Regions of attraction and recursive feasibility in Robust MPC

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CONTENTS

- MPC and Robust MPC – earlier work
- The benchmark – a lifted formulation
- Farkas’ Lemma and tube MPC
- Recursive feasibility
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PROBLEM FORMULATION

Model: \( x_{i+1} = Ax_i + Bu_i \)
Constraints: \( Fx + Gu \leq 1 \)

Minimize \( J = \sum_{i=0}^{\infty} \|x_i\|^2_Q + \|u_i\|^2_R \)

INTRACTABLE hence Receding Horizon dual mode MPC

Mode 1: \( u = Kx + c \)
Mode 2: \( u = Kx \)

Minimize over \( c \): \( J = \sum_{i=0}^{N} \left( \|x_i\|^2_Q + \|u_i\|^2_R \right) + x_N^T P x_N \)

Use \( u_0 \) and repeat at next time instant.

- \( K \) is LQ unconstraint Optimal; \( x_N^T P x_N \) is cost in Mode 2
- Mode 2 \( u = Kx \) must be feasible: \( x_N \in X_f = \{ x \mid V_f x_N \leq 1 \} \)
- \( X_f \) is invariant feasible set under \( u = Kx \)
DUAL MODE PREDICTION PARADIGM

\[ \begin{bmatrix} x_0 \end{bmatrix}_1 \]

\[ \begin{bmatrix} x_0 \end{bmatrix}_2 \]

Terminal Set

\( J_\infty = x_N^T P x_N \)

Mode 1 \( u = Kx + c \)

Mode 2 \( u = Kx \)

• Cost quadratic in \( c = [c_0 \ldots c_{N-1}]^T \)

• Minimize cost over \( c \); implement \( u_0 = Kx_0 + c_0 \)

• Repeat at next time instant
UNCERTAINTY IN MPC

Model uncertainty

\[ x_{i+1} = (A^{(0)} + \Delta)x_i + Bu_i + w_i; \]

\[ \Delta_i \in Co\{\Delta^{(j)}\}, \quad w_i \in Co\{w^{(j)}\} \]

Additive uncertainty: Use Tube MPC –

Rigid or Homothetic (fixed or scalable cross-section)

Multiplicative uncertainty:

1. No Mode 1; choose \( K \) online to satisfy LMI’s to minimize worst case cost subject to constraints and invariance.

2. Use Mode 1 but find cross sections of tubes that contain the state for all uncertainty.

PROBLEMS: (1) good for only small scale problems (e.g. \( n = 2 \)); also conservative – no Mode 1!

(2) Computation increases exponentially.
LIFTED FORMULATION

PREDICTION DYNAMICS

\[
\begin{bmatrix}
    x_{k+1} \\
    c_{k+1}
\end{bmatrix}
= \begin{bmatrix}
    \Phi & BE \\
    0 & M
\end{bmatrix}
\begin{bmatrix}
    x_k \\
    c_k
\end{bmatrix}
\quad \Leftrightarrow \quad \xi_{k+1} = \Psi \xi_k
\]

\[
E = \begin{bmatrix}
    I & 0 & \ldots & 0
\end{bmatrix}, \quad M = \begin{bmatrix}
    0 & I & 0 & \ldots \\
    0 & 0 & I & \ldots \\
    \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & \ldots & 0
\end{bmatrix}, \quad \Phi = A + BK
\]

IMPORTANT PROPERTY: Lifted dynamics are autonomous

Let \( x = z + e \); \( z = \text{nominal}, \quad e = \text{uncertain} \)

Then \( \xi = \zeta + \epsilon \) so that

\[
\zeta_{k+1} = \Psi^{(0)} \zeta_k
\]
**THE BENCHMARK**

**Proposition 1.**

\[ J_{\text{nominal}} = \sum_{i=0}^{\infty} \left\| z_{k+i} \right\|_Q^2 + \left\| u_{k+i} \right\|_R^2 = \left\| \xi_k \right\|_W^2 \]

\[ W - \Psi(0)^T W \Psi(0) = \overline{Q} = \begin{bmatrix} Q + K^T R K & K^T R E \\ E^T R K & E^T R E \end{bmatrix} \]

**Proposition 2.** Prediction feasible iff \( \xi_k \in \prod = \{ V_{\text{max}} \xi_k \leq 1 \} \)

\( \prod \) = maximal invariant set for \( \xi_{k+1} = \Psi \xi_k \)

Compute \( V_{\text{max}} \) by forward propagation (e.g. by Gilbert et al, 1991) or backward propagation (e.g. Miani and Blanchini, 2007)

Robust Lifted MPC algorithm (RLMPC): minimize \( \left\| \xi_k \right\|_W^2 \) over \( C \) such that

\[ V_{\text{max}} \xi_k \leq 1, \quad \begin{bmatrix} I & 0 \end{bmatrix} \xi_k = x_k \]
PROPERTIES OF BENCHMARK

Theorem 1. feasibility at \( k \) \( \Rightarrow \) feasibility at \( k+1 \), for \( \forall \Delta \in Co\{\Delta^{(j)}\} \)

Theorem 2. RLMPC is not outperformed by any dual mode Tube MPC in terms of the size of the Region of Attraction and closed-loop cost.

PROBLEM: For weakly contractive \( \Psi \), \( V_{\text{max}} \xi_k \leq 1 \) may imply a large number of inequalities.

SOLUTION: Use Tube MPC based on Farkas’ Lemma and introduce DOF so as to reduce suboptimality.

PROPOSITION 3 (e.g. Miani and Blanchini, 2007).

Let \( P_i = \{ x \mid F_i x \leq b_i \} \)

Then \( P_1 \subseteq P_2 \iff \exists H \geq 0 \text{ s.t. } HF_1 = F_2, \ Hb_1 \leq b_2 \)
Feasibility in robust MPC

**TUBE CROSS-SECTIONS - Inclusion**

Decompose \( x \) into nominal \( z \) and uncertain \( e \)

\[
\begin{cases}
x_{k+1} = \Phi x_k + Bc_k \\
z_{k+1} = \Phi^{(0)} z_k + Bc_k \\
e_{k+1} \in \text{Co}\{[\Phi^{(0)} + \Delta^{(j)}]e_k + \Delta^{(j)} z_k\}
\end{cases}
\]

Define Tube cross-section \( T_k = \{e_k \mid Ve_k \leq \alpha_k\} \)

**Theorem 3** (One-step ahead inclusion): \( e_k \in T_k \Rightarrow e_{k+1} \in T_{k+1} \)

If \( H^{(j)} \alpha_k + V\Delta^{(j)} z_k \leq \alpha_{k+1} \)

where \( H^{(j)} \geq 0 \) are chosen offline s.t. \( H^{(j)} V = V(\Phi^{(0)} + \Delta^{(j)}) \)

**Proof:** By Proposition 3.

**Note:** For tight inclusion \( H^{(j)} = \min_{\tilde{H}} \left\{ \left\| \tilde{H} \right\|_\infty \mid \tilde{H} V = V(\Phi^{(0)} + \Delta^{(j)}) \right\} \)

It also reduces computation through sparseness
TUBE CROSS-SECTIONS - Feasibility

Theorem 4 (Feasibility): \( e_k \in T_k \Rightarrow Fx_k + Gu_k < 1 \)

If \( H\alpha_k + (F + GH)z_k + Gc_k \leq 1 \) Mode 1

\( H\alpha_k + (F + GH)z_k \leq 1 \) Mode 2

where \( HV = F + GK \) (chosen offline)

Proof: By Proposition 3.

Note:

Tight constraint satisfaction: \( H = \min_{\tilde{H}} \{ \|\tilde{H}\|_\infty \mid \tilde{H}V = F + GK \} \)

It also reduces computation through sparseness.
CONSTRAINT SATISFACTION

MODE 1: For $k = 0, 1, \ldots, n - 1$ invoke linear conditions

$$z_{k+1} = \Phi^{(0)} z_k + Bc_k$$

$z$-dynamics

$$\alpha_{k+1} \geq H^{(j)} \alpha_k + V\Delta^{(j)} z_k$$

$\alpha$-dynamics (inclusion)

$$H\alpha_k + (F + GH)z_k + Gc_k \leq 1$$

Feasibility

MODE 2: Invoke the linear conditions

$$z_{k+1} = \Phi^{(0)} z_k$$

$$\alpha_{k+1} \geq H^{(j)} \alpha_k + V\Delta^{(j)} z_k$$

$$H\alpha_k + (F + GK)z_k \leq 1$$

For $k = N, N + 1, N + 2, \ldots$

Hence the need for a terminal set!
CONTRACTIVITY IN MODE 2

Assumption: \( \exists \lambda < 1 \) s.t. Mode 2 \( z \)-dynamics \( z^+ = \Phi z \) are \( \lambda \) - contractive

Lemma 1:
If \( \{ z \mid Vz \leq 1 \} \) is a \( \lambda \) - contractive set for \( z^+ = \Phi z \) then \( \left\| H^{(j)} \right\|_\infty \leq \lambda \)

Theorem 5.
Given \( V_f z_N \leq 1 \), \( \exists \) integer \( \mu > 0 \) s.t. \( (F + GK)x_{N+\mu+i} \leq 1 \) for all \( i > 0 \)

Proof:
The Mode 2 \( (z, \alpha) \) -dynamics are
\[
   z^+ = \Phi z
\]
\[
   \alpha^+ = \max_j (H^{(j)} \alpha + V\Delta^{(j)} z)
\]

\( \Phi, H^{(j)} \) are contractive, so beyond some finite time \( \mu \) into Mode 2, \( (z, \alpha) \) will be small enough s.t.
\[
   H\alpha_{N+\mu+i} + (F + GK)z_{N+\mu+i} \leq 1
\]
TERMINAL CONSTRAINTS

Theorem 6. A sufficiently large $\mu$ is defined by

$$0 \leftarrow \frac{\max_j \|v^{(j)}\|_\infty}{1 - \max_j \|H^{(j)}\|_\infty} \leq \gamma \leq \frac{1 - \|\tilde{f}\|_\infty}{\|H\|_\infty} \rightarrow \frac{1}{\|H\|_\infty}$$

where

$$\tilde{f} = \max_{z \in V_f \Phi^{N+\mu}, z \leq 1} \{(F + GK)z\}$$

$$\bar{v}^{(j)} = \max_{z \in V_f \Phi^{N+\mu}, z \leq 1} \{V\Delta^{(j)}z\}$$

Terminal Constraints

(Invariant set for z) $V_f z_N \leq 1$

(inclusion) $H^{(j)} \alpha_{N+i} + V\Delta^{(j)}z_{N+i} \leq \alpha_{N+i+1}; \quad i = 0, \ldots, \mu$

(feasibility) $H\alpha_{N+i} + (F + GH)z_{N+i} \leq 1; \quad i = 0, \ldots, \mu$

(recursive feasibility) $1^T \alpha_{N+\mu} \leq \gamma, \quad \alpha_{N+i} \geq 0; \quad i = 0, \ldots, \mu$
Robust Tube MPC

Algorithm: Minimize over $\mathcal{C}$ the cost $J = \| \zeta_k \|_W^2$

Subject to inclusion and feasibility constraint

**MODE 1**

\[
H^{(j)} \alpha_{k+i} + V \Delta^{(j)} z_{k+i} \leq \alpha_{k+i+1}; \quad i = 0, \ldots, N - 1
\]
\[
H \alpha_{k+i} + (F + GH) z_{k+i} + G c_{k+i} \leq 1; \quad i = 0, \ldots, N - 1
\]

Subject to Terminal Constraints

**MODE 2**

\[
V_f z_N \leq 1
\]
\[
H^{(j)} \alpha_{N+i} + V \Delta^{(j)} z_{N+i} \leq \alpha_{N+i+1}; \quad i = 0, \ldots, \mu
\]
\[
H \alpha_{N+i} + (F + GH) z_{N+i} \leq 1; \quad i = 0, \ldots, \mu
\]
\[
1^T \alpha_{N+\mu} \leq \gamma, \quad \alpha_{N+i} \geq 0, \quad i = 0, \ldots, \mu
\]

Implement $u_k = K x_k + c_k$ and repeat at next time instant.
Control Theoretic Properties of Robust Tube MPC

Theorem 7. The Robust Tube MPC algorithm is recursively feasible and asymptotically stable.

Proof: (Feasibility) By construction, the tail (the extension to the next time instant of the current optimal predictions) defines one feasible trajectory.

(Stability) The tail defines a monotonically non-increasing cost s.t.

\[
(J_{k+1})_{tail} - (J_k)_{optimal} = -(\|z_k\|^2_Q + \|u_k\|^2_R)
\]

From which we have

\[
(J_{k+1})_{optimal} - (J_k)_{optimal} \leq -(\|z_k\|^2_Q + \|u_k\|^2_R)
\]

Summing this for all k gives that the closed loop cost is bounded from above so that \(z_k, u_k\) both tend to zero with increasing k.
Tube RMPC is much “cheaper” than the benchmark but its region of attraction is almost as big.

V chosen to define maximal invariant set and has 8 rows.
State Trajectories for benchmark

- State trajectory
- Region of attraction
State Trajectories of tube RMPC
Input Responses for benchmark

- Input trajectory
- Constraint

![Graph showing input responses for benchmark]

- Vertical axis: Input $u_k$
- Horizontal axis: Time $k$
- Time range: $k = 1$ to $k = 10$
- Input values range from approximately $-1.5$ to $1.5$.
Input Response for tube RMPC
Conclusions

The lifted autonomous formulation provides a benchmark that cannot be superseded by any tube RMPC but could be “expensive”

Tube RMPC with use of Farkas’ Lemma is much “cheaper” and provides extra d.o.f. with which to shape the cross sections.

Recursive feasibility requires terminal constraints which are made possible through a bound $\gamma$ on the extra d.o.f.

The results compare favourably with the benchmark

Further, significant, economies of computation are possible via time varying bounds $\gamma$