\\
&- \frac{2a}{\sqrt{\pi}}\left[\int_0^t \{\varphi_1(y,\xi) + \varphi_2(y,\xi)\}\sqrt{t-\xi}\,\mathrm{d}\xi\right.\\
&\left. - \int_0^t \{\varphi_3(y,\xi) + \varphi_4(y,\xi)\}\sqrt{t-\xi}\,\mathrm{d}\xi\right] \qquad (3.18)
\end{aligned}$$

The functions φ_1 to φ_4 occurring in the above solution have been defined previously in Eq (3.9).

3.2.1 Particular cases (Accelerated)

The special cases we are interested in are: $\text{Pr} = 1$ and $m = 0$. The observations and solution procedure stated in the case of impulsive motion of the boundary for the case $\text{Pr} = 1$ are applicable for the accelerated motion of the boundary also. Moreover, in the non-magnetic case, since $b = 0$ for $\text{Pr} \neq 1$, the general solution (3.18) does not reduce directly to the particular solution. Thus the solution procedure has to be repeated starting from the governing equations. For these particular cases, the expressions for the velocity variable are given below:

(a) Fluids of Prandtl number unity

$$\begin{aligned}
u(y,t) \;=\;& \frac{K}{m}\left[mt - 1 + 0.5\exp(-y\sqrt{m})\,\text{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{mt}\right)\right.\\
&\left. + 0.5\exp(y\sqrt{m})\,\text{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{mt}\right) + \exp(-mt)\,\text{erf}\left(\frac{y}{2\sqrt{t}}\right)\right]\\
&+ (1-K)\left[\left(0.5t - \frac{y}{4\sqrt{m}}\right)\exp(-y\sqrt{m})\,\text{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{mt}\right)\right.\\
&\left. + \left(0.5t + \frac{y}{4\sqrt{m}}\right)\exp(y\sqrt{m})\,\text{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{mt}\right)\right]\\
&+ \frac{2G}{m}\left[\sqrt{\frac{t}{\pi}}\exp\left(-\frac{y^2}{4t}\right) - y\,\text{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right]\\
&- \frac{G}{2\sqrt{\pi}m}\int_0^t \left[\exp(-y\sqrt{m})\,\text{erfc}\left(\frac{y}{2\sqrt{\xi}} - \sqrt{m\xi}\right)\right.\\
&\left. + \exp(y\sqrt{m})\,\text{erfc}\left(\frac{y}{2\sqrt{\xi}} + \sqrt{m\xi}\right)\right](t-\xi)^{-1/2}\,\mathrm{d}\xi \qquad (3.19)
\end{aligned}$$

(b) Non-magnetic case ($\mathrm{Pr} \neq 1$)

$$\begin{aligned} u(y,t) &= \left(\frac{ay^3}{6} + \frac{y^2}{2} + ayt + t\right) \operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right) \\ &\quad - \sqrt{\mathrm{Pr}}\left(\frac{\mathrm{Pr}\, ay^3}{6} + ayt\right) \operatorname{erfc}\left(\frac{\sqrt{\mathrm{Pr}}\, y}{2\sqrt{t}}\right) \\ &\quad - \frac{\sqrt{t}}{3\sqrt{\pi}}\left(ay^2 + 3y + 4at\right) \exp\left(-\frac{y^2}{4t}\right) \\ &\quad + \frac{a\sqrt{t}}{3\sqrt{\pi}}\left(\mathrm{Pr}\, y^2 + 4t\right) \exp\left(-\frac{\mathrm{Pr}\, y^2}{4t}\right) \end{aligned} \quad (3.20)$$

3.2.2 Discussion – Accelerated case

The free convection velocity profiles caused by the accelerated motion of the bounding plate and the uniform heat flux on it, can be analyzed similar to the previous section. In this case, only the velocity profiles are different, while the temperature ones are the same as before. In Fig 6, we have shown the variation of boundary layer velocity profiles, $u(y,t)$, with the magnetic field parameter m. For a comparison with the impulsive case, we have chosen here the same set of parameter values as in Fig 2. The velocity sketches, for both modes of application of the magnetic field, bear broad similarity to the corresponding ones in the impulsive case.

In order to assess the influence of other parameters on the flow velocity for the accelerated motion of the boundary, we have shown in Figs 7 and 8 the effects of Grashoff number G and the time variable t for two different types of fluids – mercury (Fig 7) and water (Fig 8). In Fig 7, we observe that, for the range of values of G and t used, the fluid velocity exceeds its value at the boundary ($y = 0$) for small values of Pr – a feature observed in the impulsive case as well. Also, the velocity increases with G and t along the vertical sections. Comparing the profiles in Figs 7 and 8, we observe that the velocity is more sensitive to buoyancy effects for fluids of very small Prandtl numbers. For such fluids, the velocity curves attain maxima near the boundary as G is increased. The points of maxima, in turn, shift away from the bounding plate as time progresses.

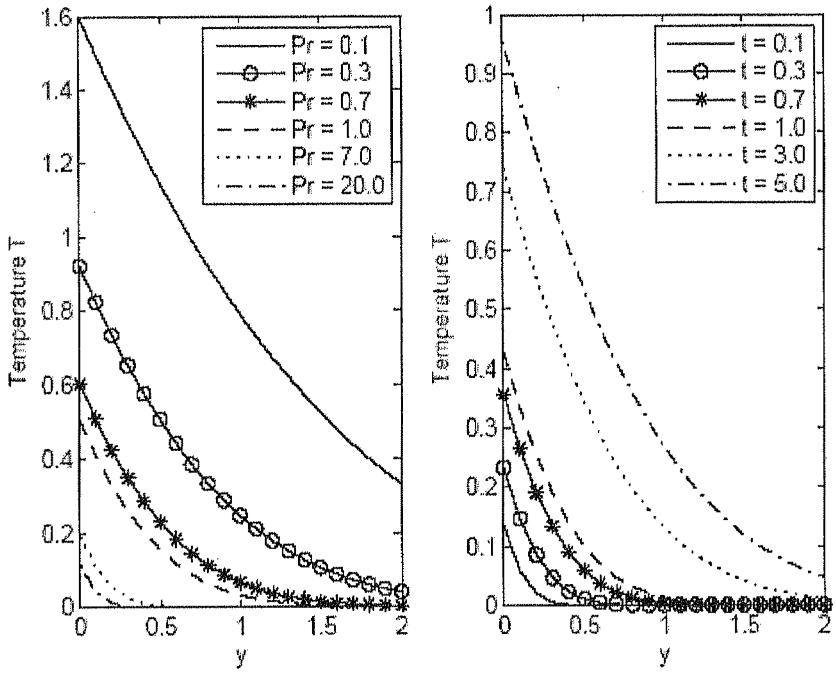

Fig 1. Temperature profiles:
Effects of Prandtl number Pr and time t.

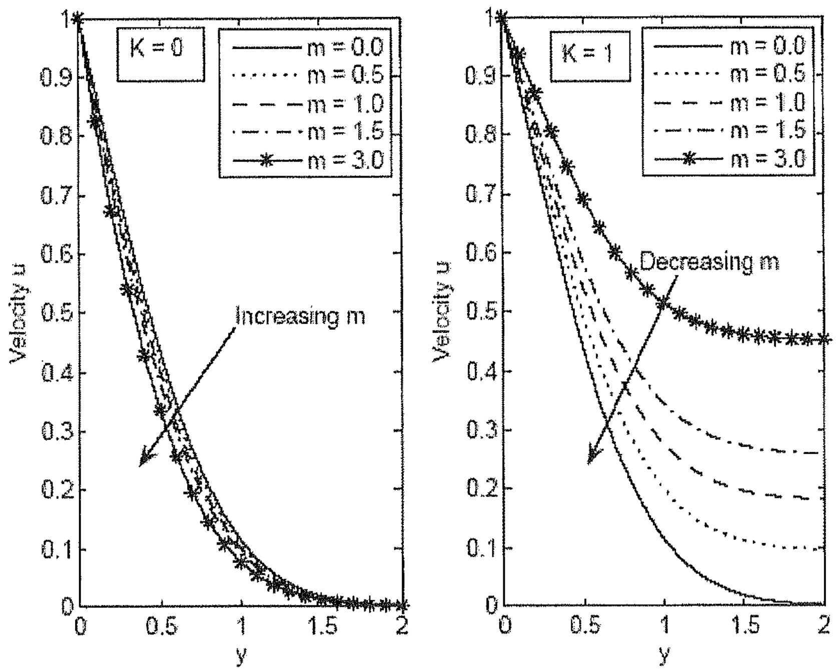

Fig 2. Velocity profiles for impulsive motion:
Effect of magnetic field parameter m.
$\mathrm{Pr} = 7.0,\ G = 5.0,\ t = 0.2$

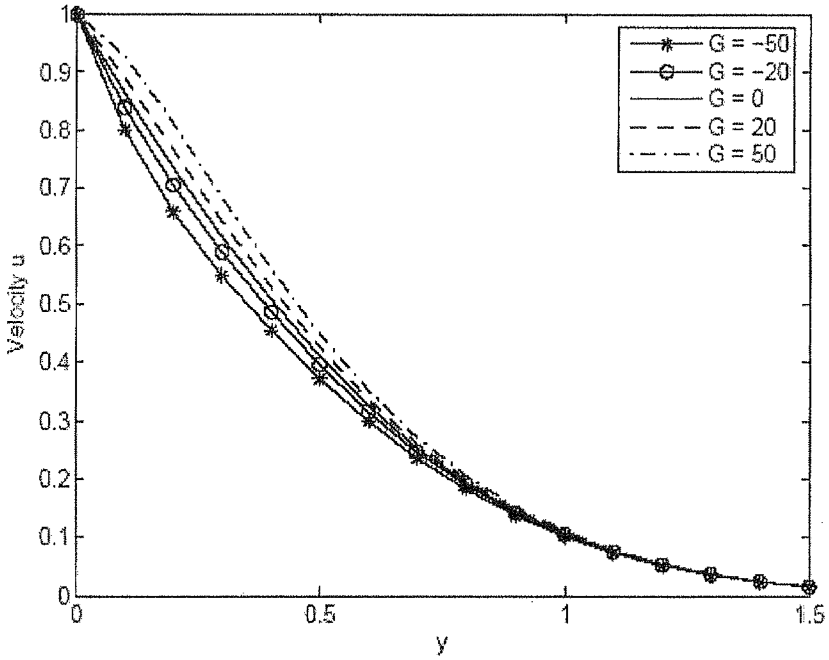

Fig 3. Velocity profiles for impulsive motion:
Effect of buoyancy parameter G.
$K = 0,\ \mathrm{Pr} = 7.0,\ t = 0.2,\ m = 0.5$

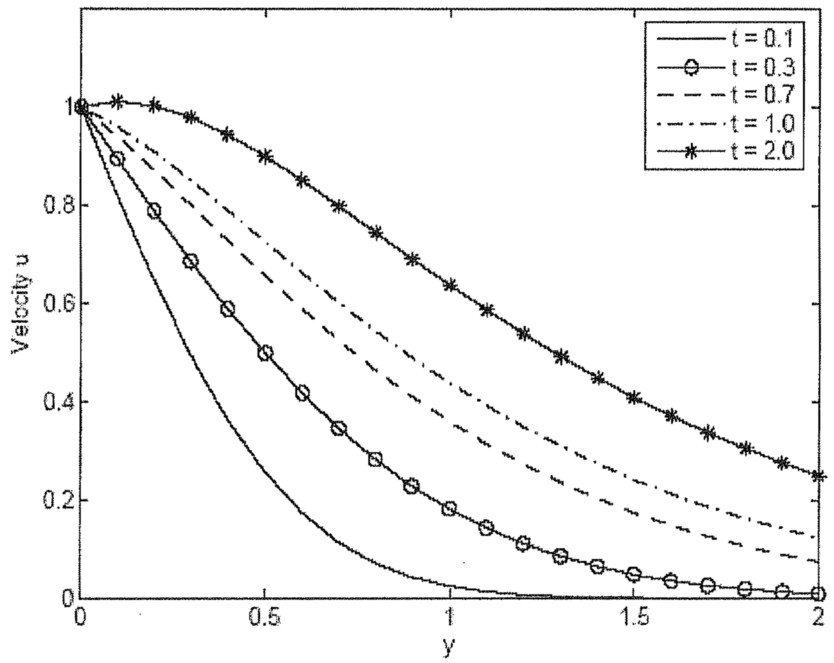

Fig 4. Developing velocity profiles for impulsive motion:
$K = 0,\ \mathrm{Pr} = 7.0,\ G = 5.0,\ m = 0.5$

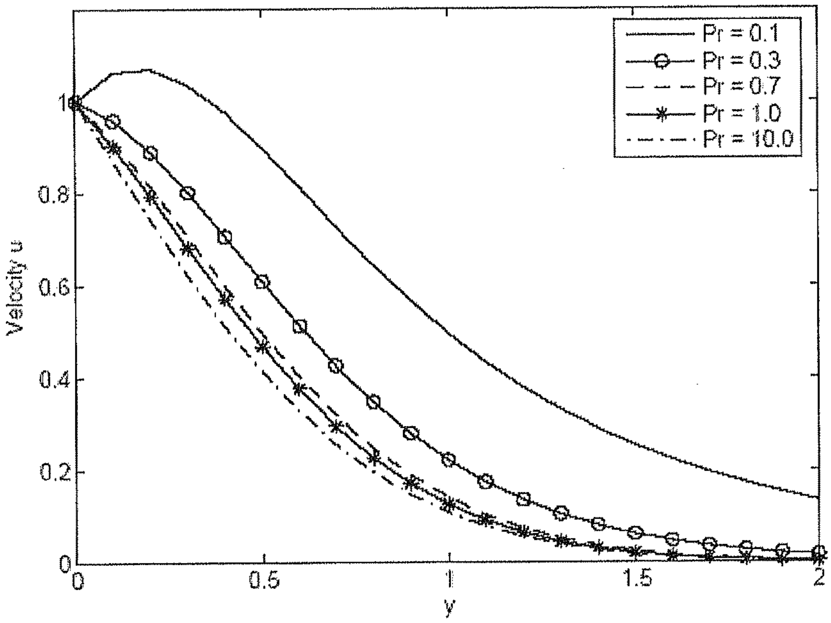

Fig 5. Velocity profiles for impulsive motion:
Effect of Prandtl number Pr.
$K = 0,\ G = 5.0,\ t = 0.2,\ m = 0.5$

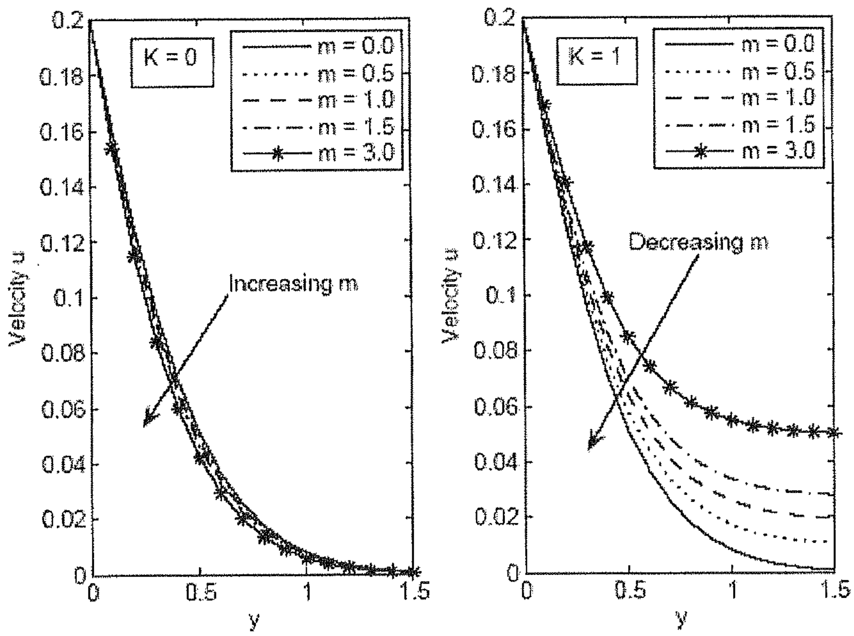

Fig 6. Velocity profiles for accelerated motion:
Effect of magnetic field parameter m.
$\mathrm{Pr} = 7.0,\ G = 5.0,\ t = 0.2,\ m = 0.5$

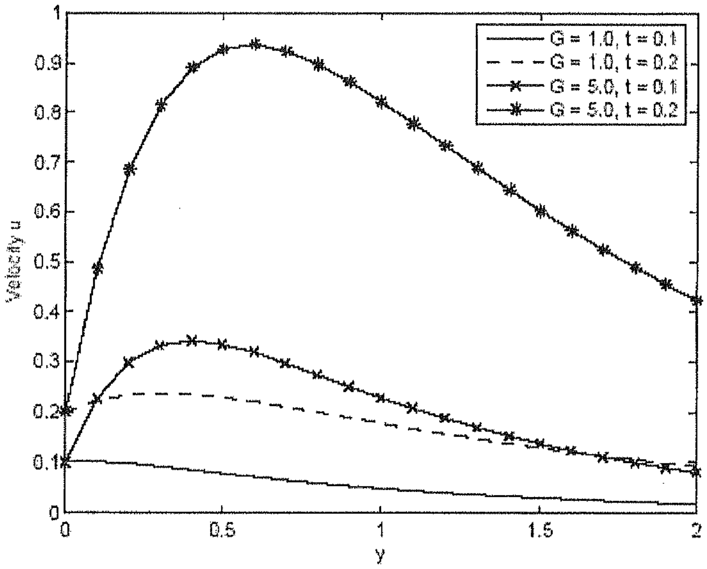

Fig 7. Velocity profiles for accelerated motion:
Effects of Buoyancy parameter G and
nondimensional time t. $K = 1$, $\Pr = 0.044$, $m = 0.5$

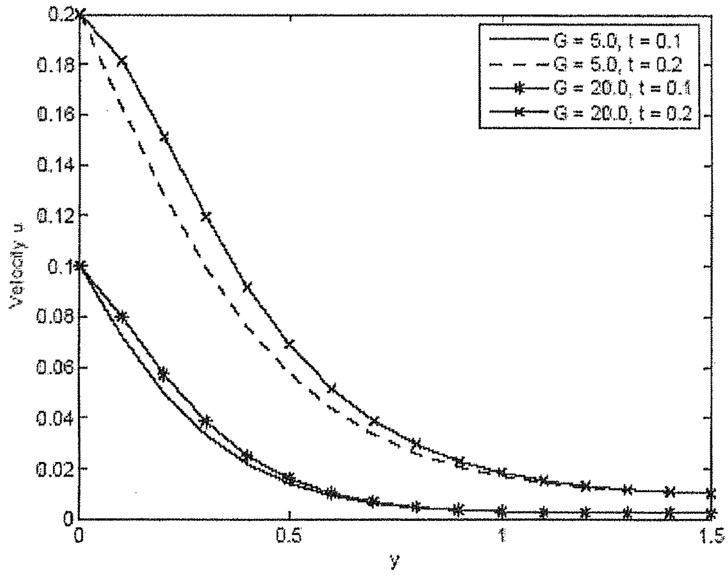

Fig 8. Velocity profiles for accelerated motion:
Effects of Buoyancy parameter G and
nondimensional time t. $K = 1$, $\Pr = 7.0$, $m = 0.5$

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International Journal of Applied Mathematics & Statistics
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

IJAMAS: September 2007, Vol. 10, No. S07, 25-37

Functions and Distributions in Spaces with Thick Points

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ABSTRACT

Euler's vision of a generalized concept of function was a forerunner of the modern concept of distribution, and his efforts to give meaning to divergent series eventually led to the concepts of asymptotic series, summability, and distributional convergence. The introduction of such suitable abstract concepts does not automatically prevent mistakes or inconsistencies resulting from careless formal reasoning. We deal with a cluster of such issues associated with the occurrence of a distributional singularity on the boundary of a domain of integration. Apparent paradoxes are resolved by introducing new classes of test functions and distributions adapted to the problems at hand; one can regard the construction as attributing internal structure to boundary points.

Keywords: distributions, test functions, thick points, Dirac delta.

2000 Mathematics Subject Classification: 46F10, 46F12.

1 Introduction

It is clear from the articles in the *Mathematics Magazine* special issue devoted to Leonhard Euler, especially those of Lützen [13] and Kline [12], that in some ways Euler's sensibilities and talents were closer to those of a creative theoretical physicist such as Richard Feynman or Paul Dirac than those of a modern rigorous mathematician. If it had not been so, many of his most important contributions would not have seen the light of day.

Like most of his contemporaries, Euler "trusted the symbols far more than logic" [12]. If an infinite series did not converge for some values of a variable, the response was not to reject it as meaningless; rather, it was taken for granted that the series had a meaning and the task was to find it. Often, formal manipulations gave useful results, whose justifications were achieved only in the twentieth-century theories of asymptotics, summability (in the sense of Cesàro and Riesz), and convergence in spaces of distributions [12].

Euler worked during the period of transition from the naive notion of a function as an algebraic formula to the modern concept of a function as an arbitrary association of dependent and independent variables. His writings can be cited on both sides of the debate [13]. Because of the limitations of nineteenth-century analysis, the new definition was a restriction as much as an extension. Euler's intuition told him that generalized functions that do not satisfy draconian requirements of smoothness or even pointwise definability are too important to be left out. His vision of a generalized calculus was vindicated, as closely as it could be, by the modern theory of distributions [13].

Powerful ideas are dangerous. To paraphrase a remark of Valentine Bargmann, it is not correct to say that the work of Laurent Schwartz justifies everything that physicists do with the Dirac delta function, because sometimes they do things that are clearly wrong. There is a spectrum of responses to this situation. The first (chosen by too many mathematicians) is to dismiss distributions as untrustworthy, a kind of pornography that should be kept out of the hands of engineering and science students. Another (adopted by many practitioners) is to rationalize after the fact whatever interpretation of the symbols gives the right answer in the problem at hand; as we shall see below, sometimes this is done in blatant contradiction to interpretations adopted in other contexts. A safer approach is to regard the delta function as a heuristic device that leads rapidly to formulas whose correctness must then be rigorously verified (e.g., by substituting a putative solution back into a differential equation). But one cannot be satisfied just with this; if distributions are unambiguously defined as linear functionals on spaces of test functions, then their properties must be unambiguous, and the mathematician should determine which formulas and calculational rules are true and why — tightening up the definitions when necessary.

2 Some puzzles

We begin with a problem that surely would have delighted Euler: Evaluate the integral

$$\int_0^\infty \cos(2kx)\,dx. \tag{2.1}$$

In the classical sense it does not converge, but nevertheless it arises naturally in the spectral theory of simple differential operators and in related applications to, for example, quantum field theory. (It is a simple analogue of integrals that arose in [3].) One expects (2.1) to make sense as a distribution in k, with $k \geq 0$. (It is essentially the orthogonality relation for the Fourier cosine transform, in which k is inherently nonnegative.) We now evaluate the integral in two very plausible ways, getting two different answers.

First, we argue that

$$\begin{aligned}\int_0^\infty \cos(2kx)\,dx &= \frac{1}{2}\int_{-\infty}^\infty \cos(2kx)\,dx \\ &= \frac{1}{2}\int_{-\infty}^\infty e^{2ikx}\,dx \\ &= \pi\delta(2k) \\ &= \frac{\pi}{2}\,\delta(k).\end{aligned} \tag{2.2}$$

Here the last two steps are well-known distributional identities, and the rest is elementary complex analysis.

On the other hand, we calculate

$$\int_0^\infty \cos(2kx)\,dx = \left.\frac{\sin(2kx)}{2k}\right|_{x=0}^{\infty} = \lim_{x\to\infty} \frac{\sin(2kx)}{2k}.$$

By definition of a distributional integral, we must evaluate this limit after integrating over a test function, $f(k)$, with support in $[0,\infty)$:

$$\lim_{x\to\infty}\int_0^\infty \frac{\sin(2kx)}{2k} f(k)\,dk = \lim_{x\to\infty}\int_0^\infty \frac{\sin u}{2u} f\left(\frac{u}{2x}\right) du = \frac{1}{2} f(0)\int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{4} f(0), \tag{2.3}$$

where the last step uses a well-known integral. That is,

$$\int_0^\infty \cos(2kx)\,dx = \frac{\pi}{4}\,\delta(k). \tag{2.4}$$

One way to resolve the conflict between (2.2) and (2.4) is to conclude *a posteriori* that the (standard!) interpretation of δ used to pass from (2.3) to (2.4) is incorrect: Instead, one should postulate that

$$\int_0^\infty f(k)\delta(k)\,dk = \frac{1}{2} f(0), \tag{2.5}$$

or, equivalently,

$$\delta(k)H(k) = \frac{1}{2}\,\delta(k). \tag{2.6}$$

Here H is the Heaviside step function; other common notations for it are u and θ.

Formulas (2.5) and (2.6) are not new to the literature [16, 2]. Vibet [16] offers a proof: Since $\frac{1}{2}\frac{d}{dk}H(k)^2 = H(k)\frac{dH}{dk} = H(k)\delta(k)$ and $H(k)^2 = H(k)$, (2.6) follows. He does not note that multiplying (2.6) by $H(k)$ and using $H(k)^2 = H(k)$ again yields the surprising equation $H(k)\delta(k) = \frac{1}{2}H(k)\delta(k)$ (and by iteration $\ldots = \frac{1}{4}H(k)\delta(k) = \ldots$); but he does assert that $H(k)^{n-1}\delta(k) = \frac{1}{n}\delta(k)$ and that this relationship is needed to solve certain engineering problems properly. (Paskusz [14] criticizes [16] similarly by concluding from (2.6) that $H(k) = \frac{1}{2}$. It may be objected (correctly) that this conclusion holds only for $k = 0$, but even that is inconsistent with a literal interpretation of $H(k)^2 = H(k)$. The real point, of course, is that the value of a distribution at one point is not well-defined, in general. It is noteworthy, however, that such pointwise definitions of $H(0)$ are used even in relatively sophisticated and accurate papers such as [2] to motivate, or at least to parametrize, rival definitions of $\delta(k)$ when 0 is an endpoint of the interval of integration. We thank R. Nevels for pointing out references [16, 14, 2].)

Clearly, a more careful analysis of definitions is needed to determine whether or not the factor $\frac{1}{2}$ does belong in (2.4), (2.5), (2.6). It will be necessary to redefine distributions, treating the point $k = 0$ in a special way.

3 Spaces with thick points

Let $a \in \mathbb{R}$. We shall define $\mathcal{D}_{*,a}$, the space of test functions with a *thick point* located at $x = a$, and $\mathcal{D}'_{*,a}$, the corresponding space of distributions. A function ϕ with domain $\mathbb{R}$ belongs to $\mathcal{D}_{*,a}$ if it has compact support, it is smooth in $\mathbb{R} \setminus \{a\}$, and at $x = a$ all its one-sided derivatives,

$$\phi^{(n)}(a \pm 0) = \lim_{x \to a^{\pm}} \phi^{(n)}(x), \quad \forall n \in \mathbb{N}, \tag{3.1}$$

exist. $\mathcal{D}_{*,a}$ has a natural topology, in which $\mathcal{D}(\mathbb{R})$ is the closed subspace where $\phi^{(n)}(a+0) = \phi^{(n)}(a-0)$, $\forall n \in \mathbb{N}$. The elements of $\mathcal{D}'_{*,a}$ are the distributions defined in the standard way as the linear functionals on this enlarged space of test functions.

One can also define in a similar way the spaces $\mathcal{A}_{*,a}$ and $\mathcal{A}'_{*,a}$ for any of the usual spaces of test functions and distributions. For instance, $\mathcal{E}'_{*,a}$ is the space of compactly supported distributions with a thick point at $x = a$, and $\mathcal{S}'_{*,a}$ the corresponding space of tempered distributions. Without loss of generality we shall take $a = 0$ and use the simpler notations $\mathcal{A}_*$ and $\mathcal{A}'_*$. It is clear that instead of one thick point one could consider a space with a finite number of thick points, or even an infinite (but discrete) set of them. Somewhat less trivial, and beyond the scope of this paper, would be the extension to distributions in several variables. In fact, the idea of considering functions and generalized functions in spaces with thick points was apparently first proposed by Blanchet and Faye [1] in the context of finite parts, pseudo-functions and Hadamard regularization studied by Sellier [15]; their analysis is aimed at the study of the dynamics of point particles in high post-Newtonian approximations of general relativity, and it thus developed in dimension 3 (which is also the natural arena for a precise reformulation of the work of Blinder [2]).

If X and Y are topological vector spaces with $X \subset Y$, the inclusion, i, being continuous, we shall denote by π the adjoint operator, $\pi = i'$, which is a projection from Y' to X'. In the case of spaces with thick points, one has $\mathcal{A} \subset \mathcal{A}_{*,a}$, and thus we have a projection $\pi\colon \mathcal{A}'_{*,a} \to \mathcal{A}'$, given explicitly as

$$\langle \pi(f), \phi \rangle_{\mathcal{A}' \times \mathcal{A}} = \langle f, \phi \rangle_{\mathcal{A}'_{*,a} \times \mathcal{A}_{*,a}}.$$

Every distribution $g \in \mathcal{A}'$ can be extended to $\mathcal{A}'_{*,a}$; that is, there exist distributions $f \in \mathcal{A}'_{*,a}$ such that $\pi(f) = g$. If f_0 is any extension, then the most general extension is given as

$$f = f_0 + \sum_{j=0}^{n} \alpha_j s_j, \tag{3.2}$$

where $s_j = s_{j,a}$ are the distributions that give the *saltus* (jump) of the jth derivative across $x = a$,

$$\langle s_j, \phi \rangle = \phi^{(j)}(a+0) - \phi^{(j)}(a-0), \tag{3.3}$$

where $n \in \mathbb{N}$, and where $\alpha_0, \dots, \alpha_n$ are arbitrary constants.

The derivative of $\phi \in \mathcal{A}_{*,a}$ is defined classically. (In particular, a saltus in ϕ does *not* generate a δ term in ϕ'.) Then we may define the derivatives of the distributions of $\mathcal{A}'_{*,a}$ by the usual duality process, $\langle f', \phi \rangle = -\langle f, \phi' \rangle$. Clearly, $\pi(f') = \pi(f)'$. Also, $s_j = (-1)^j s_0^{(j)}$.

We shall consider the one-sided delta functions at the thick point, $\delta_{\pm}(x) = \delta\big(x - (a \pm 0)\big)$, defined as

$$\big\langle \delta\big(x - (a \pm 0)\big), \phi(x) \big\rangle = \phi(a \pm 0). \tag{3.4}$$

Observe that $s_0(x) = \delta(x-(a+0)) - \delta(x-(a-0))$, and more generally $(-1)^j s_j(x) = \delta^{(j)}(x-(a+0)) - \delta^{(j)}(x-(a-0))$.
It is important to observe that the derivative formulas in the space $\mathcal{A}'_{*,a}$ can be somewhat different from the usual derivative formulas. Indeed, suppose that $f \in \mathcal{A}'_{*,a}$ is a regular distribution generated by a function that is of class C^1 in both $(-\infty, a]$ and $[a, \infty)$ but that may have a jump $[f] = f(a+0) - f(a-0)$ across the thick point. Then f can also be considered an element of the usual space of distributions $\mathcal{A}'$, and we have the well-known formula [11]

$$\frac{\overline{d}f}{dx} = \frac{df}{dx} + [f]\delta(x-a), \tag{3.5}$$

where the overbar denotes the distributional derivative and df/dx is the ordinary (classical) derivative. However, the derivative in the space $\mathcal{A}'_{*,a}$, denoted d^*f/dx, is given by the relation

$$\frac{d^*f}{dx} = \frac{df}{dx} + f(a+0)\delta_+(x) - f(a-0)\delta_-(x). \tag{3.6}$$

Naturally, (3.5) and (3.6) satisfy $\pi(d^*f/dx) = \overline{d}f/dx$. Nevertheless, if f is continuous at $x = a$, then the $\overline{d}$ derivative coincides with the ordinary derivative, but in the space $\mathcal{A}'_{*,a}$ we have

$$\frac{d^*f}{dx} = \frac{df}{dx} + f(a)s_0(x). \tag{3.7}$$

Observe, in particular, that if c is a constant then

$$\frac{d^*c}{dx} = cs_0(x). \tag{3.8}$$

In the space $\mathcal{A}'_{*,a}$ the homogenous differential equation $d^*f/dx = 0$ has only the trivial solution, while if $g \in \mathcal{A}'_{*,a}$ the equation $d^*f/dx = g$ has at most one solution. (Actually, in the spaces $\mathcal{D}'_{*,a}$ or $\mathcal{S}'_{*,a}$ the equation $d^*f/dx = g$ has exactly one solution, but in other spaces, such as $\mathcal{E}'_{*,a}$, existence requires the extra condition $\langle g, 1\rangle = 0$.)
The general form of the extensions of the Dirac delta function $\delta(x-a)$ to the thick-point space that are of order 0, that is, that do not contain derivatives of the deltas, is

$$\delta_{*,a,\lambda}(x) = \lambda\delta(x-(a+0)) + (1-\lambda)\delta(x-(a-0)), \tag{3.9}$$

where λ is any constant. The case when $\lambda = \frac{1}{2}$ gives us the only such extension,

$$\widetilde{\delta}(x-a) = \delta_{*,a,1/2}(x) = \tfrac{1}{2}\left[\delta(x-(a+0)) + \delta(x-(a-0))\right], \tag{3.10}$$

that is symmetric with respect to $x = a$.
Let us now consider multiplication in the spaces $\mathcal{A}'_{*,a}$. Any space of distributions $\mathcal{A}'$ has a corresponding Moyal algebra $\mathcal{B}$, the space of multipliers of $\mathcal{A}$ and of $\mathcal{A}'$, i.e., those smooth functions ρ that satisfy $\rho\phi \in \mathcal{A}$, $\forall\phi \in \mathcal{A}$. If $\mathcal{A} = \mathcal{D}$ then $\mathcal{B} = \mathcal{E}$; if $\mathcal{A} = \mathcal{E}$ then $\mathcal{B} = \mathcal{E}$; if $\mathcal{A} = \mathcal{S}$ then $\mathcal{B} = \mathcal{O}_M$. (For more on $\mathcal{O}_M$ and the other spaces see [10] or [8].) In the spaces with thick points, if $\rho \in \mathcal{B}_{*,a}$, then $\rho\phi \in \mathcal{A}_{*,a}$, $\forall\phi \in \mathcal{A}_{*,a}$, and thus we may define the multiplication $\rho f \in \mathcal{A}'_{*,a}$ whenever $f \in \mathcal{A}'_{*,a}$ by the formula

$$\langle \rho(x)f(x), \phi(x)\rangle = \langle f(x), \rho(x)\phi(x)\rangle. \tag{3.11}$$

On the other hand, if $\rho \in \mathcal{B}_{*,a}$ then the multiplication $\rho\phi$ belongs to $\mathcal{A}_{*,a}$ for any $\phi \in \mathcal{A}$, and thus we can define an operator of multiplication $M_\rho\colon \mathcal{A} \to \mathcal{A}_{*,a}$, and, by duality, a corresponding multiplication operator $M_\rho\colon \mathcal{A}'_{*,a} \to \mathcal{A}'$. Observe that

$$\pi(\rho f) = M_\rho(f). \tag{3.12}$$

Notice too that if $\rho_1, \rho_2 \in \mathcal{B}_{*,a}$ then we can perform the operation $\rho_1(\rho_2 f)$, which, naturally, turns out to be $(\rho_1\rho_2)f$. However, the product $M_{\rho_1}M_{\rho_2}$ is not defined. This fact is at the root of the H^2 paradoxes in Sec. 2 (see Sec. 6).

If $\rho \in \mathcal{E}$, then $\rho(x)\delta(x-a) = \rho(a)\delta(x-a)$. The corresponding formula when there are thick points is as follows:

$$\begin{aligned}\rho(x)\delta_{*,a,\lambda}(x-a) = {} & \lambda\rho(a+0)\delta\big(x-(a+0)\big) \\ & + (1-\lambda)\rho(a-0)\delta\big(x-(a-0)\big).\end{aligned} \tag{3.13}$$

Thus $M_\rho\big(\delta_{*,a,\lambda}(x-a)\big) = \big[\lambda\rho(a+0) + (1-\lambda)\rho(a-0)\big]\delta(x)$, and in particular $M_\rho\big(\widetilde{\delta}(x-a)\big) = \{\rho\}\delta(x-a)$, where $\{\rho\} = \frac{1}{2}\big(\rho(a+0)+\rho(a-0)\big)$ is the average value at the thick point.

4 The Fourier transform in spaces with thick points

The Fourier transform of tempered distributions is a much studied and well-known operator. One of the properties of the Fourier transform operator, $\mathcal{F}$, is that it is an isomorphism of the space of test functions, $\mathcal{S}$, to itself as well as an isomorphism of the space of distributions, $\mathcal{S}'$, to itself. When one considers the operator $\mathcal{F}$ in spaces of distributions that are contained in $\mathcal{S}'$, say $\mathcal{A}' \subset \mathcal{S}'$, then $\widehat{f} = \mathcal{F}(f)$ is a tempered distribution whenever $f \in \mathcal{A}'$. However, when $\mathcal{A}'$ is not a space of tempered distributions, the image $\mathcal{F}(\mathcal{A}')$ will not be a space of tempered distributions either. This is the situation in spaces with thick points, since $\mathcal{S}'_*$ is not a subspace of $\mathcal{S}'$. Another example of this situation is the study of the Fourier transform in the space $\mathcal{D}'$ done by Gel'fand and Shilov [9] ; see also [17] for a similar analysis of other integral transforms. In this article we adopt the simplest definition of the Fourier transform: $\widehat{f}(u) = \mathcal{F}\{f(x);u\}$ is given by the integral $\int_{-\infty}^{\infty} f(x)e^{ixu}\,dx$ when the integral exists and defined by duality or other methods when the integral diverges. Naturally, our results will remain valid, modulo trivial modifications, for all the variant conventions, and hence, in particular, for the inverse Fourier transform,

$$\mathcal{F}^{-1}\{f(x);u\} = (2\pi)^{-1}\mathcal{F}\{f(x);-u\}.$$

If $\phi \in \mathcal{S}_*$ then its Fourier transform $\widehat{\phi}$ is a smooth function, but it will not be of rapid decay at infinity, in general. The behavior of $\widehat{\phi}(u)$ as $|u| \to \infty$ follows from the Erdélyi asymptotic formula [4], [8, Example 79],

$$\int_{-\infty}^{\infty} \phi(x)e^{ixu}\,dx \sim \frac{c_1}{u} + \frac{c_2}{u^2} + \frac{c_3}{u^3} + \cdots, \quad |u| \to \infty, \tag{4.1}$$

where $c_{n+1} = e^{\pi i(n+1)/2}\left[\phi^{(n)}\right]$. In fact [7, Thm. 8.4.1], a smooth function ψ belongs to $\mathcal{F}(\mathcal{S}_*)$ if and only if there exist constants $c_1, c_2, c_3, \ldots$ such that $\psi(x) \sim \sum_{n=1}^{\infty} c_n x^{-n}$ as $|x| \to \infty$. Therefore, following [7, Chp. 6] we introduce the space $\mathcal{W}$ as follows.

Definition. The test-function space $\mathcal{W}$ consists of those functions $\psi \in C^\infty(\mathbb{R})$ that admit an asymptotic expansion of the type

$$\psi(x) \sim \sum_{n=1}^{\infty} c_n x^{-n} \quad \text{as } |x| \to \infty \tag{4.2}$$

for some constants $c_1, c_2, c_3, \ldots$. The space of distributions $\mathcal{W}'$ is the corresponding dual space.

We can now define the Fourier transform of the distributions of the space $\mathcal{S}'_*$.

Definition. If $f \in \mathcal{S}'_*$ then its Fourier transform $\widehat{f} = \mathcal{F}(f)$ is the element of the space $\mathcal{W}'$ defined by

$$\langle \widehat{f}(u), \psi(u) \rangle = \langle f(x), \widehat{\psi}(x) \rangle, \quad \psi \in \mathcal{W}. \tag{4.3}$$

Similarly, if $g \in \mathcal{W}'$ then its Fourier transform $\widehat{g} = \mathcal{F}(g)$ is the element of the space $\mathcal{S}'_*$ defined by

$$\langle \widehat{g}(x), \phi(x) \rangle = \langle g(u), \widehat{\phi}(u) \rangle, \quad \phi \in \mathcal{S}_* . \tag{4.4}$$

The Fourier transform is an isomorphism between the spaces $\mathcal{S}'_*$ and $\mathcal{W}'$, and between the spaces $\mathcal{W}'$ and $\mathcal{S}'_*$.

In order to understand the Fourier transform in these spaces, it is convenient to note several properties of the space $\mathcal{W}'$. This space of generalized functions was introduced in [7] to study the Hilbert transform of distributions. One of the most important characteristics of $\mathcal{W}'$ is that its elements are not distributions over $\mathbb{R}$ but rather distributions over the one-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. We denote by $\delta_{\infty,j}$ the element of $\mathcal{W}'$ given by

$$\langle \delta_{\infty,j}(u), \psi(u) \rangle = c_j \tag{4.5}$$

when $\psi \in \mathcal{W}$ has the development (4.2). Any element $g \in \mathcal{W}'$ admits a "restriction" $\pi g \in \mathcal{S}'$, but that restriction might vanish even if g does not, namely if g is "concentrated at ∞", that is, if it has the form

$$g(u) = \sum_{j=1}^{n} b_j \delta_{\infty,j}(u). \tag{4.6}$$

Each $g \in \mathcal{S}'$ admits "extensions" $\widetilde{g} \in \mathcal{W}'$, but such extensions are not unique, since we could always add a distribution of the form (4.6). Some tempered distributions admit *canonical* extensions to $\mathcal{W}'$, but there is no canonical way to extend *all* elements of $\mathcal{S}'$ to $\mathcal{W}'$. The extension problem is rather similar to the regularization problem studied in [6].

Observe that when a tempered distribution g admits a canonical extension $\widetilde{g} \in \mathcal{W}'$, then its Fourier transform $\mathcal{F}(g)$, which is an element of $\mathcal{S}'$, admits a canonical extension to the space $\mathcal{S}'_*$ of distributions over the line with a thick point at $x = 0$, and this extension is precisely $\mathcal{F}(\widetilde{g})$.

If g is a distribution of compact support, $g \in \mathcal{E}'(\mathbb{R})$, then the equation

$$\langle \widetilde{g}, \psi \rangle_{\mathcal{W}' \times \mathcal{W}} = \langle g, \psi \rangle_{\mathcal{E}' \times \mathcal{E}} \tag{4.7}$$

defines a canonical extension. On the other hand, if $g \in \mathcal{S}'$ satisfies the estimate

$$g(u) = O(|u|^\alpha) \quad (\mathrm{C}), \quad \text{as } |u| \to \infty, \tag{4.8}$$

in the Cesàro sense [5, 8], and $\alpha < 0$, then g admits a canonical extension given by the Cesàro evaluation

$$\langle \widetilde{g}, \psi \rangle_{\mathcal{W}' \times \mathcal{W}} = \langle g, \psi \rangle \ \ (\mathrm{C}), \tag{4.9}$$

which exists because $g(u)\psi(u) = O\left(|u|^{\alpha-1}\right)$ (C). Any tempered distribution g satisfies (4.8) for some $\alpha \in \mathbb{R}$ [5, 8], but if $\alpha > 0$ the extension to $\mathcal{W}'$ is not canonical but depends on k arbitrary constants if $k-1 \leq \alpha < k$ for some $k \in \{1, 2, 3, \ldots\}$, much as a primitive of order k depends on k arbitrary constants of integration.

Other tempered distributions that admit canonical extensions to $\mathcal{W}'$, obtained by analytic continuation, are the distributions u_+^α and u_-^α for $\alpha \notin \mathbb{Z}$, the combination $|\widetilde{u}|^\alpha = \widetilde{u}_+^\alpha + \widetilde{u}_-^\alpha$ for $\alpha = 0, \pm 2, \pm 4, \ldots$, and the combination $\operatorname{sgn} u \, |\widetilde{u}|^\alpha = \widetilde{u}_+^\alpha - \widetilde{u}_-^\alpha$ for $\alpha \in \mathbb{C} \setminus 2\mathbb{Z}$ [7, Sec. 6.3]. Therefore the distribution $\widetilde{u}^n$ is defined for *all* integers. In particular, the tempered distribution $1 = |u|^\alpha|_{\alpha=0}$ admits a canonical extension $\widetilde{1} = |\widetilde{u}|^\alpha|_{\alpha=0}$; this canonical extension is given by the formula

$$\left\langle \widetilde{1}, \psi(u) \right\rangle = \mathrm{p.v.} \int_{-\infty}^{\infty} \psi(u) \, du \, , \tag{4.10}$$

the principal value being taken at infinity: $\mathrm{p.v.} \int_{-\infty}^{\infty} = \lim_{A \to \infty} \int_{-A}^{A}$. Alternatively,

$$\langle \widetilde{1}, \psi(u) \rangle = \int_{-1}^{1} \psi(u) \, du + \int_{|u|>1} \left(\psi(u) - \frac{c_1}{u} \right) du. \tag{4.11}$$

5 Some Fourier transforms

We shall now give the Fourier transform of several distributions of the spaces $\mathcal{W}'$ and $\mathcal{S}'_*$. Observe that if a distribution f_0 of $\mathcal{W}'$ is an extension of a tempered distribution f of the space $\mathcal{S}'$, then the Fourier transform $\widehat{f_0}$ is an element of the space $\mathcal{S}'_*$ that extends the tempered distribution $\widehat{f}$. Similar remarks apply to the Fourier transform of the distributions of the space $\mathcal{S}'_*$.

Let us start with the computation of $\mathcal{F}\{\widetilde{\delta}(x); u\} \in \mathcal{W}'$. Observe that the equation $\langle \widetilde{\delta}(x), e^{ixu} \rangle = 1$, while correct, just tells us that $\mathcal{F}\{\widetilde{\delta}(x); u\}$ is a regularization in the space $\mathcal{W}'$ of the tempered distribution 1. Therefore, we proceed as follows:

$$\begin{aligned} \langle \mathcal{F}\{\widetilde{\delta}(x); u\}, \psi(u) \rangle &= \frac{1}{2} \left(\widehat{\psi}\,(0^+) + \widehat{\psi}\,(0^-) \right) \\ &= \frac{1}{2} \lim_{x \to 0} \left(\widehat{\psi}\,(x) + \widehat{\psi}\,(-x) \right) \\ &= \lim_{x \to 0} \int_{-\infty}^{\infty} \cos xu \, \psi(u) \, du. \end{aligned}$$

We cannot set $x = 0$ in the last integral since that would produce a divergent integral. However, we observe that $\int_{|u|>1} \cos xu \, du/u = 0$ for $x > 0$ and thus obtain with (4.11)

$$\begin{aligned} \left\langle \mathcal{F}\left\{\widetilde{\delta}(x); u\right\}, \psi(u) \right\rangle &= \lim_{x \to 0} \left\{ \int_{-1}^{1} \cos xu \, \psi(u) \, du \right. \\ &\qquad \left. + \int_{|u|>1} \cos xu \left(\psi(u) - \frac{c_1}{u} \right) du \right\} \\ &= \int_{-1}^{1} \psi(u) \, du + \int_{|u|>1} \left(\psi(u) - \frac{c_1}{u} \right) du \\ &= \langle \widetilde{1}, \psi(u) \rangle, \end{aligned}$$

so that

$$\mathcal{F}\left\{\widetilde{\delta}(x);u\right\} = \widetilde{1}\,. \tag{5.1}$$

We can compute $\mathcal{F}\{s_0(x),u\}$ in a similar fashion,

$$\begin{aligned}
\langle \mathcal{F}\{s_0(x);u\},\psi(u)\rangle &= \lim_{x\to 0^+}\left(\widehat{\psi}(x)-\widehat{\psi}(-x)\right)\\
&= 2i\lim_{x\to 0^+}\int_{-\infty}^{\infty}\sin xu\,\psi(u)\,du\\
&= 2i\lim_{x\to 0^+}\left\{\int_{-\infty}^{\infty}\sin xu\left(\psi(u)-\frac{c_1}{u}\right)du\right.\\
&\qquad\left. +c_1\int_{-\infty}^{\infty}\frac{\sin xu}{u}\,du\right\}\\
&= 2\pi i c_1\,,
\end{aligned}$$

so that

$$\mathcal{F}\{s_0(x);u\} = 2\pi i\delta_{\infty,1}(u). \tag{5.2}$$

Formulas (5.1) and (5.2) immediately give

$$\mathcal{F}\{\delta_\pm(x);u\} = \widetilde{1}\pm\pi i\delta_{\infty,1}(u), \tag{5.3}$$

where $\delta_\pm(x)=\delta(x-(0\pm 0))$. Formulas (5.3), in turn, yield the following limits in the space $\mathcal{W}'$:

$$e^{iu0^\pm} = \lim_{x\to 0^\pm} e^{iux} = \widetilde{1}\pm\pi i\delta_{\infty,1}(u). \tag{5.4}$$

If we now use the fact that $\mathcal{F}^{-1}\{f(u);x\}=(2\pi)^{-1}\mathcal{F}\{f(u);-x\}$, we obtain the formulas

$$\mathcal{F}\{\widetilde{1};x\} = 2\pi\widetilde{\delta}(x), \tag{5.5}$$

$$\mathcal{F}\{\delta_{\infty,1}(u);x\} = is_0(x). \tag{5.6}$$

Relation (5.6) can also be obtained from the Erdélyi asymptotic formula (4.1): If $\psi=\widehat{\phi}$, $\phi\in\mathcal{S}_*$, then

$$\begin{aligned}
c_1 &= \left\langle\delta_{\infty,1}(u),\psi(u)\right\rangle = \left\langle\delta_{\infty,1}(u),\widehat{\phi}(u)\right\rangle\\
&= \left\langle\mathcal{F}\{\delta_{\infty,1}(u);x\},\phi(x)\right\rangle,
\end{aligned}$$

but according to Erdélyi's formula $c_1=i\langle s_0(x),\phi(x)\rangle$, so (5.6) follows.

The usual formulas for the computation of the Fourier transforms of derivatives need to be modified in our context, since the product of a function $\psi(u)$ of the space $\mathcal{W}$ by the function u does not belong to $\mathcal{W}$, in general. Therefore, we introduce the modified multiplication operator $M_u\colon\mathcal{W}\to\mathcal{W}$ and its adjoint $M_u'\colon\mathcal{W}'\to\mathcal{W}'$ as

$$M_u(\psi) = u\psi(u)-c_1 \tag{5.7}$$

and, of course, $\langle M_u'(g),\psi\rangle=\langle g,M_u(\psi)\rangle$. Then we shall see that if $f\in\mathcal{S}_*'$, then

$$\mathcal{F}\{f'(x);u\} = -iM_u'\mathcal{F}\{f(x);u\}. \tag{5.8}$$

Indeed, if $\psi \in \mathcal{W}$, then denoting with a bar the distributional derivative in the space $\mathcal{S}'$ of tempered distributions, and by $[\phi]$ the jump of ϕ at the origin, we have by (3.5)

$$\begin{aligned}\frac{d}{dx}\widehat{\psi}(x) &= \frac{\overline{d}}{dx}\widehat{\psi}(x) - [\widehat{\psi}]\delta(x) \\ &= i\mathcal{F}\{u\psi(u); xt\} - [\widehat{\psi}]\delta(x) \\ &= \mathcal{F}\{iu\psi(u) - [\widehat{\psi}]/2\pi; x\} \\ &= \mathcal{F}\{iM_u(\psi); x\},\end{aligned}$$

and (5.8) follows by duality.
Similarly, if $g \in \mathcal{W}'$ then

$$\mathcal{F}\{M'_u g(u); x\} = -i\,\frac{d^*}{dx}\mathcal{F}\{g(u); x\}. \tag{5.9}$$

Observe that $M'_u(\delta_{\infty,j}(u)) = \delta_{\infty,j+1}(u)$. Hence by (5.6)

$$\mathcal{F}\{s_j(x); u\} = (-1)^j\mathcal{F}\{s_0^{(j)}(x); u\} = 2\pi i^{j+1}\delta_{\infty,j+1}(u), \tag{5.10}$$

$$\mathcal{F}\{\delta_{\infty,j}(u); x\} = (-i)^{j-1}s_{j-1}(x) = i^{j-1}s_0^{(j-1)}(x). \tag{5.11}$$

Notice that $M'_u(f)$ is related to the multiplication $uf(u)$, but it is not the same, even if the product is well-defined. For instance, $u\delta(u)$ vanishes, but $M'_u(\delta(u)) = -\delta_{\infty,1}(u)$ since

$$\langle M'_u\delta(u), \psi(u)\rangle = \langle \delta(u), M_u\psi(u)\rangle = \langle \delta(u), u\psi(u) - c_1\rangle = -c_1\,.$$

This gives us yet another proof of (5.6), since it yields by (3.8) that

$$\widehat{\delta_{\infty,1}}(x) = i\,\frac{d^*}{dx}\mathcal{F}\{\delta(u); x\} = i\,\frac{d^*}{dx}1 = is_0(x).$$

It is interesting to observe that if g is a tempered distribution that satisfies the estimate $g(u) = |u|^\alpha$ (C), as $|u| \to \infty$, for some $\alpha < 0$, then g can be considered as an element of $\mathcal{W}'$ in a canonical way, and its Fourier transform is the canonical extension from $\mathcal{S}'$ to $\mathcal{S}'_*$ of the usual Fourier transform $\widehat{g} \in \mathcal{S}'$. However, considering the transform in the spaces $\mathcal{W}'$ and $\mathcal{S}'_*$ may prove to be useful. For instance, let us consider the distribution $f(x) = \mathcal{F}\{\widetilde{u}^{-1}; x\}$. Using (5.9) it follows that f satisfies the differential equation $d^*f/dx = i\mathcal{F}\{\widetilde{1}; x\} = 2\pi i\widetilde{\delta}(x)$, which has a *unique* solution in $\mathcal{S}'_*$, given by $f(x) = \pi i\,\mathrm{sgn}\,x$; this is the usual Fourier transform of u^{-1}, of course.

6 Some answers

We can now address the puzzles in Section 2.
In the space $\mathcal{D}'_*$ the multiplication by H is always defined, and if $f \in \mathcal{D}'_*$ then $Hf \in \mathcal{D}'_*$ too. Observe in particular the formulas

$$H(x)\delta_{*,a,\lambda}(x) = \lambda\delta_+(x), \tag{6.1}$$

$$H(x)\delta_+(x) = \delta_+(x), \tag{6.2}$$

$$H(x)\delta_{-}(x) = 0, \tag{6.3}$$

$$H(x)\widetilde{\delta}(x) = \tfrac{1}{2}\delta_{+}(x). \tag{6.4}$$

Observe also that

$$H(x)(H(x)f(x)) = H(x)f(x), \tag{6.5}$$

since $H^2 = H$ in the space $\mathcal{E}_*$. We can also consider the multiplication projection operator $M_H : \mathcal{D}'_* \to \mathcal{D}'$. The formula

$$M_H(\widetilde{\delta}(x)) = \tfrac{1}{2}\delta(x) \tag{6.6}$$

is well-defined and correct.

The often used but ill-defined formulas (2.5) and (2.6) are loose formulations of (6.6). Indeed, in many contexts one deals with a Dirac delta function (call it $f(x)$) that is a distribution in the space $\mathcal{D}'_*$, whose projection onto $\mathcal{D}'$ is the usual Dirac delta function, and, very importantly, that is symmetric with respect to the origin, $f(-x) = f(x)$. Then if f is of the first order, we should have $f(x) = \widetilde{\delta}(x)$. Formula (6.6) then follows, and in that sense (2.5) and (2.6) are vindicated. However, the symmetry of f is not true *a priori*, and if f turns out to be $\delta_{*,a,\lambda}(x)$, then

$$M_H(\delta_{*,a,\lambda}(x)) = \lambda\delta(x), \tag{6.7}$$

which can be translated loosely as

$$\text{“}\, H(x)\delta(x) = \lambda\delta(x).\text{”} \tag{6.8}$$

Of course, one really should always use (6.6) or (6.7), not (2.6) or (6.8).

Let us reappraise the alleged proof of (2.6) in Sec. 2. The problem with it is that one must be precise and consistent in saying in which space of functions or distributions one is working. Indeed, if we understand (6.5),

$$H(x)H(x) = H(x), \tag{6.9}$$

as an equation in the test-function space $\mathcal{E}_*$, then we obtain

$$2H(x)\,\frac{dH(x)}{dx} = \frac{dH(x)}{dx}, \tag{6.10}$$

where $dH(x)/dx$ is the derivative in the space $\mathcal{E}_*$ — that is, the *ordinary* derivative of H. But that ordinary derivative is $dH(x)/dx = 0$, and thus (6.10) is true trivially because it says that "$0 = 0$". On the other hand we may consider (6.9) as an equation in the space $\mathcal{D}'_*$, but in that case the second H on the left side and the H on the right side are elements of $\mathcal{D}'_*$ while the first H is an element of $\mathcal{E}_*$. Thus it is a good idea to rewrite it as

$$H(x)\widetilde{H}(x) = \widetilde{H}(x), \tag{6.11}$$

where $\widetilde{H}$ is H as an element of $\mathcal{D}'_*$. Then denoting the derivative in $\mathcal{D}'_*$ with a star, we obtain

$$\frac{dH(x)}{dx}\,\widetilde{H}(x) + H(x)\,\frac{d^*\widetilde{H}(x)}{dx} = \frac{d^*\widetilde{H}(x)}{dx}. \tag{6.12}$$

Here the first derivative is the ordinary derivative, which is 0, while the distributional derivative in the space $\mathcal{D}'_*$ is

$$\frac{d^*\widetilde{H}(x)}{dx} = \delta_+(x), \tag{6.13}$$

and therefore we obtain (6.2), which of course is true — but contains no factor $\frac{1}{2}$. Finally, (6.9) cannot be considered in the space $\mathcal{D}'$ (because H is not in the right Moyal algebra, $\mathcal{E}$), and thus the usual distributional differentiation in $\mathcal{D}'$ is not valid. Thus it is not possible to prove in this way that λ in (6.7) must equal $\frac{1}{2}$. (In particular, it is not legal to multiply by H again and conclude that $\frac{1}{2} = \frac{1}{4}$, etc., as we were tempted to do in Sec. 2.)

Now we return to the integral (2.1). Of course it is a Fourier transform, but since it is classically divergent we need to say in which space we are working, or, what is the same, which regularization of the function 1 we are using. If we work in $\mathcal{W}'$ and, consequently, look for a result in $\mathcal{S}'_*$, it is natural because of symmetry arguments to consider the regularization $\widetilde{1}$. Hence,

$$\begin{aligned}\int_0^\infty \cos(2kx)\,dx &= \frac{1}{2}\int_{-\infty}^\infty \cos(2kx)\,dx\\ &= \frac{1}{2}\int_{-\infty}^\infty e^{2ikx}\,dx\\ &= \frac{1}{2}\mathcal{F}\{\widetilde{1}; 2k\}\\ &= \pi\widetilde{\delta}(2k)\\ &= \frac{\pi}{2}\,\widetilde{\delta}(k).\end{aligned}$$

The result $(\pi/2)\widetilde{\delta}(k)$ holds for k positive or negative. If we want the result for $k > 0$ in the space $\mathcal{S}'$, we need to apply the projection multiplication $M_H\colon \mathcal{S}'_* \to \mathcal{S}'$; that is we need to multiply by the Heaviside function. Use of (6.6) then yields

$$H(k)\int_0^\infty \cos(2kx)\,dx = \frac{\pi}{2}\,M_H\bigl(\widetilde{\delta}(k)\bigr) = \frac{\pi}{4}\,\delta(k). \tag{6.14}$$

That is, *both* (2.2) and (2.4) are correct, depending upon context! A pragmatic statement, avoiding abstract spaces, is that the definition of a delta function located at an endpoint of the interval of integration is a matter of convention ((2.5) versus the standard equation without the $\frac{1}{2}$; or, $M_H(\widetilde{\delta})$ versus δ in the notation of this section). Having chosen a convention, one must stay with it throughout a calculation. In particular, when an integral like (2.1) arises, one must be careful to evaluate it in terms of δ *using the convention chosen*. In the original application [3], it was found that the most convenient conventions were to interpret $\delta(k)$ as $2H(k)\widetilde{\delta}(k)$ (so that (2.4) is correct), but, in the conjugate variable, to interpret $\delta(x)$ as $H(x)\widetilde{\delta}(x)$. The consistency of all results could then be checked.

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International Journal of Applied Mathematics & Statistics
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

IJAMAS: September 2007, Vol. 10, No. S07, 38-49

A Slope-Bounding Procedure for Mass-Minimizing Currents

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ABSTRACT

In the higher codimensional, non-parametric context the area integrand is not a convex function, but if restricted to small enough slopes the integrand is convex. That non-convexity prevents the extension to higher codimensions of certain methods applicable to the problem of explicit determination of area-minimizing hypersurfaces. We give a simple proof that area-minimizing surfaces with small excess are graphs with small slope. We also introduce a strategy for a posteriori estimation of the excess—a strategy applicable in the presence of a rich enough family of approximating surfaces. It is hoped that it will be possible to combine the a posteriori excess estimation strategy and the slope-bounding result so as to extend to higher codimensions the explicit determination methods that are now limited to hypersurfaces.

Keywords: area-minimization, convex functions, cylindrical excess, mass-minimization.

2000 Mathematics Subject Classification: 49Q15, 26B25.

1 Introduction

While the first mathematical result regarding area-minimizing surfaces appeared 260+ years ago in Leonhard Euler's *Methodus inveniendi* (Euler, 1744), the story is still not complete, with old questions to answer and new questions always emerging.

Work appearing in (Parks, 1977b), (Parks, 1977a), (Parks, 1986), and (Parks and Pitts, 2000) has shown progress on guaranteeing that a codimension one surface that is nearly of least area is also geometrically near to a surface that is *exactly* of least area. In this paper, we begin to address this same issue for surfaces of higher codimension.

The existence theory of geometric measure theory applies to the problem of finding mass-minimizing currents [mass being the area multiplied by the multiplicity of the current] in any dimension and codimension. On the other hand, the paper of Lawson and Osserman (Lawson and Osserman, 1977) shows that there are fundamental differences between the theory of the minimal surface equation, which applies to smooth area-minimizing graphs in codimension one, and that of the minimal surface system, which applies to smooth area-minimizing graphs in higher codimensions [here we are only interested in surfaces of dimension two and higher]. While Lawson and Osserman do prove an existence result for two dimensional surfaces in $\mathbb{R}^4$, and they cite a general implicit function theorem argument applicable to small enough data in arbitrary dimensions, the discouraging facts are that, unlike the situation for the minimal surface equation, for the minimal surface system, the Dirichlet problem over a bounded, convex domain may fail to be solvable and, even when the system is solvable, the solution may fail to be unique. It seems plausible that the root cause of the observed fundamental differences between the codimension one area-minimization problem and that problem in higher codimensions is that in the former case the area integrand is convex, but in the latter setting the area integrand is non-convex. One contribution in this paper is a family of easily understood examples of that non-convexity [see (2.2), (2.3), and (2.4)] and another is a demonstration of the convexity of the integrand in the low-slope region [Theorem 2.5]. We believe that it is low-slope convexity that will need to be exploited to achieve any progress in showing that a higher codimension surface that is nearly of least area is also geometrically near to a surface that is exactly of least area.

Also in this paper, in Section 3, we give an simple argument that shows how a bound on the excess of an appropriate surface implies a bound on the slope of the surface. Finally in Section 4, we give a method that has the potential to be used to derive an estimate for the excess of a surface from the results of computing surfaces that are nearly area-minimizing.

2 The Area Integrand

We will use $\mathbf{A}[u]$ to denote the area of a surface in $\mathbb{R}^{n+m}$ that is the graph of a function $u : \Omega \to \mathbb{R}^m$, where $\Omega \subseteq \mathbb{R}^n$.

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be open. For a C^1 function $u : \Omega \to \mathbb{R}^m$ the area of the graph of u is given by*

$$\mathbf{A}[u] = \int_\Omega \sqrt{\det[\,\mathbf{I}_{n\times n} + (Du)^{t}\,(Du)\,]}\, d\mathcal{L}^n\,,$$

where $\mathbf{I}_{n\times n}$ denotes the $n \times n$ identity matrix and DU denotes the Jacobian matrix of u. Moreover, if $m = 1$, then $\det[\,\mathbf{I}_{n\times n} + (Du)^{t}\,(Du)\,] = 1 + |Du|^2$.

Proof. Define the map $\tilde{u} : \Omega \to \mathbb{R}^{n+m}$ by

$$\mathrm{e}_i \cdot \tilde{u}(x) = \begin{cases} x_i & \text{for } i = 1, 2, \ldots, n\,, \\ \mathrm{e}_{i-n} \cdot u(x) & \text{for } i = n+1, n+2, \ldots, n+m\,. \end{cases}$$

Here $\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_p$ is the standard basis in $\mathbb{R}^p$.

By the area formula [see Section 3.2.3 of (Federer, 1969)], we have

$$\mathbf{A}[u] = \int_\Omega |\,\langle \textstyle\bigwedge_n D\tilde{u},\, \mathrm{e}_1 \wedge \mathrm{e}_2 \cdots \wedge \mathrm{e}_n \rangle\,|\, d\mathcal{L}^n\,.$$

The quantity $|\,\langle \bigwedge_n D\tilde{u},\, \mathrm{e}_1 \wedge \mathrm{e}_2 \cdots \wedge \mathrm{e}_n \rangle\,|$ equals the n-dimensional area of the image of the unit n-cube under the linear mapping $D\tilde{u}$. That area is given by $\sqrt{(D\tilde{u})^{t}\,(D\tilde{u})}$ [see for instance Proposition 3.8.2 of (Krantz and Parks, 1999)]. Since

$$D\tilde{u} = \begin{pmatrix} \mathbf{I}_{n\times n} \\ Du \end{pmatrix},$$

we obtain the first conclusion.
That the area integrand in case $m = 1$ is $\sqrt{1 + |Du|^2}$ is a standard calculus fact. One can see that $\det[\mathbf{I}_{n\times n} + (Du)^{t}\,(Du)\,]$ equals $1 + |Du|^2$, as it must, by making an orthogonal change of variables so that $(Du)^{t}$ becomes parallel to a coordinate direction. □

Remark 2.1. The lemma tells us that the area integrand is the function Φ defined on $m \times n$ matrices P by

$$\Phi(P) = \sqrt{\det[\,\mathbf{I}_{n\times n} + P^{t}\,P\,]}\,. \tag{2.1}$$

Theorem 2.2. *Let m and n be positive integers. The area integrand Φ [defined in (2.1)] is convex if and only if $m = 1$ or $n = 1$.*

Proof. First we observe that, when $m = 1$ or $n = 1$, we have $\Phi(P) = \sqrt{1 + |P|^2}$, so it is straightforward to verify the convexity of the integrand.

Now we consider the case $2 \le m$ and $2 \le n$. Let p_1 and p_2 be a pair of [column] vectors in $\mathbb{R}^m$ such that

$$p_1 \cdot p_2 = 0, \quad 0 < |p_1| = |p_2| \tag{2.2}$$

and set

$$P_1 = \left(p_1,\, \mathbf{0}_{m\times(n-1)}\right), \quad P_2 = \left(\mathbf{0}_{m\times 1},\, p_2,\, \mathbf{0}_{m\times(n-2)}\right), \tag{2.3}$$

where $\mathbf{0}_{k\times\ell}$ denotes the $k \times \ell$ zero matrix. Then we have

$$\sqrt{\det[\,\mathbf{I}_{n\times n} + P_1^{t}\,P_1\,]} = \sqrt{1 + |p_1|^2} = \sqrt{1 + |p_2|^2} = \sqrt{\det[\,\mathbf{I}_{n\times n} + P_2^{t}\,P_2\,]}$$

and

$$\begin{aligned} \sqrt{\det[\,\mathbf{I}_{n\times n} + (\tfrac{1}{2}P_1 + \tfrac{1}{2}P_2)^{t}\,(\tfrac{1}{2}P_1 + \tfrac{1}{2}P_2)\,]} &= \sqrt{(1 + \tfrac{1}{4}|p_1|^2)(1 + \tfrac{1}{4}|p_2|^2)} \\ &= \left(1 + \tfrac{1}{4}|p_1|^2\right). \end{aligned}$$

Since $\sqrt{8} < t$ implies $\sqrt{1 + t^2} < 1 + \frac{1}{4}t^2$, we see that

$$\tfrac{1}{2}\Phi(P_1) + \tfrac{1}{2}\Phi(P_2) < \Phi\left(\tfrac{1}{2}P_1 + \tfrac{1}{2}P_2\right) \text{ holds for } \sqrt{8} < |p_1|\,. \tag{2.4}$$

Thus Φ is not convex. □

Definition 2.1. Let P be an $m \times n$ matrix and let $\Gamma_i(P)$ denote the coefficient of λ^i in

$$\det[\,\lambda\,\mathbf{I}_{n\times n} + P^{t}\,P\,]\,.$$

In particular, we have

$$\Gamma_1(P) = \operatorname{tr}[\,P^{t}\,P\,] = \sum_{i=1}^{m}\sum_{j=1}^{n}(P_{ij})^2 = |P|^2\,,$$

where $|P|$ denotes the Euclidean norm of P, that is, $|P| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(P_{ij})^2}$. Thus we have

$$\Phi(P) = \sqrt{1 + |P|^2 + \textstyle\sum_{i=2}^{n}\Gamma_i(P)}\,.$$

Lemma 2.3. *If*

1. $n = 2$

or

2. $3 \leq n$ *and* $|P|^2 \leq 1/(n-2)$,

then

$$\sum_{i=2}^{n} |\, \Gamma_i(P)\,| \leq n(n-1)^2\, |P|^4 .$$

Proof. Every entry in $P^{\mathrm{t}} P$ has absolute value bounded by $|P|^2$, and $\Gamma_i(P)$ is the sum of $\binom{n}{i}$ subdeterminants of $P^{\mathrm{t}} P$ of size $i \times i$. Each such $i \times i$ subdeterminant is the sum of $i!$ products of i entries of $P^{\mathrm{t}} P$. Using the assumption $|P|^2 \leq 1/(n-2)$ in case $3 \leq n$, we estimate

$$\begin{aligned} |\,\Gamma_i(P)\,| &\leq \frac{n!}{(n-i)!}\, |P|^{2i} \\ &= n(n-1)\, \underbrace{(n-2) \cdots (n-i+1)}_{i-2 \text{ factors}}\, |P|^{2i} \\ &\leq n(n-1)\, |P|^4 , \end{aligned}$$

and the result follows. □

Lemma 2.4. *If* $\mathcal{P}(X_1, X_2, \ldots, X_N)$ *is a polynomial such that*

$$\mathcal{P}(0) = \frac{\partial \mathcal{P}}{\partial X_i}(0) = \frac{\partial^2 \mathcal{P}}{\partial X_i \partial X_j}(0) = 0 \ \text{ for } 1 \leq i \leq N,\ 1 \leq j \leq N , \tag{2.5}$$

then there is an $R = R(\mathcal{P}) > 0$ *such that* $F : \mathbb{R}^N \to \mathbb{R}$ *defined by*

$$F(X) = \sqrt{1 + |X|^2 + \mathcal{P}(X)}$$

is convex on $\{\, X : |X| < R \,\}$.

Proof. Since $\mathcal{P}$ is a polynomial, (2.5) implies the existence of $c = c(\mathcal{P}) < \infty$ so that

$$|\mathcal{P}(X)| \leq c\,|X|^3, \qquad \left| \frac{\partial \mathcal{P}}{\partial X_i}(X) \right| \leq c\,|X|^2, \qquad \left| \frac{\partial^2 \mathcal{P}}{\partial X_i \partial X_j} \right| \leq c\,|X|$$

hold for all $1 \leq i, j \leq N$ and all $X \in \mathbb{R}^N$. We may assume $1 \leq c$.
We use the characterization of convexity in terms of the Hessian matrix [see Theorem 4.5 of (Rockafellar, 1970)]. We compute

$$\frac{\partial F}{\partial X_i} = [\, X_i + \tfrac{1}{2}\, \partial \mathcal{P}/\partial X_i \,] \cdot [\, 1 + |X|^2 + \mathcal{P}(X)\,]^{-1/2} .$$

$$\begin{aligned} \frac{\partial^2 F}{\partial X_i \partial X_j} &= [\, \delta_{ij} + \tfrac{1}{2}\, \partial^2 \mathcal{P}/\partial X_i \partial X_j \,] \cdot [\, 1 + |X|^2 + \mathcal{P}(X)\,]^{-1/2} \\ &\quad - [\, X_i + \tfrac{1}{2}\, \partial \mathcal{P}/\partial X_i \,] \cdot [\, X_j + \tfrac{1}{2}\, \partial \mathcal{P}/\partial X_j \,] \cdot [\, 1 + |X|^2 + \mathcal{P}(X)\,]^{-3/2} . \end{aligned}$$

Assuming that $R \leq 1$ and that R is also chosen small enough that $|\mathcal{P}| < |X|^2$ holds for $|X| < R$, we can estimate

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 F}{\partial X_i \partial X_j}\,\xi_i\,\xi_j &\geq [\,1 + |X|^2 + \mathcal{P}\,]^{-3/2} \\
&\quad \cdot \Bigg[|\xi|^2\,(1+\mathcal{P}) - (\,1 + |X|^2 + \mathcal{P}\,) \left| \textstyle\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 \mathcal{P}}{\partial X_i \partial X_j}\,\xi_i\,\xi_j \right| \\
&\qquad - |\xi|\,|X| \left| \textstyle\sum_{i=1}^{N} \frac{\partial \mathcal{P}}{\partial X_i}\,\xi_i \right| - \tfrac{1}{4} \left(\textstyle\sum_{i=1}^{N} \frac{\partial \mathcal{P}}{\partial X_i}\,\xi_i \right)^2 \Bigg] \\
&\geq 3^{-3/2} \cdot |\xi|^2 \\
&\quad \cdot \Big[\, 1 - c\,|X|^3 - (1 + |X|^2 + c\,|X|^3)\,N\,c\,|X| - \sqrt{N}\,c\,|X|^3 - \tfrac{1}{4}\,c^2\,|X|^4 \Big] \\
&\geq 3^{-3/2} \cdot |\xi|^2 \cdot (\,1 - 6\,N\,c^2\,|X|\,)\,.
\end{aligned}$$

Thus it suffices to additionally require $R < 1/(\,6\,N\,c^2\,)$. □

Theorem 2.5. *There exists $0 < R(m,n)$ such that the area integrand Φ is convex on the set of $m \times n$ matrices P with $|P| < R$.*

Proof. We apply Lemma 2.4 with $N = m \cdot n$, with $X_{i+(m-1)j} = P_{ij}$, and with $\mathcal{P}(X) = \sum_{i=2}^{n} \Gamma_i(P)$. By Lemma 2.3 we know that Lemma 2.4 is applicable. □

Remark 2.2. Let u be a function whose graph minimizes an integral of the form $\int_\Omega \mathcal{F}(Du)\,d\mathcal{L}^n$, where $\mathcal{F}$ is uniformly convex. Let v be another function with the same boundary values as u and with nearly minimal integral; that is, $\int_\Omega \mathcal{F}(Du)\,d\mathcal{L}^n \leq \int_\Omega \mathcal{F}(Du)\,d\mathcal{L}^n + \epsilon$. Provided one has bounds on $\mathrm{Lip}\,u$ and $\mathrm{Lip}\,v$, the uniform convexity of the integrand can be exploited to give an estimate in terms of ϵ for an integral norm of $|Dv - Du|$. That basic technique is used in (Parks, 1977b), where the specific argument we are describing is used in the proof of Theorem 5.1(1).

Theorem 2.5 shows us that if there is to be any hope of applying the methods described in the last paragraph to the area integrand in higher codimensions, then we will need to be able to ensure that $\mathrm{Lip}\,u$ and $\mathrm{Lip}\,v$ are small. We turn to that issue in the next section.

3 Locally Lipschitz Approximation of Currents with Small Excess

Below in the proof of Theorem 3.1 we give a simple argument—applicable to mass-minimizing currents—showing that if the excess in all vertical cylinders is small enough then the current is a Lipschitz graph, and the smaller the bound on the excess, the smaller the Lipschitz constant. While the conclusion of Theorem 3.1 can be obtained from a close examination of the proof of Lemma 3 in (Schoen and Simon, 1982), the argument given here is simplified and facilitated by the density hypothesis (3). This density hypothesis is specific to mass-minimizing currents as opposed to currents minimizing other elliptic integrals.

Definition 3.1. A thorough discussion of currents can be found in Chapter 4 of (Federer, 1969), but for the reader's convenience we provide a résumé of notation and terminology here.

1. The open $(n+m)$-dimensional ball of radius r centered at z will be denoted by $\mathbb{B}(z,r)$.

2. (a) For an n-dimensional current T [in $U \subseteq \mathbb{R}^p$] that is representable by integration, the total variation measure associated with T will be denoted by $\|T\|$. By the Riesz representation theorem, there is a $\|T\|$-measurable function $\overrightarrow{T} : U \to \bigwedge_n \mathbb{R}^p$ such that
$$T(\omega) = \int_U \langle \omega, \overrightarrow{T} \rangle \, d\|T\| \tag{3.1}$$
for each smooth n-form ω with compact support in U.
   (b) The *mass* of T is $\mathbf{M}[T] = \|T\|(U)$.
   (c) For a Borel set $A \subseteq U$, we define $T \llcorner A$ by replacing the measure $\|T\|$ in (3.1) by the measure $\|T\| \llcorner A$ defined by
$$(\|T\| \llcorner A)(V) = \|T\|(A \cap V).$$

3. The *boundary* operator on currents is the dual of the exterior differentiation operator for forms; that is, the boundary ∂T of the current T is defined by $(\partial T)(\omega) = T(d\omega)$.

4. The *push-forward* operation on currents is the dual of the pull-back operation on forms; that is, the push-forward $F_{\#}T$ of the current T by the map F is defined by $(F_{\#}T)(\omega) = T(F^{\#}\omega)$.

5. The *support* of T, denoted $\operatorname{spt} T$, is the complement of the largest open set V such that T vanishes on all smooth forms supported in V.

6. The n-dimensional current in $\mathbb{R}^n$ defined by Lebesgue measure will be denoted $\mathbf{E}^n$; it is defined by setting
$$\mathbf{E}^n(\omega) = \int_{\mathbb{R}^N} \langle \omega, \mathrm{e}_1 \wedge \mathrm{e}_2 \wedge \cdots \wedge \mathrm{e}_n \rangle \, d\mathcal{L}^n$$
for each smooth n-form ω with compact support in $\mathbb{R}^n$.

7. (a) The open n-dimensional ball in $\mathbb{R}^n$ having radius r and centered at x will be denoted by $\mathbf{B}^n(x,r)$.
   (b) The volume of $\mathbf{B}^n(0,1)$ will be denoted by Ω_n.

8. The cylinder $\mathbf{B}^n(x,r) \times \mathbb{R}^m$ will be denoted by $\mathbf{C}(x,r)$.

9. $\mathbf{p} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ will denote projection onto the first factor and $\mathbf{q} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ will denote projection onto the second factor.

10. For a current T we set
$$E(T,x,r) = r^{-n}\Big(\mathbf{M}\,[\, T \llcorner \mathbf{C}(x,r)\,] - \mathbf{M}\,[\,\mathbf{p}_{\#}(T \llcorner \mathbf{C}(x,r))\,] \Big).$$
We call $E(T,x,r)$ the *cylindrical excess* in the cylinder $\mathbf{C}(x,r)$.

The next theorem tells us that, for appropriate currents, if the excess in all cylinders is small, then the current is a Lipschitz graph. The elementary proof is based on the idea that if the slope were too steep then disjoint $(n+m)$-dimensional balls would have vertical projections that overlap more than the excess bound allows.

Theorem 3.1. *There are $\epsilon_0(n)$ and $c_0(n)$ such that, for $0 < \rho$, if the following hold*

1. $\operatorname{spt} \partial T \subseteq \mathbb{R}^{n+m} \setminus \mathbf{C}(0,\rho)$,
2. $\mathbf{p}_{\#}[T \llcorner \mathbf{C}(0,\rho)] = \mathbf{E}^n \llcorner \mathbf{B}^n(0,\rho)$,
3. $\Omega_n r^n \le \|T\|\Big(\mathbb{B}(z,r)\Big)$ *whenever* $z \in \operatorname{spt} T$ *and* $\mathbb{B}(z,r) \bigcap \operatorname{spt} \partial T = \emptyset$,
4. $E_* = \sup\Big\{ E(T,x,\delta) : x \in \mathbf{B}^n(0,\rho),\ 0 < \delta < \rho - |x| \Big\} < \epsilon_0$,

then there exists a function $g : \mathbf{B}^n(0,\rho) \to \mathbb{R}^m$ such that

$$|g(x_1) - g(x_2)| \le c_0\, E_*^{1/(2n)}\, |x_1 - x_2|\,, \quad \text{for } x_1, x_2 \in \mathbf{B}^n(0,\rho), \tag{3.2}$$

and

$$\mathbf{p}^{-1}(x) \cap \operatorname{spt} T = \{\, (z, g(z))\,\}\,, \quad \text{for } x \in \mathbf{B}^n(0,\rho). \tag{3.3}$$

Proof. Since the excess is invariant under homotheties, we may assume $\rho = 1$. Suppose $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in \operatorname{spt} T \cap \mathbf{C}(0,\sigma)$, where $\mathbf{p}(z_i) = x_i$, $\mathbf{q}(z_i) = y_i$, for $i = 1, 2$, are such that

$$|\,x_1 - x_2\,| < \tfrac{2}{3} \min\{\, 1 - |x_1|,\ 1 - |x_2|\,\}\,.$$

We set

$$a = |\,x_1 - x_2\,|\,, \quad b = |\,y_1 - y_2\,|\,,$$

and

$$c = \min\{\, 2a,\ |\,z_1 - z_2\,|\,\}\,.$$

Then we have

$$\mathbb{B}(z_1, c/2) \bigcap \mathbb{B}(z_2, c/2) = \emptyset\,.$$

Set

$$x^* = \tfrac{1}{2}(x_1 + x_2)\,,$$

so that

$$|\,x_1 - x^*\,| = |\,x_2 - x^*\,| = a/2$$

and

$$\mathbf{C}(x^*, (c+a)/2) \subseteq \mathbf{C}(x^*, 3a/2) \subseteq \mathbf{C}(0,1)\,.$$

Also, we see that

$$\begin{aligned} \mathbf{B}^n(x^*, (c-a)/2) &\subseteq \mathbf{B}^n(x_1, c/2) \bigcap \mathbf{B}^n(x_2, c/2) \\ &= \mathbf{p}\,[\,\mathbb{B}(z_1, c/2)\,] \bigcap \mathbf{p}\,[\,\mathbb{B}(z_2, c/2)\,] \end{aligned}$$

and thus that

$$\mathcal{L}^n\Big(\mathbf{p}\,[\,\mathbb{B}(z_1,c/2)\,]\bigcap\mathbf{p}\,[\,\mathbb{B}(z_2,c/2)\,]\Big)\geq\Omega_n\,2^{-n}\,(c-a)^n\,.$$

By (3) we have

$$\begin{aligned}\|T\|\Big(\,\mathbb{B}(z_1,c/2)\,\Big)+\|T\|\Big(\,\mathbb{B}(z_2,c/2)\,\Big) &\geq 2\,\Omega_n\,2^{-n}\,c^n\\ &= \mathcal{L}^n\Big(\,\mathbf{p}\,[\,\mathbb{B}(z_1,c/2)\,]\,\Big)+\mathcal{L}^n\Big(\,\mathbf{p}\,[\,\mathbb{B}(z_2,c/2)\,]\,\Big)\,.\end{aligned}$$

Using

$$\begin{aligned}&\mathcal{L}^n\Big(\,\mathbf{p}\,[\,\mathbb{B}(z_1,c/2)\,]\,\Big)+\mathcal{L}^n\Big(\,\mathbf{p}\,[\,\mathbb{B}(z_2,c/2)\,]\,\Big)\\ &\quad= \mathcal{L}^n\Big(\,\mathbf{p}\,[\,\mathbb{B}(z_1,c/2)\,]\bigcup\mathbf{p}\,[\,\mathbb{B}(z_2,c/2)\,]\,\Big)\\ &\qquad- \mathcal{L}^n\Big(\,\mathbf{p}\,[\,\mathbb{B}(z_1,c/2)\,]\bigcap\mathbf{p}\,[\,\mathbb{B}(z_2,c/2)\,]\,\Big)\,,\end{aligned}$$

we obtain

$$\begin{aligned}&E[T,x^*,(c+a)/2]\\ &\quad\geq\; 2^n\,(c+a)^{-n}\cdot\Big[\;\|T\|\Big(\,\mathbb{B}(z_1,c/2)\,\bigcup\,\mathbb{B}(z_2,c/2)\,\Big)\\ &\qquad-\mathcal{L}^n\Big(\,\mathbf{p}\,[\,\mathbb{B}(z_1,c/2)\,]\,\bigcup\,\mathbf{p}\,[\,\mathbb{B}(z_2,c/2)\,]\,\Big)\,\Big]\\ &\quad\geq\; 2^n\,(c+a)^{-n}\cdot\Big[\;\mathcal{L}^n\Big(\mathbf{p}\,[\,\mathbb{B}(z_1,c/2)\,]\bigcap\mathbf{p}\,[\,\mathbb{B}(z_2,c/2)\,]\Big)\,\Big]\\ &\quad\geq\Omega_n\,\left(\frac{c-a}{c+a}\right)^n\,.\end{aligned}$$

If $c=2a$, then we have

$$E[T,x^*,(c+a)/2]\geq 3^{-n}\,\Omega_n\,.$$

Set $\epsilon_0=3^{-n}\,\Omega_n$. If $E_*<\epsilon_0$, then we conclude that $c<2a$. For $c<2a$, we have

$$\begin{aligned}E[T,x^*,(c+a)/2] &\geq \Omega_n\,(c+a)^{-n}\,(c-a)^n\\ &= \Omega_n\,(c+a)^{-2n}\,(c^2-a^2)^n\\ &= \Omega_n\,(c+a)^{-2n}\,b^{2n}\\ &\geq \Omega_n\,(3a)^{-2n}\,b^{2n}\,,\end{aligned}$$

so we have

$$9\,\Omega_n^{-1/(2n)}\cdot\Big(\,E[T,x^*,(c+a)/2]\,\Big)^{1/(2n)}\geq\frac{b}{a}\,.$$

Setting $c_0=9\,\Omega_n^{-1/(2n)}$, we conclude that

$$|y_1-y_2|\leq c_0\,E_*^{1/(2n)}\,|x_1-x_2|\,. \tag{3.4}$$

Thus a function $g:\mathbf{B}^n(0,\rho)\to\mathbb{R}^m$ satisfying (3.3) exists and, since g satisfies the Lipschitz condition (3.4) in a neighborhood of each $x\in\mathbf{B}^n(0,1)$, we conclude that (3.2) holds. □

Remark 3.1.

1. Theorem 3.1 is the "slope-bounding procedure" of the title. Of course, the fundamental difficulty is in establishing the hypothesis (4) on the excess. Corollary 1 of Schoen and Simon's paper (Schoen and Simon, 1982) tells us that for appropriate mass-minimizing currents, if the excess in one cylinder is small enough, then in a smaller concentric cylinder, the excess in all still smaller cylinders is small, though the excess may increase by a factor. Of course the regularity theorem of (Schoen and Simon, 1982) also implies a slope bound, but the proof of that theorem is much more complicated than the proof of Theorem 3.1.

2. All the constants c_1 through c_{45} appearing in Schoen and Simon's paper (Schoen and Simon, 1982) are computable. Nonetheless, one might be discouraged by the fact that c_{20} and c_{21} [see page 425 of (Schoen and Simon, 1982)] depend on deep theorems in Morrey's treatise[1] (Morrey, 1966). For the area problem, those two constants should be obtainable using more accessible estimates based on the Poisson integral formula [as described in Section 1.4 of (Krantz, 1992)]; this is a matter we hope to discuss in future work.

3. In order to apply Theorem 3.1, one needs to be able to bound the excess in a cylinder. In the next section, we address that issue.

4 A Method for A posteriori Estimation

Little a priori information is available for the area-minimization problem in higher codimensions. Barring the discovery of new a priori estimates, it may be necessary to use a posteriori information. The next theorem gives one way in which such a posteriori information could be obtained from computational results.

Theorem 4.1. *If*

1. *A is a Borel set,*

2. *$\mathcal{F}$ is a family of currents,*

3. *$m_0 = \inf\{\, \mathbf{M}[Q] \;:\; Q \in \mathcal{F}\}$,*

4. *$T \in \mathcal{F}$ satisfies $\mathbf{M}[T] = m_0$,*

5. *$\epsilon_1 > 0$,*

6. *$\widetilde{\mathcal{F}} \subseteq \mathcal{F}$,*

7. *for each $Q \in \mathcal{F}$, there exists $P \in \widetilde{\mathcal{F}}$ such that*
   $|\mathbf{M}[Q] - \mathbf{M}[P]| \le \epsilon_1$ *and* $|\mathbf{M}[Q \llcorner A] - \mathbf{M}[P \llcorner A]| \le \epsilon_1$,

8. *(a) $S \in \widetilde{\mathcal{F}}$,*

[1] The reference list of Schoen and Simon's paper omitted [MCB], but Morrey's book is intended.

(b) $\mathbf{M}[S] \le m_0 + \epsilon_1$,

(c) $\mathbf{M}[S \llcorner A] + \epsilon_1 \ge \sup\{\, \mathbf{M}[P \llcorner A] \,:\, P \in \widetilde{\mathcal{F}},\ \mathbf{M}[P] \le m_0 + \epsilon_1 \,\}$,

then

$$\mathbf{M}[S \llcorner A] \ge \mathbf{M}[T \llcorner A] - 2\epsilon_1 \,.$$

Remark 4.1. The elements of $\widetilde{\mathcal{F}}$ should be thought of as approximating currents that might be obtained via computational results. The quantity ϵ_1 should be thought of as a measure of how well the currents in $\mathcal{F}$ can be approximated by the currents in $\widetilde{\mathcal{F}}$. Item (8) tells us that, among the approximating currents that have nearly smallest possible mass, the important ones are those that have nearly the largest possible mass in A.

Proof. By hypothesis, there exists $P \in \widetilde{\mathcal{F}}$ such that $|\,\mathbf{M}[T] - \mathbf{M}[P]\,| \le \epsilon_1$ and such that $|\,\mathbf{M}[T \llcorner A] - \mathbf{M}[P \llcorner A]\,| \le \epsilon_1$. So we have

$$\mathbf{M}[P] \le m_0 + \epsilon_1$$

and thus

$$\begin{aligned} \mathbf{M}[S \llcorner A] + \epsilon_1 &\ge \mathbf{M}[P \llcorner A] \\ &\ge \mathbf{M}[T \llcorner A] - \epsilon_1 \,. \end{aligned}$$

□

Corollary 4.2. *Suppose all the hypotheses of Theorem 4.1 hold with $A = \mathbb{B}(p, r)$. If additionally*

1. $\operatorname{spt} \partial T \bigcap \mathbb{B}(p, r) = \emptyset$,
2. $T \llcorner \mathbb{B}(p, r)$ *is mass-minimizing,*
3. $r > s = \Omega_n^{-1/n} \left(\mathbf{M}[\, S \llcorner \mathbb{B}(p, r)\,] + 2\epsilon_1 \right)^{1/n}$,

then

$$\operatorname{spt} T \bigcap \mathbb{B}(p, r - s) = \emptyset \,.$$

Proof. Suppose $r > s$. Arguing by contradiction, suppose

$$q \in \operatorname{spt} T \bigcap \mathbb{B}(p, r - s) \,.$$

Setting $t = (r - s) - |q - p| > 0$, we have $\mathbb{B}(q, s + t) \subseteq \mathbb{B}(p, r)$. Since $T \llcorner \mathbb{B}(p, r)$ is mass-minimizing and $\operatorname{spt} \partial T \bigcap \mathbb{B}(p, r) = \emptyset$, we have

$$\mathbf{M}[\, T \llcorner \mathbb{B}(q, s + t)\,] \ge \Omega_n (s + t)^n \,.$$

Applying Theorem 4.1 with $A = \mathbb{B}(p, r)$, we obtain

$$\begin{aligned} \Omega_n s^n = \mathbf{M}[\, S \llcorner \mathbb{B}(p, r)\,] + 2\epsilon_1 &\ge \mathbf{M}[\, T \llcorner \mathbb{B}(p, r)\,] \\ &\ge \mathbf{M}[\, T \llcorner \mathbb{B}(q, s + t)\,] \ge \Omega_n (s + t)^n \,, \end{aligned}$$

contradicting $t > 0$. □

Corollary 4.3. *Suppose all the hypotheses of Theorem 4.1 hold with $A = \mathbf{C}(0,\rho)$. If additionally*

1. $\operatorname{spt} \partial T \subseteq \mathbb{R}^{n+m} \setminus \mathbf{C}(0,\rho)$,
2. $\mathbf{p}_{\#}[T \llcorner \mathbf{C}(0,\rho)] = \mathbf{E}^n \llcorner \mathbf{B}^n(0,\rho)$,
3. $T \llcorner \mathbf{C}(0,\rho)$ *is mass-minimizing,*
4. $\mathbf{p}_{\#}[S \llcorner \mathbf{C}(0,\rho)] = \mathbf{E}^n \llcorner \mathbf{B}^n(0,\rho)$,

then

$$E(T,0,\rho) \le E(S,0,\rho) + 2\,\rho^{-n}\,\epsilon_1\,.$$

Proof. Applying Theorem 4.1 with $A = \mathbf{C}(0,\rho)$, we obtain

$$\mathbf{M}\,[\,S \llcorner \mathbf{C}(0,\rho)\,] + 2\epsilon_1 \ge \mathbf{M}\,[\,T \llcorner \mathbf{C}(0,\rho)\,]\,.$$

So we have

$$\begin{aligned} 2\,\rho^{-n}\,\epsilon_1 + E(S,0,\rho) &= \rho^{-n}\left(2\epsilon_1 + \mathbf{M}\,[\,S \llcorner \mathbf{C}(0,\rho)\,] - \Omega_n\,\rho^n\right) \\ &\ge \rho^{-n}\left(\mathbf{M}\,[\,T \llcorner \mathbf{C}(0,\rho)\,] - \Omega_n\,\rho^n\right) = E(T,0,\rho)\,. \end{aligned}$$

□

Remark 4.2. Corollary 4.2 allows us to conclude that any area-minimizer can be excluded from a particular ball. Corollary 4.3 has the potential to allow us to estimate the excess in a cylinder based on information gained from computational results. A small enough bound on the excess allows us to bound the slope. And in the low-slope region the integrand is uniformly convex. Of course, considerable work remains to be done to put these pieces together in an effective way.

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International Journal of Applied Mathematics & Statistics
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

IJAMAS: September 2007, Vol. 10, No. S07, 50-69

Strong Stability of Operator-Differential Equations

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ABSTRACT

The stability of first and second-order linear operator-differential equations in Hilbert spaces has been investigated. A few scales of a priori estimates of global and asymptotic stability in various norms have been obtained. Using these estimates we also construct few scales of a priori estimates of strong stability.

Keywords: perator-differential equation, global stability, asymptotical stability, strong stability.

2000 Mathematics Subject Classification: 34G10 (34D20, 35K90, 35R10, 47D06, 47N20.

1. INTRODUCTION

An abstract problem $Lu = \varphi$ is called stable if its solution u continuously depends on the input data φ, i.e. there exist a such a value $\varrho > 0$ independent of the solution and input data that for all φ, $\tilde{\varphi}$ from a certain allowable admissible set the following inequality holds

$$\|\tilde{u} - u\|_0 \le \varrho \, \|\tilde{\varphi} - \varphi\|_1 ,$$

where $\tilde{u}$ is the solution of an analogous problem with perturbed input data $L\tilde{u} = \tilde{\varphi}$, while $\|\cdot\|_0$ and $\|\cdot\|_1$ are certain norms. For linear problems this is equivalent to a priori estimate

$$\|u\|_0 \le \varrho \, \|\varphi\|_1 .$$

For evolution problems the quantity ϱ in many cases depends on the time variable t. In particular, the behavior of $\varrho(t)$ when $t \to +\infty$ is of interest. If ϱ is bounded when $t \to +\infty$ the problem is called globally stable. If $\varrho(t) \to 0$ when $t \to +\infty$ the problem is called asymptotically stable.

Another type of stability is the strong stability, or the stability of the solution u under perturbations of operator L. Let $\tilde{u}$ be the solution of equation with perturbed operator and input data $\tilde{L}\tilde{u} = \tilde{\varphi}$. A priori estimate of the strong stability have the form

$$\|\tilde{u} - u\|_0 \leq \varrho_1 \|\tilde{\varphi} - \varphi\|_1 + \varrho_2 \|\tilde{L} - L\|_2 ,$$

where $\|\cdot\|_2$ is some operator norm. In this case the problem is nonlinear, even for linear L and $\tilde{L}$, and the quantities ϱ_i depend on u and $\tilde{u}$. In the case of differential equations operator L is generally unbounded and one must have a suitable definition of its norm.

The problem of stability becomes particularly urgent in the modelling of applied problems where input data, including coefficients of differential equations, can be given roughly (as a result of observations, experimental measurements, calculations etc.). A priori estimates expressing the continuous dependence of the solution of the problem under perturbations of the right-hand side and the operator were obtained for stationary boundary value problems (see [14, 16, 19, 21]). Extensive bibliography is devoted to the stability of evolutionary differential-operator equations and operator-difference schemes (see [9, 11, 23]). In particular, strong stability of non-stationary problems were investigated in [10, 12, 13, 17, 18, 20, 24, 25].

In the present paper we report and develop the results concerning different types of stability of operator-differential equations obtained in [1, 3-6].

The layout of the paper is as follows. In Section 2 we investigate the stability of first order linear operator-differential equations and construct few scales of a priory estimates of global, asymptotical and strong stability.

In Section 3 second order operator-differential equations are considered. Here we obtained few nonstandard scales of a priory estimates of global and strong stability.

2. FIRST ORDER LINEAR OPERATOR-DIFFERENTIAL EQUATION

2.1. Preliminary Results

Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and let A be an unbounded self-adjoint positive definite linear operator with domain $\mathcal{D}(A)$ dense in H. The expression $(u,v)_A = (Au,v)$, $u,v \in \mathcal{D}(A)$, satisfies the inner product axioms. The completion of $\mathcal{D}(A)$ in the norm $\|u\|_A = (u,\, u)_A^{1/2}$ is the so-called energy space $H_A \subset H$. The expression $(u,\, v)_{A^{-1}} = (A^{-1}u,\, v)$, $u, v \in H$, also satisfies the inner product axioms. The completion of H in the norm $\|u\|_{A^{-1}} = (u,\, u)_{A^{-1}}^{1/2}$ is the space $H_{A^{-1}} \supset H$. Moreover, $H_{A^{-1}} = H_A^*$ is the dual space of H_A, the inner product $(u,\, v)$ can be continuously extended to $H_{A^{-1}} \times H_A$, and the operator A can be extended to a mapping $A : H_A \to H_{A^{-1}}$. There exists an unbounded self-adjoint positive definite linear operator $A^{1/2}$ [11, 15]. In this case $\mathcal{D}(A^{1/2}) = H_A$ and $(u,\, v)_A = (Au,\, v) = (A^{1/2}u,\, A^{1/2}v)$. The spaces $V_1 = H_A$, H and $V_1^* = H_{A^{-1}}$ form the Gelfand triple: $V_1 \subset H \subset V_1^*$. In an analogous manner, we define the spaces $V_2 = H_{A^2}$, $V_2^* = H_{A^{-2}}$, $V_3 = H_{A^3}$, $V_3^* = H_{A^{-3}}$ etc. In addition $\|u\|_{A^2} = \|Au\|$, $\|u\|_{A^{-2}} = \|A^{-1}u\|$, $\|u\|_{A^3} = \|Au\|_A$, $\|u\|_{A^{-3}} = \|A^{-1}u\|_{A^{-1}}$ etc. and $\cdots \subset V_3 \subset V_2 \subset V_1 \subset H \subset V_1^* \subset V_2^* \subset V_3^* \subset \ldots$. It will be recalled that if we consider the operator A as the map $A : V_1 \to V_1^*$, $A : V_2 \to H$ or $A : H \to V_2^*$ t5hen it naturally becomes bounded.

Let $\mu(t)$ be a real positive function defined on the interval (a, b) and $1 \leq p < +\infty$. We introduce Lebesgue space $L_p((a,b);\, \mu;\, H)$ of the functions $u = u(t)$ that map the segment $(a,b) \subset R$ to H [22], endowed with norm

$$\|u\|_{L_p((a,b);\,\mu;\,H)} = \left(\int_a^b \mu(t)\,\|u(t)\|^p\,dt\right)^{1/p}.$$

For $p = 2$ the space $L_2((a,b);\ \mu;\ H)$ is a Hilbert space with the inner product

$$(u,\ v)_{L_2((a,b);\,\mu;\,H)} = \int_a^b \mu(t)\,(u(t),\ v(t))\,dt\,.$$

For $\mu(t) \equiv 1$ we denote $L_p((a,b);\ 1;\ H) = L_p((a,b);\ H)$. We also define the space of continuous functions $C([a,b];\ H)$ with norm

$$\|u\|_{C([a,b];\,H)} = \max_{t\in[a,b]} \|u(t)\|\,,$$

and Sobolev spaces $W_p^s((a,b);\ H)$ [22], with norm

$$\|u\|^p_{W_p^s((a,b);\,H)} = \sum_{0\le i<s} |u|^p_{W_p^i((a,b);\,H)} + |u|^p_{W_p^s((a,b);\,H)}\,,$$

where

$$|u|_{W_p^i((a,b);\,H)} = \|u^{(i)}\|_{L_p((a,b);\,H)}\,, \qquad u^{(i)} = \frac{d^i u}{dt^i}\,, \qquad i = 0,1,2,\ldots,$$

$$|u|_{W_p^s((a,b);\,H)} = |u^{([s])}|_{W_p^{s-[s]}((a,b);\,H)}\,, \qquad [s]\ \text{– integer part of } s,$$

and

$$|u|^p_{W_p^\sigma((a,b);\,H)} = \int_a^b\int_a^b \frac{\|u(t)-u(t')\|^p}{|t-t'|^{1+\sigma p}}\,dt\,dt'\,, \qquad 0<\sigma<1\,.$$

2.2. Global Stability

Let us consider the abstract Cauchy problem

$$Bu' + Au = f(t), \quad t>0; \qquad u(0) = u_0, \tag{2.1}$$

where operator A satisfies the previous assumptions, B is a linear self-adjoint positive definite operator in H, $B \le A$, u_0 is the given element of H_B, $f(t) \in L_2((0,T);\ H_{A^{-1}})$ is the given function for some $T>0$ and $u(t)$ is unknown function with values in H_A.

Taking the inner product of (2.1) with $2u$ and integrating on t, after some algebra we obtain a priori (energy) estimate

$$\|u(t)\|_B^2 + \int_0^t \|u(s)\|_A^2\,ds \le \|u_0\|_B^2 + \int_0^t \|f(s)\|^2_{A^{-1}}\,ds. \tag{2.2}$$

An estimate of fractional order on t is obtained in [4, 7]

$$\int_0^t\int_0^t \frac{\|u(s)-u(s')\|_B^2}{|s-s'|^2}\,dsds' \le 3\pi\int_0^t \|u(s)\|_A^2\,ds + \pi\int_0^t \|f(s)\|^2_{A^{-1}}\,ds. \tag{2.3}$$

Applying (2.2) and (2.3) to (2.1) and equations obtained by application of BA^{-1}, AB^{-1}, $(AB^{-1})^2$,

$(AB^{-1})^3$ etc, and combining obtained results we get the following inequalities:

$$\begin{aligned}
&\|Bu(t)\|^2_{A^{-1}} + \int_0^t \|u(s)\|^2_B\, ds \le \|Bu_0\|^2_{A^{-1}} + \int_0^t \|A^{-1}f(s)\|^2_B\, ds,\\
&\|u(t)\|^2_B + \int_0^t \|u(s)\|^2_A\, ds + \int_0^t\int_0^t \frac{\|u(s)-u(s')\|^2_B}{|s-s'|^2}\, dsds' \le\\
&\qquad \le (4\pi+1)\left(\|u_0\|^2_B + \int_0^t \|f(s)\|^2_{A^{-1}}\, ds\right),\\
&\|u(t)\|^2_A + \int_0^t \|Au(s)\|^2_{B^{-1}}\, ds + \int_0^t \|u'(s)\|^2_B\, ds \le 2\left(\|u_0\|^2_A + \int_0^t \|f(s)\|^2_{B^{-1}}\, ds\right),\\
&\|Au(t)\|^2_{B^{-1}} + \int_0^t \|B^{-1}Au(s)\|^2_A\, ds + \int_0^t\int_0^t \frac{\|u'(s)-u'(s')\|^2_B}{|s-s'|^2}\, dsds' \le\\
&\qquad \le (8\pi+1)\left(\|Au_0\|^2_{B^{-1}} + \int_0^t \|B^{-1}f(s)\|^2_A\, ds + \int_0^t\int_0^t \frac{\|f(s)-f(s')\|^2_{B^{-1}}}{|s-s'|^2}\, dsds'\right),
\end{aligned}$$

etc. (2.4)

From (2.4) we obtain scale of a priori estimates:

$$\begin{aligned}
&\|u\|^2_{(0)} \equiv \|u\|^2_{C([0,t];H_{BA^{-1}B})} + \|u\|^2_{L_2((0,t);H_B)} \le 2\left(\|u_0\|^2_{BA^{-1}B} + \|f\|^2_{L_2((0,t);H_{A^{-1}BA^{-1}})}\right),\\
&\|u\|^2_{(1)} \equiv \|u\|^2_{C([0,t];H_B)} + \|u\|^2_{L_2((0,t);H_A)} + |u|^2_{W_2^{1/2}((0,t);H_B)} \le\\
&\qquad \le (4\pi+2)\left(\|u_0\|^2_B + \|f\|^2_{L_2((0,t);H_{A^{-1}})}\right),\\
&\|u\|^2_{(2)} \equiv \|u\|^2_{C([0,t];H_A)} + \|u\|^2_{L_2((0,t);H_{AB^{-1}A})} + |u|^2_{W_2^1((0,t);H_B)} \le\\
&\qquad \le 3\left(\|u_0\|^2_A + \|f\|^2_{L_2((0,t);H_{B^{-1}})}\right),\\
&\|u\|^2_{(3)} \equiv \|u\|^2_{C([0,t];H_{AB^{-1}A})} + \|u\|^2_{L_2((0,t);H_{AB^{-1}AB^{-1}A})} + |u|^2_{W_2^{3/2}((0,t);H_B)} \le\\
&\qquad \le (8\pi+2)\left(\|u_0\|^2_{AB^{-1}A} + \|f\|^2_{L_2((0,t);H_{B^{-1}AB^{-1}})} + |f|^2_{W_2^{1/2}((0,t);H_{B^{-1}})}\right),
\end{aligned} \tag{2.5}$$

etc.

Let us construct another scale of a priori estimates. Taking inner product of (2.1) with $2u$, we get

$$\frac{1}{2}\left(\|u(t)\|^2_B\right)' + \|u(t)\|^2_A = (f(t),\, u(t)) \le \|u(t)\|_B\, \|f(t)\|_{B^{-1}}\,. \tag{2.6}$$

Since

$$\left(\|u(t)\|^2_B\right)' = 2\,\|u(t)\|_B\left(\|u(t)\|_B\right)' \qquad \text{and} \qquad \|u(t)\|^2_A \ge \|u(t)\|_B\,\|u(t)\|_A,$$

from (2.6) it follows

$$\left(\|u(t)\|_B\right)' + \|u(t)\|_A \le \|f(t)\|_{B^{-1}},$$

and after integration on t

$$\|u(t)\|_B + \int_0^t \|u(s)\|_A\, ds \le \|u_0\|_B + \int_0^t \|f(s)\|_{B^{-1}}\, ds\,. \tag{2.7}$$

Applying operators BA^{-1}, AB^{-1}, $(AB^{-1})^2$, $(AB^{-1})^3$ etc. to (2.1) and using (2.7), we obtain the following scale of a priori estimates:

$$\begin{aligned}
&\|Bu(t)\|_{A^{-1}} + \int_0^t \|u(s)\|_B\, ds \le \|Bu_0\|_{A^{-1}} + \int_0^t \|f(s)\|_{A^{-1}}\, ds\,,\\
&\|u(t)\|_A + \int_0^t \|Au(s)\|_{B^{-1}}\, ds \le \|u_0\|_A + \int_0^t \|B^{-1}f(s)\|_A\, ds\,,\\
&\|Au(t)\|_{B^{-1}} + \int_0^t \|B^{-1}Au(s)\|_A\, ds \le \|Au_0\|_{B^{-1}} + \int_0^t \|AB^{-1}f(s)\|_{B^{-1}}\, ds\,,\\
&\|B^{-1}Au(t)\|_A + \int_0^t \|AB^{-1}Au(s)\|_{B^{-1}}\, ds \le \|B^{-1}Au_0\|_A + \int_0^t \|B^{-1}AB^{-1}f(s)\|_A\, ds\,,\\
&\text{etc.}
\end{aligned} \tag{2.8}$$

From (2.1) it follows

$$\|u'\|_B = \|f - Au\|_{B^{-1}} \le \|Au\|_{B^{-1}} + \|f\|_{B^{-1}} \le \|Au\|_{B^{-1}} + \|B^{-1}f\|_A\,,$$

$$\|u'\|_A \le \|B^{-1}Au\|_A + \|AB^{-1}f\|_{B^{-1}}\,.$$

Combining these inequalities with (2.7) and (2.8) we get

$$\begin{aligned}
&\|u\|_{[0]} \equiv \|u\|_{C([0,t];\,H_{BA^{-1}B})} + \|u\|_{L_1((0,t);\,H_B)} \le 2\|u_0\|_{BA^{-1}B} + 2\|f\|_{L_1((0,t);\,H_{A^{-1}})}\,,\\
&\|u\|_{[1]} \equiv \|u\|_{C([0,t];\,H_B)} + \|u\|_{L_1((0,t);\,H_A)} \le 2\|u_0\|_B + 2\|f\|_{L_1((0,t);\,H_{B^{-1}})}\,,\\
&\|u\|_{[2]} \equiv \|u\|_{C([0,t];\,H_A)} + \|u\|_{L_1((0,t);\,H_{AB^{-1}A})} + \|u'\|_{L_1((0,t);\,H_B)} \le\\
&\qquad \le 3\|u_0\|_A + 4\|f\|_{L_1((0,t);\,H_{B^{-1}AB^{-1}})}\,,\\
&\|u\|_{[3]} \equiv \|u\|_{C([0,t];\,H_{AB^{-1}A})} + \|u\|_{L_1((0,t);\,H_{AB^{-1}AB^{-1}A})} + \|u'\|_{L_1((0,t);\,H_A)} \le\\
&\qquad \le 3\|u_0\|_{AB^{-1}A} + 4\|f\|_{L_1((0,t);\,H_{B^{-1}AB^{-1}AB^{-1}})}\,,\\
&\text{etc.}
\end{aligned} \tag{2.9}$$

Scales of a priori estimates (2.5) and (2.9) express global stability of the Cauchy problem (2.1).

2.3. Asymptotical Stability

Estimates of the items $\|Bu(t)\|^2_{A^{-1}}$, $\|u(t)\|^2_B$, $\|u(t)\|^2_A$ etc. in (2.4) can be improved in the following manner. Taking inner product of equation (2.1) with $2u$ we get

$$\left(\|u(t)\|^2_B\right)' + 2\,\|u(t)\|^2_A = 2\,(f(t),\, u(t)) \le \|f(t)\|^2_{A^{-1}} + \|u(t)\|^2_A. \tag{2.10}$$

Let us consider the eigenvalue problem

$$Au = \lambda\, Bu. \tag{2.11}$$

Problem (2.11) has a countable set of eigenvalues. All eigenvalues are positive. The eigenelements are orthogonal in H_B. For each $u \in H_A$ the following inequality is valid

$$\|u\|^2_A \ge \lambda_1\, \|u\|^2_B\,, \tag{2.12}$$

where $\lambda_1 > 0$ is the minimal eigenvalue of (2.11). From (2.10) and (2.12) it follows

$$\left(\|u(t)\|^2_B\right)' + \lambda_1\, \|u(t)\|^2_B \le \|f(t)\|^2_{A^{-1}}, \tag{2.13}$$

wherefrom we get by integration

$$\|u(t)\|_B^2 \le e^{-\lambda_1 t}\Big(\|u_0\|_B^2 + \int_0^t e^{\lambda_1 s}\, \|f(s)\|_{A^{-1}}^2 \, ds \Big). \tag{2.14}$$

In an analogous way one obtains

$$\begin{aligned}
&\|u(t)\|_A^2 \le e^{-\lambda_1 t}\Big(\|u_0\|_A^2 + \int_0^t e^{\lambda_1 s}\, \|f(s)\|_{B^{-1}}^2 \, ds \Big),\\
&\|Au(t)\|_{B^{-1}}^2 \le e^{-\lambda_1 t}\Big(\|Au_0\|_{B^{-1}}^2 + \int_0^t e^{\lambda_1 s}\, \|B^{-1} f(s)\|_A^2 \, ds \Big),\\
&\|B^{-1}Au(t)\|_A^2 \le e^{-\lambda_1 t}\Big(\|B^{-1}Au_0\|_A^2 + \int_0^t e^{\lambda_1 s}\, \|AB^{-1} f(s)\|_{B^{-1}}^2 \, ds \Big),
\end{aligned} \tag{2.15}$$

etc.

Using above defined norms, inequalities (2.14)–(2.15) can be rewritten in the form of a scale of a priori estimates:

$$\begin{aligned}
&\|u(t)\|_B^2 \le e^{-\lambda_1 t}\Big(\|u_0\|_B^2 + \|f\|^2_{L_2((0,t);\, e^{\lambda_1 s}; H_{A^{-1}})} \Big),\\
&\|u(t)\|_A^2 \le e^{-\lambda_1 t}\Big(\|u_0\|_A^2 + \|f\|^2_{L_2((0,t);\, e^{\lambda_1 s}; H_{B^{-1}})} \Big),\\
&\|u(t)\|_{AB^{-1}A}^2 \le e^{-\lambda_1 t}\Big(\|u_0\|_{AB^{-1}A}^2 + \|f\|^2_{L_2((0,t);\, e^{\lambda_1 s}; H_{B^{-1}AB^{-1}})} \Big),\\
&\|u(t)\|_{AB^{-1}AB^{-1}A}^2 \le e^{-\lambda_1 t}\Big(\|u_0\|_{AB^{-1}AB^{-1}A}^2 + \|f\|^2_{L_2((0,t);\, e^{\lambda_1 s}; H_{B^{-1}AB^{-1}AB^{-1}})} \Big),
\end{aligned} \tag{2.16}$$

etc.

Coefficient at $\lambda_1 t$ in exponent can be increased to $2-\varepsilon$ estimating inner product $(f(t),\, u(t))$ in (2.10) by Schwartz inequality with $\varepsilon \in (0,2)$. In such a way, instead of (2.13) one obtains

$$\Big(\|u(t)\|_B^2 \Big)' + (2-\varepsilon)\lambda_1 \|u(t)\|_B^2 \le \frac{1}{\varepsilon} \|f(t)\|_{A^{-1}}^2 .$$

From here further follows

$$\|u(t)\|_B^2 \le e^{-(2-\varepsilon)\lambda_1 t}\Big(\|u_0\|_B^2 + \frac{1}{\varepsilon}\int_0^t e^{(2-\varepsilon)\lambda_1 s}\, \|f(s)\|_{A^{-1}}^2 \, ds \Big).$$

From (2.6) and (2.12) it follows

$$\Big(\|u(t)\|_B \Big)' + \lambda_1 \|u(t)\|_B \le \|f(t)\|_{B^{-1}}$$

and after integration on t

$$\|u(t)\|_B \le e^{-\lambda_1 t}\Big(\|u_0\|_B + \int_0^t e^{\lambda_1 s}\, \|f(s)\|_{B^{-1}} \, ds \Big).$$

Analogously one obtains

$$\begin{aligned}
&\|u(t)\|_A \le e^{-\lambda_1 t}\Big(\|u_0\|_A + \int_0^t e^{\lambda_1 s}\, \|B^{-1} f(s)\|_A \, ds \Big),\\
&\|Au(t)\|_{B^{-1}} \le e^{-\lambda_1 t}\Big(\|Au_0\|_{B^{-1}} + \int_0^t e^{\lambda_1 s}\, \|AB^{-1} f(s)\|_{B^{-1}} \, ds \Big),\\
&\|B^{-1}Au(t)\|_A \le e^{-\lambda_1 t}\Big(\|B^{-1}Au_0\|_A + \int_0^t e^{\lambda_1 s}\, \|B^{-1}AB^{-1} f(s)\|_A \, ds \Big),
\end{aligned}$$

etc.

Using above defined norms we rewrite these inequalities in the form of a scale of a priori estimates:

$$\begin{aligned}
&\|u(t)\|_B \le e^{-\lambda_1 t}\Big(\|u_0\|_B + \|f\|_{L_1((0,t);\,e^{\lambda_1 s};H_{B^{-1}})}\Big),\\
&\|u(t)\|_A \le e^{-\lambda_1 t}\Big(\|u_0\|_A + \|f\|_{L_1((0,t);\,e^{\lambda_1 s};H_{B^{-1}AB^{-1}})}\Big),\\
&\|u(t)\|_{AB^{-1}A} \le e^{-\lambda_1 t}\Big(\|u_0\|_{AB^{-1}A} + \|f\|_{L_1((0,t);\,e^{\lambda_1 s};H_{B^{-1}AB^{-1}AB^{-1}})}\Big),\\
&\|u(t)\|_{AB^{-1}AB^{-1}A} \le e^{-\lambda_1 t}\Big(\|u_0\|_{AB^{-1}AB^{-1}A} + \|f\|_{L_1((0,t);\,e^{\lambda_1 s};H_{B^{-1}AB^{-1}AB^{-1}AB^{-1}})}\Big),
\end{aligned} \tag{2.17}$$

etc.

Scales of a priori estimates (2.16) and (2.17) express asymptotical stability of the Cauchy problem (2.1).

2.4. Strong Stability

Along with problem (2.1), we consider a similar Cauchy problem with perturbed right-hand side, initial condition ad operators

$$\tilde{B}\tilde{u}' + \tilde{A}\tilde{u} = \tilde{f}(t), \quad t > 0; \qquad \tilde{u}(0) = \tilde{u}_0. \tag{2.18}$$

Following [13], we say that the problem (2.18) is strongly stable if

$$\|\tilde{u} - u\|_0 \le C_1 \|\tilde{u}_0 - u_0\|_1 + C_2 \|\tilde{f} - f\|_2 + C_3 \|\tilde{A} - A\|_3 + C_4 \|\tilde{B} - B\|_4\,, \tag{2.19}$$

where C_i are constants and $\|\cdot\|_i$ are some norms ($i = 0, 1, 2, 3, 4$).
Assume operators $\tilde{A}$, $\tilde{B}$ of perturbed problem (2.18) satisfies analogous assumptions as operators A and B and furthermore $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$, $\mathcal{D}(\tilde{B}) = \mathcal{D}(B)$. From (2.1) and (2.18) we obtain the following Cauchy problem for the error $z = \tilde{u} - u$

$$Bz' + Az = \tilde{f}(t) - f(t) - (\tilde{A} - A)\tilde{u} - (\tilde{B} - B)\tilde{u}', \quad t > 0; \qquad z(0) = \tilde{u}_0 - u_0. \tag{2.20}$$

Applying the first a priori estimate from (2.5) to (2.20) we immediately get

$$\|z\|^2_{(0)} \le 2\|\tilde{u}_0 - u_0\|^2_{BA^{-1}B} + 6\|\tilde{f} - f\|^2_{L_2((0,t);H_{A^{-1}BA^{-1}})} +$$
$$+6\|(\tilde{A} - A)\tilde{u}\|^2_{L_2((0,t);H_{A^{-1}BA^{-1}})} + 6\|(\tilde{B} - B)\tilde{u}'\|^2_{L_2((0,t);H_{A^{-1}BA^{-1}})}.$$

Further we have

$$\|(\tilde{A} - A)\tilde{u}\|^2_{L_2((0,t);H_{A^{-1}BA^{-1}})} = \int_0^t \|(\tilde{A} - A)\tilde{u}(s)\|^2_{A^{-1}BA^{-1}}\,ds =$$
$$= \int_0^t \frac{\|(\tilde{A} - A)\tilde{u}(s)\|^2_{A^{-1}BA^{-1}}}{\|\tilde{u}(s)\|^2_{\tilde{B}}}\,\|\tilde{u}(s)\|^2_{\tilde{B}}\,ds \le \|\tilde{A} - A\|^2_{H_{\tilde{B}}\to H_{A^{-1}BA^{-1}}} \int_0^t \|\tilde{u}(s)\|^2_{\tilde{B}}\,ds$$

and

$$\|(\tilde{B} - B)\tilde{u}'\|^2_{L_2((0,t);H_{A^{-1}BA^{-1}})} = \int_0^t \|(\tilde{B} - B)\tilde{u}'(s)\|^2_{A^{-1}BA^{-1}}\,ds =$$
$$= \int_0^t \frac{\|(\tilde{B} - B)\tilde{u}'(s)\|^2_{A^{-1}BA^{-1}}}{\|\tilde{u}'(s)\|^2_{\tilde{B}\tilde{A}^{-1}\tilde{B}\tilde{A}^{-1}\tilde{B}}}\,\|\tilde{u}'(s)\|^2_{\tilde{B}\tilde{A}^{-1}\tilde{B}\tilde{A}^{-1}\tilde{B}}\,ds \le$$
$$\le \|\tilde{B} - B\|^2_{H_{\tilde{B}\tilde{A}^{-1}\tilde{B}\tilde{A}^{-1}\tilde{B}}\to H_{A^{-1}BA^{-1}}} \int_0^t \|\tilde{u}'(s)\|^2_{\tilde{B}\tilde{A}^{-1}\tilde{B}\tilde{A}^{-1}\tilde{B}}\,ds =$$
$$= \|\tilde{B} - B\|^2_{H_{\tilde{B}\tilde{A}^{-1}\tilde{B}\tilde{A}^{-1}\tilde{B}}\to H_{A^{-1}BA^{-1}}} \int_0^t \|\tilde{A}\tilde{u}(u) - \tilde{f}(s)\|^2_{\tilde{A}^{-1}\tilde{B}\tilde{A}^{-1}}\,ds$$

where $\|A\|_{H_1\to H_2}=\sup_{v\in H_1}\|Av\|_{H_2}/\|v\|_{H_1}$ denotes the standard operator norm.
In such a manner, we obtained the following a priori estimate of strong stability

$$\begin{aligned}\|z\|_{(0)}^2\le 2\|\tilde u_0-u_0\|^2_{BA^{-1}B}+6\|\tilde f-f\|^2_{L_2((0,t);\,H_{A^{-1}BA^{-1}})}+\\ +C_3^{(0)}\|\tilde A-A\|^2_{H_{\tilde B}\to H_{A^{-1}BA^{-1}}}+C_4^{(0)}\|\tilde B-B\|^2_{H_{\tilde B\tilde A^{-1}\tilde B\tilde A^{-1}\tilde B}\to H_{A^{-1}BA^{-1}}},\end{aligned}\tag{2.21}$$

where

$$C_3^{(0)}=C_3^{(0)}(\tilde u)=6\,\|\tilde u\|^2_{L_2((0,t);\,H_{\tilde B})}\le 6\Big(\|\tilde u_0\|^2_{\tilde B\tilde A^{-1}\tilde B}+\|\tilde f\|^2_{L_2((0,t);\,H_{\tilde A^{-1}\tilde B\tilde A^{-1}})}\Big),$$
$$C_4^{(0)}=C_4^{(0)}(\tilde u)=6\,\|\tilde A\tilde u-\tilde f\|^2_{L_2((0,t);\,H_{\tilde A^{-1}\tilde B\tilde A^{-1}})}\le 12\Big(\|\tilde u_0\|^2_{\tilde B\tilde A^{-1}\tilde B}+2\|\tilde f\|^2_{L_2((0,t);\,H_{\tilde A^{-1}\tilde B\tilde A^{-1}})}\Big).$$

In the same way, applying the subsequent a priori estimates from the scale (2.5) to the problem (2.20), one obtains

$$\begin{aligned}&\|z\|_{(1)}^2\le(4\pi+2)\|\tilde u_0-u_0\|^2_B+3(4\pi+2)\|\tilde f-f\|^2_{L_2((0,t);\,H_{A^{-1}})}+\\&\qquad+C_3^{(1)}\|\tilde A-A\|^2_{H_{\tilde A}\to H_{A^{-1}}}+C_4^{(1)}\|\tilde B-B\|^2_{H_{\tilde B\tilde A^{-1}\tilde B}\to H_{A^{-1}}},\\&\|z\|_{(2)}^2\le 3\|\tilde u_0-u_0\|^2_A+9\|\tilde f-f\|^2_{L_2((0,t);\,H_{B^{-1}})}+\\&\qquad+C_3^{(2)}\|\tilde A-A\|^2_{H_{\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}}}+C_4^{(2)}\|\tilde B-B\|^2_{H_{\tilde B}\to H_{B^{-1}}},\\&\|z\|_{(3)}^2\le(8\pi+2)\|\tilde u_0-u_0\|^2_{AB^{-1}A}+3(8\pi+2)\|\tilde f-f\|^2_{(3),2}+\\&\qquad+C_3^{(3)}\|\tilde A-A\|^2_{(3),3}+C_4^{(3)}\|\tilde B-B\|^2_{(3),4},\end{aligned}\tag{2.22}$$

where

$$\|f\|^2_{(3),2}=\|f\|^2_{L_2((0,t);\,H_{B^{-1}AB^{-1}})}+|f|^2_{W_2^{1/2}((0,t);\,H_{B^{-1}})},$$
$$\|A\|^2_{(3),3}=\|A\|^2_{H_{\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}}}+\|A\|^2_{H_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}AB^{-1}}},$$
$$\|B\|^2_{(3),4}=\|B\|^2_{H_{\tilde B}\to H_{B^{-1}}}+\|B\|^2_{H_{\tilde A}\to H_{B^{-1}AB^{-1}}},$$
$$C_3^{(1)}=C_3^{(1)}(\tilde u)=3(4\pi+2)\|\tilde u\|^2_{L_2((0,t);\,H_{\tilde A})}\le 3(4\pi+2)\Big(\|\tilde u_0\|^2_{\tilde B}+\|\tilde f\|^2_{L_2((0,t);\,H_{\tilde A^{-1}})}\Big),$$
$$C_4^{(1)}=C_4^{(1)}(\tilde u)=3(4\pi+2)\|\tilde A\tilde u-\tilde f\|^2_{L_2((0,t);H_{\tilde A^{-1}})}\le 6(4\pi+2)\Big(\|\tilde u_0\|^2_{\tilde B}+2\,\|\tilde f\|^2_{L_2((0,t);H_{\tilde A^{-1}})}\Big),$$
$$C_3^{(2)}=C_3^{(2)}(\tilde u)=9\,\|\tilde u\|^2_{L_2((0,t);\,H_{\tilde A\tilde B^{-1}\tilde A})}\le 9\Big(\|\tilde u_0\|^2_{\tilde A}+\|\tilde f\|^2_{L_2((0,t);\,H_{\tilde B^{-1}})}\Big),$$
$$C_4^{(2)}=C_4^{(2)}(\tilde u)=9\,\|\tilde A\tilde u-\tilde f\|^2_{L_2((0,t);H_{\tilde B^{-1}})}\le 18\Big(\|\tilde u_0\|^2_{\tilde A}+2\,\|\tilde f\|^2_{L_2((0,t);\,H_{\tilde B^{-1}})}\Big),$$
$$C_3^{(3)}=C_3^{(3)}(\tilde u)\le 3\pi(8\pi+2)\Big(3\,\|\tilde u_0\|^2_{\tilde A\tilde B^{-1}\tilde A}+4\,\|\tilde f\|^2_{L_2((0,t);\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),$$
$$C_4^{(3)}=C_4^{(3)}(\tilde u)\le 6(8\pi+2)\Big(3\pi\,\|\tilde u_0\|^2_{\tilde A\tilde B^{-1}\tilde A}+4\pi\,\|\tilde f\|^2_{L_2((0,t);\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}})}+|\tilde f|^2_{W_2^{1/2}((0,t);\,H_{\tilde B^{-1}})}\Big).$$

Analogously, applying inequalities (2.9) to (2.20), we obtain the following a priori estimates of the strong stability:

$$
\begin{aligned}
\|z\|_{[0]} &\leq 2\|\tilde{u}_0 - u_0\|_{BA^{-1}B} + 2\|\tilde{f} - f\|_{L_1((0,t);\, H_{A^{-1}})} + \\
&\quad + C_3^{[0]}\|\tilde{A} - A\|_{H_{\tilde{A}} \to H_{A^{-1}}} + C_4^{[0]}\|\tilde{B} - B\|_{H_{\tilde{B}\tilde{A}^{-1}\tilde{B}} \to H_{A^{-1}}}, \\
\|z\|_{[1]} &\leq 2\|\tilde{u}_0 - u_0\|_{B} + 2\|\tilde{f} - f\|_{L_1((0,t);\, H_{B^{-1}})} + \\
&\quad + C_3^{[1]}\|\tilde{A} - A\|_{H_{\tilde{A}\tilde{B}^{-1}\tilde{A}} \to H_{B^{-1}}} + C_4^{[1]}\|\tilde{B} - B\|_{H_{\tilde{B}} \to H_{B^{-1}}}, \\
\|z\|_{[2]} &\leq 3\|\tilde{u}_0 - u_0\|_{A} + 4\|\tilde{f} - f\|_{L_1((0,t);\, H_{B^{-1}AB^{-1}})} + \\
&\quad + C_3^{[2]}\|\tilde{A} - A\|_{H_{\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}} \to H_{B^{-1}AB^{-1}}} + C_4^{[2]}\|\tilde{B} - B\|_{H_{\tilde{A}} \to H_{B^{-1}AB^{-1}}}, \\
\|z\|_{[3]} &\leq 3\|\tilde{u}_0 - u_0\|_{AB^{-1}A} + 4\|\tilde{f} - f\|_{L_1((0,t);\, H_{B^{-1}AB^{-1}AB^{-1}})} + \\
&\quad + C_3^{[3]}\|\tilde{A} - A\|_{H_{\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}} \to H_{B^{-1}AB^{-1}AB^{-1}}} + C_4^{[3]}\|\tilde{B} - B\|_{H_{\tilde{A}\tilde{B}^{-1}\tilde{A}} \to H_{B^{-1}AB^{-1}AB^{-1}}},
\end{aligned}
\tag{2.23}
$$

where

$$
\begin{aligned}
C_3^{[0]} &= C_3^{[0]}(\tilde{u}) = 2\,\|\tilde{u}\|_{L_1((0,t);\, H_{\tilde{A}})} \leq 2\Big(\|\tilde{u}_0\|_{\tilde{B}} + \|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{B}^{-1}})}\Big), \\
C_4^{[0]} &= C_4^{[0]}(\tilde{u}) = 2\,\|\tilde{A}\tilde{u} - \tilde{f}\|_{L_1((0,t);\, H_{\tilde{A}^{-1}})} \leq 2\Big(\|\tilde{u}_0\|_{\tilde{B}} + 2\,\|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{B}^{-1}})}\Big), \\
C_3^{[1]} &= C_3^{[1]}(\tilde{u}) = 2\,\|\tilde{u}\|_{L_1((0,t);\, H_{\tilde{A}\tilde{B}^{-1}\tilde{A}})} \leq 2\Big(\|\tilde{u}_0\|_{\tilde{A}} + \|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}})}\Big), \\
C_4^{[1]} &= C_4^{[1]}(\tilde{u}) = 2\,\|\tilde{A}\tilde{u} - \tilde{f}\|_{L_1((0,t);\, H_{\tilde{B}^{-1}})} \leq 2\Big(\|\tilde{u}_0\|_{\tilde{A}} + 2\,\|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}})}\Big), \\
C_3^{[2]} &= C_3^{[2]}(\tilde{u}) = 4\,\|\tilde{u}\|_{L_1((0,t);\, H_{\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}})} \leq 4\Big(\|\tilde{u}_0\|_{\tilde{A}\tilde{B}^{-1}\tilde{A}} + \|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}})}\Big), \\
C_4^{[2]} &= C_4^{[2]}(\tilde{u}) = 4\,\|\tilde{A}\tilde{u} - \tilde{f}\|_{L_1((0,t);\, H_{\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}})} \leq \\
&\qquad \leq 4\Big(\|\tilde{u}_0\|_{\tilde{A}\tilde{B}^{-1}\tilde{A}} + 2\,\|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}})}\Big), \\
C_3^{[3]} &= C_3^{[3]}(\tilde{u}) = 4\,\|\tilde{u}\|_{L_1((0,t);\, H_{\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}})} \leq \\
&\qquad \leq 4\Big(\|\tilde{u}_0\|_{\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}} + \|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}})}\Big), \\
C_4^{[3]} &= C_4^{[3]}(\tilde{u}) = 4\,\|\tilde{A}\tilde{u} - \tilde{f}\|_{L_1((0,t);\, H_{\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}})} \leq \\
&\qquad \leq 4\Big(\|\tilde{u}_0\|_{\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}} + 2\,\|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}})}\Big).
\end{aligned}
$$

Finally, applying the scales of a priori estimates (2.16) and (2.17) to the problem (2.20), after some algebra we get

$$
\begin{aligned}
&\|z(t)\|_B^2 \le e^{-\lambda_1 t}\Big(\|\tilde u_0-u_0\|_B^2+3\,\|\tilde f-f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{A^{-1}})}+\\
&\quad +C_3^{\{0\}}\|\tilde A-A\|^2_{H_{\tilde A}\to H_{A^{-1}}}+C_4^{\{0\}}\|\tilde B-B\|^2_{H_{\tilde B\tilde A^{-1}\tilde B}\to H_{A^{-1}}}\Big),\\
&\|z(t)\|_A^2 \le e^{-\lambda_1 t}\Big(\|\tilde u_0-u_0\|_A^2+3\,\|\tilde f-f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{B^{-1}})}+\\
&\quad +C_3^{\{1\}}\|\tilde A-A\|^2_{H_{\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}}}+C_4^{\{1\}}\|\tilde B-B\|^2_{H_{\tilde B}\to H_{B^{-1}}}\Big),\\
&\|z(t)\|_{AB^{-1}A}^2 \le e^{-\lambda_1 t}\Big(\|\tilde u_0-u_0\|_{AB^{-1}A}^2+3\,\|\tilde f-f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{B^{-1}AB^{-1}}})+\\
&\quad +C_3^{\{2\}}\|\tilde A-A\|^2_{H_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}AB^{-1}}}+C_4^{\{2\}}\|\tilde B-B\|^2_{H_{\tilde A}\to H_{B^{-1}AB^{-1}}}\Big),\\
&\|z(t)\|_{AB^{-1}AB^{-1}A}^2 \le e^{-\lambda_1 t}\Big(\|\tilde u_0-u_0\|_{AB^{-1}AB^{-1}A}^2+3\,\|\tilde f-f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{B^{-1}AB^{-1}AB^{-1}}})+\\
&\quad +C_3^{\{3\}}\|\tilde A-A\|^2_{H_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}AB^{-1}AB^{-1}}}+C_4^{\{3\}}\|\tilde B-B\|^2_{H_{\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}AB^{-1}AB^{-1}}}\Big),
\end{aligned}
\tag{2.24}
$$

where

$$
\begin{aligned}
C_3^{\{0\}} &= C_3^{\{0\}}(\tilde u)=3\,\|\tilde u\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde A})}\le 3t\Big(\|\tilde u_0\|^2_{\tilde A}+\|\tilde f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}})}\Big),\\
C_4^{\{0\}} &= C_4^{\{0\}}(\tilde u)=3\,\|\tilde A\tilde u-\tilde f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde A^{-1}})}\le\\
&\qquad\le 6(t+1)\Big(\|\tilde u_0\|^2_{\tilde A}+\|\tilde f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}})}\Big),\\
C_3^{\{1\}} &= C_3^{\{1\}}(\tilde u)=3\,\|\tilde u\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde A\tilde B^{-1}\tilde A})}\le 3t\Big(\|\tilde u_0\|^2_{\tilde A\tilde B^{-1}\tilde A}+\|\tilde f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),\\
C_4^{\{1\}} &= C_4^{\{1\}}(\tilde u)=3\,\|\tilde A\tilde u-\tilde f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}})}\le\\
&\qquad\le 6(t+1)\Big(\|\tilde u_0\|^2_{\tilde A\tilde B^{-1}\tilde A}+\|\tilde f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),\\
C_3^{\{2\}} &= C_3^{\{2\}}(\tilde u)=3\,\|\tilde u\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A})}\le\\
&\qquad\le 3t\Big(\|\tilde u_0\|^2_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A}+\|\tilde f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),\\
C_4^{\{2\}} &= C_4^{\{2\}}(\tilde u)=3\,\|\tilde A\tilde u-\tilde f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}})}\le\\
&\qquad\le 6(t+1)\Big(\|\tilde u_0\|^2_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A}+\|\tilde f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),\\
C_3^{\{3\}} &= C_3^{\{3\}}(\tilde u)=3\,\|\tilde u\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A})}\le\\
&\qquad\le 3t\Big(\|\tilde u_0\|^2_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A}+\|\tilde f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),\\
C_4^{\{3\}} &= C_4^{\{3\}}(\tilde u)=3\,\|\tilde A\tilde u-\tilde f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\le\\
&\qquad\le 6(t+1)\Big(\|\tilde u_0\|^2_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A}+\|\tilde f\|^2_{L_2((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),
\end{aligned}
$$

accordingly

$$\|z(t)\|_B \le e^{-\lambda_1 t}\Big(\|\tilde u_0 - u_0\|_B + \|\tilde f - f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{B^{-1}})} + \\ + C_3^{\langle 0\rangle}\|\tilde A - A\|_{H_{\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}}} + C_4^{\langle 0\rangle}\|\tilde B - B\|_{H_{\tilde B}\to H_{B^{-1}}}\Big),$$

$$\|z(t)\|_A \le e^{-\lambda_1 t}\Big(\|\tilde u_0 - u_0\|_A + \|\tilde f - f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{B^{-1}AB^{-1}})} + \\ + C_3^{\langle 1\rangle}\|\tilde A - A\|_{H_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}AB^{-1}}} + C_4^{\langle 1\rangle}\|\tilde B - B\|_{H_{\tilde A}\to H_{B^{-1}AB^{-1}}}\Big),$$

$$\|z(t)\|_{AB^{-1}A} \le e^{-\lambda_1 t}\Big(\|\tilde u_0 - u_0\|_{AB^{-1}A} + \|\tilde f - f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{B^{-1}AB^{-1}AB^{-1}})} + \\ + C_3^{\langle 2\rangle}\|\tilde A - A\|_{H_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}AB^{-1}AB^{-1}}} + C_4^{\langle 2\rangle}\|\tilde B - B\|_{H_{\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}AB^{-1}AB^{-1}}}\Big),$$

$$\|z(t)\|_{AB^{-1}AB^{-1}A} \le e^{-\lambda_1 t}\Big(\|\tilde u_0 - u_0\|_{AB^{-1}AB^{-1}A} + \|\tilde f - f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{B^{-1}AB^{-1}AB^{-1}AB^{-1}})} + \\ + C_3^{\langle 3\rangle}\|\tilde A - A\|_{H_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}AB^{-1}AB^{-1}AB^{-1}}} + \\ + C_4^{\langle 3\rangle}\|\tilde B - B\|_{H_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A}\to H_{B^{-1}AB^{-1}AB^{-1}AB^{-1}}}\Big), \tag{2.25}$$

where

$$C_3^{\langle 0\rangle} = C_3^{\langle 0\rangle}(\tilde u) = \|\tilde u\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde A\tilde B^{-1}\tilde A})} \le t\Big(\|\tilde u_0\|_{\tilde A\tilde B^{-1}\tilde A} + \|\tilde f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),$$
$$C_4^{\langle 0\rangle} = C_4^{\langle 0\rangle}(\tilde u) = \|\tilde A\tilde u - \tilde f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}})} \le \\ \le (t+1)\Big(\|\tilde u_0\|_{\tilde A\tilde B^{-1}\tilde A} + \|\tilde f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),$$
$$C_3^{\langle 1\rangle} = C_3^{\langle 1\rangle}(\tilde u) = \|\tilde u\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A})} \le \\ \le t\Big(\|\tilde u_0\|_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A} + \|\tilde f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),$$
$$C_4^{\langle 1\rangle} = C_4^{\langle 1\rangle}(\tilde u) = \|\tilde A\tilde u - \tilde f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}})} \le \\ \le (t+1)\Big(\|\tilde u_0\|_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A} + \|\tilde f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big);$$
$$C_3^{\langle 2\rangle} = C_3^{\langle 2\rangle}(\tilde u) = \|\tilde u\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A})} \le \\ \le t\Big(\|\tilde u_0\|_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A} + \|\tilde f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),$$
$$C_4^{\langle 2\rangle} = C_4^{\langle 2\rangle}(\tilde u) = \|\tilde A\tilde u - \tilde f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})} \le \\ \le (t+1)\Big(\|\tilde u_0\|_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A} + \|\tilde f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),$$
$$C_3^{\langle 3\rangle} = C_3^{\langle 3\rangle}(\tilde u) = \|\tilde u\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A})} \le \\ \le t\Big(\|\tilde u_0\|_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A} + \|\tilde f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big),$$
$$C_4^{\langle 3\rangle} = C_4^{\langle 3\rangle}(\tilde u) = \|\tilde A\tilde u - \tilde f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})} \le \\ \le (t+1)\Big(\|\tilde u_0\|_{\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A} + \|\tilde f\|_{L_1((0,t);\,e^{\lambda_1 s};\,H_{\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}\tilde A\tilde B^{-1}})}\Big).$$

The scales (2.22–2.25) can be widen by adding analogous estimates in stronger (and weaker) norms.

Remark 1. In contrast to the stability under perturbations of the initial condition and the right-hand side the problem of strong stability is nonlinear. That is why the quantities $C_3^{(i)}$, $C_4^{(i)}$, $C_3^{[i]}$, $C_4^{[i]}$, $C_3^{\{i\}}$, $C_4^{\{i\}}$, $C_3^{\langle i\rangle}$ and $C_4^{\langle i\rangle}$ in (2.21)–(2.25) depend on $\tilde u$ (and consequently on$\tilde u_0$ and $\tilde f$).

Remark 2. A priori estimates (2.21)-(2.22) can be treated as "compatible with the smoothness of data" because the same conditions are needed for $\tilde{u}_0$ and $\tilde{u}_0 - u_0$ i.e. for $\tilde{f}$ and $\tilde{f} - f$. This is not fulfilled for the scales (2.23)-(2.25) where $\tilde{u}_0$ and $\tilde{f}$ must belong to stronger spaces than $\tilde{u}_0 - u_0$ and $\tilde{f} - f$. For example, the second a priori estimate from (2.23) is satisfied if $\tilde{u}_0 - u_0 \in H_B$ and $\tilde{f} - f \in L_1((0,t);\, H_{B^{-1}})$, while the quantities $C_3^{[1]}$ and $C_4^{[1]}$ are bounded for $\tilde{u}_0 \in H_{\tilde{A}}$ and $\tilde{f} \in L_1((0,t);\, H_{\tilde{B}^{-1}\tilde{A}\tilde{B}^{-1}})$.

Remark 3. In some cases operator norms appearing in (2.21)-(2.25) can be substituted be more simples. Let for the simplicity $B = I$ be the identity operator and $\|\tilde{A} - A\|_{H_A \to H_{A^{-1}}} \le \alpha < 1$. Then the norms $\|u\|_{\tilde{A}}$ and $\|u\|_A$, $\|u\|_{\tilde{A}^{-1}}$ and $\|u\|_{A^{-1}}$, i.e. $\|\tilde{A} - A\|_{H_A \to H_{A^{-1}}}$, $\|\tilde{A} - A\|_{H_{\tilde{A}} \to H_{A^{-1}}}$, $\|\tilde{A} - A\|_{H_A \to H_{\tilde{A}^{-1}}}$ and $\|\tilde{A} - A\|_{H_{\tilde{A}} \to H_{\tilde{A}^{-1}}}$ are equivalents and the following inequalities hold

$$\sqrt{1-\alpha}\,\|u\|_A \le \|u\|_{\tilde{A}} \le \sqrt{1+\alpha}\,\|u\|_A, \qquad \frac{1}{\sqrt{1+\alpha}}\,\|u\|_{A^{-1}} \le \|u\|_{\tilde{A}^{-1}} \le \frac{1}{\sqrt{1-\alpha}}\,\|u\|_{A^{-1}},$$

$$\frac{1}{\sqrt{1+\alpha}}\,\|\tilde{A} - A\|_{H_A \to H_{A^{-1}}} \le \|\tilde{A} - A\|_{H_{\tilde{A}} \to H_{A^{-1}}} \le \frac{1}{\sqrt{1-\alpha}}\,\|\tilde{A} - A\|_{H_A \to H_{A^{-1}}},$$

$$\frac{1}{\sqrt{1+\alpha}}\,\|\tilde{A} - A\|_{H_A \to H_{A^{-1}}} \le \|\tilde{A} - A\|_{H_A \to H_{\tilde{A}^{-1}}} \le \frac{1}{\sqrt{1-\alpha}}\,\|\tilde{A} - A\|_{H_A \to H_{A^{-1}}},$$

$$\frac{1}{1+\alpha}\,\|\tilde{A} - A\|_{H_A \to H_{A^{-1}}} \le \|\tilde{A} - A\|_{H_{\tilde{A}} \to H_{\tilde{A}^{-1}}} \le \frac{1}{1-\alpha}\,\|\tilde{A} - A\|_{H_A \to H_{A^{-1}}}.$$

Analogously, if $\|\tilde{A} - A\|_{H_{A^2} \to H} \le \alpha < 1$ then

$$(1-\alpha)\,\|u\|_{A^2} \le \|u\|_{\tilde{A}^2} \le (1+\alpha)\,\|u\|_{A^2}, \qquad \frac{1}{1+\alpha}\,\|u\|_{A^{-2}} \le \|u\|_{\tilde{A}^{-2}} \le \frac{1}{1-\alpha}\,\|u\|_{A^{-2}},$$

$$(1-\alpha)\,\|\tilde{A} - A\|_{H_{\tilde{A}^2} \to H} \le \|\tilde{A} - A\|_{H_{A^2} \to H} \le (1+\alpha)\,\|\tilde{A} - A\|_{H_{\tilde{A}^2} \to H},$$

$$(1-\alpha)\,\|\tilde{A} - A\|_{H \to H_{\tilde{A}^{-2}}} \le \|\tilde{A} - A\|_{H \to H_{A^{-2}}} \le (1+\alpha)\,\|\tilde{A} - A\|_{H \to H_{\tilde{A}^{-2}}}.$$

The same inequalities follow from $\|\tilde{A} - A\|_{H \to H_{A^{-2}}} \le \alpha < 1$.

Example. Let us consider the initial-boundary value problem for the heat conducting equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) + f(x,t), \quad x \in (0,1), \quad t > 0, \quad u(0,t) = u(1,t) = 0, \quad u(x,0) = u_0(x). \tag{2.26}$$

The problem (2.26) can be reduced to the form (2.18) setting $H = L_2(0,1)$, $Au = -\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)$ and B = identity operator in $L_2(0,1)$. If the following assumptions $a \in C^1[0,1]$, $0 < c_1 \le a(x) \le c_2$ and $|a'(x)| \le c_3$ are satisfied operator A maps the set $\mathcal{D}(A) = \overset{\circ}{W}{}_2^1(0,1) \cap W_2^2(0,1)$ onto $L_2(0,1)$. It is easy to check that the inverse operator A^{-1} is given by

$$A^{-1}v(x) = -\int_0^x \frac{1}{a(x')} \int_0^{x'} v(x'')\,dx''dx' +$$
$$+ \left(\int_0^1 \frac{dx'}{a(x')}\right)^{-1} \left(\int_0^x \frac{dx'}{a(x')}\right) \left(\int_0^1 \frac{1}{a(x')} \int_0^{x'} v(x'')\,dx''dx'\right).$$

The following inequalities are valid (see [8])

$$c_0\,\|v\|^2_{W_2^1(0,1)} \le (Av,\, v) = \int_0^1 a(x)\,|v'(x)|^2\,dx \le c_2\,\|v\|^2_{W_2^1(0,1)}, \qquad v \in \overset{\circ}{W}{}_2^1\,(0,1),$$

$$c_4 \|v\|_{W_2^2(0,1)} \leq \|Av\|_{L_2(0,1)} \leq c_5 \|v\|_{W_2^2(0,1)}, \qquad v \in \overset{\circ}{W}{}_2^1(0,1) \cap W_2^2(0,1), \tag{2.27}$$

where $c_0 = c_1\pi^2/(1+\pi^2)$, while c_4 and c_5 depend on c_1, c_2 and c_3. In such a manner, $H_A = \overset{\circ}{W}{}_2^1(0,1)$, $H_{A^{-1}} = W_2^{-1}(0,1)$ i $H_{A^2} = \overset{\circ}{W}{}_2^1(0,1) \cap W_2^2(0,1)$.

Together with (2.26) let us consider the perturbed initial-boundary value problem

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial}{\partial x}\left(\tilde{a}(x)\,\frac{\partial \tilde{u}}{\partial x}\right) + \tilde{f}(x,t), \quad x \in (0,1), \ t > 0, \tilde{u}(0,t) = \tilde{u}(1,t) = 0, \quad \tilde{u}(x,0) = \tilde{u}_0(x). \tag{2.28}$$

We assume that $\tilde{a} \in C^1[0,1]$, $0 < \tilde{c}_1 \leq \tilde{a}(x) \leq \tilde{c}_2$ and $|\tilde{a}'(x)| \leq \tilde{c}_3$. Analogously as in the previous case, we set $\tilde{A}u = -\dfrac{\partial}{\partial x}\left(\tilde{a}(x)\,\dfrac{\partial u}{\partial x}\right)$ and $\tilde{B} = B =$ identity operator in $L_2(0,1)$.

It is easy to check that

$$\|\tilde{A} - A\|_{H_A \to H_{A^{-1}}} \leq C\|\tilde{a} - a\|_{C[0,1]}, \quad \|\tilde{A} - A\|_{H_{A^2} \to H} \leq C\|\tilde{a} - a\|_{C^1[0,1]}, \\ \|\tilde{A} - A\|_{H \to H_{A^{-2}}} \leq C\|\tilde{a} - a\|_{C^1[0,1]}, \tag{2.29}$$

where C is a constant.

Indeed, the first inequality in (2.29) follows from

$$\|\tilde{A} - A\|_{H_A \to H_{A^{-1}}} = \sup_{w \in H_A} \frac{\|(\tilde{A} - A)w\|_{A^{-1}}}{\|w\|_A} = \sup_{w \in H_A} \frac{|((\tilde{A} - A)w, w)|}{(Aw, w)} =$$

$$= \sup_{w \in \overset{\circ}{W}{}_2^1(0,1)} \frac{\left|\int_0^1 (\tilde{a}(x) - a(x))\,|w'(x)|^2\,dx\right|}{\int_0^1 a(x)\,|w'(x)|^2\,dx}$$

and the boundedness of $a(x)$, the second follows from the equalities

$$\|\tilde{A} - A\|_{H_{A^2} \to H} = \sup_{w \in H_{A^2}} \frac{\|(\tilde{A} - A)w\|}{\|Aw\|} = \sup_{w \in \overset{\circ}{W}{}_2^1(0,1) \cap W_2^2(0,1)} \frac{\left|\int_0^1 \left|((\tilde{a}(x) - a(x))w'(x))'\right|^2 dx\right|}{\int_0^1 \left|(a(x)w'(x))'\right|^2 dx}$$

and (2.28), while the third follows from the definition of operator norm

$$\|\tilde{A} - A\|_{H \to H_{A^{-2}}} = \sup_{w \in H} \frac{\|A^{-1}(\tilde{A} - A)w\|}{\|w\|} = \sup_{w \in L_2(0,1)} \frac{\|A^{-1}(\tilde{A} - A)w\|_{L_2(0,1)}}{\|w\|_{L_2(0,1)}},$$

representation

$$\begin{aligned} A^{-1}(\tilde{A} - A)w = \ & \frac{\tilde{a}(x) - a(x)}{a(x)}\,w(x) - \\ & - \int_0^x \left[\frac{\tilde{a}'(x') - a'(x')}{a(x')} + \frac{a'(x')\,(a(x') - \tilde{a}(x'))}{a^2(x')}\right] w(x')\,dx' + \\ & + \left(\int_0^1 \frac{dx'}{a(x')}\right)^{-1} \left(\int_0^x \frac{dx'}{a(x')}\right) \times \\ & \times \int_0^1 \left[\frac{\tilde{a}'(x') - a'(x')}{a(x')} + \frac{a'(x')\,(a(x') - \tilde{a}(x'))}{a^2(x')}\right] w(x')\,dx' \end{aligned}$$

and boundedness of $a(x)$ and $a'(x)$.
In such a way, a priori estimates of the strong stability (2.21) and (2.22) take the form

$$\|z\|^2_{(0)} \equiv \|z\|^2_{C([0,t];\,W_2^{-1}(0,1))} + \|z\|^2_{L_2((0,t)\times(0,1))} \le$$
$$\le C_1^{(0)}\|\tilde{u}_0 - u_0\|^2_{W_2^{-1}(0,1)} + C_2^{(0)}\|\tilde{f} - f\|^2_{L_2((0,t);\,W_2^{-2}(0,1))} + C_3^{(0)}\|\tilde{a} - a\|^2_{C^1[0,1]},$$
$$\|z\|^2_{(1)} \equiv \|z\|^2_{C([0,t];\,L_2(0,1))} + \|z\|^2_{W_2^{1,1/2}((0,t)\times(0,1))}$$
$$\le C_1^{(1)}\|\tilde{u}_0 - u_0\|^2_{L_2(0,1)} + C_2^{(1)}\|\tilde{f} - f\|^2_{L_2((0,t);\,W_2^{-1}(0,1))} + C_3^{(1)}\|\tilde{a} - a\|^2_{C[0,1]},$$
$$\|z\|^2_{(2)} \equiv \|z\|^2_{C([0,t];\,W_2^1(0,1))} + \|z\|^2_{W_2^{2,1}((0,t)\times(0,1))}$$
$$\le C_1^{(2)}\|\tilde{u}_0 - u_0\|^2_{W_2^1(0,1)} + C_2^{(2)}\|\tilde{f} - f\|^2_{L_2((0,t)\times(0,1))} + C_3^{(2)}\|\tilde{a} - a\|^2_{C^1[0,1]},$$

where $C_1^{(j)}$ and $C_2^{(j)}$ are absolute constants, not depending of input data, and

$$C_3^{(0)} \le C\big(\|\tilde{u}_0\|^2_{W_2^{-1}(0,1)} + \|\tilde{f}\|^2_{L_2((0,t);\,W_2^{-2}(0,1))}\big),$$
$$C_3^{(1)} \le C\big(\|\tilde{u}_0\|^2_{L_2(0,1)} + \|\tilde{f}\|^2_{L_2((0,t);\,W_2^{-1}(0,1))}\big),$$
$$C_3^{(2)} \le C\big(\|\tilde{u}_0\|^2_{W_2^1(0,1)} + \|\tilde{f}\|^2_{L_2((0,t)\times(0,1))}\big).$$

3. SECOND ORDER LINEAR OPERATOR-DIFFERENTIAL EQUATION

Let us now consider the abstract Cauchy problem for the second-order operator differential equation

$$Du''(t) + Au(t) = f(t), \quad t > 0; \qquad u(0) = u_0, \quad u'(0) = u_1, \tag{3.1}$$

where A is an unbounded self-adjoint positive definite linear operator in the Hilbert space H, D is self-adjoint positive definite linear operator in H, $D \le A$, $u_0 \in H_A$, $u_1 \in H_D$ and $f(t) \in L_2(0,T;\, H_{D^{-1}})$.
In [5] on the base of the a priori estimate

$$\|u'(t)\|^2_D + \|u(t)\|^2_A \le C\left(\|u_1\|^2_D + \|u_0\|^2_A + \int_0^t \|f(s)\|^2_{D^{-1}} ds\right)$$

the following scale of a priori estimates

$$\max_{s\in[0,t]} \|u(s)\|^2_D \le C\left(\|Du_1\|^2_{A^{-1}} + \|u_0\|^2_D + \int_0^t \|f(s)\|^2_{A^{-1}} ds\right),$$
$$\max_{s\in[0,t]} \left(\|u'(s)\|^2_D + \|u(s)\|^2_A\right) \le C\left(\|u_1\|^2_D + \|u_0\|^2_A + \int_0^t \|f(s)\|^2_{D^{-1}} ds\right)$$
$$\max_{s\in[0,t]} \left(\|u''(s)\|^2_D + \|u'(s)\|^2_A + \|Au(s)\|^2_{D^{-1}}\right) \le \tag{3.2}$$
$$\le C\left(\|u_1\|^2_A + \|Au_0\|^2_{D^{-1}} + \max_{s\in[0,t]} \|f(s)\|^2_{D^{-1}} + \int_0^t \|D^{-1}f(s)\|^2_A ds\right),$$

etc.

has been constructed.
Inequalities of the type (3.2) are usually obtained using Grönwall lemma. Therefore constant C depends on t and has the form: $C = C(t) = e^{C_1 t}$. In such a way $C(t) \to \infty$ when $t \to \infty$, so estimates (3.2) does not guarantee global stability of the problem (3.1).
In the sequel we will construct another scale of a priori estimates where C does not depends on t.

3.1. Global Stability

Taking inner product of (3.1) with $2u'$ we obtain the energy identity

$$\left(\|u'\|_D^2 + \|u\|_A^2\right)' = 2\,(f,\,u'). \tag{3.3}$$

From (3.3) follows

$$\left(\|u'\|_D^2 + \|u\|_A^2\right)' \le 2\,\|u'\|_D\,\|f\|_{D^{-1}} \le 2\left(\|u'\|_D^2 + \|u\|_A^2\right)^{1/2} \|f\|_{D^{-1}}. \tag{3.4}$$

Integrating on t we obtain

$$\left(\|u'(t)\|_D^2 + \|u(t)\|_A^2\right)^{1/2} \le \left(\|u'(0)\|_D^2 + \|u(0)\|_A^2\right)^{1/2} + \int_0^t \|f(s)\|_{D^{-1}}\,ds\,. \tag{3.5}$$

Applying operators DA^{-1}, AD^{-1}, $(AD^{-1})^2$, $(AD^{-1})^3$ etc. to equation (3.1), using inequality (3.5) and using representation

$$u^{(k+2)} = D^{-1}(f^{(k)} - Au^{(k)})\,, \qquad k = 0, 1, 2, \ldots$$

for estimation of higher order derivatives of u, we easily obtain the following scale of a priori estimates:

$$\begin{aligned}
&\|u\|_{\{0\}} \equiv \|u\|_{C([0,t];\,H_D)} \le \|u_0\|_D + \|u_1\|_{DA^{-1}D} + \|f\|_{L_1((0,t);\,H_{A^{-1}})}\,,\\
&\|u\|_{\{1\}} \equiv \|u'\|_{C([0,t];\,H_D)} + \|u\|_{C([0,t];\,H_A)} \le 2\Big(\|u_0\|_A + \|u_1\|_D + \|f\|_{L_1((0,t);\,H_{D^{-1}})}\Big),\\
&\|u\|_{\{2\}} \equiv \|u''\|_{C([0,t];\,H_D)} + \|u'\|_{C([0,t];\,H_A)} + \|u\|_{C([0,t];\,H_{AD^{-1}A})} \le\\
&\qquad \le 3\Big(\|u_0\|_{AD^{-1}A} + \|u_1\|_A + \|f\|_{L_1((0,t);\,H_{D^{-1}AD^{-1}})}\Big) + \|f\|_{C([0,t];\,H_{D^{-1}})}\,,\\
&\|u\|_{\{3\}} \equiv \|u'''\|_{C([0,t];\,H_D)} + \|u''\|_{C([0,t];\,H_A)} + \|u'\|_{C([0,t];\,H_{AD^{-1}A})} + \|u\|_{C([0,t];\,H_{AD^{-1}AD^{-1}A})} \le\\
&\qquad \le 4\Big(\|u_0\|_{AD^{-1}AD^{-1}A} + \|u_1\|_{AD^{-1}A} + \|f\|_{L_1((0,t);\,H_{D^{-1}AD^{-1}AD^{-1}})}\Big) +\\
&\qquad\quad + \|f'\|_{C([0,t];\,H_{D^{-1}})} + \|f\|_{C([0,t];\,H_{D^{-1}AD^{-1}})}\,,\\
&\text{etc.}
\end{aligned} \tag{3.6}$$

From (3.3) one obtains

$$\left(\|u'\|_D^2 + \|u\|_A^2\right)' = 2\,(f,\,u)' - 2\,(f',\,u)\,,$$

and

$$\left(\|u'\|_D^2 + \|u\|_A^2 - 2\,(u,\,f) + \|f\|_{A^{-1}}^2\right)' = 2\,(f',\,A^{-1}f) - 2\,(f',\,u) = 2\,(f',\,A^{-1}f - u)\,,$$

wherefrom follows

$$\left(\|u'\|_D^2 + \|u - A^{-1}f\|_A^2\right)' \le 2\,\|f'\|_{A^{-1}}\,\|u - A^{-1}f\|_A \le 2\,\|f'\|_{A^{-1}}\left(\|u'\|_D^2 + \|u - A^{-1}f\|_A^2\right)^{1/2}. \tag{3.7}$$

By integration on t

$$\left(\|u'(t)\|_D^2 + \|u(t) - A^{-1}f(t)\|_A^2\right)^{1/2} \le \left(\|u'(0)\|_D^2 + \|u(0) - A^{-1}f(0)\|_A^2\right)^{1/2} + \int_0^t \|f'(s)\|_{A^{-1}}ds. \tag{3.8}$$

From (3.8), analogously as in the previous case, one obtains the following scale of a priori estimates:

$$\begin{aligned}
&\|u\|_{\{0\}} \leq \|u_0\|_D + \|u_1\|_{DA^{-1}D} + \|f'\|_{L_1((0,t);\, H_{A^{-1}DA^{-1}})} + 2\,\|f\|_{C([0,t];\, H_{A^{-1}DA^{-1}})}\,,\\
&\|u\|_{\{1\}} \leq 2\Big(\|u_0\|_A + \|u_1\|_D + \|f'\|_{L_1((0,t);\, H_{A^{-1}})}\Big) + 3\,\|f\|_{C([0,t];\, H_{A^{-1}})}\,,\\
&\|u\|_{\{2\}} \leq 3\Big(\|u_0\|_{AD^{-1}A} + \|u_1\|_A + \|f'\|_{L_1((0,t);\, H_{D^{-1}})}\Big) + 4\,\|f\|_{C([0,t];\, H_{D^{-1}})}\,,\\
&\|u\|_{\{3\}} \leq 4\Big(\|u_0\|_{AD^{-1}AD^{-1}A} + \|u_1\|_{AD^{-1}A} + \|f'\|_{L_1((0,t);\, H_{D^{-1}AD^{-1}})}\Big) +\\
&\qquad +5\,\|f\|_{C([0,t];\, H_{D^{-1}AD^{-1}})} + \|f'\|_{C([0,t];\, H_{D^{-1}})}\,,
\end{aligned} \tag{3.9}$$

etc.

Let us denote $g(t) = \int_0^t f(s)\,ds$. Then $f(t) = g'(t)$ and from (3.3) follows

$$\Big(\|u'\|_D^2 + \|u\|_A^2\Big)' = 2\,(u',\, g') = 2\,(u', g)' - 2\,(u'',\, g)\,.$$

From here, using equation (3.1) one obtains

$$\Big(\|u'\|_D^2 - 2\,(u',\, g) + \|g\|_{D^{-1}}^2 + \|u\|_A^2\Big)' = 2\,(D^{-1}g',\, g) - 2\,(u'',\, g) = 2\,(D^{-1}Au,\, g)\,,$$

and

$$\Big(\|u' - D^{-1}g\|_D^2 + \|u\|_A^2\Big)' \leq 2\,\|D^{-1}g\|_A\,\|u\|_A \leq 2\,\|D^{-1}g\|_A\,\Big(\|u' - D^{-1}g\|_D^2 + \|u\|_A^2\Big)^{1/2}.$$

By integration one obtains

$$\Big(\|u'(t) - D^{-1}g(t)\|_D^2 + \|u(t)\|_A^2\Big)^{1/2} \leq \Big(\|u'(0)\|_D^2 + \|u(0)\|_A^2\Big)^{1/2} + \int_0^t \|D^{-1}g(s)\|_A\,ds \tag{3.10}$$

wherefrom, in the same manner as in the previous cases, we obtain the scale of a priori estimates:

$$\begin{aligned}
&\|u\|_{\{0\}} \leq \|u_0\|_D + \|u_1\|_{DA^{-1}D} + \|g\|_{L_1((0,t);\, H_{D^{-1}})}\,,\\
&\|u\|_{\{1\}} \leq 2\Big(\|u_0\|_A + \|u_1\|_D + \|g\|_{L_1((0,t);\, H_{D^{-1}AD^{-1}})}\Big) + \|g\|_{C([0,t];\, H_{D^{-1}})}\,,\\
&\|u\|_{\{2\}} \leq 3\Big(\|u_0\|_{AD^{-1}A} + \|u_1\|_A + \|g\|_{L_1((0,t);\, H_{D^{-1}AD^{-1}AD^{-1}})}\Big) +\\
&\qquad +\|g'\|_{C([0,t];\, H_{D^{-1}})} + \|g\|_{C([0,t];\, H_{D^{-1}AD^{-1}})}\,,\\
&\|u\|_{\{3\}} \leq 4\Big(\|u_0\|_{AD^{-1}AD^{-1}A} + \|u_1\|_{AD^{-1}A} + \|g\|_{L_1((0,t);\, H_{D^{-1}AD^{-1}AD^{-1}AD^{-1}})}\Big) +\\
&\qquad +\|g''\|_{C([0,t];\, H_{D^{-1}})} + \|g'\|_{C([0,t];\, H_{D^{-1}AD^{-1}})} + \|g\|_{C([0,t];\, H_{D^{-1}AD^{-1}AD^{-1}})}\,,
\end{aligned} \tag{3.11}$$

etc.

3.2. General Second Order Equation

Let us consider the Cauchy problem

$$Du''(t) + Bu'(t) + Au(t) = f(t), \quad t > 0; \qquad u(0) = u_0, \quad u'(0) = u_1, \tag{3.12}$$

where A, D, u_0, u_1 and $f(t)$ satisfy the assumptions from the previous section and B is nonnegative linear operator in H.

Taking inner product of (3.12) with $2u'$ we obtain the following energy identity

$$\left(\|u'\|_D^2 + \|u\|_A^2\right)' + 2\,\|u'\|_B^2 = 2\,(f,\, u')\,. \tag{3.13}$$

As since $B \geq 0$ from (3.13) follows (3.4), wherefrom after integration on t we obtain (3.5) and the second a priori estimate from (3.6). The others inequalities from (3.6) are not valid in general case of noncommutative operators A, B and D.

From (3.13) also follows (3.7), wherefrom we obtain (3.8) and the second a priori estimate from (3.9). The others inequalities from (3.9) are not valid in general case.

Let us construct the analogue of a priori estimate (3.10). For this purpose we rewrite equation (3.12) in the form

$$D(u' - D^{-1}g)' + B(u' - D^{-1}g) + Au = -BD^{-1}g$$

and take the inner product with $u' - D^{-1}g$. Further

$$\left(\|u' - D^{-1}g\|_D^2 + \|u\|_A^2\right)' + 2\,\|u' - D^{-1}g\|_B^2 = 2\,(Au,\, D^{-1}g) - 2\,(BD^{-1}g,\, u' - D^{-1}g) \leq$$
$$\leq 2\,\|u\|_A\,\|D^{-1}g\|_A + 2\,\|u' - D^{-1}g\|_D\,\|BD^{-1}g\|_{D^{-1}}$$

and

$$\left(\|u' - D^{-1}g\|_D^2 + \|u\|_A^2\right)' \leq 2\left(\|u' - D^{-1}g\|_D^2 + \|u\|_A^2\right)^{1/2}\left(\|D^{-1}g\|_A^2 + \|BD^{-1}g\|_{D^{-1}}^2\right)^{1/2}$$

After integration one obtains

$$\left(\|u'(t) - D^{-1}g(t)\|_D^2 + \|u(t)\|_A^2\right)^{1/2} \leq \left(\|u'(0)\|_D^2 + \|u(0)\|_A^2\right)^{1/2} +$$
$$+ \int_0^t \left(\|D^{-1}g(s)\|_A^2 + \|BD^{-1}g(s)\|_{D^{-1}}^2\right)^{1/2} ds\,.$$

From here we finally obtain the following analogue of the second a priori estimate from the scale (3.11)

$$\|u\|_{\{1\}} \leq 2\Big(\|u_0\|_A + \|u_1\|_D + \|g\|_{L_1((0,t);\, H_{D^{-1}AD^{-1}})} + \|g\|_{L_1((0,t);\, H_{D^{-1}BD^{-1}BD^{-1}})}\Big) +$$
$$+ \|g\|_{C([0,t];\, H_{D^{-1}})}\,.$$

The analogues of others inequalities from (3.11) can not be obtained in general case.

3.3. Strong Stability

Along the Cauchy problem (3.1) let us consider the perturbed problem

$$\tilde{D}\tilde{u}''(t) + \tilde{A}\tilde{u}(t) = \tilde{f}(t), \quad t > 0; \qquad \tilde{u}(0) = \tilde{u}_0, \quad \tilde{u}'(0) = \tilde{u}_1\,. \tag{3.14}$$

Analogously as in the case of equation of the first order, the problem (3.1) will be called strong stable if an inequality of the form

$$\|\tilde{u} - u\|_0 \leq C_1\,\|\tilde{u}_0 - u_0\|_1 + C_2\,\|\tilde{u}_1 - u_1\|_2 + C_3\,\|\tilde{f} - f\|_3 + C_4\,\|\tilde{A} - A\|_4 + C_5\,\|\tilde{D} - D\|_5\,,$$

is satisfied. Here C_i are constants and $\|\cdot\|_i$ – some norms.

Let operators $\tilde{A}$, $\tilde{D}$ of perturbed problem satisfy (3.1) the same assumptions as operators A and D and let $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$, $\mathcal{D}(\tilde{D}) = \mathcal{D}(D)$. From (3.1) and (3.14) one obtains the following Cauchy problem for the error $z = \tilde{u} - u$

$$Dz'' + Az = \tilde{f}(t) - f(t) - (\tilde{A} - A)\tilde{u} - (\tilde{D} - D)\tilde{u}'', \quad t > 0; \quad z(0) = \tilde{u}_0 - u_0; \quad z'(0) = \tilde{u}_1 - u_1. \tag{3.15}$$

Applying a priori estimates from (3.6) to (3.14), after some algebra one obtains (see [2])

$$\begin{aligned}
\|z\|_{\{0\}} &\le \|\tilde{u}_0 - u_0\|_D + \|\tilde{u}_1 - u_1\|_{DA^{-1}D} + \|\tilde{f} - f\|_{L_1((0,t);\, H_{A^{-1}})} + \\
&\quad + C_{\{0\},4}\,\|\tilde{A} - A\|_{H_{\tilde{A}} \to H_{A^{-1}}} + C_{\{0\},5}\,\|\tilde{D} - D\|_{H_{\tilde{D}\tilde{A}^{-1}\tilde{D}} \to H_{A^{-1}}}\,, \\
\|z\|_{\{1\}} &\le 2\,\|\tilde{u}_0 - u_0\|_A + 2\,\|\tilde{u}_1 - u_1\|_D + 2\,\|\tilde{f} - f\|_{L_1((0,t);\, H_{D^{-1}})} + \\
&\quad + C_{\{1\},4}\,\|\tilde{A} - A\|_{H_{\tilde{A}\tilde{D}^{-1}\tilde{A}} \to H_{D^{-1}}} + C_{\{1\},5}\,\|\tilde{D} - D\|_{H_{\tilde{D}} \to H_{D^{-1}}}\,, \\
\|z\|_{\{2\}} &\le 3\,\|\tilde{u}_0 - u_0\|_{AD^{-1}A} + 3\,\|\tilde{u}_1 - u_1\|_A + \|\tilde{f} - f\|_{\{2\},3} + \\
&\quad + C_{\{2\},4}\,\|\tilde{A} - A\|_{\{2\},4} + C_{\{2\},5}\,\|\tilde{D} - D\|_{\{2\},5}\,,
\end{aligned} \tag{3.16}$$

etc

where

$$\|f\|_{\{2\},3} = 3\,\|f\|_{L_1((0,t);\, H_{D^{-1}AD^{-1}})} + \|f\|_{C([0,t];\, H_{D^{-1}})}\,,$$

$$\|A\|_{\{2\},4} = \|A\|_{H_{\tilde{A}\tilde{D}^{-1}\tilde{A}\tilde{D}^{-1}\tilde{A}} \to H_{D^{-1}AD^{-1}}} + \|A\|_{H_{\tilde{A}\tilde{D}^{-1}\tilde{A}} \to H_{D^{-1}}}\,,$$

$$\|D\|_{\{2\},5} = \|D\|_{H_{\tilde{A}} \to H_{D^{-1}AD^{-1}}} + \|D\|_{H_{\tilde{D}} \to H_{D^{-1}}}\,,$$

$$C_{\{0\},4} = \|\tilde{u}\|_{L_1((0,t);\, H_{\tilde{A}})} \le t\Big(\|\tilde{u}_0\|_{\tilde{A}} + \|\tilde{u}_1\|_{\tilde{D}} + \|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{D}^{-1}})}\Big),$$

$$C_{\{0\},5} = \|\tilde{A}\tilde{u} - \tilde{f}\|_{L_1((0,t);\, H_{\tilde{A}^{-1}})} \le (t+1)\Big(\|\tilde{u}_0\|_{\tilde{A}} + \|\tilde{u}_1\|_{\tilde{D}} + \|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{D}^{-1}})}\Big),$$

$$C_{\{1\},4} = 2\,\|\tilde{u}\|_{L_1((0,t);\, H_{\tilde{A}\tilde{D}^{-1}\tilde{A}})} \le 2t\Big(\|\tilde{u}_0\|_{\tilde{A}\tilde{D}^{-1}\tilde{A}} + \|\tilde{u}_1\|_{\tilde{A}} + \|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{D}^{-1}\tilde{A}\tilde{D}^{-1}})}\Big),$$

$$C_{\{1\},5} = \|\tilde{A}\tilde{u} - \tilde{f}\|_{L_1((0,t);\, H_{\tilde{D}^{-1}})} \le 2(t+1)\Big(\|\tilde{u}_0\|_{\tilde{A}\tilde{D}^{-1}\tilde{A}} + \|\tilde{u}_1\|_{\tilde{A}} + \|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{D}^{-1}\tilde{A}\tilde{D}^{-1}})}\Big),$$

$$C_{\{2\},4} \le (3t+1)\Big(\|\tilde{u}_0\|_{\tilde{A}\tilde{D}^{-1}\tilde{A}\tilde{D}^{-1}\tilde{A}} + \|\tilde{u}_1\|_{\tilde{A}\tilde{D}^{-1}\tilde{A}} + \|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{D}^{-1}\tilde{A}\tilde{D}^{-1}\tilde{A}\tilde{D}^{-1}})}\Big),$$

$$\begin{aligned}
C_{\{2\},5} &\le (3t+4)\Big(\|\tilde{u}_0\|_{\tilde{A}\tilde{D}^{-1}\tilde{A}\tilde{D}^{-1}\tilde{A}} + \|\tilde{u}_1\|_{\tilde{A}\tilde{D}^{-1}\tilde{A}} + \|\tilde{f}\|_{L_1((0,t);\, H_{\tilde{D}^{-1}\tilde{A}\tilde{D}^{-1}\tilde{A}\tilde{D}^{-1}})}\Big) + \\
&\quad + \|\tilde{f}\|_{C([0,t];\, H_{\tilde{D}^{-1}})}\,.
\end{aligned}$$

Let us note, that the quantities $C_{\{i\},j}$ depends on $\tilde{u}$ (and consequently on $\tilde{u}_0$, $\tilde{u}_1$ and $\tilde{f}$) and increase with t. Analogous scales of a priori estimates of the strong stability can be obtained on the basis of (3.9) and (3.11).

ACKNOWLEDGEMENTS

The authors were supported in part by the Belarusian Republic Basic Research Foundation under grant F05-083, by the INTAS under grant 03-50-4395 and by the Serbian Ministry of Science and Environmental Protection under project 144005A.

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International Journal of Applied Mathematics & Statistics
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

IJAMAS: September 2007, Vol. 10, No. S07, 70-80

Mathematical Modeling of Wind Forces

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ABSTRACT

This paper is divided into two parts. In the first part physical meanings and some other aspects of wind and the driving forces are reviewed. There includes types of driving forces, sources for wind production, boundary layer effects, roughness and surface structure effects, and the types of wind in the nature. In the second part we first focus on the effect of surface structure, roughness and surface waviness, and several experimental, computational and mathematical investigations that have been done by the present and other authors in the past for both laminar and turbulent shear flows, are reviewed. Next, a mathematical model of moderate wind over general types of surface structures is considered and studied under some reasonable conditions. This mathematical model, which is based on a relevant system of partial differential equations and Fourier analysis, is investigated briefly here using perturbation and weakly nonlinear methods, and the mathematical features and the qualitative results are then presented. It was found, in particular, that surface structures can have significant influences on various features of the wind such as stability, instability, amplitude, scales, patterns, mode excitation and resonance.

KEY WORDS: *Wind forces, shear flow, flow forces, modeling wind, shear forces, viscous flow.*

Mathematics Subject Classification: 76Exx, 76E05, 76E30, 76D33, 76D55

1. INTRODUCTION

This paper reviews some recent studies and the subsequent results by a number of authors on the effects of surface roughness, waviness, corrugations and structures on laminar and turbulent flows and then consider a mathematical model for the flow of air over a surface structure and applies it to the case of a moderate wind and the associated driving forces. The subject of wind and its driving forces is of great interest and importance in our daily living on Earth. In particular, some mathematical and qualitative understanding of violent types of wind, such as tornado, thunderstorm and hurricane, which are reviewed briefly below, can be quite useful to gain some primary physical insight on the subject matter, which could stimulate fully future investigations, in both qualitative and quantitative aspects, of the actual physical problems in applications.

To provide the reader in mathematics areas some preliminaries on the subject matter, we first review briefly here in simple words some basic aspects of the wind and the associated fluid mechanics information. Strictly considering, wind that is experienced by anyone, is the air movement along the Earth's surface. The violent types of wind are those which require primary studies for their understanding and hopefully finding ways to control their motions. Tornados are powerful rotary (vortex) motions, which are actually due to the strong interactions between the up-flow unstable rotating warm and moist air and the overlying stable layer of air. Depending on the severity of the tornado, wind speed can vary between 42 mph to 318 mph. Here by a vortex, we mean roughly a bounded type motion whose projection on a plane normal to its core shows circular or spiral pattern. Thunderstorms are large size unstable warm and moist air that rotate and move upward. Wind shear, which is the amount of change in the wind direction or speed with respect to altitude, can make the storm stronger. Sever weather occurs when the wind shear is more significant. Hurricanes are formed from thunderstorms and are due to strong interactions between the oceanic and the atmospheric motion. They are tropical cyclones (low pressure systems), which circulate with winds exceeding about 74 mph. Heat and moisture from the warm oceanic water (above about 81 degrees Fahrenheit) is the source of energy for the hurricanes. Hurricanes usually weaken rapidly when they travel over the land or regions of colder oceanic waters, that is, regions with insufficient heat or moisture.

Winds are driven by forces, which can be thought roughly as a push or a pull of air particles. These forces can be pressure gradient forces, which are due to a change in the air pressure over a distance, centrifugal and Coriolis forces due to the Earth's rotation about its axis of rotation and frictional forces due to the Earth's surface roughness or structures such as buildings, trees, cars, bridges, etc. Frictional forces are strongest within the region near and adjacent to the Earth's surface, which is also called boundary layer region. Roughness, viscous effects, surface structures and turbulence can all be significant within the boundary layer zone. Turbulence can be thought here of mixing action in the flow field. This mixing is due to eddies of varying sizes within the air flow. Turbulence adds continuously fluctuating components of the air velocity to the flow. Turbulent velocity is irregular in time and space. Gusts of wind are the result of large-scale eddies that at times reinforce and at other times subtract from the mean wind velocity.

Shape of the mean velocity profile of the wind can depend on the degree of surface roughness. Deviations of the air flow velocity from the mean air flow velocity, like gusts, waves and fluctuations, have various discrete or continuous modes over a wide range of amplitudes, frequencies, wave numbers, length scales and time scales. Surface roughness, surface structures, such as buildings, forests, bridges, etc, surface topographies, such as hills and valleys, and thermal instability, where hot air layer is under a cold layer, for example, can all produce turbulence in air.

Each structure has its own natural frequency, which may lead to structural damage when the structure is under certain external force whose frequency matches the natural frequency leading to the so-called resonance. Also certain periodic gusts in the wind may find resonance with the natural frequency of a structure that they encounter, leading to possible destructive oscillations. For more information on the air motion, the reader is referred to Batchelor (1970) and Holton (1972).

In the following sections we first briefly review the studies that have been done by a number of authors in the past and, in particular, the mathematical modeling efforts in recent years by the present author (Riahi, 1996, 1997, 1998a-b, 1999a-b, 2000, 2001a-b, 2002), to study the effects of surface structures, roughness and surface waviness on both laminar and turbulent flows. Next, as a first step to study mathematically a wind and its driving forces, we consider a relatively simple mathematical model of a moderate wind in a layer over Earth's surface structure, analyses the problem, present the method of solution to the associated system of partial differential equation and present the results. In the last section some concluding remarks are provided.

SOME PREVIOUS RELEVANT STUDIES

Notable early works were due to Miles (1957, 1959) and Benjamin (1959). Miles studied the generation of linear two-dimensional surface waves by linear two-dimensional inviscid (frictionless) shear flows. He found, in particular, that the rate at which energy is transferred to a wave of speed c is proportional to the basic flow (original flow in the absence of wave) profile curvature at that elevation where the basic flow velocity equals c. His inviscid theory was extended later (Miles, 1959) to include the viscous effects. Benjamin (1959) studied two-dimensional linear shearing flows bounded by a two-dimensional simple-harmonic wavy surface. His main objective was to determine the normal and tangential stresses on the boundary.

More recently, Sykes (1980), Smith et al. (1981) and Hunt et al. (1988) carried out asymptotic analyses of the linear changes induced in a boundary layer flow over an undulating surface. The analyses have been of the type developed by Stewartson (1974) which inspired a number of authors including the present author (Riahi, 1977, 1978, 1981, 1994; Riahi and Vonderwell, 1994) to carry out related asymptotic analyses to describe various types of flows. Belcher et al. (1993, 1994) and Belcher and Hunt (1993) extended earlier turbulent flow analyses (Sykes, 1980; Hunt et al., 1988) to study turbulent shear flow over slowly moving waves (Belcher and Hunt, 1993) and to determine dynamics of wind waves in coupled turbulent air-water flow (Belcher et al., 1994).

Prior to Riahi (1997) few theoretical studies were done on the effect of surface structures on the shear flows despite the fact that surface corrugations and roughness elements are known (Schlichting, 1979; Morkovin, 1983; Wikinson and Malik, 1983; Waitz and Wilkinson, 1988) to affect flow instabilities and transition to turbulence. In particular, transition sites are found to shift to locations where significant roughness elements exist. Wilkinson and Malik (1983) showed that stationary disturbances can originate from isolated roughness on a rotating disk and that vortex patterns emerge only when the different wave packets have spread and filed the entire disk circumference. Other experimental studies (Muller and Bippes, 1988; Rodeztsky et al., 1994) indicated that stationary shear flow modes can be caused or manipulated by the roughness and waviness of the surface and that shear flow modes can be enhanced by selected boundary perturbations. Gong et al. (1996) studied flow over two rough rigid undulating surfaces that comprise 16 small amplitude spanwise-independent sinusoidal waves. Here and thereafter by streamwise, transverse and spanwise directions we mean, respectively the direction of the original base flow, direction perpendicular to the streamwise and average location of the surface and the direction perpendicular to the transverse and streamwise directions in a right-handed sense. Steady circulations with vorticity, which is mathematically magnitude of curl of velocity vector, aligned with the mean motion, were observed by the fourth wave, at which point the flow was approximately periodic. However, circulations were observed only above the smoother waves where the flow remained attached. With rougher waves it separated in the troughs.

There have been a number of experiments and numerical simulation studies in the past two decades or so (Zilken et al., 1977; Thorsness et al., 1978; Buckles et al., 1984; Abrams and Hanratty, 1985; Fredrick and Hanratty, 1988; Bandyopadhyay and Watson 1988; Krettenauer and Schumann, 1992; Adrian et al., 1992; Grass et al., 1993; Hudson et al., 1996; Angelis et al., 1997; Warholic, 1997; Manhart, 1999; Tomkin, 2000) that have indicated significant effects of roughness element or surface waviness on the near wall turbulent shear flow structures.

Acarlar and Smith (1987) investigated experimentally boundary-layer flow in the presence of a hemisphere on the wall. They discovered two types of vortices: one a standing-type vortex, which was due to the presence of the hemisphere, and the other were non-standing type vortices, which induced low-speed motion near the wall. Grass et al. (1993) investigated experimentally turbulent flow over both smooth and rough boundaries. They found vortices over the rough boundary. Over large roughness elements, the dominant spanwise wavelength of the streamwise velocity fluctuations was found to be the same as the spanwise spacing of the roughness elements. Angelis et al. (1997) studied computationally turbulent flow over a wavy wall. The velocity fluctuations in the streamwise direction were decreased in a region near the wall. Quasi-streamwise vortices were found to initiate in the region downstream of the troughs of the wavy wall. The extent of these vortices was found to scale with the wavelength of the wavy wall. Tomkins (2000) investigated experimentally turbulent flow in the presence of a full array of hemispheres on the wall. The increase in wall friction over the rough surface was found to cause a decrease in the turbulence fluctuations. The roughness elements with larger height affected the flow more significantly than those with smaller height. Vortices were observed to occur much more frequently over the roughness elements than over the smooth surface. Vortices generated by the roughness elements were found to take on the scale of the roughness elements. Also streamwise length-scales were found to be reduced due to the presence of roughness elements.

Some relevant research works, which were carried out by the present author in recent years, are described briefly as follows. Riahi (1997) investigated stability of rotating disk boundary layer flow over a simple rough surface using weakly nonlinear method (Drazin and Reid, 1980) where the value of the controlling stability parameter (Reynolds number R) was closed to its critical value R_c below which no linear instability of the basic state primary motion is possible. It was found, in particular, that certain new types of secondary solutions can become stable only for particular forms of the amplitude and length scale of the surface roughness. Here and thereafter by secondary motion we mean compoment of the total motion that is arisen as the result of the instability of the original basic flow. Riahi (1999) and Riahi (2001a) used asymptotic and scaling analyses at high R ($R>>1$) to investigate theoretically channel flow over a wavy wall for flow in laminar regime and over a rough wall for turbulent flow regime, respectively. It was found, in particular, that certain wavy or rough boundaries are stabilizing, while some other types may be destabilizing. Riahi (2001, 2002) investigated the effect of a boundary corrugated surface on the laminar shear flow. He used perturbation and weakly nonlinear approaches to determine the results about the effects of such boundary on the flow. He found, in particular, that a resonance condition can lead to preference of a flow solution, provided the wave number vector of such solution is the same as the wave number vector of one of the mode generating the shape of the corrugated boundary.

MATHEMATICAL MODEL

We consider the governing partial differential equation for momentum and mass conservation (Batchelor, 1970) for a planar shear flow layer of air, which we refer to as wind, of average depth d over a surface structure. The air layer is assumed to be rotating with constant angular velocity Ω about its axis of rotation, which is in the transverse direction perpendicular to the top plane of the average depth of the layer. Compressibility effect of a layer of wind of moderate strength is expected to be negligible. Even for wind at speed of 42 mph, which corresponds to the speed of a low speed tornado, the Mach number is about $0.05<<1$. At any rate, we regard the layer of air here as a layer of incompressible fluid with uniform density, as such idealization has been assumed before in the case of the atmosphere (Batchelor, 1970). The rotational effect due to the rotating Earth introduces two fictitious Coriolis and centrifugal forces to act on the air layer. However, in a fluid of uniform density the centrifugal force can be written as gradient of a scalar and be combined with the pressure-gradient term in the momentum equation (Batchelor, 1970).

Using a Cartesian system of coordinate x^*, y^*, z^*, with $z^*=0$ being the averaged location of the surface structure, we assume that the coordinate system is embedded into the air layer and is moving with angular velocity Ω about the z^*-axis. We non-dimensionalize the governing partial differential equations by using d, U, d/U and ρU as scales for length, velocity, time and modified pressure, respectively. Here U is an appropriate velocity scale, which is chosen here to be the maximum wind speed, and ρ is the uniform density of the air. The non-dimensional form of the governing equation (Batchelor, 1970) then can be written in the form

$$(\partial/\partial t+\mathbf{u}.\nabla)\mathbf{u}= -\nabla P+(1/R)(T\mathbf{u}\times\mathbf{z}+\nabla^2\mathbf{u}), \quad (1a)$$

$$\nabla.\mathbf{u}=0, \quad (1b)$$

Here $\mathbf{u}=(u, v, w)$ is the velocity vector, P is the modified pressure, t is the non-dimensional time variable, $\mathbf{z}$ is a unit vector in the positive direction of the z-axis,

$T=\Omega d^2/\nu$ is a rotational parameter and $R=Ud/\nu$ is the Reynolds number, where ν is the kinematic viscosity. In general the non-dimensional dependent variables $\mathbf{u}$ and P are functions of the non-dimensional independent variables x, y, z and t.

We shall measure the strength of the wind in terms of its maximum speed U only, so that the layer thickness d is assumed to be fixed. We designate the original basic wind velocity vector and the modified pressure gradient, which exist initially in the absence of any perturbation, by $\mathbf{u}_b$ and ∇P_b, respectively, and assume that they vary at most with respect to the z-variable. We also assume that the height δ in the z-direction of the surface structure is small ($\delta<<1$). We now write $\mathbf{u}=\mathbf{u}_b+\mathbf{u}'$, $P=P_b+P'$, where $\mathbf{u}'$ and P' are the velocity vector and the modified pressure for the perturbations superimposed on the basic flow. The governing partial differential equations for the perturbation variables can then be obtained by using the above expressions for $\mathbf{u}$ and P in the equations (1a, b) and subtracting from the resulting equations the corresponding equations for the basic flow. The resulting equations are

$$(\partial/\partial t+\mathbf{u}_b.\nabla)\mathbf{u}'+\mathbf{u}'.\nabla\mathbf{u}_b+\nabla P'+(1/R)(T\mathbf{z}\times\mathbf{u}'-\nabla^2\mathbf{u})= -\mathbf{u}'.\nabla\mathbf{u}', \quad (2a)$$

$$\nabla.\mathbf{u}'=0. \quad (2b)$$

The boundary conditions for (2a, b) are

$$\mathbf{u}'= -\Sigma_{m=1} [(\delta h)^m /m!](\partial^m/\partial z^m)(\mathbf{u}_b+\mathbf{u}') \text{ at } z=0, \quad (3a)$$

$$\mathbf{u}'=0 \text{ at } z=1, \quad (3b)$$

where h(r) is the shape function for the surface structure, which is, in general, a function of the horizontal variable vector $\mathbf{r}=(x, y)$, and δh is the structure height. We have assumed that the boundary conditions $\mathbf{u}_u$ and $\mathbf{u}_l$ for the total flow velocity vector $\mathbf{u}$ at the upper and lower boundaries, respectively, are prescribed constants. Conditions (3a, b) are obtained by the consideration that the base flow velocity vector assumes the value $\mathbf{u}_l$ and $\mathbf{u}_u$ at z =0 and z =1, respectively, since the structure on the lower boundary introduces simply surface perturbations which contribute to the perturbation system only. The terms in the right-hand side of (3a) arise simply by the contributions of the higher-order terms in a Taylor-series expansion about z =0 of the total velocity vector which are due to the surface structure.

We shall assume that the shape function for the surface structure can be represented by

$$h(\mathbf{r})=\sum_{n=-M}^{M} A_n E_n \equiv \sum_{n=-M}^{M} A_n \exp(i\mathbf{j}_n.\mathbf{r}), \tag{4}$$

where $i=\sqrt{-1}$, M is a positive integer, which may be sufficiently large for particular structures, and the horizontal wave number vectors of the structure satisfy the properties

$$\mathbf{j}_{-n}=-\mathbf{j}_n. \tag{5}$$

The amplitude coefficients A_n satisfy the condition

$$A_n^* = A_{-n}, \tag{6}$$

where "asterisk" indicates complex conjugate. Conditions (5)-(6) ensure that expression (4) for h is real.

From the previous studies (Riahi, 2001b), it is anticipated that effective surface structure can be possible if $\mathbf{k}_m=\mathbf{j}_n$ for at least some m and n. Here $\mathbf{k}_m$ is the wave number vector of the wind. We consider first the so-called critical regime where $R\approx R_c$ and $\varepsilon=\delta^{1/3}$

(Riahi, 1995), where ε is the magnitude of the amplitude of the perturbation motion. Later we briefly cover investigations for some other cases where ε could be of different order from that for $\delta^{1/3}$.

Applying the weakly nonlinear theory (Drazin and Reid, 1980), we define the slowly varying time τ

$$\tau=\delta^{2/3}\, t, \tag{7}$$

and pose expansions for $\mathbf{u}$, P and R in powers of $\lambda\equiv\delta^{1/3}$

$$(\mathbf{u}, P, R)=(\mathbf{0}, 0, R_0)+\lambda(\mathbf{u}_1, P_1, R_1)+\lambda^2(\mathbf{u}_2, P_2, R_2)+\dots \tag{8}$$

Using (7)-(8) into (2)-(3) and disregarding the quadratic perturbation terms, we find the linear problem whose system admits solution of the form

$$(\mathbf{u}_1, P_1)=\sum_{n=-N}^{N} [\mathbf{u}_{1n}(z), P_{1n}(z)]B_n F_n, \tag{9}$$

where

$$F_n=\exp(i\mathbf{k}_n.\mathbf{r}), \tag{10}$$

The amplitude function $B_n(\tau)$ satisfies condition of the form (6) that was satisfied by A_n. The horizontal wave number vectors $\mathbf{k}_n=(\alpha_n, \beta_n)$ of the perturbation satisfy the condition of the form (5) that was satisfied by $\mathbf{j}_n$. The coefficient functions $\mathbf{u}_{1n}$ and P_{1n} satisfy the condition of the form (6) and are obeyed by a system, which can be derived by using (9) in the linear system. This system of ordinary differential equations and similar types of systems, which are involved in the present study, will not be given here but will be reported elsewhere.

The weakly nonlinear method requires us to consider the adjoint linear system whose solution can be written in the form

$$(\mathbf{u}_1^{\wedge}, P_1^{\wedge})=\sum_{n=-N}^{N} [\mathbf{u}_{1n}^{\wedge}(z), P_{1n}^{\wedge}(z)]B_n F_n, \tag{11}$$

where the above z-dependent coefficients satisfy the condition of the form (6) and are obeyed by a system, which will not be given here but will be reported elsewhere.

Following Riahi (2001b), the solution to order λ^2 can be written. Forming the solvability condition, which states that non-homogeneities must be orthogonal to the solution of the adjoint linear system (Drazin and Reid, 1980), for the order λ^2 system, we find

$$R_1=0. \tag{12}$$

In the order λ^3, the boundary conditions at $z=0$ are no longer zero, in general, as can be seen from (3a). The solvability condition for this order then yield the following system of ordinary differential equations for B_n

$$(d/d\tau - R_2 a_n - c_n |B_n|^2 - \sum_{m,\, m\neq\pm n} e_{mn}|B_m|^2)B_n = b_n \sum_m A_m \langle F_n^* E_m \rangle,$$

$$(n=-N, \ldots, -1, 1, \ldots N), \tag{13}$$

where the angular bracket < > denotes an average over the wind layer. The expressions for the constant coefficients in (13), which are generally lengthy, will not be given here since the qualitative results of the present problem do not require the use of the specific form of these coefficients, but they will be reported elsewhere. It should be noted that these coefficients are, in general, functions of the rotational parameter T. For Earth, Ω has the value of about 7.3 (10^{-5}) rad/s (Batchelor, 1970). The kinematic viscousity ν of air at about 59 degrees Fahrenheit is about 1.46 (10^{-5}) m^2/s (Roberson and Crowe, 1980). So for a layer of fixed height d, $T=5.105\, d^2$ per $meter^2$. Since the present paper considers and seeks only qualitative features of the wind over a fairly general form of the surface structure, where the amplitude coefficients and the wave number vectors in (4) can take almost any assigned values for any given surface structure and accordingly affect the coefficients in (13) in so many different ways, the constant value of T turns out to be rather of minor significance in the present qualitative study.

To distinguish the physically realizable solutions(s) among all the solutions of (13), the stability of B_m $(m=-N, \ldots, -1, 1, \ldots, N)$ with respect to disturbances $C_m(\tau)$ are investigated. The system of the ordinary differential equations for the time-dependent disturbances is given by

$$(d/d\tau - R_2 a_n - 2c_n|B_n|^2)C_n = c_n B_n^2 C_n^* + \sum_{m,\, m\neq\pm n} e_m(|B_m|^2 C_n + B_m B_n C_m^* + B_n B_m^* C_m),$$

$$(n=-N, \ldots, -1, 1, \ldots, N), \tag{14}$$

where C_n also satisfies the condition of the form (6).

It can be seen from (13)-(14) that the surface structure affects the wind solutions directly as the source term in (13), while the surface structure affects the disturbances indirectly through the wind solutions. It is seen from (13) that the structure can be effective only if there is at least one m for which $\mathbf{j}_m=\mathbf{k}_n$, otherwise the surface structure contribution term in (13) vanishes.

We studied solutions for (13) for particular low values of N and given values of A_n in the effective surface structure case and found that such solutions yield negative growth rate based on (14) for sufficiently large $|R_2|$ and $R_2<0$ implying stable secondary wind solution, which is a solution of the perturbation superimposed on the basic wind, and here is due to the surface structure. On the other hand, unstable secondary wind solution results for sufficiently large $R_2>0$.

There are different types of preferred single- or multi-mode secondary wind solutions that can be possible for effective surface structure. For example, suppose there are two surface structured wave number vectors $\mathbf{j}_1$ and $\mathbf{j}_2$, which are equal to two wave number vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively, on the neutral stability curve for the stationary modes for one given $R_0>R_c$. Then the preferred secondary wind can consist of a double-mode along the two wave number vectors exhibiting, in general, an irregular rectangular pattern. The pattern can be irregular since it is z-dependent and the magnitudes of the two vectors are not generally equal.

Finally, let us consider briefly other particular $\varepsilon=\delta^{1/n}$ cases for $n\neq1/3$ (n=a positive integer). For n>3, the magnitude of the surface structure is so small that all the main results follow, up to order λ^3, without consideration of the boundary structure. For n=1, the trivial result that the structure control the linear secondary solution follows and will not be presented here. For n=2, one need to define τ like

$$\tau=\delta^{0.5}\,t. \tag{15}$$

In the order $\delta^{0.5}$, the same linear problem as the one presented earlier follows. In the order δ, the preference of steady secondary solution can be followed by the surface structure as in the earlier case discussed above. The steady solutions for B_n (n=-N, ..., -1, 1, ..., N) are then determined in terms of A_n and R_1, which is now non-zero, and these solutions are stable with respect to C_m for sufficiently large $|R_1|$ and $\varepsilon R_1<0$. The solvability condition in the order δ then determines R_1, R_2 and B_n in terms of A_m, and the preferred solutions are those which are stable and correspond to the smallest value of R.

CONCLUDING REMARKS

The results of the present qualitative investigation of the effect of surface structure on the wind, which was based on particular mathematical model, indicate that the Earth's surface structures can have significant influence on the amplitude, scales, pattern and secondary mode excitation of the wind. Stationary surface structures, such as those considered here, were found to affect the spatial features of the wind. It should be noted that even though the secondary wind solutions, can, in general, be due to both steady and time-dependent modes, the stationary surface structure can be effective only on the secondary steady modes of the wind.

The question of how a secondary solution of the wind can be preferred and stable by the surface structure, can be answered by the results of the present study, which can complement earlier results in thermal convection studies (Riahi, 1995). The secondary solution in the present studied case of resonant wavelength excitation for the steady modes can become stable and preferred if it can make R as small as possible, includes at least a mode with the same phase angle and the wave number vector as the one of the steady mode of the surface structure, which has the largest possible value of the amplitude.

The present study was restricted to stationary surface structures, discrete stationary modes and the so-called resonant wave number excitation case (Riahi, 2001b). Further extensions of the present study, which will be left to future studies, include qualitative studies for cases of moving surface structures, non-discrete (Riahi, 1996) stationary and non-stationary modes and non-resonant wave number excitation cases.

In application areas the next important extensions of the present study are to consider wind at much higher values of R in turbulent regime (Riahi, 2001a) and making use of sets of data for actual structures on land, such as buildings, bridges, vehicles, etc, in order to determine qualitative as well as quantitative results on the effects of specific surface structures on more realistic cases of wind of higher degrees of strength. Such studies for both stationary and moving surface structure extensions, which will be left to future work, could provide us further understanding of the way we could hopefully suppress or at least control the strong wind motions, which are destructive and quite undesirable to form and operate especially in cities and other populated areas.

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International Journal of Applied Mathematics & Statistics
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

IJAMAS: September 2007, Vol. 10, No. S07, 81-97

Mathematical Modeling of Flow Control Using Magnetic Fluid and Field

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ABSTRACT

In this paper some previous studies related to those for flow control aspects of the hydrodynamics of magnetic fluid and field are reviewed first and recent mathematical approaches for the corresponding systems of partial differential equations are discussed. Next, mathematical modeling of a two-fluid system, with one fluid being a non-conducting ferrofluid and another fluid being a regular non-ferrofluid is considered. A numerical method is used to solve the mathematical system for the problem, which, in particular, captures the flow structure in the two-fluid system. The convective flow velocities and the heat fluxes were determined for various values of the parameters of the problem. Certain aspects of the magnetic fluids and fields were found to be useful for convective flow control, which is important in many application areas including microgravity space applications. In particular, under some conditions, the surface force that can exist at the interface between the two fluid zones was found to reduce the magnitude of the flow velocity and instabilities that may occur in the two-fluid system.

KEY WORDS : *Flow control, magnetic fluid, magnetic field, magnetic convection, hydro-magnetic flow.*

Mathematics Subject Classification: 76Rxx, 76W05, 76E06, 76E17, 76D55

1. INTRODUCTION

Systems of immiscible fluid layers can be found in a number of applications in areas such as materials processing and convective heat and mass transfer. In the area of materials processing we can provide examples such as use of encapsulants in float zone crystal growth process and a buffer layer in industrial Czochralski crystal growth process to prevent convection due to the surface tension gradient force. In the microgravity and space-processing realm, the exploration of other planets requires the development of enabling technologies in several fronts. The reduction in the gravity level poses unique challenges for fluid handling and heat transfer applications. By controlling (curtailing or augmenting) the buoyant and thermocapillary convection, the latter is the dominant convective flow in a microgravity environment, significant advantages can be achieved by pursuing space based processing.

The present paper is based on our recent research studies, which consider thermal convective flow and its control using magnetic fluids and fields in a two immiscible fluid-layer system subjected to an imposed temperature gradient, which is assumed to be parallel to the averaged location of the interface between the two layers. Using an external magnetic field one can essentially dial in a volumetric force – gravity level, on the magnetic fluid and thereby affect the system thermo-fluid behavior.

Our recent investigation of the present problem has been based on the following preamble: (i) Convection control (flow reduction or enhancement) is an important problem with many practical applications; (ii) In terrestrial applications such as heat exchanges where augmented heat transfer is desirable in order to achieve higher efficiencies, gravity induced convection due to thermal buoyancy force assists fluid flow and heat transfer. In reduced-gravity situations heat transfer enhancement is important and is solely achieved using forced flow configurations; (iii) In semiconductor crystal growth applications, however, convection can cause undesirable motions in the melt resulting in higher defect densities, improper mixing resulting in reduced inhomogeneity (dopant distribution) and therefore non uniform properties (electrical) of the grown crystals. Experiments conducted in space have yielded promising results for alloy-semiconductors that are traditionally difficult to grown on Earth because of wide disparities in the material properties (densities) of the components; (iv) In containerless processing of materials that is used to significantly reduce the thermal and contact stresses and impurity incorporation in the grown crystal, significant flows due to Marangoni (thermocapillary) and solutocapillary convection plaque the growth process and the resulting crystal quality. Again different methods of convection control such as liquid encapsulation, static and dynamic magnetic field approaches have been used to reduce the so-called convective contamination in the system. The encapsulant also serves to reduce the evaporation of a volatile component in the crystal growth, for example, Boron oxide encapsulant is used in GaAs crystal growth to reduce the volatile As from escaping the system and thereby affecting the crystal stoichiometry.

While a level of convection control can be realized by using magnetic means, usually through Lorentz dissipation, the force reduces as the system flow reduces so that in order to approach diffusion limited conditions ($Pe < 1$, where Pe is the Peclet number, a ratio of the system thermal to viscous transport), large magnetic fields (~1 Tesla) are required. Dynamic magnetic fields such as rotating magnetic fields (rmf) and traveling magnetic fields (tmf) are also being studied for this purpose. It has been shown that significantly reduced magnetic field strengths (~3 orders of magnitude smaller) are required for rmf damping than a static one. Magnetic damping is non-intrusive and its intensity can be controlled externally. A few studies are ongoing that utilize a magnetic field gradient and the variation of the material susceptibility with temperature to control fluid flow in a system. In this approach, the magnetic field-field gradient product acts as a body force similar to gravity and can be tailored to either curtail or enhance the fluid flow. This can be likened to prescribing a desired gravity-level on the system and is a powerful experimental tool for simulating variable and reduced gravity environments.

We have been interested to use a magnetic fluid (ferrofluid) as a control element in our system. In conjunction with a tailored magnetic field (orientation and intensity) it can provide an extra parameter space for flow control. In our recent studies we have had the objective that we can control flow in a system driven by thermal buoyancy induced convection (terrestrial and planetary scenarios, no free surface). While real-time control through feedback is ultimately realizable, our method of approach has been to study the fluid mechanics in a passive sense (investigate the system flow in the presence of a parametric variation of the magnetic force). Guided by the technological applications, we have been interested to control flows driven by pure thermocapillary forces (zero-g) and also in instances when thermal buoyancy driven and Marangoni flows are simultaneously present, and, in addition, we also wanted to be able to control flow instabilities in the system driven by thermal – buoyancy, surface and interfacial tension driven flows. We think that our approach can be used for practical applications such as heat exchangers (in order to improve heat transfer) germane to the present space exploration program in the United States and possibly elsewhere, such as interplanetary and lunar missions where gravity is reduced.

The main components of our studies in this topic have been those, which are briefly given here as follows. For pure buoyancy driven flow studies, we utilize a working fluid that is magnetically responsive (ferrofluid or aqueous solutions of paramagnetic salts). For working fluids that are not magnetically responsive (weak diamagnetic properties) we made use of an additional fluid, such as a regular fluid like oil or water, to form an immiscible double-layer system configuration. For the latter, we determine the flow solutions in both layers for both a flat and an undeformable interface and for deformed interface cases. We have carried out stability analysis for a few cases so far to determine the range of the parameter values under which steady or time dependent solutions are stable and thus preferred. Our goal has been to determine ultimately conditions under which such solutions are most realistic and search for the conditions on the magnetic fields and fluids where convection is maximally reduced (halt condition). For the deformed interface cases, we wanted to examine the interface dynamics and the roles played by the magnetic fields with different strength and orientation on the flow intensity, flow patterns, convective instability, surface tension, due to interface and free surface, and fluid viscosity. The magnetic fluid properties are affected by a magnetic field. Our goal also have been to determine eventually the dependence of viscosity and surface tension on the magnetic fields, the type of the flow patterns and structures generated by the interface and free surface dynamics, the types of flow features affected or otherwise by the magnetic fields and the magnetic fluid gradient, characteristics of the most critical disturbances destabilizing the flow at critical values of the parameters, optimum conditions for the magnetic damping of convection, operational regimes based on the best damping technique, conditions for the preference of particular flow states, types of the instability process that transition between two flow states involves, range of the parameters where multiple states are possible, bifurcations leading to different flow states and most importantly conditions on the magnetic fields and fluids where the flow in the liquid layer can be suppressed significantly.

An appropriate mathematical modeling and computational studies of flow in a two immiscible fluid-layer system can be of value to follow the technological research objectives in the sense that such research seeks to find ways to control thermocapillary convection in such a fluid flow system with a low gravity level, which can be achieved by employing a tailored magnetic field in the presence of a magnetic fluid layer. In addition, the results of such research studies in this area are fundamental as well and can fall under the space research areas of interest since they include investigations that arrange appropriate gravity levels to study the effects of surface tension gradient forces on fluid behavior in order to expand the current knowledge of understanding or to promote innovation in the terrestrial technologies.

Our studies, which have been encompassed both mathematical modeling and computational activities, seek to uncover important scientific results about the flows, which are affected by the properties of the magnetic fluids in the presence of magnetic fields of different strengths and orientations. Such magnetic fluids' properties will be explored further in future to determine more scientific understanding of the dependence of the surface tension and viscosity with respect to the magnetic fields.

Furthermore, as the space-research documents have been referring to interfacial phenomena as one of the research areas of interest to space technology and therefore of technological interest, the roles played by the magnetic fields and the magnetic fluids on the free surface and interface dynamics and morphologies have been of interest to be explored and eventually optimum conditions for the preference of the horizontal and flat interface and free surface need to be determined.

2. SOME PREVIOUS RELEVANT STUDIES

The topic of our research studies is multi-faceted and complex. It involves interface dynamics involving immiscible fluid layers with deformable free and interfacial layers; interactions between system flows such as buoyancy driven and thermocapillary convection; and magnetic effects and magnetic fluids. In the following sub-sections, an overview of relevant research studies in each of these areas is presented.

2.1. Convection in immiscible fluid layers

There have been a number of studies on the effects of convection in systems of immiscible fluid layers subjected to imposed temperature gradients (Szekely and Todd 1971; Simanovskii 1979; Bourde and Simonovskii 1979; Knight and Palmer 1983; Kimura et al.1985; Sparrow et al.1986; Myrum et al.1986; Nepomnyashchy and Simanovskii1990; Simonovskii et al.1992; Liu and Roux 1992; Georis et al. 1993; Ramachandran 1993; Doi and Koster 1993; Prakash and Koster 1993, 1994a, 1994b, 1994c; Georis et al.1994; Georis and Legros 1995; Kats-Demianets et al.1997a,b; Georis et al.1997; Nepomnyashchy and Simanovskii1997; Kliakhandler et al.1998; Xu and Zebib1998; Nepomnyashchy and Simanovskii 1999; Georis et al.1999; Kliakhandler and Nepomnyashchy 1999; Hamed and Florian 2000; Monti2001; Velarde and Nepomnyashchy 2001; Smith et al. 2002). Ramachandran (1993) investigated numerically the effects of buoyancy and surface tension gradient forces on thermal convection in a system with two horizontal immiscible fluids subjected to an imposed lateral temperature gradient. The investigated flow system consisted of a lighter fluid layer on top of a heavier fluid layer, and both layers were contained in a two-dimensional open cavity. Both upper free surface and the interface between the two fluid layers were assumed to be flat and undeformable in his calculations. Ramachandran (1993) solved the governing system of equations and boundary conditions by using a control volume-based finite difference scheme for two cases of immiscible fluids. The main results were that steady-state calculations predicted dramatically different flows when interfacial tension effects were included, and complex flow patterns, with induced secondary flows, were found in both of the fluid layers. Doi and Koster (1993) investigated analytically and numerically two-dimensional pure thermocapillary convection in two immiscible fluid layers with an upper free surface. Both the free surface and the interface were assumed to be horizontal flat with zero deformation. Their results are briefly as follows. First, based on some approximations, they found an analytical solution in the steady state for infinite horizontal extent of the layers. Under a zero gravity environment, four different flow profiles exist which are controlled by a parameter, λ. This parameter is ratio of the temperature rate of change of the interfacial tension between the two layers to the temperature rate of change of the surface tension of the upper layer. They found three 'halt conditions' which stop the flow motion in the lower layer. They identified the technologically relevant halt condition as $\lambda = 0.5$. Next, they studied numerically the effects of the vertical end walls on the flow. They determined conditions on the flow parameters under which the above halt condition can be valid, and they showed that for $0 < \lambda < 0.2$, thermocapillary convection can greatly be suppressed in the encapsulated liquid layer at some higher Marangoni number. A direct numerical simulation of thermal convection in an enclosed cavity filled by three immiscible fluid layers and subjected to an imposed temerature gradient parallel to the interfaces were carried out by Georis et al.(1997), where the deformations of the interfaces were neglected. These authors found, in particular, an essential influence of the nonlinear effects for particular range of the parameters, such as Rayleigh and Marangoni numbers, and that depending on the value of the Rayleigh number, the flow intensity was observed in different layers

2.2. Deformed interface/free surface cases

In regard to the investigated cases of convective flows in systems with deformed interface and/or deformed free surface, kinematic and dynamic conditions need to be satisfied on the deformed surfaces (Davis 1987), which were treated either by applying some approximations or by numerical means (Sen and Davis 1982; Smith 1986; Hjellming and Walker 1987; Riahi and Walker1989; Lie, Riahi & Walker 1989; Hamed & Florian 2000). Sen and Davis (1982) studied steady thermocapillary flows in two-dimensional slot with an imposed temperature gradient along the free surface. They used an asymptotic theory in the limit of small aspect ratio A of the slot to determine the fluid and thermal field and the interfacial shapes and found that deformation of an interface between the liquid flow in the slot and the ambient passive gas was small of order A. Hjellming and Walker (1987) investigated analytically the melt motion due to buoyancy and thermocapillarity in a Czochralski crystal puller with an axial magnetic field. They found, in particular, that thermocapillarity, which becomes progressively more dominant as the crystal grows and the melt depth decreases, is sensitive to changes in the amount of heat lost through the part of the deformed free surface adjacent to the crystal. Riahi and Walker (1989) investigated the deformed free surface stability and shape of the melt during the float zone crystal growth process with the presence of the electromagnetic body force due to a radio-frequency induction coil. They found that this force pinches the float zone and produces a smaller minimum radius, relative to the feed rod and crystal radii, and for coil, which is close to the free surface, a sufficiently strong electromagnetic force destabilizes the free surface of the zone. Lie, Riahi and

Walker (1989) investigated buoyancy driven flow, which was due to the temperature gradient in the melt of a float zone, and the surface tension driven flow, which was due to the non-uniform temperature distribution along the deformed free surface of the zone, in the presence of a strong axial magnetic field. The non-cylindrical deformed shape of the free surface of the zone was found to have a profound effect on the melt motion. Their results indicated that the regions near the free surface were controlled mainly by the thermocapillarity, while the inner region was dominated by the buoyancy driven flow. Hamed and Florian (2000) investigated computationally two-dimensional Marangoni convection in a cavity with differentially heated sidewalls. Their study took into account the complete effects of the deformed interface between the liquid layer and a passive gas. They found that under certain parameter regime, multiple states with steady and oscillatory flows were possible and transition between the steady and the oscillatory states appeared to involve a nonlinear instability process.

2.3. Hele-Shaw cell systems

There have been a number of studies of flow of two immiscible fluids in a Hele-Shaw cell (Saffman and Taylor 1958; Zeybek and Yortsos 1991; McCloud and Maher 1995; Gondret and Rabaud 1997; Miranda and Widom 2000), which provides a simple mathematical and experimental model for theoretical and experimental studies of the two-layer systems in order to hopefully gain further physical understanding of the qualitative aspects of related but more complex flow pattern-evolution problems. Most of the studies of the two-layer flow in the Hele-Shaw cell have been focused on viscous fingering that was first studied by Saffman and Taylor (1958) who considered two immiscible viscous fluids moving in the narrow space between two parallel plates of a Hele-Shaw cell. They demonstrated that air driven into glycerin in such a cell could form a steady pattern in the form of a single finger of air, and the flow equations admit a family of solutions one of which agrees well with the experimental observation. McCloud and Maher (1995) reviewed experimental perturbations to Saffman-Taylor (S-T) flow problem, which provides a simple case of nonlinear interfacial pattern formation. In a number of experiments perturbations have been added to the S-T problem in order to learn more about interface dynamics and about the roles played by the material properties on the dynamical evolution of the resulting flow patterns. Zeybek and Yortsos (1991) studied theoretically and experimentally the long waves in parallel flow in Hele-Shaw cells. To study interface dynamics, they first derived the linear dispersion relation using normal mode approach (Drazin and Reid 1981) and then determined the solutions. Next, they used asymptotic analysis to determine the nonlinear evolution of small amplitude and long wave disturbances. They found that such disturbances are governed by Kortweg-de-Vries and Airy equations (Whitham 1974; Drazin and Reid 1981). For a symmetric case, experimental evidence supported the theory, including the propagation of solitary waves (Whithan1974). Gondret and Rabaud (1997) studied experimentally the parallel flow in a Hele-Shaw cell of two immiscible fluids, a gas and a liquid layer, driven by an imposed pressure gradient. They observed that the interface destabilized above a critical value of the gas flow at which waves grew and propagated along the cell. Their theoretical prediction based on a linear stability analysis agreed with their experimental results.

2.4. Flow instability problems

We already referred to a number of investigations for the deformed interface or deformed free surface effects on the convective flow, which involved flow instabilities under certain conditions. Here some other notable studies in the past that involved flow instabilities are briefly described. Smith and Davis (1983a) investigated instabilities in thermocapillary liquid layers by considering a horizontal liquid layer subjected to an imposed temperature gradient along the layer, which led to the flow motion due to thermocapillarity. They carried out linear stability analysis of disturbances superimposed on their detected basic flow solutions and found, in particular, that the dynamic state of the flow is then susceptible to two types of thermal-convective instabilities of either stationary longitudinal rolls, which involve the classical Marangoni instability (Pearson 1958), or unsteady hydrothermal waves, which derive their energy from the horizontal temperature gradient. In the second part of this study (Smith and Davis 1983b) the authors found, in a particular, that for a particular linearized case, the thermal field decoupled from the hydrodynamic field and the instability of the basic flow set up by the thermocapillarity is a purely isothermal one. Xu and Zebib (1998) investigated numerically oscillatory two- and three-dimensional thermocapillary convective flows in a rectangular cavity and in a box, respectively, and determined, in particular, the character and stability of such flows. For two-dimensional cases, they used a finite-volume based scheme to determine the solutions, while in three-dimensional cases a finite-volume-based primitive variable solver was used. Kliakhandler and Nepmnyashchy (1999) investigated theoretically and numerically instabilities of thermocapillary flow in three-layer systems. They derived linear and weakly nonlinear equations for the evolution of the interface and carried out linear stability analysis. They found, in particular, new types of long-wavelength instabilities, which persisted at arbitrary small Reynolds number. Hoyas et al. (2002) investigated thermo-convective instabilities in a fluid within a cylindrical annulus heated laterally. They considered a particular domain in the parameter space where buoyancy force dominated over the surface tension gradient force. They found a nonlinear basic flow computationally and carried out a linear stability calculation of the base flow. They found, in particular, that there existed stationary bifurcations to radial rolls and oscillatory bifurcations to hydrothermal waves.

2.5. Effects of magnetic fields and fluids

There have been a number of studies on the application of the effect of a magnetic field on the flow in a liquid layer (Hjellming and Walker 1987; Riahi and Walker 1989; Lie, Riahi and Walker 1989; Lie, Walker and Riahi 1990; Morthland and Walker 1996, 1997a, 1997b) or in a ferrofluid layer (Miranda and Widom 2000; Zahn 2001; Leslie and Ramachandran 2001). Applications of a strong magnetic field on flow of melt in Czochralski or float zone crystal growth processes (Hjellming and Walker 1987; Lie, Riahi and Walker 1989; Lie, Walker and Riahi 1989, 1991) led to significantly reduced effects of the inertial terms in the momentum equation and consequently the fluid flow was significantly weakened. Miranda and Widom (2000) carried out a linear stability analysis for parallel flow in a Hele-Shaw cell when one fluid was ferrofluid and a magnetic field was applied. They found that the magnetic field may provide a new mechanism for destabilizing the interface in the absence of the inertial effects. They determined the magnetic correction to the dispersion relation and suggested that parallel flow of ferrofluids can be a novel system for investigating soliton interactions. Leslie and Ramachandran (2001) pointed out the importance of establishing a solute concentration gradient in a magnetic field under microgravity conditions. In particular, an appropriate magnetic field gradient acting on the ferrofluid flow can counteract the effect of Earth's gravity effectively producing suitable low gravity conditions in an experimental laboratory. Ramachandran and Leslie (2001) investigated numerically magnetic susceptibility effects and Lorentz force damping in diamagnetic fluids. They found, in particular, that convection damping of 50% observed in the experiment can be attributed to Lorentz force damping effects and higher level of flow reduction was possible by exploiting the fluid diamagnetic susceptibility variations.

2.6. Moving surfaces

In dealing with moving surfaces a variety of computational methods have been developed, which can be classified basically into two categories: moving-grid and fixed-grid methods. The moving-grid method is a Lagrange-type method for treating the free surface as the boundary of a moving surface-fitted grid (Floryan and Rasmussen 1989). However, when the grids are highly distorted because of strongly deformed free surface, then rezoning or re-meshing becomes necessary, which could lead to excessive numerical diffusion if frequent rezoning is done. An important fixed-grid type method, which is based on the surface capturing approach (Iafrati et al. 2001; Yue et al. 2003), is the so-called level set method (LSM) (Osher and Sethian 1988), which has been used in a number of problems in applications including those in solidification and crystal growth areas (Kim et al. 2000; Smereka 2000). In this method one defines a function $\phi(x, y, z, t)$, called level set, with some degree of smoothness (Yue et al. 2003) that represents the interface at $\phi = 0$. The level sets are advected by the local velocity field. The interface can be captured at any time by locating the zero level set, which alleviates the burden of increasing grid resolution at the interface in many other numerical methods. The LSM provides convenient features for handling topological merging, breaking and self intersecting of interfaces, and information about the interface, such as orientation and curvature can be conveniently obtained as well, so that surface tension can be accurately estimated (Yue et al. 2003). In addition, using LSM, extension from two to three dimensions can be done easily.

3. MATHEMATICAL MODELING AND RESULTS

Our research studies have been based on a system of partial differential equations appropriate for the description of motion of the magnetic fluids. These equations are well known as the equations for conservation of mass, momentum and heat and Maxwell equations for magnetohydrodynamics. They are given in the book of Rosensweig (1985). Details of the mathematical formulation based on such equations are provided in Bhattacharjee (2005). We considered the already described two-layer system to be based on these equations, and following Rosensweig (1985), we assumed that the operating conditions are within the limit where the magnetization vector $\boldsymbol{M}$ of the fluid does not depend on the temperature T. We restrict our modeling to linearly magnetizable fluids, so that

$$\boldsymbol{B}=\mu_0(\boldsymbol{H}+\boldsymbol{M})=\mu\boldsymbol{H}, \qquad (1)$$

where $\boldsymbol{B}$ is the magnetic induction vector, $\boldsymbol{H}$ is the applied magnetic field vector, $\boldsymbol{M}=\chi\boldsymbol{H}$, χ is the volumetric susceptibility and $\mu=\mu_0(1+\chi)$. We then simplified the forces, which have contributions in the momentum equations such as the magnetic body force per unit volume, the force of gravity per unit volume and the surface force due to the surface tension at the interface.

Following Rosensweig (1985), the magnetic body force vector per unit volume $\boldsymbol{f}_m$ can be written in the following form

$$\boldsymbol{f}_m = -\nabla\{\mu_0 \int_0^H [\partial(M\theta)/\partial\theta] dH\} + \mu_0 M \nabla H, \qquad (2a)$$

where H is the magnitude of $\boldsymbol{H}$, M is the magnitude of $\boldsymbol{M}$ and θ is the specific volume. The expression for this force contains pressure-like-variables, which are obtained by expanding the first term in the right-hand-side in (2a). Simplifying the resulting expression, we find

$$\boldsymbol{f}_m = -(H^2/2)\nabla\mu. \qquad (2b)$$

As can be seen from (2a) and (2b), the magnetic force in the present model is due to presence of magnetic fluid, which is a source for the magnetization, presence of the magnetic field and non-uniform value of the magnetic permeability across the interface, which required presence of two fluid with different permeability.

Although the force due to the surface tension acts on the interface boundary between the two layers, we found it convenient for the numerical simulation of the resulting mathematical model to include such surface force in the momentum equation. The moving interface dynamically satisfies a jump condition in the normal stress between the two fluid, which is $\sigma\kappa\boldsymbol{n}$, where $\boldsymbol{n}$ is a unit vector normal to the interface, κ is the curvature of the interface and σ is the surface tension, which is assumed to be a function of temperature T. Applying a Taylor series expansion of the surface tension about a reference temperature T_0 and keeping only the first two terms in such expansion, which is generally appropriate, we have

$$\sigma = \sigma_0 - \gamma(T - T_0), \qquad (3a)$$

where σ_0 is the value of the surface tension at the reference temperature and $\gamma = -d\sigma/dT$ evaluated at the reference temperature. Following the method of approach of Brackbill et al.(1992), the surface tension force $\boldsymbol{f}_s$ can be incorporated into the momentum equation as a body force in the form

$$\boldsymbol{f}_s = \sigma\kappa\boldsymbol{n}\delta(\boldsymbol{x}), \qquad (3b)$$

where $\boldsymbol{x}$ is the position vector on the interface and $\delta(\boldsymbol{x})=1$ on the interface and 0 elsewhere.

The force of gravity per unit volume is given by

$$\boldsymbol{f}_g = -\rho g \boldsymbol{y}, \qquad (4a)$$

where g is acceleration due to gravity, ρ is the density of the fluid and is a unit vector in the positively upward vertical direction. Boussinesq approximation that neglects density variation with respect to the temperature everywhere except in the force of gravity, is adopted here so that the density in (4a) is given by

$$\rho = \rho_0[1 - \beta(T - T_0)], \qquad (4b)$$

where ρ_0 is the value of the density at the reference temperature T_0 and β is the coefficient of thermal expansion.

In this study we restrict our mathematical modeling to the cases of non-conducting ferrofluids. The Maxwell's equations are then reducing to the following equations:

$$\nabla.\boldsymbol{B}=0,\ \nabla\times\boldsymbol{H}=0. \quad (5a, b)$$

Using the equation (5b), we can define a magnetic potential functionΦ, which satisfies

$$\boldsymbol{H}=\nabla\Phi. \quad (5c)$$

Using this relation, (1) and (5a), we find

$$\nabla.\mu\nabla\Phi=0. \quad (5d)$$

The system of partial differential equations under the present study then consists of the equation (2b), (3a, b), (4a, b), (5d, c) and the following equations for the conservation of mass, momentum and heat, respectively:

$$\nabla.\boldsymbol{u}=0, \quad (6)$$

$$\rho(\partial/\partial t+\boldsymbol{u}.\nabla)\boldsymbol{u}= -\nabla P+\nabla.(\eta\nabla\boldsymbol{u})+\boldsymbol{f}_m+\boldsymbol{f}_s+\boldsymbol{f}_g, \quad (7)$$

$$(\partial/\partial t+\boldsymbol{u}.\nabla)(\rho C_p T)=\nabla.(k\nabla T), \quad (8)$$

where $\boldsymbol{u}$ is the velocity vector, P is the pressure, η is the dynamic viscosity, k is the coefficient of thermal conductivity, C_p is the specific heat and t is the time variable.

The physical domain of the two-layer system is basically a square box filled by the two-fluid layer. Only two-dimensional flow case is considered in these studies at present since no two-dimensional results or adequate computational codes are known for the present problem. The boundary conditions that we applied for the above system of equations are zero components of the velocity vector on the left, right, upper and lower walls of the square box. Both left and right walls are assumed to be isothermal, so that T=0 at the left wall and $T=\Delta T$ at the right wall, where ΔT is a prescribed temperature on the hot wall. Upper and lower walls are assumed to be adiabatic, so that the derivative of T with respect the variable in the direction perpendicular to each of these walls is zero ($\partial T/\partial n=0$). The boundary conditions that are chosen for the magnetic potentialΦ, are zero value of the rate of change of Φ with respect to the horizontal variable x on the walls and prescribed values for the rate of change of Φ with respect to the vertical variable y on the walls, as provided in Bhattacharjee (2005).

The already described system of partial differential equations is then appropriately non-dimensionalized, which lead to a number of non-dimensional parameters for the problems the main of which are the Reynolds number R_e, which is a measure of the inertial force relative to the viscous force, the Weber number W_e, which is a measure of the inertial force relative to that of the surface tension, the Grashof number G_r, which is a measure of the buoyancy force relative to the viscous force, the magnetic Bond number B_{om}, which is a measure of the magnetization force to that of surface tension, the Prandtl number P_r, which is the ratio of the kinematic viscosity to the thermal diffusivity, the Froude number F_r, which is a measure of the inertial force relative to the force of gravity and the Marangoni number M_a, which is measure of the surface tension gradient force to that of viscous or thermal diffusion.

The solutions to the system of equations and the boundary conditions that were already described were determined numerically for various values of the parameters. The computation was based on our newly developed numerical method, which is a combination of a level-set method and a volume of fluid method, and it is described in details in Bhattacharjee (2005).

The application of the mathematical model considered here is for the two immiscible fluid layers contained in the square box, and is subjected to a temperature gradient applied in the horizontal direction. The lower fluid is a ferrofluid (water based), while the upper fluid is a diamagnetic fluid, oil. The system is subjected to an external magnetic field gradient. The aspect ration is kept constant, and the wall boundaries are solid. A schematic of the setup is shown in Figure 1.

Some of the results of the numerical solutions of the prescribed system of partial differential equations for the two-layer system are described briefly in the following paragraphs. We determined the results for the maximum magnitude of the flow velocity in the system, the maximum of the magnitude of the local velocity near the interface and both the conductive and convective heat transport across the interface.

First, we consider zero-gravity case and investigated variations of flow velocity and heat fluxes with respect to two parameters and keeping the rest of the parameters fixed. The maximum of the magnitude of both global and local velocity was found to decrease with increasing the magnetic Bond number. This result indicate beneficial effect of the magnetic fluid and field, which were modeled in the present study, to reduce and control the flow velocity that may be needed to prevent growth of disturbances in the flow system. Variations of the magnitude of the maximum global velocity with respect to the magnetic Bond number were determined for different values of the Marangoni number. It was found that the magnitude of such velocity decreases with increasing the Bond number only if the Marangoni number is not too large, say of order 10 or smaller other wise such magnitude increases with the Bond number if the Marangoni number is too large, say of order 1000 or higher. This result indicates a competition between the stabilizing effects of the magnetic field versus a destabilizing effect of surface tension gradient force. For smaller magnitudes of the surface tension gradient force, the stabilizing effect of the magnetic field dominates over the destabilizing effect of the surface tension gradient, while the opposite holds if the magnitude of the surface tension gradient is sufficiently large. However, our results for the variation of the magnitude of maximum local velocity versus the Bond number for different values of the Marangoni number indicate that such magnitude decreases with the Bond number if the Bond number is not too small, which indicate dominating stabilizing effect of the magnetic field. Similar results are obtained for the heat flux, where again the stabilizing effect of the magnetic field dominates over the destabilizing effect of the surface tension gradient. The results about the velocity contours in both of the fluid layer indicated a complex competition between the stabilizing effect of the magnetic field and the destabilizing effect of surface tension gradient in the sense that on further increasing the Bond number, the flow goes through an instability to bifurcate into a sort of laminar flow with a multi-cellular structure, provided the Marangoni number is not too large, while the flow is in a new laminar state with a different multi-cellular structure if the Marangoni number is sufficiently large, say of order 1000 or so. The variations of the magnitude of the maximum local velocity with respect to the Weber number are presented in Figure 2. It can be seen from this figure the stabilizing effect of the magnetic field throughout the range considered for the Weber number, but such stabilization is enhances for lower values of the Weber number, where the stabilizing effect of the surface tension is higher.

Next, we determined the results in the presence of gravity. We again considered the effects of two parameters on the flow velocity and heat flux for given values of the other parameters. Although the magnitude of the maximum velocity again decreases with increasing the magnetic field and for different values of the Froude numbers, the flow patterns in both layers are found to be more smooth and similar to rolls-types of structures.

4. CONCLUDING REMARKS

The results of the present study indicates that the interface properties of a magnetic fluid in the presence of a vertical magnetic field can be used to reduce and control the convective flow in a fluid system within certain non-trivial regimes of the magnetic Bond number and the Marangoni number. The convective flow is either due to the surface tension gradient force alone or due to this force plus that of buoyancy.

In the present study of our mathematical model for the prescribed two-layer system, we carried out computation of a two-dimensional version of the flow system. However, the realistic problems in applications are, in general, in three-dimensional space, and we hope that our present two-dimensional results could stimulate future three-dimensional studies of such problem that could lead to more effective means to control the flow velocity and the associated instabilities.

One important future study of the present mathematical model, which could relay heavily on the results of future experiments, is dependence of the surface tension and viscosity on the magnetic fields and fluids, which will need to be measured first from the experiments and will then be implemented in the computational and modeling aspects of the present problem to determine such roles of the magnetic fields and fluids and their implications on the interface dynamics, flow patterns and other flow features of the two-layer system.

The results of the present study about the usefulness of the presence of the magnetic fluid and field in a two-layer system could also stimulate future theoretical studies of the two-layer system. The basic theoretical approaches could include those such as scaling, asymptotic analysis, perturbations, multiple scales, weakly nonlinear and stability analysis, which have been used by the second author in a number of convection and crystal growth problems in the past including Riahi (1993, 1994, 1996, 1999, 2001, 2002) and the cited references. One could first consider certain parameter regimes or restricted conditions where analytical approaches such as multiple scaling, asymptotic analyses and integration can become possible. For example, for strong magnetic field case, scaling and asymptotic approaches due to Hjellming and Walker (1987), Lie, Walker and Riahi (1990) and Morthland and Walker (1997a, b) can be feasible, while for small aspect ratio case, scaling and asymptotic analysis due to Sen and Davis (1982) becomes feasible, and for infinite horizontal layers in the steady state, integration approach due to Doi and Koster (1993) becomes appropriate. For those analytical solutions that one could determine for the case of infinite horizontal layers, one could use numerical methods, such as those used by Doi and Koster (1993), to find their analog solutions in finite layers bounded by the lateral walls. Next, one could consider certain restricted conditions where analytical approaches such as approximation, perturbation, stability analyses, weakly nonlinear theory and multiple scaling can be possible in order to study the interface dynamics. For example, for long waves, multiple scales approach can become feasible (Zeybek and Yortsos 1991), for semi-infinite layers, perturbation and approximations become relevant (Miranda and Widom2000), and for large surface tension, stability and weakly nonlinear analyses can become appropriate (Kliakhandler and Nepomnyashchy 1999). Under certain conditions, proper scaling of the flow variables and the parameters can reduce the governing system to appropriate form, which will lead to evolution equations for the deformed interfaces as in the work due to Riahi (1993).

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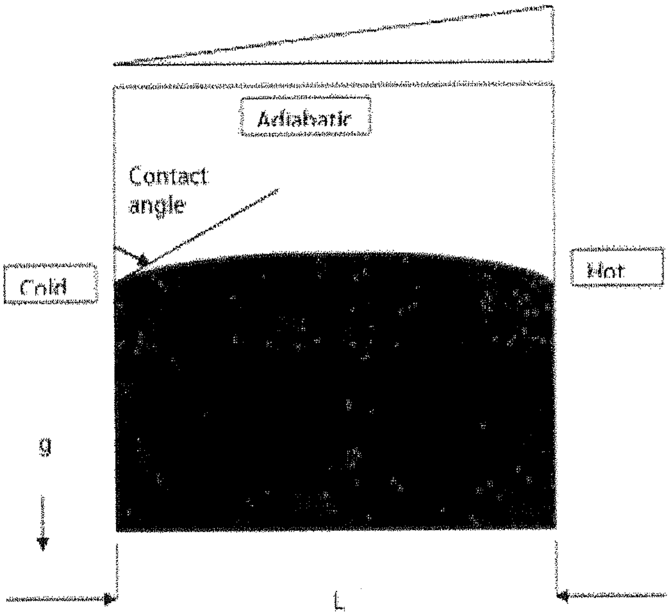

Figure 1. The Two-layer system

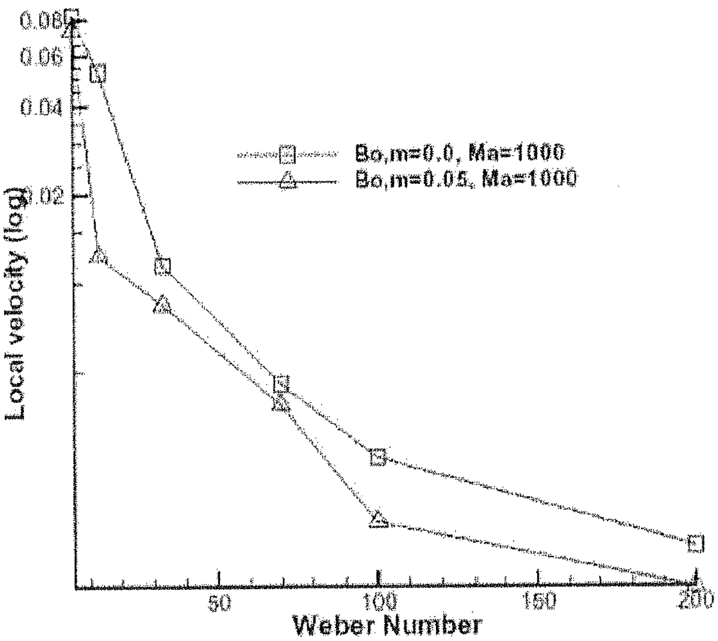

Figure2. Weber number effect

International Journal of Applied Mathematics & Statistics
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

IJAMAS: September 2007, Vol. 10, No. S07, 98-115

Separation of Variables Solution of PDE via Sinc Methods

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ABSTRACT

In their 1953 text of Morse & Feshbach, list 13 regions of when the three dimensional Laplace and Helmholtz partial differential equation (PDE), can be solved via use of separation of variables, i.e., via use of one–dimensional methods. They describe explicit transformations which make such solutions possible.
In this paper we state precise assumptions on the PDE, its piecewise smooth curvilinear spacial boundary and the boundary conditions, i.e., assumptions of analyticity in each variable, which are satisfied, in essence, whenever calculus is used to model the PDE. Under these assumptions we are able to prove that the approximate solution of the PDE has similar analyticity properties. By combining this analyticity assumption with novel Sinc convolution methods, we are able to solve the PDE to arbitrary uniform accuracy via use of a relatively small sequence of one dimensional matrix operations.
The proofs of the above claims are lengthy, and we therefore present such proofs only for PDE in two dimensions. Proofs for the case of three dimensional will be published elsewhere.

Keywords: Sinc methods, Sinc convolution, separation of variables.

2000 Mathematics Subject Classification: 65M99, 65M70, 65N99.

1 INTRODUCTION AND SUMMARY

In (Morse and Feshbach, 1953) Morse & Feshbach discuss the possibility of separation of variables when solving three dimensional Laplace and Helmholtz equations. They conclude that there are essentially only 13 cases of when this is possible. In such cases, it is possible to solve 3-d problems via use of one dimensional methods.

In this paper we illustrate the solution of partial differential equations (PDE) via BIE (boundary integral equation) method, and we make assumptions on analyticity that enable us to construct exponentially convergent and uniformly accurate solutions via use of relatively few "one–dimensional" matrix operations, i.e., via *separation of variables.* The procedure enables us to solve PDE without use of the large matrices required by other (finite difference, finite element, spectral, etc.) methods that are based on approximation of the derivatives of the PDE. As a result, we able to solve nearly every linear and many nonlinear PDE in d variables with

$\mathcal{O}(|\log(\varepsilon)|^{2d+2})$ time complexity, even in the presence of unknown–type integrable singularities on the boundary.

The proofs of the above claims are lengthy, and in this paper we present such proofs only for two dimensional Poisson Dirichlet and Neumann problems. Detailed proofs for all other cases will appear elsewhere (Stenger, 2006).

In §2 below, we give a brief summary of Sinc methods. To this end, we summarize in §2.1 the known one dimensional Sinc methods that are relevant to this paper, and we also present a novel one dimensional approximation which we require for extension of Sinc methods to solve PDE over curvilinear regions in two dimensions. In §3 we demonstrate separability, including definitions of analyticity, our method of subdividing regions to preserve analyticity, our method of extension of convolutions to curvilinear regions, proofs that the integration operators that represent solutions correctly map our original spaces back to similar ones, and that the solutions have the requisite analyticity features enabling exponentially accurate approximation via one-dimensional Sinc methods, even in the presence of unknown-type singularities.

2 SINC METHODS

One dimensional Sinc methods in essence enable the approximation of every operation of calculus, including approximate: interpolation, differentiation, definite and indefinite integration, definite and indefinite convolution, more general convolutions, evaluation and inversion of Laplace transforms, Hilbert transforms, analytic continuation, solution of ODE and integral equations.
All of these are derived via use of the *Cardinal expansion*,

$$\begin{aligned} C(f,h)(x) &= \sum_{k=-\infty}^{\infty} f(k\,h)\,S(k,h)(x) \\ S(k\,,h)(x) &= \operatorname{sinc}\left(\frac{x}{h}-k\right) \\ \operatorname{sinc}(x) &= \frac{\sin(\pi\,x)}{\pi\,x}\,. \end{aligned} \tag{2.1}$$

The function $C(f,h)$ is replete with many identities obtained via use of calculus operations on $C(f,h)$, such as differentiation, integration, orthogonality, delta function-like behavior of Sinc functions, Fourier transforms, Hilbert transforms, etc. These identities become highly accurate approximations if f is not analytic in the entire complex plane. A convenient choice for a region of analyticity of f is the strip

$$D_d = \{z \in \mathbf{C} : |\Im(z)| < d\}\,,$$

where $\mathbf{C}$ denotes the complex plane. For example if f is analytic and bounded by M in D_d, then $|f(x) - C(f,h)(x)| < M\,exp(-\pi\,d/h)$ for all $x \in \mathbf{R}$.

A conformal map φ of another simply connected region

$$\mathcal{D} = \{z \in \mathbf{C} : |\arg(\varphi(z))| < d\}$$

onto D_d automatically yields methods of interpolation (as well as other formulas of approximation) over a contour $\Gamma = \varphi^{-1}(\mathbf{R})$.

2.1 Some Basic Definitions

We introduce here, some general notions of Sinc methods.

Definition 2.1. Let φ denote a smooth one–to–one transformation of a domain $\mathcal{D}$ in the complex plane $\mathbf{C}$,let $\psi = \varphi^{-1}$ denote the inverse function of φ, and let a curve Γ lying in $\mathcal{D}$ with end–points a and b be defined by

$$\Gamma = \{z \in \mathbf{C} : z = \psi(u),\ u \in \mathbf{R}\}. \tag{2.2}$$

Note that $\varphi(a) = -\infty$, and $\varphi(b) = \infty$.
Given φ, ψ and a positive number h, we define the *Sinc points* z_k by

$$z_k = z_k(h) = \psi(kh),\ k = 0, \pm 1, \pm 2, \dots, \tag{2.3}$$

and we let ρ denote the function

$$\rho(z) = e^{\varphi(z)}. \tag{2.4}$$

Observe that $\rho(z)$ increases from 0 to ∞ as z traverses Γ from a to b.

Corresponding to positive numbers α and d, let $\mathbf{L}_{\alpha,d}(\varphi)$ denote the family of all functions F that are analytic and uniformly bounded in $\mathcal{D}$, such that

$$F(z) = \begin{cases} \mathcal{O}\left(\rho(z)^{\alpha}\right) & \text{as} \quad z \to a\,, \\ \\ \mathcal{O}\left(\rho(z)^{-\alpha}\right) & \text{as} \quad z \to b\,. \end{cases} \tag{2.5}$$

Another important family of functions is $\mathbf{M}_{\alpha,d}(\varphi)$, where for simplicity of formulas to represent functions in this class, we assume that $0 < \alpha \le 1$, and $0 < d < \pi$. It consists of all those functions F defined on Γ such that

$$G = F - L\,F \in \mathbf{L}_{\alpha,d}(\varphi)\,, \tag{2.6}$$

and where $L\,F$ is defined by

$$L\,F(x) = \frac{F(a) + \rho(x)\,F(b)}{1 + \rho(x)}. \tag{2.7}$$

The Hilbert transform $\mathcal{S}\,f$ referred to in the next theorem is defined in §2.11 below.

Theorem 2.1. *Let $d' \in (0, d)$. The spaces $\mathbf{L}_{\alpha,d}(\varphi)$ and $\mathbf{M}_{\alpha,d}(\varphi)$ have the following properties:*

1. *If $f \in \mathbf{M}_{\alpha,d}(\varphi)$, then $f'/\varphi' \in \mathbf{L}_{\alpha,d'}(\varphi)$;*

2. *If* $f'/\varphi' \in \mathbf{L}_{\alpha,d}(\varphi)$, *then* $f \in \mathbf{M}_{\alpha,d}(\varphi)$;

3. *If* $f \in \mathbf{L}_{\alpha,d}(\varphi)$, *then* $\int_\Gamma |\varphi'(x)\, f(x)|\, dx < \infty$;

4. *If* $f \in \mathbf{L}_{\alpha,d}(\varphi)$, *then* $Sf \in \mathbf{M}_{\alpha,d'}(\varphi)$.

Definition 2.2. For given a positive integer N and for a given function u defined on Γ, we define a diagonal matrix $D(u)$ and a column vector $V(u)$ by

$$\begin{aligned} D(u) &= \operatorname{diag}\left[u(z_{-N}), \ldots, u(z_N)\right] \\ V(u) &= \left(u(z_{-N}), \ldots, u(z_N)\right)^T \end{aligned} \tag{2.8}$$

where $z_j = \varphi^{-1}(jh)$ denote the Sinc points. Set

$$\begin{aligned} h &= \left(\frac{\pi d}{\alpha N}\right)^{1/2}, \\ \gamma_j &= S(j,h) \circ \varphi, \quad j = -N, \ldots, N, \\ \omega_j &= \gamma_j, \quad j = -N+1, \ldots, N-1, \\ \omega_{-N} &= \frac{1}{1+\rho} - \sum_{j=-N+1}^{N} \frac{1}{1+e^{jh}} \gamma_j, \\ \omega_N &= \frac{\rho}{1+\rho} - \sum_{j=-N}^{N-1} \frac{e^{jh}}{1+e^{jh}} \gamma_j, \\ \varepsilon_N &= N^{1/2} e^{-(\pi \alpha d N)^{1/2}}. \end{aligned} \tag{2.9}$$

The ω_j are the basis functions.
In addition, let $\|\cdot\|$ denote the uniform norm defined on Γ, i.e., $\|f\| = \sup_{x \in \Gamma} |f(x)|$.
For given f defined in Γ, we can now form the *Sinc* approximation, using the basis $\mathbf{w} = (\omega_{-N}, \ldots, \omega_N)$,

$$f \approx \mathbf{w}\, V\, f. \tag{2.10}$$

2.2 Sinc Interpolation

(See (Kowalski, Sikorski and Stenger, 1995), (Stenger, 2006))

Theorem 2.2. *If* $F \in \mathbf{M}_{\alpha,d}(\varphi)$, *then there exists a constant* C *that is independent of* $N > 0$, *such that*

$$\|F - \mathbf{w}\, V\, F\| \le C\, \varepsilon_N. \tag{2.11}$$

Remark: The numbers α and d that we introduced in Definition 2.1 above might not be easily determined. However, under our assumptions of analyticity which are introduced in §3.1, such positive numbers always exist. Then, if instead of selecting h as in (2.9), we take $h = c/N^{1/2}$, with c some constant independent of N, then (2.11) is replaced by

$$\|F - \mathbf{w}\, V\, F\| \le C\, \exp\left(-c'\, N^{1/2}\right), \tag{2.12}$$

with C and c' positive constants that are independent of N, and where it may be shown that $c' = \max(\pi\, d/c\,,\ \alpha\, c)$, i.e., we are still guaranteed exponential convergence.

We next illustrate the Sinc approximation of several important operations of calculus on a function f.

2.3 Sinc Collocation

The following theorem states that if the error of approximation of a function f at Sinc points is small, then the error at all other points on Γ of the Sinc approximation using these erroneous values is not much larger.

Theorem 2.3. *If $f \in \mathrm{M}_{\alpha,d}(\varphi)$, and $\mathbf{c} = (c_{-M}, \ldots, c_N)^T$ is a complex vector of order m, such that for some positive number δ that is independent of N,*

$$\left(\sum_{j=-N}^{N} |f(z_j) - c_j|^2 \right)^{1/2} < \delta\,, \tag{2.13}$$

then

$$\|f - \mathbf{w}\,\mathbf{c}\| < C\varepsilon_N + \delta, \tag{2.14}$$

with $C\,\varepsilon_N$ as in Theorem 2.2 above.

2.4 Approximation of Derivatives

Let $D(u)$ be defined as in Definition 2.2. Then

$$D\left(\left(\frac{h}{\varphi'} \right)^k \right) V\, f^{(k)} \approx A^{(k)}\, V\, f\,, \tag{2.15}$$

where in the notation of Definition 2.2, $A^{(k)}$ is the square matrix of order m, with $(i,j)^{th}$ element $(\omega_j)^{(k)}(z_i)$.

2.5 Sinc Quadrature

$$\int_\Gamma f(x)\, dx \approx (V\,(h/\varphi'))^T\, V\, f\,. \tag{2.16}$$

Several algorithms exist based on this formula ((see (Sikorski, Schwing and Stenger, 1984) and (Stenger, 2006)).

2.6 Sinc Indefinite Integration

Let numbers σ_k and e_k be defined by

$$\sigma_k = \int_0^k \mathrm{sinc}(x)\, dx, \quad k \in \mathbf{Z}, \quad \mathrm{and} e_k = 1/2 + \sigma_k, \tag{2.17}$$

and define an $m \times m$ (*Toeplitz*) matrix $I^{(-1)} = [e_{i-j}]$, with e_{i-j} denoting the $(i,j)^{th}$ element of $I^{(-1)}$. Let us also define operators $\mathcal{J}^+$, $\mathcal{J}^-$, $\mathcal{J}_m^+$ and $\mathcal{J}_m^-$, and matrices A^+ and A^- of order m by

$$\begin{aligned}
(\mathcal{J}^+ f)(x) &= \int_a^x f(t)\,dt\,, \\
(\mathcal{J}^- f)(x) &= \int_x^b f(t)\,dt\,, \\
(\mathcal{J}_m^+ f)(x) &= \mathrm{w}(x)\,A^+\,V\,f\,, \quad A^+ = h\,I^{(-1)} D(1/\varphi')\,, \\
(\mathcal{J}_m^- f)(x) &= \mathrm{w}(x)\,A^-\,V\,f\,, \quad A^- = h\,(I^{(-1)})^T D(1/\varphi')\,,
\end{aligned} \tag{2.18}$$

where the dependence of $\mathcal{J}_m^{\pm}$ on m is through the dependence of $A^{\pm}$, V and Sinc basis w on m. We then have the approximations

$$V\,\mathcal{J}^+\,f \approx A^+\,V\,f \quad \text{and} \quad V\,\mathcal{J}^-\,f \approx A^-\,V\,f\,. \tag{2.19}$$

2.7 Sinc Indefinite Convolution

(Stenger, 1995a). The model integrals which we can approximate are

$$p(x) = \int_a^x f(x-t)\,g(t)\,dt, \quad \text{and} \quad q(x) = \int_x^b f(t-x)\,g(t)\,dt\,, \tag{2.20}$$

where $x \in \Gamma$. In presenting these convolution results, we shall assume that $\Gamma = (a,b) \subseteq \mathbf{R}$.

To evaluate these convolutions, we require that the "Laplace transform",

$$\mathcal{F}(s) = \int_E f(t)\,e^{-t/s}\,dt$$

with $E \supseteq (0, b-a)$, should exists for all $s \in \Omega_+ \equiv \{s \in \mathbf{C} : \Re s > 0\}$. We then get the following approximations (see (Stenger, 1995a), (Stenger, 1995b), §4.6):

$$V\,p \approx \mathcal{F}(A^+)\,V\,g \quad \text{and} \quad V\,q \approx \mathcal{F}(A^-)\,V\,g\,. \tag{2.21}$$

We remark that the eigenvalue–eigenvector decomposition $A^+ = XSX^{-1}$ with S a diagonal matrix enables us to evaluate $\mathcal{F}(A^+)$ and similarly for $\mathcal{F}(A^-)$.

2.8 More General 1–d Convolutions

(Stenger, 2006). The following more general one dimensional convolution plays an important role to enable separation of variables in solution of two dimensional PDE over curvilinear regions via use of Sinc methods.

$$r(x) = \int_a^x k(x-t,t)\,dt\,, \quad x \in (a,b), \tag{2.22}$$

At the outset, we require the use of the "Laplace transform"

$$K(s,t) = \int_0^c k(x,t)\,\exp(-x/s)\,dx\,, \quad c \geq (b-a)\,.$$

Upon setting $A^+ = X\,S\,X^{-1}$, $S = \mathrm{diag}(s_{-N}\,,\,\ldots\,,\,s_N)$, $X = [x_{ij}]$ and $X^{-1} = [x^{ij}]$, we get the following approximation of $r(x)$ at a Sinc point z_n, for $n = -N\,,\,\ldots\,,\,N$:

$$p(z_n) = \sum_{k=-N}^{N} x_{nk} \sum_{\ell=-N}^{N} x^{k\ell} K(s_k\,,z_\ell)\,. \tag{2.23}$$

2.9 Hilbert Transforms

The Hilbert transform of a function f taken over the contour Γ is defined by

$$(\mathcal{S}\,f)(x) = \frac{P.V.}{\pi\,i}\int_\Gamma \frac{g(t)}{t-x}\,dt\,. \tag{2.24}$$

Several Sinc methods are known for approximating Hilbert transforms (Stenger, 2006). We cite here the following one, due to Yamamoto (Yamamoto, 2006, to appear).

Let $A^\pm$ are defined corresponding to an interval $(a,b) \subseteq \mathbf{R}$ as in (2.18) above. Then we have the approximation

$$V\,(\mathcal{S}\,f) \approx \frac{1}{\pi\,i}\left(\log(A^-) - \log(A^+)\right)\,V\,f\,. \tag{2.25}$$

2.10 Analytic Continuation

Let φ and ψ be defined as in Definition 2.1 above, and set $\mathcal{D}^+ = \{\psi(u+iv) : u \in \mathbf{R},\ v \geq 0\,,\ \text{and } \psi(u+iv) \text{ is well defined}\}$. We then define the following functions:

$$\begin{aligned}
\sigma_j(z) &= \Im\left\{\frac{e^{i\pi[\varphi(z)-jh]/h}-1}{\pi[\varphi(z)-jh]/h}\right\}, \quad j \in \mathbf{Z}, \\
\tau_a(z) &= \left[1 - \frac{\Im\varphi(z)}{\pi}\right]\Re\left\{\frac{1}{1+\rho(z)}\right\} - \frac{\Re\varphi(z)}{\pi}\Im\left\{\frac{1}{1+\rho(z)}\right\}, \\
\tau_b(z) &= \left[1 - \frac{\Im\varphi(z)}{\pi}\right]\Re\left\{\frac{\rho(z)}{1+\rho(z)}\right\} - \frac{\Re\varphi(z)}{\pi}\Im\left\{\frac{\rho(z)}{1+\rho(z)}\right\}, \\
\delta_j &= \sigma_j, \quad -M < j < N, \\
\delta_{-N} &= \tau_a - \sum_{j=-M+1}^{N}\frac{1}{1+e^{jh}}\sigma_j, \\
\delta_N &= \tau_b - \sum_{j=-M}^{N-1}\frac{e^{jh}}{1+e^{jh}}\sigma_j\,.
\end{aligned} \tag{2.26}$$

The functions δ_j may be shown to be harmonic in $\mathcal{D}^+$. Moreover, if $\zeta \in \Gamma$, then $\lim_{z\to\zeta, z\in\mathcal{D}^+}\delta_j(z) = \omega_j(\zeta)$, where the basis $\{\omega_j\}$ is defined in Definition 2.2.

3 ANALYTICITY AND SEPARATION OF VARIABLES

Our discussion of this section involves linear PDE, that are modeled via use of calculus, which enables us to postulate that the coefficients of such PDE are analytic in each variable with all other variables fixed. In such circumstances, we claim that Sinc methods enable a separation of variables solution of such elliptic PDE. There are two criteria which enable us to prove this claim:

- Our assumption of analyticity of the coefficients of linear PDE; and
- Our method of solution, involving combining the boundary integral equation technique (BEM) with *Sinc convolution*.

We remark here, that other methods solve PDE via approximation of the highest derivatives of the PDE, and they thus require the storage of large matrices. Indeed, other effective Sinc methods have also used the approach of approximating the highest derivative of the PDE (see (Parker, 1997), (Lund and Bowers, 1992), (El-Gamel and Zayed, 2002)). On the other hand, our most recent methods enable solutions of PDE without approximation of the highest derivatives. The one dimensional Sinc convolution procedure of (2.21) above extends readily to multidimensional rectangular regions, enabling a highly efficient and accurate approximation of multidimensional Green's function convolution integrals via the use of a relatively small number of multiplications of one dimensional matrices i.e., via *separation of variables*. The success of this endeavor requires the multidimensional "Laplace transforms" of Green's functions, and to this end, we were very fortunate in being able to obtain explicit expressions of the multidimensional "Laplace transforms" of every free space Green's functions known to us (Stenger, 2006).

3.1 Multidimensional Analyticity

We again restrict our discussion to problems of at most dimension 2.

(a) Arcs, Curves, and Contours.

An *analytic arc* Γ in the plane $\mathbf{R}^2$ is defined by a mapping

$$\Gamma = \{\bar{\rho} = (x, y) \in \mathbf{R}^2 : x = \xi(t)\,,\ y = \eta(t)\,,\ \ 0 \le t \le 1\}\,, \tag{3.1}$$

where, for some positive constant d, the functions ξ and η are analytic and bounded in the eye–shaped region

$$\mathcal{E} = \left\{ z \in \mathbf{C} : \left| \arg\left(\frac{z}{1-z}\right)\right| < d \right\}\,, \tag{3.2}$$

real–valued on the interval $[0, 1]$, and such that $\dot{\xi}(t)^2 + \dot{\eta}(t)^2$ exists, is positive and bounded for all $t \in [0, 1]$. Here the "dot" indicates differentiation with respect to t, e.g., $\dot{\xi}(t) = d\xi(t)/dt$.

For some positive integer n, a *curve* $\mathcal{B}^{(1)}$ in $\mathbf{R}^2$ is defined to be the union of n *arcs* Γ_j, with $\rho_j : [0, 1] \to \Gamma_j$, i.e.,

$$\partial B^{(1)} = \bigcup_{j=1}^{n} \Gamma_j\,, \tag{3.3}$$

and such that $\bar{\rho}_j(1) = \bar{\rho}_{j+1}(0)$, for $j = 1, 2, \ldots, n-1$. We have thus introduced an *orientation* with a curve.

A *contour* is a *curve* for which $\bar{\rho}_n(1) = \bar{\rho}_1(0)$. Contours $\mathcal{B}^{(1)}$ will frequently be boundaries of two dimensional regions, $\mathcal{B}^{(2)}$ and in such cases we shall require that the interior angles at the junctions of two arcs are positive. We do not otherwise restrict the size of the interior angles (Stenger and Schmidtlein, 1999).

(b) One Dimensional Function Spaces. Let $\mathbf{X}^{(1)}$ denote the family of all functions g defined on a curve $\mathcal{B}^{(1)}$, with $\mathcal{B}^{(1)}$ defined as above, and with g having the following property: g restricted to Γ_j is a function g_j, and moreover, if $\phi_j : [0,1] \to \Gamma_j$, then $\gamma_j \equiv g_j \circ \phi_j$ is analytic on $\mathcal{E}$.

While we shall require analyticity in $\mathcal{E}$ for all our spaces $\mathbf{X}^{(1)}$, depending on a particular problem we shall add additional requirements in order to define $\mathbf{X}^{(1)}$ more specifically, e.g., for purposes of quadrature or indefinite integration of g over $\mathcal{B}^{(1)}$, we would require $\gamma_j/\varphi' \in \mathbf{L}_{\alpha,d}(\varphi)$, where $\gamma_j = g_j \circ \bar{\phi}_j$, and where $\varphi(t) = \log(t/(1-t))$; and when dealing with Hilbert transforms, and requiring that the Hilbert transform taken over $\mathcal{B}^{(1)}$ should map $\mathbf{X}^{(1)}$ back into $\mathbf{X}^{(1)}$, we would let $\mathbf{X}^{(1)}$ denote the family of all functions g defined on $\mathcal{B}^{(1)}$, such that $\gamma_j \in \mathbf{M}_{\alpha,d}(\varphi)$, and such that g is continuous on $\mathcal{B}^{(1)}$ (so that g is of class $\mathbf{Lip}_\alpha$ on $\mathcal{B}^{(1)}$).

(c) The Space $\mathbf{X}^{(2)}$ in Two Dimensions. Let us briefly motivate the type of analyticity we shall require in more than one dimension. For the two dimensional case, if $F = F(x,y)$ is a given function defined on a rectangular region $\mathcal{B}^{(2)} = [a,b] \times [c,d]$, if T^x and T^y are one–dimensional operations of calculus in the variables x and y respectively, if $T^x(N_1)$ and $T^y(N_2)$ are their corresponding Sinc approximations, i.e., operations which typically commute with one another, we can bound the error, $T^x\,T^y\,F - T^x(N_1)\,T^y(N_2)\,F$ in the following (usual) manner,

$$
\begin{aligned}
&\|T^x\,T^y\,F - T^x(N_1)\,T^y(N_2)\,F\| \\
&\le \|T^y\,(T^x\,F - T^x(N_1))\,F\| + \|T^x(N_1)\,(T^y\,F - T^y(N_2))\,F\|\,, \qquad (3.4)\\
&\le \|T^y\|\,\|T^x\,F - T^x(N_1)\,F\| + \|T^x(N_1)\|\,\|T^y\,F - T^y(N_2)\,F\|\,.
\end{aligned}
$$

The third line of this inequality shows that we can still get the exponential convergence of one dimensional Sinc approximation, provided that $F \in \mathbf{X}^{(2)}$, i.e., provided that there exist positive constants c_1 and c_2, such that

1. For each fixed $y \in [c,d]$, function $F(\cdot\,,y)$ of one variable belongs to the appropriate space $\mathbf{X}^{(1)}$ to enable an error of the form

$$\mathcal{O}\left(\exp(-c_1\,N_1^{1/2})\right)\;;$$

and

2. The norm $\|T^y\|$ is bounded,

and dually, provided that

1. For each fixed $x \in [a, b]$, the one dimensional function $F(x\,,\cdot)$ satisfies the appropriate one-dimensional conditions to enable an error of the form

$$\mathcal{O}\left(\exp(-c_2\,N_2^{1/2})\right);$$

and

2. The norm $\|T^x_{N_2}\|$ is bounded.

If the region $\mathcal{B}^{(2)}$ is not rectangular, we assume that the boundary of the region is a contour $\mathcal{B}^{(1)}$ of the type described above, and moreover, that the region housing the solution can be represented as the union of rotations of a finite number n of regions of the form

$$\mathcal{B}_J^{(2)} = \{(x, y) : a1_J < x < b1_J,\ a2_J(x) < y < b2_J(x)\}, \tag{3.5}$$

for $J = 1\,,\ 2\,,\ \ldots\,,\ n$, i.e.,

$$\mathcal{B}^{(2)} = \bigcup_{J=1}^{n} \mathcal{B}_J^{(2)} \tag{3.6}$$

with the property that any two regions $\mathcal{B}_J^{(2)}$ and $\mathcal{B}_K^{(2)}$ with $K \neq J$ share at most a common arc.

In the above definition of $\mathcal{B}_J^{(2)}$, $a1_J$ and $b1_J$ are constants, while with $\beta = b1_J - a1_J$, $a2_J(a1_J + \beta\,t) : [0, 1] \to \mathbf{R}^2$ and $b2_J(a1_J + \beta\,t) : [0, 1] \to \mathbf{R}^2$ represent analytic arcs of the type defined above. Such regions $\mathcal{B}_J^{(2)}$ can easily be represented as transformations of a square $Q^2 = [0, 1] \times [0, 1]$ via the transformation

$$\begin{aligned} x &= a1_J + (b1_J - a1_J)\,\xi \\ y &= a2_J(x) + (b2_J(x) - a2_J(x))\,\eta\,. \end{aligned} \tag{3.7}$$

Now, let $f = f(x, y)$ be a function defined on $\mathcal{B}^{(2)}$, such that f restricted to $\mathcal{B}_J^{(2)}$ is f_J. We thus get a function F_J defined on Q^2, after use of a transformation of the above type, from Q^2 to $\mathcal{B}_J^{(2)}$. We define $\mathbf{X}^{(2)}$ to be the family of all functions $F_J = F_J(\xi\,, \eta)$ such that $F_J(\cdot\,, \eta) \in \mathbf{X}^{(1)}$ for each fixed $\eta \in [0, 1]$, and dually, such that $F_J(\xi\,, \cdot) \in \mathbf{X}^{(1)}$ for each fixed $\xi \in \mathbf{X}^{(1)}$. This property is guaranteed, e.g., if $f_J \circ \bar{\rho} \in \mathbf{X}^{(1)}$ for every analytic arc $\bar{\rho}$ of finite length lying in in the closure of $\mathcal{B}_J^{(2)}$.

We can then solve the problem over Q^2, via a Sinc approximation to get a "mapped" solution or approximate solution which can be represented by

$$U^J(\xi,\eta) = \sum_{i=-M}^{M} \sum_{j=-N}^{N} U_{i,j}^J \, \omega_i^{(1)}(\xi)\, \omega_j^{(2)}(\eta) \tag{3.8}$$

on Q^2, where the functions $\omega_j^{(k)}$ are defined in (2.9) above.

To get back to approximation on $\mathcal{B}_J^{(2)}$ we simply solve for ξ and η in (3.7), to get $\xi = (x - a1)/(b2 - b1)$ and $\eta = (y - a2(x))/(b2(x) - a2(x))$.

3.2 Analyticity and Separation of Variables

We skip the proofs of one-dimensional analyticity, which have already been discussed in (Stenger, 1995b), §7.1 & §7.2.

(a) *Dirichlet Problems Two Dimensions*. We discuss here the problem

$$\begin{aligned} &\nabla^2 u(\bar{r}) = 0\,, \quad \bar{r} \in \mathcal{B}^{(2)}, \\ &u(\bar{r}) = g(\bar{r})\,,\ \bar{r} \in \mathcal{B}^{(1)} = \partial \mathcal{B}^{(2)}\,, \end{aligned} \tag{3.9}$$

where $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ are defined in §3.1 above.

The solution of (3.9) is, of course, a harmonic functions, whenever g is such that a solution exists. But for simplicity of proofs, we shall shall only admit all functions g that are continuous on $\mathcal{B}^{(1)}$, and such that if we denote by g_J the function g restricted to Γ_J, where $\bar{\phi}_J : [0,1] \to \Gamma_J$, then $g_j \circ \bar{\phi}_J \in \mathbf{M}_{\alpha,d}(\varphi)$, with $\varphi(t) = \log(t/(1-t))$. We shall refer to this space of functions as $\mathbf{X}^{(1)}$. We then prove that the solution u of (3.9) belongs to $\mathbf{X}^{(2)}$.

Let $\zeta \in \mathcal{B}^{(1)} = \partial \mathcal{B}^{(2)}$, and for given $F \in \mathbf{X}^{(1)}$, let us denote the Hilbert transform of F taken over $\mathcal{B}^{(1)}$ by

$$(\mathcal{S}F)(\zeta) = \frac{P.V.}{\pi\, i} \int_{\mathcal{B}^{(1)}} \frac{F(t)}{t-\zeta}\, dt\,. \tag{3.10}$$

Let v denote a conjugate harmonic function of the solution u to (3.9), and let $f = u + iv$ denote a function that is analytic in $D = \left\{z = x + i\,y \in \mathbf{C} : (x,y) \in \mathcal{B}^{(2)}\right\}$. Such a function f is given by

$$f(z) = \frac{1}{\pi\, i} \int_{\mathcal{B}^{(1)}} \frac{\mu(\tau)}{\tau - z}\, d\tau \tag{3.11}$$

where μ is a real valued function on $\mathcal{B}^{(1)}$, which is to be determined.

Upon letting $z \to \zeta$, with ζ not a corner point of Γ, and taking real parts, we get the equation

$$\mu(\zeta) + (K\,\mu)(\zeta) = g(\zeta)\,, \tag{3.12}$$

with $K\mu = \Re \mathcal{S}\,\mu$.

The integral equation operator K defined by $K\,u = \Re\,\mathcal{S}\,u$ arises for nearly every integral equation that is used for constructing conformal maps. It has been shown in (Gaier, 1964), Ch. I, §3] that this operator K has a simple eigenvalue 1, for which the corresponding eigenfunction is also 1. Furthermore, the other eigenvalues λ such that the equation $K\,v = \lambda\,v$ has non–trivial solutions v are all less than 1 in absolute value.

Writing $\kappa = (K-1)/2$ we can rewrite (3.12) as follows:

$$\mu(\zeta) + (\kappa\,\mu)(\zeta) = g(\zeta)/2\,. \tag{3.13}$$

Since the norm of κ is less than one in magnitude, the series

$$\sum_{p=0}^{\infty}(-1)^p\,\kappa^p\,g \tag{3.14}$$

converges to the unique solution of (3.12). Since $g \in \mathbf{Lip}_\alpha(\Gamma)$, we know that (see (Gakhov, 1966)) $\mathcal{S}\,g \in \mathbf{Lip}_\alpha(\Gamma)$, so that $\kappa\,g \in \mathbf{Lip}_\alpha(\Gamma)$. It thus follows by Theorem 2.1, that $\kappa : \mathbf{X}^{(1)} \to \mathbf{X}^{(1)}$. The sum (3.14) thus converges to a function $\mu \in \mathbf{X}^{(1)}$.

That is, if $\zeta_J : [0,1] \to \Gamma_J$, it follows, by Theorem 2.1, that the function f defined in (3.11) above is not only analytic in the region $\mathcal{B}^{(2)}$ but in the larger region

$$\mathcal{B}_{\mathcal{E}} \equiv \mathcal{B}^{(2)} \bigcup_{J=1}^{n} \zeta_J^{-1}(\mathcal{E})\,, \tag{3.15}$$

with $\mathcal{E}$ defined as in (3.2).

This means of course, that if γ is any bounded analytic arc in the closure of the region $\mathcal{B}^{(2)}$, then $u \circ \gamma(t)$ is an analytic function of t in $\mathcal{E}$, and moreover, it belongs to $\mathbf{M}_{\alpha,d}(\varphi)$. That is, $u \in \mathbf{X}^{(2)}$.

Setting $\mu|_{\Gamma_J} = \mu^J$, and denoting the Sinc points of Γ_J by $\{\zeta_k^J\}$, and in the notation of (2.26), denoting the functions δ_k defined on Γ_J by δ_k^J, it follows that an accurate approximation to the solution u in $\mathcal{B}^{(2)}$ is given by the expression

$$U(z) \approx u_N(z) = \sum_{J=1}^{n}\ \sum_{k=-N_J}^{N_J} \mu^J\left(\zeta_k^J\right)\,\delta_k^J(z)\,, \tag{3.16}$$

where in this expression, $z = x + i\,y$, and where we have set $U(z) = u(x, y)$. Indeed, this expression itself may by used iteratively to determine the numbers μ_k^J, instead of solution of (3.12) via e.g., the used of the approximation (2.25) of the Hilbert transform (2.24).

It thus follows that the one dimensional methods of §2 can be used to solve two dimensional Dirichlet problems. That is, we can obtain a solution via simple one dimensional methods, i.e., via separation of variables.

(b) *Neumann Problems in Two Dimensions*. We consider here, the solution to the boundary value problem

$$\begin{aligned} &\nabla^2 v(\bar{r}) = 0\,, \quad \bar{r} \in \mathcal{B}^{(2)}, \\ &\frac{\partial v}{\partial \mathbf{n}} = \gamma \quad \text{on} \ \ \gamma = \mathcal{B}^{(1)} = \partial \mathcal{B}^{(2)}\,, \end{aligned} \tag{3.17}$$

where $\mathbf{n}$ denotes the unit outward normal at points of smoothness of $\mathcal{B}^{(1)}$. Let γ_J denote the restriction of γ to $\mathcal{B}_J^{(1)}$, set $\gamma_J(t) = \gamma_J(\bar{\rho}_J(t))$, where $\bar{\rho}_J$ is defined as for (3.1). Then we have $\gamma_J \in \mathbf{M}_{\alpha,d}(\varphi)$. If v denotes the solution to (3.17), and if u denotes the function conjugate to v, we have $u = \mathcal{S}\,v$, and also, $v = \mathcal{S}\,u$, where $\mathcal{S}$ denotes the Hilbert transform defined as in (2.24). Furthermore, the Cauchy–Riemann equations imply that

$$\frac{\partial v}{\partial \mathbf{n}} = -\frac{\partial u}{\partial \mathbf{t}} = -\gamma\,, \tag{3.18}$$

where $\mathbf{t}$ denotes the unit tangent at points of smoothness of $\mathcal{B}^{(1)}$. Given γ, we can thus determine $g(z) = \int_a^z h(t)\,dt$, where the integrations are taken along $\gamma = \mathcal{B}^{(1)}$, and where we can accurately carry out such indefinite integrations via use of Sinc indefinite integration of §2.6 along each arc $\mathcal{B}_J^{(1)}$ of $\mathcal{B}^{(1)}$. We can thus solve a Dirichlet problem to determine a function ν on $\mathcal{B}^{(1)}$, as we did above to determine μ, and having determined ν, we can get $\mu = \mathcal{S}\,\nu$ via the Sinc approximation procedure based on (2.25). We can then approximate v in the interior of $\mathcal{B}$ via the procedure of (3.16).

(c) *Solution of a Poisson Problem on $\mathcal{B}^{(2)}$*. A particular solution to Poisson's equation

$$\nabla^2 w(\bar{r}) = -f(\bar{r})\,, \quad \bar{r} \in \mathcal{B}^{(2)}\,, \tag{3.19}$$

is given by

$$w(\bar{r}) = \int_{\mathcal{B}^{(2)}} \mathcal{G}(x - \xi\,,\, y - \eta)\, f(\xi\,,\, \eta)\, d\xi\, d\eta\,, \tag{3.20}$$

where $\bar{r} = (x\,, y)$, and where $\mathcal{G}$ denotes the Green's function

$$\mathcal{G}(x, y) = \frac{1}{2\,\pi} \log\left\{\frac{1}{\sqrt{x^2 + y^2}}\right\}\,. \tag{3.21}$$

It will now be assumed that $f \in X^{(2)}$. A solution to (3.19) subject to given boundary conditions can be obtained by first obtaining the above particular solution and then adding a solution of (3.9).

Analyticity of w. We let $\mathbf{X}^{(2)}$ be defined as in the above discussion following (3.4). Let us assume that $\mathcal{B}^{(2)}$ has been subdivided into a union of a finite number of regions $\mathcal{B}_J^{(2)}$ as in (3.5)-(3.6) above, such that, with f_J the function f restricted to $\mathcal{B}_J^{(2)}$, and then transformed to a function F_J via use of the transformation (3.7), the resulting function belongs to $\mathbf{X}^{(2)}$. More specifically, we assume here simply that $\mathbf{X}^{(1)} = \mathbf{M}_{\alpha,d}(\varphi)$. The above integral expressing $w(\bar{r})$ can then be written in the form

$$w(\bar{r}) = \sum_{K=1}^{n} \int_{\mathcal{B}_K^{(2)}} \mathcal{G}(x - x', \, y - y') \, f(x', \, y') \, dx' \, dy' . \tag{3.22}$$

Now, consider first the case when $\bar{r} \in \mathcal{B}_J^{(2)}$, and we are integrating over $\mathcal{B}_K^{(2)}$, with $J \neq K$. If $\mathcal{B}_J^{(2)}$ and $\mathcal{B}_K^{(2)}$ do not have a common boundary, then, after transformation of both $(x,y) \to (\xi\,,\eta)) \in Q^2$ and $(x',y') \to (\xi',\eta') \in Q^2$ via (3.7), (each of these, via a different transformation) we arrive at an integral of the form

$$\Omega_{JK}(\xi,\eta) = \int_{Q^2} G_{JK}(\xi,\xi',\eta,\eta') F_J(\xi',\eta') \, \mathcal{J}(\xi',\eta') d\xi' \, d\eta' \tag{3.23}$$

where G_{JK} is the transformed Green's function, F_J is the transformed f, and $\mathcal{J}$ is the Jacobian of the transformation (3.7). We then find that $G_{JK}(\cdot\,,\xi'\,,\eta\,,,\eta')$ belongs to $\mathbf{M}_{\alpha,d}(\varphi)$ for each fixed $\xi'\,,\eta$ and η' on $[0,1]$, and $G(\xi\,,\cdot\,,\eta,\eta')$ belongs to $\mathbf{M}_{\alpha,d}(\varphi)$ for each fixed $\xi\,,\eta$ and η' on $[0,1]$. It thus follows, that the transformed Green's function belongs to $\mathbf{M}_{\alpha,d}(\varphi)$ as a function of ξ, for each fixed $\xi'\,,\eta$ and η' on $[0,1]$, and this is also true as a function of η, for each fixed ξ, ξ', and η' on $[0,1]$. It thus follows, that $\Omega_{JK} \in \mathbf{X}^{(2)}$.

If $J \neq K$ but $\mathcal{B}_J^{(2)}$ and $\mathcal{B}_K^{(2)}$ do have a common boundary, then $\mathcal{G}(x - \xi\,, \; y - \eta)$ is not uniformly bounded. In this case we again transform over Q^2 as above. However, instead of integrating over Q^2, we integrate (in theory) over

$$Q_\ell^2 = \{(\xi',\eta') : 1/4^\ell \le \xi' \le 1 - 1/4^\ell \,, \;\; \ell = 1\,, \; 2\,, \; \ldots\,,\}$$

to get a sequence of functions $\{\Omega_{JK}^\ell\}_{\ell=1}^\infty$. Setting $u^\ell(\eta) = \Omega_{JK}^\ell(\cdot,\eta)$, it is clear that the sequence $\{u^\ell(\eta)\}_1^\infty$ is an infinite sequence of analytic functions of ξ that is uniformly bounded on $\mathcal{E}$ for all fixed $\eta \in [0,1]$. By Montel's theorem, this sequence has a convergent subsequence. It is clear, also, that the sequence $\{u^\ell(\eta)\}_1^\infty$ converges for every fixed $\xi \in [1/4\,,3/4]$, so that in particular, it converges at an infinite sequence of discrete points on $[1/4\,,3/4]$ that has a limit point on $[1/4\,,3/4]$. It thus follows by Vitali's theorem (see (Stenger, 1995b), Corollary 1.1.17]) that the sequence $\{u^\ell(\eta)\}_1^\infty$ converges to a function of ξ that is analytic and bounded on $\mathcal{E}$ for all fixed $\eta \in [0,1]$. Similar statements can be made for the sequence of functions $\{v^\ell(\xi)\}_1^\infty$, where $v^\ell(\xi) = \Omega_{JK}^\ell(\xi,\cdot)$. It follows, therefore, that the sequence of functions $\{\omega_{JK}^\ell\}_{\ell=1}^\infty$ converges to a function $\Omega \in \mathbf{X}^{(2)}$.

Due to lack of space, we skip the proof (which is not unduly different from the above) for the case when $J = K$, since it is similar to the proof analyticity discussed in §6.5.4 of (Stenger, 1995b).

Separation of Variables. Let G denote the "Laplace transform" of $\mathcal{G}$, i.e.,

$$G(s\,,\,\sigma) = \int_0^\infty \int_0^\infty \mathcal{G}(x,y)\,\exp\left(-\frac{x}{s} - \frac{y}{\sigma}\right)\,dx\,dy\,. \tag{3.24}$$

It may be shown (Stenger, 2006), that this function G is explicitly given by

$$\begin{aligned} G(u,v) &= \int_0^\infty \int_0^\infty \exp\left(-\frac{x}{u} - \frac{y}{v}\right)\,\mathcal{G}(x,y)\,dx\,dy \\ &= \left(\frac{1}{u^2} + \frac{1}{v^2}\right)^{-1} \cdot \\ &\quad \cdot \left(-\frac{1}{4} + \frac{1}{2\pi}\left(\frac{v}{u}\,(\gamma - \log(v)) + \frac{u}{v}\,(\gamma - \log(u))\right)\right) \end{aligned} \tag{3.25}$$

The Case of a Rectangular Region $\mathcal{B}$. Let us first consider the case of $\mathcal{B} = (0,1)\times(0,1)$.

For purposes of indefinite integration on $(0,1)$, we take $\varphi(z) = \log(z/(1-z))$ and then select the basis $\mathbf{w}$ as defined in (2.9)–(2.10), and in the notation of Definition 2.2, we select N_x, h_x, N_y, h_y, Sinc points

$$\left\{z_j^{(1)}\right\}_{-N_x}^{N_x}, \quad \left\{z_k^{(2)}\right\}_{-N_y}^{N_y},$$

and we also set $m_x = 2\,N_x + 1$, and $m_y = 2\,N_y + 1$. We use the notation of (2.21) for integration with respect to x, $A^+ = h\,I^{(-1)}D(1/\varphi')$, and $A^- = h\,(I^{(-1)})^T D(1/\varphi')$, and similarly for integration with respect to y, in which case we replace $A^\pm$ by $B^\pm$. Let us diagonalize these matrices, in the form $A^+ = X_+\,S\,X_+^{-1}$, $A^- = X_-\,S\,X_-^{-1}$, $B^+ = Y_+\,T\,Y_+^{-1}$, and $B^- = Y_-\,T\,Y_-^{-1}$. Here S and T are diagonal matrices, with $S = \text{diag}\,(s_{-N_x}\,,\,\ldots\,,\,s_{N_x})$ and similarly, with $T = \text{diag}\left(\sigma_{-N_y}\,,\,\ldots\,,\,\sigma_{N_y}\right)$. For convenience, we also set $X_\pm^{-1} = Xi_\pm$ and $Y_\pm^{-1} = Yi_\pm$.

The above integral expression (3.23) for Ω_{JK} can be written as a sum of four indefinite product integrals in the form

$$w = \left(\int_0^x \int_0^y + \int_0^x \int_y^1 + \int_x^1 \int_0^y + \int_x^1 \int_y^1\right)\,G\,g d\xi\,d\eta. \tag{3.26}$$

Each of these four product integrals can be evaluated via a repetition of the one dimensional formula for p (and/or q) defined in (2.21), e.g., if F denotes the "Laplace transform" of f, then, with an operator $\mathbf{V}$ defined as in (2.8), set $\mathbf{V}p = \mathbf{p}$ and $Vg = \mathbf{g}$ so that $\mathbf{p} = X^+\,F(S)\,Xi^+\,\mathbf{g}$, where $F(S) = \text{diag}\,(s_{-N_x}\,,\,\ldots\,,\,s_{N_x})$. Thus, for example, letting F and G denote the $m_x \times m_y$ matrices, setting $F = \left[f\left(z_j^{(1)}\,,\,z_k^{(2)}\right)\right]$ and $G = [G(s_j\,,\,\sigma_k)]$, respectively, and by applying this one dimensional procedure to the above integral $\int_x^1 \int_0^y$ we get an $m_x \times m_y$ matrix $W = [W_{j,k}]$, such that $W_{j,k} \approx \int_{z_j^{(1)}}^1 \int_0^{z_k^{(2)}}$ that is given by the *Matlab* statement

```
W = X^-*(G.*(Xi^-*G*(Y^+).'))*(Yi^+).';
```

Indeed, in view of (3.22) above, we can approximate the function $w(x, y)$ of (3.26) at the Sinc points $(x, y) = \left((z_j^{(1)}, z_k^{(2)}\right)$ as a sum of four such statements. That is, if the matrix $W = [W_{j,k}]$ is the sum of four such statements, then we get the accurate approximation $w\left(z_j^{(1)}, z_k^{(2)}\right) \approx W_{j,k}$.

The Case of a Curvilinear $\mathcal{B}$. We consider here the case when $\mathcal{B} = \bigcup_{J=1}^n \mathcal{B}_J$ is a curvilinear region, with each $\mathcal{B}_J$ of the form of (3.5). In this more complicated situation, we proceed as follows:

We transform each of these regions $\mathcal{B}_J$ to the unit square, via use of transformations (and possibly, also rotations) of the form (3.5). For sake of simplifying our argument, we also take $N_x = N_y = N$, $h_x = h_y = h$, and $m = m_x = m_y$, so that the Sinc grid takes he form $(\xi_i, \eta_j) = (z_i, z_j)$, where $\{z_j\}_{-N}^N$ are Sinc points on $(0, 1)$.

If $\bar{r}_{ij} = \bar{r}(z_i, z_j) \in \mathcal{B}_J$ and we are integrating over $\mathcal{B}_K$, with $K \neq J$, we simply use repeated Sinc quadrature to approximate each of the $n - 1$ integrals

$$P_{JK}(\bar{r}_{ij}) = \int\int_{\mathcal{B}_K} H(\bar{r}_{ij}, \bar{\rho}) d\bar{\rho} \tag{3.27}$$

for $K = 1, \ldots, n$, $K \neq J$, where H is the integral in (3.22), after transformation of the variable $\bar{r}$ over (a possibly rotated) $\mathcal{B}_J$ and the variable $\bar{r}'$ over (a possibly rotated) $\mathcal{B}_K$.

Let us now consider the case when $K = J$. The double integral (3.30) taken over the region $\mathcal{B}_K$ can again be expressed in terms of four indefinite integrals, all of which can be dealt with via the same procedure. For example, one such integral is

$$w(x, y) = \int\int_{a_1}^x \int_{a_2(x')}^y G(x - x', y - y') g(x', y') dy' dx'. \tag{3.28}$$

The argument that $w \in \mathbf{X}^{(2)}$ on $\mathcal{B}_K^{(2)}$ whenever $g \in \mathbf{X}^{(2)}$ on $\mathcal{B}_K^{(2)}$ is similar to that in §6.5 of (Stenger, 1995b), and is omitted here for sake of space.

By applying the transformation (3.7) we arrive at the integral

$$W(\xi, \eta) = \\ \int_0^\xi \int_0^\eta G(\alpha(\xi - \xi'), \beta(\eta - \eta')) g(x', y') \alpha \beta \, d\eta' \, d\xi' .b \tag{3.29}$$

where, by (3.7), $x' = a_1 + \alpha\,\xi'$ and $y' = a_2(x') + \beta\,\eta'$. We now use the usual two dimensional "Laplace transform" of the Green's function, namely,

$$\hat{G}(s,\sigma) = \int_0^\infty \int_0^\infty G(x,y)\, e^{-x/s - y/\sigma}\, dy\, dx\,. \qquad (3.30)$$

In terms of this notation, we thus get

$$\int_0^\infty \int_0^\infty G(\alpha\,\xi\,,\ \beta\,\eta)\, e^{-\xi/s-\eta/\sigma} \alpha\,\beta\, d\xi\, d\eta = \hat{G}(\alpha\, s,\ \beta\,\sigma) \qquad (3.31)$$

Upon fixing the variable η, we see that the inner integral with respect to ξ is of the form $r(x)$ in (2.22) above. By approximating this via use of (2.23) and then similarly approximating the integral with respect to η, we get the approximation

$$\begin{aligned} & W(\xi_m\,, \eta_n) \\ & \equiv \mathrm{r}(x_m, y_n) \approx \sum_{i=-M_1}^{N_1} x_{mi} \sum_{j=M_1}^{N_1} x^{ij} \sum_{k=-M_2}^{N_2} y_{n,k} \sum_{\ell=-M_2}^{N_2} y^{k\ell}\,. \\ & \quad \cdot \hat{G}(\alpha\, s_i\,,\ \beta(\xi_j)\,\sigma_k)\, g(\alpha\,\xi_j\,,\ \beta(\xi_j)\,\eta_\ell)\,. \end{aligned} \qquad (3.32)$$

We have thus been able to approximate the integral (3.29) via one dimensional methods, using the "Laplace transform" of the free space Green's function. Moreover the approximation of each of the other three indefinite integral expression approximations of the integral over $\mathcal{B}_K^{(2)}$ proceeds in exactly the same manner.

3.3 Other PDE

The discussions of these, e.g., the two dimensional Helmholtz equation, the two dimensional wave and heat problems is similar, but is omitted due to space. See (Stenger, 2006) for details of these as well as discussions of PDE in more than two dimensions.

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International Journal of Applied Mathematics & Statistics
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

IJAMAS: September 2007, Vol. 10, No. S07, 116-131

Fourier Type Analysis and Applications to Quantum Mechanics

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ABSTRACT

We discuss Fourier type analysis originating from quantum mechanics. The usual Fourier transform is an example of our Fourier type analysis. Our Fourier type analysis is suitable for differential operators in bounded or unbounded domains with variable coefficients. Here some variable coefficients are singular. We construct an integral transform U, which is a generalized Fourier transform. We define spaces of Sobolev type using our transform, and show an embedding theorem for each space. Our embedding theorem is a generalization of the Sobolev embedding theorem. We apply our results both to partial differential equations in bounded or unbounded domains with singular variable coefficients and to quantum mechanics.

Keywords: Fourier type analysis, integral transform, a generalized Fourier transform, space of Sobolev type, embedding theorem, partial differential equation in bounded or unbounded domains with singular variable coefficients, quantum mechanics.

2000 Mathematics Subject Classification: 42B10, 44A20, 46E35, 81V99.

1 Introduction

Let $-\infty \le a_j < b_j \le \infty \quad (j = 1, 2, \ldots, d)$ and let f_j be a diffeomorphism of (a_j, b_j) onto $\mathbb{R}$:

$$\xi_j = f_j(x_j), \qquad \xi_j \in \mathbb{R}, \qquad x_j \in (a_j, b_j) \qquad (j = 1, 2, \ldots, d)$$

with

$$f_j(c_j) = 0, \qquad a_j < c_j < b_j \qquad (j = 1, 2, \ldots, d).$$

Set

$$\widetilde{x}_j = f_j^{-1}(-\xi_j)$$

and set

$$\Omega = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d), \quad x = (x_1, x_2, \ldots, x_d) \in \Omega, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d.$$

We deal with the following operator with a singular variable coefficient:

$$\mathcal{D}_j = \frac{1}{f_j'}\frac{\partial}{\partial x_j} - \frac{f_j''}{2\left(f_j'\right)^2} - q_j \frac{\sqrt{|f_j'|}}{f_j} R_j \frac{1}{\sqrt{|f_j'|}}, \qquad q_j > -\frac{1}{2}. \tag{1.1}$$

Here the reflection R_j is given by

$$R_j g(\xi_1, \ldots, \xi_{j-1}, \xi_j, \xi_{j+1}, \ldots, \xi_d) = g(\xi_1, \ldots, \xi_{j-1}, -\xi_j, \xi_{j+1}, \ldots, \xi_d),$$

where g is a function of ξ. Hence

$$\begin{aligned} & R_j v(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d) \\ = & R_j v\left(f_1^{-1}(\xi_1), \ldots, f_{j-1}^{-1}(\xi_{j-1}), f_j^{-1}(\xi_j), f_{j+1}^{-1}(\xi_{j+1}), \ldots, f_d^{-1}(\xi_d)\right) \\ = & v\left(f_1^{-1}(\xi_1), \ldots, f_{j-1}^{-1}(\xi_{j-1}), f_j^{-1}(-\xi_j), f_{j+1}^{-1}(\xi_{j+1}), \ldots, f_d^{-1}(\xi_d)\right) \\ = & v(x_1, \ldots, x_{j-1}, \widetilde{x}_j, x_{j+1}, \ldots, x_d). \end{aligned}$$

The expression for our operator then becomes

$$\begin{aligned} \mathcal{D}_j u(x) = & \frac{1}{f_j'(x_j)} \frac{\partial u}{\partial x_j}(x) - \frac{f_j''(x_j)}{2 f_j'(x_j)^2} u(x) \\ & - q_j \frac{\sqrt{\left|f_j'(x_j)\right|}\, u(x_1, \ldots, x_{j-1}, \widetilde{x}_j, x_{j+1}, \ldots, x_d)}{f_j(x_j)\sqrt{\left|f_j'(\widetilde{x}_j)\right|}}. \end{aligned}$$

Remark 1.1. Our operator is a linear differential operator *in a bounded or unbounded domain* Ω. Moreover, its coefficients are *variable coefficients, and the last one is singular* since $f_j(c_j) = 0$.

We denote by f_j the multiplication by f_j, and regard it as an operator in $L^2(\Omega)$. We also denote by y_j the multiplication by y_j $(y_j \in \mathbb{R})$, and regard it as an operator in $L^2(\mathbb{R}^d)$. We construct an integral transform U that transforms $\mathcal{D}_j$ into the multiplication by iy_j $(i = \sqrt{-1})$ for each j $(j = 1, \ldots, d)$. Our transform is associated both with $\mathcal{D}_j$ and with f_j, and they satisfy Wigner's commutation relations [31] in quantum mechanics:

$$\{\mathcal{D}_j, [f_j, \mathcal{D}_j]\} = -2\mathcal{D}_j, \qquad \{f_j, [f_j, \mathcal{D}_j]\} = -2f_j,$$

where $\{A, B\} = AB + BA$. So our Fourier type analysis originates from quantum mechanics. Using our transform U we define spaces of Sobolev type, and show an embedding theorem for each space. Our embedding theorem is a generalization of the Sobolev embedding theorem. We apply our results both to partial differential equations *in bounded or unbounded domains with singular variable coefficients* and to quantum mechanics.

We now give an example. Let $a_j = -\infty$, $b_j = \infty$, $f_j(x_j) = x_j$ and $q_j = 0$ for all $j = 1, 2, \ldots, d$. Then, by (1.1),

$$\mathcal{D}_j = \frac{\partial}{\partial x_j} \qquad (j = 1, 2, \ldots, d),$$

and hence our transform U reduces to the Fourier transform in this case (see Remark 3.1 below). Therefore, our transform can be regarded as a generalized Fourier transform, and the Fourier transform is an example of our Fourier type analysis.

For $n_j = 0, 1, 2, 3, \ldots$ $(j = 1, 2, \ldots, d)$, set (cf. [12, [(4.31)]OKa and [13, (23.80)]OKa2)

$$\begin{cases} u_{2n_j}^{(j)}(x_j) = K_{n_j}^{q_j+\frac{1}{2}} \sqrt{\left|f_j'(x_j)\right|}\, |f_j(x_j)|^{q_j}\, L_{n_j}^{q_j-\frac{1}{2}}(f_j(x_j)^2) \exp\left(-\frac{f_j(x_j)^2}{2}\right), \\ u_{2n_j+1}^{(j)}(x_j) = K_{n_j}^{q_j+\frac{3}{2}} \sqrt{\left|f_j'(x_j)\right|}\, f_j(x_j)\, |f_j(x_j)|^{q_j}\, L_{n_j}^{q_j+\frac{1}{2}}(f_j(x_j)^2) \exp\left(-\frac{f_j(x_j)^2}{2}\right). \end{cases}$$

Here $K_n^\nu = (-1)^n\sqrt{n!/\Gamma(n+\nu)}$ with Γ the gamma function, and L_n^ν is a generalized Laguerre polynomial. Set

$$u_{n_1 n_2 \dots n_d}(x) = \prod_{j=1}^{d} u_{n_j}^{(j)}(x_j)$$

and let V be the set of finite linear combinations of $u_{n_1 n_2 \dots n_d}$'s. The fact that the set $\{u_n^{(j)}\}_{n=0}^\infty$ is a complete orthonormal set of $L^2(a_j,\, b_j)$ [18] immediately implies the following.

Lemma 1.1. *The set $\{u_{n_1 n_2 \dots n_d}\}_{n_1,\, n_2,\, \dots,\, n_d}$ is a complete orthonormal set of $L^2(\Omega)$. Consequently, V is dense in $L^2(\Omega)$.*

We denote by $(\,\cdot\,,\,\cdot\,)_{L^2(\Omega)}$ the inner product of $L^2(\Omega)$:

$$(u_1,\, u_2)_{L^2(\Omega)} = \int_\Omega u_1(x)\,\overline{u_2(x)}\,dx, \quad u_1,\, u_2 \in L^2(\Omega),$$

and by $\|\cdot\|_{L^2(\Omega)}$ the norm $\|\cdot\|_{L^2(\Omega)} = \sqrt{(\,\cdot\,,\,\cdot\,)_{L^2(\Omega)}}$. We also denote by $(\,\cdot\,,\,\cdot\,)_{L^2(\mathbb{R}^d)}$ the inner product of $L^2(\mathbb{R}^d)$, and by $\|\cdot\|_{L^2(\mathbb{R}^d)}$ the norm $\|\cdot\|_{L^2(\mathbb{R}^d)} = \sqrt{(\,\cdot\,,\,\cdot\,)_{L^2(\mathbb{R}^d)}}$.
For simplicity we assume that for all $j = 1,\, 2,\, \dots,\, d$,

$$f_j'(x_j) > 0.$$

2 Essential selfadjointness of the operator $-i\mathcal{D}_j$

In this section we show essential selfadjointness of the operator $-i\mathcal{D}_j$ in $L^2(\Omega)$.
Set

$$\Omega' = \Omega \setminus [\{x \in \Omega : x_1 = c_1\} \cup \{x \in \Omega : x_2 = c_2\} \cup \cdots \cup \{x \in \Omega : x_d = c_d\}]. \tag{2.1}$$

Let $\phi \in C_0^\infty(\Omega')$. An integration by parts gives

$$\begin{aligned}
&\left(u_{n_1 \dots n_{j-1} n_j n_{j+1} \dots n_d},\, -i\mathcal{D}_j\phi\right)_{L^2(\Omega)} \\
=\;& \left(-i\mathcal{D}_j u_{n_1 \dots n_{j-1} n_j n_{j+1} \dots n_d},\, \phi\right)_{L^2(\Omega)} \\
=\;& \left(-i\sqrt{\frac{n_j - 1 + a_{n_j}}{2}}\, u_{n_1 \dots n_{j-1}(n_j-1) n_{j+1} \dots n_d} + i\sqrt{\frac{n_j + b_{n_j}}{2}}\, u_{n_1 \dots n_{j-1}(n_j+1) n_{j+1} \dots n_d},\, \phi\right)_{L^2(\Omega)}.
\end{aligned}$$

Here, $a_{n_j} = 1$ and $b_{n_j} = 2E_0^{(j)}$ when n_j is an even number, while $a_{n_j} = 2E_0^{(j)}$ and $b_{n_j} = 1$ when n_j is an odd number, and $E_{n_j}^{(j)} = n_j + q_j + \frac{1}{2}$.
We then have the following.

Lemma 2.1. *Let $-i\mathcal{D}_j$ denote the adjoint operator of $(-i\mathcal{D}_j) \restriction C_0^\infty(\Omega')$. Then*

$$\begin{aligned}
&-i\mathcal{D}_j u_{n_1 \dots n_{j-1} n_j n_{j+1} \dots n_d} \\
=\;& -i\sqrt{\frac{n_j - 1 + a_{n_j}}{2}}\, u_{n_1 \dots n_{j-1}(n_j-1) n_{j+1} \dots n_d} + i\sqrt{\frac{n_j + b_{n_j}}{2}}\, u_{n_1 \dots n_{j-1}(n_j+1) n_{j+1} \dots n_d}.
\end{aligned}$$

Lemma 2.2. *Let $N = 1, 2, 3, \dots$. Then*

$$\left\|(-i\mathcal{D}_j)^N u_{n_1 \dots n_j \dots n_d}\right\|_{L^2(\Omega)} \le \sqrt{2^{N-1} E_{n_j}^{(j)} E_{n_j+1}^{(j)} \cdots E_{n_j+N-1}^{(j)}} \qquad (n_1,\, \dots,\, n_d = 0, 1, 2, \dots).$$

Proof. We show the lemma by induction on N. A straightforward calculation gives

$$\left\|-i\mathcal{D}_j u_{\dots n_j \dots}\right\|^2_{L^2(\Omega)} = \frac{n_j - 1 + a_{n_j}}{2} + \frac{n_j + b_{n_j}}{2} = E^{(j)}_{n_j},$$

which implies that the lemma is true for $N = 1$. Suppose that it is true for N. Since

$$(-i\mathcal{D}_j)^{N+1} u_{\dots n_j \dots} = -i\sqrt{\frac{n_j - 1 + a_{n_j}}{2}}\,(-i\mathcal{D}_j)^N u_{\dots (n_j-1)\dots} + i\sqrt{\frac{n_j + b_{n_j}}{2}}\,(-i\mathcal{D}_j)^N u_{\dots (n_j+1)\dots},$$

it follows that

$$\begin{aligned} &\left\|(-i\mathcal{D}_j)^{N+1} u_{\dots n_j \dots}\right\|^2_{L^2(\Omega)} \\ \leq\ & (n_j - 1 + a_{n_j})\left\|(-i\mathcal{D}_j)^N u_{\dots (n_j-1)\dots}\right\|^2_{L^2(\Omega)} + (n_j + b_{n_j})\left\|(-i\mathcal{D}_j)^N u_{\dots (n_j+1)\dots}\right\|^2_{L^2(\Omega)} \\ \leq\ & 2^N E^{(j)}_{n_j} E^{(j)}_{n_j+1} \cdots E^{(j)}_{n_j+N}. \end{aligned}$$

Thus the lemma is true for $N+1$, and hence for $N = 1, 2, 3, \dots$. □

Proposition 2.3. *Let $-i\mathcal{D}_j$ be as in Lemma 2.1. Then the operator $(-i\mathcal{D}_j) \upharpoonright V$ is essentially selfadjoint.*

Proof. It follows immediately from Lemma 2.1 that $(-i\mathcal{D}_j) \upharpoonright V$ is symmetric. By Lemma 2.2, every $u_{n_1 \dots n_d}$, and hence every element of V is an analytic vector for $(-i\mathcal{D}_j) \upharpoonright V$. Nelson's analytic vector theorem [11] (see also Goldstein [5, p.73]G or Reed and Simon [19, pp.200–203]RS) thus proves its essential selfadjointness. □

3 An integral transform

In this section we construct an integral transform that transforms our operator $\mathcal{D}_j$ into the multiplication by iy_j $(y_j \in \mathbb{R}, i = \sqrt{-1})$ for each j $(j = 1, \dots, d)$.
Set (cf. [12, (3.22)]OKa and [13, (23.39)]OKa2)

$$\varphi^{(j)}(y_j, x_j) = \frac{\sqrt{|y_j f_j(x_j) f_j'(x_j)|}}{2}\left\{ J_{q_j-1/2}(|y_j f_j(x_j)|) + i\,\mathrm{sgn}(y_j f_j(x_j)) J_{q_j+1/2}(|y_j f_j(x_j)|)\right\},$$

where J_ν denotes the Bessel function of the first kind. Set

$$\varphi(y, x) = \prod_{j=1}^{d} \varphi^{(j)}(y_j, x_j),$$

where $y = (y_1, y_2, \dots, y_d) \in \mathbb{R}^d$. We consider the following integral transform:

$$Uu(y) = \int_\Omega \overline{\varphi(y, x)}\, u(x)\, dx, \qquad u \in V.$$

By a straightforward calculation,

$$Uu_{n_1 \dots n_d}(y) = \prod_{j=1}^{d} \int_{a_j}^{b_j} \overline{\varphi^{(j)}(y_j, x_j)}\, u^{(j)}_{n_j}(x_j)\, dx_j = (-i)^{n_1 + \dots + n_d} \underline{u}_{n_1 \dots n_d}(y), \tag{3.1}$$

where

$$\underline{u}_{n_1\dots n_d}(y) = \prod_{j=1}^{d} \underline{u}_{n_j}^{(j)}(y_j), \qquad \underline{u}_{n_j}^{(j)}(y_j) = \frac{u_{n_j}^{(j)}\left(f_j^{-1}(y_j)\right)}{\sqrt{f_j'\left(f_j^{-1}(y_j)\right)}}.$$

Let $\underline{V}$ be the set of finite linear combinations of $\underline{u}_{n_1\dots n_d}$'s. In [26, Lemma 1.1]Wa7 it is shown that the set $\{\underline{u}_{n_1\dots n_d}\}_{n_1,\dots,n_d}$ is a complete orthonormal set of $L^2(\mathbb{R}^d)$ and that $\underline{V}$ is dense in $L^2(\mathbb{R}^d)$ as a result. Combining this fact with (3.1) yields the following.

Lemma 3.1. *The transform U is a linear operator from V onto $\underline{V}$, and satisfies*

$$(Uu_1, Uu_2)_{L^2(\mathbb{R}^d)} = (u_1, u_2)_{L^2(\Omega)}, \qquad u_1, u_2 \in V.$$

Consequently, U is one-to-one. The inverse U^{-1} is expressed as

$$U^{-1}v(x) = \int_{\mathbb{R}^d} \varphi(y, x)v(y)\, dy, \qquad v \in \underline{V}.$$

This immediately implies the following.

Theorem 3.2. *The transform U is a unitary operator from $L^2(\Omega)$ to $L^2(\mathbb{R}^d)$.*

Lemma 2.1 together with (3.1) gives the following.

Lemma 3.3.

$$U\left(-i\mathcal{D}_j\right)U^*v = y_j v, \qquad v \in \underline{V}.$$

Note that the multiplication operator y_j is essentially selfadjoint on $\underline{V}$ (see [26, Theorem 2.6]Wa7).

Proposition 3.4. *Let $-i\mathcal{D}_j$ be the selfadjoint operator in $L^2(\Omega)$ given by Proposition 2.3 and let y_j be the selfadjoint operator in $L^2(\mathbb{R}^d)$. Then*

$$U\left(-i\mathcal{D}_j\right)U^* = y_j.$$

Proof. We first show $U\left(-i\mathcal{D}_j\right)U^* \supset y_j$. As mentioned just above, the multiplication operator y_j is essentially selfadjoint on $\underline{V}$. So, for $v \in D(y_j)$, there is a sequence $\{v_n\}_n \subset \underline{V}$ such that

$$v_n \longrightarrow v, \quad y_j v_n \longrightarrow y_j v \quad \text{in} \quad L^2(\mathbb{R}^d).$$

Since U^* is continuous (see Theorem 3.2),

$$U^*v_n \longrightarrow U^*v, \quad U^*y_j v_n \longrightarrow U^*y_j v \quad \text{in} \quad L^2(\Omega).$$

Combining Proposition 2.3 with Lemma 3.3 yields

$$U^*v \in D(-i\mathcal{D}_j) \quad \text{and} \quad -i\mathcal{D}_j U^*v = U^*y_j v,$$

where $D(-i\mathcal{D}_j)$ is the domain of the selfadjoint operator $-i\mathcal{D}_j$. Hence $U\left(-i\mathcal{D}_j\right)U^* \supset y_j$. Since $U\left(-i\mathcal{D}_j\right)U^*$ and y_j are both selfadjoint operators, the result thus follows. □

A straightforward calculation gives the following.

Lemma 3.5.

$$U^*v(x) = \operatorname*{l.i.m.}_{\Lambda\to\infty} \int_{|y|\le\Lambda} \varphi(y,\, x)v(y)\, dy, \qquad v \in L^2(\mathbb{R}^d).$$

Remark 3.1. Let $a_j = -\infty$, $b_j = \infty$, $f_j(x_j) = x_j$ and $q_j = 0$ for all $j = 1, 2, \ldots, d$. Then

$$\mathcal{D}_j = \frac{\partial}{\partial x_j}, \qquad \varphi(y,\, x) = \frac{e^{i(y_1x_1+y_2x_2+\cdots+y_dx_d)}}{(2\pi)^{d/2}}, \qquad (y,\, x) \in \mathbb{R}^d \times \mathbb{R}^d \qquad (j = 1, 2, \ldots, d),$$

and hence our transform U reduces to the Fourier transform in this case by Lemma 3.5. Therefore, our transform can be regarded as a generalized Fourier transform.

Remark 3.2. We constructed our transform on the basis of the study of the Hankel transform. Kilbas and Borovco [7] considered a more general integral transform including the Hankel transform.

4 Spaces of Sobolev type

In this section we define spaces of Sobolev type using our generalized Fourier transform and discuss some of their properties.

Definition 4.1. For $\nu \ge 0$,

$$\mathcal{H}^\nu(\Omega) = \left\{ u \in L^2(\Omega) : \int_{\mathbb{R}^d} (1 + |y|^2)^\nu\, |Uu(y)|^2\, dy < \infty \right\}.$$

A straightforward calculation gives that each $\mathcal{H}^\nu(\Omega)$ is a Hilbert space with inner product

$$(u_1,\, u_2)_{\mathcal{H}^\nu(\Omega)} = \int_{\mathbb{R}^d} (1 + |y|^2)^\nu\, Uu_1(y)\, \overline{Uu_2(y)}\, dy, \qquad u_1,\, u_2 \in \mathcal{H}^\nu(\Omega)$$

and norm $|\, u\,|_{\mathcal{H}^\nu(\Omega)} = \sqrt{(u,\, u)_{\mathcal{H}^\nu(\Omega)}}\,.$

Remark 4.1. Let $a_j = -\infty$, $b_j = \infty$, $f_j(x_j) = x_j$ and $q_j = 0$ for all $j = 1, 2, \ldots, d$. Our transform U then reduces to the Fourier transform as mentioned above, and hence $\mathcal{H}^\nu(\Omega)$ to the usual Sobolev space $H^\nu(\mathbb{R}^d)$ in this case.

Definition 4.1 immediately implies the following.

Corollary 4.1. (A) $\mathcal{H}^0(\Omega) = L^2(\Omega)$.

(B) $\mathcal{H}^{\nu'}(\Omega) \subset \mathcal{H}^\nu(\Omega), \qquad \nu' \ge \nu.$

(C) $|\, u\,|_{\mathcal{H}^\nu(\Omega)} \le |\, u\,|_{\mathcal{H}^{\nu'}(\Omega)}, \qquad u \in \mathcal{H}^{\nu'}(\Omega), \qquad \nu' \ge \nu.$

(D) *Let $|y|^\nu$ be the selfadjoint, multiplication operator and let $D(|y|^\nu)$ be its domain. Then* $U\mathcal{H}^\nu(\Omega) = D(|y|^\nu)$.

We need the following to prove our embedding theorem in the next section.

Lemma 4.2. *Suppose $\nu > \frac{d}{2}$. Let $k_j \ge 0$ $(j = 1, 2, \ldots, d)$ and let $0 \le k_1+k_2+\cdots+k_d < \nu-\frac{d}{2}$. Then, for $u \in \mathcal{H}^\nu(\Omega)$,*

$$|y_1|^{k_1}|y_2|^{k_2}\cdots|y_d|^{k_d}Uu \in L^1(\mathbb{R}^d).$$

Proof. By the Schwarz inequality,

$$\int_{\mathbb{R}^d} |y_1|^{k_1}\cdots|y_d|^{k_d}\, |Uu(y)|\, dy \le \sqrt{\int_{\mathbb{R}^d} \frac{|y_1|^{2k_1}\cdots|y_d|^{2k_d}}{(1+|y|^2)^\nu}\, dy}\; |\, u\,|_{\mathcal{H}^\nu(\Omega)} < \infty.$$

The lemma follows. □

5 An embedding theorem of Sobolev type

As is well known, the usual Sobolev embedding theorem tells us only about smoothness of each element. On the other hand, our embedding theorem tells us both about smoothness of u $(u \in \mathcal{H}^\nu(\Omega))$ and about continuity of $u/(f_1^{\beta_1}\cdots f_d^{\beta_d})$, as is shown below. Here, β_j are nonnegative integers. So our embedding theorem is a generalization of the Sobolev embedding theorem.

Definition 5.1. For $\beta_j = 0, 1, 2, \dots$ $(j = 1, \dots, d)$,

$$S^{\beta}_{f_1\dots f_d}(\Omega) = \left\{ u \in C(\Omega) : \frac{u}{f_1^{\beta_1}\cdots f_d^{\beta_d}} \in C(\Omega),\ \beta = \beta_1 + \cdots + \beta_d \right\}.$$

Remark 5.1. If $u \in S^{\beta}_{f_1\dots f_d}(\Omega)$, then all the functions $u/(f_1^{\beta_1}\cdots f_d^{\beta_d})$ $(\beta = \beta_1 + \cdots + \beta_d)$ are continuous on Ω.

The following is our embedding theorem.

Theorem 5.1. *Let $q_j \geq 0$ for all $j = 1, \dots, d$ and let $\nu > d/2$. Then*

$$\mathcal{H}^\nu(\Omega) \subset C^\alpha(\Omega) \cap S^{\beta}_{f_1\dots f_d}(\Omega),$$

where α and β are nonnegative integers satisfying

$$\alpha < \min(\gamma_1, \dots, \gamma_d, \nu - d/2), \qquad \beta < \min(\delta_1, \dots, \delta_d, \nu - d/2)$$

with

$$\gamma_j = \begin{cases} \infty & (q_j = 0, 2, 4, 6, \dots), \\ q_j & (otherwise), \end{cases} \qquad \delta_j = \begin{cases} q_j + 1 & (q_j = 0, 2, 4, 6, \dots), \\ q_j & (otherwise). \end{cases}$$

Proof. Let $u \in \mathcal{H}^\nu(\Omega)$. By Lemma 3.5,

$$u(x) = \lim_{n\to\infty} \int_{|y|\leq n} \varphi(y, x) Uu(y)\, dy \qquad (\text{a.e. } x \in \Omega).$$

Here, n are positive integers. Lemma 4.2 implies that the integrand just above is integrable on $\mathbb{R}^d$. Therefore,

$$u(x) = \sqrt{f_1'(x_1)\cdots f_d'(x_d)} \int_{\mathbb{R}^d} \phi_1(y_1 f_1(x_1)) \cdots \phi_d(y_d f_d(x_d))\, Uu(y)\, dy \qquad (\text{a.e. } x \in \Omega),$$

where

$$\phi_j(y_j f_j(x_j)) = \frac{\sqrt{|y_j f_j(x_j)|}}{2} \left\{ J_{q_j - 1/2}(|y_j f_j(x_j)|) + i\,\mathrm{sgn}(y_j f_j(x_j)) J_{q_j + 1/2}(|y_j f_j(x_j)|) \right\}.$$

We denote the integral just above by $\widetilde{u}(x)$:

$$\widetilde{u}(x) = \int_{\mathbb{R}^d} \phi_1(y_1 f_1(x_1)) \cdots \phi_d(y_d f_d(x_d))\, Uu(y)\, dy.$$

In what follows we study smoothness of $\widetilde{u}$ together with continuity of $\widetilde{u}/(f_1^{\beta_1}\cdots f_d^{\beta_d})$.

Let

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}, \qquad \alpha = \alpha_1 + \cdots + \alpha_d.$$

We formally differentiate $\widetilde{u}$ α times:

$$D^\alpha \widetilde{u}(x) = \sum_{k_1=0}^{\alpha_1} \cdots \sum_{k_d=0}^{\alpha_d} \int_{\mathbb{R}^d} \left[\prod_{j=1}^{d} \left\{ F_{k_j}(x_j) \frac{\partial^{k_j} \phi_j}{\partial (y_j f_j)^{k_j}} (y_j f_j(x_j)) \right\} \right] y_1^{k_1} \cdots y_d^{k_d} Uu(y)\, dy.$$

Here, F_{k_j} are smooth functions of x_j. By Lemma 4.2, the integrand just above is integrable on $\mathbb{R}^d$. So $\widetilde{u}$ is α times differentiable. Moreover, the function

$$\xi \longmapsto \int_{\mathbb{R}^d} \left[\prod_{j=1}^{d} \left\{ \frac{\partial^{k_j} \phi_j}{\partial (y_j \xi_j)^{k_j}} (y_j \xi_j) \right\} \right] y_1^{k_1} \cdots y_d^{k_d} Uu(y)\, dy$$

is continuous on $\mathbb{R}^d$ (see [26, Lemma 5.4]Wa7). Here, ξ and ξ_j are those in Introduction. Thus $D^\alpha \widetilde{u}$ is continuous on Ω.

The function

$$\xi \longmapsto \int_{\mathbb{R}^d} \frac{\phi_1(y_1\xi_1)}{(y_1\xi_1)^{\beta_1}} \cdots \frac{\phi_d(y_d\xi_d)}{(y_d\xi_d)^{\beta_d}} y_1^{\beta_1} \cdots y_d^{\beta_d} Uu(y)\, dy$$

is continuous on $\mathbb{R}^d$ (see [26, Lemma 5.6]Wa7), and hence $\widetilde{u}/(f_1^{\beta_1} \cdots f_d^{\beta_d})$ is continuous on Ω. □

Remark 5.2. Let $a_j = -\infty$, $b_j = \infty$, $f_j(x_j) = x_j$ and $q_j = 0$ for all $j = 1, 2, \ldots, d$. As mentioned above, our transform U and our space $\mathcal{H}^\nu(\Omega)$ then reduce to the Fourier transform and to the Sobolev space $H^\nu(\mathbb{R}^d)$, respectively. Moreover, α is a nonnegative integer satisfying $\alpha < \nu - d/2$, and $\beta = 0$ in this case. Our embedding theorem thus reduces to the usual Sobolev embedding theorem:

$$H^\nu(\mathbb{R}^d) \subset C^\alpha(\mathbb{R}^d), \qquad \alpha < \nu - d/2.$$

So our embedding theorem is a generalization of the Sobolev embedding theorem. See also [23, 24, 26, 18] for related material.

6 Applications to partial differential equations

Our Fourier type analysis can treat partial differential equations *in bounded or unbounded domains with singular variable coefficients*, where our operators $\mathcal{D}_j$ appear. For simplicity we confine ourselves to treating such partial differential equations that the operator

$$\Delta_{\mathcal{D}} = \sum_{j=1}^{d} \mathcal{D}_j^2 \tag{6.1}$$

appears. We assume that $q_j \geq 0$ for all $j = 1, \ldots, d$.

Remark 6.1. The operator $\Delta_{\mathcal{D}}$ reduces to the usual Laplacian Δ when $a_j = -\infty$, $b_j = \infty$, $f_j(x_j) = x_j$ and $q_j = 0$ for all $j = 1, 2, \ldots, d$.

A straightforward calculation gives

$$
\begin{aligned}
&\mathcal{D}_j^2 u(x) \\
&= \frac{1}{f_j'(x_j)^2}\frac{\partial^2 u}{\partial x_j^2}(x) - \frac{2f_j''(x_j)}{f_j'(x_j)^3}\frac{\partial u}{\partial x_j}(x) + \frac{5f_j''(x_j)^2 - 2f_j'(x_j)f_j'''(x_j)}{4f_j'(x_j)^4}u(x) \\
&\quad - q_j \frac{\sqrt{f_j'(x_j)}}{f_j(x_j)\, f_j'(\widetilde{x}_j)^{3/2}}\frac{\partial u}{\partial x_j}(\ldots, x_{j-1}, \widetilde{x}_j, x_{j+1}, \ldots) \\
&\quad - q_j \frac{1}{f_j(x_j)\sqrt{f_j'(x_j)}}\frac{\partial}{\partial x_j}\left(\frac{u(\ldots, x_{j-1}, \widetilde{x}_j, x_{j+1}, \ldots)}{\sqrt{f_j'(\widetilde{x}_j)}}\right) \\
&\quad + q_j \frac{\sqrt{f_j'(x_j)}}{f_j(x_j)}\left\{\frac{1}{f_j(x_j)} + \frac{f_j''(\widetilde{x}_j)}{2f_j'(\widetilde{x}_j)^2}\right\}\frac{u(\ldots, x_{j-1}, \widetilde{x}_j, x_{j+1}, \ldots)}{\sqrt{f_j'(\widetilde{x}_j)}} + q_j^2 \frac{u(x)}{f_j(x_j) f_j(\widetilde{x}_j)},
\end{aligned}
$$

where $\widetilde{x}_j = f_j^{-1}(-\xi_j)$ (see Introduction).

Definition 6.1. We fix j. For $|a_j|, |b_j| < \infty$, let $c_j = (a_j + b_j)/2$. Here, $c_j = f_j^{-1}(0)$. Set $(x_{jR} + x_j)/2 = c_j$, where $a_j < x_{jR} < b_j$. A function u on (a_j, b_j) (resp. on Ω) is said to be an even function with respect to $x_j = c_j$ if $u(x_{jR}) = u(x_j)$ (resp. if $u(\ldots, x_{j-1}, x_{jR}, x_{j+1}, \ldots) = u(x)$). A function u on (a_j, b_j) (resp. on Ω) is said to be an odd function with respect to $x_j = c_j$ if $u(x_{jR}) = -u(x_j)$ (resp. if $u(\ldots, x_{j-1}, x_{jR}, x_{j+1}, \ldots) = -u(x)$).

Suppose that f_j is an odd function with respect to $x_j = c_j$. Then $x_{jR} = \widetilde{x}_j$, and hence $f_j(\widetilde{x}_j) = -f_j(x_j)$ and $f_j'(\widetilde{x}_j) = f_j'(x_j)$. The expressions for $\mathcal{D}_j u$ and $\mathcal{D}_j^2 u$ therefore become somewhat simpler:

$$
\begin{aligned}
\mathcal{D}_j u(x) &= \frac{1}{f_j'(x_j)}\frac{\partial u}{\partial x_j}(x) - \frac{f_j''(x_j)}{2f_j'(x_j)^2}u(x) - q_j \frac{u(x_1, \ldots, x_{j-1}, \widetilde{x}_j, x_{j+1}, \ldots, x_d)}{f_j(x_j)}, \\
\mathcal{D}_j^2 u(x) &= \frac{1}{f_j'(x_j)^2}\frac{\partial^2 u}{\partial x_j^2}(x) - \frac{2f_j''(x_j)}{f_j'(x_j)^3}\frac{\partial u}{\partial x_j}(x) + \frac{5f_j''(x_j)^2 - 2f_j'(x_j)f_j'''(x_j)}{4f_j'(x_j)^4}u(x) \\
&\quad + q_j \frac{u(x_1, \ldots, x_{j-1}, \widetilde{x}_j, x_{j+1}, \ldots, x_d)}{f_j(x_j)^2} - q_j^2 \frac{u(x)}{f_j(x_j)^2}.
\end{aligned}
$$

Moreover, if u is an even function with respect to $x_j = c_j$, then

$$
u(x_1, \ldots, x_{j-1}, \widetilde{x}_j, x_{j+1}, \ldots, x_d) = u(x).
$$

Hence

$$
\begin{aligned}
\mathcal{D}_j u(x) &= \frac{1}{f_j'(x_j)}\frac{\partial u}{\partial x_j}(x) - \frac{f_j''(x_j)}{2f_j'(x_j)^2}u(x) - q_j \frac{u(x)}{f_j(x_j)}, \\
\mathcal{D}_j^2 u(x) &= \frac{1}{f_j'(x_j)^2}\frac{\partial^2 u}{\partial x_j^2}(x) - \frac{2f_j''(x_j)}{f_j'(x_j)^3}\frac{\partial u}{\partial x_j}(x) + \frac{5f_j''(x_j)^2 - 2f_j'(x_j)f_j'''(x_j)}{4f_j'(x_j)^4}u(x) \\
&\quad + \frac{q_j(1 - q_j)}{f_j(x_j)^2}u(x).
\end{aligned}
$$

The variable $\widetilde{x}_j$ disappears in the expressions for $\mathcal{D}_j u(x)$ and $\mathcal{D}_j^2 u(x)$. A similar argument holds for an odd function u.

Application 1. We first deal with the following problem in $L^2(\Omega)$ with singular variable coefficients:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \Delta_{\mathcal{D}} u(t, x), \quad t > 0, \quad x \in \Omega, \\ u(0, x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{6.2}$$

where $\Delta_{\mathcal{D}}$ is given by (6.1) and $u_0 \in L^2(\Omega)$. When $a_j = -\infty$, $b_j = \infty$, $f_j(x_j) = x_j$ and $q_j = 0$ for all $j = 1, 2, \ldots, d$, this problem reduces to the initial value problem for the usual heat equation. See Watanabe and Watanabe [22], and Watanabe [25] for related material. For applications of transform methods to partial differential equations, see e.g. Duffy [3].

Remark 6.2. If f_j is an odd function with respect to $x_j = c_j$ and if u_0 is an even function (resp. an odd function) with respect to $x_j = c_j$, then the solution $u(t, \cdot)$ of the problem is also an even function (resp. an odd function) with respect to $x_j = c_j$. So, in this case, the variable $\widetilde{x}_j$ disappears in (6.2).

Let us look for $u(t, \cdot) \in \mathcal{H}^2(\Omega)$ satisfying the problem (6.2). Since our transform U transforms $\Delta_{\mathcal{D}}$ with domain $\mathcal{H}^2(\Omega)$ into the multiplication operator $-|y|^2$ (Proposition 3.4), the operator $\Delta_{\mathcal{D}}$ with domain $\mathcal{H}^2(\Omega)$ generates an analytic semigroup $\{\exp(t\Delta_{\mathcal{D}}) : t > 0\}$ on $L^2(\Omega)$. Combining Corollary 4.1 with Theorem 5.1 thus yields the following.

Corollary 6.1. *Let $m = 1, 2, 3, \ldots$. For $u_0 \in L^2(\Omega)$, there is a unique solution*

$$u \in C\left([0, \infty); L^2(\Omega)\right) \cap C^1\left((0, \infty); \mathcal{H}^{2m}(\Omega)\right)$$

of the problem (6.2) *such that*

$$u(t, \cdot) = \exp(t\Delta_{\mathcal{D}})\, u_0 \in C^{\alpha}(\Omega) \cap S^{\beta}_{f_1 \cdots f_d}(\Omega),$$

where α and β are nonnegative integers satisfying

$$\alpha < \min(\gamma_1, \ldots, \gamma_d), \qquad \beta < \min(\delta_1, \ldots, \delta_d),$$

and γ_j, δ_j are those in Theorem 5.1.

We now try to write the solution in an explicit form. Let Ω' be as in (2.1). Suppose $u_0 \in C_0^{\infty}(\Omega')$ for simplicity. By Proposition 3.4, our transform U turns (6.2) into

$$\begin{cases} \dfrac{dv}{dt} = -|y|^2 v, \quad t > 0, \\ v(0) = U u_0, \end{cases}$$

where $v = Uu$. Therefore, $v(t) = e^{-t|y|^2}\, U u_0$. A straightforward calculation gives

$$u(t, x) = \int_{\mathbb{R}^d} \varphi(y, x) e^{-t|y|^2}\, U u_0(y)\, dy, \qquad U u_0(y) = \int_{\Omega} \overline{\varphi(y, \xi)} u_0(\xi)\, d\xi.$$

Combining Fubini's theorem with Lemma 6.2 just below then yields

$$\begin{aligned} u(t, x) \;=\; & \int_{\Omega} \frac{u_0(\xi)}{(4t)^d} \exp\left[-\frac{1}{4t} \sum_{j=1}^{d} \left\{f_j(x_j)^2 + f_j(\xi_j)^2\right\}\right] \times \\ & \times \sqrt{\left|f_1(x_1) f_1'(x_1) f_1(\xi_1) f_1'(\xi_1) \cdots f_d(x_d) f_d'(x_d) f_d(\xi_d) f_d'(\xi_d)\right|} \times \\ & \times \prod_{j=1}^{d} \left\{ I_{q_j - \frac{1}{2}}\left(\frac{|f_j(x_j) f_j(\xi_j)|}{2t}\right) + \operatorname{sgn}\left(f_j(x_j) f_j(\xi_j)\right) I_{q_j + \frac{1}{2}}\left(\frac{|f_j(x_j) f_j(\xi_j)|}{2t}\right) \right\} d\xi, \end{aligned} \tag{6.3}$$

where I_ν is a modified Bessel function.

Lemma 6.2.

$$\int_0^\infty e^{-a^2y^2}\, y\, J_\nu(py)\, J_\nu(qy)\, dy = \frac{1}{2\,a^2}\,\exp\left(-\frac{p^2+q^2}{4\,a^2}\right)\, I_\nu\left(\frac{pq}{2\,a^2}\right),$$

where $\Re(\nu) > -1$ *and* $|\arg a| < \pi/4$.

For a proof of this lemma, see e.g. [4, (23), p.51]E.

Corollary 6.3. *Suppose* $u_0 \in C_0^\infty(\Omega')$. *Let* u *be the solution of the problem* (6.2) *given by Corollary 6.1. Then the solution is explicitly given by* (6.3).

Application 2. We second deal with the following problem in $L^2(\Omega)$ with singular variable coefficients:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta_{\mathcal{D}} u(t,x), & t \in \mathbb{R}, \quad x \in \Omega, \\ u(0,x) = u_0(x), \quad \dfrac{\partial u}{\partial t}(0,x) = u_1(x), & x \in \Omega, \end{cases} \tag{6.4}$$

where $u_0 \in \mathcal{H}^2(\Omega)$ and $u_1 \in \mathcal{H}^1(\Omega)$. When $a_j = -\infty$, $b_j = \infty$, $f_j(x_j) = x_j$ and $q_j = 0$ for all $j = 1, 2, \dots, d$, this problem reduces to the initial value problem for the usual wave equation. Since $-\Delta_{\mathcal{D}}$ with domain $\mathcal{H}^2(\Omega)$ and $\sqrt{-\Delta_{\mathcal{D}}}$ with domain $\mathcal{H}^1(\Omega)$ are both nonnegative selfadjoint operators, the problem (6.4) is well-posed. Therefore, $u(t,\cdot) \in \mathcal{H}^2(\Omega)$. The following is an immediate consequence of Theorem 5.1.

Corollary 6.4. *Suppose* $u_0 \in \mathcal{H}^2(\Omega)$ *and* $u_1 \in \mathcal{H}^1(\Omega)$. *Then the problem* (6.4) *is well-posed. Moreover, if* $d < 4$, *then the solution satisfies*

$$u(t,\cdot) \in C^\alpha(\Omega) \cap S^\beta_{f_1 \dots f_d}(\Omega),$$

where α *and* β *are nonnegative integers satisfying*

$$\alpha < \min(\gamma_1, \dots, \gamma_d, 2 - d/2), \qquad \beta < \min(\delta_1, \dots, \delta_d, 2 - d/2),$$

and γ_j, δ_j *are those in Theorem 5.1.*

Application 3. We finally deal with the following problem in $L^2(\Omega)$ with singular variable coefficients. Given a $g \in L^2(\Omega)$ we look for a solution $u \in \mathcal{H}^2(\Omega)$ satisfying

$$\Delta_{\mathcal{D}} u(x) - \lambda^2 u(x) = g(x) \qquad \text{in} \quad \Omega, \tag{6.5}$$

where $\lambda > 0$.

Remark 6.3. When $a_j = -\infty$, $b_j = \infty$, $f_j(x_j) = x_j$ and $q_j = 0$ for all $j = 1, 2, \dots, d$, (6.5) reduces to the elliptic equation:

$$\Delta u(x) - \lambda^2 u(x) = g(x) \qquad \text{in} \quad \mathbb{R}^d.$$

Corollary 6.5. *For $g \in L^2(\Omega)$, there is a unique solution $u \in \mathcal{H}^2(\Omega)$ of the problem (6.5), and the estimate*

$$| u |_{\mathcal{H}^2(\Omega)} \leq C \| g \|_{L^2(\Omega)}$$

holds for some constant $C > 0$, independent of the solution u. Consequently, the solution u continuously depends on the data g. Moreover, if $d < 4$, then

$$u \in C^\alpha(\Omega) \cap S^\beta_{f_1 \cdots f_d}(\Omega),$$

where α and β are those in Corollary 6.4.

Proof. As mensioned above, $-\Delta_\mathcal{D}$ with domain $\mathcal{H}^2(\Omega)$ is a nonnegative selfadjoint operator, and hence the number $-\lambda^2$ is in the resolvent set of the operator $-\Delta_\mathcal{D}$. Therefore, $-\Delta_\mathcal{D} + \lambda^2$ is a linear one-to-one operator from $\mathcal{H}^2(\Omega)$ onto $L^2(\Omega)$. Thus, for $g \in L^2(\Omega)$ there is a unique solution $u \in \mathcal{H}^2(\Omega)$ of the problem (6.5). A straightforward calculation gives

$$\| g \|_{L^2(\Omega)} \leq (1 + \lambda^2) \, | u |_{\mathcal{H}^2(\Omega)} \, .$$

Therefore, $-\Delta_\mathcal{D} + \lambda^2$ is a bounded linear one-to-one operator from $\mathcal{H}^2(\Omega)$ onto $L^2(\Omega)$. Hence $\left(-\Delta_\mathcal{D} + \lambda^2\right)^{-1}$ is continuous. So

$$| u |_{\mathcal{H}^2(\Omega)} \leq C \| g \|_{L^2(\Omega)}$$

holds for some constant $C > 0$, independent of the solution u. Our embedding theorem (Theorem 5.1) immediately implies

$$u \in C^\alpha(\Omega) \cap S^\beta_{f_1 \cdots f_d}(\Omega),$$

where α and β are those in Corollary 6.4. □

7 Applications to quantum mechanics

Our operators appear in many quantum mechanical systems.

Application 4. Let $a_j = -\infty$, $b_j = \infty$ and $q_j = 0$ for all $j = 1, 2, \ldots, d$. In this case, Ω coincides with $\mathbb{R}^d$. The function $f : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$f : x = (x_1, x_2, \ldots, x_d) \longmapsto \xi = (\xi_1, \xi_2, \ldots, \xi_d), \qquad \xi_j = f_j(x_j)$$

gives rise to a point transformation in quauntum mechanics. This is because a point transformation in quauntum mechanics is defined to be a C^3-diffeomorphism of $\mathbb{R}^d$ onto $\mathbb{R}^d$. For more details, see Ohnuki and Watanabe [16], where we first define a point transformation in quauntum mechanics from the view point of operator theory, and point out that point transformations in quauntum mechanics are quite different from those in classical mechanics. Our operator $-i\mathcal{D}_j$ then corresponds to one of the new momentum operators given by the point transformation, and becomes a selfadjoint operator in $L^2(\mathbb{R}^d)$ by Proposition 2.3.

Application 5. Let $a_j = 0$, $b_j = \infty$, $f_j(x_j) = \ln x_j$ and $q_j = 0$ for all $j = 1, 2, \ldots, d$. In this case, Ω coincides with $(0, \infty) \times \cdots \times (0, \infty)$. Our operator $-i\mathcal{D}_j$ then corresponds to one of the dilatation operators in quantum mechanics and also corresponds to the generator of the dilation operator appearing in wavelets analysis (see e.g. [6, 8, 10]. It becomes a selfadjoint operator in $L^2(\Omega)$ by Proposition 2.3. Moreover, when $d = 1$, our transform U coincides with the Mellin transform. See Ohnuki and Watanabe [17] for more details.

Application 6. Let $d = 2$. Let $a_1 = 0$, $b_1 = \pi$, $f_1(x_1) = -\ln\tan(x_1/2)$ and $q_1 = 0$. Let $a_2 = -\pi/2$, $b_1 = \pi/2$, $f_2(x_2) = \frac{1}{2}\ln\frac{1+\sin x_2}{1-\sin x_2}$ and $q_2 = 0$. Our operator $-i\mathcal{D}_j$ $(j = 1, 2)$ in this case corresponds to one of the momentum operators appearing in quantum mechanics on S^1 based on Dirac Formalism [2, 14]. Each of our operators becomes a selfadjoint operator by Proposition 2.3. See [28, 29, 30] and Soltani [20] for related material.

Application 7. Let $a_j = -\infty$, $b_j = \infty$, $f_j(x_j) = x_j$ for all $j = 1, 2, \ldots, d$. In this case, Ω coincides with $\mathbb{R}^d$. Our operator $-i\mathcal{D}_j$ then corresponds to one of the momentum operators of a bose-like oscillator governed by Wigner's commutation relations [31], and becomes a selfadjoint operator in $L^2(\mathbb{R}^d)$ by Proposition 2.3. See Yang [32], Ohnuki and Kamefuchi [12, 13], Ohnuki and Watanabe [15] for related material.

Application 8. Let a_j, b_j and $f_j(x_j)$ be as in application 7. We now consider the motion of a quantum mechanical particle under the inverse square potential $k/(|x|^2)$, where k is a constant and $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$.
The Hamiltonian for such a particle is given by

$$H = -\Delta + \frac{k}{|x|^2}\,, \tag{7.1}$$

where Δ is the Laplacian. The Hamiltonian (7.1) for $d = 1$ appears in the two-body problems of the Calogero model [1], the Calogero-Moser model [1, 9] and the Sutherland model [21]. Each model describes a quantum mechanical system of many identical particles in one dimension with long-range interactions, and has attracted considerable interest because it is exactly solvable. The author [25] explicitly obtained the form of the solution of the initial value problem for the Schrödinger equation with the Hamiltonian (7.1) in the case $d = 1$.
To show that there exists the motion of such a particle under the inverse square potential, we need to prove the selfadjointness of the Hamiltonian (7.1). It may happen that a particle falls down to the origin (the center of the inverse square potential). In this case, the spectrum of the Hamiltonian is not bounded below and there does not exist its motion. Hence the spectrum of the Hamiltonian should be bounded below as long as there exists its motion. So we also need to prove that the spectrum of the selfadjoint Hamiltonian is bounded below.
Since a_j, b_j and $f_j(x_j)$ are those in application 7, the domain Ω' (see (2.1)) coincides with

$$\Omega' = \mathbb{R}^d \setminus \left[\left\{x \in \mathbb{R}^d : x_1 = 0\right\} \cup \left\{x \in \mathbb{R}^d : x_2 = 0\right\} \cup \cdots \cup \left\{x \in \mathbb{R}^d : x_d = 0\right\}\right] \tag{7.2}$$

and our operator becomes

$$\mathcal{D}_j = \frac{\partial}{\partial x_j} - \frac{q_j}{x_j} R_j\,,$$

where $q_j > -1/2$ and

$$R_j v(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d) = v(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_d).$$

We decompose a function $\phi \in L^2(\mathbb{R}^d)$ as follows:

$$\phi(x) = \sum_{\sigma_1=+,-} \sum_{\sigma_2=+,-} \cdots \sum_{\sigma_d=+,-} \phi_{\sigma_1\sigma_2\ldots\sigma_d}(x), \tag{7.3}$$

where $\sigma_1 = +$ if $\phi_{\sigma_1\sigma_2\ldots\sigma_d}$ is an even function with respect to x_1, $\sigma_1 = -$ if $\phi_{\sigma_1\sigma_2\ldots\sigma_d}$ is an odd function with respect to x_1, and so on. So, for example, $\phi_{++\cdots+}$ is an even function with respet to each x_j $(j = 1, 2, \ldots, d)$, $\phi_{--\cdots-}$ is an odd function with respect to each x_j $(j = 1, 2, \ldots, d)$, and $\phi_{-++\cdots+}$ is an odd function with respcet to x_1 and is an even function with respect to each x_j $(j = 2, 3, \ldots, d)$.
Suppose that k satisfies $-d/4 < k < 0$. This represents the attractive force toward the origin. Since

$$\frac{1}{|x|^2} \le \frac{1}{x_j^2},$$

it follows from (7.3) that for $\phi \in C_0^\infty(\Omega')$,

$$\begin{aligned}\left(\phi, \left(-\Delta + \frac{k}{|x|^2}\right)\phi\right)_{L^2(\mathbb{R}^d)} &\ge \left(\phi, \left(-\Delta + \frac{k}{d}\sum_{j=1}^{d}\frac{1}{x_j^2}\right)\phi\right)_{L^2(\mathbb{R}^d)} \\ &= \sum_{\sigma_1,\ldots,\sigma_d}\left(\phi_{\sigma_1\ldots\sigma_d}, \left(-\Delta + \frac{k}{d}\sum_{j=1}^{d}\frac{1}{x_j^2}\right)\phi_{\sigma_1\ldots\sigma_d}\right)_{L^2(\mathbb{R}^d)} \\ &= \sum_{\sigma_1,\ldots,\sigma_d}\left(\phi_{\sigma_1\ldots\sigma_d}, -\sum_{j=1}^{d}\mathcal{D}_j^2\phi_{\sigma_1\ldots\sigma_d}\right)_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Here, q_j satisfies $q_j(q_j - 1) = k/d$ (resp. $q_j(q_j + 1) = k/d$) when $\sigma_j = +$ (resp. when $\sigma_j = -$). We also note here that $q_j(q_j \pm 1) > -1/4$ and that

$$\mathcal{D}_j^2 = \frac{\partial^2}{\partial x_j^2} - \frac{q_j}{x_j^2}(q_j - R_j).$$

Hence, as long as $-(1/4) < (k/d)$, the operator $-\Delta + k/|x|^2$ restricted to $C_0^\infty(\Omega')$ is nonnegative by Proposition 3.4. Therefore, it is a nonnegative symmetric operator, and hence there exists its Friedrichs extension. When $k \ge 0$, there clearly exists its Friedrichs extension. For the Friedrichs extension, see e.g. Reed and Simon [19, Theorem X.23]RS.

Corollary 7.1. *Let Ω' be as in (7.2). If $k > -d/4$, then there exists the Friedrichs extension of the operator $(-\Delta + k/|x|^2)$ restricted to $C_0^\infty(\Omega')$. The lower bound of the spectrum of this selfadjoint operator is zero. Consequently, there exists the motion of a quantum mechanical particle under the inverse square potential $k/(|x|^2)$ as long as $k > -d/4$.*

Remark 7.1. Compared with other methods, ours gives the best possible value of k when $1 \le d \le 4$. For more details, see Watanabe [27].

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