Landau Problem

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- In 1913 , E. Landau (Proc. London Math. Soc. 13 (1913) , no.2 , 43-49) initiated the following extremum problem for twice differentiable functions .

  Landau Extremum Problem : The sharp inequality between the supremum-norms of derivatives of twice differentiable functions f such that : $\|f'\|^2 \le 4\|f\|\,\|f''\|$ holds with norm referring to the space $C[0,\infty]$.

  If f is a real - valued function defined on $R = (-\infty, \infty)$, $\|f\| = \sup\{|f(x)| : x \in R\}$ and f is twice differentiable and both *f* and f'' are bounded, J. Hadamard (Comptes Rendus Acad. Sci. Paris 41 (1914), 68-72) achieved the best possible constant 2 in this case. For $C(-\infty, \infty)$, A. N. Kolmogorov (Ucen. Zap. Moskov. Gos. Unive. , Mat. 30 , (1939) , 3-16 ; Amer. Math. Soc. Transl. 4 , New York , (1949) , 233-243) established the above inequality with the same constant 2 and generalized this inequality to derivatives of order higher than 2 . Besides , R. R. Kallman & G. C. Rota ("Inequalities , II" (O. Shisha , Ed.), Academic Press , New York , (1970) , 187-192) demonstrated that the constant 4 , is true also for a semigroup of linear contractions. Moreover , H. Kraljevic & S. Kurepa (Glas. Mat. 5 (1970) , 109-117) established the constant 4/3 for a strongly continuous cosine function of linear contractions with an infinitesimal generator . In addition , Z. Ditzian (Aequat. Math. 12 (1975) , 145-151) achieved the constant 2 for a group of linear isometries . For a real-valued function *f* defined on $(0, \infty)$, define $\|f\| = \left(\int_0^\infty f^2(x)dx\right)^{\frac{1}{2}}$. If f is twice differentiable and both f and f'' are bounded , G. H. Hardy ; J. E. Littlewood ; and G. Polya (Proc. London Math. Soc. 25 (1926) , no. 2 , 265-282 ; "Inequalities" , (1934) Cambridge , Univ. Press , England) showed the above inequality with 2 the best possible constant . Moreover , these three authors (1934) showed that the general inequality $\|f^{(k)}\|^n \le \|f\|^{n-k}\|f^{(n)}\|^k$, $0 < k < n$ holds with 1 the best possible constant , if f is a real-valued function on $(-\infty, \infty)$ and $\|f\| = \left(\int_{-\infty}^\infty f^2(x)dx\right)^{\frac{1}{2}}$ as well as f is n-differentiable and both f and $f^{(n)}$ are bounded . This extremum problem is interesting in operator theory and approximation theory , as well .