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Contents

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IJAMAS: June 2007, Vol. 9, No. J07, 6-7

PREFACE

This Euler's commemorating volume entitled :
Functional Equations , Integral Equations, Differential Equations and Applications (**F. I. D. A**), is a forum for exchanging ideas among eminent mathematicians and physicists, from many parts of the world, as a tribute to the tri-centennial birthday anniversary of Leonhard Paul Euler (April 15,1707 A.D., b. in Basel – September 18, 1783 A.D., d. in St. Petersburg).
This 998 pages long collection is composed of outstanding contributions in mathematical and physical equations and inequalities and other fields of mathematical, physical and life sciences.
In addition, this anniversary volume is unique in its target, as it strives to represent a broad and highly selected participation from across and beyond the scientific and technological country regions. It is intended to boost the cooperation among mathematicians and physicists working on a broad variety of pure and applied mathematical areas.
Moreover, this new volume will provide readers and especially researchers with a detailed overview of many significant insights through advanced developments on Euler's mathematics and physics. This transatlantic collection of mathematical ideas and methods comprises a wide area of applications in which equations, inequalities and computational techniques pertinent to their solutions play a core role.
Euler's influence has been tremendous on our everyday life, because new tools have been developed, and revolutionary research results have been achieved , bringing scientists of exact sciences even closer, by fostering the emergence of new approaches, techniques and perspectives.
The central scope of this commemorating 300birthday anniversary volume is broad, by deeper looking at the impact and the ultimate role of mathematical and physical challenges, both inside and outside research institutes, scientific foundations and organizations.
We have recently observed a more rapid development in the areas of research of Euler worldwide. Leonhard P. Euler (1707-1783) was actually the most influential mathematician and prolific writer of the eighteenth century, by having contributed to almost all the fundamental fields of mathematics and mathematical physics. In calculus of variations, according to C. Caratheodory, Euler's work: *Methodus inveniendi lineas curves…*(1740 A.D.) was one of the most beautiful works ever written. Euler was dubbed *Analysis Incarnate* by his peers for his incredible ability. He was especially great from his writings and that produced more academic work on mathematics than anyone. He could produce an entire new mathematical paper in about thirty minutes and had huge piles of his works lying on his desk. It was not uncommon to find Analysis Incarnate ruminating over a new subject with a child on his lap.
This volume is suitable for graduate students and researchers interested in functional equations, integral equations and differential equations and would make an ideal supplementary reading or independent study research text.
This item will also be of interest to those working in other areas of mathematics and physics. It is a work of great interest and enjoyable read as well as unique in market.

This Euler's volume (F. I. D. A.) consists of six (6) issues containing various parts of contemporary pure and applied mathematics with emphasis to Euler's mathematics and physics.
It contains sixty eight (68) fundamental research papers of one hundred one (101) outstanding research contributors from twenty seven (27) different countries.
In particular, these contributors come from:
Algerie (1 contributor); Belgique (2); Bosnia and Herzegovina (2); Brazil (2); Bulgaria (3); China (9); Egypt (1); France (3); Greece (2); India (8); Iran (3); Italy (1); Japan (7); Korea (7); Morocco (3); Oman (2); Poland (3); R. O. Belarus (8); Romania (2); Russia (3); Saudi Arabia (1); Serbia and Montenegro (5); The Netherlands (3); U. A. Emirates (1);U. K. (2); U. S. A. (15); Uzbekistan (2).

First Issue (F. E. I.) consisting of 14 research papers, 181 pages long, contains various parts of ***Functional Equations and Inequalities*,**
namely:
Euler's Life and Work, Ulam stability, Hyers – Ulam stability and Ulam – Gavruta - Rassias stability of functional equations, Euler – Lagrange type and Euler – Lagrange – Rassias quadratic mappings in Banach and Hilbert spaces, Aleksandrov and isometry Ulam stability problems, stability of Pexider and Drygas functional equations, alternative of fixed point, and Hyers - Ulam stability of differential equations.

Second Issue (MT. PDE) consisting of 9 research papers, 117 pages long, contains various parts of ***Mixed Type Partial Differential Equations*,**
namely:
Tricomi - Protter problem of nD mixed type partial differential equations, solutions of generalized Rassias' equation, degenerated elliptic equations, mixed type oblique derivative problem, Cauchy problem for Euler – Poisson - Darboux equation, non - local boundary value problems, non-uniqueness of transonic flow past a flattened airfoil, multiplier methods for mixed type equations.
Third Issue (F . D . E.) consisting of 9 research papers, 146 pages long, contains various parts of ***Functional and Differential Equations*,**
namely:
Iterative method for singular Sturm - Liouville problems, Euler type boundary value problems in quantum mechanics, positive solutions of boundary value problems, controllability of impulsive functional semi-linear differential inclusions in Frechet spaces, asymptotic properties of solutions of the Emden-Fowler equation, comparison theorems for perturbed half-linear Euler differential equations, almost sure asymptotic estimations for solutions of stochastic differential delay equations, difference equations inspired by Euler's discretization method, extended oligopoly models.
Fourth Issue (D. E. I.) consisting of 9 research papers, 160 pages long, contains various parts of ***Differential Equations and Inequalities*,**
namely:
New spaces with wavelets and multi-fractal analysis, mathematical modeling of flow control and wind forces, free convection in conducting fluids, distributions in spaces, strong stability of operator–differential equations, slope – bounding procedure, sinc methods and PDE, Fourier type analysis and quantum mechanics.
Fifth Issue (DS. IDE.) consisting of 9 research papers, 159 pages long, contains various parts of ***Dynamical Systems and Integro - Differential Equations*,**
namely:
Semi-global analysis of dynamical systems, nonlinear functional-differential and integral equations, optimal control of dynamical systems, analytical and numerical solutions of singular integral equations, chaos control of classes of complex dynamical systems, second order integro-differential equation, integro-differential equations with variational derivatives generated by random partial integral equations, inequalities for positive operators, strong convergence for a family of non-expansive mappings.
Sixth Issue (M. T. A.) consisting of 18 research papers, 231 pages long, contains various parts of ***Mathematical Topics and Applications*,**
namely:
Maximal subgroups and theta pairs in a group, Euler constants on algebraic number fields, characterization of modulated Cox measures on topological spaces, hyper-surfaces with flat r-mean curvature and Ribaucour transformations, Leonhard Euler's methods and ideas live on in the thermodynamic hierarchical theory of biological evolution, zeroes of L-series in characteristic p,
Beck's graphs, best co-positive approximation function, Convexity in the theory of the Gamma function, analytical and differential – algebraic properties of Gamma function, Ramanujan's summation formula and related identities, ill – posed problems, zeros of the q-analogues of Euler polynomials, Eulerian and other integral representations for some families of hyper-geometric polynomials, group C*-algebras and their stable rank, complementaries of Greek means to Gini means, class of three- parameter weighted means, research for Bernoulli's inequality.

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John Michael Rassias

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An Iterative Method for the Computation of Eigenpairs of Singular Sturm-Liouville Problems

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ABSTRACT

We propose an iterative method coupled with simple shooting for computing the eigenpairs of singular Sturm-Liouville problems. The problem is first transformed into a regular initial value problem which is then solved using standard numerical methods. The accuracy of the proposed iterative method per step is the same as that of the IVP solver. Numerical examples will be presented to demonstrate the efficiency of the method.

Keywords: Eigenpairs, Sturm-Liouville, Singular B. V. P's.

2000 Mathematics Subject Classification: 65N25.

1 Introduction

It is well known that the study of Sturmian theory is an important aid in solving numerous problems in mathematical physics, thermodynamics, electrostatics, quantum mechanics and statistics. These sciences are rich source of the eigenvalue problems that are termed ”Sturm-Liouville” two point boundary value problems of the form

$$-\left(py'\right)' + qy = \lambda wy \tag{1.1}$$

on some interval $(a,\ b)$ of the real line subject to some boundary conditions. Very often and in applications singularities are encountered at one or more points in that interval. For example equations having the form given in (1.1) appears when separation of variables is attempted on the heat equation in a solid sphere or the electrostatic potential in the sphere. The source of the singularity in this case is the vanishing of the function p at the endpoints.

When applied to the singular problems of the form (1.1), standard numerical methods designed for regular ordinary differential equations will have difficulties in terms of convergence and suffer from a loss of accuracy. In some cases they may even fail to converge [2]. For that reason, special numerical methods have been proposed to handle the singular problem. Such proposed methods converge, some with a lesser order of convergence due to the singularity and some under various assumptions on the data, may retain the standard rate of convergence in the regular case. We refer to [3, 6, 8] and the references therein for more on this point.

Many authors considered the Sturm-Liouville problem both the singular and the regular cases. For example polynomial Sturm-Liouville problems were considered by [1], singular problems with rational coefficients was considered by [10] while regular and singular problems were studied extensively by [11, 12, 13, 16, 17]. Some theoretical results can be found for example [2, 7, 14, 15].

In this work we propose an iterative method, coupled with simple shooting for the numerical solution of (1.1). In each step of the iteration we solve a regular ODE and, therefore, may use standard numerical methods. Thus the accuracy of the proposed iterative method per step is the same as the accuracy of the numerical ODE solver used, see [8].

The outline of the paper is as follows. In Section 2 some preliminary results are given to set up the terminology and state the class of the boundary value problems to be numerically treated. We discuss the applicability of the simple shooting method to this class of singular boundary value problems in Section 3. In Section 4 we introduce the iterative method and discuss its justification. Some numerical examples are given in the last section.

2 Preliminaries

Before considering the Sturm-Liouville eigenvalue problem further, it is worth mentioning that a general second order linear ordinary differential equation of the form

$$P(x)u'' + Q(x)u' + (R(x) + \lambda)u = 0$$

can be transformed to Sturm-Liouville problem which has the form

$$-\left(p(x)u'\right)' + [q(x) + \lambda w(x)]u = 0. \tag{2.1}$$

This can be done by first dividing by $P(x)$ and then multiplying by the integrating factor given by $e^{\int^x \frac{Q}{P} dt}$.

It is amazing how much information one can infer about the eigenvalues and eigenfunctions of a Surm-Liouville problem without actually solving the differential equation explicitly. Thus from very general and simple considerations, one can discover that the eigenvalues are real, are discrete if the domain is finite, have a lowest member, increase without limit, and that the corresponding eigenfunctions are orthogonal to each other, oscillate, oscillate more rapidly the larger the eigenvalue, to mention just a few pieces of useful information. In practice this kind of information is quite often the primary thing of interest. In other words, the philosophy quite often is that one verifies that a certain system is of the Surm-Liouville types, thus having at one's immediate disposal a concomitant list of properties of the system, properties whose qualitative nature is quite sufficient to answer the questions one had about the system in the first place. The standard existence theory for solutions of the differential equation given by (2.1) can be found in for example Reid [15] and Naimark [14]

In considering this equation (2.1), which can be written in the form

$$\ell(u) = \lambda w(x)u \text{ on } I, \tag{2.2}$$

where

$$\ell(u) = -\left(p(x)u'\right)' + q(x)u$$

$$I = (a,b), \qquad -\infty < a < b < \infty,$$

we will make the following assumptions:

$$\begin{aligned} p,q,w &: I \to \mathbb{R}. & (2.3)\\ p^{-1},q,w &\in L_{loc}(I) & (2.4)\\ p(x) &> 0, w(x) > 0 \text{ almost everywhere on } I. & (2.5)\end{aligned}$$

We mean by $L_{loc}(I)$ the set of real-valued functions on I which are Lebesgue integrable on each compact subintervals of I. When additional assumptions are needed they will be specified. Standard terminology associated with singular differential operators will be used here.

Definition 2.1. The endpoint $a \in I$ is said to be regular if a is finite and $p^{-1}, q, w \in L[a,c]$ for some $c \in (a,b)$. An endpoint is called singular if it is not singular.

Note that this definition implies that an end point is singular if it is infinite or the end point is finite but one of p^{-1}, q, w is not in $L[a,c]$ in any neighborhood of the endpoint.
By a solution of (2.2) we mean a function u and pu' are both absolutely continuous on all compact subintervals of I (the classical derivative u' may not be absolutely continuous) and in this case (2.2) holds a.e. on I and its left hand side is defined a.e. on I. The reader is referred to for details. Throughout this paper we will assume that the endpoint a is *regular* and the endpoint b to be *singular, limit circle* (LC). Under the foregoing assumptions, all solutions of (2.2) are $L^2_w(I)$. For $x \in I$ and $H, Q : I \to \mathbb{C}$ define the "generalized" Wronskian by

$$W_x(H,Q) = \begin{vmatrix} H(x) & Q(x) \\ p(x)H'(x) & p(x)Q'(x) \end{vmatrix}. \tag{2.6}$$

This Wonskian is needed since for all solutions H, Q, we have the well known Green's formula

$$\int_c^d (H\ell(Q) - Q\ell(H)) = W_x(H,Q)|_c^d; \;\; c,\; d \in I.$$

Note that if $W_x(H,Q) = 1$, then H and Q are in $L^2_w(I)$. If we let

$$U = \begin{pmatrix} u_1 \\ pu_2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1/p \\ q & 0 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix},$$

then

$$\begin{pmatrix} H(x) & Q(x) \\ p(x)H'(x) & p(x)Q'(x) \end{pmatrix}$$

is a fundamental solution matrix of

$$U' = (P - \lambda W)U.$$

Having this, choose real valued solutions $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ satisfying the initial conditions

$$\begin{aligned} \theta(a,\lambda) &= p(a)\varphi'(a,\lambda) = 1, & (2.7)\\ p(a)\theta'(a,\lambda) &= \varphi(a,\lambda) = 0, & (2.8)\end{aligned}$$

and $W_x(\theta, \varphi) = 1$ for all $x \in I$. Such choice is possible since for example $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ can be two linearly independent solutions of (2.2) for any real λ. Now, there exists a meromorphic function $m(\lambda)$, known as the Weyle's *m*-function, with simple poles on the real axis such that the linear combination

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\varphi(x, \lambda)$$

satisfies

$$W_{b^-}(u, \psi) = 0, \tag{2.9}$$

where u is a solution of,

$$(\ell - \lambda w)u = wf$$

for any $f \in L^2_w(I)$ as long as λ is not a pole of $m(\lambda)$. Condition (2.9) is the only boundary condition that can be assumed at b, similar discussions can be found in [8], [11] and [12], an arbitrary boundary condition may be imposed at a. As mentioned before, assuming that $\lambda = 0$ is not a pole of $m(\lambda)$ is equivalent to the solvability of (2.2) subject to (2.9). This means this assumption is needed and hence assumed from now on. As a result the boundary value problem can be stated as follows:
For a given $f \in L^2_w(I)$, find $u \in L^2_w(I)$ such that

$$\begin{aligned} \ell(u) &= wf \qquad \text{on } I, \\ u(a) &= A \quad W_{b^-}(u, \psi) = 0. \end{aligned} \tag{2.10}$$

3 The Shooting Method

The shooting method constructs a solution of a boundary value problem by iteratively solving a sequence of initial value problems where the initial conditions are modified in each step. Since the problem we are considering is linear, the shooting method usually converges in two iterations. We will review the simple shooting method for solving systems of ODEs. In particular, we will present a version of the method that is particularly useful to us in the next section.
Suppose we are required to solve a boundary value problem of the form

$$Y' = PY + G \quad \text{on } I \tag{3.1}$$

where $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $P \in M_2(L_{\ell oc}(I))$ and $G \in M_{2,1}(L_{\ell oc}(I))$, together with the boundary conditions

$$y_1(a) = \alpha, \quad y_2(b) = \gamma. \tag{3.2}$$

The shooting method starts by solving the initial value problem (IVP) consisting of (3.1) together with the initial conditions

$$Y(a) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.3}$$

for arbitrary β. Then we try to fix β such that the boundary condition $y_2(b) = \gamma$ is satisfied. In the linear case, β can be exactly determined. We will use the notation $Y(x, \eta)$ for the solution

of the IVP (3.1) such that $Y(a,\eta)=\eta$. Assuming all the necessary conditions for the existence and uniqueness of solutions of the IVP (3.1), (3.3), β can be determined as follows.
Let β be an arbitrary real number

1. Set $\eta_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

2. Solve the IVPs

$$\left.\begin{array}{l} Y' = PY, \\ Y(a) = e_i, \quad i = 1,2 \end{array}\right\} \tag{3.4}$$

where e_i is the i-th standard unit vector in $\mathbb{R}^2$.

2.1 Set

$$\Phi = (Y(b,e_1)\ Y(b,e_2)) \tag{3.5}$$

3. Solve

$$\left.\begin{array}{l} Y' = PY + G, \\ Y(a) = 0 \end{array}\right\}. \tag{3.6}$$

3.1 Set

$$C = \Phi\eta_0 + Y(b,0). \tag{3.7}$$

4. Set $\overline{\eta} = \begin{pmatrix} \alpha \\ \overline{\beta} \end{pmatrix}$, $\overline{C} = \begin{pmatrix} \overline{\alpha} \\ \gamma \end{pmatrix}$ where $\overline{\alpha}, \overline{\beta}$ are computed from

$$\Phi(\overline{\eta} - \eta_0) = \overline{C} - C \tag{3.8}$$

5. The solution of the boundary value problem (3.1), (3.2) is given by

$$Y(x) = (Y(x,e_1)Y(x,e_2))\overline{\eta} + Y(x,0). \tag{3.9}$$

Now $Y(x)$ solves (3.1) since if we differentiate both sides of (3.9) and using (3.4) and (3.6) we get

$$\begin{aligned} Y'(x) &= P(Y(x,e_1)Y(x,e_2))\overline{\eta} + PY(x,0) + G \\ &= P[(Y(x,e_1)Y(x,e_2))\overline{\eta} + Y(x,0)] + G \\ &= PY(x) + G \end{aligned}$$

i.e. $Y(x)$ is a solution of (3.1). To satisfy the boundary conditions; that is, $y_1(a)=\alpha, y_2(b)=\gamma$. Using the initial conditions in (3.4), (3.5) and (3.6) we get

$$Y(a) = (Y(a,e_1)Y(a,e_2)\overline{\eta} = \overline{\eta}.$$

Hence $y_1(a)=\alpha$. Using (3.9), (3.5), (3.7) and (3.8) we get

$$\begin{aligned} Y(b) &= \Phi\overline{\eta} + Y(b,0) \\ &= \Phi(\overline{\eta} - \eta_0) + C \\ &= \overline{C}, \end{aligned}$$

so that $y_2(b)=\gamma$. This means to obtain the solution of the boundary value problem (3.1), (3.2) will requires solving the three initial value problems given in (3.4) and (3.6).

To apply the shooting method to the singular problem, we proceed as in the regular shooting method; that is, find the solution of the initial value problem

$$\begin{aligned} \ell(u) &= wf \qquad \text{on } I, \\ u(a) &= A \quad pu'(a) = s \end{aligned} \tag{3.10}$$

and try to fix s such that $W_{b^-}(u,\psi) = 0$. To be able to use an initial value solver like Runge-Kutta method, transform (3.10) into a first order system that has the form

$$\begin{aligned} u_1' &= u_2 - \frac{p'}{p}u_1 \\ u_2' &= \frac{q}{p}u_1 - wf \end{aligned}$$

subject to the initial conditions

$$u_1(a) = pA, \quad u_2(a) = s.$$

Using the method of variation of parameters, we can write the solution of (3.10) as

$$u(x) = A\psi(x) + s\varphi(x) + \varphi(x)\int_a^x \psi(t)f(t)w(t)dt + \psi(x)\int_x^b \varphi(t)f(t)w(t)dt.$$

(Here, $\psi(x) = \psi(x,0)$, ... etc.) Therefore,

$$\begin{aligned} W_{b^-}(u,\psi) &= AW_{b^-}(\psi,\psi) + sW_{b^-}(\varphi,\psi) + W_{b^-}(\varphi,\psi)\int_a^b \psi(t)f(t)w(t)dt \\ &= -s - \int_a^b \psi(t)f(t)w(t)dt. \end{aligned}$$

Thus, letting $\tau = -W_{b^-}(u,\psi) - s$ and u_1 the solution of (3.10) with s replaced by τ we get $W_{b^-}(u,\psi) = 0$. In other words, u_1 solves the boundary value problem (2.10) and the shooting method converges in two steps.

4 The Iterative Method

We will employ the shooting method presented in the previous section to reduce the BVP (2.10) to a regular IVP given by (3.10) away from the singular the singular point b. When near b, the solution may become unbounded or oscillatory. This poses a difficult problem which needs curing. Notice that for a certain class of problems, while the solution u may become infinite or oscillates, the quantity pu remains finite or vanishes at b. We will take advantage of this fact to develop an iterative method for solving the IVP given by (3.10). Assuming that the function p is differentiable on I. Introduce the change of variables $y_1 = pu$ and $y_2 = pu'$ and rewrite the boundary value problem as the system

$$Y' = MY + Nu + F, \tag{4.1}$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} p' \\ q \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 \\ -f \end{pmatrix}. \tag{4.2}$$

Then, assuming that u is known for the moment, equation (4.1) together with the initial condition

$$Y(a) = \eta \equiv \begin{pmatrix} pM \\ \beta \end{pmatrix}, \tag{4.3}$$

where β is arbitrary, form a regular initial value problem amenable to numerical methods for regular IVPs. The boundary value problem (2.10) is equivalent to the following problem: Find $u,\ \beta$ and Y such that

(a) Y satisfies (4.1), (4.3)

(b) $y_2(b^-) = 0$

(c) $y_1(x) = p(x)u(x)$ for all $x \in I$.

According to the setting (4.1), (4.3), we will derive the iterative method for the solution of the IVP. To do so rewrite (4.1) as

$$\left(\overline{e}^{Mx}Y\right)' = \overline{e}^{Mx}(Nu + F) \tag{4.4}$$

and let

$$z = e^{-Mx}Y.$$

Then (3.1), (3.3) are transformed into

$$\left.\begin{aligned} z' &= P(x)z + \tilde{F}, \\ z(a) &= \tilde{\eta} \end{aligned}\right\} \tag{4.5}$$

where

$$\begin{aligned} P(x) &= \frac{1}{p(x)} e^{-Mx} N(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{Mx} = \begin{pmatrix} \frac{p'-xq}{p} & \frac{xp'-x^2q}{p} \\ \frac{q}{p} & \frac{xq}{p} \end{pmatrix}, \\ \tilde{F} &= e^{-Mx}F \end{aligned}$$

and

$$\tilde{\eta} = e^{-Ma}\eta.$$

Thus our iterative method is equivalent to

$$\begin{aligned} z_0(a) &= \tilde{\eta} \\ z'_{n+1} &= p(x)z_n + \tilde{F} \\ z_{n+1}(a) &= \tilde{\eta} \end{aligned}$$

which in turn is equivalent to

$$z_{n+1}(x) = \tilde{\eta} + \int_a^x [p(s)z_n(s) + \tilde{F}(s)]ds, \qquad x \in I. \tag{4.6}$$

This discussion leads to the following Algorithm:
Suppose u_0 is an initial guess satisfying the initial conditions at α and ε is a given tolerance.
(1) For $n = 0, 1, 2, \ldots$ do

(2) Solve

$$\left.\begin{array}{l} Y'_{n+1} = AY_{n+1} + Bu_n + F, \\ Y_{n+1}(a) = \eta \end{array}\right\}$$

(3) If $\|Y_{n+1} - Y_n\| < \varepsilon$, then a solution is obtained. Stop;
else, set $n = n + 1$, $u_n = \frac{1}{p} y_1^{(n)}$ and go to (2).

Finally, convergence of the scheme given by (4.6) follows from the following theorem whose proof can be found in [16] p. 5.

Theorem 4.1. *Let $I = [a, b), p(t) \in M_2(L_{\ell oc}(I))$, $\tilde{F} \in M_{2,1}(L_{l0c}(I))$. Then the iterative process (4.6) converges uniformly on any compact subinterval of I to the unique solution of the IVP (4.5).*

One final remark is that from a computational point of view, the singularity of the problem (3.10) is dealt with in a purely algebraic way, namely, in solving the equation

$$pu_{n+1} = y_{1,n+1}$$

for u_{n+1}.

Since $p(b) = 0$ and $u(b^-)$ exists, then $y_1(b^-) = 0$. Therefore, in order to find the value of $u_{n+1}(b)$ one may use an extrapolation method of order comparable to the order of the numerical method used to solve the IVP numerically. The same approach as in the previous remark can be used to find values of u in case $p(x)$ has zeros inside the interval I. Again since this is done algebraically, the cost is minimal.

5 Numerical Results:

In this section, we will present some numerical examples to illustrate the theory developed in the previous sections. The initial value problems involved will be solved using Runge-Kutta Schemes of order 4 (RK-4) and in some cases of order 2 (RK-2).

Example 1. (The Legendre Equation). This is the equation

$$-\left(\left(1 - x^2\right) y'(x)\right)' + \frac{1}{4} y(x) = \lambda y(x); \quad 0 < x < 1,$$

with $I = (0,\ 1)$. For $\lambda = \frac{1}{4}$ two linearly independent solutions are (see (2.7) and (2.8))

$$\theta(x) = 1 \text{ and } \phi(x) = \ln \frac{1 + x}{1 - x}; \quad x \epsilon I$$

with boundary conditions

$$y(0) = 0 \text{ and } W(y, \varphi)(1^-) = 0.$$

This is the classical case whose eigenfunctions are the classical Legendre polynomials and whose eigenvalues are known to be

$$\lambda_n = n(n + 1) + \frac{1}{4}; \quad n = 0, 1, 2, 3, ...$$

Table 4.1 shows the computed eigenvalues with the value of the Wronskian using the iterative method in addition to results of Baily et al [4, 5] using an advanced code SLEIGN2.

n	$\lambda-$iterative	$W(y,\varphi)(1^-)$	$\lambda-$SLEIGN2
0	0.2500000	0.0	0.2500000
1	2.250027	6.897943-8	2.250000
2	6.250152	3.605173-8	6.250000

The computed values were done with a step size $h = \frac{1}{16}$ and the initial value problems were solved using Runge-Kutta of order 4. The iterative method also produced eigenvalues for $n > 3$ but we reported the first 3 for the sake of comparison.

Example 2. The Latzko equation: It has the form

$$-\frac{1}{x^7}(py')' + y = \lambda y;\ 0 < x < 1$$

with $p(x) = \left(1 - x^7\right)$. This is a historical differential equation associated with a heat conduction problem studied for the first time by Latzko in 1920. For details on the problem see Fichera [9]. The endpoint 0 is clearly regular while the endpoint 1 is singular since ${}_0^1(1 - x^7)^{-1}dx = \infty$. For boundary conditions we can take

$$\theta(x) = 1 \text{ and } \phi(x) = \ln(\frac{1}{x-1}),\quad I = (0, 1).$$

This means the boundary conditions are

$$y(0) = 0;\ W(y, \psi)(1^-) = 0.$$

We solve the above singular boundary value problem for the eigenvalues λ using Algorithm 1.

n	$\lambda-$iterative	$\lambda-$SLEIGN2	$\lambda-$Durfee	$W(y,\psi)(1^-)$
0	8.727820	8.7274702	8.72747	3.634088-8
1	152.4451	152.423014	152.423	6.131883-7
2	435.1673	435.060768	435.060	-3.656848-7
3	855.9708	855.681700	855.680	-9.60201-7

Again the computed values were done with a step size $h = \frac{1}{16}$ and the initial value problems were solved using Runge-Kutta of order 4. The solutions corresponding to the first two eigenvalues are shown in figures 5.1 and 5.2.

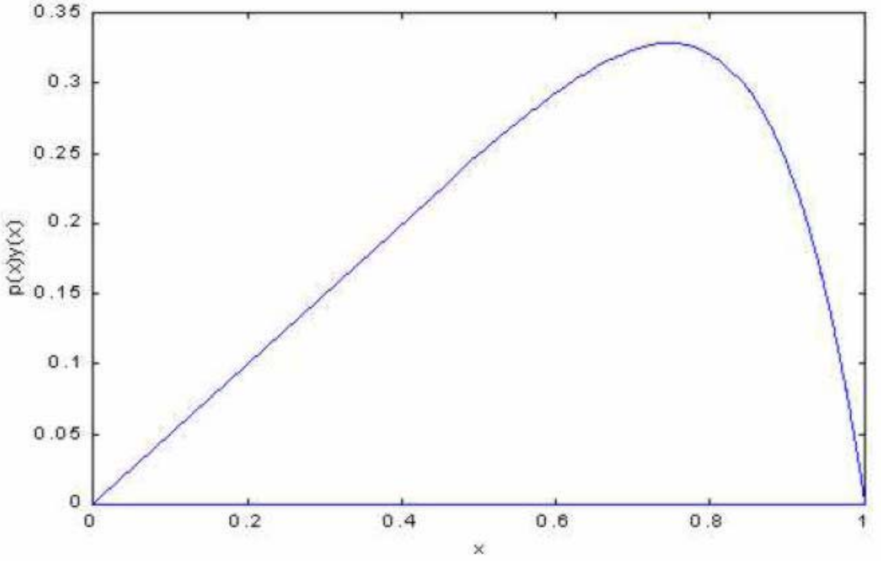

Figure 1: (5.1)

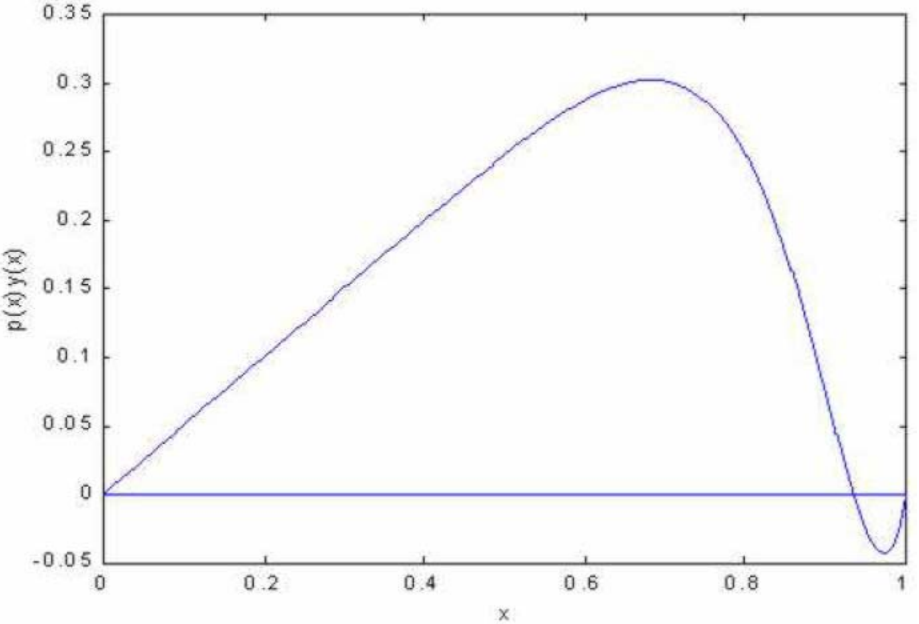

Figure 2: (5.2)

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Euler-type Boundary Value Problems in Quantum Calculus

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ABSTRACT

We study a boundary value problem consisting of a second-order q-difference equation together with Dirichlet boundary conditions. Separation of variables leads us to an eigenvalue problem for a second-order Euler q-difference equation. We determine the exact number of eigenvalues.

Keywords: Euler–Cauchy dynamic equation, quantum calculus.

2000 Mathematics Subject Classification: 39A13, 34K10, 65N25.

1 Introduction

While in ordinary calculus we study differential equations and in discrete calculus we study difference equations, we study so-called q-difference equations in quantum calculus. Let

$$q > 1 \quad \text{and} \quad \mathbb{T} = \{q^k : k \in \mathbb{N}_0\}. \tag{1.1}$$

For a function $u : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, we define the *Jackson derivatives* of u with respect to the first and the second variable, respectively, by

$$u_x(x,t) = \frac{u(qx,t) - u(x,t)}{(q-1)x} \quad \text{and} \quad u_t(x,t) = \frac{u(x,qt) - u(x,t)}{(q-1)t}.$$

Let $N \in \mathbb{N}$. In this paper, we consider the boundary value problem

$$x^2 u_{xx} = t^2 u_{tt}, \quad u(1,t) = u(q^N, t) = 0, \tag{1.2}$$

which is a second-order q-difference equation together with Dirichlet boundary conditions. For material on quantum calculus we refer to the monograph by Kac and Cheung (Kac and Cheung 2002), the paper by Bohner and Ünal (Bohner and Ünal 2005), and the books about dynamic equations on time scales by Bohner and Peterson (Bohner and Peterson 2001, Bohner and Peterson 2003).

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Now we rewrite the second-order q-difference equation in (1.2) as a second-order q-recursion relation. By the definition of the Jackson derivative, the second-order partial of u with respect to x is given by

$$u_{xx}(x,t)=\frac{u_x(qx,t)-u_x(x,t)}{(q-1)x}. \tag{1.3}$$

When we expand equation (1.3), we obtain

$$u_{xx}(x,t)=\frac{\frac{u(q^2x,t)-u(qx,t)}{(q-1)qx}-\frac{u(qx,t)-u(x,t)}{(q-1)x}}{(q-1)x}.$$

Similarly we can compute the partial u_{tt}. Thus, the second-order q-difference equation in (1.2) is equivalent to

$$x^2\left(\frac{u(q^2x,t)-(q+1)u(qx,t)+qu(x,t)}{x^2}\right)=t^2\left(\frac{u(x,q^2t)-(q+1)u(x,qt)+qu(x,t)}{t^2}\right),$$

i.e.,

$$u(q^2x,t)-(q+1)u(qx,t)+qu(x,t)=u(x,q^2t)-(q+1)u(x,qt)+qu(x,t). \tag{1.4}$$

The setup of this paper is as follows. In the next section, we use separation of variables to arrive at a certain eigenvalue problem. Finally, in Section 3 we determine the eigenvalues and the number of eigenvalues of the resulting eigenvalue problem. An example is given as well.

2 Separation of Variables

We let $u(x,t)=f(x)g(t)$ so that $u(qx,t)=f(qx)g(t)$ and $u(q^2x,t)=f(q^2x)g(t)$. This is also applied to the terms $u(x,qt)$ and $u(x,q^2t)$. When we substitute these values into the partial q-difference equation (1.4), we get

$$f(q^2x)g(t)-(q+1)f(qx)g(t)+qf(x)g(t)=f(x)g(q^2t)-(q+1)f(x)g(qt)+qf(x)g(t). \tag{2.1}$$

Now we divide each side of (2.1) by $f(x)g(t)$ to gather like terms and then set both sides equal to a constant λ to arrive at

$$\frac{f(q^2x)-(q+1)f(qx)+qf(x)}{f(x)}=\frac{g(q^2t)-(q+1)g(qt)+qg(t)}{g(t)}=\lambda. \tag{2.2}$$

Hence, from (1.2) and (2.2), the eigenvalue problem for f is

$$f(q^2x)-(q+1)f(qx)+(q-\lambda)f(x)=0, \quad f(1)=f(q^N)=0. \tag{2.3}$$

The second-order q-difference equation in (2.3) is an Euler–Cauchy q-difference equation as studied in (Bohner and Ünal 2005). We let $f(x)=\alpha^{\log_q x}$, which in return gives us

$$f(qx)=\alpha^{\log_q qx}=\alpha f(x) \quad \text{and} \quad f(q^2x)=f(q(qx))=\alpha f(qx)=\alpha^2 f(x).$$

Now we make these substitutions into the Euler–Cauchy equation in (2.3) and get

$$\alpha^2 f(x)-(q+1)\alpha f(x)+(q-\lambda)f(x)=0.$$

The characteristic equation therefore reads

$$\alpha^2 - (q+1)\alpha + (q-\lambda) = 0. \tag{2.4}$$

We solve (2.4) for α and get

$$\alpha = \frac{q+1 \pm \sqrt{(q+1)^2 - 4(q-\lambda)}}{2} = \frac{q+1}{2} \pm \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda}.$$

Hence we let

$$\alpha_1 = \frac{q+1}{2} + \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda} \quad \text{and} \quad \alpha_2 = \frac{q+1}{2} - \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda}.$$

We distinguish the following three cases:

$$\text{Case I.} \quad \lambda > -\left(\frac{q-1}{2}\right)^2;$$

$$\text{Case II.} \quad \lambda = -\left(\frac{q-1}{2}\right)^2;$$

$$\text{Case III.} \quad \lambda < -\left(\frac{q-1}{2}\right)^2.$$

The general solution of the Euler–Cauchy equation in (2.3) for each case is found in (Bohner and Ünal 2005) as follows:

$$\text{Case I:} \quad f(x) = c_1\alpha_1^{\log_q x} + c_2\alpha_2^{\log_q x}, \tag{2.5}$$

$$\text{Case II:} \quad f(x) = (c_1 \ln x + c_2)\left(\frac{q+1}{2}\right)^{\log_q x}, \tag{2.6}$$

$$\text{Case III:} \quad f(x) = |\alpha|^{\log_q x}(c_1 \cos(\theta \log_q x) + c_2 \sin(\theta \log_q x)), \tag{2.7}$$

where $\theta = \arccos\left(\frac{\operatorname{Re}\alpha}{|\alpha|}\right)$ and c_1, $c_2 \in \mathbb{R}$.

3 Finding Eigenvalues

Our next step is to look at the three different cases and thus find the eigenvalues of (2.3).

Case I

We apply the first Dirichlet condition $f(1) = 0$ to (2.5) to obtain

$$f(1) = c_1\alpha_1^{\log_q 1} + c_2\alpha_2^{\log_q 1} = c_1 + c_2 = 0 \quad \text{so that} \quad c := c_1 = -c_2.$$

Now we use the relationship between c_1 and c_2 and apply it to the general solution and then use the other Dirichlet condition $f(q^N) = 0$ to find

$$0 = f(q^N) = c\left(\alpha_1^{\log_q q^N} - \alpha_2^{\log_q q^N}\right) = c\left(\alpha_1^N - \alpha_2^N\right).$$

Since $c = 0$ results in the trivial solution, we shall discuss

$$\alpha_1^N = \alpha_2^N.$$

This can occur if $\alpha_1 = \alpha_2$ or (for even N) if $\alpha_1 = -\alpha_2$. First, $\alpha_1 = \alpha_2$ implies

$$\frac{q+1}{2} + \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda} = \frac{q+1}{2} - \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda},$$

i.e.,

$$2\sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda} = 0 \quad \text{so that} \quad \lambda = -\left(\frac{q-1}{2}\right)^2,$$

which is not a valid value for λ in Case I. Next, $\alpha_1 = -\alpha_2$ implies

$$\frac{q+1}{2} + \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda} = -\frac{q+1}{2} + \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda}$$

which results in $q = -1$, contradicting (1.1). Thus there are no eigenvalues in this case.

Case II

Now we look at the case where $\lambda = -(\frac{q-1}{2})^2$ and use equation (2.6) with the first Dirichlet condition to find

$$0 = f(1) = (c_1 \ln 1 + c_2)\left(\frac{q+1}{2}\right)^{\log_q 1} = c_2.$$

We let $c := c_1$ and apply the second Dirichlet condition, which gives

$$0 = f(q^N) = c \ln q^N \left(\frac{q+1}{2}\right)^{\log_q q^N} = cN \ln q \left(\frac{q+1}{2}\right)^N.$$

For this to be true either $c = 0$, $N = 0$, $q = 1$, or $q = -1$, which would all not lead to any eigenvalues. Hence there are no eigenvalues in this case also.

Case III

Finally we look at the case where $\lambda < -(\frac{q-1}{2})^2$ and use equation (2.7) with the first Dirichlet condition to find

$$0 = f(1) = |\alpha|^{\log_q 1}(c_1 \cos(\theta \log_q 1) + c_2 \sin(\theta \log_q 1)) = c_1.$$

We let $c := c_2$ and apply the other Dirichlet condition to obtain

$$0 = f(q^N) = |\alpha|^{\log_q q^N}(c \sin(\theta \log_q q^N)) = c|\alpha|^N \sin(\theta N). \tag{3.1}$$

Note now that

$$|\alpha| = \sqrt{q - \lambda} \quad \text{and} \quad \theta = \arccos\left(\frac{\operatorname{Re}\alpha}{|\alpha|}\right) = \arccos\left(\frac{q+1}{2\sqrt{q-\lambda}}\right).$$

Therefore we obtain from (3.1) that

$$0 = f(q^N) = c\sqrt{q-\lambda}^N \sin(\theta N). \tag{3.2}$$

When looking at (3.2), we can see that $\lambda = q$ would work, but this is not in the range of values we are looking at for this case. So we consider the only other possible solution, which is $\sin(\theta N) = 0$. This leads us to $\theta_m N = m\pi$, which gives us the values λ_m, $m \in \mathbb{N}_0$, where

$$\arccos\left(\frac{q+1}{2\sqrt{q-\lambda_m}}\right) = \frac{m\pi}{N}. \tag{3.3}$$

Solving for λ_m provides

$$\lambda_m = q - \left(\frac{q+1}{2\cos\left(\frac{m\pi}{N}\right)}\right)^2 \tag{3.4}$$

for $m = 1, \ldots, (N-2)/2$ if N is even and $m = 1, \ldots, (N-1)/2$ if N is odd. Hence we arrive at the following main result of this paper.

Theorem 3.1. *Let* $N \in \mathbb{N}$. *The problem* (2.3) *has exactly*

$$\left\lfloor \frac{N-1}{2} \right\rfloor, \quad \textit{where} \quad \lfloor\cdot\rfloor \textit{ denotes the greatest integer function},$$

eigenvalues, and they can be calculated from the formula (3.4)*. The corresponding eigenfunctions are given by*

$$f(t) = \sqrt{q-\lambda_m}^{\log_q t} \sin\left(\frac{m\pi \log_q t}{N}\right). \tag{3.5}$$

Example 3.2*.* Let $N = 6$. For $m = 0$, f given by (3.5) is trivial, so this case does not lead to an eigenvalue. For $m = 1$, the eigenvalue and corresponding eigenfunction is

$$\lambda_1 = \frac{-q^2+q-1}{3} \quad \text{and} \quad f_1(t) = \left(\frac{q+1}{\sqrt{3}}\right)^{\log_q t} \sin\left(\frac{\pi \log_q t}{6}\right).$$

For $m = 2$, the eigenvalue and corresponding eigenfunction is

$$\lambda_2 = -(q^2+q+1) \quad \text{and} \quad f_2(t) = (q+1)^{\log_q t} \sin\left(\frac{\pi \log_q t}{3}\right).$$

Next, $m = 3$ would imply by (3.3) that $q = -1$, which therefore does not lead to an eigenvalue. Similarly, (3.3) implies that $m = 4$ and $m = 5$ leads to $q < -1$. Further values of m result in repetition of the above arguments. Hence there are only two eigenvalues in this case as given above. In particular, if $q = 2$, then the eigenvalues are -1 and -7.

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Positive Solutions of Third Order Boundary Value Problems

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ABSTRACT

We investigate the existence of positive solutions of third order boundary value problems with changing sign Carathéodory nonlinearities of the form. We provide simple sufficient conditions on the nonlinearity f in order to obtain a priori bounds on solutions of a one-parameter family of problems, related to the original one. We then rely on the topological transversality theorem to prove the existence of positive solutions of the given problem. As a byproduct, we shall obtain a multiplicity result.

Keywords: Third order differential equations, three-point boundary value problems, a priori bound on solutions, positive solutions, Granas topological transversality theorem.

2000 Mathematics Subject Classification: 34B15, 34B18.

1 Introduction

In this paper, we are concerned with the existence of positive solutions of three-point boundary value problems for third order differential equations,

$$u'''(t) = f(t, u(t), u'(t), u''(t)) \qquad 0 < t < 1 \tag{1}$$

$$u(0) = u'(a) = u(1) = 0, \;\; 0 < a < 1. \tag{2}$$

Problems of this type arise naturally in the description of physical phenomena, where only positive solutions, i.e. solutions u satisfying $u(t) > 0$ for all $t \in (0,1)$, are meaningful. It is well known that Krasnoselskii's fixed point theorem in a cone (see [14],) has been instrumental in proving existence of positive solutions of problem (1), (2).
In the last decade or so, several papers have been devoted to the study of positive solutions to third order differential equations with two-point or three-point boundary conditions. Most of the previous works assume that f is nonnegative, depends only on u, and some other conditions. See for instance[1], [3], [4], [5],[8], [9], [12], [14], [15] and [16]. It should be pointed out that even in the case of second order boundary problems, only few papers have dealt with changing

sign nonlinearities, that also depend on the first derivative of u. We refer the interested reader to [2] and [6].
Our aim, in this paper, is to establish sufficient conditions on the nonlinearity f that will allow us to obtain a priori bounds on solutions of a one-parameter family of problems related to (1), (2). We, then, rely on the topological transversality theorem of Granas (see [10] for definitions and details) to prove the existence of at least one positive solution of problem (1), (2).
Our assumptions are simple and more general than the conditions found in the literature. In fact, we obtain a multiplicity result as a byproduct of our main result, with no extra assumptions. We exploit the fact that the nonlinearity changes sign with respect to its second argument. We do not rely on cone preserving mappings, and the sign of the Green's function, see see [4], of the corresponding linear homogeneous problem plays no role in our analysis. We assume, however, the existence of positive upper and lower solutions. For general results, not necessarily on positive solutions, see [7], [9] and [11].

2 Preliminaries

Let I denote the real interval $[0,1]$. $AC(I)$ is the Banach space of real-valued absolutely continuous functions on I, equipped with the norm $||u||_0 := max\{|u(t)|; t \in I\}$. For $k = 1,2,\ldots AC^k(I)$ is the Banach space of absolutely continuous functions defined on I together with their derivatives up to order k, with the norm $||u||_{(k)} = \|u\|_0 + \|u'\|_0 + \ldots + \|u^{(k)}\|_0$. $AC_0^2(I)$ is the space of functions $u \in AC^2(I)$ satisfying $u(0) = u'(a) = u(1) = 0;$ $L^1(I)$ is the space of Lebesgue integrable functions on I with its usual norm.
We say that $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function, if f satisfies the following conditions
(i) $f(t,.,.,.)$ is continuous for almost every $t \in I$,
(ii) $f(.,u,v,w)$ is measurable for all $(u,v,w) \in \mathbb{R}^3$,
(iii) for every $R > 0$, there exists $h_R \in L^1(I)$ such that $|u| + |v| + |w| \leq R$ implies $|f(t,u,v,w)| \leq h_R(t)$ for almost all $t \in I$.
Since our arguments are based on the topological transversality theorem we consider the following one-parameter family of problems

$$u'''(t) = \lambda f(t,u(t),\ u'(t),\ u''(t)) \quad 0 < t < 1 \tag{3}$$

$$u(0) = u'(a) = u(1) = 0,\ \ 0 < a < 1 \tag{4}$$

for $0 \leq \lambda \leq 1$.

For $\lambda = 0$ problem (3), (4) has only the trivial solution.
Proof. Obvious.
It follows that the corresponding Green's function $G(t,s)$ exists. To construct $G(t,s)$ we proceed as follows (for a general nth order problem see [13]). Let $u_j(t),\ 1 \leq j \leq 3$ be solutions of $y''' = 0$

such that the following boundary conditions are satisfied

$$\begin{aligned} u_1(0) = 1, \quad & u_1'(a) = 0,\ u_1(1) = 0 \\ u_2(0) = 0, \quad & u_2'(a) = 1,\ u_2(1) = 0 \\ u_3(0) = 0, \quad & u_3'(a) = 0,\ u_3(1) = 1. \end{aligned}$$

Simple computations give

$$u_1(t) = 1 - \frac{t^2 - 2at}{1 - 2a}, \ \ u_2(t) = \frac{-t^2 + t}{1 - 2a}, \ \ u_3(t) = \frac{t^2 - 2at}{1 - 2a}.$$

On the other hand consider the function $v(t,s) := \dfrac{(t-s)^2}{2}$. Then $\dfrac{\partial^3 v}{\partial t^3} = 0$.

Let $v_1(s) := v(0,s),\ \ v_2(s) := v(a,s)$ and $v_3(s) := v(1,s)$, so that

$$v_1(s) = \frac{s^2}{2},\ v_2(s) = \frac{(a-s)^2}{2},\ v_3(s) = \frac{(1-s)^2}{2}.$$

Now, let $\varphi(t,s) = u_1(t)v_1(s) + u_2(t)v_2(s) + u_3(t)v_3(s)$. One can easily show that $\varphi(.,s)$ is a solution of $y''' = 0$, for each fixed s. Moreover $\varphi(0,s) = v(0,s), \varphi'(a,s) = v(a,s)$ and $\varphi(1,s) = v(1,s)$. It follows from the uniqueness of solutions of a linear homogeneous boundary value problem that $\varphi(t,s) = v(t,s)$, that is

$$u_1(t)v_1(s) + u_2(t)v_2(s) + u_3(t)v_3(s) = \frac{(t-s)^2}{2},\ \ \forall\,(t,s)\ \in\ I^2.$$

We define $G(t,s)$ as follows.

For $0 \le s \le a$, we let

$$G(t,s) = \begin{cases} -u_2(t)v_2(s) - u_3(t)v_3(s) & 0 \le t \le s \\ \\ u_1(t)v_1(s) & s \le t \le a \end{cases}$$

and for $a \le s \le 1$, we let

$$G(t,s) = \begin{cases} -u_3(t)v_3(s) & a \le t \le s \\ \\ u_1(t)v_1(s) + u_2(t)v_2(s) & s \le t \le 1. \end{cases}$$

If $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function, then $u \in AC^2(I)$ is a solution of (3), (4) if and only if u is a solution of the integral equation

$$u(t) = \lambda \int_0^1 G(t,s) f(s, u(s), u'(s), u''(s)) ds. \tag{5}$$

Assume that $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function. Then, the operator $T : AC_0^2 \to AC_0^2$, defined by $Tu(t) = \int_0^1 G(t,s) f(s, u(s), u'(s), u''(s)) ds,$ is continuous and completely continuous.

Proof. (i) T is continuous. For, let $u_n \to u$ in $AC_0^2(I)$. Then, for $k = 0, 1, 2, u_n^{(k)} \to u^{(k)}$ uniformly on I. Given that f is continuous with respect to its second, third and fourth arguments, and for

$k = 0, 1$, $\dfrac{\partial^k}{\partial t^k} G(.,.)$ is uniformly continuous on $I \times I$, and also $\dfrac{\partial^2}{\partial t^2} G(.,.)$ is continuous on $I \times I$ except on the set $\{(t,t); t \in I\}$, which has measure zero, we have

$$\frac{\partial^k}{\partial t^k} G(t,s) f(s, u_n(s), u_n'(s), u_n''(s)) \to \frac{\partial^k}{\partial t^k} G(t,s) f(s, u(s), u'(s), u''(s)), \quad \text{as } n \to +\infty,$$

for almost every $s \in I$, and $k = 0, 1, 2$. Now, there exists $M > 0$, independent of $n \in \mathbb{N}$, such that $|u_n| + |u_n'| + |u_n''| \leq M$ for all $n \in \mathbb{N}$. Hence, there is $h_M \in L^1(I)$ such that for all $n \in \mathbb{N}$, we have

$$\left| f(s, u_n(s), u_n'(s), u_n''(s)) \right| \leq h_M(s) \text{ for almost all } s \in I.$$

The Lebesgue dominated convergence theorem implies that, as $n \to +\infty$

$$\int_0^1 \frac{\partial^k}{\partial t^k} G(t,s) f(s, u_n(s), u_n'(s), u_n''(s)) ds \to \int_0^1 \frac{\partial^k}{\partial t^k} G(t,s) f(s, u(s), u'(s), u''(s)) ds.$$

This shows that for $k = 0, 1, 2$, and all $t \in I$

$$(Tu_n)^{(k)}(t) \to (Tu)^{(k)}(t) \quad \text{as } n \to +\infty.$$

Therefore

$$\|Tu_n\|_{(2)} \to \|Tu\|_{(2)} \quad \text{as } n \to +\infty.$$

(ii) T is completely continuous. For, let $B = B(0, r)$ be a ball in $AC_0^2(I)$, and let $u \in B$. Then

$$\|u\|_{(2)} \leq r.$$

Since f is a Carathéodory function there exits $h_r \in L^1(I)$ such that

$$\left| f(t, u(t), u'(t), u''(t)) \right| \leq h_r(t) \text{ for almost all } t \in I.$$

Thus

$$\|Tu\|_{(2)} \leq \max\{|G(t,s)| + \left|\frac{\partial}{\partial t} G(t,s)\right| + \left|\frac{\partial^2}{\partial t^2} G(t,s)\right|; (t,s) \in I^2\} \|h_r\|_{L^1}.$$

This shows that $T(B)$ is uniformly bounded. To show that $T(B)$ is equicontinuous, let $u \in B$ and $0 < t_1 < t_2 < 1$. Then

$$Tu(t_2) - Tu(t_1) = \int_0^1 [G(t_2, s) - G(t_1, s)] f(s, u(s), u'(s), u''(s)) ds$$

so that

$$\begin{aligned} |Tu(t_2) - Tu(t_1)| &\leq \int_0^1 |G(t_2, s) - G(t_1, s)| h_r(s) ds \\ &\leq \max_{s \in I} |G(t_2, s) - G(t_1, s)| \ \|h_r\|_{L^1}. \end{aligned}$$

Since $G(.,.)$ is uniformly continuous on $I \times I$, it follows that $|Tu(t_2) - Tu(t_1)| \to 0$ whenever $|t_2 - t_1| \to 0$. The conclusion follows from Arzela-Ascoli's theorem.

3 Topological Transversality Theory

In this section, we recall the most important notions and results related to the topological transversality theory due to Granas. See [10] for the details of the theory.
Let X be a Banach space, $\mathcal{C}$ a convex subset of X and U an open set in $\mathcal{C}$.

(i) $g : X \to X$ is compact if $\overline{g(X)}$ is compact.

(ii) $H : [0,1] \times X \to X$ is a compact homotopy if H is a homotopy and for all $\lambda \in [0,1]$, $H(\lambda, \cdot) : X \to X$ is compact.

(iii) $g : \overline{U} \to \mathcal{C}$ is called admissible if g is compact and has no fixed points on $\Gamma = \partial U$.

Let $\mathcal{M}_\Gamma(\overline{U}, \mathcal{C})$ denote the class of all admissible maps from $\overline{U}$ to $\mathcal{C}$.

(iv) A compact homotopy H is admissible if, for each $\lambda \in [0,1]$, $H(\lambda, \cdot)$ is admissible.

(v) Two mappings g and h in $\mathcal{M}_\Gamma(\overline{U}, \mathcal{C})$ are homotopic if there is an admissible homotopy $H : [0,1] \times \overline{U} \to \mathcal{C}$ such that $H(0, \cdot) = g$ and $H(1, \cdot) = h$.

(vi) $g \in \mathcal{M}_\Gamma(\overline{U}, \mathcal{C})$ is called inessential if there is a fixed point free compact map $h : \overline{U} \to \mathcal{C}$ such that $g|_\Gamma = h|_\Gamma$. Otherwise, g is called essential.

Let d be an arbitrary point in U and $g \in \mathcal{M}_\Gamma(\overline{U}, \mathcal{C})$ be the constant map $g(x) \equiv d$. Then g is essential.
$g \in \mathcal{M}_\Gamma(\overline{U}, \mathcal{C})$ is inessential if and only if g is homotopic to a fixed point free compact map.
Let g, $h \in \mathcal{M}_\Gamma(\overline{U}, \mathcal{C})$ be homotopic maps. Then g is essential if and only if h is essential.

4 Main Results

Consider the nonlinear problem (3), (4)

$$\begin{cases} u'''(t) = \lambda\, f(t, u(t),\ u'(t),\ u''(t)), & t \in (0,1) \\ u(0) = u'(a) = u(1) = 0, & 0 < a < 1. \end{cases}$$

The nonlinearity $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is an L^1-Carathéodory function and satisfies

(H1) There exist positive functions $\alpha, \beta \in AC_0^2(I)$ such that $\alpha \le \beta$, $\alpha' \le \beta'$ and

(i) $\alpha'''(t) \ge f\left(t, \alpha(t), \alpha'(t), \alpha''(t)\right)$, $\beta'''(t) \le f\left(t, \beta(t), \beta'(t), \beta''(t)\right)$,
(ii) $f\left(t, \alpha(t), \alpha'(t), 0\right) < 0 < f\left(t, \beta(t), \beta'(t), 0\right)$, $\forall t \in I$,
(iii) $f(t, \beta(t), v, w) \le f(t, u, v, w) \le f(t, \alpha(t), v, w)$ for $(t, v, w) \in I \times \mathbb{R}^2$ and $\alpha(t) \le u \le \beta(t)$.

(H2) There exist $Q \in L^1(I; \mathbb{R}_+)$ and $\Psi : [0, +\infty) \to (0, +\infty)$ continuous and nondecreasing with $\dfrac{1}{\Psi}$ integrable over bounded intervals and $\int_0^{+\infty} \dfrac{d\sigma}{\Psi(\sigma)} > \|Q\|_{L^1}$, such that $|f(t, u, v, w)| \le Q(t)\Psi\left(|w|\right)$, $\quad \forall\, t \in I, \alpha \le u \le \beta, \alpha' \le v \le \beta', w \in \mathbb{R}$.

Remark 1. By $\alpha \leq u \leq \beta$ it is meant $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in I$.

Assume (H1) and (H2) are satisfied. Then (1), (2) has at least one positive solution, $u \in [\alpha, \beta]$, with $u' \in [\alpha', \beta']$.
Proof. It will be given in several steps. For $x, y, z \in \mathbb{R}$ with $x \leq z$ let $\delta(x, y, z) = \max(x, \min(y, z))$. Notice that $x \leq \delta(x, y, z) \leq z$ for all $y \in \mathbb{R}$.
Our arguments are based on the topological transversality theory.
For $\lambda \in [0, 1]$, consider the following one-parameter family of problems

$$\begin{cases} u'''(t) = \lambda F(t, u(t),\ u'(t),\ u''(t)) \quad 0 < t < 1 \\ u(0) = u'(a) = u(1) = 0,\ 0 < a < 1 \end{cases} \tag{6.λ}$$

where

$$F(t, u(t), u'(t), u''(t)) = f(t, \delta(\alpha(t), u(t), \beta(t)), \delta(\alpha'(t), u'(t), \beta'(t)), u''(t)).$$

Notice that (6.0) has only the trivial solution, see Lemma 2.1. Hence, we shall consider only the case $0 < \lambda \leq 1$.
Step.1. Consider $K_1 := \max(\|\alpha'\|_0, \|\beta'\|_0)$. All solutions of $(6.\lambda)$ satisfy $|u'(t)| \leq K_1$ and $|u(t)| \leq K_1$ for all $t \in I$.
Let u be a possible solution of $(6.\lambda)$. Then (H1) implies that $|u'(t)| \leq K_1$ for all $t \in I$. Suppose on the contrary that there is a $\tau \in I$ such that $|u'(\tau)| > K_1$, which implies that $u'(\tau) > K_1$ or $u'(\tau) < -K_1$. We consider only the first case, the second case can be handled similarly. Then there exists $t_0 \in I$ such that

$$\max_{t \in I} u'(t) := u'(t_0) > K_1.$$

If $t_0 \in (0, 1)$ then $u''(t_0) = 0$ and $u'''(t_0) \leq 0$. Then, it follows from (H1) that

$$0 \geq u'''(t_0) = \lambda F(t_0, u(t_0),\ u'(t_0),\ u''(t_0)) =$$

$$\lambda f(t_0, \delta(\alpha(t_0), u(t_0), \beta(t_0)), \delta(\alpha'(t_0), u'(t_0), \beta'(t_0)), u''(t_0)) =$$

$$\lambda f(t_0, \delta(\alpha(t_0), u(t_0), \beta(t_0)), \beta'(t_0), 0)$$

$$\geq \lambda f(t_0, \beta(t_0)), \beta'(t_0)), 0) > 0.$$

This is a contradiction.
If $t_0 = 0$, then $u''(0) = 0$ and $u'''(0+) \leq 0$. It follows that $0 \geq u'''(0+) > \lambda f(0, 0, \beta'(0), 0) > 0$.
This is a contradiction.
Similarly, if $t_0 = 1$, then we will reach a contradiction.
Therefore,

$$u'(t) \leq K_1 \text{ for all } t \in I.$$

Similarly, we can prove that $-K_1 \leq u'(t)$ for all $t \in I$.

Hence, we have shown that any solution u of $(6.\lambda)$ is such that

$$\left|u'(t)\right| \leq K_1, \ \forall t \in I. \tag{7}$$

Since $u(0) = 0$, a simple integration gives

$$|u(t)| \leq \int_0^1 \left|u'(s)\right| ds \leq K_1.$$

Hence

$$|u(t)| \leq K_1 \ \forall t \in I. \tag{8}$$

Step.2. A priori bound on the second derivative u'', of solutions u of $(6.\lambda)$.
Define $K_2 > 0$ by the formula $\int_0^{K_2} \dfrac{ds}{\Psi(s)} > \|Q\|_{L^1}$ (this is possible because of the property of Ψ).
We want to show that $|u''(t)| \leq K_2$ for all $t \in I$.
Suppose, on the contrary that there exists $\tau_1 \in I$ such that $|u''(\tau_1)| > K_2$. Then, there exists an interval $[\mu, \xi] \subset [0, 1]$ such that one of the following situations occur

(i) $u''(\mu) = 0, \ u''(\xi) = K_2, \ \ 0 < u''(t) < K_2 \qquad \mu < t < \xi,$
(ii) $u''(\mu) = K_2, \ u''(\xi) = 0, \ \ 0 < u''(t) < K_2 \qquad \mu < t < \xi,$
(iii) $u''(\mu) = 0, \ u''(\xi) = -K_2, \ -K_2 < u''(t) < 0 \qquad \mu < t < \xi,$
(iv) $u''(\mu) = -K_2, \ u''(\xi) = 0, \ -K_2 < u''(t) < 0 \qquad \mu < t < \xi.$

We study the first case. The other cases can be handled in a similar way. We have

$$u'''(t) \leq Q(t)\Psi\left(u''(t)\right), \qquad \mu \leq t \leq \xi.$$

This implies

$$\frac{u'''(t)}{\Psi\left(u''(t)\right)} \leq Q(t) \quad \text{ for } \ \mu \leq t \leq \xi.$$

An integration from μ to ξ, and a change of variables lead to

$$\int_0^{K_2} \frac{ds}{\Psi(s)} \leq \|Q\|_{L^1}.$$

This clearly contradicts the definition of K_2. Taking into consideration the four cases above, we see that

$$\left|u''(t)\right| \leq K_2 \quad \text{for all } t \in I.$$

Let

$$K_3 := \max\{K_2, \left\|\alpha''\right\|_0, \left\|\beta''\right\|_0\}.$$

Then, any solution u of $(6.\lambda)$ is such that its second derivative u'' will satisfy the a priori bound

$$\left|u''(t)\right| \leq K_3 \quad \text{for all } t \in I. \tag{9}$$

From Step.1 and Step.2 above and the fact that f is an L^1-Carathéodory function, we deduce that there exists a positive constant K, such that

$$\|u\|_{(2)} \leq K \tag{10}$$

for any solution u of $(6.\lambda)$.

Step.3. Existence of solutions of $(6.\lambda)$. It follows from (5) that problem $(6.\lambda)$ is equivalent to

$$u(t) = \lambda \int_0^1 G(t,s)F(s,u(s),\, u'(s),\, u''(s))ds. \tag{11}$$

Define $H : [0,1] \times AC_0^2(I) \to AC_0^2(I)$ by

$$H(\lambda, u)(t) = \lambda \int_0^1 G(t,s)F(s,u(s),\, u'(s),\, u''(s))ds \qquad \text{for all } \lambda \in [0,1],\ t \in I. \tag{12}$$

Let

$$U := \{u \in AC_0^2(I); \quad \|u\|_{(2)} < 1 + K\},$$

where K is the constant from (10).

Lemma 2.2 implies that $H(\lambda, .) : \overline{U} \to AC_0^2(I)$ is compact. It is clear from Steps 1 and 2 above and the choice of U that there is no $u \in \partial U$ such that $H(\lambda, u) = u$ for $\lambda \in [0,1]$. It is also clear that $H(.,.)$ is uniformly continuous in λ.

Therefore, $H(\lambda, .) : \overline{U} \to AC_0^2(I)$ is an admissible homotopy between the constant map $H(0,.) = 0$ and the compact map $H(1,.)$. Since $0 \in U$, we have that $H(0,.)$ is essential. By the topological transversality theorem of Granas, $H(1,.)$ is essential. This implies that it has a fixed point in U, and this fixed point is a solution of (6.1). Since solutions of (6.1) are solutions of (1), (2), we conclude that (1), (2) has at least one solution $u_0 \in U$.

Step.4. We show that $\alpha'(t) \leq u_0'(t)$ for all $t \in I$. A simple integration will then give $\alpha(t) \leq u_0(t)$ for all $t \in I$.

Suppose, on the contrary, that there is $\eta \in I$ such that $\alpha'(\eta) > u_0'(\eta)$. Let $\alpha'(c) - u_0'(c) := \max\{\alpha'(t) - u_0'(t); t \in I\}$. Then, $\alpha'(c) > u_0'(c)$, $\alpha''(c) = u_0''(c)$ and $\alpha'''(c) \leq u_0'''(c)$. This implies the following contradiction,

$$\begin{aligned}
&0 \geq \alpha'''(c) - u_0'''(c) \geq \\
&\ f\left(c, \alpha\left(c\right), \alpha'\left(c\right), \alpha''\left(c\right)\right) - f(c, \delta(\alpha\left(c\right), u_0\left(c\right), \beta\left(c\right)), \delta(\alpha'\left(c\right), u_0'\left(c\right), \beta'\left(c\right)), u_0''\left(c\right)) \\
&\ = f\left(c, \alpha\left(c\right), \alpha'\left(c\right), \alpha''\left(c\right)\right) - f(c, \delta(\alpha\left(c\right), u_0\left(c\right), \beta\left(c\right)), \alpha'\left(c\right), \alpha''\left(c\right)) > 0.
\end{aligned}$$

The last inequality follows from the fact that $\alpha\left(c\right) \leq \delta(\alpha\left(c\right), u_0\left(c\right), \beta\left(c\right)) \leq \beta\left(c\right)$ and assumption (H1)(iii).

In a similar way, we prove that $u_0'(t) \leq \beta'(t)$ for all $t \in I$, which will imply that $u_0(t) \leq \beta(t)$ for all $t \in I$.

This completes the proof of the main result.

Remark 2. It is possible to obtain a uniqueness result if we assume, in addition to (H1) and (H2), the following condition

(H3) There exists $\ell > 0$, such that $f(t, u_1, v_1, w) - f(t, u_2, v_2, w) > \ell(v_1 - v_2)$ for $\alpha' \leq v_2 \leq v_1 \leq \beta'$, all $t \in I$, $w \in \mathbb{R}$, and $\alpha \leq u_1, u_2 \leq \beta$.

Assume that the conditions (H1), (H2) and (H3) hold. Then Problem (1), (2) has a unique positive solution $u \in [\alpha, \beta]$ with $u' \in [\alpha', \beta']$.

Proof. Theorem 4.1 guarantees the existence of at least one solution $u \in [\alpha, \beta]$ with $u' \in [\alpha', \beta']$.

Suppose there are two solutions $u_1, u_2 \in [\alpha, \beta]$. We exhibit that $u_1'(t) = u_2'(t)$ for all $t \in I$. A simple integration will give $u_1(t) = u_2(t)$ for all $t \in I$. Suppose on the contrary that $u_1'(\xi) \neq u_2'(\xi)$ for some $\xi \in I$. Let $z(t) := u_1'(t) - u_2'(t)$ for all $t \in I$. Suppose, first, that $z(\xi) > 0$. Let $z(\eta) = \max\{z(t); t \in I\}$. Then, $z(\eta) > 0$, $z'(\eta) = 0$ and $z''(\eta) \leq 0$.

Since

$$z''(t) = u_1'''(t) - u_2'''(t) = f(t, u_1(t), u_1'(t), u_1''(t)) - f(t, u_2(t), u_2'(t), u_2''(t))$$

and

$$z'(\eta) = u_1''(\eta) - u_2''(\eta) = 0$$

we have

$$\begin{aligned} 0 \geq z''(\eta) &= f(t, u_1(\eta), u_1'(\eta), u_1''(\eta)) - f(t, u_2(\eta), u_2'(\eta), u_2''(\eta)) = \\ &= f(t, u_1(\eta), u_1'(\eta), u_1''(\eta)) - f(t, u_2(\eta), u_2'(\eta), u_1''(\eta)) > \ell\, z(\eta) > 0. \end{aligned}$$

This is a contradiction.

Similarly, if we assume $z(\xi) < 0$, we will arrive at a contradiction.

Therefore

$$u_1'(t) = u_2'(t) \text{ for all } t \in I.$$

This yields

$$u_1(t) = u_2(t) \text{ for all } t \in I,$$

proving uniqueness.

5 Multiplicity of Solutions

In this section we use Theorem 4.1 to get multiplicity of solutions of problem (1), (2).

Assume $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function and satisfies

(H4) there are sequences $\{\alpha_j\}, \{\beta_j\}$ of positive functions in $AC_0^2(I)$ such that for all $j = 1, 2, ...$

(i) $0 \leq \alpha_j \leq \beta_j \leq \alpha_{j+1}$, and $\alpha_j' \leq \beta_j' \leq \alpha_{j+1}'$,

(ii) $\alpha_j'''(t) \geq f(t, \alpha_j(t), \alpha_j'(t), \alpha_j''(t))$, $\beta_j'''(t) \leq f(t, \beta_j(t), \beta_j'(t), \beta_j''(t))$,

(iii) $f(t, \alpha_j(t), \alpha_j'(t), 0) < 0 < f(t, \beta_j(t), \beta_j'(t), 0)$, $t \in I$,

(iv) $f(t, \beta_j(t), v, w) \leq f(t, u, v, w) \leq f(t, \alpha_j(t), v, w)$ for $(t, v, w) \in I \times \mathbb{R}^2$, $\alpha_j(t) \leq u \leq \beta_j(t)$,

(v) The condition (H2) holds on $I \times [\alpha_j, \beta_j] \times [\alpha_j', \beta_j'] \times \mathbb{R}$.

Then, Problem (1), (2) has infinitely many positive solutions u_j such that $\alpha_j \leq u_j \leq \beta_j$, and $\alpha_j' \leq u' \leq \beta_j'$.

6 Example

Consider the following problem

$$\begin{cases} u'''(t) = \phi(t)\,(1 + u''(t)^2)(1 + \cos(u'(t))g(u(t)) & 0 < t < 1 \\ u(0) = u'(a) = u(1) = 0 \end{cases} \tag{13}$$

where $\phi \in L^1(I)$, $\phi(t) \geq 0$ for all $t \in I$, and $g : \mathbb{R} \to \mathbb{R}$ is continuous and has an infinite number of positive simple zeros. This is the case if we assume the existence of an increasing sequence $\{a_j\}_{j\in\mathbb{N}}$ of positive numbers such that

$$g(a_j)\; g(a_{j+1}) < 0 \qquad \text{for } j = 0, 1, ...$$

None of the results in the previously published works can be applied to problem (13). However, f, defined by

$$f(t, u, v, w) = \phi(t)\,(1 + w^2)(1 + \cos v)g(u)$$

satisfies condition (**H4**) of our Theorem 5.1.

Hence Problem (13) has infinitely many positive solutions.

Remark 3. A typical example for g is $g(u) = \sin u$, whose positive zeros form an infinite sequence $\{n\pi; n = 1, 2, ...\}$.

It is clear that the differential equation

$$u'''(t) = \phi(t)\,(1 + u''(t)^2)(1 + \cos(u'(t))\sin(u(t))$$

has infinitely many positive solutions, $u_n(t) = n\pi,\ \ n \geq 1.$

The function f, defined by

$$f(t, u, v, w) = \phi(t)\,(1 + w^2)(1 + \cos v)\sin u \qquad 0 \leq t \leq 1$$

changes sign infinitely many times. In fact, we have

$$f(t, \alpha_j, v, w) < 0 \quad \text{for} \quad \alpha_j = \left(\frac{3}{2} + 2j\right)\pi, \quad j = 0, 1, 2, ...$$

and

$$f(t, \beta_j, v, w) > 0 \quad \text{for} \quad \beta_j = \left(\frac{5}{2} + 2j\right)\pi, \quad j = 0, 1, 2, ...$$

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Controllability of Nondensely Defined Impulsive Functional Semilinear Differential Inclusions in Fréchet Spaces

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ABSTRACT

Frigon nonlinear alternative for multivalued admissible contractions in Fréchet spaces, combined with integral semigroup theory, is used to investigate the controllability of some classes of impulsive semilinear functional and neutral functional differential inclusions in Fréchet spaces.

Key words and phrases: *Impulsive functional differential inclusions, controllability, integral solution, fixed point, Fréchet space, admissible contraction.*

AMS (MOS) Subject Classifications: 34G25, 34K25, 93B05

1 Introduction

This paper is concerned with an application of a recent Frigon nonlinear alternative, for admissible contraction maps in Fréchet spaces (Frigon, 2002), to the existence of integral solutions of first order controllability associated with impulsive semilinear functional and neutral functional differential inclusions in Fréchet spaces. In Section 3, we will consider first order controllability for the impulsive semilinear functional differential inclusion,

$$y'(t) - Ay(t) \in F(t, y_t) + (Bu)(t), \quad a.e.\ t \in J := [0, \infty) \setminus \{t_1, t_2, \ldots\}, \tag{1.1}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, 2, \ldots, \tag{1.2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{1.3}$$

where $r > 0,\ F : J \times \mathcal{D} \to \mathcal{P}(E)$ is a multivalued map with compact values ($P(E)$ is the family of all nonempty subsets of E), $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator on E, B is a bounded linear operator from $\overline{D(A)}$ into $\overline{D(A)}$, $\mathcal{D} = \{\psi : [-r, 0] \to \overline{D(A)} : \psi$ is continuous everywhere except for a countable number of points $\bar{t}$ at which $\psi(\bar{t}^-)$ and $\psi(\bar{t}^+)$ exist, $\psi(\bar{t}^-) = \psi(\bar{t})$ and $\sup_{\theta \in [-r,0]} |\psi(\theta)| < \infty\}$, $(0 < r < \infty),\ 0 = t_0 < t_1 < \ldots < t_m < \ldots,\ \lim_{n\to\infty} t_n = \infty$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k - h)$, and $I_k \in C(\overline{D(A)}, \overline{D(A)}),\ k \in \{1, 2, \ldots\}$. In Section 4 we study the first order controllability of impulsive semilinear neutral functional differential inclusions of the form,

$$\frac{d}{dt}[y(t) - g(t, y_t)] - Ay(t) \in F(t, y_t) + (Bu)(t), \quad \text{a.e. } t \in [0, \infty) \setminus \{t_1, t_2, \ldots\}, \tag{1.4}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \ \ k = 1, \ldots, \tag{1.5}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{1.6}$$

where $F,\ A, I_k$ and ϕ d are as in problem (1.1)–(1.3), and $g : J \times \mathcal{D} \to \overline{D(A)}$.

Impulsive differential and partial differential equations have become more important in recent years in some mathematical models of real phenomena, especially in control, biological or medical domains; see the monographs of Lakshmikantham *et al* (Lakshmikantham and Simeonov, 1989) and Samoilenko and Perestyuk (Samoilenko and Perestyuk, 1995), the papers of Ahmed (Ahmed, 2001; Ahmed, 2000) and Liu (Liu, 1999), and the survey paper by Rogovchenko (Rogovchenko, 1997) and the references therein. In the case where $I_k = \overline{I}_k \equiv 0,\ k = 1, \ldots$, and A is a densely defined linear operator generating a semigroup, the controllability of differential inclusions with different conditions was studied by Benchohra *et al* (Benchohra and Ntouyas, 2003a; Benchohra and Ntouyas, 2003b), Balachandran and Manimegolai (Balachandran and Manimegalai, 2002) and Li and Xue (Li and Xue, 2003). Very recently, Guo *et al* (Guo and Li, 2004) initiated the study of controllability of impulsive evolution inclusions with nonlocal conditions, where they considered a class of first order evolution inclusions with a convex valued right side. For the non-convexity of the right side, the controllability of first order impulsive functional differential inclusions, with a fixed number of impulses, was studied by Benchohra *et al* (Benchohra and Ouahab, 2004). As we know, the investigation of many properties of solutions for a given equation, such as stability or oscillation, needs its guarantee of global existence. Thus, it is important and necessary to establish sufficient conditions for global existence of solutions for impulsive differential equations. For the case where $A = B \equiv 0$, the global existence results for impulsive differential equations and inclusions with different conditions were studied by Benchohra *et al* (Benchohra and Ouahab, (in press)a), Cheng and Yan (Cheng and Yan, 2001), Graef and Ouahab (Graef and Ouahab, (in press)), Guo (Guo, 1999; Guo, 2002), Guo and Liu (Guo and Liu, 1996), Henderson and Ouahab (Henderson and Ouahab, 2005b), Marino *et al* (Marino and Muglia, 2004), Ouahab (Ouahab, (in press)), Stamov and Stamova (Stamov and Stamova, 1996), Weng (Weng, 2002) and Yan (Yan, 1997; Yan, 1999).

On infinite intervals, and still when A is a densely defined linear operator generating C_0-semigroup families of linear bounded operators and F is a single valued map, the problems (1.1)-(1.3) and (1.4)-(1.6) were studied by Arara, Benchohra and Ouahab (Arara and Ouahab, 2003) by means of the nonlinear alternative for contraction maps in Fréchet spaces due to Frigon and Granas (Frigon and Granas, 1998). For the case where the impulses are absent, $I_k = \overline{I}_k \equiv 0,\ k = 1, \ldots$, an application of a recent nonlinear alternative due to Frigon (Frigon, 2002) was applied by Benchohra and Ouahab (Benchohra and Ouahab, (in press)b).

Recently, the existence of integral solutions on compact intervals for the problem (1.1), (1.3) with periodic boundary conditions in a Banach space was considered by Ezzinbi and Liu (Ezzinbi and Liu, 2002). For more details on nondensely defined operators and the concept of integrated semigroups, we refer to the monograph (Ahmed, 2001) and to the papers (Arendt, 1987a; Arendt, 1987b; Busenberg and Wu, 1992; Da Prato and Sinestrari, 1987; Neubrander, 1988; Thieme, 1990). Very recently, global exact controllability for semilinear differential inclusions with nondensely defined operators was studied by Henderson and Ouahab (Henderson and Ouahab, 2005a). For more details and examples on nondensely defined operators, we refer to the survey paper by Da Prato and Sinestrari (Da Prato and Sinestrari, 1987) and the paper by Ezzinbi and Liu (Ezzinbi and Liu, 2002).

Our goal here is to give existence results for the above problems by using this nonlinear alternative for multivalued admissible contractions in Fréchet spaces, and to extend some results considered very recently by Arara *et al* (Arara and Ouahab, 2003), Benchohra *et al* (Benchohra and Ouahab, 2004), Benchohra and Ouahab (Benchohra and Ouahab, (in press)b), and Henderson and Ouahab (Henderson and Ouahab, 2005a).

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $[0, b]$ be a interval in $\mathbb{R}$, and $C([0, b], E)$ is the Banach space of all continuous functions from $[0, b]$ into E with the norm,

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le b\}.$$

$B(E)$ is the Banach space of all bounded linear operators from E into E with norm,

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

A measurable function $y : [0, b] \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, Yosida (Yosida, 1980)).
$L^1([0, b], E)$ denotes the Banach space of functions $y : [0, b] \longrightarrow E$ which are Bochner integrable and normed by,

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

Definition 2.1 ((Arendt, 1987a))**.** We say that a family $\{S(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is an integrated semigroup family if,

(1) $S(0) = 0.$

(2) $t \to S(t)$ is strongly continuous.

(3) $S(s)S(t) = \int_0^s (S(t + r) - S(r))dr$ for all $t, s \geq 0.$

Definition 2.2 ((Kellermann and Hieber, 1989))**.** An operator A is called a generator of an integrated semigroup, if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ ($\rho(A)$ is the resolvent set of A), and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of bounded operators such that $S(0) = 0$ and $(\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt$ exists for all λ, with $\lambda > \omega$.

Lemma 2.1 ((Arendt, 1987a))**.** *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then, for all $x \in E$ and $t \geq 0$,*

$$\int_0^t S(s)xds \in D(A) \quad \textit{and} \quad S(t)x = A\int_0^t S(s)xds + tx.$$

Definition 2.3. We say that a linear operator A satisfies the "Hille-Yosida condition" if there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$, and

$$\sup\{(\lambda - \omega)^n |(\lambda I - A)^{-n}| : n \in \mathbb{N}, \lambda > \omega\} \leq M.$$

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from (Arendt, 1987a), $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$ and $(S'(t))_{t\geq 0}$ is a C_0- semigroup on $\overline{D(A)}$.

Here and hereafter, we assume,

(H1) A satisfies the Hille-Yosida condition.

Let $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A. Then we have the following.

Theorem 2.2 ((Arendt, 1987a), (Kellermann and Hieber, 1989))**.** *Let $f : [0, T] \to E$ be a continuous function. Then, for $y_0 \in \overline{D(A)}$, there exists a unique continuous function $y : [0, T] \to E$ such that,*

(i) $\int_0^t y(s)ds \in D(A), \quad t \in [0, T],$

(ii) $y(t) = y_0 + A\int_0^t y(s)ds + \int_0^t f(s)ds, \ \ t \in [0,T],$

(iii) $|y(t)| \leq Me^{\omega t}(|y_0| + \int_0^t e^{-\omega s}|f(s)|ds), \quad t \in [0,T].$

Moreover, y satisfies the variation of constants formula,

$$y(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)f(s)ds, \quad t \geq 0. \tag{2.1}$$

Let $B_\lambda = \lambda R(\lambda, A)$, where $R(\lambda, A) := (\lambda I - A)^{-1}$. Then for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$. As a consequence, if y satisfies (2.1), then

$$y(t) = S'(t)y_0 + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s)ds, \quad t \geq 0.$$

Throughout this paper we will use the following notations. $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$. We denote by $D_\alpha, \ \alpha \in \bigwedge$, the Hausdorff pseudo-metric induced by d_α; that is, for $A, B \in \mathcal{P}(X)$,

$$D_\alpha(A,B) = \inf\Big\{\varepsilon > 0 : \forall x \in A, \ \forall y \in B, \ \exists \bar{x} \in A, \ \bar{y} \in B \text{ such that } d_\alpha(x,\bar{y}) < \varepsilon, d_\alpha(\bar{x},y) < \varepsilon\Big\}$$

with $\inf \emptyset = \infty$. In the particular case where X is a complete locally convex space, we say that a subset $A \subset X$ is bounded if $D_\alpha(\{0\}, A) < \infty$ for every $\alpha \in \bigwedge$. More details can be found in (Frigon, 2002).

Definition 2.4. A multivalued map $F : X \to \mathcal{P}(E)$ is called an admissible contraction with constants $\{k_\alpha\}_{\alpha\in\bigwedge}$ if, for each $\alpha \in \bigwedge$, there exists $k_\alpha \in (0,1)$ such that,

i) $D_\alpha(F(x), F(y)) \leq k_\alpha d_\alpha(x,y)$ for all $x, y \in X$.

ii) For every $x \in X$ and every $\varepsilon \in (0,\infty)^{\bigwedge}$, there exists $y \in F(x)$ such that

$$d_\alpha(x,y) \leq d_\alpha(x, F(x)) + \varepsilon_\alpha \ for \ every \ \alpha \in \bigwedge.$$

Lemma 2.3. *(Nonlinear Alternative, (Frigon, 2002)). Let E be a Fréchet space and U an open neighborhood of the origin in E, and let $N : \overline{U} \to \mathcal{P}(E)$ be an admissible multivalued contraction. Assume that N is bounded. Then one of the following statements holds:*

(C1) N has at least one fixed point.

(C2) There exist $\lambda \in [0,1), \ n \in$ IN, and $x \in \partial U$ such that $x \in \lambda N(x)$.

For applications of Lemma 2.3, we consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by,

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b),\ d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

In what follows, we will assume that the function $F : [0,\infty)\times\mathcal{D} \to \mathcal{P}(E)$ is an L^1_{loc}- Carathéodory function; that is,

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathcal{D}$;

(ii) $x \longmapsto F(t,x)$ is continuous for almost all $t \in [0,\infty)$;

(iii) For each $q > 0$, there exists $h_q \in L^1_{loc}([0,\infty), \mathbb{R}_+)$ such that

$$\|F(t,x)\| \le h_q(t) \quad \text{for all } \|x\| \le q, \text{ for each } x \in \mathcal{D} \text{ and for almost all } t \in [0,\infty).$$

For more details on multivalued maps, we refer the reader to the books of Casting and Valaadin (Castaing and Valadier, 1977), Deimling (Deimling, 1992), Gorniewicz (Gorniewicz, 1999), Hu and Papageorgiou (Hu and Papageorgiou, 1997), Kamenskii (Kamenskii and Zecca, 2001) and Tolstonogov (Tolstonogov, 2000).

3 Controllability of Impulsive FDIs

In this section we shall establish sufficient conditions for the controllability of the first order functional semilinear differential inclusions (1.1)-(1.3). We shall first consider the space,

$$\begin{aligned} PC &= \{y : [0,\infty) \to \overline{D(A)} \mid y_k \in C(J_k, \overline{D(A)}), k = 0,\ldots,m, \text{ and there exist } y(t_k^-) \\ &\quad \text{and } y(t_k^+), k = 1,\ldots,m, \text{ with } y(t_k^-) = y(t_k)\}. \end{aligned}$$

Set

$$\Omega = \{y : J_1 \to \overline{D}(A) : y \in \mathcal{D} \cap \Omega\},\ J_1 = [-r,0] \cup J.$$

Definition 3.1. A function $y \in \Omega$ is said to be an integral solution of (1.1)-(1.3) if there exists a function $v \in L^1(J, \overline{D(A)})$ such that $v(t) \in F(t, y_t)$ a.e $t \in [0,\infty)$, and

$$y(t) = S'(t)\phi(0) + A\int_0^t y(s)ds + \int_0^t v(s)ds + \int_0^t Bu(s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)),$$

$\int_0^t y(s)ds \in \overline{D(A)},$ for $t \in [0,\infty)$, and $y(t) = \phi(t),\ t \in [-r,0]$.

Before stating and proving our main result of this section, we give also the definition of controllability on the interval J_1.

Definition 3.2. The system (1.1)–(1.3) is said to be infinite controllable on the interval J_1, if for every initial function $\phi \in \mathcal{D}$ and every $y_1 \in \overline{D(A)}$, and for each $n \in \mathbb{N}$, there exists a control $u \in L^2([0,t_n], U)$, such that the integral solution $y(t)$ of (1.1)–(1.3) satisfies $y(t_n) = y_1$.

Let us introduce the following hypotheses:

(H2) $\phi(0) \in \overline{D(A)}$ and there exist constants c_k, such that $|I_k(y)| \le c_k$, for each $y \in E$.

(H3) $F : [0,\infty) \times \mathcal{D} \longrightarrow \mathcal{P}(\overline{D(A)})$ is a nonempty compact-valued, multivalued map, and for all $R > 0$, there exists $l_R \in L^1_{loc}([-r,\infty), \mathbb{R}_+)$ such that

$$H_d(F(t,x), F(t,\overline{x})) \le l_R(t)\|x - \overline{x}\| \text{ for all } x,\ \overline{u} \in \mathcal{D} \text{ with } \|x\|, \|\overline{x}\| \le R,$$

and

$$d(0, F(t,0)) \le l_R(t) \text{ for a.e. } t \in J.$$

(H4) There exist a continuous nondecreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t,u)\| \le p(t)\psi(\|u\|_{\mathcal{D}}) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{D},$$

with

$$\int_c^\infty \frac{du}{u + \psi(u)} = \infty.$$

(H5) For every $n > 0$, the linear operator $W : L^2(J_n, U) \to E$ $(J_n = [0, t_n])$ defined by,

$$Wu = \int_0^{t_n} S'(t_n - s)Bu(s)ds,$$

has an invertible operator W^{-1} which takes values in $L^2(J_n, U)\backslash KerW$, and there exist positive constants $\overline{M},\ \overline{M}_1$ such that $\|B\| \le \overline{M}$ and $\|W^{-1}\| \le \overline{M}_1$.

(H6) There exist constants $b_k \ge 0$, $k = 1, \ldots, m$, such that

$$\|I_k(y) - I_k(x)\| \le b_k\|x - \overline{x}\| \text{ for each } x, \overline{x} \in \overline{D(A)}.$$

Remark 3.1. For the construction of W, see (Carmichael and Quinn, 1984-1985).

Theorem 3.1. *Assume that hypotheses (H1)-(H6) hold. If* $\sum_{k=1}^{\infty} Mb_k < 1$, *then the IVP (1.1)–(1.3) has at least one integral solution.*

Proof. We begin by defining a family of semi-norms on Ω, thus rendering Ω into a Fréchet space. Let τ be sufficiently large. Then, for each $n \in \mathbb{N}$, we define in Ω the semi-norms by

$$\|y\|_n = \sup\{e^{-|\omega|t - \tau L_n(t)}|y(t)| : -r \le t \le t_n\},$$

where $L_n(t) = \int_0^t \widetilde{l}_n(s)ds$ and

$$\widetilde{l}_n(t) = \max\{t_n\overline{M}M^2\overline{M}_1 e^{|\omega|t_n} l_n(t), \overline{M}M^2\overline{M}_1 e^{|\omega|t_n}, Ml_n(t)\}.$$

Thus $\Omega = \bigcup_{n\ge1} \Omega_n$, where

$$\Omega_n = \{y : [-r, t_n] \to \overline{D(A)} : y \in \mathcal{D} \cap PC_n(J, \overline{D(A)})\},$$

and $PC_n = \{y : [0, t_n] \to \overline{D(A)}$ such that $y(t)$ is continuous everywhere except for some t_k at which $y(t_k^-)$ and $y(t_k^+)$ exist, and $y(t_k^-) = y(t_k)$, $k = 1, 2, \ldots, n-1\}$. Then Ω is a Fréchet space with the family of semi-norms $\{\| \cdot \|_n\}$.

Now, using hypothesis (H5) for each $y(\cdot)$ and

$$v \in S_{F,y} = \{v \in L^1(J, \overline{D(A)}) : v(t) \in F(t, y_t) \; a.e \; t \in J\},$$

define the control,

$$\begin{aligned} u_y(t) &= W^{-1}[y_1 - S'(t_n)\phi(0) - \lim_{\lambda \to +\infty} \int_0^{t_n} S'(t_n - s) B_\lambda v(s) ds \\ &\quad - \sum_{0<t_k<t} S'(t_n - t_k) I_k(y(t_k^-))](t). \end{aligned}$$

Now, transform the problem (1.1)–(1.3) into a fixed point problem. Consider the operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by,

$$N(y) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ S'(t)\phi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)v(s)ds & \\ +\frac{d}{dt}\displaystyle\int_0^t S(t-s)(Bu_y)(s)ds & \\ +\displaystyle\sum_{0<t_k<t} S'(t-t_k) I_k(y(t_k^-)), & \text{if } t \in J, \end{cases} \right\}$$

where $v \in S_{F,y}$. Clearly, the fixed points of the operator N are solutions of the problem (1.1)–(1.3).

Let y be a possible solution of the problem (1.1)–(1.3). Given $n \in \mathbb{N}$ and $t \le t_n$, then $y \in N(y)$, and there exists $v \in S_{F,y}$ such that for each $t \in [0, \infty)$, we have

$$\begin{aligned} |y(t)| &\le Me^{|\omega|t}|\phi(0)| + Me^{|\omega|t}\int_0^t e^{-|\omega|s} p(s)\psi(\|y_s\|)ds \\ &\quad + Me^{|\omega|t}\int_0^t e^{-|\omega|s}\|(Bu_y(s)\|ds + Me^{|\omega|t}\sum_{k=1}^n e^{-|\omega|t_k} c_k \\ &\le Me^{|\omega|t}|\phi(0)| + Me^{|\omega|t}\int_0^t e^{-|\omega|s} p(s)\psi(\|y_s\|)ds \\ &\quad + M\overline{M}e^{|\omega|t}\overline{M}_1 t_n\Big(\|y_1\| + Me^{|\omega|t_n}|\phi(0)| + Me^{|\omega|t_n}\sum_{k=1}^n e^{-|\omega|t_k} c_k \\ &\quad + Me^{|\omega|t_n}\int_0^t e^{-|\omega|s} p(s)\psi(\|y_s\|)ds\Big) + Me^{|\omega|t}\sum_{k=1}^n e^{-|\omega|t_k} c_k. \end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \;\; 0 \le t \le t_n.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, t_n]$, then by the previous inequality we have for $t \in [0, t_n]$,

$$\begin{aligned} e^{-|\omega|t}\mu(t) &\leq M|\phi(0)| + M\sum_{k=1}^{n} e^{-\omega t_k}c_k + \overline{M}M\overline{M}_1 t_n\|y_1\| \\ &+ \overline{M}M\overline{M}_1 t_n[Me^{|\omega|t_n}|\phi(0)| + Me^{|\omega|t_n}\sum_{k=1}^{n} e^{-|\omega|t_k}c_k] \\ &+ \int_0^t e^{-|\omega|s}m(t)\psi(\mu(s))ds, \end{aligned}$$

where

$$m(t) = [M + \overline{M}M\overline{M}_1 t_n e^{|\omega|t_n}]p(t).$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\mu(t) \leq e^{\omega t}v(t) \text{ for all } t \in [0, t_n],$$

and

$$v(0) = \left(\overline{M}M\overline{M}_1 t_n\|y_1\| + [\overline{M}M^2\overline{M}_1 t_n + M]|\phi(0)| + [\overline{M}M^2\overline{M}_1 t_n + M]\sum_{k=1}^{n} e^{-|\omega|t_k}c_k\right),$$

and

$$v'(t) = Me^{-|\omega|t}m(t)\psi(\mu(t)), \ \ t \in [0, t_n].$$

Using the increasing character of ψ we get

$$v'(t) \leq Me^{-|\omega|t}m(t)\psi(e^{|\omega|t}v(t)) \ \text{ a.e. } \ t \in [0, t_n].$$

Then for each $t \in [0, t_n]$ we have

$$\begin{aligned} (e^{|\omega|t}v(t))' &= |\omega|e^{|\omega|t}v(t) + v'(t)e^{|\omega|t} \\ &\leq |\omega|e^{|\omega|t}v(t) + Mp(t)\psi(e^{|\omega|t}v(t)) \\ &\leq m(t)[e^{|\omega|t}v(t) + \psi(e^{|\omega|t}v(t))], \ \ t \in [0, t_n]. \end{aligned}$$

Thus

$$\int_{v(0)}^{e^{|\omega|t}v(t)} \frac{du}{u + \psi(u)} \leq \int_0^{t_n} m(s)ds < \int_{v(0)}^{\infty} \frac{du}{u + \psi(u)}.$$

Consequently, there exists a constant d_n such that $v(t) \leq d_n, \ t \in [0, t_n]$, and hence $\|y\|_n \leq \max(\|\phi\|, d_n) := K_n$. Set

$$U = \{y \in \Omega : \|y\|_n < K_n + 1 \text{ for all } n \in \mathbb{N}\}.$$

Clearly, U is a open subset of Ω.

We shall show that $N : \overline{U} \to \mathcal{P}(\Omega)$ is an admissible contraction operator; that is, there exists $\gamma < 1$, such that

$$H_d(N(y), N(\overline{y})) \le \gamma \|y - \overline{y}\|_n \text{ for each } y, \overline{y} \in PC_n.$$

Let $y, \overline{y} \in PC_n$ and $h \in N(y)$. Then there exists $v(t) \in F(t, y_t)$, so that

$$\begin{aligned} h(t) &= S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)v(s)ds \\ &\quad + \frac{d}{dt}\int_0^t S(t-s)(Bu_y)(s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)). \end{aligned}$$

By (H3) and (H6), it follows that

$$H_d(F(t, y_t), F(t, \overline{y}_t)) \le l_n(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}.$$

Hence, there is $w \in F(t, \overline{y}_t)$ such that

$$|v(t) - w| \le l_n(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}, \quad t \in J_n.$$

Consider $U : J_n \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : |v(t) - w| \le l_n(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable (see Proposition III.4 in (Castaing and Valadier, 1977)), there exists a function $t \to \overline{v}(t)$, which is a measurable selection for V. So, $\overline{v}(t) \in F(t, \overline{y}_t)$ and

$$|v(t) - \overline{v}(t)| \le l_n(t)\|y_t - \overline{y}_t\|_{\mathcal{D}} \text{ for each } t \in J_n.$$

Let us define, for each $t \in J_n$,

$$\begin{aligned} \overline{h}(t) &= S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)\overline{v}(s)ds \\ &\quad + \frac{d}{dt}\int_0^t S(t-s)(Bu_{\overline{y}})(s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(\overline{y}(t_k^-)). \end{aligned}$$

Then we have

$$\begin{aligned} \|h(t) - \overline{h}(t)\| &= \Big\| \frac{d}{dt}\int_0^t S(t-s)[(Bu_y)(s) - (Bu_{\overline{y}})(s)]ds \\ &\quad + \frac{d}{dt}\int_0^t S(t-s)[v(s) - \overline{v}(s)]ds \\ &\quad + \sum_{0<t_k<t_n} S'(t-t_k)[I_k(y(t_k)) - I_k((\overline{y}(t_k)))] \Big\| \\ &\le Me^{|\omega|t}\int_0^t e^{-|\omega|s}\|B\|\|u_y(s) - u_{\overline{y}}(s)\|ds \\ &\quad + Me^{|\omega|t}\int_0^t l_n(s)e^{-|\omega|s}\|y_s - \overline{y}_s\|ds \\ &\quad + Me^{\omega t}\sum_{k=1}^n e^{-\omega t_k}\|I_k(y(t_k)) - I_k((\overline{y}(t_k)))\| \end{aligned}$$

$$
\begin{aligned}
\leq\ & t_n\overline{M}_1M^2\overline{M}e^{|\omega|t}\int_0^t e^{-|\omega|s}e^{|\omega|t_n}\|v(s)-\overline{v}(s)\|ds\\
&+\overline{M}_1M^2\overline{M}e^{|\omega|t}\int_0^t\sum_{k=1}^n e^{|\omega|(t_n-t_k)}b_k\|y(t_k)-\overline{y}(t_k)\|ds\\
&+e^{|\omega|t}\int_0^t e^{-|\omega|s}l_n(s)\|y_s-\overline{y}_s\|ds\\
&+Me^{|\omega|t}\sum_{k=1}^n e^{-|\omega|t_k}b_k\|y(t_k)-\overline{y}(t_k)\|\\
\leq\ & e^{|\omega|t}\int_0^t t_n\overline{M}_1M^2\overline{M}e^{\omega t_n}l_n(s)e^{-|\omega|s}\|y_s-\overline{y}_s\|ds\\
&+e^{\omega t}\int_0^t\overline{M}_1M^2\overline{M}e^{|\omega|t_n}\sum_{k=1}^n e^{|\omega|t_k}b_k\|y(s)-\overline{y}(s)\|ds\\
&+e^{|\omega|t}\int_0^t Ml_n(s)e^{-|\omega|s}\|y_s-\overline{y}_s\|_{\mathcal{D}}ds\\
&+e^{|\omega|t}\sum_{k=1}^n Mb_ke^{-|\omega|t_k}\|y(s)-\overline{y}(s)\|\\
\leq\ & e^{|\omega|t}\int_0^t\widetilde{l}_n(s)e^{-|\omega|s}\|y_s-\overline{y}_s\|ds\\
&+e^{\omega t}\int_0^t\widetilde{l}_n(s)\sum_{k=1}^n e^{|\omega|t_k}b_k\|y(s)-\overline{y}(s)\|ds\\
&+e^{|\omega|t}\int_0^t\widetilde{l}_n(s)e^{-|\omega|s}\|y_s-\overline{y}_s\|_{\mathcal{D}}ds\\
&+e^{|\omega|t}\sum_{k=1}^n Mb_ke^{-|\omega|t_k}\|y(s)-\overline{y}(s)\|\\
\leq\ & e^{|\omega|t}\int_0^t\widetilde{l}_n(s)e^{\tau L_n(s)}e^{-|\omega|s}e^{-\tau L_n(s)}\|y_s-\overline{y}_s\|ds\\
&+e^{\omega t}\int_0^t\widetilde{l}_n(s)e^{\tau L_n(s)}\sum_{k=1}^n e^{-|\omega|t_k}e^{-\tau L_n(s)}b_k\|y(t_k)-\overline{y}(t_k)\|ds\\
&+e^{|\omega|t}\int_0^t\widetilde{l}_n(s)e^{\tau L_n(s)}e^{-|\omega|s}e^{-\tau L_n(s)}\|y_s-\overline{y}_s\|_{\mathcal{D}}ds\\
&+e^{|\omega|t}\sum_{k=1}^n Mb_ke^{\tau L_n(t)}e^{-\tau L_n(t)}e^{-|\omega|t_k}\|y(t_k)-\overline{y}(t_k)\|\\
\leq\ & e^{|\omega|t}\int_0^t(e^{\tau L_n(s)})'ds\|y-\overline{y}\|_n+e^{\omega t}\int_0^t(e^{\tau L_n(s)})'ds\sum_{k=1}^n b_k\|y-\overline{y}\|_n\\
&+e^{|\omega|t}\int_0^t(e^{\tau L_n(s)})'ds\|y-\overline{y}\|_n+e^{|\omega|t}\sum_{k=1}^n Mb_ke^{\tau L_n(t)}\|y-\overline{y}\|_n\\
\leq\ & e^{|\omega|t+\tau L_n(t)}\frac{2}{\tau}\|y-\overline{y}\|_n+e^{|\omega|t+L_n(t)}\sum_{k=1}^n\frac{b_k}{\tau}\|y-\overline{y}\|_n\\
&+e^{|\omega|t+\tau L_n(t)}\sum_{k=1}^n Mb_k\|y-\overline{y}\|_n.
\end{aligned}
$$

Thus,

$$e^{-|\omega|t-\tau L_n(t)}\|h(t)-\overline{h}(t)\| \le \left(\frac{2}{\tau}+\sum_{k=1}^{n}\frac{b_k}{\tau}+\sum_{k=1}^{n}Mb_k\right)\|y-\overline{y}\|_n.$$

Therefore,

$$\|h-\overline{h}_n\|_n \le \left(\frac{2}{\tau}+\sum_{k=1}^{n}\frac{b_k}{\tau}+\sum_{k=1}^{n}Mb_k\right)\|y-\overline{y}\|_n.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y),N(\overline{y})) \le \left(\frac{2}{\tau}+\sum_{k=1}^{n}\frac{b_k}{\tau}+\sum_{k=1}^{n}Mb_k\right)\|y-\overline{y}\|_n.$$

Let $y\in\Omega$, and $\varepsilon\in(0,\infty)$. Consider $N:\Omega_n\to\mathcal{P}_{cl}(\Omega_n)$ given by,

$$N(y)=\left\{h\in\Omega_n : h(t)=\begin{cases}\phi(t), & \text{if } t\in[-r,0],\\ S'(t)\phi(0)+\frac{d}{dt}\int_0^t S(t-s)v(s)ds & \\ +\frac{d}{dt}\int_0^t S(t-s)(Bu_y)(s)ds & \\ +\sum_{0\le t_k<t} S'(t-t_k)I_k(y(t_k)), & \text{if } t\in[0,n],\end{cases}\right\}$$

where $v\in S^n_{F,y}=\{h\in L^1([0,t_n],\overline{D(A)}) : v\in F(t,y_t)\ a.e.\ t\in[0,t_n]\}$. By (H3) and (H5)-(H6), and since F is compact valued, we can prove that, for every $y\in\Omega_n$, $N(y)\in\mathcal{P}_{cp}(\Omega_n)$ and there exists $y_*\in\Omega_n$ such that $y_*\in N(y_*)$.

Let $h\in\Omega_n$, $\overline{y}\in\overline{U}$ and $\varepsilon>0$. Assume that $y_*\in N(\overline{y})$. Then we have

$$\begin{aligned}\|\overline{y}(t)-y_*(t)\| &\le \|\overline{y}(t)-h(t)\|+\|y_*(t)-h(t)\|\\ &\le \|\overline{y}-N\overline{y}\|_n e^{-\omega t-\tau L_n(t)}+\|y_*(t)-h(t)\|.\end{aligned}$$

Since h is arbitrary, we may suppose that $h\in B(y_*,\varepsilon)=\{h\in\Omega_n : \|h-y_*\|_n\le\varepsilon\}$. Therefore,

$$\|\overline{y}-y_*\|_n \le \|\overline{y}-N\overline{y}\|_n+\varepsilon.$$

If $y_*\notin N(\overline{y})$, then $\|y_*-N(\overline{y})\|\neq 0$. Since $N(\overline{y})$ is compact, there exists $x\in N(\overline{y})$ such that $\|y_*-N(\overline{y})\|=\|y_*-x\|$. Then we have

$$\begin{aligned}\|\overline{y}(t)-x(t)| &\le \|\overline{y}(t)-h(t)\|+\|x(t)-h(t)\|\\ &\le \|\overline{y}-N\overline{y}\|_n e^{-|\omega|t-\tau L_n(t)}+\|x(t)-h(t)\|.\end{aligned}$$

Thus,

$$\|\overline{y}-x\|_n \le \|\overline{y}-N\overline{y}\|_n+\varepsilon.$$

So, N is an admissible contraction operator by Lemma 2.3, and N has a fixed point y, which is an integral solution to (1.1)–(1.2). □

4 CONTROLLABILITY OF IMPULSIVE NEUTRAL FDIS

In this section we study the problem (1.4)–(1.6). We give first the definition of an integral solution of the problem (1.4)–(1.6).

Definition 4.1. A function $y \in \Omega$ is said to be an integral solution of (1.4)–(1.6), if there exists $v \in L^1(J, \overline{D(A)})$ such that $v(t) \in F(t, y_t)$ a.e. $t \in J$, and $y(t)$ satisfies the integral equation,

$$\begin{aligned} y(t) &= \phi(0) - g(0, \phi(0) + g(t, y_t) + A\int_0^t y(s)ds + \int_0^t v(s)ds \\ &\quad + \int_0^t (Bu)(s)ds + \sum_{\leq t_k < t} S'(t - t_k)I_k(y(t_k^-)), \end{aligned}$$

and $y(t) = \phi(t),\ t \in [-r, 0],\ \displaystyle\int_0^t y(s)ds \in D(A),\ t \in [0, \infty)$.

Definition 4.2. The system (1.4)–(1.6) is said to be controllable on the interval $[-r, \infty)$, if for every continuous initial function $\phi \in \mathcal{D}$ and every $y_1 \in \overline{D(A)}$, and for each $n \in \mathbb{N}$, there exists a control $u \in L^2([0, t_n], U)$, such that the integral solution $y(t)$ of (1.4)–(1.6) satisfies $y(t_n) = y_1$.

For our next result, we will invoke some hypotheses unique to the problem.

(A1) There exist constants $0 \leq d_1 < 1$ and $d_2 \geq 0$ such that

$$|g(t, u)| \leq d_1\|u\|_{\mathcal{D}} + d_2 \ \ t \in [0, \infty),\ u \in \mathcal{D}.$$

(A2) For each $R > 0$, there exists a function $c_R^* > 0$ such that

$$|g(t, x) - g(t, \overline{x})| \leq c_R^*\|x - \overline{x}\|_{\mathcal{D}},\ t \in [0, \infty),\ x, \overline{x} \in \mathcal{D},\ \text{with } \|x\|,\ \|\overline{x}\| \leq R.$$

Let $\overline{L}_n(t) = \displaystyle\int_0^t \widehat{l}_n(s)ds$ where

$$\widehat{l}_n(t) = \max\{t_n\overline{M}M^2\overline{M}_1 e^{|\omega|t_n}l_n(t), \overline{M}M^2\overline{M}_1 e^{|\omega|t_n}, \overline{M}M\overline{M}_1, Ml_n(t)\}.$$

For each $n \in \mathbb{N}$ we define in Ω the semi-norms by

$$\|y\|_n = \sup\{e^{-|\omega|t - \tau\overline{L}_n(t)}|y(t)| : -r \leq t \leq t_n\}.$$

Then Ω is a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}$.

Theorem 4.1. *Assume (H1)–(H6) and (A1)-(A2) are satisfied. If, for each* $n > 0$*, we have* $c_n^* + M\displaystyle\sum_{k=1}^{\infty} b_k < 1$*, then the IVP (1.4)-(1.6) is controllable on* $[-r, \infty)$.

Proof. By using hypothesis (H5), for each $y(\cdot)$ and

$$v \in S_{F,y} = \{v \in L^1(J, \overline{D(A)}) : v(t) \in F(t, y_t)\ a.e\ t \in J\},$$

define the control,

$$\begin{aligned} u_y(t) &= W^{-1}\left[y_1 - S'(t_n)[\phi(0) - g(0,\phi(0))] - g(t_n, y_{t_n})\right. \\ &\qquad \left. - \lim_{\lambda\to+\infty}\int_0^{t_n} S'(t_n - s)B_\lambda v(s)ds - \sum_{0<t_k<t} S'(t_n - t_k)I_k(y(t_k^-))\right](t). \end{aligned}$$

Transform the problem (1.4)–(1.6) into a fixed point problem. Consider the operator $N_1 : \Omega \to \mathcal{P}(\Omega)$ defined by,

$$N_1(y) := \left\{ h \in \Omega : h(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } t \in [-r, 0], \\ S'(t)[\phi(0) - g(0,\phi(0))] + g(t, y_t) & \\ +\dfrac{d}{dt}\displaystyle\int_0^t S(t-s)v(s)\,ds & \\ +\dfrac{d}{dt}\displaystyle\int_0^t S(t-s)(Bu_y)(s)ds & \\ +\displaystyle\sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) & \text{if } t \in [0,\infty)\ , v \in S_{F,y}. \end{array}\right. \right\}$$

Remark 4.1. It is clear that the fixed points of N_1 are integral solutions to (1.4)–(1.6).

Let y be a possible solution of the problem (1.4)–(1.6). Given $n \in \mathbb{N}$ and $t \le t_n$, then $y \in N_1(y)$, and there exists $v \in S_{F,y}$ such that for each $t \in [0,\infty)$, we have

$$\begin{aligned} |y(t)| &\le Me^{|\omega|t}[\|\phi(0)\| + \|g(,\phi(0))\|] + d_1\|y_t\|_{\mathcal{D}} + d_2 \\ &\quad + Me^{|\omega|t}\int_0^t e^{-|\omega|s}p(s)\psi(\|y_s\|)ds \\ &\quad + Me^{|\omega|t}\int_0^t e^{-|\omega|s}\|(Bu_y(s)\|ds + Me^{|\omega|t}\sum_{k=1}^n e^{-|\omega|t_k}c_k \\ &\le Me^{|\omega|t}[\|\phi(0)\| + \|g(0,\phi(0))\|] + d_1\|y_t\|_{\mathcal{D}} + d_2 \\ &\quad + Me^{|\omega|t}\int_0^t e^{-|\omega|s}p(s)\psi(\|y_s\|)ds + M\overline{M}e^{|\omega|t}\overline{M}_1 t_n\int_0^t d_1\|y_{t_n}\|ds \\ &\quad + M\overline{M}e^{|\omega|t}\overline{M}_1 t_n\Big(\|y_1\| + Me^{|\omega|t_n}[\|\phi(0)\| + \|g(0,\phi(0))\| + d_2 \\ &\quad + Me^{|\omega|t_n}\sum_{k=1}^n e^{-|\omega|t_k}c_k \\ &\quad + Me^{|\omega|t_n}\int_0^t e^{-|\omega|s}p(s)\psi(\|y_s\|)ds\Big) + Me^{|\omega|t}\sum_{k=1}^n e^{-|\omega|t_k}c_k. \end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\},\quad 0 \le t \le t_n.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, t_n]$, then by the previous inequality, we have for $t \in [0, t_n]$,

$$\begin{aligned}
\mu(t) &\leq e^{|\omega|t_n} M[\|\phi(0)\| + \|g(0,\phi(0))\|] + d_1\mu(t) + d_2 \\
&+ Me^{|\omega|t} \sum_{k=1}^{n} e^{-\omega t_k} c_k + M\overline{M} e^{|\omega|t_n} \overline{M}_1 t_n \int_0^t d_1 \mu(s) ds \\
&+ M\overline{M} e^{|\omega|t_n} \overline{M}_1 t_n \Big(\|y_1\| + Me^{|\omega|t_n} [\|\phi(0)\| + \|g(0,\phi(0))\| \\
&+ d_2 + Me^{|\omega|t_n} \sum_{k=1}^{n} e^{-|\omega|t_k} c_k + Me^{|\omega|t_n} \int_0^t e^{-|\omega|s} p(s)\psi(\mu(s)) ds \Big) \\
&+ Me^{|\omega|t_n} \int_0^t e^{-|\omega|s} p(s)\psi(\mu(s)) ds \\
&\leq e^{|\omega|t_n} M[\|\phi(0)\| + \|g(0,\phi(0))\|] + d_1\mu(t) + d_2 + Me^{|\omega|t_n} \sum_{k=1}^{n} e^{-\omega t_k} c_k \\
&+ M\overline{M} e^{|\omega|t_n} \overline{M}_1 t_n \Big(\|y_1\| + Me^{|\omega|t_n} [\|\phi(0)\| + \|g(0,\phi(0))\| \\
&+ d_2 + Me^{|\omega|t_n} \sum_{k=1}^{n} e^{-|\omega|t_k} c_k \Big) + \int_0^t m(s)\psi(\mu(s)) ds.
\end{aligned}$$

Then

$$\begin{aligned}
\mu(t) &\leq \frac{1}{1-d_1} \left(e^{|\omega|t_n} M[\|\phi(0)\| + \|g(0,\phi(0))\|] + d_2 + Me^{|\omega|t_n} \sum_{k=1}^{n} e^{-\omega t_k} c_k \right) \\
&+ \frac{M\overline{M} e^{|\omega|t_n} \overline{M}_1 t_n}{1-d_1} \Big(\|y_1\| + Me^{|\omega|t_n} [\|\phi(0)\| + \|g(0,\phi(0))\|] \Big) \\
&+ \frac{M\overline{M} e^{|\omega|t_n} \overline{M}_1 t_n}{1-d_1} \left(d_2 + Me^{|\omega|t_n} \sum_{k=1}^{n} e^{-|\omega|t_k} c_k \right) + \int_0^t m(s)[\mu(s) + \psi(\mu(s))] ds,
\end{aligned}$$

where

$$m(t) = \frac{1}{1-d_1} [M^2 \overline{M} M \overline{M}_1 t_n e^{2|\omega|t_n}] e^{-|\omega|t} p(t) + Me^{|\omega|t_n} e^{-|\omega|t} p(t) + M\overline{M} e^{|\omega|t_n} \overline{M}_1 t_n d_1.$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\mu(t) \leq v(t) \text{ for all } \quad t \in [0, t_n],$$

and

$$\begin{aligned}
v(0) &= \frac{1}{1-d_1} \left(e^{|\omega|t_n} M[\|\phi(0)\| + \|g(0,\phi(0))\|] + d_2 + Me^{|\omega|t_n} \sum_{k=1}^{n} e^{-\omega t_k} c_k \right) \\
&+ \frac{M\overline{M} e^{|\omega|t_n} \overline{M}_1 t_n}{1-d_1} \Big(\|y_1\| + Me^{|\omega|t_n} [\|\phi(0)\| + \|g(0,\phi(0))\| + d_2 \\
&+ Me^{|\omega|t_n} \sum_{k=1}^{n} e^{-|\omega|t_k} c_k \Big),
\end{aligned}$$

and

$$v'(t) = m(t)[\mu(t) + \psi(\mu(t))], \;\; t \in [0, t_n].$$

Using the increasing character of ψ we get

$$v'(t) \leq m(t)\psi(v(t)) \;\; \text{a.e.} \;\; t \in [0, t_n].$$

Then for each $t \in [0, t_n]$, we have

$$\int_{v(0)}^{v(t)} \frac{du}{u + \psi(u)} \leq \int_0^{t_n} m(s)ds < \int_{v(0)}^{\infty} \frac{du}{u + \psi(u)}.$$

Consequently, there exists a constant d_n such that $v(t) \leq \overline{d}_n, \; t \in [0, t_n]$, and hence $\|y\|_n \leq \max(\|\phi\|, \overline{d}_n) := \overline{K}_n$. Set

$$U_1 = \{y \in \Omega : \|y\|_n < \overline{K}_n + 1 \text{ for all } n \in \mathbb{N}\}.$$

Clearly, U_1 is a open subset of Ω. As in Theorem 3.1, we can show that $N_1 : \overline{U}_1 \to \mathcal{P}(\Omega)$ is an admissible contraction operator. From the choice of U_1 there is no $y \in \partial U_1$ such that $y \in \lambda N_1(y)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative (Frigon, 2002), we deduce that N_1 has at least one fixed point which is an integral solution to (1.4)–(1.6). □

5 AN EXAMPLE

As an application of our results, we consider the following impulsive partial functional differential inclusion,

$$\frac{\partial z(t,x)}{\partial t} - d\Delta z(t,x) \in Q(t, z(t-r,x)) + (Bu)(t), \; t \in [0,\infty) \setminus \{t_1, t_2, \ldots\}, \; x \in \Gamma, \tag{5.1}$$

$$b_k z(t_k^-, x) = z(t_k^+, x) - z(t_k^-, x), \; k = 1, 2, \ldots, \; x \in \partial\Gamma, \tag{5.2}$$

$$z(t,x) = 0, \; t \in [0,\infty) \setminus \{t_1, t_2, \ldots\}, x \in \partial\Gamma, \tag{5.3}$$

$$z(t,x) = \phi(t,x), \;\; -r \leq t \leq 0, \;\; x \in \overline{\Gamma}, \tag{5.4}$$

where $d, r, b_k > 0$, Γ is a bounded open in $\mathbb{R}^n$ with regular boundary $\partial\Gamma$, B is as in (1.1), $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $\phi \in \mathcal{D}([-r,0] \times \overline{\Gamma}, \mathbb{R}^n) = \{\psi : [-r,0] \times \overline{\Gamma} \to \mathbb{R}^n : \psi$ is continuous everywhere except for a countable number of points $\bar{t}$ at which $\psi(\bar{t}^-,.)$ and $\psi(\bar{t}^+,.)$ exist, $\psi(\bar{t}^-,.) = \psi(\bar{t},.)$, and $\sup_{(\theta,x)\in[-r,0]\times\overline{\Gamma}} |\psi(\theta,x)| < \infty\}$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \ldots < t_m < \ldots$, $z(t_k^+, x) = \lim_{(h,x)\to(0^+,x)} z(t_k + h, x)$, and $z(t_k^-, x) = \lim_{(h,x)\to(0^-,x)} z(t_k - h, x)$, $I_k(z) = b_k z \;\; k \in \{1, 2, \ldots\}$, $Q : [0,\infty) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map with compacts values, and there exist constants $k_p > 0$ such that

$$H_d(Q(t,x),Q(t,y)) \leq k_q\|x-y\| \text{ for all } x,y \in [0,\pi], \quad t\in[0,\infty),$$

and

$$H_d(0,Q(t,0)) \leq k_q.$$

Consider $E = C(\overline{\Gamma}, \mathbb{R}^n)$ the Banach space of continuous functions on $\overline{\Gamma}$ with values in $\mathbb{R}^n$. Define the linear operator A on E by $Az = d\Delta z$, where

$$D(A) = \{z \in E : \Delta z \in E \text{and } z|_{\partial\Gamma} = 0\}.$$

Now we have

$$\overline{D(A)} = \{z \in E : z|_{\partial\Gamma} = 0\} \neq E.$$

It is well known from (Da Prato and Sinestrari, 1987) that Δ satisfies the properties,

(i) $(0,\infty) \subset \rho(\Delta)$,

(ii) $\|R(\lambda,\Delta)\| \leq \dfrac{1}{\lambda}$, for some $\lambda > 0$.

It follows that Δ satisfies (H1), and hence it generates an integrated semigroup $(S(t))_t,\ t \geq 0$, and that $|S'(t)| \leq e^{-\mu t}$, for $t \geq 0$ and some constant $\mu > 0$. Let

$$F(t,w_t)(x) = Q(t, w(t-x)),\ 0 \leq x \leq \pi.$$

Then problem (5.1)-(5.4) takes the abstract form (1.1)-(1.3). We can easily see that all hypotheses of Theorem 3.1 are satisfied. Hence from Theorem 3.1 the problem (5.1)-(5.4) has at last one integral solution on $[-r,\infty)$.

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Asymptotic Properties of Nonoscillatory Proper Solutions of the Emden-Fowler Equation

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ABSTRACT

The paper is an overview of the selected results of the authors devoted to a solution of two problems by I. T. Kiguradze concerning to existence of nonoscillatory proper Kneser and strongly increasing solutions of the Emden – Fowler equation. Some new exact conditions of existence and asymptotic estimations of such solutions are given.

Keywords: Emden – Fowler equation, nonoscillatory solutions, asymptotic properties, strongly increasing solutions, Kneser solutions.

2000 Mathematics Subject Classification: 34C05, 34C11.

1 Introduction and preliminaries

The paper contains a survey of the authors' results devoted to solutions of two well known I. T. Kiguradze's problems on the existence of nonoscillatory infinitely continuable to the right strongly increasing solutions and Kneser solutions vanishing at infinity of the following Emden – Fowler type equation

$$u^{(n)} = p(t)|u|^{\lambda}\operatorname{sign} u, \quad 0 < \lambda \neq 1, \quad n \geq 2, \quad t > a > 0, \tag{1.1}$$

with a locally integrable function p, that is nonzero on a set of positive measure in every neighbourhood of $+\infty$.

1.1 Basic definitions and notations

Definition 1.1. A solution u of the equation (1.1) is called proper if it is infinitely continuable to the right and for all sufficiently large t satisfies condition $\sup\{|u(s)| : s > t\} > 0$; it is called singular otherwise (Kiguradze and Chanturia, 1990; Kiguradze and Chanturia, 1993).

Definition 1.2. A maximally prolonged to the right solution $u : [t_0, t_u) \to \mathbf{R}$ of the equation (1.1) is called nonoscillatory if it has no zero on some interval $[t_1, t_u) \subset [t_0, t_u)$, and oscillatory otherwise (Kiguradze and Chanturia, 1990; Kiguradze and Chanturia, 1993).

We shall consider nonoscillatory solutions of the equation (1.1) of various types such as proper Kneser solutions vanishing at infinity, singular Kneser solutions of the first type, strongly increasing solutions and singular solutions of the second type.

Definition 1.3. A solution $u: [t_0, +\infty) \to \mathbf{R}$ of the equation (1.1) satisfying conditions

$$(-1)^i u^{(i)}(t)u(t) > 0, \quad i = 1, \ldots, n-1, \quad t \in [t_0, t^*), \quad \lim_{t \to t^*} u(t) = 0, \tag{1.2}$$

is called a proper Kneser solution vanishing at infinity if $t^* = +\infty$, and a singular solution of the first type, if $t* < +\infty$ and $u(t) \equiv 0$ for $t \geq t^* > t_0$ (Kiguradze and Chanturia, 1990; Kiguradze and Chanturia, 1993).

Definition 1.4. A solution $u: [t_0, t_u) \to \mathbf{R}$ of the equation (1.1) satisfying conditions

$$u^{(i)}(t)u(t) > 0, \quad i = 1, \ldots, n-1, \quad t \in [t_0, t_u), \quad \lim_{t \to t_u - 0} |u^{(n-1)}(t)| = +\infty, \tag{1.3}$$

is said to be: 1) a strongly increasing solution, if $t_u = +\infty$; 2) a singular solution of the second type if $t_u < +\infty$ (Kiguradze and Chanturia, 1990; Kiguradze and Chanturia, 1993).

1.2 Problem statements

The second order equation (1.1) with power function p for the first time has arose at the beginning of XX century in astrophysical researches of R. Emden and then was explicitly investigated by R. Fowler, R. Bellman, G. Sansone and many other authors. The general case of the equation (1.1) has the rich history which was comprehensively stated by I. T. Kiguradze in his monograph (Kiguradze and Chanturia, 1990; Kiguradze and Chanturia, 1993). The major results in studying of asymptotic properties of solutions of this equation and their various generalizations also belong to I. T. Kiguradze and representatives of his school.

1.2.1 Kneser solutions

Let us assume the condition

$$(-1)^n p(t) \geq 0, \quad t \geq 0, \quad \lambda \in (0, 1) \tag{1.4}$$

holds.

The basic results here are the following two general assertions

The first Kiguradze's theorem (Kiguradze, 1975) *Let (1.4) hold. The equation (1.1) with function p, differing from zero on a set of positive measure, has Kneser singular solutions of the first type. Moreover, if $(-1)^n p(t) > ct^{-n} > 0$, $t > t_0$, then each solution of the (1.2) type of the equation (1.1) is a singular solution of the first type.*

Kvinikadze condition (Kvinikadze, 1978) *Let function $p: [0, +\infty) \to$ **R** satisfy conditions (1.4) and*

$$\int_t^{+\infty} |p(\tau)|\tau^{n-1} d\tau < +\infty. \tag{1.5}$$

Then equation (1.1) has proper vanishing at infinity Kneser solutions $u : [0, +\infty) \to \boldsymbol{R}$ and each of them admits the asymptotic estimation

$$|u(t)| < c \left[\int_t^{+\infty} |p(\tau)|(\tau - t)^{n-1} d\tau \right]^{1/(1-\lambda)}$$

for large t.

In connection with the sufficient condition (1.5) the following problem has arisen.

The first Kiguradze's problem (Kiguradze, 1975) *Let $0 < \lambda < 1$. Suppose that that the function p satisfies the condition (1.4) and is not identical zero in every neighbourhood of $+\infty$. Is the condition (1.5) necessary for the equation (1.1) to have at least one proper solution of the (1.2) type?*

The solution of this problem and some new necessary conditions of the existence of specified solutions are given in the Section 2 of the paper.

1.2.2 Strongly increasing solutions

Let $\lambda > 1$ and function $p : [0, +\infty) \to [0, +\infty)$ is no zero on a set of positive measure in any neighbourhood of $+\infty$.
The basic results here are the following two statements.

The second Kiguradze theorem (Kiguradze, 1975) *Equation (1.1) with parameter $\lambda > 1$ and function $p : [0, +\infty) \to [0, +\infty)$, that is nonzero on a set of positive measure, has a $n-$parametrical set of singular solutions of the second type. If $p(t) > ct^{-1-(n-1)\lambda} > 0,\, t > t_0$ then this equation has no any proper solutions of the (1.3) type.*

Kiguradze – Kvinikadze's condition (Kiguradze and Kvinikadze, 1982) *Equation (1.1) with parameter $\lambda > 1$ and function $p : [0, +\infty) \to [0, +\infty)$, differing from zero on a set of positive measure in any neighbourhood of $+\infty$ and satisfying the condition*

$$J(a, +\infty) \equiv \int_a^{+\infty} p(\tau)\tau^{(n-1)\lambda} d\tau < +\infty, \tag{1.6}$$

has $(n-1)-$parametrical set of nonoscillatory strongly increasing solutions and each of them in some neighbourhood of $+\infty$ supposes the estimation

$$|u^{(n-1)}(t)| \geq c\Big(\int_t^{+\infty} \tau^{(n-1)\lambda} p(\tau) d\tau \Big)^{1/(1-\lambda)}.$$

In connection with these results the following problem has arisen

The second Kiguradze's problem (Kiguradze, 1983; Kiguradze, 1978) *Let $\lambda > 1$. Suppose that that the non-negative function p is not identical zero in every neighbourhood of $+\infty$. Is the* *condition (1.6) necessary for the equation (1.1) to have at least one proper solution of the (1.3) type?*

The Section 3 gives both the solution of this problem and some new exact necessary existence conditions and asymptotic estimations for such solutions.

2 Kneser solutions

Let $\lambda \in (0,\, 1)$ and function $p : [0,\, +\infty) \to [0,\, +\infty)$ is nonzero on a set of positive measure in every neighbourhood of $+\infty$. The results on studying nonoscillatory proper Kneser solutions (1.2) vanishing at infinity of the Emden – Fowler type equation (1.1) satisfying the condition (1.4) are stated here in terms of convergence of the integral (Izobov, 1985b)

$$J_\mu(p, \varphi) \equiv \int\limits_1^{+\infty} |p(\tau)|^\mu \varphi(\tau) d\tau$$

with various constants $\mu > 0$ and piecewise continuous functions $\varphi : [0, +\infty) \to (0, +\infty)$. Initial here is the article (Izobov, 1985b) in which the solution of the first Kiguradze's problem for the second order equation (1.1) is given and integral conditions of the existence or the absence of proper Kneser solutions to the equation (1.1) of an arbitrary order are received. This work contains also an algorithm of simultaneous building both the second order Emden – Fowler equation with admissible piecewise constant function p and its corresponding Kneser solution. In the work (Izobov and Rabtsevich, 1990) which has appeared later the solution of the first Kiguradze problem for the equation of an arbitrary order $n \geq 2$ is received.
A sufficient condition of the absence of proper Kneser solutions for the equation (1.1) of an arbitrary order establishes the following theorem.

Theorem 2.1. *(Izobov, 1985b) Equation (1.1) with the parameter $\lambda \in (0,\, 1)$ and the function p satisfying (1.4) and the condition*

$$\int\limits_1^{+\infty} |p(\tau)|^\mu \tau^{n_2\nu - 1} d\tau = +\infty, \quad \nu < \mu \in (0, n_1^{-1}), \tag{2.1}$$

where $n_i = n_i(n) \equiv (n-i)\lambda + i$, $i = 1,\, 2$, has no proper nonoscillatory Kneser solutions (1.2.)

Proof. For any positive number $\alpha < (2n_1)^{-1} \min\{1 - n_1\mu,\, 1 - \lambda\}$ we introduce other numbers as follows $\mu_1 = \alpha + \lambda n_1^{-1}$, $\mu_2 = -\alpha + n_1^{-1}$, if $n = 2$; $\mu_{k+1} = k\beta + \lambda n_1^{-1}$, $k = 0,\, 1,\, \ldots, n-2$, $\mu_n = -\alpha + n_1^{-1}$, where $\beta = 2\alpha/((n-1)(n-2))$, if $n > 2$. These numbers satisfy the conditions $\lambda\mu < \lambda\mu_n < \mu_1 < \ldots < \mu_n$, $\sum\limits_{i=1}^{n} \mu_i = 1$.
Let us rewrite the condition (2.1) in the more convenient and equivalent form

$$\int\limits_1^{+\infty} \tau^{-1-\varepsilon} \left[|p(\tau)|\tau^{n_2}\right]^\mu d\tau = +\infty, \quad \varepsilon = n_2(\mu - \nu). \tag{2.1_1}$$

It is clear, the divergence of the integral (2.1_1) implies also the divergence of the integral

$$\int_1^{+\infty} \tau^{-1-n\delta}\,[|p(\tau)|\tau^{n_2}]^{\mu_n}\,d\tau = +\infty, \quad \delta \in (0, \varepsilon/n). \tag{2.1_2}$$

Assume now, the equation (1.1) has a proper Kneser solution $u : [t_0, +\infty) \to$ **R**. Then, according to (1.2), the inequalities $u^{(i)}(t)u^{(i+1)}(t) < 0$ hold for $i = 0,\, 1,\, \ldots\, n-2$ and $t \geq t_0 \geq 1$. Without loss of generality it is possible to suppose, that the solution u satisfies conditions

$$(-1)^i u^{(i)}(t) > 0, \quad i = 0,\, 1,\, \ldots\, n-1, \quad t \geq t_0. \tag{2.2}$$

Since the obvious representation $u^{(i)}(t) = u^{(i)}(t_0) + \int\limits_{t_0}^{t} u^{(i+1)}(\tau)d\tau, \quad t \geq t_0$, and the inequalities (2.2) are valid, then the inequality

$$|u^{(i+1)}(t)|(t - t_0) \leq |u^{(i)}(t_0)|, \quad t \geq t_0, \tag{2.3_1}$$

holds for all $t \geq t_0 \geq 0$, and, hence, the estimations

$$|u^{(i+1)}(t)| \leq |u^{(i)}(t_0)|(2/t), \quad i = 0,\, 1,\, \ldots\, n-2, \quad t \geq t_1 = 2t_0, \tag{2.3_2}$$

are valid also.

For the function $v(t) \equiv (-1)^{[n/2]} \prod\limits_{i=0}^{n-1} u^{(i)}(t) > 0, \quad t \geq t_1$, we calculate their derivative and then using estimation (2.3_2), the inequality (Beckenbach and Bellman, 1960)

$$\sum_{i=1}^{n} \beta_i x_i \geq \prod_{i=1}^{n} x_i \beta_i, \quad x_i \geq 0, \quad \beta_i > 0, \quad \sum_{i=1}^{n} \beta_i = 1,$$

and the choice of numbers μ_i and α we finally find the upper estimation for this derivative in the form

$$v'(t) = -\sum_{k=0}^{n-1} |u^{(k+1)}(t)| \sum_{i=0,\neq k}^{n-1} |u^{(i)}(t)| \leq -c \prod_{k=0}^{n-1} \left[|u^{(k+1)}(t)| \prod_{i=0,\neq k}^{n-1} |u^{(i)}(t)| \right]^{\mu_{k+1}}$$

$$\leq -c|p(t)|^{\mu_n} v(t) u^{\lambda\mu_n - \mu_1}(t) \prod_{i=1}^{n-1} |u^{(i)}(t)|^{\mu_i - \mu_{i+1}} \leq -c|p(t)|^{\mu_n} t^{\mu_n - \mu_1 - n\delta} v^{1-\delta}(t), \quad t \geq t_1. \tag{2.4}$$

The number δ here is chosen to satisfy the condition $\delta < \min\{\varepsilon/n, \lambda\alpha, -2\alpha + (1-\lambda)n_1^{-1}\}$ and also to the condition $\delta < \beta$ in the case $n > 2$, c is a universal positive constant.

Since for the exponent t in (2.4) the inequalities $\mu_n - \mu_1 - n\delta \geq (1-\lambda)n_1^{-1} - 2\alpha - n\delta = -1 - n\delta + (n-2)\lambda\alpha + n_2\mu_n \geq -1 - n\delta + n_2\mu_n$ hold, the inequality $v'(t) \leq -f(t)v^{1-\delta}(t), \quad f(t) \equiv ct^{-1-n\delta}[|p(t)|t^{n_2}]^{\mu_n}, \quad t \geq t_1$ also is true. Integrating it from t_1 up to t, we have the estimation $v^{\delta}(t) - v^{\delta}(t_1) \leq -\delta \int\limits_{t_1}^{t} f(\tau)d\tau, \quad t \geq t_1$, where in accordance to the (2.1_2) right term tends to $-\infty$ as $t \to +\infty$. This fact contradicts to the inequality $v(t) > 0$ for all $t \geq t_1$ derived from initial assumptions. This contradiction proves the theorem. □

Corollary 2.2. *(Izobov and Rabtsevich, 1990) The statement of the theorem 2.1 remains valid after replacement of the condition (2.1) by the weaker condition*

$$\int_1^{+\infty} |p(\tau)|^{\mu} \tau^{n\nu-1} d\tau = +\infty, \quad \nu < \mu \in (0, n_1^{-1}). \tag{2.1$_3$}$$

Proof. Putting $t_0 = t/2$ in the estimation (2.3_1), yields the obvious inequalities $|u^{(i+1)}(t)| \le 2|u^{(i)}(t/2)|/t$, $i = 0, 1, \dots n-2$, $t \ge t_1 = 2t_0$. Their sequential application allows to specify the estimation (2.3_2) as follows

$$|u^{(i)}(t)| \le 2^{(i-1)i/2}|u(t_0)|t^{-i}, \quad i = 1, \dots n-1, \quad t \ge 2^{n-1}t_0.$$

Using the obtained estimation we get the inequality $\prod_{i=1}^{n-1} |u^{(i)}(t)|^{\mu_i - \mu_{i+1}} \ge \text{const} \cdot t^{n\mu_n - 1} > 0$ for the product in (2.4), which together with (2.1_3) proves the corollary. □

Corollary 2.3. *(Izobov and Rabtsevich, 1990) Let* $\lambda \in (0, 1)$ *and the condition (1.4) hold. If the equation (1.1) has a proper Kneser solution of the (1.2) type, then*

$$\int_1^{+\infty} |p(\tau)|^{\mu} \tau^{n\nu-1} d\tau < +\infty \quad \textit{for all} \quad \nu < \mu \in (0, n_1^{-1}(n)).$$

The condition (2.1) and the weaker condition (2.1_3) are exact with respect to the parameter μ. This fact establishes [1]

Theorem 2.4. *(Izobov, 1985b; Izobov and Rabtsevich, 1990) For any natural* $n \ge 2$, *piecewise continuous function* $\varphi : [0, +\infty) \to (0, +\infty)$ *and numbers* $\mu \ge 1/n_1(2)$ *in the case* $n = 2$ *and* $\mu > 1/n_1(n)$ *in the case* $n > 2$ *there exists such piecewise continuous function* $p(t) \equiv p_{n,\varphi,\mu}(t)$, *satisfying (1.4) and the condition*

$$\int_1^{+\infty} |p(\tau)|^{\mu} \varphi(\tau) d\tau = +\infty, \tag{2.5}$$

that the equation (1.1) has proper Kneser solutions of the (1.2.) type.

Proof. The proof of this theorem consists of two parts. In the first part will be constructed the second order equation with function p satisfying the condition (2.5) at $\mu = 1/n_1(2)$ which has proper Kneser solutions. In the second part similar construction is made in the case $\mu \in (1/n_1(n), +\infty)$, leaving thus not investigated the case $\mu = 1/n_1(n)$, $n > 2$.

1. As it was marked above, from the proof of this theorem for the second order equation (1.1) presented in (Izobov, 1985b) we shall consider the most important case $\mu = 1/n_1(2) = (1 + \lambda)^{-1}$. The construction of the desired piecewise continuous function p and the proper Kneser solution u of the equation (1.1) with this function we will realise the step-by-step procedure with the increasing number of steps at each of the following stage.

[1] Without loss of generality it is assumed that the function $\varphi(t) > 0$ has the exact lower boundary $\inf_{\tau \le t \le \tau+1} \varphi(t) \equiv \varphi_\tau > 0$ on the every unit segment $[\tau, \tau + 1]$.

Let's consider the constructions of the k-th stage, that acts on some segment $[t_k, t_{k+1}]$ which length is ≥ 1. Pick the auxiliary constants and the functions of the natural argument $i \geq 3$ as follows: 1) the number $\gamma \in (0,\, 1)$ and the corresponding number $R \equiv \sum_{j=3}^{+\infty} \gamma^{j-3} j^{\lambda/(1-\lambda)} < +\infty$; 2) the number $\alpha \in (1,\, 1+\lambda)$ and the functions

$$a_i = i^{-1} \prod_{j=2}^{i-1} (1 - j^{-1-\lambda})^{(1-\lambda)/(1+\lambda)}, \quad 1 > i a_i > a_0 \in (0,\, 1),$$

$$b_i = (i^{\alpha} a_i^{1+\lambda})^{1/(1-\lambda)} > b_{i+1}, \quad 1 > b_i^{(1+\lambda-\alpha)/(1-\lambda)} > b_0 = a_0^{(1+\lambda)/(1-\lambda)} \in (0,\, 1),$$

for $i \geq 3$. The introduced functions satisfy the inequality

$$b_i - b_{i+1} \geq b_i^{\lambda} a_i^{1+\lambda}, \quad i \geq i_{\alpha}, \tag{2.6}$$

which was proved in (Izobov, 1985b). Let at the initial moment $t = t_k$ the following inequalities fulfill

$$0 < u(t_k), \quad 0 < (\gamma - 1) b_0^{-1} R u(t_k) \dot{u}^{-1}(t_k) < 1. \tag{2.7}$$

At the i-th step of the current stage considered on some segment $[\tau_{i-1},\, \tau_i]$, $i \geq 3$, $\tau_2 = t_k$, realize the equalities

$$u(\tau_i) = \gamma u(\tau_{i-1}) > 0, \quad \dot{u}(\tau_i) = b_i b_{i-1}^{-1} \dot{u}(\tau_{i-1}) < 0, \quad i \geq 3 \tag{2.8}$$

for the solution u of the equation (1.1) with the function

$$p(t) = p_i \equiv \frac{1+\lambda}{2} \, \frac{1 - (b_i/b_{i-1})^2}{1 - \gamma^{1+\lambda}} \, \frac{\dot{u}^2(\tau_{i-1})}{u^{1+\lambda}(\tau_i)}.$$

From the relations (2.8) and the inequality $(\gamma - 1) u(\tau_{i-1}) = \int_{\tau_{i-1}}^{\tau_i} \dot{u}(\tau) d\tau \leq \dot{u}(\tau_i)(\tau_i - \tau_{i-1})$ $= b_i b_{i-1}^{-1} \dot{u}(\tau_{i-1})(\tau_i - \tau_{i-1})$ an estimation for the length of the segment $[\tau_{i-1},\, \tau_i]$ $\tau_i - \tau_{i-1}$ $\leq (1-\gamma)\gamma^{i-3} b_i^{-1} u(t_k) |\dot{u}(t_k)|^{-1}$, $i \geq 3$ follows. This estimation together with the condition (2.7) leads to inequalities

$$\tau_i - t_k \leq (1-\gamma) \frac{u(t_k)}{|\dot{u}(t_k)|} \sum_{j=3}^{i} \gamma^{j-3} b_j^{-1} < (1-\gamma) b_0^{-1} R \frac{u(t_k)}{|\dot{u}(t_k)|} < 1, \quad i \geq 3. \tag{2.9}$$

On the other hand, the following opposite estimation for length of the segment $[\tau_i,\, \tau_{i+1}]$

$$\tau_{i+1} - \tau_i \geq (1 - b_{i+1} b_i {-} 1) p_{i+1}^{-1} |\dot{u}(\tau_i)| u^{-\lambda}(\tau_i) \tag{2.10}$$

follows from the evident representation $\dot{u}(\tau_{i+1}) - \dot{u}(\tau_i) = \int_{\tau_i}^{\tau_{i+1}} p_{i+1} u^{\lambda}(\tau) d\tau$.

For some $l \geq 3$ by the equalities (2.8) and by the inequalities (2.6), (2.9) and (2.10) we can receive now the desired estimation of the integral:

$$\int_{\tau_k}^{\tau_{l+1}} p^{\mu}(\tau) \varphi(\tau) d\tau \geq \varphi_{t_k} \sum_{i=2}^{l} p_{i+1}^{\mu} (\tau_{i+1} - \tau_i) \geq \varphi_{t_k} \sum_{i=2}^{l} p_{i+1}^{-\lambda/(1+\lambda)} (\tau_{i+1} - \tau_i)$$

$$\geq \left(\frac{1-\gamma^{1+\lambda}}{1+\lambda}\right)^{\frac{\lambda}{1+\lambda}} |\dot{u}(t_k)|^{\frac{1-\lambda}{1+\lambda}} \sum_{i=2}^{l} a_i \geq a_0 c(k) \sum_{i=i_\alpha}^{l} i^{-1} \to +\infty$$

as $l \to +\infty$. Therefore there exists a number $l(k) > i_\alpha$ such that

$$\int_{t_k}^{\theta_k} p^\mu(\tau)\varphi(\tau)d\tau \geq 1, \quad \theta_k \equiv \tau_{l(k)+1}. \tag{2.11}$$

At the penultimate step of the considered k-th stage, that acts on the segment $[\theta_k, \eta_k]$, using the function

$$p(t) = p_1 \equiv \frac{1+\lambda}{2}\, \frac{1-\beta_1^2}{1-\beta_0^{1+\lambda}}\, \frac{\dot{u}^2(\theta_k)}{u^{1+\lambda}(\theta_k)}$$

we realize equalities $u(\eta_k) = \beta_0 u(\theta_k) > 0, \quad \dot{u}(\eta_k) = \beta_1 \dot{u}(\theta_k) < 0$ with those constants $\beta_0,\, \beta_1 \in (0,\, 1)$ such that the inequality

$$u(\eta_k)|\dot{u}(\eta_k)|^{-1} \geq 2 \tag{2.12}$$

is fulfilled.

On the last step of the current stage we shall put $p(t) \equiv 0$ on the interval $(\eta_k, t_{k+1}]$. We shall choose the end point t_{k+1} of this interval (and the start moment of the following $(k+1)$-th stage) to satisfy to the following conditions: inequalities (2.7) with replacement k on $k+1$ and the condition $0 < u(t_{k+1}) < \gamma u(\eta_k)/2$. The last condition together with the inequality (2.12) ensures also the second necessary alongside with (2.11) the estimation $t_{k+1} - \eta_k \geq 1$.

So, we have constructed the piecewise constant function $p(t) \geq 0$ on the interval $(t_k,\, t_{k+1}]$ with length ≥ 1, such that the inequality

$$\int_{t_k}^{t_{k+1}} p^\mu(\tau)\varphi(\tau)d\tau \geq 1, \quad \mu = 1/(1+\lambda),$$

holds and the solution u of the second order equation (1.1) admits the estimation

$$0 < u(t), \quad \dot{u}(t) < 0, \quad t \in (t_k,\, t_{k+1}], \quad u(t_{k+1}) < \gamma u(t_k), \quad |\dot{u}(t_{k+1})| < \gamma |\dot{u}(t_k)|.$$

To complete the proof of the theorem in the considered case it is enough to extend these constructions to the all semiaxis $t \geq 0$.

2. This part of the proof (Izobov and Rabtsevich, 1990) also has the constructive character. The construction of the necessary equation (1.1) of the order $n > 2$ with the function p satisfying the condition (2.5) with parameter $\mu \in (1/n_1(n), +\infty)$ such that it has the proper solution u of the (1.2) type we perform by a number of steps. Each of the stages consists of two steps. The algorithm is arranged such that the end point of the interval on which the constructions of the previous stage operate, serves as the beginning point for the interval at the following stage.

Thus, let at the moment t_{2k}, that is the end point for the previous k-th stage of the given constructions the following conditions are fulfilled:

1) for the function φ the estimation

$$\varphi(t) \geq \varphi_k = \mathrm{const} > 0, \quad t \in [t_{2k},\, t_{2k} + (\delta+1)|\beta_k|]; \tag{2.13}$$

holds for some positive number δ to be determine below (see the lemma 2.1), which depends on the number n only;
2) for the solution u the equalities

$$u^{(i)}(t_{2k}) = \alpha_k \beta_k^{n-i-1} \quad (i = 0, \ldots, n-1), \quad |\beta_k| < (n+1)^{-1}, \tag{2.14}$$

are true, in which and everywhere below in this part of the proof the elements of the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ satisfy the inequalities $(-1)^{n-1}\alpha_k > 0, \quad \beta_k < 0.$
Now we give the needed constructions for the $(k+1)$-th stage.

Step 1. The following Lemma is a base for the proposed constructions.

Lemma 2.1 *Let arbitrary numbers $n \geq 2,\, \alpha_k,\, \beta_k$ be chosen along with the segment $[t_{2k},\, t_{2k+1}]$ the length of which is not less $\delta|\beta_k|$, where $\delta = \delta(n) \geq n$. Then for all α_{k+1} and β_{k+1} which are close enough to zero, there exists a function p of the (1.4) type and a solution u defined on this segment such that*

$$(-1)^i u^{(i)}(t) > 0, \quad u^{(i)}(t_j) = \alpha_j \beta_j^{n-i-1} \quad (i = 0, \ldots, n-1, \quad j = 2k,\, 2k+1). \tag{2.15}$$

Following this lemma, put $t_{2k+1} = t_{2k} + \delta|\beta_k|$ and choose the numbers α_{k+1} and β_{k+1} such that the following condition

$$\varphi_k |\alpha_{k+1}|^{(1-\lambda)\mu} |\beta_{k+1}|^{1-\mu n_1(n)} \exp[\mu(1-\lambda)] > 1 \tag{2.16}$$

holds. This is possible according to the assumption $\mu n_1(n) > 1$.

Step 2. It is easy to check, that the function $y(t) = \alpha_{k+1}\beta_{k+1}^{n-1}\exp[\beta_{k+1}^{-1}(t - t_{2k+1})]$ is the solution of the equation $y^{(n)} = \beta_{k+1}^{-n} y$, satisfying the conditions $y^{(i)}(t_{2k+1}) = \alpha_{k+1}\beta_{k+1}^{n-i-1}$, $0 < (-1)^i y^{(i)}(t), \quad t > t_{2k+1},\, i = 0, \ldots, n-1$. Then the equation (1.1) with the function $p(t) = \beta_{k+1}^{-n} y^{1-\lambda}(t)$ has the solution $u(t) = y(t), t > t_{2k+1}$. Choose the moment t_{2k+2} satisfying the condition (2.13) where k is replaced by $k+1$ and such that $t_{2k+2} > t_{2k} + 1$. By monotonicity of the function p, the equalities $t_{2k+1} = t_{2k} + \delta|\beta_k|$ and the conditions (2.15), (2.16) the following estimation of the integral $\int\limits_{t_{2k+1}}^{t_{2k+2}} |p(\tau)|^{\mu}\varphi(\tau)d\tau > 1$ holds and it ensures the validity of the condition (2.5).
Thus the constructions of the $(k+1)$-th stage, acting on the segment $[t_{2k}, t_{2k+2}]$ the length of which is > 1, are completed. In accordance with the conditions (2.13) and (2.14) where k is replaced by $k+1$ the moment t_{2k+2} can serve as the beginning point for the following stage of the proposed constructions. The theorem is completely proved. □

The first author of this paper proved a stronger statement in the form

Theorem 2.5. *For any natural number $n \geq 2$ and piecewise continuous function $\varphi : [0, +\infty) \to (0, +\infty)$ there exists a piecewise continuous function $p(t) \equiv p_{n,\varphi}(t)$, satisfying the inequality (1.4) and the condition (2.5) for every $\mu \geq n_1^{-1}(2)$ in the case $n = 2$ and $\mu > n_1^{-1}(n)$ in the case $n > 2$ such that the n-th order equation (1.1) has proper Kneser solutions.*

I.T. Kiguradze extended the approach used in (Izobov, 1985b). He received both more exact necessary existence condition of such solutions and their asymptotic estimations.

Theorem 2.6. *(Kiguradze and Chanturia, 1993) Let the equation (1.1) has a proper nonoscillatory Kneser vanishing at infinity solution* u. *Then for every number* $\mu \in [2/(n_1+n),\, 1/n_1)$, $n_1 = n_1(n)$,

$$\lim_{t\to+\infty} t^{\nu} \int\limits_t^{+\infty} \tau^{n\mu-\nu-1}|p(\tau)|\mu d\tau = 0, \tag{2.17}$$

where $\nu = 1-\mu n_1$, *and the following estimation*

$$|u(t)| \geq \gamma_0(n,\lambda,\mu)\left[\int\limits_t^{+\infty}(\tau-t)^{n-2}\left(\int\limits_\tau^{+\infty}|p(x)|^{\mu}dx\right)^{(n-1)/(n\mu-1)} d\tau\right]^{(n\mu-1)/((n-1)(1-\lambda)\mu)} \tag{2.18}$$

holds in some neighbourhood of $+\infty$

After improvements of some parameters in (Rabtsevich, 1993) this statement was modified to the form:

Theorem 2.7. *Let the equation (1.1) has a proper nonoscillatory Kneser solution* u *vanishing at infinity. Then for any numbers* $\mu \in [1/n,\, 1/n_1)$ *and* $\nu \in (0,\, \sigma_0]$, *where* $\sigma_0 = \min\{n\mu-1,\, 1-n_1\mu\}$, *the equality (2.17) holds and the estimation (2.18) is valid in some neighbourhood of* $+\infty$.

The second author of this paper proved also the following statements (Rabtsevich, 2000c).

Theorem 2.8. *If the equation (1.1) has a proper nonoscillatory Kneser solution* u *vanishing at infinity then for any nonincreasing function* $\varphi(t) > 0$, *satisfying the condition* $t|\dot\varphi(t)|/\varphi(t) \leq 1$ *for all* $t > t_0$ *and any numbers* $\mu \in (0, 1/n_1)$ *and* $\varepsilon > 0$ *the following equality*

$$\lim_{t\to+\infty} \Phi_{\mu,\varepsilon}(\varphi(t)) = 0$$

holds, and, since some moment $t_u > t_0$, *the estimate* $s_{n-1}(t) > \gamma[\Phi_{\mu,\varepsilon}(\varphi(t))]^{1/((1-\lambda)\mu)}$ *is valid, where* $\Phi_{\mu,\varepsilon}(\varphi(t)) = \varphi^{-\varepsilon}(t)\int\limits_t^{+\infty}(p(\tau)\tau^{n-1})^{\mu}\varphi^{\varepsilon}(\tau)(|\dot\varphi(\tau)|/\varphi(\tau))^{1-\mu}d\tau$, $s_{n-1}(t) = \sum\limits_{i=0}^{n-1}|u^{(i)}(t)|t^i/i!$ *and* γ *is a positive constant depending on* $n,\ \lambda,\ \mu$ *only.*

Theorem 2.9. *In the case* $0 < (-1)^n p(t) \leq ct^{-n}$, $t > t_0$, *the condition (1.6) is necessary and sufficient for proper Kneser solutions* $u : [0, +\infty) \to$ ***R*** *vanishing at infinity of the equation (1.1) to exist and each of them admits the asymptotic estimation*

$$|u(t)| > c\left[\int\limits_t^{+\infty}|p(\tau)|\tau^{n-1}d\tau\right]^{1/(1-\lambda)}$$

in some neighbourhood of $+\infty$.

3 Strongly increasing solutions

In this paragraph some results concerning the study of strongly increasing solutions (1.3) of the equation (1.1) with parameter $\lambda > 1$ and nonnegative locally integrable function p are given in terms of divergence of the integral $J_\mu(p,\varphi)$ (its definition see in previous Section). These results are connected with the sufficient condition (1.6) for the equation (1.1) to have proper unbounded solutions and are initiated by the I. T. Kiguradze's problem about the necessity of this condition. This problem was stated by him in the survey (Kiguradze, 1978) and at the 6-th session of the joint seminar by name I. G. Petrovsky and the Moscow mathematical society (Kiguradze, 1983).

The fact, that the condition (1.6) is not a necessary one for the equation (1.1) to have strongly increasing solutions, is the simple consequence of the following theorem.

Theorem 3.1. *(Izobov, 1984c; Izobov, 1984a; Izobov, 1984b; Izobov, 1985a) For any number $\mu \geq 1/n$ and piecewise continuous function $\varphi : [0, +\infty) \to (0, +\infty)$ there exists piecewise continuous function $p : [0, +\infty) \to [0, +\infty)$ satisfying the condition $J_\mu(p, \varphi) = +\infty$, such that equation (1.1) with parameter $\lambda > 1$ has a n-parametrical family of strongly increasing solutions with initial values*

$$0 \neq U(0) \equiv (u(0),\, u^{(1)}(0), \dots,\, u^{(n-1)}(0)) \in X_1 \equiv \{x \in \boldsymbol{R}^n \, : \, 0 \leq x_i \leq 1\}. \tag{3.1}$$

Proof. Put $\delta_{1/n} = 1/2$ and $\delta_\mu = \exp\left\{[(1+n)\mu - 1](1 - n\mu)^{-1} \ln 2\right\}$ at $n\mu > 1$. For some numbers $\theta > 0$ and $\delta \in (0,\, \delta_\mu)$ we introduce the time moments as

$$\tau_{k+1} = \tau_k + \delta^{1+k}, \quad \tau_1 = \theta, \quad k = 1, \dots, m = [\delta^{-1}], \tag{3.2}$$

and for the interval $(\tau_1, \tau_{m+1}]$ the length which satisfies $\tau_{m+1} - \tau_1 < 2\delta^2 < 1/2$ define the function p by equality

$$p^\mu(t) = \delta^{-k} \varphi^{-1}(t), \quad t \in (\tau_k, \tau_{k+1}], \quad k = 1, \dots, m. \tag{3.3}$$

For this function the evident inequality

$$\int\limits_{\tau_1}^{\tau_{m+1}} p^\mu(\tau)\varphi(\tau) d\tau = m\delta > 2/3 \tag{3.4}$$

is fulfilled. Let's receive now the upper estimation of the integral

$$\begin{aligned} \frac{1}{(n-1)!} \int\limits_{\tau_1}^{\tau_{m+1}} p(\tau)(\tau_{m+1} - \tau)^{n-1} d\tau &\leq \frac{1}{(n-1)!} \sum_{k=1}^{m} (\tau_{m+1} - \tau_k)^{n-1} \int\limits_{\tau_k}^{\tau_{k+1}} p(\tau) d\tau \\ &\leq \frac{1}{(n-1)!} \delta^n (1-\delta)^{1-n} \varphi_\theta^{-1/\mu} \sum_{k=1}^{m} \delta^{k(n-1/\mu)} \leq 2m\delta^n \varphi_\theta^{-1/\mu} \leq 2\delta^{n-1} \varphi_\theta^{-1/\mu}, \end{aligned} \tag{3.5}$$

where without loss of generality it is assumed $\varphi_\theta > 0$ for all $\theta \geq 0$ where $\varphi_\theta \equiv \inf \varphi(\tau) > 0$, $\tau \in [\theta, \theta+1]$.

The required piecewise continuous function $p(t) \geq 0$ we shall build by step-to-step procedure. At the first step choose the needed values $\theta = \theta_1 > \theta_0 = 0$, $\delta = \delta_1$ by putting $p(t) \equiv 0$ on the segment $[\theta_0,\, \theta_1]$, where the moment θ_1 is unknown. This moment $t = \theta_1$ must be so large to provide the inequalities

$$\theta_1 - \theta_0 > 4(n-1)!, \quad \delta_1 \equiv \frac{1}{4\lambda} \overline{u}^{1-\lambda}(\theta_1)(1 + \varphi_{\theta_1}^{-1/\mu})^{-1} < \delta_\mu, \tag{3.6}$$

where $\overline{u}(\theta_1) \equiv \sup\{u(\theta_1) : U(\theta_0) \in X_1\} > 1.$
For the chosen $\theta = \theta_1$ and $\delta = \delta_1$ calculate the moment τ_k and number $m_1 = [\delta_1^{-1}]$ by the formula (3.2) ($[\cdot]$ means the integer part of number) and then define the function p by formula (3.3).
Establish now that every solution u of such defined equation (1.1) with initial conditions (3.1) exists on the whole segment $[\theta_0, \theta_2]$, $\theta_2 = \tau_{m_1+1}$. Let's suppose to the contrary that some solution u exists on the interval $[\theta_0, t*)$, $t^* \in (\theta_1, \theta_2]$ and $u(t) \uparrow +\infty$ at $t \to t^* - 0$. From the representation $u^{(k)}(t) = \sum\limits_{i=k}^{n-1} u^{(i)}(\theta_0)(t-\theta_0)^{i-k}/(i-k)!$, $k \in \{0, 1, \ldots, n-1\}$, $t \in [\theta_0, \theta_1]$, by the first condition (3.6) we obtain the inequality $4u^{(l)}(\theta_1) < u^{(k)}(\theta_1)$ for any $0 \le k < l \le n-1$. For derivative of this solution the following representation

$$\dot{u}(t) = \sum_{i=1}^{n-1} u^{(i)}(\theta_1)(t-\theta_1)^{i-1}/(i-1)! + \frac{1}{(n-2)!}\int\limits_{\theta_1}^{t} p(\tau)(t-\tau)^{n-2}u^{\lambda}(\tau)d\tau, \quad t \in [\theta_1, t*),$$

holds. Due to the inequalities $4\alpha_1[u(\theta_1)] \equiv 4 \max\limits_{1\le i\le n-1} u^{(i)}(\theta_1) < u(\theta_1)$, $t^* - \theta_1 < 2\delta_1^2 < 1/2$ and (3.5) we have the estimations

$$\dot{u}(t) \le \left[2\alpha_1[u(\theta_1)]u^{-\lambda}(\theta_1) + \frac{1}{(n-2)!}\int\limits_{\theta_1}^{t} p(\tau)(t-\tau)^{n-2}d\tau\right] u^{\lambda}(t)$$

$$< \left[u^{1-\lambda}(\theta_1) + \frac{1}{(n-2)!}\int\limits_{\theta_1}^{t} p(\tau)(t-\tau)^{n-2}d\tau\right] u^{\lambda}(t), \quad t \in [\theta_1, t*).$$

Integration the last differential inequality from θ_1 up to t ($u(t) > 0$ for $t \in [\theta_1, t*)$) yields the inequality

$$R(t) \equiv 1 - (\lambda-1)\left[t - \theta_1 + \frac{u^{\lambda-1}(\theta_1)}{(n-1)!}\int\limits_{\theta_1}^{t} p(\tau)(t-\tau)^{n-1}d\tau\right] \le \left[\frac{u(\theta_1)}{u(t)}\right]^{\lambda-1}, \quad t \in [\theta_1, t^*). \quad (3.7)$$

On the other hand, by virtue of the condition (3.6) and inequalities (3.5) we have the estimation

$$R(t) \ge 1-(\lambda-1)\left[2\delta_1^2 + 2\overline{u}^{\lambda-1}(\theta_1)\delta_1^{n-1}\varphi_{\theta_1}^{-1/\mu}\right] > 1 - 4\overline{u}^{\lambda-1}(\theta_1)(1+\varphi_{\theta_1}^{-1/\mu}) = \lambda^{-1}, \quad t \in [\theta_1, t*),$$

which contradicts to the fact that right part of the inequality (3.7) tends to $+0$ as $t \to t^* - 0$.
Thus, all solutions u of the equation (1.1) with the initial conditions $U(\theta_0) \in X_1$ exist on whole segment $[\theta_0, \theta_2]$. The constructions of the first step are completed.
Let's assume now, the function $p(t) \ge 0$ is constructed on the segment $[\theta_0, \theta_{2k}]$ the length of which is greater $4k(n-1)!$. Then all solutions of the equation (1.1) with initial conditions (3.1) exist on this segment. Now we realize the constructions that are needed to design the admissible function $p(t)$ for the $(k+1)$-th step. For this purpose similar to the first step we shall choose the moments $\theta = \theta_{2k+1}$ and $\delta = \delta_{k+1}$. Put $p(t) \equiv 0$ on the interval $(\theta_{2k}, \theta_{2k+1}]$. The moment $t = \theta_{2k+1}$ we shall take so large that conditions

$$\theta_{2k+1} - \theta_{2k} > 4(n-1)!, \quad \delta_{k+1} \equiv \frac{1}{4\lambda}\overline{u}^{1-\lambda}(\theta_{2k+1})(1+\varphi_{\theta_{2k+1}}^{-1/\mu})^{-1} < \delta_\mu, \qquad (3.6_1)$$

have been satisfied where $\overline{u}(\theta_{2k+1}) \equiv \sup\{u(\theta_{2k+1}) : U(\theta_0) \in X_1\}$.
It is obvious, as well as on the first step, at such θ_{2k+1} for any solution u of the equation (1.1) with initial values (3.1) the inequality $4\alpha_1[u(\theta_{2k+1})] < u(\theta_{2k+1})$ is true. For the chosen θ and δ calculate the number m_{k+1} and the corresponding moments τ_i, $i = 1, \ldots, m_{k+1}$ by formulas (3.2). Then define function $p(t)$ according to the equality (3.3). All solutions of the equation (1.1) with the initial values (3.1) exist on the whole segment $[\theta_{2k}, \theta_{2k+2}]$, $\theta_{2k+2} = \tau_{m_{k+1}} + 1$. The proof of this fact repeats the similar proof of the first step with replacement of θ_0, θ_1 and δ_1 by θ_{2k}, θ_{2k+1} and δ_{k+1}, respectively.
It remains to note that inequalities of the type (3.4) guarantee the validity of the condition $J_\mu(p, \varphi) = +\infty$. The theorem 3.1 is proved. □

At $\mu = 1$ and $\varphi(t) = t^{(n-1)\lambda}$ this theorem gives the solution of the second Kiguradze problem (Izobov, 1984a; Izobov, 1984b; Izobov, 1985a):

Corollary 3.2. *The inequality (1.6) is not a necessary condition for the n-th order equation (1.1) with parameter $\lambda > 1$ and function $p : [0, +\infty) \to [0, +\infty)$ to have strongly increasing solutions.*

The necessary condition for existence of strongly increasing solutions of the n-th order equation (1.1) follows from the next theorem (Izobov, 1984a; Izobov, 1984b; Izobov, 1985a).

Theorem 3.3. *The equation (1.1) with function $p(t) \geq 0$ satisfying condition*

$$J_\mu[p, \tau^{n_1\nu-1}] = +\infty, \quad \nu < \mu \in (0,\, 1/n), \tag{3.8}$$

has no proper solutions of the (1.3) type.

Proof. For $\beta = n^{-2}\min\{1 - \lambda^{-1},\, 1 - n\mu\}$ define numbers $\mu_{k+1} = 1/n$ at $n = 2k+1$, $\mu_i = -\mu_{n-i+1} + 2/n = ([n/2] + 1 - i)\beta + 1/n,\ \ 1 \leq i \leq [n/2]$, satisfying the conditions $\lambda\mu_n > \mu_1 > \ldots > \mu_n > \mu$, $\sum\limits_{i+1}^{n} \mu_i = 1$, and pick the number α by means of inequalities $0 < \alpha < \beta\min\{1,\, \lambda - 1\}$, $n^2\alpha \leq \varepsilon = n_1(\mu - \nu)$. From the condition (3.8) follows the equality

$$J_\mu[p, \tau^{n_1\mu-1-\varepsilon}] = +\infty. \tag{3.9}$$

Since the integral with function $t^{-\varepsilon-1}[p(t)t^{n_1}]^\gamma$ converges on the set $T \equiv \{t \geq 1 : p(t)t^{n_1} \leq 1\}$ for any $\gamma \geq 0$, then from divergence of the integral (3.9) under proper choice of numbers μ_n and α follows a divergence of the integral

$$J_{\mu_n}[\tau^{n_1\mu_n - n\alpha - 1}] = +\infty. \tag{3.10}$$

Now, assume to the contrary that the equation (1.1) with the function $p(t) \geq 0$ satisfying the condition (3.10) has a proper solution of the (1.3) type. Then due to the well known relationship between the arithmetic and geometric average, obvious estimations $u^{(i)}(t) > ct^{n-i-1}$, $c > 0$, $i = 0, 1, \ldots, n-1$, $t > t_2$, and the choice of numbers μ_i and α for the derivative of the auxiliary function $v(t) = \prod\limits_{i=0}^{n-1} u^{(i)}(t)$ we have the estimations

$$\dot{v}(t) = \sum_{k=0}^{n-1} u^{(k+1)}(t) \prod_{i=0, \neq k}^{n-1} u^{(i)}(t) \geq c \prod_{k=0}^{n-1} \left[u^{(k+1)}(t) \prod_{i=0, \neq k}^{n-1} u^{(i)}(t) \right]^{\mu_{k+1}} \geq$$

$$\geq cv(t)p^{\mu_n}(t)u^{\lambda\mu_n-\mu_1}(t)\prod_{i=1}^{n-1}[u^{(i)}(t)]^{\mu_i-\mu_{i+1}} \geq cp^{\mu_n}(t)t^{(n-1)\lambda\mu_n+\mu_n-n\alpha-1}v^{1+\alpha}(t), \quad t > t_2.$$

From here by the condition (3.10) follows the existence of a such moment t^*, for which $v(t) \to +\infty$ as $t \to t^* - 0$, that contradicts to our assumption. The theorem is proved. □

Remark 3.1. The accuracy of the condition (3.8) with respect to parameter μ follows from the theorem 3.1. This accuracy of (3.8) with parameter ν is confirmed by the consideration of the function $p(t) = t^{-n_1}\ln^{\varepsilon-1/\mu} t$, $\varepsilon \in (0,\, 1/2)$, $\mu \in (0,\, 1/n)$, $t \geq e$, satisfying both to the constructed integral condition and the sufficient condition (1.6).

Corollary 3.4. *(Izobov, 1984a; Izobov, 1984b; Izobov, 1985a) If the equation (1.1) has a proper solution of the (1.3) type then the function p satisfies condition $J_\mu[p, \tau^{n_1\nu-1}] < +\infty$ for every parameters $\nu < \mu \in (0,\, 1/n)$.*

Corollary 3.5. *(Izobov, 1984a; Izobov, 1984b; Izobov, 1985a) In the class $P = \{p : 0 \leq p(t) \leq c_p(1 + tm_p),\, t \geq 0\}$ of piecewise continuous functions p with a power majorant the condition $J_{1/n}[t^\alpha] = +\infty$ at $\alpha < (\lambda - 1)(1 - n^{-1})$ guaranties the absence of any proper solutions of the (1.3) type for the equation (1.1)*

To the theorems 3.1 and 3.2 above it had preceded the following theorem proved in (Izobov, 1984c) for the second order equation (1.1) with the concrete admissible value of parameter ν.

Theorem 3.0. *The second order equation (1.1) with an arbitrary function $p(t) \geq 0$, satisfying the condition*

$$\int_a^{+\infty} t^{-2}\left[p(t)t^{\lambda+3}\right]^\mu = +\infty \tag{3.11}$$

with $\mu \in (0,\, 1/2)$ has no proper solutions of the (1.3) type. Nevertheless, for any $\mu > 1/2$ there exists the piecewise continuous function $p(t) \geq 0$ satisfying conditions (3.11) and $J(a, +\infty) = +\infty$ such that the equation (1.1) has a two-parametrical family of proper solutions of the (1.3) type. In the special case $\mu = 1/2$ there exist both satisfying the condition (3.11) functions $p(t) \geq 0$, for which the equation (1.1) has no unbounded proper solutions, and functions $p(t) \geq 0$ satisfying (1.6), for which this equation (1.1) has a two-parametrical family of such solutions.

Corollary 3.6. *For any piecewise continuous function $\varphi(t) \uparrow +\infty$ as $t \uparrow +\infty$ there exists a piecewise continuous function p satisfying the condition*

$$\int_a^{+\infty} \varphi(\tau)\sqrt{p(\tau)}\tau^{(\lambda-1)/2}d\tau = +\infty$$

such that the equation (1.1) has a two-parametrical family of proper solutions of the (1.3) type. Nevertheless, there are also such continuous functions $\varphi(t) \downarrow 0$ at $t \uparrow +\infty$ and $p(t) \geq 0$ satisfying the condition (3.11), that this equation has also the same family of solutions .

Note, that in (Izobov, 1984c) linear extension of the integral (1.6) $J_1[p, \tau^{(n-1)\lambda}/\varphi(\tau)]$ with a non-decreasing positive function φ (see formulated below the theorem 3.3) was firstly considered.

By means of a recurrent algorithm of parallel construction of the function p and the solution possessing the needed properties the following theorems are proved. They also solve the second Kiguradze problem.

Theorem 3.7. *(The case $n = 2$ see in (Izobov, 1984c) and $n > 2$ see in (Izobov and Rabtsevich, 1987b; Rabtsevich, 1986; Rabtsevich, 1990c; Rabtsevich, 1990b). For any piecewise continuous function $\varphi(t) > 1$ nondecreasing and unbounded on the semiaxis $t \geq a$ and for number $\alpha > 1$ there exists a function $p(t) \geq 0$ satisfying the condition $J_1[p, \tau^{(n-1)\lambda}/\varphi(\tau)] = +\infty$ such that the equation (1.1) has n-parametrical family of strongly increasing solutions possessing additional differential property*

$$0 < u^{(n-1)}(t)/u(t) < (n-1)!\varphi^{\alpha}(t)t^{1-n} \tag{3.12}$$

in some neighbourhood of $+\infty$.

Theorem 3.8. *($n = 2$ see in (Izobov, 1984a), $n > 2$ see in (Izobov and Rabtsevich, 1987a; Rabtsevich, 1990c; Rabtsevich, 1990b). There exist piecewise continuous function $p(t) \geq 0$ and monotonous unbounded function $\varphi(t) > 1$, united by the equality $J_1[p, \tau^{(n-1)\lambda}/\varphi(\tau)] = +\infty$ and such that the equation (1.1) has n-parametrical family of strongly increasing solutions, possessing the additional differential property (3.12) with parameter $\alpha = 1$ in some neighbourhood of $+\infty$.*

The case $\alpha < 1$ is stated by the next theorem

Theorem 3.9. *($n = 2$ see in (Izobov, 1984c), $n > 2$ see in (Rabtsevich, 1986)) Let $\varphi(t) > 1$ be an arbitrary nondecreasing function. Then the equation (1.1) with any function $p(t) \geq 0,$ satisfying the condition $J_1[p, \tau^{(n-1)\lambda}/\varphi(\tau)] = +\infty,$ does not have solutions with property (3.12) for $\alpha < 1.$*

This theorem strenthes the following assertion.

Theorem 3.10. *(Izobov and Rabtsevich, 1987a; Rabtsevich, 1990c; Rabtsevich, 1990b) Let $\varphi(t) > 1$ be an arbitrary nondecreasing function possessing a power majorant. Then the equation (1.1) with function $p(t) \geq 0,$ satisfying the condition $J_1[p, \tau^{(n-1)\lambda}/\varphi(\tau)] = +\infty,$ has no any solutions with property (3.12) at $\alpha = 1.$*

From the theorems 3.1 and 3.3 in obvious way follows that the condition (1.6) is not obliged to be carried out in the case of presence at the equation (1.1) of the proper solutions of the (1.3) type. Moreover, the subintegral expression must be nonlinear with respect to function $p(t)$ in any prospective integral condition of the absence of nonoscillatory unbounded solutions for the equation (1.1). Nevertheless, the set of functions $p,$ defined by the linear on p condition $J_1[p, \tau^{(n-1)\lambda}/\varphi(\tau)] < +\infty$ such that the equation (1.1) has no proper solutions of the (1.3) type is not empty. It is established by the following theorem.

Theorem 3.11. *(Izobov and Rabtsevich, 1987a; Rabtsevich, 1990c; Rabtsevich, 1990b) For any piecewise continuous unbounded (not necessarily monotonous) on semiaxis $t \geq 0$ function $\varphi(t) > 0$ there exists a piecewise continuous function $p(t) > 0$ satisfying the condition $J_1[p, \tau^{(n-1)\lambda}/\varphi(\tau)] < +\infty$ such that the equation (1.1) has no proper solutions of the (1.3.) type.*

Also, I. T. Kiguradze has received new necessary conditions of the existence of proper solutions of the (1.3) type and their a priori upper asymptotic estimations, too.

Theorem 3.12. *(Kiguradze and Chanturia, 1993) Let the equation (1.1) has a proper solution of the (1.3) type. Then for all* $\mu \in [2/(n+n_1),\, 1/n)$ *the equality*

$$\lim_{t\to+\infty} t^{1-n\mu} \int_t^{+\infty} p^{\mu}(\tau)\tau^{[n+n_1]\mu-2}d\tau = 0$$

holds and the inequality

$$|u(t)| \le \gamma(n,\lambda,\mu)\Big(t^{n\mu-1}\int_t^{+\infty} p^{\mu}(\tau)d\tau\Big)^{1/((1-\lambda)\mu)}$$

is fulfilled in some neighbourhood of $+\infty$*.*

Next, in the papers (Rabtsevich, 1994; Rabtsevich, 1996) a more exact condition and an estimation are received in the maximal area of change of parameters, using the monotony and differential properties of the auxiliary functions $v_{\delta,i}(t) = u^{(i)}(t)t^{i-n+1+\delta}$ $(i=0,\,\dots,\,n),\quad \delta>0.$

Theorem 3.13. *If the equation (1.1) has a strongly increasing solution* $u,$ *then for all* $\mu\in(0,1/n),\ \sigma>0$ *the condition*

$$\lim_{t\to+\infty} F_{\mu\,\sigma}(t)=0, \quad F_{\mu\,\sigma}(t)\equiv t^{\sigma}\int_t^{+\infty} p^{\mu}(s)s^{n_1\mu-1-\sigma}\,ds,$$

holds and the estimation

$$|u(t)|\le ct^{n-1}\left(F_{\mu\,\sigma}(t)\right)^{1/(1-\lambda)\mu}$$

is valid in some neighbourhood of $+\infty$*.*

Nevertheless, the function $p(t)=t^{-n_1}\varphi(t)$ with the piecewise continuous function $\varphi(t)\to+0$ as $t\to+\infty$ does not satisfy any of maintained above necessary conditions rather the equation (1.1) with the $\varphi(t)=\ln^{\gamma}t$ for $\gamma\in(-1,0)$ has no any such solutions. This fact for the case $n=2$ is established by

Theorem 3.14. *(Izobov, 1996a; Izobov, 1996b) Let the function* $p_\mu(t)\equiv t^{-1}[p(t)t^{1+\lambda}]^{\mu}$ *satisfy the condition*

$$J_{\varepsilon\mu\alpha}(p)\equiv\int_{t_0}^{+\infty} t^{-\varepsilon}p_\mu(t)\exp\alpha\int_{t_0}^{t} p_\mu(\tau)\Big(\int_{t_0}^{\tau} p(\xi)\xi^{\lambda}d\xi\Big)^{\mu_m}d\tau dt=+\infty,\quad t_0>0,$$

for some $\varepsilon>0,\ \alpha>0,\ \mu\in(0,1/2)$ *and* $\mu_n\equiv\min\{1-2\mu,(\lambda-1)^{\mu}\}$*. Then the equation (1.1) has no proper nonoscillatory unbounded solutions .*

Due to the condition (1.6) the equation (1.1) with the function $p(t)\le\mathrm{const}\times t^{-1-\lambda}\ln^{\gamma}t,$ $\gamma<-1,\ t\ge t_0,$ has a two-parametrical family of proper nonoscillatory unbounded solutions. It is natural to assume, that this equation with function $p(t)\ge\mathrm{const}\times t^{-1-\lambda}\ln^{\gamma}t,\ \gamma>-1,$ $t\ge t_0,$ has no any solutions specified above. This particular statement contains the following theorem

Theorem 3.15. *(Izobov, 1996a; Izobov, 1996b) Let for some numbers* $\varepsilon > 0,\ \alpha > 0,$ $\mu \in (0, 1/2)$, $q \in \boldsymbol{N}$ *and* $0 < \omega < \Omega < 1$ *the condition*

$$J_{\varepsilon\mu\alpha q}(p) \equiv \int\limits_{t_0}^{+\infty} t^{-\varepsilon} p_\mu(t) \exp \alpha \Big\{ \int\limits_{t^\Omega}^{t} p_\mu(\xi) d\xi \prod_{i=1}^{q} \Big(\int\limits_{t^{\Omega\omega(i)}} t^{\omega(i)} p(\xi) \xi^\lambda d\xi \Big)^{\mu(\lambda-1)\lambda^{i-1}} \Big\} dt = +\infty$$

holds, where $\omega(i) \equiv \omega^i$. *Then the equation (1.1) has no proper nonoscillatory unbounded solutions .*

Corollary 3.16. *Equation (1.1) with function* $p(t) \geq ct^{-1-\lambda} \ln^\gamma t,\ c > 0,\ \gamma > -1,\ t \geq t_0 > 1,$ *has no proper nonoscillatory unbounded solutions.*

Remark 3.2. An example of the function for which the integral $J_{\varepsilon\mu\alpha}(p)$ diverges for small $\mu > 0$ and converges for small $1/2 - \mu > 0$ is the function $p(t) = t^{-1-\lambda} \ln^\gamma t$ where $\gamma = (1 - \lambda)/(2\lambda)$.

Remark 3.3. Splitting the segment $[t_0, t]$ by points $\tau_i = \tau_i(t) = t\omega^i$ and $\theta_i = \theta_i(t) = t^{\Omega\omega^i}$, $0 < \omega < \Omega < 1,\ i = 0, 1, \ldots, q,$ that used in the theorem 3.11, is adapted to catch the logarithmic component of the function p. Using other methods of splitting of the segment leads to the more general in comparison with the theorem 3.11 criterion of the absence of unbounded proper solutions of the equation (1.1).

The following two criterion are proved in the same way:

Theorem 3.17. *(Izobov, 1996b) Let for functions* $p(t)$ *and* $p_\mu(t, n) \equiv t^{-1}[p(t)t^{n_1}]^\mu$ *the condition*

$$\int\limits_{t_0}^{+\infty} t^{-\varepsilon} p_\mu(t, n) \exp \alpha \int\limits_{t_0}^{t} p_\mu(\tau, n) \Big(\int\limits_{t_0}^{\theta\tau} p(\xi) \xi^{(n-1)\lambda} d\xi \Big)^{\mu(\lambda-1)} d\tau dt = +\infty$$

holds for some $\varepsilon > 0,\ \alpha \geq 0,\ \mu \in (0, 1/n)$ *and* $\theta \in (0, 1)$. *Then the equation (1.1) has no unbounded nonoscillatory solutions .*

Theorem 3.18. *(Izobov, 1996b) Let for some numbers* $\varepsilon > 0,\ \alpha \geq 0,\ \mu \in (0, 1/n)$ *and* $0 < \omega$ $< \Omega < 1$ *the condition*

$$\int\limits_{t_0}^{+\infty} t^{-\varepsilon} p_\mu(t, n) \exp \alpha \Big\{ \int\limits_{t^\Omega}^{t} p_\mu(\xi, n) d\xi \prod_{l \in L(t)} \Big(\int\limits_{t^{\Omega\omega(l)}}^{t^{\omega(l)}} p_1(\xi, n) d\xi \Big)^{\mu(\lambda-1)\lambda^{l-1}} \Big\} dt = +\infty,$$

holds, where $\omega(l) \equiv \omega^l$ *and* $L(t) = \big\{1, \ldots, 1 + \big[\,|\ln \omega|^{-1} \ln(\ln t / \ln t_0)\big]\big\},\ t_0 \geq e$. *Then the equation (1.1) has no proper unbounded nonoscillatory solutions .*

The theorem 3.13 presents an exact condition: function $p(t) \geq \mathrm{const} \times t^{-n_1} \ln^\gamma t,\ t \geq t_0,$ with number $\gamma \geq -1$ satisfies to this condition while according to the Kiguradze – Kvinikadze condition (1.6) the equation (1.1) with non-negative function $p(t) \leq \mathrm{const} \times t^{-n_1} \ln^\gamma t,\ t \geq t_0,$ at $\gamma < -1$ already has $(n - 1)$-parametrical family of unbounded solutions. Then the second author shown that the more general necessary conditions of existence of those solutions and their estimations can be obtained by introduction of auxiliary functions $v_{\varphi,i}(t) = u_i(t)t^{1-n}\varphi(t)$ with any nondecreasing function $\varphi(t) > 0$.

For an arbitrary function $\varphi : [a, +\infty) \to [0, +\infty)$ introduce the following sets $A_\varphi(t) = \{x > a :$ $(x - t)\varphi(x) > 0\},\ A_\varphi = A_\varphi(a);\ \varphi_1(t) = \min\{t^{-1}, \dot\varphi(t)/\varphi(t)\}$ for $t > a$. The following general theorem is true.

Theorem 3.19. *(Rabtsevich, 1997b; Rabtsevich, 1997a) Let u be the solution of the problem (1.1), (1.3), and $\varphi : [0, +\infty) \to [0, +\infty)$ be any nondecreasing absolutely continuous function having the limited variation on the set $[a, +\infty) \setminus A_p$. Then for any numbers $\nu \in [0, 1)$, $\mu \in (2(1-\nu)/(n+n_1), (1-\nu)/n)$ and $\varepsilon > 0$ the equality*

$$\lim_{t\to+\infty} F_{\nu,\mu,\varepsilon}(\varphi(t)) = 0, \quad F_{\nu,\mu,\varepsilon}(\varphi(t)) \equiv \varphi^{\varepsilon}(t) \int_t^{+\infty} \frac{(p(\tau)\tau^{(n-1)\lambda})^{\mu}}{\varphi^{\varepsilon}(\tau)} \left(\frac{\dot\varphi(\tau)}{\varphi(\tau)}\right)^{\nu} \varphi_1^{1-\mu-\nu}(\tau)d\tau \quad (3.13)$$

and, since some moment $t_u > a$, the estimation

$$|u(t)| < \gamma t^{n-1} [F_{\nu,\mu,\varepsilon}(\varphi(t))]^{1/((1-\lambda)\mu)}$$

hold, where γ is a positive constant depending of n, λ, μ.

A slight modification of this theorem with smaller amount of parameters is given as follows

Theorem 3.20. *(Rabtsevich, 1997b; Rabtsevich, 1997a) Let u be the solution of the problem (1.1), (1.3), $\varphi : [0, +\infty) \to [0, +\infty)$ be any nondecreasing absolutely continuous function having the limited variation on the set $[a, +\infty) \setminus A_p$ and satisfying almost everywhere on A_p to the condition $t\dot\varphi(t)/\varphi(t) \le 1$. Then for any numbers $\mu \in (0, 1/n)$ and $\varepsilon > 0$ the equality*

$$\lim_{t\to+\infty} G_{\mu,\varepsilon}(\varphi(t)) = 0, \quad G_{\mu,\varepsilon}(\varphi(t)) \equiv \varphi^{\varepsilon}(t) \int_t^{+\infty} (p(\tau)\tau^{(n-1)\lambda})^{\mu} \varphi^{-\varepsilon}(\tau) (\dot\varphi(\tau)/\varphi(\tau))^{1-\mu} d\tau \quad (3.13_1)$$

is fulfilled and, since some moment $t_u > a$, the estimation

$$|u(t)| < \gamma t^{n-1} [G_{\mu,\varepsilon}(\varphi(t))]^{1/((1-\lambda)\mu)}$$

holds, where γ is a positive constant depending on n, λ, μ.

By means of (3.13) it is found a wide class of functions p (in particular, having a power majorant) for which the (1.6) is the necessary and sufficient condition of solvability of the mentioned above problem (1.1), (1.3). Then, the question on necessary and sufficient conditions of existence of strongly increasing solutions of the equation (1.1) with function $p(t) < ct^{-n_1}, \quad t > a$, partially studied in (Rabtsevich, 1990c; Rabtsevich, 1990b), completely considers the following theorem

Theorem 3.21. *(Rabtsevich, 1997b) If the equation (1.1) with function p satisfying the condition*

$$\delta p(t) t^{n_1} < 1, \quad t > a, \quad \delta > 0, \quad (3.14)$$

has a strongly increasing solution, then the condition (1.6) holds.
In addition, if the condition $\delta p(t)t^{n_1} < J(t; +\infty) < 1$, $t > a$, $\delta > 0$, holds then each strongly increasing solution u of the equation (1.1) admits the estimation

$$0 < c_* < |u(t)| t^{1-n} J^{-1/(\lambda-1)}(t; +\infty) < c^* < +\infty$$

in some neighbourhood $+\infty$, where the constants c_* and c^* depend of n and λ only.

The complete solution of the problem stated in (Izobov, 1996a) about the existence of strongly increasing solutions of the equation (1.1) with function p monotonously decreasing on the semi-axis $t > a$, which defined by the equality

$$p(t) = ct^{-n_1}(l_{k-1}(t))^{-1}(\ln_k t)^{-1-\sigma}, \quad \sigma \in \mathbf{R}, \quad k \in \mathbf{N}, \tag{3.15}$$

where $\ln_0 t = t$, $\ln_{k+1} t = \ln(\ln_k t)$, $l_k(t) = \prod_{i=0}^{k} \ln_i t$, gives the following

Corollary 3.22. *Let the function p be defined by the equality (3.15). In the case $\sigma \geq 0$ all solutions of the type (1.3) of the equation (1.1) are the singular solutions of the second type. If $\sigma \leq 0$ then the equation (1.1) has strongly increasing solutions and each of them admits two-sided asymptotic estimation $c_* < |u(t)|t^{1-n}(\ln_k t)^{\sigma/(\lambda-1)} < c^*$ where the constants c_* and c^* depend of n and λ only.*

The case in which the function p is a nonmonotone or unbounded function is considered by the following

Theorem 3.23. *(Rabtsevich, 1997b) Let the function p satisfies the inequality $p(t)t^{n_1}f(t) < \delta J_f(a;t)$ for all $t \in A_p$ where $\delta > 0$ and function $f : A_p \to [0,1]$. If the equation (1.1) has a proper solution of the (1.3) type then integral condition $J_f(a;+\infty) < +\infty$ holds, where $J_f(a;t) \equiv \int_a^t p(\tau)\tau^{(n-1)\lambda}f(\tau)d\tau$.*

In the case of divergence of the integral $J_f(a;+\infty)$ the following statements are valid:

Corollary 3.24. *Let the integral $J_f(a;+\infty)$ diverges. If the equation (1.1) has a strongly increasing solution, then for any $\delta > 0$ the set $M_\delta = \{t > a : p(t)t^{(n-1)\lambda}f(t) > \delta J_f(a;t)/t\}$ has subsets of a positive measure in every neighbourhood of $+\infty$ and the integral condition*

$$\int_{M_\delta} p(t)t^{n_1}f(t)dt < +\infty$$

holds.

One of the more general situations describes

Theorem 3.25. *(Rabtsevich, 1997b) Let the function p satisfies the condition $p(t)t^{n_1}f(t) \leq \psi(J_f(a;t))$ for all $t \in A_p$ where $\psi : \boldsymbol{R}_+ \to [1,+\infty)$ and $f : A_p \to [0,1]$ are the given functions. If the equation (1.1) has a proper solution of the (1.3) type then the integral condition*

$$\int_a^{+\infty} p(t)t^{n_1}f(t)/\psi(J_f(a;t))dt = +\infty$$

holds.

Moreover, if $\int_1^{+\infty} \frac{dx}{x\psi(x)} = +\infty$ then the integral $J_f(a;+\infty)$ converges.

Remark 3.4. Many of the listed above results are extended to the more general equations and systems of the Emden – Fowler type in the papers (Rabtsevich, 1990a; Rabtsevich, 1993; Rabtsevich, 2000d; Rabtsevich, 1998). Asymptotic properties of nonoscillatory singular solutions of the Emden – Fowler equation was studied in (Rabtsevich, 2000b; Rabtsevich, 2000a; Rabtsevich, 2001; Rabtsevich, 2003).

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Comparison Theorems for Perturbed Half-linear Euler Differential Equations

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ABSTRACT

We consider perturbed half-linear Euler differential equations

(A) $\Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}\,[\alpha + f(t)]\varphi(x) = 0\,,$

(B) $\Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}\,[\alpha + g(|x|)]\varphi(x) = 0\,,$

where $\alpha > 0$, $\varphi(u) = |u|^{\alpha-1}u$ and $E(\alpha) = \alpha^\alpha/(\alpha+1)^{\alpha+1}$. Our objective is to establish comparison principles which enable us to deduce the oscillation or nonoscillaiton of (B) from that of (A), and vice versa.

Keywords. Comparison, Oscillation, Half-linear Euler Differential Equation

MSC (Mathematical Subject Classification): 34C10, 34C15

1. INTRODUCTION

The oscillatory and nonoscillatory behavior of the Euler differential equation $x'' + \gamma\,t^{-2}\,x = 0$ has been well analyzed. It is known that all nontrivial solutions of this equation are oscillatory

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for $\gamma > 1/4$ and nonoscillatory for $\gamma \le 1/4$. Consider the half-linear differential equation

$$\Big(\varphi(x')\Big)' + \frac{\alpha\,\gamma}{t^{\alpha+1}}\,\varphi(x) = 0, \quad t > 0\,, \tag{1.1}$$

where $\alpha > 0$ and γ are constants and $\varphi(x) = |x|^{\alpha-1}\,x$, which may well be called the half-linear (or generalized) Euler differential equation. The asymptotic behavior of (1.1) is investigated by Elbert in [3]. It is established that the value

$$E(\alpha) = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}},$$

plays a crusical role in determining the oscillatory behavior of solutions of the equation (1.1). Namely, for $\gamma > E(\alpha)$ all nontrivial solutions of (1.1) are oscillatory, while for $\gamma \le E(\alpha)$ all nontrivial solutions of (1.1) are nonoscillatory. It should be noticed that this is not the single case of similarity between the second order linear differential equation $(p(t)x'(t))' + q(t)x(t) = 0$ and the half-linear differential equation $\Big(\varphi(x')\Big)' + q(t)\,\varphi(x) = 0$. Generally, there is a striking similarity between these two types of differential equations. This similarity was first observed for the first time by Elbert [2], who extended the classical Sturmian comparison and separation theorems for the linear differential equation to the corresponding half-linear differential equation. Among numerous papers, we choose to refer to the papers [6] - [11].

Elbert and Schneider [5] considered a perturbed version of the equation (1.1)

$$\Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}\,[\alpha + \delta(t)]\varphi(x) = 0\,, \tag{1.2}$$

where $\delta(t)$ is a continuous function on (t_0, ∞) for some $t_0 > 0$. They proved that under the condition that

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{\delta(s)}{s}\,ds$$

exists as a finite number and

$$\lim_{t\to\infty} \int_{t}^{\infty} \frac{\delta(s)}{s}\,ds \ge 0 \ \text{ for } \ t \ge t_1 > t_0\,,$$

in the superlinear case, i.e. for $\alpha > 1$, the nonoscillation of the linear differential equation

$$z''(t) + \delta(e^s)\,z(t) = 0 \tag{1.3}$$

implies that of the half-linear differential equation (1.2), whereas in the sublinear case, i.e. for $0 < \alpha < 1$, the nonoscillation of the half-linear equation (1.2) implies that of the linear differential equation (1.3). Moreover, under the same condition for the function $\delta(t)$ they proved the following oscillation criterion for the equation (1.2).

Theorem A. *The equation* (1.2) *is oscillatory if*

$$\liminf_{t\to\infty} t \int_t^\infty \delta(e^\eta)\, d\eta > \frac{\alpha+1}{2} \tag{1.4}$$

and nonoscillatory if

$$\limsup_{t\to\infty} t \int_t^\infty \delta(e^\eta)\, d\eta < \frac{\alpha+1}{2}\,. \tag{1.5}$$

In order to prove this theorem they used Riccati technique combined with the notion of the principal solution of the half-linear equation introduced by Elbert and Kusano [4] and its asymptotic form.

Our objective here is to compare the half-linear differential equation (1.2) with its "nonlinear"perturbation of the form

$$\Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}\{\alpha + g(|x|)\}\varphi(x) = 0, \tag{1.6}$$

establishing comparison principles which enable us to deduce the oscillation or nonoscillation of (1.6) from that of (1.2), and vice versa. For this purpose we need to give a more compact and straightforward proof of Theorem A, which will be our first task in this paper.

This paper is organized as follows. In Section 2 we prove some auxiliary lemmas which will be used in the proofs of our main results. After providing an alternative proof of Theorem A in Section 3, we establish in Section 4 the main comparison theorems connecting (1.2) with (1.6). Some examples illustrating our main results will also be presented.

We note that in the proofs of our main results including Theorem A we are going to use the Schauder-Tychonoff fixed point theorem, for whose formulation and proof we refer the reader to the book of Coppel [1] (pp. 9-10).

2. AUXILIARY LEMMAS

In this section we collect auxiliary lemmas which will be used later.

The first lemma due to Elbert [2] is a nonoscillation principle presenting a close connection between second order half-linear differential equations and the first order nonlinear differential equations called the generalized Riccati equation.

Lemma 0.1.. *The half-linear differential equation*

$$\Big(\varphi(x')\Big)' + q(t)\varphi(x) = 0$$

is nonoscillatory if and only if the generalized Riccati equation

$$u' + q(t) + \alpha\,|u|^{1+1/\alpha} = 0$$

has a solution defined for all sufficiently large t.

Lemma 0.2.. *The function*

$$F_\alpha(\varrho) = |\varrho|^{1+\frac{1}{\alpha}} - \varrho + E(\alpha), \quad \varrho \in R \tag{2.1}$$

has the following properties:

(i) *it is nonnegative for all $\varrho \in R$;*

(ii) $F_\alpha(\varrho) = 0$ *if and only if* $\varrho = D(\alpha) = \left(\dfrac{\alpha}{\alpha+1}\right)^\alpha$*;*

(iii) $$\lim_{\varrho \to D(\alpha)} \frac{F_\alpha(\varrho)}{(\varrho - D(\alpha))^2} = \frac{1}{2\alpha\, D(\alpha)}\,.$$

PROOF: Since

$$F_\alpha{}'(\varrho) = \frac{\alpha+1}{\alpha}\,|\varrho|^{1/\alpha}\,\text{sgn}\,\varrho - 1\,, \quad F_\alpha{}'(D(\alpha)) = 0\,,$$
$$F_\alpha{}''(\varrho) = \frac{\alpha+1}{\alpha^2}|\varrho|^{\frac{1}{\alpha}-1}\,, \quad F_\alpha{}''(D(\alpha)) = \frac{1}{\alpha D(\alpha)}\,,$$

we conclude that $F_\alpha(\varrho)$ takes its absolute minimum at $\varrho_0 = D(\alpha)$. Accordingly, $F_\alpha(\varrho) \geq F_\alpha(D(\alpha)) = 0$ for all $\varrho \in R$ and we see that $F_\alpha(\varrho) = 0$ if and only if $\varrho = D(\alpha)$. Moreover, by L'Hospital's rule we have

$$\lim_{\varrho \to D(\alpha)} \frac{F_\alpha(\varrho)}{(\varrho - D(\alpha))^2} = \lim_{\varrho \to D(\alpha)} \frac{F'_\alpha(\varrho)}{2(\varrho - D(\alpha))} = \lim_{\varrho \to D(\alpha)} \frac{F''_\alpha(\varrho)}{2} = \frac{1}{2\,\alpha D(\alpha)}\,,$$

completing the proof of the lemma. $\triangle$

Lemma 0.3.. *Let $x(t)$ be a positive function on $[t_0,\infty)$ satisfying*

$$\Big(\varphi(x')\Big)' + \frac{\alpha\, E(\alpha)}{t^{\alpha+1}}\,\varphi(x) \le 0\,,\quad t\ge t_0\,. \tag{2.2}$$

Then

$$\lim_{t\to\infty} x(t)=\infty,\qquad \lim_{t\to\infty} x'(t)=0 \tag{2.3}$$

and

$$\lim_{t\to\infty} t\,\frac{x'(t)}{x(t)} = \frac{\alpha}{\alpha+1}\,. \tag{2.4}$$

PROOF: From the inequality (2.2) it is obvious that $x'(t)$ is decreasing on $[t_0,\infty)$. Using the fact that if a function $x(t)\in C^2[t_0,\infty)$ satisfies $x'(t)<0$ and $x''(t)<0$ for all large t, then $x(t)\to-\infty$ as $t\to\infty$, we conclude that $x'(t)>0$ for all $t\ge t_0$. Since $x'(t)$ is positive and decreasing it tends to a finite limit $x'(\infty)\ge 0$. If we integrate (2.2) from t to ∞, we get

$$\Big(x'(t)\Big)^{\alpha} \ge \Big(x'(\infty)\Big)^{\alpha} + \alpha\, E(\alpha)\int_t^{\infty} \frac{x^{\alpha}(s)}{s^{\alpha+1}}\,ds,\quad t\ge t_0\,, \tag{2.5}$$

from which, using the increasing property of $x(t)$, we see that

$$\Big(x'(t)\Big)^{\alpha} \ge \frac{E(\alpha)\,x^{\alpha}(t_0)}{t^{\alpha}},\quad t\ge t_0\,,$$

or

$$x'(t)\ge \Big(E(\alpha)\Big)^{1/\alpha}\frac{x(t_0)}{t},\qquad t\ge t_0\,.$$

Integrating again the above inequality over $[t_0,t]$, we get

$$x(t)\ge x(t_0) + \Big(E(\alpha)\Big)^{1/\alpha} x(t_0)\,\log\frac{t}{t_0}\,,\quad t\ge t_0\,,$$

from which it follows that $\lim\limits_{t\to\infty} x(t)=\infty$.

Suppose now that $x'(\infty)>0$. Then, $\lim\limits_{t\to\infty} x(t)/t = x'(\infty)$, so that there exists constant $c>0$ such that $x(t)\ge c\,t$ for $t\ge t_0$. Then, from (2.5), we have

$$\Big(x'(t_0)\Big)^{\alpha} > \alpha\, E(\alpha)\int_{t_0}^{\infty}\frac{x^{\alpha}(s)}{s^{\alpha+1}}\,ds \ge \alpha\, E(\alpha)\, c^{\alpha}\int_{t_0}^{\infty}\frac{ds}{s} = \infty\,.$$

This is impossible, so that we prove that $\lim\limits_{t\to\infty} x'(t)=0$.

Now, for $t\ge t_0$, we define

$$f(t) = -\left[\Big(\varphi(x')\Big)' + \frac{\alpha\, E(\alpha)}{t^{\alpha+1}}\,\varphi(x)\right]\ge 0,\qquad \Phi(t) = t^{\alpha+1}\,\frac{f(t)}{x^{\alpha}(t)}\ge 0\,.$$

Then, (2.2) can be rewritten in the form

$$\Big(\varphi(x')\Big)' + \frac{1}{t^{\alpha+1}}\Big(\alpha\, E(\alpha) + \Phi(t)\Big)\,\varphi(x) = 0\,, \quad t \geq t_0\,. \tag{2.6}$$

The function $u(t)$ defined for $t \geq t_0$ with

$$u(t) = \left(\frac{x'(t)}{x(t)}\right)^{\alpha}\,,$$

satisfies the generalized Riccati equation

$$u'(t) + \alpha\big(u(t)\big)^{\frac{\alpha+1}{\alpha}} + \frac{1}{t^{\alpha+1}}\big(\alpha\, E(\alpha) + \Phi(t)\big) = 0\,, \quad t \geq t_0\,. \tag{2.7}$$

Since, by (2.3), $u(t) \to 0$ as $t \to \infty$, from (2.7) we obtain

$$u(t) = \alpha \int_t^\infty \Big(u(s)\Big)^{\frac{\alpha+1}{\alpha}}\, ds + \int_t^\infty \frac{\Phi(s)}{s^{\alpha+1}}\, ds + \frac{E(\alpha)}{t^\alpha}\,, \quad t \geq t_0\,. \tag{2.8}$$

Accordingly, $\Big(u(t)\Big)^{\frac{\alpha+1}{\alpha}} \in L^1[t_0, \infty)$.

If we put $v(t) = t^\alpha\, u(t)$, from (2.7) we have

$$v'(t) + \frac{\alpha}{t}\, F_\alpha\big(v(t)\big) + \frac{\Phi(t)}{t} = 0\,, \quad t \geq t_0, \tag{2.9}$$

where the function $F_\alpha(\varrho)$ is defined by (2.1). Also, from (2.8) we obtain the following Riccati integral equality for the function $v(t)$

$$v(t) = \alpha\, t^\alpha \int_t^\infty \frac{(v(s))^{\frac{\alpha+1}{\alpha}}}{s^{\alpha+1}}\, ds + t^\alpha \int_t^\infty \frac{\Phi(s)}{s^{\alpha+1}}\, ds + E(\alpha)\,, \quad t \geq t_0\,. \tag{2.10}$$

By Lemma 0.2, we have $F_\alpha(v(t)) \geq 0$, so that from (2.9) we see that

$$v'(t) + \frac{\Phi(t)}{t} \leq 0, \quad t \geq t_0. \tag{2.11}$$

Consequently, $v(t)$ is positive and decreasing, so that there exists $\lim_{t\to\infty} v(t) = V < \infty$. Integrating (2.11) over $[t_0, \infty)$ we conclude that $\Phi(t)/t \in L^1[t_0, \infty)$, so that

$$\lim_{t\to\infty} t^\alpha \int_t^\infty \frac{\Phi(s)}{s^{\alpha+1}}\, ds = 0.$$

We now let $t \to \infty$ in (2.10), obtaining

$$V = V^{\frac{\alpha+1}{\alpha}} + E(\alpha) \quad \text{i.e.} \quad F_\alpha(V) = 0\,.$$

Applying Lemma 0.2 (ii), we conclude that $V = D(\alpha)$. Accordingly,

$$\lim_{t\to\infty}\left(t\,\frac{x'(t)}{x(t)}\right)^{\alpha} = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha},$$

which proves (2.4). $\triangle$

Lemma 0.4.. *If a positive function $x(t)$ satisfies (2.2) for all large t, then for any $\varepsilon > 0$ we have*

$$\lim_{t\to\infty} t^{\varepsilon-\frac{\alpha}{\alpha+1}}x(t) = \infty \quad \textit{and} \quad \lim_{t\to\infty} t^{-\varepsilon-\frac{\alpha}{\alpha+1}}x(t) = 0\,. \tag{2.12}$$

PROOF: Let $x(t)$ be a positive function satisfying (2.2). Then, we have (2.4) by Lemma 0.3, which implies that

$$t\,\frac{x'(t)}{x(t)} = \frac{\alpha}{\alpha+1} + \delta(t)\,, \quad \lim_{t\to\infty}\delta(t) = 0\,. \tag{2.13}$$

If we denote $\sigma = \frac{\alpha}{\alpha+1}$ and integrate (2.13) over $[t_0, t]$, we get

$$x(t) = x(t_0)\,\text{exp}\left(\int_{t_0}^{t}\frac{\sigma+\delta(s)}{s}\,ds\right), \quad t \ge t_0\,. \tag{2.14}$$

For any $\lambda \in \mathbb{R}$, we have $t^{\lambda} = \text{exp}(\lambda\log t) = t_0^{\lambda}\,\text{exp}\Big(\lambda\int_{t_0}^{t}\frac{ds}{s}\Big)$, which combining with (2.14) yields

$$t^{\lambda}\,x(t) = c_1\,\text{exp}\left(\int_{t_0}^{t}\frac{\sigma+\lambda+\delta(s)}{s}\,ds\right), \quad t \ge t_0\,,$$

where $c_1 = t_0^{\lambda}\,x(t_0)$. If we now take $\lambda = -\sigma+\varepsilon$ or $\lambda = -\sigma-\varepsilon$, we get

$$\begin{aligned} t^{-\sigma+\varepsilon}\,x(t) &= c_1\,\text{exp}\left(\int_{t_0}^{t}\frac{\varepsilon+\delta(s)}{s}\,ds\right), \quad t \ge t_0\,,\\ t^{-\sigma-\varepsilon}\,x(t) &= c_1\,\text{exp}\left(\int_{t_0}^{t}\frac{\delta(s)-\varepsilon}{s}\,ds\right), \quad t \ge t_0\,. \end{aligned}$$

Letting $t\to\infty$ and noting that $\delta(t)\to 0$ as $t\to\infty$, we have the desired conclusion (2.12). $\triangle$

3. OSCILLATION AND NONOSCILLATION OF THE PERTURBED HALF-LINEAR EULER EQUATION

The purpose of this section is to present a proof of Theorem A, different from that of Elbert and Schneider [5], for the half-linear perturbed Euler differential equation (1.2). We begin by transforming (1.2) into the following equivalent form

$$\text{(E)} \qquad \Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}\left[\alpha + \frac{f(t)}{(\log t)^2}\right]\varphi(x) = 0, \quad t \ge a$$

where $f : [a, \infty) \to (0, \infty)$ is a continuous function. Then, Theorem A formulated for (E) reads as follows.

Theorem 3.5.. *The equation* (E) *is oscillatory if*

$$\liminf_{t\to\infty} \log t \int_t^\infty \frac{f(s)}{s(\log s)^2}\, ds > \frac{\alpha+1}{2}\,, \tag{3.1}$$

and nonoscillatory if

$$\limsup_{t\to\infty} \log t \int_t^\infty \frac{f(s)}{s(\log s)^2}\, ds < \frac{\alpha+1}{2}\,. \tag{3.2}$$

PROOF: Suppose that the equation (E) has a positive solution $x(t)$ on $[t_0, \infty)$. Since (E) is of the form (2.6), as in the proof of Lemma 2.3, we see that the function $u(t)$ and $v(t)$ defined by

$$u(t) = \left(\frac{x'(t)}{x(t)}\right)^\alpha \quad \text{and} \quad v(t) = t^\alpha u(t)\,,$$

satisfy (2.7), (2.8), (2.9) and (2.10) with $\Phi(t) = E(\alpha)\dfrac{f(t)}{(\log t)^2}$.
Furthermore, we define $U(t) = v(t) - D(\alpha)$ and $V(t) = U(t) \cdot \log t$. Then, according to (2.9), $U(t)$ satisfies

$$U'(t) + \frac{\alpha}{t}\, F_\alpha(U(t) + D(\alpha)) + \frac{E(\alpha)\, f(t)}{t(\log t)^2} = 0, \quad t \ge t_0\,. \tag{3.3}$$

Since $\lim_{t\to\infty} U(t) = 0$ by Lemma 0.3, integrating (3.3) from t to ∞ we obtain

$$U(t) = \alpha \int_t^\infty F_\alpha\big(U(s) + D(\alpha)\big)\frac{ds}{s} + \int_t^\infty \frac{E(\alpha)\, f(s)}{s(\log s)^2}\, ds, \ \ t \ge t_0\,, \tag{3.4}$$

or

$$V(t) = \log t \int_t^\infty \frac{\alpha}{s}\, F_\alpha\left(\frac{V(s)}{\log s} + D(\alpha)\right) ds + \log t \int_t^\infty \frac{E(\alpha)\, f(s)}{s(\log s)^2}\, ds, \ \ t \ge t_0\,.$$

Let ε be an arbitrary positive number such that $0 < \varepsilon < \frac{1}{2\alpha D(\alpha)}$. Since $U(t) \to 0$ as $t \to \infty$, using the property (iii) of the function $F_\alpha(\varrho)$ in Lemma 0.2, we see that there exists $\delta > 0$ and some $t_1 \ge t_0$, such that $0 < U(t) < \delta$ and

$$F_\alpha\big(U(t) + D(\alpha)\big) \ge \left(\frac{1}{2\alpha D(\alpha)} - \varepsilon\right) U^2(t) \tag{3.5}$$

for all $t \ge t_1$. This combined with (3.3) yields

$$U'(t) + \left(\frac{1}{2\alpha D(\alpha)} - \varepsilon\right) \frac{\alpha}{t}\, U^2(t) \le 0, \quad t \ge t_1\,. \tag{3.6}$$

Integrating (3.6) on $[t_1, t]$ and then taking the upper limit as $t \to \infty$, we have

$$\limsup_{t\to\infty} U(t) \cdot \log t \leq 2D(\alpha) \text{ or } \limsup_{t\to\infty} V(t) < \infty\,. \tag{3.7}$$

Combining (3.4) and (3.5), we obtain

$$U(t) \geq \left(\frac{1}{2\alpha D(\alpha)} - \varepsilon\right)\alpha \int_t^\infty \frac{U^2(s)}{s}\,ds + \int_t^\infty \frac{E(\alpha)\,f(s)}{s(\log s)^2}\,ds, \quad t \geq t_1\,,$$

which implies, for all $t \geq t_1$, that

$$V(t) \geq \left(\frac{1}{2\alpha D(\alpha)} - \varepsilon\right)\alpha \log t \int_t^\infty \frac{V^2(s)\,ds}{s(\log s)^2} + \log t \int_t^\infty \frac{E(\alpha)\,f(s)}{s(\log s)^2}\,ds\,. \tag{3.8}$$

Put $V = \liminf_{t\to\infty} V(t)$, $V \in [0, \infty)$. Taking the lower limit as $t \to \infty$ in (3.8), we find that V satisfies

$$V \geq \left(\frac{1}{2\alpha D(\alpha)} - \varepsilon\right)\alpha\, V^2 + \liminf_{t\to\infty} \log t \cdot \int_t^\infty \frac{E(\alpha)\,f(s)}{s(\log s)^2}\,ds\,,$$

which is a quadratic inequality of the form $K\,V^2 - V + L \leq 0$, with

$$K = \left(\frac{1}{2\alpha D(\alpha)} - \varepsilon\right)\alpha, \quad L = \liminf_{t\to\infty} \log t \cdot \int_t^\infty \frac{E(\alpha)\,f(s)}{s(\log s)^2}\,ds.$$

Since $K > 0$, the above inequality is satisfied only if $1 - 4\,K\,L \geq 0$, or $L \leq 1/4K$, that is,

$$\liminf_{t\to\infty} \log t \int_t^\infty \frac{f(s)}{s(\log s)^2}\,ds \leq \frac{1}{E(\alpha)\left(\frac{2}{D(\alpha)} - 4\alpha\,\varepsilon\right)}\,.$$

Since ε is arbitrary, it follows that

$$\liminf_{t\to\infty} \log t \int_t^\infty \frac{f(s)}{s(\log s)^2}\,ds \leq \frac{\alpha+1}{2}\,,$$

which contradicts the assumption (3.1). This proves the oscillatory part of the theorem.

Our next task is to show that the equation (E) is nonoscillatory if condition (3.2) is satisfied. Assume that (3.2) holds. Then, there exist $0 < \eta < \frac{\alpha+1}{2}$ and some $t_1 \geq t_0$, such that

$$\limsup_{t\to\infty} \log t \int_t^\infty \frac{f(s)}{s(\log s)^2}\,ds \leq \eta \text{ for } t \geq t_1\,. \tag{3.9}$$

With the choice of $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{\frac{\alpha+1}{2} - \eta}{2\alpha\,D(\alpha)\,\eta}\,,$$

the quadratic equation

$$\alpha\left(\frac{1}{2\alpha D(\alpha)}+\varepsilon\right)X^2 - X + \eta\, E(\alpha) = 0 \tag{3.10}$$

has two distinct positive roots. Denote the smaller root by σ. Using once more the property (iii) of the function $F_\alpha(\varrho)$ in Lemma 0.2, we see that there exists $\delta > 0$ such that $0 \le \varrho - D(\alpha) < \delta$ implies

$$F_\alpha(\varrho) \le \left(\frac{1}{2\alpha D(\alpha)}+\varepsilon\right)(\varrho - D(\alpha))^2 . \tag{3.11}$$

Let $T > t_1$ be so large that

$$\frac{\sigma}{\log t} < \delta \;\text{ for }\; t \ge T . \tag{3.12}$$

Let $C[T,\infty)$ be the set of all continuous functions $v : [T,\infty) \to \mathbb{R}$ with the topology of uniform convergence on compact subintervals of $[T,\infty)$. Define the set $\Psi \subset C[T,\infty)$ and the operator $\mathcal{G} : \Psi \to C[T,\infty)$ by

$$\Psi = \{v \in C[T,\infty) \,:\, 0 \le v(t) \le \sigma,\ t \ge T\}$$

and

$$\mathcal{G}v(t) = \log t \int_t^\infty \frac{\alpha}{s}\, F_\alpha\left(\frac{v(s)}{\log s} + D(\alpha)\right) ds + \log t \int_t^\infty \frac{E(\alpha)\, f(s)}{s(\log s)^2}\, ds, \quad t \ge T .$$

For every $v \in \Psi$, using (3.9), (3.10), (3.11) and (3.12), we have

$$\begin{aligned} 0 \le \mathcal{G}v(t) &\le \log t \int_t^\infty \frac{\alpha}{s}\left(\frac{1}{2\alpha D(\alpha)}+\varepsilon\right)\left(\frac{v(s)}{\log s}\right)^2 ds + \log t \int_t^\infty \frac{E(\alpha)\, f(s)}{s(\log s)^2}\, ds \\ &\le \alpha\left(\frac{1}{2\alpha D(\alpha)}+\varepsilon\right)\sigma^2 + E(\alpha)\,\eta = \sigma , \quad t \ge T . \end{aligned}$$

Thus, $\mathcal{G}(\Psi) \subset \Psi$. Let $\{v_n\}$ be a sequence in Ψ which converges to $v_0 \in \Psi$ in the topology of $C[T,\infty)$. Using the Lebesgue dominated convergence theorem, we see that

$$\int_t^\infty \frac{\alpha}{s}\, F_\alpha\left(\frac{v_n(s)}{\log s} + D(\alpha)\right) ds \;\text{ converges to }\; \int_t^\infty \frac{\alpha}{s}\, F_\alpha\left(\frac{v_0(s)}{\log s} + D(\alpha)\right) ds$$

uniformly on $[T,\infty)$, which shows that $\mathcal{G}$ is a continuous operator. Since $\mathcal{G}(\Psi) \subset \Psi$ and Ψ is uniformly bounded on $[T,\infty)$, it is evident that $\mathcal{G}(\Psi)$ is also uniformly bounded on $[T,\infty)$. Moreover, we have

$$\begin{aligned} 0 \le (\mathcal{G}v)'(t) = \frac{1}{t} \int_t^\infty \frac{\alpha}{s}\, F_\alpha\left(\frac{v(s)}{\log s} + D(\alpha)\right) ds + \frac{1}{t} \int_t^\infty \frac{E(\alpha)\, f(s)}{s(\log s)^2}\, ds \\ - \alpha\, \frac{\log t}{t}\, F_\alpha\left(\frac{v(t)}{\log t} + D(\alpha)\right) - \frac{E(\alpha)\, f(t)}{t\, \log t} , \end{aligned}$$

which ensures that $\mathcal{G}(\Psi)$ is locally equicontinuous on $[T,\infty)$. This implies, via the Ascoli-Arzela lemma, that $\mathcal{G}(\Psi)$ is relatively compact in $C[T,\infty)$.

Therefore, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled, so that there exists $v_1 \in \Psi$ such that $\mathcal{G}v_1 = v_1$. Accordingly, $v_1(t)$ satisfies on $[T,\infty)$ the integral equation

$$v_1(t) = \log t \int_t^\infty \frac{\alpha}{s} F_\alpha\bigg(\frac{v_1(s)}{\log s} + D(\alpha)\bigg)\, ds + \log t \int_t^\infty \frac{E(\alpha)\, f(s)}{s(\log s)^2}\, ds\,.$$

Then the function $u_1(t)$ defined by

$$u_1(t) = \frac{1}{t^\alpha}\bigg(\frac{v_1(t)}{\log t} + D(\alpha)\bigg), \quad t \geq T\,,$$

satisfies the equation

$$u_1' + \alpha\, u_1^{\frac{\alpha+1}{\alpha}} + \frac{E(\alpha)}{t^{\alpha+1}}\left[\alpha + \frac{f(t)}{(\log t)^2}\right] = 0, \quad t \geq T\,. \tag{3.13}$$

Since (3.13) is the generalized Riccati equation associated with the equation (E), we conclude from Lemma 0.1 that (E) is nonoscillatory. This proves Theorem A. △

4. COMPARISON THEOREMS

Our main results developed in this section answer the question as to the similarity in the oscillatory behavior existing between the half-linear differential equation

$$\Big(\varphi(x')\Big)' + \frac{1}{t^{\alpha+1}}\Big(\alpha\, E(\alpha) + f(t)\Big)\varphi(x) = 0\,, \quad t \geq a\,, \tag{4.1}$$

and its “nonlinear”perturbation of the form

$$\Big(\varphi(x')\Big)' + \frac{1}{t^{\alpha+1}}\Big(\alpha\, E(\alpha) + f(|x|^\delta)\Big)\varphi(x) = 0\,, \quad t \geq a\,. \tag{4.2}$$

Theorem 4.1.. *Let the function $f:[a,\infty) \to (0,\infty)$ be continuous. Suppose that there exists a constant $L > 0$ such that $f(t)$ is nonincreasing for $t \geq L$ and*

$$t^\alpha f\Big(t^{\frac{\alpha+1}{\alpha}}\Big) \text{ is nondecreasing for all } t \geq L > 0. \tag{4.3}$$

If the equation (4.1) is nonoscillatory, then there exists a nonoscillatory solution of the equation (4.2) for every $\delta > \frac{\alpha+1}{\alpha}$.

PROOF: Let $X(t)$ be a positive solution of the equation (4.1) on $[t_0, \infty)$ and let ε be any constant such that $0 < \varepsilon < \frac{\alpha}{\alpha+1}$. Then, $X'(t)$ is positive and decreasing, and since $X(t)$ satisfies (2.2), we have by lemma 0.3

$$\lim_{t\to\infty} X(t) = \infty \text{ and } \lim_{t\to\infty} X'(t) = 0 . \tag{4.4}$$

By applying Lemma 0.4, we also have

$$\lim_{t\to\infty} t^{\varepsilon - \frac{\alpha}{\alpha+1}} X(t) = \infty .$$

Consequently, there exists $T \geq \max\{t_0, L\}$ such that

$$X(t) > t^{\frac{\alpha}{\alpha+1} - \varepsilon} \text{ for } t \geq T . \tag{4.5}$$

Denote $\mu = \frac{\alpha+1}{\alpha - \varepsilon(\alpha+1)}$ and notice that $\mu > \frac{\alpha+1}{\alpha}$. Then, by the nonincreasing property of $f(t)$, from (4.5), we obtain

$$f(t) > f\Big(\big[X(t)\big]^{\mu}\Big), \quad t \geq T . \tag{4.6}$$

Integrating the equation (4.1) twice, first from t to ∞ and then from T to t, and using (4.4), we obtain

$$X(t) = X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(\xi)\Big) X^{\alpha}(\xi)\, d\xi \right\}^{1/\alpha} ds , \tag{4.7}$$

for $t \geq T$, which, combined with (4.6), yields

$$X(t) \geq X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f\big(X^{\mu}(\xi)\big)\Big) X^{\alpha}(\xi)\, d\xi \right\}^{\frac{1}{\alpha}} ds , \tag{4.8}$$

for $t \geq T$.

Define the set $\Omega \subset C[T, \infty)$ and the operator $\mathcal{F}_1 : \Omega \to C[T, \infty)$ by

$$\Omega = \{x \in C[T, \infty) \,:\, X(T) \leq x(t) \leq X(t),\ t \geq T\} \tag{4.9}$$

$$\mathcal{F}_1 x(t) = X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f\Big(\big[x(\xi)\big]^{\mu}\Big)\Big) x^{\alpha}(\xi)\, d\xi \right\}^{\frac{1}{\alpha}} ds, \quad t \geq T .$$

Because of (4.8), using the assumption (4.3), we see that

$$X(T) \leq \mathcal{F}_1\, x(t) \leq X(t) \text{ for } t \geq T, \text{ i.e. } \mathcal{F}_1\, x \in \Omega \text{ for } x \in \Omega .$$

Using the Lebesgue dominated convergence theorem it can be shown that $\mathcal{F}_1$ is continuous mapping. We now show that $\mathcal{F}_1(\Omega)$ is relatively compact in $C[T, \infty)$. From the Ascoli-Arzela lemma it suffices to verify that it is locally uniformly bounded and locally equicontinuous on $[t_0, \infty)$. Let $t^* > T$ be fixed. If $x \in \Omega$, then $X(T) \le x(t) \le X(t) \le X(t^*)$ for $t \in [T, t^*]$. This shows that $\mathcal{F}_1(\Omega)$ is uniformly bounded on $[T, t^*]$. Also, if $x \in \Omega$, then for all $t \in [T, t^*]$

$$\begin{aligned} 0 \le (\mathcal{F}_1\, x)'(t) &= \left\{ \int_t^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f\Big(\big[x(\xi)\big]^{\,\mu} \Big) \Big) x^\alpha(\xi)\, d\xi \right\}^{1/\alpha} \\ &\le \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f\Big(\big[X(\xi)\big]^{\,\mu} \Big) \Big) X^\alpha(\xi)\, d\xi \right\}^{1/\alpha} \\ &\le \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(\xi) \Big) X^\alpha(\xi)\, d\xi \right\}^{1/\alpha} = X'(t) \le X'(t^*)\,. \end{aligned}$$

This shows that $\mathcal{F}_1(\Omega)$ is equicontinuous on $[T, t^*]$. Consequently, $\mathcal{F}_1(\Omega)$ is a relatively compact subset of $C[T, \infty)$.

Therefore, by the Schauder-Tychonoff fixed point theorem, there exists an element $y \in \Omega$ such that $\mathcal{F}_1\, y = y$, or equivalently

$$y(t) = X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f\Big(\big[y(\xi)\big]^{\,\mu} \Big) \Big) y^\alpha(\xi)\, d\xi \right\}^{\frac{1}{\alpha}} ds\,, \quad t \ge T\,.$$

Differentiating this integral equation, we find that $y(t)$ is a solution of the differential equation

$$\Big(\varphi(y') \Big)' + \frac{1}{t^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(|y|^\mu) \Big) \varphi(y) = 0\,, \quad t \ge T,$$

which implies that equation (4.2) possesses a nonoscillatory solution for all $\delta > \frac{\alpha+1}{\alpha}$. $\triangle$

Theorem 4.2.. *Consider the equations*

$$\Big(\varphi(x'\,) \Big)' + \frac{1}{t^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(|x|) \Big) \varphi(x) = 0\,, \quad t \ge a\,, \tag{4.10}$$

and

$$\Big(\varphi(x'\,) \Big)' + \frac{1}{t^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(t^\delta) \Big) \varphi(x) = 0\,, \quad t \ge a\,. \tag{4.11}$$

where the function $f : [a, \infty) \to (0, \infty)$ is continuous. Let there exist an $L > 0$ such that $f(t)$ is nonincreasing for $t \ge L$. If the equation (4.10) has a nonoscillatory solution, then the equation (4.11) is nonoscillatory for every $\delta > \frac{\alpha}{\alpha+1}$.

PROOF: Let $X(t)$ be a positive solution of the equation (4.10) on $[t_0, \infty)$. Let $\varepsilon > 0$ be an arbitrary constant. By Lemma 0.4, we have

$$\lim_{t \to \infty} t^{-\varepsilon - \frac{\alpha}{\alpha+1}} X(t) = 0\,,$$

so that, there exists $T \geq \max\{t_0, L\}$ such that

$$L \leq X(t) < t^{\frac{\alpha}{\alpha+1}+\varepsilon} \ \text{ for } \ t \geq T\,. \tag{4.12}$$

Denote $\mu = \frac{\alpha}{\alpha+1} + \varepsilon > \frac{\alpha}{\alpha+1}$. Integrating (4.10) from t to ∞, we have

$$X'(t) = \left\{ \int_t^{\infty} \frac{1}{s^{\alpha+1}} \Big(\alpha\, E(\alpha) + f\big(X(s)\big) \Big) X^{\alpha}(s)\, ds \right\}^{\frac{1}{\alpha}}, \quad t \geq T\,. \tag{4.13}$$

We then integrate (4.13) on $[T, t]$ and obtain

$$X(t) = X(T) + \int_T^t \left\{ \int_s^{\infty} \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f\big(X(\xi)\big) \Big) X^{\alpha}(\xi)\, d\xi \right\}^{\frac{1}{\alpha}} ds\,, \tag{4.14}$$

for all $t \geq T$. Therefore, from (4.12) and (4.14), using the nonincreasing nature of $f(t)$, we have the integral inequality

$$X(t) \geq X(T) + \int_T^t \left\{ \int_s^{\infty} \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f\big(\xi^{\mu}\big) \Big) X^{\alpha}(\xi)\, d\xi \right\}^{1/\alpha} ds\,, \quad t \geq T\,.$$

We now define the operator $\mathcal{F}_2 : \Omega \to C[T, \infty)$, Ω is being given by (4.9), as follows

$$\mathcal{F}_2\, x(t) = X(T) + \int_T^t \left\{ \int_s^{\infty} \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f\big(\xi^{\mu}\big) \Big) x^{\alpha}(\xi)\, d\xi \right\}^{1/\alpha} ds\,, \quad t \geq T\,.$$

Then it can be verified as in the proof of Theorem 4.1 that $\mathcal{F}_2$ sends Ω continuously into a relatively compact subset of Ω. It follows from the Schauder-Tychonoff fixed point theorem that $\mathcal{F}_2$ has a fixed point z in Ω, which satisfies

$$z(t) = X(T) + \int_T^t \left\{ \int_s^{\infty} \frac{1}{\xi^{\alpha+1}} \Big(\alpha\, E(\alpha) + f\big(\xi^{\mu}\big) \Big) z^{\alpha}(\xi)\, d\xi \right\}^{1/\alpha} ds\,, \ \ t \geq T.$$

Since $z(t)$ a positive solution of the half-linear equation

$$\Big(\varphi(z')\Big)' + \frac{1}{t^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(t^{\mu}) \Big) \varphi(z) = 0\,,$$

we conclude via the Sturm comparison theorem that the perturbed Euler equation (4.11) is nonoscillatory for all $\delta > \frac{\alpha}{\alpha+1}$. $\triangle$

Moreover, we are able to prove a comparison theorem between the two nonlinear second order differential equations of the form (4.10), namely

$$\text{(A)} \qquad \Big(\varphi(x')\Big)' + \frac{1}{t^{\alpha+1}} \Big(\alpha\, E(\alpha) + f(|x|) \Big) \varphi(x) = 0\,,$$

$$\text{(B)} \qquad \Big(\varphi(x')\Big)' + \frac{1}{t^{\alpha+1}}\Big(\alpha\, E(\alpha) + g(|x|)\Big)\varphi(x) = 0\,,$$

where $f,\, g : (0,\infty) \to (0,\infty)$ are continuous functions.

Theorem 4.3.. *Let there exists a constant $L > 0$ such that*

$$g(\xi) \text{ is nonincreasing, } \xi^\alpha\, g(\xi) \text{ is nondecreasing for } \ \xi \geq L\,, \tag{4.15}$$

$$f(\xi) \geq g(\xi) \ \text{ for } \ \xi \geq L\,. \tag{4.16}$$

If the equation (A) *has a nonoscillatory solutions, then so does the equation* (B)*.*

PROOF: Let $X(t)$ be a positive solution of the equation (A) on $[t_0,\infty)$. Then, since

$$\Big(\varphi\big(X'(t)\big)\Big)' + \frac{\alpha\, E(\alpha)}{t^{\alpha+1}}\,\varphi(X(t)) = -\frac{f\big(|X(t)|\big)}{t^{\alpha+1}}\,\varphi(X(t)) \leq 0, \quad t \geq t_0\,,$$

by Lemma 0.3 we have

$$\lim_{t\to\infty} X(t) = \infty, \quad \lim_{t\to\infty} X'(t) = 0\,,$$

so that there exists some $T > t_0$, such that $X(t) \geq L$ for all $t \geq T$. Proceeding as in the proof of Theorem 4.2 we find that $X(t)$ satisfies the integral equation (4.14), which in view of (4.16) leads to the following inequality holding for $t \geq T$

$$X(t) \geq X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}}\Big(\alpha\, E(\alpha) + g\big(X(\xi)\big)\Big) X^\alpha(\xi)\, d\xi \right\}^{\frac{1}{\alpha}} ds\,. \tag{4.17}$$

Define the set $\Omega \subset C[T,\infty)$ by (4.9) and the operator $\mathcal{F}_3 : \Omega \to C[T,\infty)$ by

$$\mathcal{F}_3\, x(t) = X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}}\Big(\alpha\, E(\alpha) + g\big(x(\xi)\big)\Big) x^\alpha(\xi)\, d\xi \right\}^{1/\alpha} ds, \ \ t \geq T\,.$$

Because of (4.17), using the assumption (4.15), we have

$$X(T) \leq \mathcal{F}_3\, x(t) \leq X(t) \ \text{ for all } t \geq T, \ \text{ i.e. } \ \mathcal{F}_3\, x \in \Omega \text{ for } x \in \Omega\,.$$

In addition, it can be proved in a routine manner that $\mathcal{F}_3$ is a continuous mapping and that the set $\mathcal{F}_3(\Omega)$ is relatively compact in $C[T,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem, there exists an element $x \in \Omega$ such that $\mathcal{F}_3\, x = x$, which is equivalent to

$$x(t) = X(T) + \int_T^t \left\{ \int_s^\infty \frac{1}{\xi^{\alpha+1}}\Big(\alpha\, E(\alpha) + g\big(x(\xi)\big)\Big) x^\alpha(\xi)\, d\xi \right\}^{1/\alpha} ds\,.$$

This function $x(t)$ provides a positive solution of the equation (B). This completes the proof. $\triangle$

Our main results developed in the previous sections will be illustrated by the following two examples.

. Consider the equation

$$\Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}\left[\alpha + \frac{\lambda}{(\log t)^{\beta}}\right]\varphi(x) = 0, \quad t \geq a\,, \tag{4.18}$$

where $\alpha,\ \beta,\ \lambda$ are positive constants. By Theorem 3.5 we conclude that:

(i) if $\beta < 2$, then (4.18) is oscillatory for all $\lambda > 0$;

(ii) if $\beta = 2$, then (4.18) is oscillatory for all $\lambda > \frac{\alpha+1}{2}$ and nonoscillatory for all $\lambda \leq \frac{\alpha+1}{2}$;

(iii) if $\beta > 2$, then (4.18) is nonoscillatory for all $\lambda > 0$.

We now use Theorem 4.1 and 4.2 to obtain fairly good information about the oscillatory and nonoscillatory behavior of solutions of the nonlinear differential equation

$$\Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}\left[\alpha + \frac{\mu}{(\log|x|)^{\beta}}\right]\varphi(x) = 0, \quad t \geq a\,. \tag{4.19}$$

(i) If $\beta > 2$, Theorem 4.1 implies that (4.19) has a nonoscillatory solution for all $\mu > 0$;

(ii) If $\beta = 2$, Theorem 4.1 implies that (4.19) has a nonoscillatory solution for all $\mu < \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$;

(iii) If $\beta = 2$, we claim that all solutions of (4.19) are oscillatory for every $\mu > \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$. Assume to the contrary that the equation (4.19) has a positive solution for some $\mu > \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$. Applying Theorem 4.2, we see that the half-linear equation

$$\Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}\left(\alpha + \frac{\mu}{\delta^2}\,\frac{1}{(\log t)^2}\right)\varphi(x) = 0 \tag{4.20}$$

is nonoscillatory for all $\delta > \frac{\alpha}{\alpha+1}$. But this is impossible, since in this case we have $\frac{\mu}{\delta^2} > \frac{\alpha+1}{2}$ for all δ sufficiently close to $\frac{\alpha}{\alpha+1}$, and this forces the equation (4.20) to be oscillatory (see the statement (ii) for (4.18)).

Similarly, we have the following statement

(iv) If $\beta < 2$, every solution of (4.19) are oscillatory for all $\mu > 0$.

Consider the nonlinear differential equation

$$\Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}\left[\alpha + \frac{\mu}{(\log|x|)^2} + \frac{\nu}{(\log|x|\cdot\log\log|x|)^2}\right]\varphi(x) = 0 \tag{4.21}$$

where $\alpha,\ \mu,\ \nu$ are positive constants.

By comparing (4.21) with the case $\beta = 2$ of the equation (4.19) on the basis of the comparison theorem, Theorem 4.3, we conclude that:

(i) If $\mu < \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$, then (4.21) has a nonoscillatory solution for all $\nu > 0$;

(ii) If $\mu > \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$, then all solutions of (4.21) are oscillatory for all $\nu > 0$;

Indeed, let $\mu < \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$. Choose μ_0 such that

$$\mu < \mu_0 < \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$$

and put

$$f(\xi) = \frac{\mu_0}{(\log\xi)^2}, \quad g(\xi) = \frac{\mu}{(\log\xi)^2} + \frac{\nu}{(\log\xi\cdot\log\log\xi)^2}\,.$$

From Example 4.1 the equation (4.19) with $\beta = 2$ and $\mu = \mu_0$ has a nonoscillatory solution. Noting that, for ξ large enough, $f(\xi) \ge g(\xi)$, $g(\xi)$ is decreasing and $\xi^\alpha g(\xi)$ is increasing, applying Theorem 4.3, we conclude that the equation (4.21) possesses a nonoscillatory solution.

Let $\mu > \frac{\alpha+1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2$. Assume to the contrary that (4.21) has a nonoscillatory solution for some $\nu > 0$. We now use Theorem 4.2 to assert that the half-linear equation

$$\Big(\varphi(x')\Big)' + \frac{E(\alpha)}{t^{\alpha+1}}\left(\alpha + \frac{\mu}{\delta^2}\,\frac{1}{(\log t)^2} + \frac{\nu}{(\log t^\delta\cdot\log\log t^\delta)^2}\right)\varphi(x) = 0$$

is nonoscillatory for all $\delta > \frac{\alpha}{\alpha+1}$. Since $\nu > 0$, the Strum comparison theorem for half-linear differential equations implies that (4.20) is nonoscillatory for all $\delta > \frac{\alpha}{\alpha+1}$. However, as we have observed in Example 4.1, this is impossible. This establishes the truth of the statement (ii).

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Almost Sure Asymptotic Estimations for Solutions of Stochastic Differential Delay Equations

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ABSTRACT

The classical Khasminskii theorem (see [7]) on the non-explosion solutions of stochastic differential equations (SDEs) is very important since it gives a powerful test for SDEs to have non-explosion solutions without the linear growth condition. Recently, we [15] established a more general Khasminskii-type test for stochastic differential delay equations (SDDEs) which covers a wide class of highly non-linear SDDEs. We also give some interesting and useful moment estimations. This paper is a continuation of our earlier one and the main aim is to establish almost sure asymptotic estimations.

Key words: *Brownian motion, stochastic differential delay equation, Itô's formula, Gronwall inequality, asymptotic estimation.*

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1 Introduction

One of the main problems in the study of stochastic differential equations is to find sufficient conditions for the existence and uniqueness of the solution. The classical existence-and-uniqueness theorem requires that the coefficients of the underlying SDEs obey the local Lipschitz condition and the linear growth condition (see e.g. [1, 4, 7, 17, 18]). However, there are many SDEs that do not satisfy the linear growth condition. Therefore, lot of researches have been devoted to proving the existence and uniqueness by replacing the linear growth condition with more general conditions. One of the most powerful results is the Khasminskii test [7] for the SDEs to have non-explosion solutions without the satisfaction of the linear growth condition.

In this paper we will consider an n-dimensional stochastic differential delay equation (SDDE)

$$dx(t) = f(x(t), x(t-\tau), t)dt + g(x(t), x(t-\tau), t)dB(t), \tag{1.1}$$

on $t \geq 0$ with initial data $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R^n)$. Here

$$f : R^n \times R^n \times R_+ \to R^n \quad \text{and} \quad g : R^n \times R^n \times R_+ \to R^{n\times m}$$

and τ is a positive constant. It is also known that if both f and g obey the local Lipschitz condition and the linear growth condition, then the SDDE has a unique global solution (see e.g. [8, 9, 11, 12, 16]). Nevertheless, a lot of SDDEs appeared in engineering, finance and population dynamics etc. (see e.g. [2, 3, 5, 14]) do not satisfy the linear growth condition. In 2002, Mao [13] established the following useful result (the notations used will be explained in Section 2 below).

Theorem 1.1. ***(Mao's test)*** *Let both coefficients f and g be locally Lipschitz continuous on $R^n \times R^n \times R_+$. Assume moreover that there is a function $V \in C^{2,1}(R^n \times [-\tau,\infty); R_+)$ and a positive constant K such that*

$$\lim_{|x|\to\infty} \inf_{0\le t<\infty} V(x,t) = \infty, \tag{1.2}$$

and

$$\begin{aligned} LV(x,y,t) &:= V_t(x,t) + V_x(x,t)f(x,y,t) + \frac{1}{2}\text{trace}[g^T(x,y,t)V_{xx}(x,t)g(x,y,t)] \\ &\le K[1 + V(x,t) + V(y,t-\tau)], \quad \forall (x,y,t) \in R^n \times R^n \times R_+. \end{aligned} \tag{1.3}$$

Then the SDDE (1.1) has a unique global solution on $t \in [-\tau,\infty)$ for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau,0]; R^n)$.

Theorem 1.2. *[15, Theorem 2.4] Let both coefficients f and g be locally Lipschitz continuous on $R^n \times R^n \times R_+$. Assume moreover that there are two functions $V \in C^{2,1}(R^n \times [-\tau,\infty); R_+)$ and $U \in C(R^n \times [-\tau,\infty); R_+)$ as well as two positive constants λ_1 and λ_2 such that*

$$\lim_{|x|\to\infty} \inf_{0\le t<\infty} V(x,t) = \infty, \tag{1.4}$$

and

$$LV(x,y,t) \le \lambda_1[1 + V(x,t) + V(y,t-\tau) + U(y,t-\tau)] - \lambda_2 U(x,t), \tag{1.5}$$

$\forall (x,y,t) \in R^n \times R^n \times R_+$. *Then the SDDE (1.1) has a unique global solution on $t \in [-\tau,\infty)$ for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau,0]; R^n)$. Moreover, the solution has the properties that*

$$EV(x(t),t) < \infty \quad and \quad E\int_0^t U(x(s),s)ds < \infty \tag{1.6}$$

for any $t \ge 0$.

As demonstrated in [15], Theorem 1.2 does not only give sufficient conditions for many highly non-linear SDDEs to have unique global solutions but also gives many interesting and useful moment estimations for the solutions. In this paper, we will concentrate on the almost sure asymptotic estimations for the solutions.

2 Preliminary Results

Throughout this paper, unless otherwise specified, we use the following notation. Let $|\cdot|$ be the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^TA)}$. Let $R_+ = [0,\infty)$ and

$\tau > 0$. Denote by $C([-\tau, 0]; R^n)$ the family of continuous functions from $[-\tau, 0]$ to R^n with the norm $\|\varphi\| = \sup_{-\tau \le \theta \le 0} |\varphi(\theta)|$. Denote by $C(R^n \times [-\tau, \infty]; R_+)$ the family of continuous functions from $R^n \times [-\tau, \infty]$ to R_+. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space.

As a standing hypothesis we will impose the following local Lipschitz condition on the coefficients of the SDDE (1.1).

Assumption 2.1. For each integer $i \ge 1$ there is a positive constant K_i such that

$$|f(x, y, t) - f(\bar{x}, \bar{y}, t)|^2 \vee |g(x, y, t) - g(\bar{x}, \bar{y}, t)|^2 \le K_i(|x - \bar{x}|^2 + |y - \bar{y}|^2)$$

for those $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le i$ and any $t \in R_+$.

Let $C^{2,1}(R^n \times [-\tau, \infty); R_+)$ denote the family of all continuous non-negative functions $V(x, t)$ defined on $R^n \times [-\tau, \infty)$ such that they are continuously twice differentiable in x and once in t. Given $V \in C^{2,1}(R^n \times [-\tau, \infty); R_+)$, we define the operator $LV : R^n \times R^n \times R_+ \to R$ by (1.3), where

$$V_t(x, t) = \frac{\partial V(x, t)}{\partial t}, \quad V_x(x, t) = \Big(\frac{\partial V(x, t)}{\partial x_1}, \cdots, \frac{\partial V(x, t)}{\partial x_n}\Big),$$

$$V_{xx}(x, t) = \Big(\frac{\partial^2 V(x, t)}{\partial x_i \partial x_j}\Big)_{n\times n}.$$

Let us emphasize that LV is thought as a single notation and is defined on $R^n \times R^n \times R_+$ while V is only defined on $R^n \times [-\tau, \infty)$.

The following result from [10, Lemma 2.1] will be of great use in the paper.

Lemma 2.2. *Let $M(t)$, $t \ge 0$ be a continuous real-valued local martingale with initial value $M(0) = 0$. Let $\langle M(t), M(t)\rangle$ be its quadratic variation. Let $\rho > 1$ be a number while $\{\tau_k\}_{k\ge 1}$ and $\{\gamma_k\}_{k\ge 1}$ be two sequences of positive numbers such that $\tau_k \to 0$ as $k \to \infty$. Then for almost all $\omega \in \Omega$, there is a random integer $k_0 = k_0(\omega)$ such that for all $k \ge k_0$,*

$$M(t) \le \frac{\gamma_k}{2}\langle M(t), M(t)\rangle + \frac{\rho}{\gamma_k}\log k \quad \textit{on } 0 \le t \le \tau_k.$$

3 Almost Sure Asymptotic Estimations

Let us now begin to discuss the almost sure asymptotic estimations for the solution of the SDDE (1.1). Let us emphasize that the additional conditions imposed in each theorem below plus the standing Assumption 2.1 will guarantee the existence of the unique global solution of the SDDE (1.1) as shown later in the *Appendix*. Hence we will only prove the asymptotic estimations in this section.

Theorem 3.1. *Let Assumption 2.1 hold. Assume that there are real numbers $\alpha_1 \ge 0$, $\alpha_2 > \alpha_3 \ge 0$, $r \ge 0$, $\theta > 2 + r$ and $K > 0$ such that*

$$x^T f(x, y, t) \le \alpha_1 - \alpha_2|x|^\theta + \alpha_3|y|^\theta \tag{3.1}$$

and

$$|g(x, y, t)|^2 \le K(1 + |x|^r + |y|^r) \tag{3.2}$$

for all $(x, y, t) \in R^n \times R^n \times R_+$*. Then the unique global solution* $x(t)$ *of equation (1.1) has the property that*

$$\lim_{t\to\infty} \frac{|x(t)|}{\sqrt[p]{\log t}} = 0 \quad a.s. \tag{3.3}$$

for every $p \in [2, \theta - r)$*.*

Proof. Fix any $p \in [2, \theta - r)$ and choose $\gamma > 0$ sufficiently small for

$$\alpha_2 > \frac{\alpha_3(p - 2 + \theta e^{p\gamma\tau})}{p - 2 + \theta}. \tag{3.4}$$

Define

$$V(x, t) = e^{p\gamma t}\, |x|^p, \quad (x, t) \in R^n \times R_+.$$

It is easy to show that

$$\begin{aligned} LV(x, y, t) &\leq p\gamma e^{p\gamma t}|x|^p + pe^{p\gamma t}|x|^{p-2}x^T f(x, y, t) \\ &+ \tfrac{1}{2}p(p-1)e^{p\gamma t}|x|^{p-2}|g(x, y, t)|^2. \end{aligned}$$

By Itô's formula, we then have

$$\begin{aligned} &V(x(t), t) - V(x(0), 0) \\ \leq &\int_0^t e^{p\gamma s}\Big[p\gamma|x(s)|^p + p|x(s)|^{p-2}x^T(s)f(x(s), x(s-\tau), s) \\ + &\frac{1}{2}p(p-1)|x(s)|^{p-2}|g(x(s), x(s-\tau), s)|^2\Big]ds + M(t), \end{aligned} \tag{3.5}$$

where

$$M(t) = p\int_0^t e^{p\gamma s}|x(s)|^{p-2}x^T(s)g(x(s), x(s-\tau), s)dB(s).$$

It is easy to see that $M(t)$ is a real-valued continuous local martingale vanishing at 0 and its quadratic variation is given by

$$\begin{aligned} \langle M(t), M(t)\rangle &= p^2\int_0^t e^{2p\gamma s}|x(s)|^{2(p-2)}|x^T(s)g(x(s), x(s-\tau), s)|^2 ds \\ &\leq p^2\int_0^t e^{2p\gamma s}|x(s)|^{2(p-1)}|g(x(s), x(s-\tau), s)|^2 ds. \end{aligned}$$

Now, assigning $\beta > 0$ and $\rho = 2$ arbitrarily and applying Lemma 2.2 with $\tau_k = k$ and $\gamma_k = \beta e^{-p\gamma k}$, we observe that for almost all $\omega \in \Omega$ there exists a random integer $k_0(\omega)$ such that for all $k \geq k_0$ and $0 \leq t \leq k$,

$$\begin{aligned} M(t) &\leq \frac{1}{2}\beta e^{-p\gamma k}\langle M(t), M(t)\rangle + (\beta^{-1}e^{p\gamma k})2\log k \\ &\leq (\beta^{-1}e^{p\gamma k})2\log k \\ &+ \frac{1}{2}\beta e^{-p\gamma k}p^2\int_0^t e^{2p\gamma s}|x(s)|^{2(p-1)}|g(x(s), x(s-\tau), s)|^2 ds. \end{aligned}$$

Substituting this into (3.5) yields that, with probability one, if $k \geq k_0$ and $0 \leq t \leq k$, then

$$\begin{aligned} |x(t)|^p e^{p\gamma t} &\leq |x(0)|^p + (\beta^{-1} e^{p\gamma k})(2\log k) \\ &+ \int_0^t e^{p\gamma s} \, [\, p\gamma |x(s)|^p + p|x(s)|^{p-2} x^T(s) f(x(s), x(s-\tau), s) \\ &+ \frac{1}{2}\, p(p-1)\, |x(s)|^{p-2}\, |g(x(s), x(s-\tau), s)|^2 \,] ds \\ &+ \frac{1}{2}\beta e^{-p\gamma k} p^2 \int_0^t e^{2p\gamma s} |x(s)|^{2(p-1)} |g(x(s), x(s-\tau), s)|^2 ds. \end{aligned}$$

By conditions (3.1) and (3.2) we hence have that with probability one, if $k \geq k_0$ and $0 \leq t \leq k$,

$$\begin{aligned} e^{p\gamma t}|x(t)|^p &\leq |x(0)|^p + (2\log k)\,(\beta^{-1} e^{p\gamma k}) \\ &+ \int_0^t e^{p\gamma s} \, [\, p\gamma |x(s)|^p + p|x(s)|^{p-2}(\alpha_1 - \alpha_2 |x(s)|^\theta + \alpha_3 |x(s-\tau)|^\theta) \\ &+ \frac{1}{2} p(p-1) K |x(s)|^{p-2} (1 + |x(s)|^r + |x(s-\tau)|^r)] ds \\ &+ \frac{1}{2}\,\beta p^2 \int_0^t e^{p\gamma s} |x(s)|^{2(p-1)} K (1 + |x(s)|^r + |x(s-\tau)|^r) ds. \end{aligned} \tag{3.6}$$

Recall the well-known Young's inequality (see e.g. [6]):

$$a^c b^{1-c} \leq ca + (1-c)b, \quad \forall a, b \geq 0,\ c \in [0,1]. \tag{3.7}$$

The application of this inequality gives

$$\begin{aligned} |x(s)|^{p-2}|x(s-\tau)|^\theta &\leq \frac{p-2}{p-2+\theta}|x(s)|^{p-2+\theta} + \frac{\theta}{p-2+\theta}|x(s-\tau)|^{p-2+\theta}, \\ |x(s)|^{p-2}|x(s-\tau)|^r &\leq \frac{p-2}{p-2+r}|x(s)|^{p-2+r} + \frac{r}{p-2+r}|x(s-\tau)|^{p-2+r}, \\ |x(s)|^{2(p-1)}|x(s-\tau)|^r &\leq \frac{2(p-1)}{2(p-1)+r}|x(s)|^{2(p-1)+r} + \frac{r}{2(p-1)+r}|x(s-\tau)|^{2(p-1)+r}. \end{aligned}$$

Substituting these into (3.6) implies that, with probability one, if $k \geq k_0$ and $0 \leq t \leq k$,

$$\begin{aligned} e^{p\gamma t}|x(t)|^p &\leq |x(0)|^p + (2\log k)(\beta^{-1} e^{p\gamma k}) \\ &+ \int_0^t e^{p\gamma s} \Big\{ p\gamma |x(s)|^p + \alpha_1 p |x(s)|^{p-2} - \alpha_2 p |x(s)|^{p-2+\theta} \\ &+ \alpha_3 p \frac{p-2}{p-2+\theta} |x(s)|^{p-2+\theta} + \alpha_3 p \frac{\theta}{p-2+\theta} |x(s-\tau)|^{p-2+\theta} \\ &+ \frac{1}{2} p(p-1) K |x(s)|^{p-2} + \frac{1}{2} p(p-1) K |x(s)|^{p-2+r} \\ &+ \frac{1}{2} p(p-1) K \frac{p-2}{p-2+r} |x(s)|^{p-2+r} + \frac{1}{2} p(p-1) K \frac{r}{p-2+r} |x(s-\tau)|^{p-2+r} \\ &+ \frac{1}{2}\beta p^2 K |x(s)|^{2(p-1)} + \frac{1}{2}\beta p^2 K |x(s)|^{2(p-1)+r} \\ &+ \frac{1}{2}\beta p^2 K \frac{2(p-1)}{2(p-1)+r} |x(s)|^{2(p-1)+r} + \frac{1}{2}\beta p^2 K \frac{r}{2(p-1)+r} |x(s)|^{2(p-1)+r} \Big\} ds. \end{aligned}$$

In view of the fact that

$$\int_0^t e^{p\gamma s} |x(s-\tau)|^w ds \ \leq\ \int_{-\tau}^0 e^{p\gamma(s+\tau)} |x(s)|^w ds + \int_0^t e^{p\gamma(s+\tau)} |x(s)|^w ds$$

for $w = (p-2+\theta),\ (p-2+r)$ and $[2(p-1)+r]$ respectively, we obtain that, with probability one, if $k \ge k_0$ and $0 \le t \le k$,

$$\begin{aligned} e^{p\gamma t}|x(t)|^p &\le C + (2\log k)(\beta^{-1}e^{p\gamma k}) \\ &+ \int_0^t e^{p\gamma s}\Big\{p(\alpha_1 + \frac{1}{2}p(p-1)K)|x(s)|^{p-2} + p\gamma|x(s)|^p \\ &+ \frac{1}{2}p(p-1)K(1+\frac{p-2}{p-2+r} + \frac{r}{p-2+r}e^{p\gamma\tau})|x(s)|^{p-2+r} \\ &+ \frac{1}{2}\beta p^2 K|x(s)|^{2(p-1)} \\ &+ \frac{1}{2}\beta p^2 K(1+\frac{2(p-1)}{2(p-1)+r} + \frac{r}{2(p-1)+r}e^{p\gamma\tau})|x(s)|^{2(p-1)+r} \\ &- p(\alpha_2 - \frac{\alpha_3}{p-2+\theta}(p-2+\theta e^{p\gamma\tau}))|x(s)|^{p-2+\theta}\Big\}ds, \end{aligned}$$

where

$$\begin{aligned} C = |x(0)|^p + \int_{-\tau}^0 e^{p\gamma(s+\tau)}p\Big[\frac{\alpha_3\theta}{p-2+\theta}|x(s)|^{p-2+\theta} + \frac{r(p-1)K}{2(p-2+r)}|x(s)|^{p-2+r} \\ + \frac{r\beta pK}{2(2(p-1)+r)}|x(s)|^{2(p-1)+r}\Big]ds. \end{aligned}$$

Recalling (3.4) as well as $\theta > 2+r$ and $p \in [2, \theta - r)$, it is easy to see that the following polynomial is bounded upper by, say Λ, in R^n. That is, for all $x \in R^n$,

$$\begin{aligned} p(\alpha_1 + \frac{1}{2}p(p-1)K)|x(s)|^{p-2} + p\gamma|x|^p \\ +\frac{1}{2}p(p-1)K(1+\frac{p-2}{p-2+r} + \frac{r}{p-2+r}e^{p\gamma\tau})|x|^{p-2+r} \\ +\frac{1}{2}\beta p^2 K|x|^{2(p-1)} + \frac{1}{2}\beta p^2 K(1+\frac{2(p-1)}{2(p-1)+r} \\ +\frac{r}{2(p-1)+r}e^{p\gamma\tau})|x|^{2(p-1)+r} \\ -p(\alpha_2 - \frac{\alpha_3}{p-2+\theta}(p-2+\theta e^{p\gamma\tau}))|x|^{p-2+\theta} \\ \le \Lambda. \end{aligned}$$

Hence, with probability one, if $k \ge k_0$ and $0 \le t \le k$, then

$$\begin{aligned} e^{p\gamma t}|x(t)|^p &\le C + (2\log k)(\beta^{-1}e^{p\gamma k}) + \int_0^t e^{p\gamma s}\Lambda ds \\ &\le C + (2\log k)(\beta^{-1}e^{p\gamma k}) + \frac{\Lambda}{p\gamma}e^{p\gamma t}. \end{aligned}$$

In particular, with probability one, if $k \ge k_0$ and $k-1 \le t \le k$, we have

$$|x(t)|^p \le Ce^{-p\gamma(k-1)} + (2\log k)(\beta^{-1}e^{p\gamma}) + \frac{\Lambda}{p\gamma},$$

which gives

$$\frac{|x(t)|^p}{\log t} \le \Big\{Ce^{-p\gamma(k-1)} + (2\log k)(\beta^{-1}e^{p\gamma}) + \frac{\Lambda}{p\gamma}\Big\} \times [\log(k-1)]^{-1}.$$

Letting $t \to \infty$ (forcing $k \to \infty$) we obtain that

$$\limsup_{t\to\infty} \frac{|x(t)|^p}{\log t} \ \le \ (e^{p\gamma})(2\beta^{-1}) \quad a.s.$$

That is

$$\limsup_{t\to\infty} \frac{|x(t)|}{\sqrt[p]{\log t}} \ \le \ (e^{\gamma})\sqrt[p]{2\beta^{-1}} \quad a.s.$$

Finally, letting $\beta \to \infty$ we have

$$\limsup_{t\to\infty} \frac{|x(t)|}{\sqrt[p]{\log t}} \le 0 \quad a.s.$$

and hence the required assertion (3.3) must hold. □

The above theorem shows that almost every sample path of the solution will not grow faster than $\sqrt[p]{\log t}$. The following theorem further reveals that the average in time of almost every sample path of the solution is bounded.

Theorem 3.2. *Assume that all conditions of Theorem 3.1 hold except (3.1) which is replaced by the following more general condition*

$$x^T f(x,y,t) \le \alpha_1 - \alpha_2|x|^\theta + \alpha_3|y|^\theta - U(x) + U(y), \tag{3.8}$$

wherer $U \in C(R^n; R_+)$. Then the unique global solution $x(t)$ of equation (1.1) has the property that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t |x(s)|ds \le \tilde{\Lambda} \quad a.s. \tag{3.9}$$

where $\tilde{\Lambda}$ is a positive constant independent of the initial data.

Before we prove this theorem, let make a quick remark. It is easy to see that if we set $U \equiv 0$ then condition (3.8) reduces to condition (3.1). Hence under the conditions of Theorem 3.1, the solution of equation (1.1) also has property (3.9) in addition to (3.3).

Proof. Define

$$V(x,t) := V(x) = |x|^2, \quad (x,t) \in R^n \times R_+.$$

Obviously, the operator LV has the form

$$LV(x,y,t) = 2x^T f(x,y,t) + |g(x,y,t)|^2, \quad (x,y,t) \in R^n \times R^n \times R_+.$$

Hence, by the Itô formula,

$$\begin{aligned} & |x(t)|^2 - |x(0)|^2 \\ = \ & \int_0^t [2x^T(s) f(x(s), x(s-\tau), s) + |g(x(s), x(s-\tau), s)|^2]ds + M(t), \end{aligned} \tag{3.10}$$

where

$$M(t) = 2\int_0^t x^T(s) g(x(s), x(s-\tau), s)dB(s),$$

is a real-valued continuous local martingale vanishing at 0. By condition (3.2) we compute its quadratic variation

$$\begin{aligned}\langle M(t), M(t)\rangle &= 4\int_0^t |x^T(s)g(x(s), x(s-\tau), s)|^2 ds \\ &\le 4\int_0^t |x(s)|^2|g(x(s), x(s-\tau), s)|^2 ds \\ &\le 4K\int_0^t |x(s)|^2(1+|x(s)|^r+|x(s-\tau)|^r)ds.\end{aligned}$$

By Lemma 2.2 with $\rho = 2,\ \gamma_k = 2$ and $\tau_k = k$, we see that for almost all $\omega \in \Omega$, there is a random integer $k_0 = k_0(\omega)$ such that

$$M(t) \le \log k + 4K\int_0^t |x(s)|^2(1+|x(s)|^r+|x(s-\tau)|^r)ds \tag{3.11}$$

for $0 \le t \le k$ whenever $k \ge k_0$. But, by Young's inequality (3.7),

$$|x(s)|^2|x(s-\tau)|^r \le \frac{2}{2+r}|x(s)|^{2+r} + \frac{r}{2+r}|x(s-\tau)|^{2+r}.$$

Hence,

$$\begin{aligned}&\int_0^t |x(s)|^2(1+|x(s)|^r+|x(s-\tau)|^r)ds \\ \le\ &\int_0^t \Big[\ |x(s)|^2 + (1+\frac{2}{2+r})|x(s)|^{2+r} + \frac{r}{2+r}|x(s-\tau)|^{2+r}\ \Big]ds \\ \le\ &\frac{r}{2+r}\int_{-\tau}^0 |x(s)|^{2+r}ds + \int_0^t [|x(s)|^2 + 2|x(s)|^{2+r}]ds\end{aligned}$$

for

$$\int_0^t |x(s-\tau)|^{2+r}ds \le \int_{-\tau}^0 |x(s)|^{2+r}ds + \int_0^t |x(s)|^{2+r}ds.$$

Therefore, we see from (3.11) that, with probability one,

$$M(t) \le \log k + \frac{4Kr}{2+r}\int_{-\tau}^0 |x(s)|^{2+r}ds + 4K\int_0^t [\ |x(s)|^2 + 2|x(s)|^{2+r}\ ]ds$$

for $0 \le t \le k$ whenever $k \ge k_0$. Substituting this into (3.10) and then using condition (3.8) we obtain that, with probability one,

$$\begin{aligned}|x(t)|^2 \le\ & |x(0)|^2 + \frac{4Kr}{2+r}\int_{-\tau}^0 |x(s)|^{2+r}ds + \log k \\ &+ \int_0^t \big[2\alpha_1 - 2\alpha_2|x(s)|^\theta + 2\alpha_3|x(s-\tau)|^\theta - 2U(x(s)) + 2U(x(s-\tau)) \\ &\quad + K(1+|x(s)|^r+|x(s-\tau)|^r) + 4K|x(s)|^2 + 8K|x(s)|^{2+r}\ \big]ds \\ \le\ & C + \log k + \int_0^t \big[\ (2\alpha_1+K) - 2(\alpha_2-\alpha_3)|x(s)|^\theta \\ &\quad + 2K|x(s)|^r + 4K|x(s)|^2 + 8K|x(s)|^{2+r}\ \big]ds\end{aligned}$$

for $0 \le t \le k$ whenever $k \ge k_0$, where

$$C = |x(0)|^2 + \int_{-\tau}^0 [\ \frac{4Kr}{2+r}|x(s)|^{2+r} + 2\alpha_3|x(s)|^\theta + K|x(s)|^r + 2U(x(s))\ ]ds.$$

Consequently, with probability one,

$$(\alpha_2-\alpha_3)\int_0^t |x(s)|^\theta ds \;\le\; C+\log k+\int_0^t \Big[\, (2\alpha_1+K)-(\alpha_2-\alpha_3)|x(s)|^\theta +2K|x(s)|^r+4K|x(s)|^2+8K|x(s)|^{2+r}\,\Big]ds$$

for $0 \le t \le k$ whenever $k \ge k_0$. Recalling that $\alpha_2 > \alpha_3$ and $\theta > 2+r$, we see that there is a positive constant Λ such that

$$(2\alpha_1+K)+4K|x|^2+2K|x|^r+8K|x|^{2+r}-(\alpha_2-\alpha_3)|x|^\theta \le \Lambda, \quad \forall x \in R^n.$$

So, with probability one,

$$(\alpha_2-\alpha_3)\int_0^t |x(s)|^\theta ds \;\le\; C+\log k+\Lambda t$$

for $0 \le t \le k$ whenever $k \ge k_0$. Hence, for almost all $\omega \in \Omega$, if $(k-1) \le t \le k$ and $k \ge k_0$,

$$\frac{1}{t}\int_0^t |x(s)|^\theta ds \;\le\; \frac{1}{\alpha_2-\alpha_3}\Big[\frac{C+\log k}{k-1}+\Lambda\Big].$$

Letting $t \to \infty$ yields

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t |x(s)|^\theta ds \le \frac{\Lambda}{\alpha_2-\alpha_3} \quad a.s.$$

However, it is easy to show by the Hölder inequality that

$$\frac{1}{t}\int_0^t |x(s)|ds \le \Big(\frac{1}{t}\int_0^t |x(s)|^\theta ds\Big)^{\frac{1}{\theta}}.$$

Thus

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t |x(s)|ds \le \sqrt[\theta]{\frac{\Lambda}{\alpha_2-\alpha_3}} := \tilde{\Lambda} \quad a.s.$$

as required. □

Theorem 3.3. *Let Assumption 2.1 hold. Assume that there is a function $U \in C(R^n; R_+)$ and two positive constants $\alpha,\ K$ such that*

$$2x^T f(x,y,t) \le \alpha - U(x) + U(y) \tag{3.12}$$

and

$$|g(x,y,t)|^2 \le K \tag{3.13}$$

for all $(x,y,t) \in R^n \times R^n \times R_+$. Then the unique global solution of equation (1.1) has the property that

$$\limsup_{t\to\infty}\frac{|x(t)|}{\sqrt{2t\log\log t}} \le \sqrt{eK} \quad a.s. \tag{3.14}$$

Proof. By the Itô formula,

$$\begin{aligned}|x(t)|^2 - |x(0)|^2 &= \int_0^t (2x^T(s)f(x(s),x(s-\tau),s) + |g(x(s),x(s-\tau),s)|^2)ds \\ &+ 2\int_0^t x^T(s)g(x(s),x(s-\tau),s)dB(s) \\ &\le \int_0^t [\alpha + K - U(x(s)) + U(x(s-\tau))]ds + M(t) \\ &\le (\alpha+K)t + \int_{-\tau}^0 U(x(s))ds + M(t),\end{aligned}$$

where

$$M(t) = 2\int_0^t x^T(s)g(x(s),x(s-\tau),s)dB(s),$$

is a real-valued continuous local martingale vanishing at 0 and

$$\langle M(t), M(t)\rangle \le 4K\int_0^t |x(s)|^2 ds.$$

Assign $\beta > 0$ and $\rho > 1$ arbitrarily. Applying Lemma 2.2 with $\gamma_k = \beta\rho^{-k}$ and $\tau_k = \rho^k$, we see that for almost all $\omega \in \Omega$, there is a random integer $k_0 = k_0(\omega)$ such that

$$M(t) \le \beta^{-1}\rho^{k+1}\log k + 2K\beta\rho^{-k}\int_0^t |x(s)|^2 ds$$

for $0 \le t \le \rho^k$ whenever $k \ge k_0$. Hence, with probability one,

$$|x(t)|^2 \le \xi + \beta^{-1}\rho^{k+1}\log k + (\alpha+K)\rho^k + 2\beta\rho^{-k}K\int_0^t |x(s)|^2 ds$$

for $0 \le t \le \rho^k$ whenever $k \ge k_0$, where $\xi = |x(0)|^2 + \int_{-\tau}^0 U(x(s))ds$. By the well-known Gronwall inequality, then have that, with probability one,

$$|x(t)|^2 \le [\ \xi + (\alpha+K)\rho^k + \beta^{-1}\rho^{k+1}\log k\ ]e^{2\beta K},$$

for $0 \le t \le \rho^k$ whenever $k \ge k_0$. In particular, with probability one, if $\rho^{k-1} \le t \le \rho^k$ and $k \ge k_0$, then

$$\frac{|x(t)|^2}{2t\log\log t} \le \frac{[\xi + (\alpha+K)\rho^k + \beta^{-1}\rho^{k+1}\log k]e^{2\beta K}}{2\rho^{k-1}\log((k-1)+\log\rho)}.$$

Letting $t \to \infty$ yields

$$\limsup_{t\to\infty}\frac{|x(t)|^2}{2t\log\log t} \le \frac{\rho^2}{2\beta}e^{2\beta K} \quad a.s.$$

Finally, letting $\rho \to 1$ and setting $\beta = \frac{1}{2K}$ we obtain the assertion

$$\limsup_{t\to\infty}\frac{|x(t)|}{\sqrt{2t\log\log t}} \le \sqrt{eK} \quad a.s.$$

as desired. □

Theorem 3.4. *Let Assumption 2.1 hold. Assume there are non-negative constants* $\alpha_1, \alpha_2, \alpha_3$ *and* K *such that* $\alpha_2 + \alpha_3 > 0$*,*

$$x^T f(x, y, t) \leq \alpha_1 + \alpha_2|x|^2 + \alpha_3|y|^2 \tag{3.15}$$

and

$$|g(x, y, t)|^2 \leq K \tag{3.16}$$

for all $(x, y, t) \in R^n \times R^n \times R_+$*. Then the unique global solution* $x(t)$ *of equation (1.1) has the property that*

$$\lim_{t\to\infty} \frac{|x(t)|}{e^{(\alpha_2+\alpha_3)t}\sqrt{\log\log t}} = 0 \quad a.s. \tag{3.17}$$

Proof. Define

$$V(x, t) = e^{-2(\alpha_2+\alpha_3)t}|x|^2, \quad (x, t) \in R^n \times R_+.$$

Hence the operator LV has the form

$$\begin{aligned} LV(x, y, t) &= -2(\alpha_2 + \alpha_3)e^{-2(\alpha_2+\alpha_3)t}|x|^2 + 2e^{-2(\alpha_2+\alpha_3)t}x^T f(x, y, t) \\ &+ e^{-2(\alpha_2+\alpha_3)t}|g(x, y, t)|^2. \end{aligned}$$

By Itô's Formula and the conditions we compute

$$\begin{aligned} & e^{-2(\alpha_2+\alpha_3)t}|x(t)|^2 - |x(0)|^2 \\ = & \int_0^t e^{-2(\alpha_2+\alpha_3)s}\big[-2(\alpha_2 + \alpha_3)|x(s)|^2 + 2x^T(s)f(x(s), x(s-\tau), s) \\ & \qquad + |g(x(s), x(s-\tau), s)|^2\big]ds + M(t) \\ \leq & \int_0^t e^{-2(\alpha_2+\alpha_3)s}\big[-2\alpha_3|x(s)|^2 + 2\alpha_3|x(s-\tau)|^2 + (2\alpha_1 + K)\big]ds + M(t), \end{aligned}$$

where

$$M(t) = 2\int_0^t e^{-2(\alpha_2+\alpha_3)s}x^T(s)g(x(s), x(s-\tau), s)dB(s).$$

Noting

$$\int_0^t e^{-2(\alpha_2+\alpha_3)s}|x(s-\tau)|^2 ds \leq \int_{-\tau}^0 |x(s)|^2 ds + \int_0^t e^{-2(\alpha_2+\alpha_3)s}|x(s)|^2 ds,$$

we have

$$e^{-2(\alpha_2+\alpha_3)t}|x(t)|^2 \leq C + M(t), \tag{3.18}$$

where

$$C = |x(0)|^2 + 2\alpha_3\int_{-\tau}^0 |x(s)|^2 ds + \frac{2\alpha_1 + K}{2(\alpha_2 + \alpha_3)}.$$

Clearly, $M(t)$ is a real-valued local martingale vanishing at 0 and

$$\langle M(t), M(t)\rangle \leq 4K\int_0^t e^{-4(\alpha_2+\alpha_3)s}|x(s)|^2 ds.$$

Let $\delta > 1$ be arbitrary. Applying Lemma 2.2 with $\rho = 2$, $\gamma_k = 1$ and $\tau_k = 2^{k^\delta}$, we observe that for almost all $\omega \in \Omega$, there is a random integer $k_0 = k_0(\omega)$ such that

$$\begin{aligned} M(t) &\leq \frac{1}{2}\langle M(t), M(t)\rangle + 2\log k \\ &\leq 2\log k + 2K\int_0^t e^{-4(\alpha_2+\alpha_3)s}|x(s)|^2 ds \end{aligned}$$

for $0 \leq t \leq 2^{k^\delta}$ whenever $k \geq k_0$. Substituting this into (3.18) gives that, with probability one,

$$e^{-2(\alpha_2+\alpha_3)t}|x(t)|^2 \leq C + 2\log k + 2K\int_0^t e^{-2(\alpha_2+\alpha_3)s}[e^{-2(\alpha_2+\alpha_3)s}|x(s)|^2]ds$$

for $0 \leq t \leq 2^{k^\delta}$ whenever $k \geq k_0$. Applying the Gronwall inequality, we get that, with probability one,

$$\begin{aligned} e^{-2(\alpha_2+\alpha_3)t}|x(t)|^2 &\leq [\, C + 2\log k\,]\exp[2K\int_0^t e^{-2(\alpha_2+\alpha_3)s}ds] \\ &\leq [\, C + 2\log k\,]\exp(\frac{K}{\alpha_2+\alpha_3}) \end{aligned}$$

for $0 \leq t \leq 2^{k^\delta}$ whenever $k \geq k_0$. Hence, with probability one, if $2^{(k-1)^\delta} \leq t \leq 2^{k^\delta}$ and $k \geq k_0$,

$$\frac{|x(t)|^2}{e^{2(\alpha_2+\alpha_3)t}\log\log t} \leq [\, C + 2\log(k)\,] \times \exp(\frac{K}{\alpha_2+\alpha_3}) \times [\delta\log(k-1) + \log\log 2]^{-1}.$$

Letting $t \to \infty$ implies

$$\limsup_{t\to\infty} \frac{|x(t)|^2}{e^{2(\alpha_2+\alpha_3)t}\log\log t} \leq \frac{2}{\delta}\exp(\frac{K}{\alpha_2+\alpha_3}).$$

Since $\delta > 1$ is arbitrary, we must have that

$$\lim_{t\to\infty} \frac{|x(t)|^2}{e^{2(\alpha_2+\alpha_3)t}\log\log t} = 0 \quad a.s.$$

which is the same as the required assertion. □

4 Examples

Let us now discuss four examples to illustrate our theory. In the first two example, the SDDEs are scalar and the Brownian motion $B(t)$ is scalar too.

Example 4.1. Let

$$f(x,y,t) = 1 - 2x^5 + y^5 \quad \text{and} \quad g(x,y,t) = 1 + x + y, \quad (x,y,t) \in R \times R \times R_+.$$

It is easy to show that

$$xf(x,y,t) \leq 1.5 - \frac{8}{6}x^6 + \frac{5}{6}y^6 \quad \text{and} \quad |g(x,y,t)|^2 \leq 3(1 + |x|^2 + |y|^2).$$

By Theorems 3.1 and 3.2, we can conclude that for any given initial data $\{x(s) : -\tau \leq s \leq 0\} \in C([-\tau,0];R)$, the unique global solution of the SDDE

$$dx(t) = [1 - 2x^5(t) + x^5(t-\tau)]dt + [1 + x(t) + x(t-\tau)]dB(t)$$

obeys that

$$\lim_{t\to\infty} \frac{|x(t)|}{\sqrt[p]{\log t}} = 0 \quad a.s$$

for any $p \in [2,4)$, and

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t |x(s)|ds \le \tilde{\Lambda} \quad a.s.$$

where $\tilde{\Lambda}$ is a positive constant independent of the initial data.

Example 4.2. Let

$$f(x,y,t) = 1 - 2x^5 + y^5 - x^7 + y^7 \quad \text{and} \quad g(x,y,t) = 1 + x + y$$

for $(x,y,t) \in R \times R \times R_+$. It is easy to show that

$$xf(x,y,t) \le 1.5 - \frac{8}{6}x^6 + \frac{5}{6}y^6 - \frac{7}{8}x^8 + \frac{7}{8} \quad \text{and} \quad |g(x,y,t)|^2 \le 3(1+|x|^2+|y|^2).$$

By Theorem 3.2 we can conclude that for any given initial data $\{x(s) : -\tau \le s \le 0\} \in C([-\tau,0];R)$, the unique global solution of the SDDE

$$dx(t) = [1 - 2x^5(t) + x^5(t-\tau) - x^7(t) + x^7(t-\tau)]dt + [1 + x(t) + x(t-\tau)]dB(t)$$

obeys that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t |x(s)|ds \le \tilde{\Lambda} \quad a.s$$

where $\tilde{\Lambda}$ is a positive constant independent of the initial data.

Example 4.3. Let us now consider an n-dimensional linear SDDE

$$dx(t) = [a + A_1x(t) + A_2(t-\tau)]dt + A_3dB(t), \tag{4.1}$$

where $a \in R^n$, $A_1, A_2 \in R^{n\times n}$ and $A_3 \in R^{n\times m}$ while $B(t)$ is an m-dimensional Brownian motion. Denote by $\|A_1\|$ the operator norm of A_1, namely $\|A_1\| = \sup\{|A_1x| : x \in R^n,\ |x| = 1\}$ and similarly, $\|A_2\|$ and $\|A_3\|$. Fix any $\varepsilon > 0$ and compute, for $x, y \in R^n$,

$$\begin{aligned} x^T[a + A_1x + A_2y] &\le |a||x| + \|A_1\||x|^2 + \|A_2\||x||y| \\ &\le \frac{|a|^2}{4\varepsilon} + (\varepsilon + \|A_1\| + \tfrac{1}{2}\|A_2\|)|x|^2 + \tfrac{1}{2}\|A_2\||y|^2. \end{aligned}$$

By Theorem 3.4, we can conclude that for any given initial data $\{x(s) : -\tau \le s \le 0\} \in C([-\tau,0];R^n)$, the unique global solution of the linear SDDE (4.1) obeys that

$$\lim_{t\to\infty} \frac{|x(t)|}{e^{(\varepsilon+\|A_1\|+\|A_2\|)t}\sqrt{\log\log t}} = 0 \quad a.s. \tag{4.2}$$

for any $\varepsilon > 0$.

Example 4.4. Let us still consider the linear SDDE (4.1). Assume that the symmetric matrix $A_1 + A_1^T$ is negative-definite and

$$-\tfrac{1}{2}\lambda_{\max}(A_1 + A_1^T) > \|A_2\|, \tag{4.3}$$

where $\lambda_{\max}(A_1 + A_1^T) < 0$ is the largest eigen-value of $A_1 + A_1^T$. Let

$$\varepsilon = -\tfrac{1}{2}\lambda_{\max}(A_1 + A_1^T) - \|A_2\| > 0.$$

Compute, for $x, y \in R^n$,

$$\begin{aligned} x^T[a + A_1 x + A_2 y] &\le |a||x| + \tfrac{1}{2}x^T(A_1 + A_1^T)x + \|A_2\||x||y| \\ &\le \frac{|a|^2}{4\varepsilon} + \Big[\varepsilon + \tfrac{1}{2}\lambda_{\max}(A_1 + A_1^T) + \tfrac{1}{2}\|A_2\|\Big]|x|^2 + \tfrac{1}{2}\|A_2\||y|^2 \\ &= \frac{|a|^2}{4\varepsilon} - \tfrac{1}{2}\|A_2\||x|^2 + \tfrac{1}{2}\|A_2\||y|^2. \end{aligned}$$

By Theorem 3.3, we can conclude that for any given initial data $\{x(s) : -\tau \le s \le 0\} \in C([-\tau, 0]; R^n)$, the unique global solution of the linear SDDE (4.1) obeys that

$$\limsup_{t\to\infty} \frac{|x(t)|}{\sqrt{2t\log\log t}} \le \|A_3\|\sqrt{e} \quad a.s. \tag{4.4}$$

5 APPENDIX

In this appendix, we will show that under the conditions of any theorem in Section 3, the SDDE (1.1) has a unique global solution. In fact, under the conditions of Theorem 3.4, the existence and uniqueness of the solution follows easily from Mao's test (i.e. Theorem 1.1) by using $V(x,t) = |x|^2$. We also note that the conditions of Theorem 3.2 are more general than those of either Theorem 3.1 or Theorem 3.3 so we only need to show the existence and uniqueness of the solution under the conditions of Theorem 3.2. In this case, we still let $V(x,t) = |x|^2$. Compute the operator LV:

$$\begin{aligned} LV(x,y,t) &= 2x^T f(x,y,t) + |g(x,y,t)|^2 \\ &\le 2[\alpha_1 - \alpha_2|x|^\theta + \alpha_3|y|^\theta - U(x) + U(y)] + K(1 + |x|^r + |y|^r). \end{aligned}$$

By the Young inequality, it is easy to show that

$$K(1 + |x|^r + |y|^r) \le C_1 + \alpha_2(|x|^\theta + |y|^\theta)$$

where C_1 and following C_2 etc. are all positive constants. Hence

$$\begin{aligned} LV(x,y,t) &= C_2 + (a_2 + 2\alpha_3)|y|^\theta + 2U(y) - (\alpha_2|x|^\theta + 2U(x)) \\ &\le C_3[1 + (\alpha_2|y|^\theta + 2U(y))] - (\alpha_2|x|^\theta + 2U(x)). \end{aligned}$$

By Theorem 1.2, we see easily that the SDDE (1.1) has a unique global solution.

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A Class of Second Order Difference Equations Inspired by Euler's Discretization Method

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ABSTRACT

In this paper we study a multiparameter, nonlinear second order difference equation that is motivated by the Euler discretization of derivatives in the autonomous, second order differential equation derived from Newton's second law in mechanics. Our objective is mainly to analyze qualitative properties of the second order difference equation such as convergence, periodicity and chaos. With proper restrictions, two different semiconjugate factorizations facilitate our work.

Keywords: Euler's forward method, difference equation, global stability, persistent oscillations, periodicity, chaos, semiconjugates

2000 Mathematics Subject Classification: 39A10, 39A11.

1 Introduction

Euler's simple method of rendering derivatives discrete in time has, over the centuries led to interesting classes of difference equations that have inspired a significant amount of research. The bulk of this research has been done during the past 30 years when digital computing has been available and increasingly accessible.
We start with the differential equation

$$x'' = \phi(x, x') \tag{1.1}$$

of classical mechanics. Using Euler's forward difference method (1.1) may be transformed into a second order difference equation. The time axis is made discrete as $t_0, t_1, t_2, \ldots$ with a fixed step size τ so that for each $n = 0, 1, 2, \ldots$ we have $t_{n+1} - t_n = \tau$. Then we estimate the first and second derivatives of the function $x(t)$ using forward differences as

$$x'(t_n) \approx \frac{x(t_n+\tau)-x(t_n)}{\tau} = \frac{x_{n+1}-x_n}{\tau},$$
$$x''(t_n) \approx \frac{x'(t_n+\tau)-x'(t_n)}{\tau} = \frac{1}{\tau}\left[\frac{x_{n+2}-x_{n+1}}{\tau} - \frac{x_{n+1}-x_n}{\tau}\right].$$

Inserting these into (1.1) yields

$$\frac{1}{\tau}\left[\frac{x_{n+2}-x_{n+1}}{\tau} - \frac{x_{n+1}-x_n}{\tau}\right] = \phi\left(x_n, \frac{x_{n+1}-x_n}{\tau}\right). \tag{1.2}$$

This is the Euler discretization of (1.1) with a fixed step size. For sufficiently small τ and a wide range of functions ϕ Eq.(1.2) gives good estimates of the solutions of (1.1) over a chosen time interval $[a,b]$, in which case $t_0 = a$ and $t_N = b$ where N is the largest index that one would consider. For more details on Euler's and other methods for solving differential equations a standard numerical analysis text such as [4] may be consulted.

In this paper we consider a slightly more general form of (1.2) that is capable of producing a much richer variety of asymptotic behavior through parameter adjustments. Our discussion is focused on the asymptotics of that general difference equation rather than on estimating solutions of (1.1) using (1.2).

Relabling x_{n+1}/τ as y_n and rearranging terms in Equation (1.2) gives

$$y_{n+1} = 2y_n - y_{n-1} + \tau\phi(\tau y_{n-1}, y_n - y_{n-1}). \tag{1.3}$$

This is a special case of the second order difference equation

$$x_{n+1} = ax_n + bx_{n-1} + f(x_{n-1}, x_n - cx_{n-1}) \tag{1.4}$$

where the parameters a, b, c are given real numbers and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given function. This equation may also be written succinctly as

$$x_{n+1} = F(x_n, x_{n-1})$$

which is reminicent of (1.1) but with

$$F(u,v) = au + bv + f(v, u - cv). \tag{1.5}$$

It may be mentioned in passing that difference equations may be used directly as equations of motion in mechanics. An interesting study of this approach is given in [7].

2 General Concepts and Results

Each fixed point or equilibrium $\bar{x}$ of Eq.(1.4) is given by the equation

$$\bar{x} = a\bar{x} + b\bar{x} + f(\bar{x}, (1-c)\bar{x})$$

or equivalently,

$$(1-a-b)\bar{x} = f(\bar{x}, (1-c)\bar{x}). \tag{2.1}$$

For example, if f is a homogeneous function of degree k, i.e. $f(tu,tv) = t^k f(u,v)$ then the origin $\bar{x} = 0$ is a fixed point (if the domain of f contains it) and for $k \neq 1$ another isolated fixed point

$$\bar{x} = \left[\frac{1-a-b}{f(1,1-c)}\right]^{1/(k-1)}$$

may exist provided that the various quantities are well defined. For $k = 1$ if $1-a-b = f(1,1-c)$ then all points on the diagonal (and in the domain of f) are fixed and thus none are isolated; otherwise, origin is the unique isolated fixed point if it is in the domain of f. We refer the reader to texts such as [1], [6], [12], [13] and [17] for basic background material, including the definitions of stability, asymptotic stability and instability for fixed points and cycles of difference equations.

2.1 Global stability

Let $\bar{x}$ be an isolated fixed point of (1.4) and let F be the function defined in (1.5). If f is continuously differentiable, then so is F and through linearization it may be shown that $\bar{x}$ is locally stable if

$$\left|\frac{\partial F}{\partial u}(\bar{x},\bar{x})\right| < 1 + \frac{\partial F}{\partial v}(\bar{x},\bar{x}) < 2.$$

If $\bar{x}$ is the only fixed point of (1.4) then we also have the following general result in which f is only assumed to be continuous.

Theorem 1. *Assume that f is continuous on $\mathbb{R}^2$ and define $g(u,v) = f(v, u-cv)$. If $\bar{x}$ is the only fixed point of (1.4) and there is $\delta \in (0,1)$ such that*

$$|a| + |b| + \delta < 1$$

and

$$|g(u,v) - g(\bar{x},\bar{x})| \leq \delta \max\{|u-\bar{x}|, |v-\bar{x}|\}, \quad (u,v) \in \mathbb{R}^2 \tag{2.2}$$

then $\bar{x}$ is globally asymptotically stable.

Proof. Note that

$$\begin{aligned}|F(u,v) - \bar{x}| &= |F(u,v) - F(\bar{x},\bar{x})| \\ &\leq |a||u-\bar{x}| + |b||v-\bar{x}| + |f(v,u-cv) - f(\bar{x},(1-c)\bar{x})| \\ &\leq (|a|+|b|+\delta)\max\{|u-\bar{x}|, |v-\bar{x}|\}.\end{aligned}$$

Therefore, by Corollary 4.3.5 in [17] $\bar{x}$ is globally asymptotically stable.

Example 1. Consider the difference equation

$$x_{n+1} = 0.5x_{n-1} + 1 + \sqrt[3]{x_n + 1}. \tag{2.3}$$

which is a special case of (1.4) with $a = c = 0$, $b = 0.5$ and $f(v,u) = 1 + \sqrt[3]{u+1}$. Equation (2.3) has a unique fixed point $\bar{x}$ with a value of approximately 5.8. Inequality (2.2) holds for

(2.3) because

$$\begin{aligned}|g(u,v)-g(\bar{x},\bar{x})| &= |f(v,u)-f(\bar{x},\bar{x})| \\ &= |\sqrt[3]{u+1}-\sqrt[3]{\bar{x}+1}| \\ &= \frac{|u-\bar{x}|}{\left|\sqrt[3]{(u+1)^2}+\sqrt[3]{u+1}\sqrt[3]{\bar{x}+1}+\sqrt[3]{(\bar{x}+1)^2}\right|}\end{aligned}$$

and using the approximate value of $\bar{x}$ we find that the denominator of the fraction above exceeds 5/2 for all real u, so we may set $\delta = 2/5 = 0.4$ in Theorem 1 and conclude that the unique fixed point $\bar{x}$ of (2.3) is globally asymptotically stable.

Remark. Condition (2.2) in particular holds if f is a contraction on the plane, i.e.

$$|f(x,y)-f(s,t)| \leq \gamma \max\{|x-s|,|y-t|\}, \quad \gamma < \frac{1}{1+|c|}. \tag{2.4}$$

If (2.4) holds then

$$\begin{aligned}|g(u,v)-g(\bar{x},\bar{x})| &= |f(v,u-cv)-f(\bar{x},(1-c)\bar{x})| \\ &\leq \gamma \max\{|v-\bar{x}|,|u-cv-(1-c)\bar{x}|\} \\ &\leq \gamma \max\{|v-\bar{x}|,|u-\bar{x}|+c|v-\bar{x}|\} \\ &\leq \gamma(1+|c|)\max\{|u-\bar{x}|,|v-\bar{x}|\}\end{aligned}$$

and (2.2) follows if $\gamma(1+|c|)<1$.
Inequality (2.2) is essentially weaker than (2.4) because the latter inequality is assumed to hold globally whereas the former only requires sufficient flatness of the graph of g near the point $(\bar{x},\bar{x})$. For instance, in Example 1 above the function $f(v,u)$ is not a contraction on the plane (in fact, f is not differentiable when $u=-1$), but near the point $(\bar{x},\bar{x})$ the cylindrical surface $1+\sqrt[3]{u+1}$ flattens out significantly. See [17, Sec.4.3] for further general remarks with regard to the geometric aspects of Theorem 1.

2.2 Persistent oscillations

Suppose that $\bar{x}$ is an isolated fixed point of F. Then the following inequalities imply that both eigenvalues of the linearization of (1.4) have modulus greater than 1 (see, e.g. [17, p.168]):

$$\left|\frac{\partial F}{\partial v}(\bar{x},\bar{x})\right| > 1, \quad \left|\frac{\partial F}{\partial v}(\bar{x},\bar{x})-1\right| > \left|\frac{\partial F}{\partial u}(\bar{x},\bar{x})\right|. \tag{2.5}$$

These inequalities and a basic result from [17, p.166] imply the next theorem. We say that a bounded solution of (1.4) will *oscillate persistently* if it has at least two distinct limit points.

Theorem 2. *Suppose that inequalites (2.5) hold at an isolated fixed point $\bar{x}$ of (1.4) and further, the equation*

$$a\bar{x}-bv+f(v,\bar{x}-cv)=\bar{x} \tag{2.6}$$

has no real solution $v \neq \bar{x}$. Then all bounded, non-constant solutions of (1.4) oscillate persistently.

Example 2. In Equation (1.4) let $c = 1$ and assume that $f(v, u - cv) = g(u - v) - bv$ for some real function g so that (1.4) takes the form

$$x_{n+1} = ax_n + g(x_n - x_{n-1}). \tag{2.7}$$

We make the following additional assumptions:
(a) $0 \leq a < 1$;
(b) g is continuous, nondecreasing and bounded below on $\mathbb{R}$;
(c) There is $\alpha \in (0, 1)$ and $t_0 > 0$ such that $g(t) \leq \alpha t$ for all $t > t_0$;
Then (2.7) has a unique fixed point $\bar{x} = g(0)/(1 - a)$ and all of its solutions are bounded ([17, T4.1.1]). If we also assume that:
(d) g is continuously differentiable at 0 with $g'(0) > 1$,
then every solution of (2.7) with at least one initial value different from $\bar{x}$ oscillates persistently. To prove this last assertion, we note with regard to (2.5) that if $F(u, v) = au + g(u - v)$ then

$$\left|\frac{\partial F}{\partial v}(\bar{x}, \bar{x})\right| = g'(0) > 1,$$
$$\left|\frac{\partial F}{\partial v}(\bar{x}, \bar{x}) - 1\right| = |-g'(0) + 1| > |a + g'(0)| = \left|\frac{\partial F}{\partial u}(\bar{x}, \bar{x})\right|.$$

Further, Eq.(2.6) takes the form

$$g(\bar{x} - v) = g(0)$$

whose only solution by assumptions (b) and (d) is $v = \bar{x}$. Thus by Theorem 2 all nontrivial solutions of (2.7) oscillate persistently. A specific example of g that satisfies (a)-(d) above is $g(t) = \tan^{-1} \beta t$ with $\beta > 1$.

3 Semiconjugate factorizations

A second order equation such as (1.4) may be viewed as a mapping of the two dimensional space upon unfolding in vector form. Such an equation in principle admits factorizations into two mappings of the real line ([17]). When a difference equation is stated in scalar form as (1.4) is, we may obtain the semiconjugate factors through substitutions. We obtain our first (of two) such factorization by subtracting the term cx_n from both sides of (1.4) and rearrange terms to obtain

$$x_{n+1} - cx_n = (a - c)x_n + bx_{n-1} + f(x_{n-1}, x_n - cx_{n-1}).$$

We now make two assumptions:

> **(SC1)** f *is linear in the first coordinate,* i.e., $f(u, v) = du + g(v)$ where d is a real number (possibly 0) and g is a function.

Then the terms on the right hand side of the preceding expression may be rearranged to give

$$x_{n+1} - cx_n = (a - c)x_n + (b + d)x_{n-1} + g(x_n - cx_{n-1}). \tag{3.1}$$

Our second assumption is as follows:

(SC2) *The constants* a, b, c, d *satisfy*

$$b + d = c(c - a). \tag{3.2}$$

Note that the constant values in Eq.(1.3) namely, $a = 2$, $b = -1$, $c = 1$ satisfy condition (3.2) if we assume hypothesis (SC1) above with $d = 0$. The case $d = 0$ corresponds to the function ϕ in (1.1) being "space independent".

Under assumptions (SC1) and (SC2), we substitute $t_n = x_n - cx_{n-1}$ into Eq.(3.1) and obtain the equivalent system of first order difference equations

$$t_{n+1} = (a - c)t_n + g(t_n) \tag{3.3a}$$

$$x_{n+1} = cx_n + t_{n+1} \tag{3.3b}$$

These two first order equations represent the first semiconjugate factorization of (1.4) that we discuss here. We may call this type of factorization *semiconjugacy by sums*. For reference, we note that under the assumptions (SC1) and (SC2) Eq.(1.4) takes the following form:

$$x_{n+1} = ax_n + c(c - a)x_{n-1} + g(x_n - cx_{n-1}). \tag{3.4}$$

The second type of semiconjugate factorization requires the following assumption:

(SC3) f *is homogeneous of degree one, i.e.* $f(tu, tv) = tf(u, v)$ *for all real values of* t *for which* f *is defined.*

Examples of mappings that satisfy (SC3) include linear maps $f(u, v) = \alpha u + \beta v$ as well as the following:

$$|\alpha u + \beta v|, \quad \sqrt{\alpha u^2 + \beta uv + \gamma v^2}, \quad \frac{\alpha u^2 + \beta uv + \gamma v^2}{\delta u + \xi v}$$

under suitable domain restrictions where necessary. Under (SC3) we may divide both sides of (1.4) by x_n to obtain

$$\begin{aligned}\frac{x_{n+1}}{x_n} &= a + b\frac{x_{n-1}}{x_n} + \frac{1}{x_n} f(x_{n-1}, x_n - cx_{n-1}) \\ &= a + b\frac{x_{n-1}}{x_n} + \frac{x_{n-1}}{x_n} f(1, \frac{x_n}{x_{n-1}} - c).\end{aligned}$$

In the preceding expression we substitute

$$r_n = \frac{x_n}{x_{n-1}} \tag{3.5}$$

to obtain

$$r_{n+1} = a + \frac{b + f(1, r_n - c)}{r_n}.$$

Note that this is a first order difference equation that together with (3.5) gives the following factorization of (1.4) that we call *semiconjugacy by ratios*:

$$r_{n+1} = a + \frac{b + f(1, r_n - c)}{r_n} \tag{3.6a}$$

$$x_{n+1} = r_{n+1}x_n \tag{3.6b}$$

The essential or structural difference between (3.3) and (3.6) is in their second equations (3.3b) and (3.6b), respectively. These latter equations translate the dynamics of fibers given by equations (3.3a) and (3.6a) in different ways into behaviors for solutions of (1.4). For instance, even if in both (3.3a) and (3.6a) all solutions converge to a unique fixed point, the resulting behaviors for (1.4) will be quite different in the two cases because (3.3b) and (3.6b) give different outcomes.

We note that *these two semiconjugate types are essentially complementary, becasuse if both of the assumptions (SC1) and (SC3) hold then f is linear, which reduces (1.4) to a linear equation*. It may also be mentioned that equations (3.3) and (3.6) are examples of "triangular" systems; these types of systems have been studied at a general level for their periodic structure; see, e.g. [2] and [11]. Since we are dealing with somewhat specific systems, we can obtain substantial information (some of which go beyond periodicity) without having to appeal to the more general results.

3.1 Semiconjugacy by sums

Throughout this section, we assume that (SC1) and (SC2) hold. We demonstrate that solutions of Eq.(3.4) exhibit a wide variety of dynamic behaviors ranging from periodic to chaotic. Solutions of Eq.(3.3a) are orbits, or sequences of iterates $\{t_n\} = \{h^n(t_0)\}$ where

$$h(t) = (a - c)t + g(t), \quad t_0 = x_0 - cx_{-1}.$$

For each given sequence of real numbers $\{t_n\}$, the general solution of (3.3b) is

$$x_n = c^n x_0 + \sum_{j=1}^{n} c^{n-j} t_j, \quad n \geq 1. \tag{3.7}$$

For nontriviality, we assume that $c \neq 0$ in the sequel. The sum in (3.7) is of convolution type but here the sequence $\{t_n\}$ is rarely given in explicit form. Often we only know some of the qualitative features of $\{t_n\}$ as a solution of (3.3a), e.g. whether it is stable or periodic. We use (3.7) to translate those qualitative properties into properties of solutions of Eq.(3.4).

Lemma 1. *Assume that* $|c| \neq 1$.
(a) *Let* $\{t_n\}$ *be a periodic sequence of real numbers with period* p, *and let* $\{\tau_0, \ldots, \tau_{p-1}\}$ *be one cycle of* $\{t_n\}$. *If*

$$\xi_i = \frac{1}{1 - c^p} \sum_{j=0}^{p-1} c^{p-j-1} \tau_{(i+j) \bmod p} \quad i = 0, 1, \ldots, p-1 \tag{3.8}$$

then the solution $\{x_n\}$ *of Eq.(3.3b) with* $x_0 = \xi_0$ *and* $t_1 = \tau_0$ *has period* p *and* $\{\xi_0, \ldots, \xi_{p-1}\}$ *is a cycle of* $\{x_n\}$.
(b) *If for a given sequence* $\{t_n\}$ *of real numbers Eq.(3.3b) has a solution* $\{x_n\}$ *of period* p *then* $\{t_n\}$ *is periodic with period* p.
Proof. (a) With $x_0 = \xi_0$ and $t_1 = \tau_0$ we find that

$$x_1 = cx_0 + t_1 = c\xi_0 + \tau_0$$

which upon using (3.8) for ξ_0 gives

$$x_1 = \frac{c}{1-c^p}\left(\sum_{j=0}^{p-1} c^{p-j-1}\tau_j\right) + \tau_0 = \frac{1}{1-c^p}\left(\sum_{j=0}^{p-2} c^{p-j-1}\tau_{j+1} + \tau_0\right) = \xi_1$$

Proceeding in an inductive fashion, we show in this way that $x_i = \xi_i$ for $i = 0, \ldots, p-1$. Next, we show that $x_p = x_0$. Using (3.7) we have

$$x_p = c^p\xi_0 + \sum_{j=0}^{p-1} c^{p-j-1}\tau_j = \frac{c^p}{1-c^p}\sum_{j=0}^{p-1} c^{p-j-1}\tau_j + \sum_{j=0}^{p-1} c^{p-j-1}\tau_j = \xi_0 = x_0.$$

Hence $\{x_n\}$ is a solution with period p, as claimed.

(b) Suppose that for a given sequence $\{t_n\}$ of real numbers, the corresponding solution of (3.3b) is periodic with period p. Let $t_1 = x_1 - cx_0$ and from (3.3b) obtain

$$t_{p+1} = x_{p+1} - cx_p = x_1 - cx_0 = t_1.$$

It follows that $\{t_n\}$ is periodic with period p.

Theorem 2 (periodic solutions) *Assume that $|c| \neq 1$ and let $\{t_n\}$ be a periodic solution of the first order equation (3.3a) with prime period p. If $\{\tau_0, \ldots, \tau_{p-1}\}$ is one cycle of $\{t_n\}$ then (3.4) has a solution $\{x_n\}$ of prime period p with a cycle $\{\xi_0, \ldots, \xi_{p-1}\}$ given by (3.8).*

Proof. By Lemma 1(a) the periodic sequence $\{t_n\}$ generates a periodic solution of (3.3b). By construction, this periodic solution is also a solution of (3.4) if $\{t_n\}$ is a solution of (3.3a). It remains to show that p is the prime or minimal period. Let q be the prime period of $\{x_n\}$ so that $q \leq p$. Then by Lemma 1(b) $\{t_n\}$ has period $q \geq p$ since p is the prime period for $\{t_n\}$. Therefore, $q = p$.

The periodic orbits in iterates of a continuous one dimensional map of an interval satisfy the following ordering known as the Sharkovski ordering of cycles; see [5], [17], [20].

$$3 \rhd 5 \rhd 7 \rhd \cdots 2\cdot 3 \rhd 2\cdot 5 \rhd \cdots 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd \cdots 2^k \rhd 2^{k-1} \rhd \cdots \rhd 2 \rhd 1.$$

In particular, if a continuous mapping has an orbit with period 3, then it has periodic orbits with all possible periods. The following is an immediate consequence of Theorem 2.

Corollary 1. (coexisting periods) *If Eq.(3.3a) has a solution of period 3 (e.g. satisfies the Li-Yorke conditions; see[14]) then Eq.(3.4) has periodic solutions with all possible periods that are arranged in the Sharkovski ordering*.

Lemma 2 (boundedness) *Let $|c| < 1$. If $\{t_n\}$ is a bounded sequence with $|t_n| \leq B$ for some $B > 0$, then the corresponding solution $\{x_n\}$ for Eq.(3.3b) is also bounded and $|x_n| < |x_0| + B/(1-|c|)$ for all $n \geq 1$. Further, there is a positive integer N such that*

$$|x_n| \leq 1 + \frac{B}{1-|c|} \quad \textit{for all } n \geq N. \tag{3.9}$$

Proof. From (3.7) we obtain

$$|x_n| \leq |c|^n|x_0| + \sum_{j=1}^{n} |c|^{n-j}B < |x_0| + B\sum_{k=0}^{\infty} |c|^k = |x_0| + \frac{B}{1-|c|}.$$

From the preceding inequalities it also follows that if n is large enough, then $|c|^n|x_0| \leq 1$ from which (3.9) follows.

In the literature, the term "chaotic" usually indicates non-periodic, oscillatory behavior that is sensitive to initial values. See [5], [14], [15] and [17] for some background on this concept. In particular, Theorem 3.3.3 in [17] implies the following:

Theorem 3 (chaotic behavior) *Assume that $|c| < 1$ and that the first order equation (3.3a) is chaotic within an invariant closed interval $[A, B]$ on the line. Then the second order equation (3.4) is chaotic in the following invariant compact, convex set in the plane*

$$\{(u,v) : cu + A \leq v \leq cu + B\} \cap \left[-1 - \frac{\max\{|A|,|B|\}}{1-|c|}, 1 + \frac{\max\{|A|,|B|\}}{1-|c|}\right]^2.$$

Example 3. Consider the one parameter family of rational second order equations

$$x_{n+1} = \frac{6x_n^2 - 5x_n x_{n-1} + x_{n-1}^2 - \alpha(2x_n - x_{n-1}) + 4}{4x_n - 2x_{n-1}}, \quad 0 < \alpha < 4 \qquad (3.10)$$

which is obtained from (3.4) by setting $a = 3/2$, $c = 1/2$ and $g(t) = 1/t - \alpha/2$. We may write the mapping h in (3.3a) as

$$h(t) = t - \frac{\alpha}{2} + \frac{1}{t}.$$

h has a positive fixed point $\bar{t} = 2/\alpha$ and a positive global minimum value of $h_{\min} = 2 - \alpha/2$ at $t = 1,$ so $h(t) > 0$ for all $t > 0$ if $0 < \alpha < 4.$ If $\alpha \leq 2\sqrt{2}$ then the fixed point $\bar{t}$ attracts all positive iterates of $h.$ To see this, we note that if

$$h^2(t) = h(h(t)) = t + \frac{1}{t} - \alpha + \frac{t}{t^2 - (\alpha/2)t + 1}$$

then

$$h^2(t) - t = \frac{-\alpha[t - (2/\alpha)][t^2 - (\alpha/2)t + 1/2]}{t[t^2 - (\alpha/2)t + 1]}$$

Since both of the quadratic terms in the preceding expression are positive for all t if $\alpha \leq 2\sqrt{2}$ it follows that $h^2(t) > t$ for $0 < t < 2/\alpha$ so by Theorem 2.1.2 of [17] $\bar{t} = 2/\alpha$ is a global attractor of all positive orbits of $h.$ Thus when $\alpha \leq 2\sqrt{2}$, Eq.(3.10) has a fixed point $\bar{x} = \bar{t}/(1-c) = 4/\alpha$ (this can also be computed directly from (3.10)) which attracts all positive solutions of (3.10). The attractivity of $\bar{x}$ can be established directly using (3.7) or by observing that the invariant fiber $v = u/2 + \bar{t}$ is attracting and all points on this fiber approach $\bar{x}.$

If $\alpha > 2\sqrt{2}$ then it is not hard to see that all iterates of h will eventually enter and remain in the invariant interval $[2-\alpha/2, h(2-\alpha/2)]$ and with increasing value of $\alpha,$ a sequence of bifurcations of periodic orbits ensues that progresses through the Sharkovski ordering to lead to a period 3 orbit at about $\alpha = 3.48.$ Since for each point (u, v) in the plane, $v - u/2 = t,$ it follows that for $2 < \alpha < 4,$ each solution of Eq.(3.10) eventually enters the invariant bounded set

$$\left\{(u,v) : \frac{u}{2} + 2 - \frac{\alpha}{2} \leq v \leq \frac{u}{2} + h\left(2 - \frac{\alpha}{2}\right)\right\} \cap (0, 1 + 2h(2-\alpha/2)]^2 \qquad (3.11)$$

For α up to 3.48, each period-p orbit of h uniquely generates a period-p trajectory of (3.10) in the set (3.11) according to (3.8). For instance, solving the equation

$$t = h^2(t) = t + \frac{1}{t} - \alpha + \frac{2t}{2t^2 - \alpha t + 2}$$

for $\alpha > 2\sqrt{2}$ yields the period-2 orbit

$$\tau_0 = \frac{\alpha - \sqrt{\alpha^2 - 8}}{4}, \quad \tau_1 = \frac{\alpha + \sqrt{\alpha^2 - 8}}{4}.$$

Now using (3.8) we obtain a period-2 solution of (3.10) as

$$\xi_0 = \frac{c\tau_0 + \tau_1}{1 - c^2} = \frac{3\alpha + \sqrt{\alpha^2 - 8}}{6},$$
$$\xi_1 = \frac{c\tau_1 + \tau_0}{1 - c^2} = \frac{3\alpha - \sqrt{\alpha^2 - 8}}{6}.$$

On the other hand, if α is close enough to 4, e.g. $\alpha > 3.48$, then the trajectories of (3.10) will exhibit sensitivity to initial conditions and undergo nonperiodic oscillations within the set (3.11).

Remark. As the preceding results show, when $|c| < 1$ then Eq.(3.4) rather faithfully duplicates the qualitative behavior of solutions of Eq.(3.3a). When $|c| > 1$ then it is evident from (3.7) that solutions of (3.4) are typically unbounded and thus any bounded solutions (including periodic ones) of (3.4) that correspond to bounded behavior in (3.3a) must be unstable. Therefore, different qualitative behaviors will be exhibited by (3.4) and (3.3a) when $|c| > 1$.
The relationship between the solutions of (3.4) and (3.3a) in cases $c = \pm 1$ is also different from $|c| \neq 1$. It is worth noting that if ϕ is linear in its first coordinate, then (1.2) becomes a special case of Eq.(3.4) with $c = 1$ (upon re-scaling the mapping ϕ). With $c = 1$ (3.7) changes into a sum (or discrete integral) so the nature of solutions of (3.4) will be qualitatively different from that of the solutions of (3.3a). In particular, a periodic solution of (3.3a) with a cycle $\{\tau_0, \ldots, \tau_{p-1}\}$ can translate into periodic solutions of (3.4) if and only if $\sum_{i=0}^{p-1} \tau_i = 0$. Thus there is a significant loss of periodicity in the second order equation. For more details on the case $c = 1$ in certain special cases of Eq.(3.4) see [16], [17] and [19].

3.2 Semiconjugacy by ratios

In this section we assume only that (SC3) holds, i.e., f is homogeneous of degree 1. We do not put any further restrictions such as (3.2) on the coefficients a, b, c in Eq.(1.4). The solutions of (1.4) under (SC3) exhibit a very different type of behavior than was the case with (SC1) and (SC2). In order to avoid singularities in (3.6), solutions of (1.4) that contain zero may be singled out and treated differently.
Since for each given solution $\{r_n\}$ of (3.6a) the corresponding solution of (1.4) is obtained from (3.6b) as

$$x_n = r_n r_{n-1} \cdots r_0 x_{-1} \tag{3.12}$$

the following result is easy to establish.

Theorem 4. *Let x_0, x_{-1} be given initial values with $x_{-1} \neq 0$ and let $\{r_n\}$ be a solution of Eq.(3.6a) with $r_0 = x_0/x_{-1}$. Assume that r_n is a real number for all $n \geq 0$ (e.g. r_n is contained in an invariant set of (3.6a) which does not contain 0). Then the following is true:*
(a) If there is $n_0 \geq 0$ such that $|r_n| < 1$ for all $n \geq n_0$ then the corresponding solution $\{x_n\}$ of (1.4) converges to 0.

(b) If there is $n_0 \geq 0$ such that $|r_n| > 1$ for all $n \geq n_0$ then the corresponding solution $\{x_n\}$ of (1.4) is unbounded.
(c) If $\{r_n\}$ converges to a cycle $\{\rho_1, \ldots, \rho_p\}$ with $\rho_1\rho_2 \cdots \rho_p = 1$ then $\{x_n\}$ converges to a periodic solution of (1.4) with period p.
(d) If $\{x_n\}$ converges to a nonzero value, then the infinite product $\prod_{n=0}^{\infty} r_n$ is convergent; in particular, there are disjoint, infinite sets of positive integers K_0 and K_1 such that $|r_n| < 1$ (or $|x_n| < |x_{n-1}|$) for $n \in K_0$ and $|r_n| > 1$ (or $|x_n| > |x_{n-1}|$) for $n \in K_1$.

The following corollary illustrates the various points made in Theorem 4 as well as the fact that semiconjugates can sometimes be useful in the derivation of solutions in quantitatively explicit form. If we set $\beta = a - \alpha c$ for arbitrary a, c then the difference equation given in next corollary is a version of (1.4) with $b = 0$ and

$$f(v, u - cv) = \frac{\alpha(u - cv)^2}{v} + \alpha c(u - cv).$$

Corollary 2. *Consider the following rational difference equation*

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1}}{x_{n-1}}, \quad x_{-1} \neq 0 \tag{3.13}$$

(a) *If for $n \geq 0$ we set $x_{n+1} = 0$ when $x_{n-1} = x_n = 0$ then the general solution of (3.13) is given in explicit form as*

$$x_n = x_0 \prod_{k=1}^{n} \left[\left(\frac{x_0}{x_{-1}} - \frac{\beta}{1 - \alpha} \right) \alpha^k + \frac{\beta}{1 - \alpha} \right], \quad \alpha \neq 1 \tag{3.14a}$$

$$x_n = x_0 \prod_{k=1}^{n} \left(\frac{x_0}{x_{-1}} + \beta k \right), \quad \alpha = 1. \tag{3.14b}$$

(b) *If $|\alpha| < 1$ and $|\beta| < 1 - \alpha$ then every solutions of (3.13) converges to zero.*
(c) *If $|\alpha| > 1$ or $|\beta| > 1 - \alpha$ then almost all solutions of (3.13) are unbounded.*
(d) *If $|\alpha| = 1$ then certain solutions of (3.13) are periodic with period 2, hence bounded and not converging to zero. There are also both unbounded solutions and solutions converging to zero in this case*.
Proof. (a) Since the function on the right hand side of (3.13) is homogeneous of degree 1 in x_n and x_{n-1}, we may divide by x_n and use the r_n notation to obtain

$$r_{n+1} = \alpha r_n + \beta. \tag{3.15}$$

The solution of this linear first order equation is easily obtained and then transformed into the appropriate form in (3.14) using (3.12) to complete the proof.
(b) If $|\alpha| < 1$ then every solution of (3.15) converges to the unique fixed point $\beta/(1 - \alpha)$. This point has absolute value less than unity if $|\beta| < 1 - \alpha$ so by Theorem 4(a) every solution of (3.13) converges to zero.
(c) If $|\alpha| > 1$ then almost every solution of (3.15) is unbounded exponentially, where as if $|\alpha| < 1$ but $|\beta| > 1 - \alpha$ then every solution of (3.15) converges to the fixed point $\beta/(1 - \alpha)$ with magnitude greater than unity. In either case, the conclusion follows upon an application of Theorem 4(b).

(d) In this case it is more efficient to use Eq.(3.14). If $\alpha = 1$ and $\beta \neq 0$ then using (3.14b) it is clear that every solution of (3.13) is unbounded whereas if $\beta = 0$ then solutions with initial values satisfying $|x_0| \leq |x_{-1}|$ are bounded (periodic if $x_0 = -x_{-1} \neq 0$).
If $\alpha = -1$ then consider solutions with initial values satisfying $x_0/x_{-1} = 1 + \beta/2$. For these solutions (3.14a) reduces to

$$x_n = x_0 \prod_{k=1}^{n} \left[(-1)^k + \frac{\beta}{2} \right] = K_n \left(\frac{\beta^2}{4} - 1 \right)^{[n/2]}$$

where $[n/2]$ is the greatest integer less than or equal to $n/2$ and $K_n = x_0$ if n is even and $K_n = x_0(\beta/2 - 1)$ if n is odd. It follows that if $|\beta| \leq 2\sqrt{2}$ then $\{x_n\}$ is bounded whereas if $\beta > 2\sqrt{2}$ then $\{x_n\}$ is unbounded. In particular, if $\beta = 0, \pm 2\sqrt{2}$ then $\{x_n\}$ is periodic with period 2.

To give further applications of ratios, the next two results are quoted from the literature ([9], [10], [18]) concerning the following equation

$$x_{n+1} = |\alpha x_n - \beta x_{n-1}| \tag{3.16}$$

which is a special case of Eq.(1.4) where $a = b = 0$ and $f(v, u - cv) = \alpha|u - cv|$ if we define $c = \beta/\alpha$. In this case, Eq.(3.6a) takes the form

$$r_{n+1} = \left| \alpha - \frac{\beta}{r_n} \right| \tag{3.17}$$

Theorem 5. [18] *Let* $\alpha = \beta = 1$ *in* (3.16) *and let* $\mathbb{Q}^+$ *denote the set of all non-negative rational numbers*.
(a) *If* $x_0/x_{-1} \in \mathbb{Q}^+$ *or* $x_{-1} = 0$ *then the corresponding solution* $\{x_n\}$ *of* (3.16) *has period 3 eventually and for all large* n *its cycles are* $\{0, \alpha, \alpha\}$ *where* $\alpha > 0$.
(b) *If* $x_0/x_{-1} \notin \mathbb{Q}^+$ *then the corresponding solution* $\{x_n\}$ *of* (3.16) *converges to zero*.
(c) *Equation* (3.17) *has a period-*p *solution or a* p*-cycle* $\{\rho_1, \ldots, \rho_p\}$ *for every* $p \neq 3$. *These* p*-cycles are given as*

$$\rho_1 = \frac{1 + \sqrt{5}}{2}, \ \rho_2 = \frac{\sqrt{5} - 1}{\sqrt{5} + 1} \quad (p = 2)$$
$$\rho_1 = \frac{1}{2} \left[y_{p-4} + \sqrt{y_{p-4}^2 + 4y_{p-4}y_{p-1}} \right], \ \rho_k = \frac{y_{k-4}\rho_1 - y_{k-2}}{y_{k-3} - y_{k-5}\rho_1}, \ 2 \leq k \leq p, \ (p \geq 4)$$

where y_n *is the* n*-th Fibonacci number; i.e.,* $y_{n+1} = y_n + y_{n-1}$ *for* $n \geq -2$ *where we define*

$$y_{-3} = -1, \ y_{-2} = 1.$$

(d) *If* $\{\rho_1, \ldots, \rho_p\}$ *is a periodic solution of* (3.17) *then for the corresponding solution* $\{x_n\}$ *of* (3.16) *it is true that*

$$x_n = x_0 \rho^{n/p}, \quad \text{if } n/p \text{ is an integer}$$
$$x_n \leq x_0 \alpha \rho^{n/p}, \quad \text{otherwise}$$

where

$$\rho = \prod_{i=1}^{p} \rho_i < 1, \quad \alpha = \max\{r_1, \ldots, r_p\}\rho^{-(1-1/p)} > 1.$$

Theorem 6. [9] (a) *Eq.*(3.16) *has a positive period-2 solution if and only if*

$$\beta^2 - \alpha^2 = 1, \quad \alpha > 0.$$

Further, these period-2 solutions are confined to the pair of lines $y = r_1 x$ *and* $y = r_2 x$ *in phase space, where the slopes* r_1, r_2 *are given by*

$$r_1 = \frac{\beta - 1}{\alpha}, \quad r_2 = \frac{\beta + 1}{\alpha}.$$

On the other hand, the only period-2 solutions of (3.16) *that pass through the origin occur at* $\alpha = 0$ *where* $\beta = 1$.
(b) *Eq.*(3.16) *has a positive period-3 solution if and only if*

$$\alpha^3 + \alpha\beta - \beta^3 = 1, \quad \alpha > 1. \tag{3.18}$$

Further, these period-3 solutions are confined to the three lines $y = r_i x$ *in phase space where for* $i = 1, 2, 3$*, the slopes* r_i *are given by*

$$r_1 = \frac{\alpha\beta + 1}{\alpha^2 + \beta}, \quad r_2 = \frac{\beta^2 - \alpha}{\alpha\beta + 1}, \quad r_3 = \frac{\beta + \alpha^2}{\beta^2 - \alpha}. \tag{3.19}$$

On the other hand, the only period-3 solutions of (3.16) *that pass through the origin occur at* $\alpha = 1$ *where* $\beta = 1$ *also (see Theorem... above).*
(c) *Let* $\beta = 1$. *Then there is a strictly increasing sequence of parameter values* $\{\alpha_p\}$, $p \geq 3$, *such that*

$$\alpha_3 = 1 \quad \text{and} \quad \lim_{p\to\infty} \alpha_p = 2$$

and for each $p = 3, 4, 5, \ldots$ *the particular solution* $\{x_n\}$ *of* (3.16) *with initial values* $x_{-1} = 1$, $x_0 = \alpha_p$ *is periodic with period* p.

It is a curious fact that the behaviors of solutions of (3.13) are considerably simpler than those of (3.16). This is not easy to understand through a direct comparison of the two second order difference equations which seem to have little in common except that they are both 2-parameter difference equations. However, the essential difference becomes apparent when we contrast the relatively simple dynamics of the linear mapping (3.15) with the much more complex dynamics of the first order mapping (3.17).

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Extended Oligopoly Models and Their Asymptotical Behavior

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ABSTRACT

We will investigate the different extensions of the Cournot oligopoly model including models with intertemporal demand interaction, including production adjustment cost, pollution treatment cost sharing, and cost interaction. The local asymptotic stability of the steady state is examined in all cases and the stability conditions are compared to the classical Cournot model.

KEY WORDS: *Dynamic economic systems, oligopoly, asymptotic stability*

AMS Mathematic Subject Classification: 91A40, 93A30

INTRODUCTION

Cournot oligopolies, their different variants, extensions and generalizations play an important role in the literature of mathematical economics. The early results up to the mid 70s are summarized in Okuguchi (1976), and their multi-product extensions with application studies are discussed in Okuguchi and Szidarovszky (1999). In the earliest studies the existence and uniqueness of the Nash-equilibrium of static oligopolies was the central issue, and later the attention has been turned to the asymptotical properties of the dynamic extensions. Puu and Sushko (2002) give a comprehensive summary of the most important more recent findings. All models discussed in the literature so far, are based on certain simplifying assumptions, which made the simple, analytic examination possible.

In this paper we will drop some of the simplifying assumptions of earlier models and will focus on more sophisticated models. In particular, we will consider models with intertemporal demand interaction, production adjustment costs, pollution treatment cost sharing and also with cost interaction. The simplified versions of some of the models to be discussed in this paper have been introduced earlier in Okuguchi and Szidarovszky (1999) as well as in the forthcoming book of Bischi et al. (2008). This paper develops as follows. Oligopolies with intertemporal demand interaction will be discussed in Section 2 followed by models with production adjustment costs. Section 4 will focus on oligopolies with pollution treatment cost sharing, and Section 5 will discuss the case of cost interaction among the firms. Conclusions will be drawn in the last section of the paper.

OLIGOPOLIES WITH INTERTEMPORAL DEMAND INTERACTION

In this section we will examine the affect of intertemporal demand interaction. In the case of durable goods the demand at any time period depends on the price and the demands of earlier periods. Even in the case of nondurable goods taste or habit formation has effect on future demands.

Let *N* be the number of firms producing a single item. Let x_k be the output of firm *k* and $C_k\left(x_k\right)$ its cost. Market saturation, habit formation, etc. of earlier time periods are condensed into a variable *Q* which is assumed to follow the dynamic rule

$$Q(t+1)=H\left(\sum_{k=1}^{N}x_k(t),Q(t)\right), \tag{1}$$

where H is a real valued function on $\left[0,\sum_{k=1}^{N}L_k\right]\times R$ with L_k being the capacity limit of firm *k*.

The price function *f* is assumed to depend on both the total output of the industry and the current value of parameter *Q*. So the profit of firm *k* can be given as

$$\Pi_k=x_k f\left(x_k+S_k,Q\right)-C_k\left(x_k\right), \tag{2}$$

where $S_k=\sum_{l\neq k}x_l$ is the output of the rest of the industry.

In this section we assume that functions *f* and C_k $\left(1\leq k\leq N\right)$ are twice continuously differentiable, and

(A) $f_x'<0$;

(B) $f_x'+x_k f_{xx}''<0$;

(C) $f_x'-C_k''<0$

for all *k* and feasible values of x_k, S_k and Q.

Under these conditions Π_k is strictly concave in x_k, so with fixed values of S_k and Q there is a unique best response of firm *k*, since the feasible set for x_k is the compact set $\left[0,L_k\right]$. It can be given as follows:

$$R_k\left(S_k,Q\right)=\begin{cases}0 & \text{if } f\left(S_k,Q\right)-C_k'\left(0\right)\le 0\\ L_k & \text{if } L_k f_x'\left(L_k+S_k,Q\right)+f\left(L_k+S_k,Q\right)-C_k'\left(L_k\right)\ge 0\\ x_k^* & \text{otherwise,}\end{cases} \tag{3}$$

where x_k^* is the unique solution of equation

$$f\left(x_k+S_k,Q\right)+x_k f_x'\left(x_k+S_k,Q\right)-C_k'\left(x_k\right)=0 \tag{4}$$

in interval $\left(0,L_k\right)$. In the first two cases of (3) the partial derivatives of R_k are zeros, except the boundary points. Implicit differentiation shows that in the third case

$$r_k=\frac{\partial R_k}{\partial S_k}=-\frac{f_x'+x_k f_{xx}''}{2f_x'+x_k f_{xx}''-C_k''} \quad \text{and} \quad \overline{r}_k=\frac{\partial R_k}{\partial Q}=-\frac{f_Q'+x_k f_{xQ}''}{2f_x'+x_k f_{xx}''-C_k''}. \tag{5}$$

Assumptions (B) and (C) imply that

$$-1<r_k<0 \tag{6}$$

as it is usual in the theory of discrete concave oligopolies. In addition, assume that

(D) $f_Q'+x_k f_{xQ}''\le 0$

for all feasible values of x_k, S_k and Q. Then

$$\overline{r}_k\le 0. \tag{7}$$

Let $a_k>0$ denote the speed of adjustment of firm *k* and assume that the firms change their outputs in the direction towards their best responses. This dynamism can be mathematically described by the discrete system

$$x_k\left(t+1\right)=x_k\left(t\right)+a_k\left(R_k\left(\sum_{l\ne k}x_l\left(t\right),Q\left(t\right)\right)-x_k\left(t\right)\right) \quad \left(1\le k\le N\right) \tag{8}$$

$$Q\left(t+1\right)=H\left(\sum_{k=1}^{N}x_k\left(t\right),Q\left(t\right)\right). \tag{9}$$

A vector $\left(\overline{x}_1,\ldots,\overline{x}_N,\overline{Q}\right)$ is a steady state of this system if and only if for all *k*,

$$\overline{x}_k=R_k\left(\sum_{l\ne k}\overline{x}_l,\overline{Q}\right)$$

and

$$\overline{Q}=H\left(\sum_{k=1}^{N}\overline{x}_k,\overline{Q}\right).$$

The local asymptotical stability of this system can be examined by linearization. The Jacobian of the system at the steady state can be written as follows:

$$\boldsymbol{J} = \begin{pmatrix} 1-a_1 & a_1 r_1 & \cdots & a_1 r_1 & a_1 \overline{r}_1 \\ a_2 r_2 & 1-a_2 & \cdots & a_2 r_2 & a_2 \overline{r}_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_N r_N & a_N r_N & \cdots & 1-a_N & a_N \overline{r}_N \\ h & h & \cdots & h & \overline{h} \end{pmatrix},$$

where *h* and $\overline{h}$ denote the partial derivatives of *H* with respect to *x* and *Q* at the steady state. The steady state is locally asymptotically stable if all eigenvalues of $\boldsymbol{J}$ are inside the unit circle. For the sake of simplicity assume symmetric firms, then $a_1 = \ldots = a_N = a$, $r_1 = \ldots = r_N = r$, and $\overline{r}_1 = \ldots = \overline{r}_N = \overline{r}$. Then the eigenvector equation of $\boldsymbol{J}$ has the special form

$$(1-a)u_k + ar\sum_{l \neq k} u_l + a\overline{r}v = \lambda u_k \quad (1 \leq k \leq N) \tag{10}$$

$$h\sum_{k=1}^{N} u_k + \overline{h}v = \lambda v. \tag{11}$$

Let $U = \sum_{k=1}^{N} u_k$, then equation (10) can be rewritten as

$$arU + a\overline{r}v + (1-a-ar-\lambda)u_k = 0. \tag{12}$$

Assume first that $\lambda = 1-a-ar$, then this eigenvalue is inside the unit circle if

$$a < \frac{2}{1+r}. \tag{13}$$

Otherwise $u_1 = \ldots = u_N = u$, and (10) and (11) simplify as

$$\begin{aligned} &\left(1-a+(N-1)ar-\lambda\right)u + a\overline{r}v = 0 \\ &hNu + \left(\overline{h}-\lambda\right)v = 0. \end{aligned} \tag{14}$$

Nontrivial solution exists if and only if

$$\mathrm{Det}\begin{pmatrix} 1\text{-}a\left(1+(1-N)r\right)-\lambda & a\overline{r} \\ hN & \overline{h}-\lambda \end{pmatrix} = 0.$$

This is a quadratic equation:

$$\lambda^2 + \lambda\left(-1-\overline{h}+az\right) + \left(\overline{h} - hNa\overline{r} - a\overline{h}z\right) = 0, \tag{15}$$

where we use the simplifying notation $z = 1+(1-N)r$. Notice that (6) implies that $1 < z < N$. The roots of the quadratic equation are inside the unit circle if and only if

$$\overline{h}(1-az) - hNa\overline{r} < 1 \tag{16}$$

$$-1-\overline{h}+az+\overline{h}(1-az) - hNa\overline{r} + 1 > 0 \tag{17}$$

$$1+\bar{h}-az+\bar{h}\left(1-az\right)-hNa\bar{r}+1>0. \tag{18}$$

These relations can be rewritten as

$$a\left[\bar{h}z+hN\bar{r}\right]>\bar{h}-1 \tag{19}$$

$$a\left[z\left(1-\bar{h}\right)-hN\bar{r}\right]>0 \tag{20}$$

and

$$a\left[z\left(1+\bar{h}\right)+hN\bar{r}\right]<2\left(1+\bar{h}\right). \tag{21}$$

It is reasonable to assume that $-1<\bar{h}<1$ and h>0. Then we have the following cases

(i) If $\bar{h}z+hN\bar{r}<0$, then (19) holds, if

$$a<\frac{\bar{h}-1}{\bar{h}z+hN\bar{r}}. \tag{22}$$

Notice that relation (20) always holds, since the multiplier of *a* in the left hand side is positive. If $z\left(1+\bar{h}\right)+hN\bar{r}\leq 0$, then (21) also holds for all *a*>0, otherwise it holds if

$$a<\frac{2\left(1+\bar{h}\right)}{z\left(1+\bar{h}\right)+hN\bar{r}}. \tag{23}$$

So in this case the steady state is locally asymptotically stable if *a* is sufficiently small.

(ii) If $\bar{h}z+hN\bar{r}\geq 0$, then (19) is always valid. Since $z\left(1-\bar{h}\right)-hN\bar{r}>0$, inequality (20) is also true for all *a*>0. If $z\left(1+\bar{h}\right)+hN\bar{r}\leq 0$, then (21) is also satisfied, otherwise it holds if (23) holds.

So in this case the steady state is either always locally asymptotically stable, or it is when the value if *a* is sufficiently small.

In comparison to the classical case without intertemporal demand interaction notice that in that case

$$f'_Q=f''_{xQ}=h=\bar{h}=\bar{r}=0,$$

so from (15) the eigenvalues are $\lambda_1=0$ and $\lambda_2=1+\bar{h}-az=1-a\left(1+\left(1-N\right)r\right)$. The first eigenvalue is always inside the unit circle, and the second is there if and only if

$$a<\frac{2}{1+\left(1-N\right)r}=\frac{2}{z}. \tag{24}$$

Since this inequality is stronger than (13), this is the stability condition for the classical case. Simple algebra shows that

$$\frac{2}{z}\leq\frac{2\left(1+\bar{h}\right)}{z\left(1+\bar{h}\right)+hN\bar{r}}$$

and equality occurs if and only if $\bar{r}=0$. Therefore in this sense, intertemporal demand interaction makes the system more stable, since the stability region for *a* becomes larger.

OLIGOPOLIES WITH PRODUCTION ADJUSTMENT COST

In this section we consider again an *N*-firm single-product oligopoly, when the price depends on only the total production level of the industry. However, the firms face additional cost if they increase their production levels compared to their outputs in the previous time period. Let $t+1\geq 1$ be any time period, then the profit of firm *k* can be written as

$$\Pi_k = x_k f\left(x_k+S_k\right)-C_k\left(x_k\right)-A_k\left(x_k-x_k\left(t\right)\right), \tag{25}$$

where $S_k=\sum_{l\neq k} x_l$ is the output of the rest of the industry as before. Assume that functions f, C_k and A_k are twice continuously differentiable, furthermore,

(E) $f'<0,\ C_k'\geq 0,\ A_k'\geq 0$;

(F) $f'+x_k f''<0$;

(G) $f'-C_k''<0,\ \ A_k''\geq 0$

for all *k* and feasible values of x_k, $x_k\left(t\right)$ and S_k.

Under these conditions Π_k is strictly concave in x_k, and in the case of finite capacity limits of all firms, there is a unique best response of firm *k*, which will be now denoted by $R_k\left(S_k, x_k\left(t\right)\right)$. It can be defined similarly to (3), and by implicit differentiation it is easy to see that

$$r_k=\frac{\partial R_k}{\partial S_k}=-\frac{f'+x_k f''}{2f'+x_k f''-C_k''-A_k''} \tag{26}$$

and

$$\bar{r}_k=\frac{\partial R_k}{\partial x_k\left(t\right)}=-\frac{A_k''}{2f'+x_k f''-C_k''-A_k''}. \tag{27}$$

Assumptions (E)—(G) imply that

$$-1<r_k<0\leq \bar{r}_k<1 \tag{28}$$

and

$$-1<r_k-\bar{r}_k \tag{29}$$

for all *k*. In this case the dynamic system (8)—(9) is modified as follows:

$$x_k\left(t+1\right)=x_k\left(t\right)+a_k\left(R_k\left(\sum_{l\neq k} x_l\left(t\right), x_k\left(t\right)\right)-x_k\left(t\right)\right) \tag{30}$$

for *k*=1,2,…,*N*. A vector $\left(\bar{x}_1,\ldots,\bar{x}_N\right)$ is a steady state of this system if and only if for all *k*,

$$\overline{x}_k = R_k\left(\sum_{l\neq k}\overline{x}_l, \overline{x}_k\right).$$

The Jacobian of system (30) now has the special structure

$$\boldsymbol{J} = \begin{pmatrix} 1-a_1\left(1-\overline{r}_1\right) & a_1 r_1 & \cdots & a_1 r_1 \\ a_2 r_2 & 1-a_2\left(1-\overline{r}_2\right) & \cdots & a_2 r_2 \\ \vdots & \vdots & & \vdots \\ a_N r_N & a_N r_N & \cdots & 1-a_N\left(1-\overline{r}_N\right) \end{pmatrix}.$$

For the sake of simplicity we consider only symmetric firms, when $a_1 = \ldots = a_N = a$, $r_1 = \ldots = r_N = r$ and $\overline{r}_1 = \ldots = \overline{r}_N = \overline{r}$. Then the eigenvector equation has the special form

$$\left(1-a\left(1-\overline{r}\right)\right)u_k + ar\sum_{l\neq k} u_l = \lambda u_k. \tag{31}$$

Let $U = \sum_{k=1}^{N} u_k$ as before, then

$$arU + \left(1-a\left(1-\overline{r}\right) - ar - \lambda\right)u_k = 0. \tag{32}$$

Assume first that $\lambda = 1-a\left(1-\overline{r}\right) - ar$, then this eigenvalue is inside the unit circle if

$$a\left(r-\overline{r}+1\right) < 2,$$

and since the multiplier of *a* is positive (by relation (29)), it holds if *a* is sufficiently small:

$$a < \frac{2}{r-\overline{r}+1}. \tag{33}$$

Otherwise $u_1 = \ldots = u_N = u$, and (32) has the special form:

$$\left(1-a\left(1-\overline{r}\right)+\left(N-1\right)ar-\lambda\right)u = 0,$$

and nontrivial solution exits if and only if

$$\lambda = 1-a\left(1-\overline{r}-\left(N-1\right)r\right).$$

Notice that the multiplier of *a* is always positive, since $1-\overline{r}-\left(N-1\right)r = \left(1+r-\overline{r}\right)-Nr$ with both terms being positive. So λ is inside the unit circle if

$$a < \frac{2}{1+r-\overline{r}-Nr}. \tag{34}$$

That is, if the value of *a* is sufficiently small. Notice that since $r < 0$, (34) is more restrictive than (33), so inequality (34) is the stability condition. Notice also that in the absence of production adjustment cost $\overline{r} = 0$, in that case the stability condition is relation (24), and since $\overline{r} \geq 0$, (34) is less restrictive than (24) unless $\overline{r} = 0$. That is, production adjustment costs usually make the system more stable.

OLIGOPOLIES WITH POLLUTION TREATMENT COST SHARING

Pollution emerges in many industries as the result of the manufacturing process. In this section we will examine *N*-firm single-product oligopolies with the additional assumption that the firms treat jointly the pollution and share the treatment cost in proportion to their share in the total output. Therefore the profit of firm *k* is

$$\Pi_k = x_k f\left(x_k + S_k\right) - C_k\left(x_k\right) - x_k \frac{T\left(x_k + S_k\right)}{x_k + S_k}, \tag{35}$$

where *f* is the price function, C_k is the cost function of firm *k* as before, and *T* is the total pollution treatment cost function. By introducing the notation

$$G\left(x_k + S_k\right) = \frac{T\left(x_k + S_k\right)}{x_k + S_k} \quad \text{and} \quad F\left(x_k + S_k\right) = f\left(x_k + S_k\right) - G\left(x_k + S_k\right) \tag{36}$$

we have

$$\Pi_k = x_k F\left(x_k + S_k\right) - C_k\left(x_k\right), \tag{37}$$

where Π_k has the form of the payoff functions of classical oligopolies in which *f* is replaced by *F*.

By assuming that functions $f,\ G,\ C_k\ \left(1 \le k \le N\right)$ are twice continuously differentiable (*G*(0) is taken as the finite limit of this function at zero), from (5) we see that

$$r_k = \frac{\partial R_k}{\partial S_k} = -\frac{F_x' + x_k F_{xx}''}{2F_x' + x_k F_{xx}'' - C_k''}, \tag{38}$$

where $R_k\left(S_k\right)$ is the best response of firm *k*. The corresponding dynamic system and its steady state can be presented similarly to the previously discussed cases. Assume that

(H) $F_x' < 0$;

(I) $F_x' + x_k F_{xx}'' < 0$;

(J) $F_x' - C_k'' < 0$

for all *k* and feasible values of x_k and S_k, then clearly

$$-1 < r_k < 0, \tag{39}$$

and the stability condition is inequality (24) for symmetric firms. Notice that the stability region for *a* becomes larger when *r* increases. Therefore pollution treatment cost sharing makes the system more stable if

$$-\frac{f_x' + x_k f_{xx}''}{2f_x' + x_k f_{xx}'' - C_k''} < -\frac{f_x' - G_x' + x_k f_{xx}'' - x_k G_{xx}''}{2f_x' - 2G_x' + x_k f_{xx}'' - x_k G_{xx}'' - C_k''}$$

which is the case, when

$$x\left(G'f'' - f'G''\right) + C''\left(G' + xG''\right) < 0\,, \tag{40}$$

where we assume symmetric firms again. The conditions given earlier do not imply this relation, so it depends on the particular choice of the functions involved in this model.

OLIGOPOLIES WITH COST INTERACTION

Here we assume that the firms hire manpower, purchase supplies and materials from the same market, so the cost of each firm depends on its own production level as well as on the total production level of the rest of the industry. This cost interaction has not been considered in earlier models. So the profit of firm *k* is as follows:

$$\Pi_k = x_k f\left(x_k + S_k\right) - C_k\left(x_k, S_k\right), \tag{41}$$

where f, S_k are as before, and C_k is the modified cost function of firm *k*.

The corresponding dynamic system and its steady state can be presented similarly to the previously discussed cases.

With fixed value of S_k, we have

$$\frac{\partial \Pi_k}{\partial x_k} = f\left(x_k + S_k\right) + x_k f'\left(x_k + S_k\right) - C'_{kx}\left(x_k, S_k\right) \tag{42}$$

and

$$\frac{\partial^2 \Pi_k}{\partial x_k^2} = 2f'\left(x_k + S_k\right) + x_k f''\left(x_k + S_k\right) - C''_{kxx}\left(x_k, S_k\right).$$

So if we assume that

(K) $f' < 0$;

(L) $f' + x_k f'' < 0$;

(M) $f' - C''_{kxx} < 0$

for all *k* and feasible values of x_k and S_k, then Π_k is strictly concave in x_k, and if all firms have finite capacity limits, then there is a unique best response $R_k\left(S_k\right)$ of each firm *k*. By implicit differentiation

$$r_k = \frac{\partial R_k}{\partial S_k} = -\frac{f' + x_k f'' - C''_{kxs}}{2f' + x_k f'' - C''_{kxx}}\,, \tag{43}$$

where the denominator is always negative. Assuming again symmetric firms, from the previous sections we know that cost interaction makes the system more stable if the value of *r* increases. This is the case, when $C''_{xs} < 0$,

We finally note that the payoff functions (41) reduce to (35) by the special selection

$$C_k\left(x_k, S_k\right) = C_k\left(x_k\right) + x_k \cdot \frac{T\left(x_k + S_k\right)}{x_k + S_k},$$

so models with pollution treatment cost sharing are special cases of the models of this section.

CONCLUSIONS

In this paper four extensions of the classical Cournot model were introduced and their local asymptotical stability was examined.

In the first case we assumed the presence of intertemporal demand interaction in the market as the result of market saturation, taste or habit formation etc. In the second case we assumed that any increase in the output level during any time period results in an additional cost. In the third case the firms treat pollution jointly and share the cleaning cost in proportion to their share in the total output. In the fourth case we assumed that the cost of each firm depends on the firm's own output as well as on the output of the rest of the industry.

For the sake of simplicity we considered only symmetric firms and presented the stability conditions in term of the common speed of adjustment of the firms. In all cases the condition requires its value to be sufficiently small. This stability region increases by intertemporal demand interaction in the market as well as by the presence of production adjustment costs. Conditions were derived in the other two cases to guarantee the increase of the stability region. These conditions depend on the particular choice of the cost and price functions.

In the discussions of this paper we assumed symmetric firms for mathematical convenience only. The general case can be investigated in a similar manner based on the methodology being discussed in Bischi et al. (2008).

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