An alternative treatment of the problem of image formation of an object through plane or spherical interfaces

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The problem of the virtual image of an object being itself in a different medium than that of the observer is discussed. In dealing with the above problem, plane or spherical refraction surfaces are replaced by lenses. By using this "trick," one can first study the lens and then proceed to study images formed by an interface. This is the reverse of what most textbooks do. This analysis is simple and suitable for students of introductory physics courses.

I. INTRODUCTION

The problem of determining the location and nature of the image of an object that is in a different optical medium than that of the observer has been studied previously. It can be found in many books of introductory physics.[1-3] However, the presentation is almost always incomplete, partly because of the complexity of the required mathematical analysis. Authors avoid a thorough discussion of the subject. As a result, students usually leave with an impression that does not correspond to reality. The conception, for example, that the virtual image of an object submerged in a liquid is always at a single location is wrong and misleading.

Bartlett *et al.*[4] in a recent article in this Journal have given a detailed theoretical and experimental analysis of the subject. They have concluded that there is no unique location of the virtual image. The location depends on the angle from which the object is viewed and the orientation of the two eyes of the observer, which means that the image is astigmatic.

In the present work we have approached the problem differently. Lenses were used instead of plane refraction surfaces for the interface between the two adjacent media. Our conclusions are similar to those of Ref. 4. Inasmuch as the mathematical demands of dealing with problems concerning lenses are not too high, our analysis offers the advantage of simplicity, which makes it appropriate for students of introductory physics.

II. ANALYSIS

A. Case 1

Let us derive the nature and location of the image I of a point source S, which is situated at a distance Y below the plane surface xx of a medium of index of refraction n_1 when the observer O is in the adjacent medium of index n_2. We assume that $n_1 > n_2$. The plane of Fig. 1(a) is the plane of incidence that contains the point S and is perpendicular to the plane boundary xx between the two media.

Let us imagine the spherical surface of radius Y having its center at the point S. Imagine that the medium of index n_1 in the interior of this surface is replaced by a medium of index n_2 as it is shown in Fig. 1(b). All rays leaving point source S are perpendicular to the spherical interface and so they pass into the medium n_1 without any deviation, and this is independent of the index of reflection of the medium in the sphere centered at S. These rays deviate when they meet the plane surface xx with an angle of incidence in the range 0°–90°. It is clear from Fig. 1 that the image of S through a plane refraction surface is identical to that formed by a lens with parameters $\{R_1 = Y, R_2 = \infty, n_1\}$, which is surrounded by a medium n_2, where the object distance is Y.

Assuming paraxial rays, which in our case implies that the corresponding equivalent lens behaves as a thin one, we can use the lensmaker's formula

$$\frac{1}{f} = \left(\frac{n_1}{n_2} - 1\right)\left(\frac{1}{(-Y)} - \frac{1}{\infty}\right),$$

from which we have the focal length

$$f = -[n_2/(n_1 - n_2)]Y. \quad (1)$$

Since in our case $n_1 > n_2$, Eq. (1) leads to $f < 0$. This means that the assumed lens is a diverging one. If Y' is the distance of the virtual image, then

$$1/Y + 1/Y' = 1/f. \quad (2)$$

Eliminating f between Eqs. (1) and (2), we finally have

$$Y' = -(n_2/n_1)Y. \quad (3)$$

The last result means that the virtual image is located at a distance $(n_2/n_1)Y$ underneath the plane surface xx.

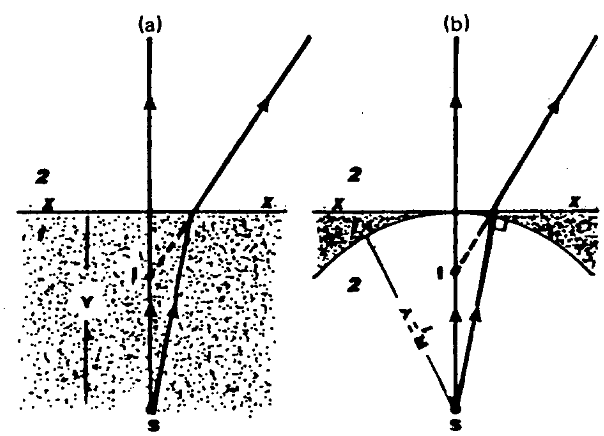

Fig. 1. (a) The formation of the image I of a point source S through a plane interface. S is located in a medium whose index of refraction is n_1 and the observer in a medium of index n_2 with $n_1 > n_2$. Additionally, the position of the observer is close to the perpendicular drawing from S toward the separating surface of the two media in order that only the vertical diverging of the rays be significant. (b) The geometrical representation of the equivalent problem of (a) where the plane interface has been substituted by a diverging lens.

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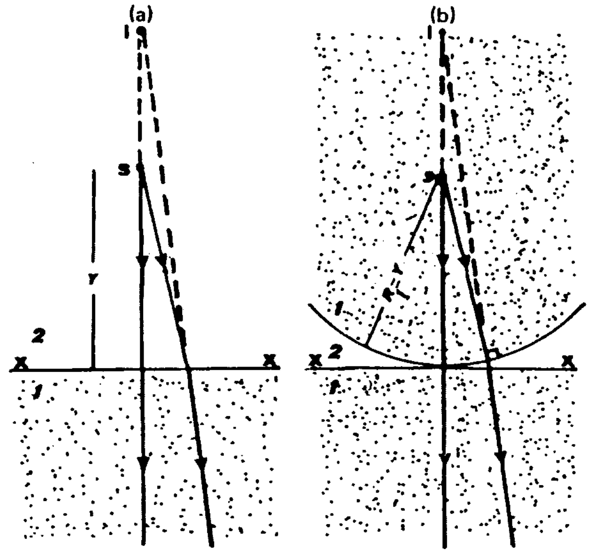

Fig. 2. The formation of the image of an object through a plane interface with the observer placed within the medium of larger index of refraction. (a) and (b) correspond to Figs. 1(a) and 1(b), respectively.

B. Case 2

We shall study now the image of a point source located within a medium that is characterized by a smaller refraction index than that which surrounds the observer. A similar analysis as in case 1 led to the conclusion that the plane surface xx of Fig. 2(a) and the lens $\{R_1 = Y, R_2 = \infty, n_2\}$, being in an environment with refraction index n_1 of Fig. 2(b), are equivalent optical systems as regards the formation of the image I.

The focal length of the lens is

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{(-Y)} + \frac{1}{\infty}\right),$$

from which we have

$$f = [n_1/(n_1 - n_2)]Y. \tag{4}$$

In relations (1) and (4), n_1 and n_2 appear interchanged. This is justified by the fact that in the first case the lens is made from material n_1 surrounding material n_2, and in the second case the opposite happens.

The fact that f is a positive quantity means that the planoconcave lens of Fig. 2(b) behaves like a converging one. From Eq. (4) we obtain $f/Y = n_1/(n_1 - n_2) > 1$, which means that the image I is always virtual.

C. Case 3

Our method could be easily used for spherical interfaces as well. Figure 3(b) presents the optically equivalent system of Fig. 3(a) as regards the formation of the image through a curved surface separating the two media. The focal length of the lens in Fig. 3(b) is given by

$$\frac{1}{f} = \left(\frac{n_1}{n_2} - 1\right)\left(\frac{1}{(-Y)} - \frac{1}{R}\right),$$

for the case of paraxial rays. For the special arrangement with $n_1 = 1$, $n_2 = 2$, $R = 10$ cm, and $Y = 20$ cm, we calculate $f = \frac{40}{3}$ cm. This value indicates that the lens behaves as a converging one. The fact that $Y > f$ means that the image will be a real one and its location Y' could be found from the relation $1/Y + 1/Y' = 1/f$. We estimate $Y' = 40$ cm.

III. DISCUSSION

The three previous examples lead to the conclusion that a plane or a spherical refraction surface can be generally substituted for an optically equivalent lens as far as the image of a point source is concerned. In this sense, defects appearing in the image could be studied in the same way that we study the errors usually caused by lenses, such as spherical aberration, color dispersion, and astigmatism. When, for example, the observer is near the vertical ($\theta_1 \simeq 0°$) from S to the interface surface, the rays which enter his or her eyes have a refraction angle $\theta \simeq 0°$. Obviously, one can consider these rays as passing through the central area of the equivalent "liquid lens." In this case we have to do with the image formed from a thin lens illuminated by paraxial rays, and of course the image will be without errors located at a depth $Y' = (n_1/n_2)Y$ below the separating surface of the two media. When the observer is far away from the vertical, the rays entering his or her eyes pass from parts of the "liquid lens" that cannot be considered as parts of a thin lens. Nor could the rays be considered as paraxial. Hence we have to consider the image formed from a thick lens illuminated from nonparaxial rays. In this case errors are introduced. It is expected[5] that the virtual image of the point source will be an astigmatic one. The location of the different components of this astigmatic image depend upon the refraction angle θ_2. So it is known that when the angle of incidence is $\theta = \arcsin(n_1/n_2)$, then the angle of refraction θ is 90°. Thus the rays emerging with $\theta = 90°$ give a virtual image formed at zero distance from the interface. On the other hand, rays

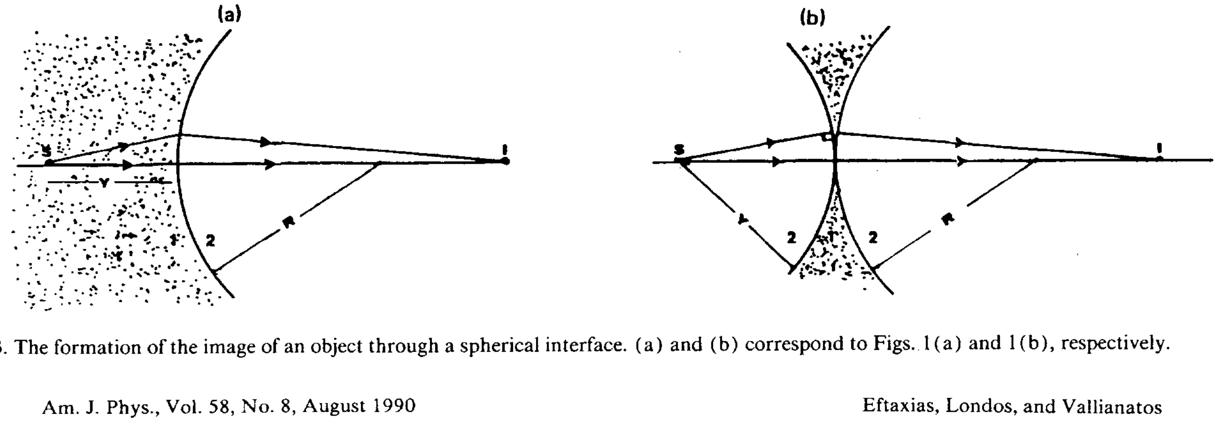

Fig. 3. The formation of the image of an object through a spherical interface. (a) and (b) correspond to Figs. 1(a) and 1(b), respectively.

emerging with $\theta_2 \simeq 0°$ give a virtual image at a distance $Y/(n_1/n_2)$ below the interface. The quantity Y'/Y therefore varies between $1/(n_1/n_2)$ and 0.

IV. CONCLUSIONS

In conclusion, we have dealt with the subject of the image formation of an object through interfaces in a different way than usually appears in the literature. More specifically, this way is the reverse of what most textbooks do. We suggest that first we have to study the lens and then the interface since the latter could be related to the former. The essential point of our approach is not only simplicity, but also self-sufficiency. This method seems more suitable for students in introductory physics courses than the conventional method. Moreover, the problem of the defect in the image formation can more easily and understandably be treated through the errors that appear in the cases of lenses than at discrete refracting surfaces.

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Simple forms for equations of rays in gradient-index lenses

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The calculation of the shape of an optical ray in a gradient-index lens is often greatly simplified by the use of formalism in which the equation governing the ray assumes the form of Newton's law of motion. A new justification of this method is presented, and the method is applied to a gradient-index lens with cylindrical symmetry.

I. INTRODUCTION

An article published recently in this Journal[1] discussed an especially simple version of the optical–mechanical analogy: The equation governing a ray of light in a medium of varying index of refraction can be cast into the form of Newton's law of motion.

This formulation of geometrical optics has both theoretical and practical advantages. On the theoretical side, it presents the optical–mechanical analogy in a much simpler form than usual. (The traditional expression of the analogy relies upon the Hamilton–Jacobi equation and a rather esoteric formulation of the laws of mechanics.) From the practical point of view, the "$F = ma$" formulation of geometrical optics offers substantial calculational advantages. Many problems involving gradient-index media are solved more easily with this approach than with any other.

The present article serves two purposes: (1) It offers a new proof of this formulation of geometrical optics. The new proof relies less on calculation and more on physical argument and, it is to be hoped, makes the physical basis of the analogy clearer. (2) The formalism of "$F = ma$" optics is applied to examples somewhat more complex than those treated in the earlier article. In particular, one example treats a case in which the ray is three-dimensional, i.e., not confined to a plane. The facility with which the equations governing the ray may be obtained and solved is a strong argument for the utility of this method of calculation.

II. THE FORMALISM

Many problems in gradient-index optics can be solved most easily by means of a formalism in which the equation governing the optical ray assumes the form of Newton's second law:[2]

$$\frac{d^2\mathbf{x}}{da^2} = \nabla\left(\frac{1}{2}n^2\right). \tag{1}$$

The position $\mathbf{x}$ of a light pulse moves along a ray through a region of varying index of refraction $n(\mathbf{x})$. The equation governing the motion of the light along the ray takes the form of Eq. (1) when we use as independent variable, not the time t, but a stepping parameter a that is defined by the relation

$$\left|\frac{d\mathbf{x}}{da}\right| = n. \tag{2}$$

In Eq. (1), the optical analog of the potential energy is $-\frac{1}{2}n^2$ and the optical analog of the mass is the number 1. With the identifications

$$t \to a,$$
$$m \to 1,$$
$$\mathbf{x}(t) \to \mathbf{x}(a),$$
$$U(\mathbf{x}) \to -\tfrac{1}{2}n^2(\mathbf{x}),$$

Eq. (1) is formally equivalent to Newton's law of motion,

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