Exact controllability
for Maxwell’s equations
in bianisotropic media

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joint work with
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After the early experiments on **Optical Activity**\(^1\) by F. Arago (1811), J.-B. Biot (1812) and A. Fresnel (1822), A. L. Cauchy (1842) presented the first mathematical work on the laws of circular polarisation, while L. Pasteur (1848) explained this phenomenon essentially by introducing Geometry into Chemistry (thus originating the branch nowadays called Stereochemistry), and K. F. Lindman (1914 - works published in 1920 and 1922) was the first to demonstrate the effect of a **chiral**\(^2\) medium on electromagnetic waves.

The revival in the interest of complex media in Electromagnetics emerged in the mid 1980s, motivated and assisted by vast technological progress, especially at microwave frequencies.

\(^1\)The ability of a material to rotate the plane of polarisation of a beam of light passing through it.

\(^2\)The mirror image of a right-handed object is otherwise the same as the original, but it is left-handed (the original object cannot be superposed upon its mirror image).
Already in the beginning of the 21st century, the related publications within the Applied Physics and Electrical Engineering communities were calculated in more than 3000 papers.

Related books:

- G. Kristensson (in progress).
Mathematical Work on Complex Media – 1: Frequency Domain Problems (time-harmonic fields)

• (As far as I know) the first publication on chiral media is by Petri Ola (1994).
• Simultaneous/independent work from the mid 1990s by the groups at
  • CMAP, École Polytechnique (Palaiseau): Jean-Claude Nédélec, Habib Ammari, and their collaborators.
  • the Department of Mathematics of the National and Kapodistrian University of Athens: Christodoulouso Athanasiadis, $\S$, and later on collaborators in various places.
• From the late 1990s, in addition to the above, many researchers enter the field; indicatively (in alphabetical, non-chronological, order) some names: A. Boutet de Monvel, G. Costakis, P. Courilleau, S. Dimitroula, T. Gerlach, S. Heumann, T. Horsin, H. Kiili, V. Kravchenko, S. Li, P. A. Martin, S. R. McDowall, M. Mitrea, R. Potthast, D. Shepelsky, C. Skourogiannis, S. Vänskä, ...
Mathematical Work on Complex Media – 2: Time Domain Problems

From the early 2000s attention is focussed on the time domain, as well. Problems on the solvability, the homogenisation, and the controllability of IBVPs for the Maxwell equations, supplemented with nonlocal in time, linear constitutive relations (describing the so-called bianisotropic media), are studied.


- The biggest part of this work deals with deterministic bianisotropic media.
- But problems regarding stochastic bianisotropic media are also studied.

The Maxwell System

Electromagnetic phenomena are specified by 4 (vector) quantities: the electric field $E$, the magnetic field $H$, the electric flux density $D$ and the magnetic flux density $B$. The inter-dependence between these quantities is given by the celebrated Maxwell system,

$$
curl H(t, x) = \partial_t D(t, x) + J(t, x),
$$
$$
curl E(t, x) = -\partial_t B(t, x),
$$

where $J$ is the electric current density. All fields are considered for $x \in O \subset \mathbb{R}^3$ and $t \in \mathbb{R}$, $O$ being a domain with appropriately smooth boundary. These equations are the so called Ampère’s law and Faraday’s law, respectively. In addition to the above, we have the two laws of Gauss

$$
\text{div} D(t, x) = \rho(t, x),
$$
$$
\text{div} B(t, x) = 0,
$$

where $\rho$ is the density of the (externally impressed) electric charge.
The initial conditions are considered to be of the form

\[ E(0, x) = E_0(x), \ H(0, x) = H_0(x), \ x \in \mathcal{O}. \] (3)

We consider the “perfect conductor” boundary condition

\[ n(x) \times E(t, x) = 0, \ x \in \partial \mathcal{O}, \ t \in I, \] (4)

where \( I \) is a time interval, and \( n(x) \) denotes the outward normal on \( \partial \mathcal{O} \).

Constitutive relations must also be introduced, that are in general of the form

\[ D = D(E, H), \ B = B(E, H). \] (5)
The Six Vector Notation

To express the system in more compact form, we use the six-vector notation:

- the electromagnetic flux density $d := (D, B)^{tr}$,
- the electromagnetic field $u := (u_1, u_2)^{tr} := (E, H)^{tr}$,
- the current $j := (-J, 0)^{tr}$,
- the initial state $u_0 := (E_0, H_0)^{tr}$,

where the superscript $tr$ denotes transposition.

Introduce also the Maxwell operator

$$M := \begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix}.$$  \hspace{1cm} (6)
The Maxwell System as an IVP

The constitutive relations are now modelled by an operator $\mathcal{L}$ and are understood as the functional equation

$$d = \mathcal{L}u.$$ 

The properties of this operator reflect the physical properties of the medium in question.
So the Maxwell system can be written as an IVP for an abstract evolution equation

$$(\mathcal{L}u)'(t) = Mu(t) + j(t), \text{ for } t > 0,$$

$$u(0) = u_0.$$  \hspace{1cm} (7)

The prime stands for the time derivative.
The evolution equation in the above IVP is an inhomogeneous neutral functional differential equation.
To state the postulates that govern the evolution of the e/m field in a complex medium we follow a system-theoretic approach (Ioannidis (PhD; 2006) / Ioannidis, Kristensson, §), in the sense that we consider the e/m field $u$ as the cause, and the e/m flux density $d$ as the effect.

**Postulates (plausible physical hypotheses)**

- **Determinism:** For every cause there exists exactly one effect.
- **Linearity:** The effect is produced linearly by its cause.
- **Causality:** The effect cannot precede its cause.
- **Locality in space:** A cause at any particular spatial point produces an effect only at this point and not elsewhere.
- **Time–translation invariance:** If the cause is advanced (or delayed) by some time interval, the same time-shift occurs for the effect.

Compliance with these postulates dictates the form of the operator $\mathcal{L}$. 
The Constitutive Relations for Bianisotropic Media

The general form of $\mathcal{L}$, consistent with the above physical postulates, turns to be a continuous operator having the convolution form

$$d(t, x) = (\mathcal{L}u)(t, x) = A_{or}(x)u(t, x) + \int_{0}^{t} G_{d}(t - s, x)u(s, x)\, ds$$  \hspace{1cm} (8)$$

$$A_{or}(x) := \begin{pmatrix} \varepsilon(x) & \xi(x) \\ \zeta(x) & \mu(x) \end{pmatrix}, \quad G_{d}(t, x) := \begin{pmatrix} \varepsilon_{d}(t, x) & \xi_{d}(t, x) \\ \zeta_{d}(t, x) & \mu_{d}(t, x) \end{pmatrix}. \hspace{1cm} (9)$$

Each $A_{or}(\cdot)$, $G_{d}(t, \cdot)$ defines a multiplication operator in the state space. The above constitutive equation is abbreviated as

$$d = A_{or}u + G_{d} \ast u.$$  \hspace{1cm} (10)$$

The local in space part $A_{or}$ (optical response operator) of $\mathcal{L}$ models the instantaneous response of the medium. The nonlocal in space part $G_{d} \ast$ of $\mathcal{L}$ models the dispersion phenomena; $G_{d}$ is called the susceptibility kernel.
Several alternative approaches to the solvability of the IVP for the Maxwell system

\[(Lu)'(t) = Mu(t) + j(t) \quad \text{for } t > 0,\]
\[u(0) = u_0,\]

supplemented with the constitutive relations for dissipative bianisotropic media

\[(Lu)(t, x) = A_{or}(x)u(t, x) + \int_0^t G_d(t - s, x)u(s, x)\, ds,\]

can be considered, e.g., semigroups, evolution families, the Faedo–Galerkin method.

We adopt the former, based on the semigroup generated by the Maxwell operator. Then the convolution terms are treated as perturbations of this semigroup.
The choice of the semigroup approach is reasonable since
- the semigroup (group actually) generated by the Maxwell operator is very well studied,
- the kernels in the convolution terms are known to be physically small, therefore, it is plausible to consider them as perturbations.

These approaches have been used in different variations by Bossavit, Griso and Miara / Ciarlet and Legendre / Ioannidis, Kristensson and Liaskos, Ioannidis, Kristensson and Yannacopoulos.

For the integrodifferential equation (11) a variety of different types of solutions can be defined, regarding spatial - or temporal - regularity. We totally skip their technical descriptions here.

So a well-posedness result can be stated
Well-posedness

Theorem

Under suitable regularity assumptions on the data, (11) is weakly / mildly / strongly / classically well-posed.

The underlying space is $H_0(\text{curl}, \mathcal{O}) \times H(\text{curl}, \mathcal{O})$, where

- $H(\text{curl}, \mathcal{O}) := \{ u \in (L^2(\mathcal{O}))^3 : \text{curl } u \in (L^2(\mathcal{O}))^3 \}$.
- For bounded $\mathcal{O}$, $H_0(\text{curl}, \mathcal{O})$ is the space $\{ u \in H(\text{curl}, \mathcal{O}) : n \times u|_{\partial\mathcal{O}} = 0 \}$.

The first component of the underlying space incorporates the perfect conductor boundary condition for the electric field.
Controllability Concepts - The Cauchy problem

Let $H, U$ be Hilbert spaces, $A$ an (unbounded) operator $D(A) \subset H \rightarrow H$ which is assumed to generate a strongly continuous semigroup of operators $(S(t))_{t \geq 0}$ and $B$ a bounded operator $U \rightarrow H$.

Consider the Cauchy problem

\[ z' = Az + Bu, \quad z(0) = z_0, \quad (13) \]

where $z : [0, T] \rightarrow H$, $u \in L^2(0, T; U)$ and $z_0 \in H$.

Recall that if $z_0 \in D(A)$ and $u \in W^{1,1}(0, T; U)$, then (13) admits a unique classical solution $z \in C([0, T]; D(A)) \cap C^1([0, T]; H)$ given by the Duhamel (variation of constants) formula

\[ z(t) = S(t)z_0 + \int_0^t S(t - s)Bu(s)ds, \quad \forall t \in [0, T]. \]

Even if only $z_0 \in H$ and $u \in L^1(0, T; U)$, the above formula is still meaningful and defines the mild solution of (13).
Notions of controllability

**Definition (Exact Controllability)**

The Cauchy problem (13) is said to be exactly controllable in time $T > 0$, if for any $z_0, z_T \in H$ there exists a $u \in L^2(0, T; U)$ such that the solution $z(t)$ of (13) fulfills $z(T) = z_T$.

**Definition (Null Controllability)**

The Cauchy problem (13) is said to be null controllable in time $T > 0$, if for any $z_0, z_T \in H$ there exists a $u \in L^2(0, T; U)$ such that the solution $z(t)$ of (13) fulfills $z(T) = 0$.

**Definition (Approximate Controllability)**

The Cauchy problem (13) is said to be approximately controllable in time $T > 0$, if for any $z_0, z_T \in H$ and for any $\epsilon > 0$ there exists a $u \in L^2(0, T; U)$ such that the solution $z(t)$ of (13) satisfies $\|z(T) - z_T\|_H < \epsilon$. 
Introducing the operator $L_T : L^2(0, T; U) \rightarrow H$ by

$$L_T u := \int_0^T S(T - s)Bu(s)ds$$

we have the following characterisations for these concepts:

- Exact controllability in time $T \iff R(L_T) = H$
- Null controllability in time $T \iff S(T)H \subset R(L_T)$
- Approximate controllability in time $T \iff \overline{R(L_T)} = H$
Finite dimensions - ODEs

Let us note that in finite dimensions, i.e., for the initial value problem for a linear vector ODE (such problems arose already in the 1770s in relation to J. Watt’s steam engine) of the form

\[ z' = Az + Bu, \quad z(0) = z_0, \]

where \( z : [0, T] \rightarrow \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and \( z_0 \in \mathbb{R}^n \) and \( A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m} \), with \( m \leq n \), the three above concepts of controllability are equivalent, and further they are equivalent to a purely algebraic condition, the celebrated Kalman rank condition (\( \text{rank}[B, AB, \cdots, A^{n-1}B] = n \)). Therefore the time \( T \) plays no role at all!

- Even for finite dimensional systems, controllability is not always achieved (consider, e.g., the system \( z'_1 = z_1 + u, \quad z'_2 = z_2 \); clearly \( u \) does not influence the trajectory \( z_2(t) \), so this system is uncontrollable).
Passing to PDEs the situation is more tricky!

- No algebraic test for controllability.
- The control time is important for hyperbolic PDEs.
- The converses of
  - exact controllability $\implies$ null controllability
  - exact controllability $\implies$ approximate controllability
are not true in general.
- However, if $A$ generates a continuous group, then
  - exact controllability $\iff$ null controllability.

This result applies to the wave equation, but not to the heat equation. The finite speed of propagation and the evolution of singularities that are characteristics of hyperbolic PDEs and the infinite speed of propagation and the smoothing effects that are characteristics of parabolic PDEs affect the controllability.
Controllability tests

Let $B^*$ denote the adjoint of $B$, and $S^*(t)$ be the semigroup generated by $A^*$.

**Theorem (Observability Inequality)**

The Cauchy problem (13) is exactly controllable in time $T > 0$, if and only if there exists a constant $c > 0$ such that

$$\int_0^T \| B^* S^*(t)y_0 \|_U^2 \, dt \geq c \| y_0 \|_H^2, \quad \forall y_0 \in H. \quad (14)$$

The inequality (14) is called an observability inequality. It means that the map $y_0 \mapsto B^* S^*(\cdot)y_0$ is boundedly invertible, i.e. that it is possible to recover complete information about the initial state $y_0$ by measuring on $[0, T]$ the output $B^* [S^*(t)y_0]$ (observability property).
Controllability tests - cont’d

There is a weaker version of OI:

**Theorem (Weak Observability Inequality)**

*The Cauchy problem (13) is exactly controllable in time $T > 0$, if and only if there exists a constant $c > 0$ such that*

\[
\int_0^T \|B^* S^* (t)y_0\|^2_U dt \geq c \|S^* (t)y_0\|^2_H, \quad \forall y_0 \in H. \tag{15}
\]

The inequality (15) is a weak observability inequality: only $S^* (t)y_0$ can be recovered, not $y_0$.

- Regarding approximate controllability, there is an - in general - weaker characterisation, the so-called **Unique Continuation Property**.
The theoretical basis of HUM is, roughly speaking, the observation that if one has uniqueness of solutions of a linear evolution system in a Hilbert space, it is possible to introduce a Hilbert space norm $\| \cdot \|_H$ - based on the uniqueness property - in such a way that the dual system is exactly controllable to the dual space $H^*$. The exact controllability problem is thereby transferred to the problem of identifying (or otherwise characterising) the couple $H, H^*$. The latter is essentially a problem in PDEs when the original evolution system is a distributed parameter system: can \textit{a priori} estimates of $\| \cdot \|_H$ be obtained in terms of norms in spaces which are both intrinsic to the given problem and which are readily identifiable?
We associate to the Cauchy problem

\[ z' = Az + Bu , \; z(0) = 0 , \]  

its adjoint problem (uncontrolled, backward in time)

\[ y' = -A^* y , \; y(T) = y_T . \]  

For any \( y_T \in H \), the solution \( y \) of (17) reads \( y(t) = S^*(T - t)y_T \).

The following key identity holds

\[ (z(T), y_T)_H = \int_0^T (u, B^* y)_U \, dt \]

A simple proof of the exact controllability characterisation in terms of the observability inequality is based on the study of the operator \( \Lambda : H \to L^2(0, T; U) \), which is the inverse of the restriction of \( L_T \) \( (= \int_0^T S(T - s)Bu(s)ds) \) to \((\ker L_T)\perp \). The operator \( \Lambda \) (which is bounded and actually furnishes the control) serves - in view of the above key identity - to prove the observability inequality.
This procedure is the basis of HUM, essentially reducing exact controllability to studying an uncontrolled system (the adjoint problem), to considering an observation \( (B^*y) \) and to establishing an observability inequality (via the operator \( \Lambda \)). The underlying idea is the classical duality between controllability of a system and observability of its dual system.

- HUM is a constructive method
- the existence of one control implies the existence of many controls - HUM gives the one of minimal \( L^2 \)-norm
- the identification of the spaces between which \( \Lambda \) acts is a crucial and not easy task
- boundary data may not be adequately smooth - “hidden regularity” saves the day!
- interpretation of weak solution - method of transposition
Several methods to prove observability inequality. The principal ones are

- **Multipliers Method** - some time is required for the control to transfer information due to finite speed of propagation - non-optimal result, explicit constant
- **Carleman Estimates** - time irreversibility creates difficulties - non-optimal result - allows perturbations
  - Global Carleman Estimates are inequalities of the form
    \[
    \int \int_{\mathcal{O} \times (0,T)} \varphi^2 |\varphi|^2 \, dx \, dt \leq C \int \int_{\mathcal{O}_o \times (0,T)} \varphi^2 |\varphi|^2 \, dx \, dt,
    \]
    where \( \mathcal{O}_o \subset \subset \mathcal{O} \) and \( \varphi = \varphi(x, t) \) is continuous and strictly positive.
- **Microlocal Analysis** - propagation of Sobolev regularity or microlocal defect measures - non explicit constant - necessary and sufficient condition
  - Microlocalisation is a process which combines the standard techniques of localisation and Fourier transform: one localises not only in the space variable \( x \), but as well in the Fourier transform variable \( p \). The resulting space of the variables \((p,x)\) is called phase space and is a symplectic manifold. The symplectic geometry in this space and the corresponding Hamiltonian dynamical systems are then used to study the original PDE problems.
Controllability Problems for the Maxwell equations


On the contrary, controllability questions for complex media have only recently started being addressed. Work by Courilleau and Horsin / Courilleau, Horsin and S / Horsin and S / Horsin, S and Yannacopoulos / S and Yannacopoulos / Nicaise.
Internal Controllability

The governing equation

$$(A_{or} u + G_d \ast u)' = Mu + j,$$  (18)

can be simplified if we assume that $G_d(t, x)$ is weakly differentiable with respect to the temporal variable. Then we may differentiate the convolution integral, and by multiplying to the right by $A_{or}^{-1}$ we get

$$u' = M_A u + G_A \ast u + J_A,$$  (19)

where

$$G_A := -A_{or}^{-1}G'_d, \quad M_A := A_{or}^{-1}M, \quad J_A := A_{or}^{-1}j,$$

and we have assumed that $G_d(0, x) = 0$.

The boundary conditions, as well as the divergence free character of the electromagnetic field, can be included in the definition of the operator $M$ in appropriately selected function spaces.
We now assume that we have access to an *internal control* $v$, that acts on the system. The action of the control $v$ on the state of the system is modelled, via the so-called *control to state operator* $B$, by the evolution equation

$$u' = M_A u + J_A + G_A \star u + B v. \quad (20)$$

The problem of controllability can now be stated as follows:
Given $T > 0$, an initial condition $u(0) = U_0$ and a final condition $u(T) = U_T$, can we find a control procedure $v^*(\cdot)$ such that the solution of the system (20) with $v(\cdot) = v^*(\cdot)$ satisfies $u(0) = U_0$ and $u(T) = U_T$?

In the sequel we describe an HUM-based perturbative (fixed-point scheme) approach (S and Yannacopoulos) to study the exact internal controllability of (20).

- In the case of time-harmonic fields, the approximate controllability problem has also been studied (Horsin and S).
First we consider the achiral case and formally work as follows:

- **Step 1**
  Solve the uncontrolled system with initial condition $U_0$:

  \[
  V' = M_A V + J_A, \\
  V(0) = U_0,
  \]

  to obtain $V(T)$.

- **Step 2**
  Consider the map $\Lambda$, which connects the final states of the adjoint backward and forward systems, and solve the operator equation

  \[
  \Lambda(\Phi_T^b) = U_T - V(T),
  \]

  to obtain $\Phi_T^b \in H_1$ ($H_1$ is an appropriate subset of the state space $H$).
**Step 3**
Solve the backward adjoint equation with final condition $\Phi^b_T$:

$$
\Phi' = M_A \Phi, \\
\Phi(T) = \Phi^b_T,
$$

(23)
to obtain the solution $\Phi^b(\cdot)$.

**Step 4**
The required control is $v^b = \Phi^b$.

**Step 5**
The desired path is given by the solution of

$$
u' = M_A u + J_A + B v,$$

(24)with control $v^b = \Phi^b$. 
The whole problem reduces to the study of the mapping $\Lambda$ and the solvability of the operator equation (22). If this equation has a solution for any $U_T$, then, provided that the above auxiliary problems are well posed - which is true in the appropriate functional setting - it is possible to find a control procedure $\nu(\cdot)$ that allows us to steer the system (24) from any initial state $U_0$ to any final state $U_T$ within the required time $T$. In this case we say that the system is exactly controllable. The control procedure constructed above is a minimal-norm control 3.

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3In general the controllability problem does not have a unique solution unless an extra condition is required, that of choosing the control of minimal norm. This coincides with the control constructed by the Hilbert Uniqueness Method described above.
From the previous discussion, we see that exact controllability of the system is equivalent to the invertibility of the operator $\Lambda$ on all of $\mathbb{H}_1$.

The Hilbert Uniqueness Method essentially consists in trying to define an appropriate Hilbert space $\mathbb{H}_1$, endowed with the proper norm, so as to guarantee the invertibility of $\Lambda$.

In our case it turns out that $\mathbb{H}_1$ must be $\mathcal{X}_M$, where

$$\mathcal{X}_M := (H_0(\text{curl}, \mathcal{O}) \cap H(\text{div}0, \mathcal{O})) \times (H(\text{curl}, \mathcal{O}) \cap H_0(\text{div}0, \mathcal{O})).$$
Proofs’ ingredients

- $M_A$ generates a $C_0$-group
- Continuity + Coercivity $\rightarrow$ Invertibility of $\Lambda$
- Forward problem: estimate of the sup-norm of the solution in terms of the $L^2$-norm of its time derivative
- Backward problem: estimate of the $X_M$-norm of the final value in terms of the $X_M$-norm of the solution
The General Problem - Chirality Included

Having treated the achiral problem, we can consider the “full” problem, treating the convolution terms as a perturbation. By an argument based on the Schauder fixed-point theorem we end up with

**Theorem**

Let $u_0 \in \mathcal{X}_M$ and $J_A \in W^{1,1} := C([0, T]; \mathbb{X}) \cap W^{1,1}([0, T], \mathbb{X})$. In addition let $\mathcal{B} \in \mathcal{L}(W^{1,1}, W^{1,1})$ be invertible. Then the full Maxwell system

$$u' = M_A u + G_A * u + J_A + \mathcal{B} v,$$

is exactly controllable.

- The approximate controllability problem has been studied (Horsin, S and Yannacopoulos) in the stochastic case.
There is a strong connection between the study of controllability\(^4\) (including optimal controllability) and the treatment of certain classes of inverse problems.

In particular, for PDEs an (alphabetical) incomplete authors list would certainly comprise:


- Linear Problems $\leftrightarrow$ Duality Techniques
- Nonlinear Problems $\leftrightarrow$ A variety of methods

\(^4\)and its dual notion of observability
Consider a Banach space $X$ and a closed linear operator $A$ with a dense domain; $A$ is supposed to be the generator of a strongly continuous semigroup. Then the following type of inverse source problem, for first order abstract ODEs, can be considered (Prilepko, Orlovsky, Vasin): find the source term $F$ so that the two-point problem

$$x'(t) = Ax(t) + F(t), \ x(0) = x_0, \ x(T) = x_1,$$

is uniquely solvable.
The controllability result obtained in the previous section can serve as a method of proving the unique solvability of the corresponding inverse source problem (determination of the externally imposed current) for the considered model (bianisotropic media) for the Maxwell equations.

Eventually, it turns out that

\[ J = -\varepsilon (B \nu^b)^{(1)} - \xi (B \nu^b)^{(2)} \]
Thank you for your patience!

- But not really “The End” of my talk.
- See next slide...
As we can all see, today, 24 years later, Grigoris is much more good looking!!!