Tricomi Problem

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- In 1923 , F. G. Tricomi (Atti Accad. Naz. Lincei , 14 (1923) , 133-247) initiated the work on boundary value problems for partial differential equations of mixed type and related equations of variable type . The Tricomi equation $yu_{xx} + u_{yy} = 0$ plays a central role in the mathematical analysis of transonic flow . As the simplest equation with that property , it provides a useful mathematical model of the transition from subsonic to supersonic speeds in aerodynamics .

Tricomi Problem or *Problem T* : consists in finding a function $u = u(x, y)$ which satisfies the *Tricomi equation* : $yu_{xx} + u_{yy} = 0 \quad (*)$ in a mixed domain D which is simply connected and bounded by a Jordan (non-selfintersecting) "elliptic" arc g_1 (for y > 0) with endpoints $O = (0,0)$ and $A = (1, 0)$ and by the "real or hyperbolic" characteristics g_2 : $x + \frac{2}{3}(-y)^{3/2} = 1$, g_3 : $x - \frac{2}{3}(-y)^{3/2} = 0$ of $(*)$ (for y < 0) satisfying the characteristic equation : $y(dy)^2 + (dx)^2 = 0$ such that these characteristics meet at a point P (for y < 0) and u assumes prescribed continuous boundary values $u = \varphi(s)$ on g_1 and $u = \psi(x)$ on g_3 $(**)$.

In 1935 , S. Gellerstedt (Doctoral Thesis , Uppsala , 1935 ; Jbuch Fortschritte Math. 61 (1935) , 1259) generalized the *problem T* by replacing the coefficient y of u_{xx} in the above equation (*) by $\operatorname{sgn}(y)|y|^m$, $m > 0$. In 1945 , F. I. Frankl (Izv. Akad. Nauk SSSR ser. mat. 9 ; Bull. de l'Acad. des Sci. de l' URSS , 9 (1945) , no.2 , 121-143) established a generalization of the problem T for the *Chaplygin equation* : $K(y)u_{xx} + u_{yy} = 0$ with $K(y) > 0$ for $y > 0$; < 0 for $y < 0$; $K(0) = 0$. We note that this equation was established in 1904 by S. A. Chaplygin ("On Gas Jets" , Scientific Annals of the Imperial University of Moscow , Publication no.21 , 1904 ; translation : Brown Univ. , R. I. , 1944) .

Frankl Problem or *Problem F* : consists in finding a function $u = u(x, y)$ which satisfies the Chaplygin equation : $K(y)u_{xx} + u_{yy} = 0 \ (**)$ in a mixed domain D which is simply connected and bounded by a Jordan "elliptic" arc g_1 (for y > 0) with endpoints $O = (0,0)$ and $A = (1,0)$, by the real characteristic g_2 : $x = \int_0^y \sqrt{-K(t)}\,dt + 1$ of $(**)$ (for y < 0) satisfying the characteristic equation : $K(y)(dy)^2 + (dx)^2 = 0$ and by the non-characteristic g_3' emanating from the point O , lying inside the characteristic triangle OAP and intersecting the characteristic g_2 at most once (g_3' may coincide with the “real” characteristic g_3 : $x = -\int_0^y \sqrt{-K(t)}\,dt$ of $(**)$ (for y < 0) near the point O) and assuming prescribed continuous boundary values $u = \varphi(s)$ on g_1 and $u = \psi(x)$ on g_3' .

F. I. Frankl (in 1945) initiated a new stage in the theory of equations of mixed type . In particular , he established the uniqueness of the solution of the above Problem F in the case where *the Frankl condition* : $F(y) = 1 + 2\left(K/K'\right)' > 0$, for $y < 0$ holds with derivative $K'(y) > 0$. Note that this condition is equivalent to the convexity of $(-K)^{-1/2}$ for $y < 0$. M. A. Lavrentjev and A. V. Bitsadze (Dokl. Akad. Nauk. SSSR 70 , 3 , 1950 , 373-376) suggested the well-known *Bitsadze - Lavrentjev model* with a discontinuous $K = \mathrm{sgn}(y)$. According to M. H. Protter (Bull. Amer. Math. Soc. , 1 (1979), no. 3 , 534-538) the task of forming a single comprehensive theory for mixed type equations in two dimensions appears formidable ; the development in three and more dimensions is even more remote . M. H. Protter (J. Rat. Mech. & Anal. 2 (1953) , no. 1 , 107-114) improved the above Frankl condition . Besides , Protter (J. Rat. Mech. & Anal. 3 (1954) , no. 4 , 435-446) was the first to consider the case in three dimensions . These boundary value problems are important in fluid dynamics .