Structured distance-to-singularity problems arising in robust control

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Main Points

- The structured singular value problem
- Quadratic Integer programming problem
- Convex relaxation and duality gap reduction
- General characteristics of approach
- Conclusions and further work

Distance to singularity and ssv - 1

- Let $M \in \mathcal{C}^{n\times n}$, $\det(M) \neq 0$ (w.l.g.). Define $\mathbf{\Delta} \subseteq \mathcal{C}^{n\times n}$:

$$\mathbf{\Delta} = \{\text{diag}[\delta_1 I_{r_1}, \dots, \delta_s I_{r_s}, \Delta_{s+1}, \dots, \Delta_{s+f}] :$$

$$\delta_i \in \mathcal{C}, \Delta_{s+j} \in \mathcal{C}^{m_j \times m_j}, 1 \leq i \leq s, 1 \leq j \leq f\}$$

where $\sum_{i=1}^{s} r_i + \sum_{j=1}^{f} r_i = n$.

- Define the "*structured distance to singularity*" of M:

$$r_{\mathbf{\Delta}}(M) = \min\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \det(I - \Delta M) = 0\}$$

and the "*structured singular value*" of M:

$$\mu_{\mathbf{\Delta}}(M) = r_{\mathbf{\Delta}}^{-1}(M)$$

Distance to singularity and ssv - 2

Problem has applications in:

- *Robust control:* Structured distance problems to stability, controllability radius, etc.
- *Numerical Analysis:* Structured condition number, approximate GCD of polynomials, pseudo-spectra of operators, etc.

Example: Suppose that $a(s)$ and $b(s)$ are monic relative-prime polynomials with $\partial a(s) = m$ and $\partial b(s) = n$. What is the minimal magnitude perturbation in their coefficients such that the perturbed polynomials have a common root?

Distance to singularity and ssv - 3

Define Sylvester matrix:

$$R_{m,n}(a,b) = \begin{pmatrix} 1 & \alpha_{m-1} & \alpha_{m-2} & \cdots & \alpha_0 & 0 & \cdots & 0 \\ 0 & 1 & \alpha_{m-1} & \cdots & \alpha_1 & \alpha_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{m-1} & \cdots & \alpha_1 & \alpha_0 \\ 1 & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_0 & 0 & \cdots & 0 \\ 0 & 1 & \beta_{n-1} & \cdots & \beta_1 & \beta_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \beta_{n-1} & \cdots & \beta_1 & \beta_0 \end{pmatrix}$$

Then:

$$(a,b) \text{ co-prime} \;\Leftrightarrow\; \text{Rank}(R_{m,n}(a,b)) = m+n$$

Distance to singularity and ssv - 4

"Perturbed" Sylvester matrix:

$$R_{m,n}(a,b) + \begin{pmatrix} 0 & \delta_{m-1} & \delta_{m-2} & \cdots & \delta_0 & 0 & \cdots & 0 \\ 0 & 0 & \delta_{m-1} & \cdots & \delta_1 & \delta_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \delta_{m-1} & \cdots & \delta_1 & \delta_0 \\ 0 & \epsilon_{n-1} & \epsilon_{n-2} & \cdots & \epsilon_0 & 0 & \cdots & 0 \\ 0 & 0 & \epsilon_{n-1} & \cdots & \epsilon_1 & \epsilon_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \epsilon_{n-1} & \cdots & \epsilon_1 & \epsilon_0 \end{pmatrix}$$

and hence the uncertainty in this case has "structure". Hence original problem is equivalent to the computation of a s.s.v:

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Theorem: Define $\gamma = \max\{\ |\delta_0|, \ldots, |\delta_{m-1}|,\ |\epsilon_0|, \ldots, |\epsilon_{n-1}|\ \}$ and $\gamma^* = \min\{\gamma : (a,b)$ have a common root$\}$. Then,

$$\gamma^* = \frac{1}{\mu_{\mathbf{\Delta}}(-ZR_{m,n}^{-1}(a,b)\Theta)}$$

where

$$\mathbf{\Delta} = \{\text{diag}(\delta_{m-1}I_n,\ \ldots, \delta_0 I_n, \epsilon_{n-1}I_m,\ \ldots, \epsilon_0 I_m)\}$$

$$\Theta = \left(\begin{array}{ccc|ccc} I_n & \cdots & I_n & O_{n,m} & \cdots & O_{n,m} \\ \hline O_{m,n} & \cdots & O_{m,n} & I_m & \cdots & I_m \end{array}\right)$$

$$Z' = \left(\begin{array}{ccc|ccc} (Z_{nm}^0) & \cdots & (Z_{nm}^{m-1}) & (Z_{mn}^0) & \cdots & (Z_{mn}^{n-1}) \end{array}\right)$$

and

$$Z_{nm}^k = \begin{pmatrix} O_{n,k+1} & I_n & O_{n,m-k-1} \end{pmatrix}'$$

for $k = 0, 1, \ldots, m-1$.

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- In general, the computation of $\mu_{\mathbf{\Delta}}$ is a non-convex NP-hard problem.
- To compute upper bound first note that:

$$\begin{aligned}
\mu_{\mathbf{\Delta}}^{-1}(M) &= \min\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\} \\
&= \min\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \det(M^{-1} - \Delta) = 0\} \\
&\geq \min\{\|\Delta\| : \det(M^{-1} - \Delta) = 0\} \\
&= \|M\|^{-1}
\end{aligned}$$

so that $\mu_{\mathbf{\Delta}}(M) \leq \|M\|$.

- To tighten bound define:

$$\mathbf{D} = \{D = D^* > 0 : D\Delta = \Delta D \;\forall\; \Delta \in \mathbf{\Delta}\}$$

Then for any $D \in \mathbf{D}$, $\Delta \in \mathbf{\Delta}$: $D^{1/2}\Delta = \Delta D^{1/2}$.

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- Hence:

$$\begin{aligned}\mu_{\boldsymbol{\Delta}}^{-1}(M) &= \min\{\|\Delta\| : \Delta \in \boldsymbol{\Delta}, \det(I - M\Delta) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \boldsymbol{\Delta}, \det(I - MD^{-1/2}\Delta D^{1/2}) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \boldsymbol{\Delta}, \det(I - D^{1/2}MD^{-1/2}\Delta) = 0\} \\ &= \mu_{\boldsymbol{\Delta}}^{-1}(D^{1/2}MD^{-1/2}) \geq \|D^{1/2}MD^{-1/2}\|^{-1}\end{aligned}$$

- We then have relaxation bound:

$$\mu_{\boldsymbol{\Delta}}(M) \leq \min_{D \in \mathbf{D}} \|D^{1/2}MD^{-1/2}\| := \bar{\mu}_{\boldsymbol{\Delta}}(M)$$

Since:

$$\|D^{1/2}MD^{-1/2}\| \leq \gamma \Leftrightarrow \gamma^2 D - M^* D M \geq 0$$

the computation of $\bar{\mu}_{\boldsymbol{\Delta}}(M)$ now reduces to a convex programming problem.

Reducing the duality gap: $\bar{\mu}_{\mathbf{\Delta}} - \mu_{\mathbf{\Delta}}$ (1)

- Solve: $\mu_0 = \min_{D \in \mathbf{D}} \|D^{1/2} M D^{-1/2}\|$ and let D_o be a minimizer.
- Re-define: $M \leftarrow \mu_0^{-1} D_o^{1/2} M D_o^{-1/2}$ so that: $\bar{\mu}_{\mathbf{\Delta}}(M) = \|M\| = 1$.
- Let $M \in \mathcal{C}^{n \times n}$ have (ordered) SVD:

$$M = U \Sigma V^* = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} I_m & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix}$$

  where $\Sigma_2 := \mathrm{diag}(\sigma_{m+1}, \ldots, \sigma_n)$, $\sigma_{m+1} < 1$.

- <u>Theorem:</u> Let $M = U_1 V_1^* + U_2 \Sigma_2 V_2^*$ with

$$U_1 = \begin{pmatrix} A_1 \\ \vdots \\ A_s \\ E_1 \\ \vdots \\ E_f \end{pmatrix}, \quad V_1 = \begin{pmatrix} B_1 \\ \vdots \\ B_s \\ H_1 \\ \vdots \\ H_f \end{pmatrix}$$

compatibly partitioned with $\mathbf{\Delta}$. Let:

$$\nabla_M = \{\text{diag}(P_1^\eta, \ldots, P_s^\eta, p_{s+1}^\eta I_{m_1}, \ldots, p_{s+f-1}^\eta I_{m_f-1}, 0_{m_r}) \\ : \eta \in \mathbf{C}^m, \|\eta\| = 1\}$$

where

$$P_i^\eta = A_i \eta \eta^* A_i^* - B_i \eta \eta^* B_i^*, \;\; p_{s+j}^\eta = \eta^*(E_j^* E_j - H_j^* H_j)\eta$$

Then:

$$\mu_{\mathbf{\Delta}}(M) = \bar{\mu}_{\mathbf{\Delta}}(M) \;\; \Leftrightarrow \;\; 0 \in \text{co}(\nabla_M)$$

- Note: Multiplicity of largest s.v. of M is $m = 1 \Rightarrow 0 \in \text{co}(\nabla_M)$. Therefore assume that $m > 1$.

Reducing the duality gap: $\bar{\mu}_{\mathbf{\Delta}} - \mu_{\mathbf{\Delta}}$ (2)

- Consider perturbation matrix $\Delta \in \mathcal{C}^{n\times n}$:

$$\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}, \quad \Delta_{11} \in \mathcal{C}^{m\times m}$$

- Solve, or at least calculate upper bounds:

$$\phi_1 := \max\{\rho(V_1^* \Delta U_1) : \Delta \in \mathcal{B}\mathbf{\Delta}\} \leq \bar{\phi}_1 \leq 1$$

$$\phi_2 := \max\{\|V_1^* \Delta U_1\| : \Delta \in \mathcal{B}\mathbf{\Delta}\} \leq \bar{\phi}_2 \leq 1$$

Reducing the duality gap: $\bar{\mu}_{\mathbf{\Delta}} - \mu_{\mathbf{\Delta}}$ (3)

Then:

$$
\begin{aligned}
r_{\mathbf{\Delta}} &= \mu_{\mathbf{\Delta}}^{-1}(M) \\
&= \min\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \det(I - \Delta M) = 0\} \\
&= \min\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \det(I - \Delta U \Sigma V^*) = 0\} \\
&= \min\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \det(I - V^* \Delta U \Sigma) = 0\} \\
&= \min\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \det(\Sigma^{-1} - V^* \Delta U) = 0\} \\
&\geq \min\{\|\Delta\| : \rho(\Delta_{11}) \leq \bar{\phi}_1 \|\Delta\|, \|\Delta_{11}\| \leq \bar{\phi}_2 \|\Delta\|, \det(\Sigma^{-1} - \Delta) = 0\} \\
&:= \hat{r}_{\mathbf{\Delta}}(M) := \hat{\mu}_{\mathbf{\Delta}}^{-1}(M)
\end{aligned}
$$

Reducing the duality gap: $\bar{\mu}_{\mathbf{\Delta}} - \mu_{\mathbf{\Delta}}$ (4)

- $\hat{\mu}_{\mathbf{\Delta}}(M)$ can can be obtained analytically and $\hat{\mu}_{\mathbf{\Delta}}(M) \leq \bar{\mu}_{\mathbf{\Delta}}(M)$.
- Calculation of ϕ_1 ($\bar{\phi}_1$): Since

$$\begin{aligned}\phi_1 = \max_{\Delta \in \mathcal{B}\mathbf{\Delta}} \rho(V_1^* \Delta U_1) &= \max_{\Delta \in \mathcal{B}\mathbf{\Delta}} \rho(\Delta U_1 V_1^*) \\ &:= \max_{\Delta \in \mathcal{B}\mathbf{\Delta}} \rho(\Delta M_0) \\ &= \mu_{\mathbf{\Delta}}(M_0)\end{aligned}$$

  this is a ssv problem *of reduced rank*.

- Calculation of ϕ_2 ($\bar{\phi}_2$): Use convex relaxation (LMI):

$$\phi_2^2 = \max_{\Delta \in \mathcal{B}\mathbf{\Delta}} \|V_1^* \Delta U_1\|^2 \leq \min_{D \in \mathbf{D}, D - V_1 V_1^* \geq 0, \gamma I - U_1{}^* D U_1 \geq 0} \gamma = \bar{\phi}_2^2$$

- Lemma: $\phi_2 = \bar{\phi}_2$; $\inf_{D \in \mathbf{D}} \|D^{1/2} M D^{-1/2}\| = 1 \Rightarrow \phi_2 = \bar{\phi}_2 = 1$.

Sequence of relaxation problems

General problem:

$$\gamma_{\mathbf{\Delta_{11}}} = \min\{\|\Delta\|, \det(\Sigma^{-1} - \Delta) = 0, \Delta_{11} \in \mathbf{\Delta_{11}}\}$$

where:

- $\mathbf{\Delta_{11}} = \{\delta \in \mathcal{C} : |\delta| \leq \phi\}, 0 \leq \phi \leq 1 \ (m = 1)$
- $\mathbf{\Delta_{11}} = \{0_{m \times m}\}$
- $\mathbf{\Delta_{11}} = \{\Delta_{11}\}, \|\Delta_{11}\| \leq 1, 1 \neq \lambda(\Delta_{11})$
- $\mathbf{\Delta_{11}} = \{\Delta_{11} : \|\Delta_{11}\| \leq 1, 1 \notin \lambda(\Delta_{11})\}$
- $\mathbf{\Delta_{11}} = \{\Delta_{11} : \rho(\Delta_{11}) \leq \phi_1, \|\Delta_{11}\| \leq \phi_2\}$

Breaching the convex bound

Theorem: Let $0 \leq \bar{\phi}_1 \leq \bar{\phi}_2 \leq 1$. Then

$$\hat{\mu}_{\mathbf{\Delta}}^{-1} := \min\{\|\Delta\| : \det(\Sigma^{-1} - \Delta) = 0, \rho(\Delta_{11}) \leq \bar{\phi}_1 \|\Delta\|, \|\Delta_{11}\| \leq \bar{\phi}_2 \|\Delta\|\}$$

$$= \min\left\{\gamma > 1 : \|f(\gamma, \Psi)\| = \sigma_{m+1}^{-1}\right\}$$

where: $f(\gamma, \Psi) = (\gamma I - \Psi)(\gamma^{-1} I - \Psi)^{-1}$ and

$$\Psi_{ij} = \left\{ \begin{array}{cc} 0, & j < i \\ \phi_1, & j = i \\ -\left(\frac{\phi_1}{\phi_2}\right)^{j-i-1} \frac{\phi_2^2 - \phi_1^2}{\phi_2}, & j > i \end{array} \right\}$$

Also: (i) $\bar{\phi}_1 < 1 \Rightarrow \mu_{\mathbf{\Delta}}(M) < 1$ and $\hat{\mu}_{\mathbf{\Delta}}(M) < 1$ (ii) $\mu_{\mathbf{\Delta}}(M) = \hat{\mu}_{\mathbf{\Delta}}(M)$ iff $\exists \Delta \in \mathbf{\Delta}$ s.t.

$$V_1^* \Delta U_1 = \hat{\mu}_{\mathbf{\Delta}}(M) W^* \Psi W,$$

for unitary W. $\square$

Note: Ψ is Toeplitz upper triangular. For example, if $m = 4$,

$$\Psi = \begin{bmatrix} \phi_1 & \alpha & \beta & \gamma \\ 0 & \phi_1 & \alpha & \beta \\ 0 & 0 & \phi_1 & \alpha \\ 0 & 0 & 0 & \phi_1 \end{bmatrix}$$

with

$$\alpha = \frac{\phi_2^2 - \phi_1^2}{\phi_2}, \beta = -\frac{\phi_1}{\phi_2}\frac{\phi_2^2 - \phi_1^2}{\phi_2}, \gamma = (\frac{\phi_1}{\phi_2})^2\frac{\phi_2^2 - \phi_1^2}{\phi_2}$$

Note: The calculation of $\hat{\mu}_{\boldsymbol{\Delta}}(M)$ is a $4m \times 4m$ e-value problem. Set $a = a_{m+1}$, and $\Psi_s = \Psi + \Psi^*$. Then $a = \|(\gamma I - \Psi)(\gamma^{-1} I - \Psi)^{-1}\| \Rightarrow$

$$\det \left(\gamma I - \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ a^2 I - a^2 \Psi_s & (a^2 - 1)\Psi^*\Psi & \Psi_s & \end{bmatrix} \right) = 0$$

Hence $\hat{\mu}_{\boldsymbol{\Delta}}^{-1}$ is the smallest real e-value larger than one of a $4m \times 4m$ matrix.

P2: Quadratic Integer Programming problem

- QIP Problem has the form:

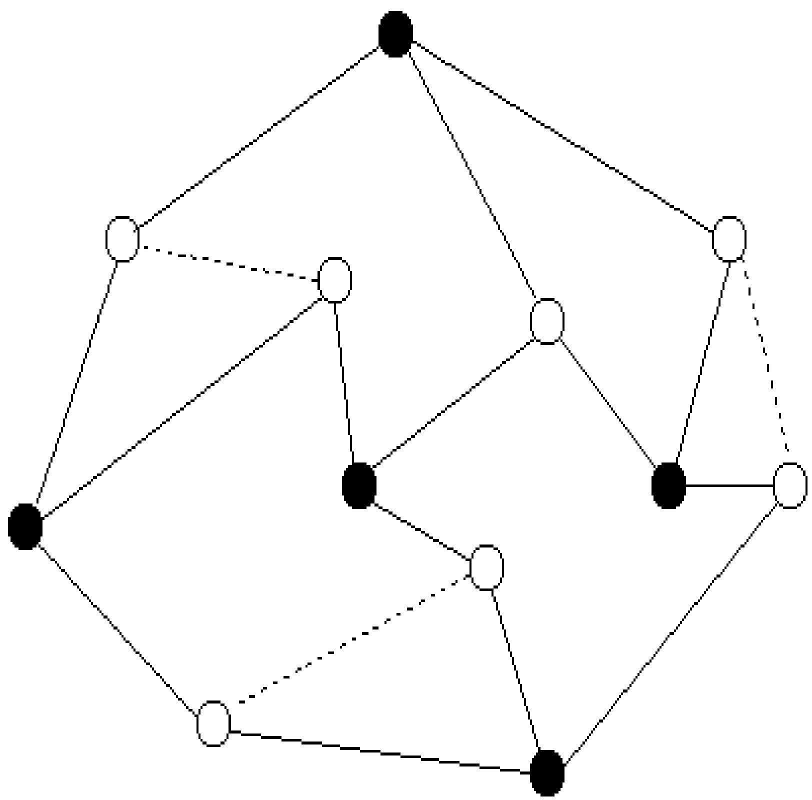
$$\gamma := \max\{x'Qx : x \in \{-1,1\}^n\}$$

  where $Q = Q'$ of mixed inertia in general.

- This needs 2^n (or 2^{n-1}) function evaluations which grows exponentially in n.

- Classical NP-hard problem with many applications in Control, Statistics, Graph Theory ("max-cut"), etc.

Example: Max-cut problem 1



Given a graph defined by:

- Vertex set $V = \{1, 2, \ldots, n\}$
- Edge set $E = \{(i, j) : i, j \in V, i \neq j\}$.
- For $S \subseteq V$ capacity is number of edges connecting a node in S to a node not in S.
- Max-Cut problem: Find $S \subseteq V$ with maximum capacity

Example: Max-cut problem 2

- Define adjacency matrix:

$$Q_{ij} = 1 \qquad \text{if } (i,j) \in E$$
$$= 0 \qquad \text{otherwise}$$

- Specify cut by vector $x \in \mathcal{R}^n$:

$$x_i = 1 \qquad \text{if } i \in S$$
$$= -1 \qquad \text{otherwise}$$

- If $(i,j) \in S$, $1 - x_i x_j = 2$. Hence capacity:

$$c(x) = \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} (1 - x_i x_j) Q_{ij}$$

- Thus max-cut can be formulated as: $\min\{x' Q x : x \in \{-1, 1\}^n\}$.

QIP: Convex relaxation - 1

- Let $D = \text{diag}(D)$, $D - Q \geq 0$. Then for all $x \in \{-1, 1\}^n$:

$$x'Qx = \underbrace{x'Dx - \text{trace}(D)} - x'(D-Q)x + \text{trace}(D)$$

$$= -x'(D-Q)x + \text{trace}(D) \quad (x_i^2 = 1)$$

$$\leq \text{trace}(D) \quad (D - Q \geq 0)$$

- This results in the convex relaxation:

$$\gamma = \max\{x'Qx : x \in \{-1,1\}^n, Q = Q'\}$$

$$\leq \min\{\text{trace}(D) : D = \text{diag}(D), D - Q \geq 0\}$$

$$:= \bar{\gamma}$$

- Let D_o be the (unique) optimal solution and define V such that $R(V) = \text{Ker}(D_o - Q)$, $V^*V = I_r$. If $r = 1$ duality gap is zero.

QIP: Convex relaxation - 2

- Solve RRQIP problem:

$$\gamma_r = \frac{1}{n}\max\{x'VV'x : x \in \{-1,1\}^n\}$$

  Then: $\gamma = \bar{\gamma} \Leftrightarrow \gamma_r = 1$.

- Provided $\gamma_r < 1$,

$$\gamma < \bar{\gamma} - n(1-\gamma_r)\lambda_+(D_o - Q) < \bar{\gamma}$$

  where

$$\lambda_+(D_o - Q) := \min_{\{i:\lambda_i(D_o-Q)>0\}} \lambda_i(D_o - Q)$$

- Note that Rank$(VV') = r < n$ (typically $r \ll n$).
- Reduced-rank problem has complexity $O(n^r)$ and can be solved efficiently using theory of zonotopes, hyperplane arrangements and linear programming [Avis, Fukuda].

General characteristics of method - 1

- μ-problem:

  $$\mu^{-1} = \min_{\Delta \in \mathbf{\Delta}, \det(I - \Delta M) = 0} \|\Delta\|$$

  is NP-hard.
- Formulate and solve dual: Get (μ_0, D_o) s.t.

  $$\min_{D \in \mathbf{D}} \|D^{1/2} M D^{-1/2}\|$$

  $$M \leftarrow \mu_0^{-1} D_o^{1/2} M D_o^{-1/2}$$

- QIP problem:

  $$\gamma := \max_{x \in \{-1,1\}^n} x' Q x$$

  is NP-hard
- Formulate and solve dual: Get $(\bar{\gamma}, D_o)$ s.t.

  $$\bar{\gamma} = \min_{D = \text{diag}(D), D - Q \geq 0} \text{trace}(D)$$

  s.t. $D = \text{diag}(D)$, $D - Q \geq 0$

General characteristics of method - 2

- Sufficient condition for zero duality gap: $\mu = \bar{\mu}$ if $m :=$ null$(I - MM^*) = 1$.
- Duality gap can be breached if we can solve a rank-m μ problem (or get a bound less than 1): $\max_{\Delta \in \mathcal{B}\Delta} \rho(\Delta U_1 V_1^*)$
- Improved bound obtained via the solution of an eigenvalue problem of dimension $4m \times 4m$.

- Sufficient condition for zero duality gap: $\gamma = \bar{\gamma}$ if $r :=$ null$(D_o - Q) = 1$.
- Duality gap can be breached if we can solve a rank-r QIP problem of the form $\gamma_r = \frac{1}{n} \max_{x \in \{-1,1\}^n} x' V V' x$.
- Improved bound can be obtained by calculating the eigenvalues of a symmetric matrix.

Conclusions and Open Issues

- Development of efficient algorithms for low-rank μ problems.
- Exploit similarities in solutions. Can method be generalized to a large class of problems involving convex relaxations (e.g. new class of primal/dual algorithms)?
- Can approach be combined with probabilistic relaxation methods?