The spacetime exterior to rotating neutron stars and the possibility of constraining the equation of state.

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- An analytical spacetime for the exterior of NS,
- A few words on multipole moments,
- Using QPOs to constrain the EOS.

An analytical spacetime for the exterior of NS.

When doing Astrophysics around compact objects (accretion disks, jets), we have to specify the geometry of the space time.

Usual choices for the background geometry:

- Schwarzschild geometry,
- Kerr geometry,
- Approximate Spacetimes (slow rotation)
- Numerical spacetimes.

Choosing an analytical spacetime.

There is a large variety of analytical solutions of axisymmetric vacuum spacetimes (Ernst, Manko and Sibgatullin).

Such an analytical solution, appropriately matched to a neutron star, could be used to describe the stationary properties of the spacetime.

(trajectories, location of ISCO, Ω of circular orbits, epicyclic frequencies Ω_ρ, Ω_z)

If we decide to use analytical solutions, there are three issues:

- Choosing an analytical solution from the variety,
- Matching the solution to the compact object,
- Comparing the analytical to the numerical spacetime.

Choosing an analytical axisymmetric solution: The Two Soliton.

The stationary and axisymmetric spacetime in vacuum:

$$ds^2 = -f\,(dt - \omega d\phi)^2 + f^{-1}\left[e^{2\gamma}\left(d\rho^2 + dz^2\right) + \rho^2 d\phi^2\right], \quad (1)$$

Einsteins field equations in vacuum reduce to the Ernst equation

$$Re(\mathcal{E})\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot\nabla\mathcal{E}, \qquad (2)$$

where the Ernst potential is a complex function, $\mathcal{E} = f + i\psi$.

We have a general procedure for generating solutions of the Ernst equation (Manko and Sibgatullin):

First, choose an ansatz for the Ernst potential on the axis of the form:

$$\mathcal{E}(\rho = 0, z) = e(z) = \frac{P(z)}{R(z)}, \tag{3}$$

where, $P(z), R(z)$ are polynomials of order n in z with complex coefficients. Then the algorithm will generate a solution that will depend on the parameters and form of the potentials.

The Two Soliton solution (Manko):

$$e(z) = \frac{(z - M - ia)(z + ib) - k}{(z + M - ia)(z + ib) - k} \tag{4}$$

M, a are the mass and the reduced angular momentum $\frac{J}{M}$. All the parameters are real for equatorial symmetry.

Multipolar structure (Fodor):

$$\begin{aligned} M_0 &= M, \quad M_1 = 0, \quad M_2 = -(a^2 - k)M, \quad M_3 = 0, \\ J_0 &= 0, \quad J_1 = aM, \quad J_2 = 0, \\ J_3 &= -(a^3 - (2a - b)k)M, \quad J_4 = 0. \end{aligned} \tag{5}$$

Matching the solution: The multipole moments.

For a numerical spacetime of a neutron star model, it is possible to evaluate its mass moments M, $Q, ...$ and angular momentum moments J, $S_3, ...$

These numerically evaluated moments can be used to impose the matching conditions to the analytical spacetime. The first four nonzero multipole moments of the Two Soliton solution are:

$$\begin{aligned} M_0 = M, \qquad & J_1 = aM, \\ M_2 = -(a^2 \quad - \quad & k)M, \\ J_3 = -(a^3 \quad - \quad & (2a-b)k)M, \end{aligned} \tag{6}$$

Comparing the analytical to the numerical: Criteria.

Criteria that are characteristic of the geometric structure of the spacetime (metric, geodesics).

Related to properties of the spacetime that could be measured in astrophysical phenomena.

Criteria:

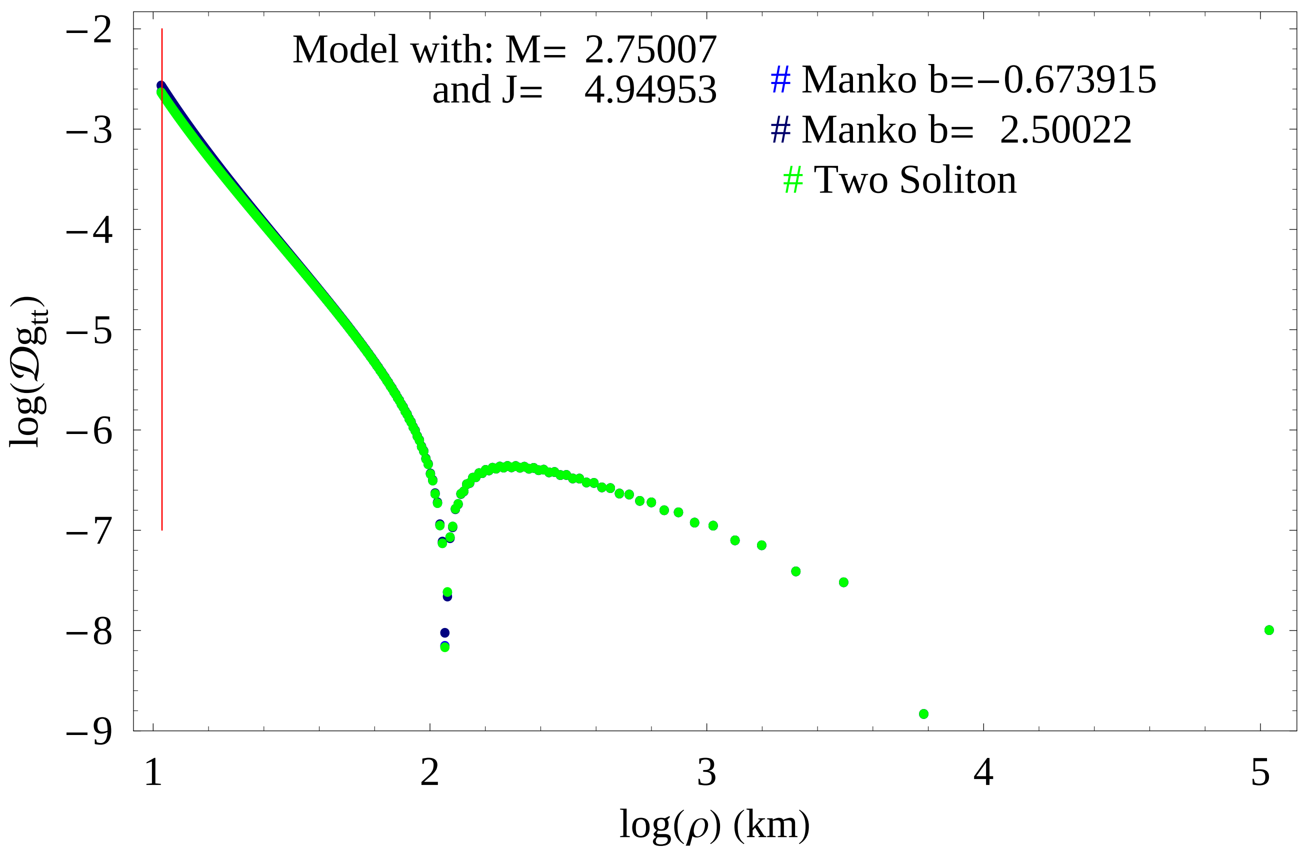
- How well the components of the analytic and the numerical metric compare,

- Comparison of the innermost stable circular orbit (ISCO),

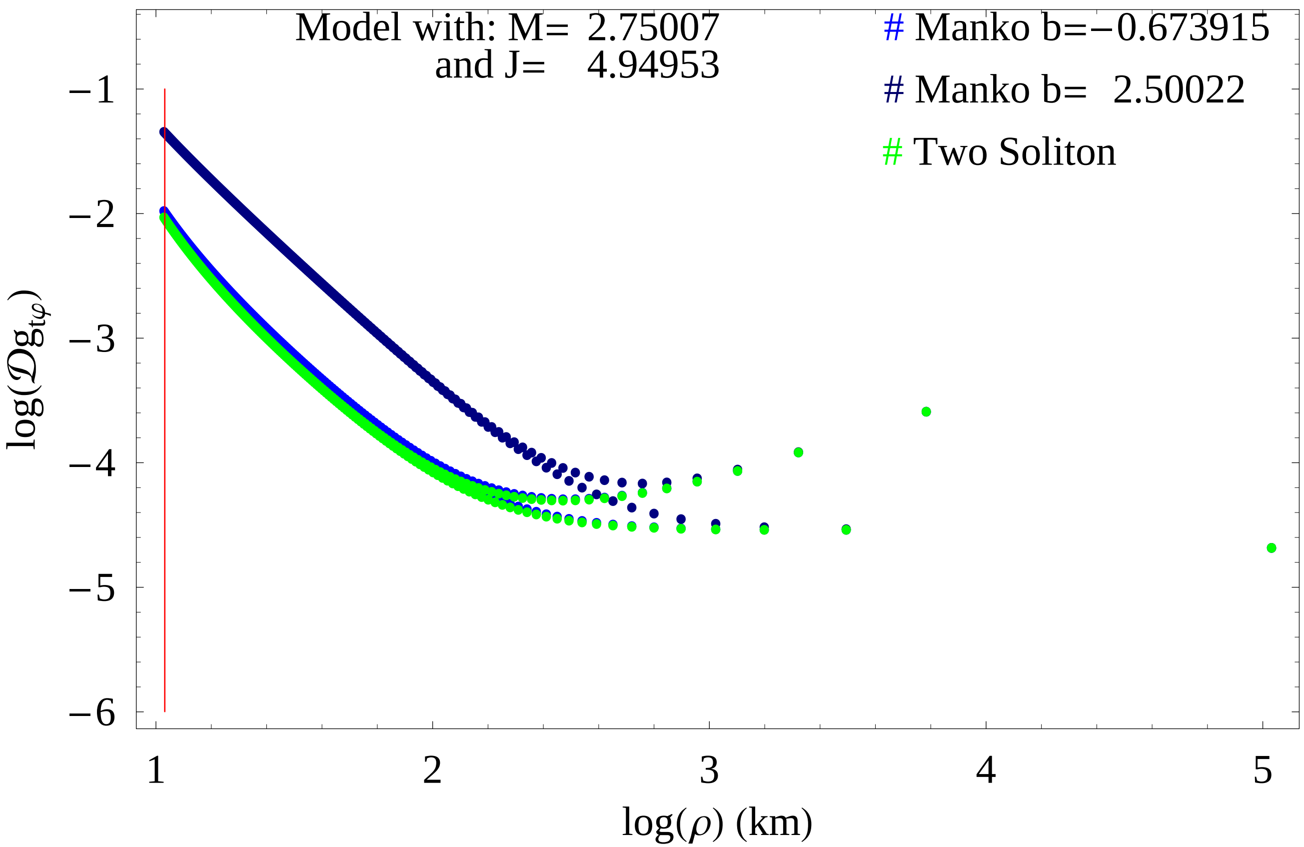
- Comparison of the rotation frequency of circular orbits on the equatorial plane,

- Comparison of the epicyclic frequencies Ω_ρ, Ω_z.

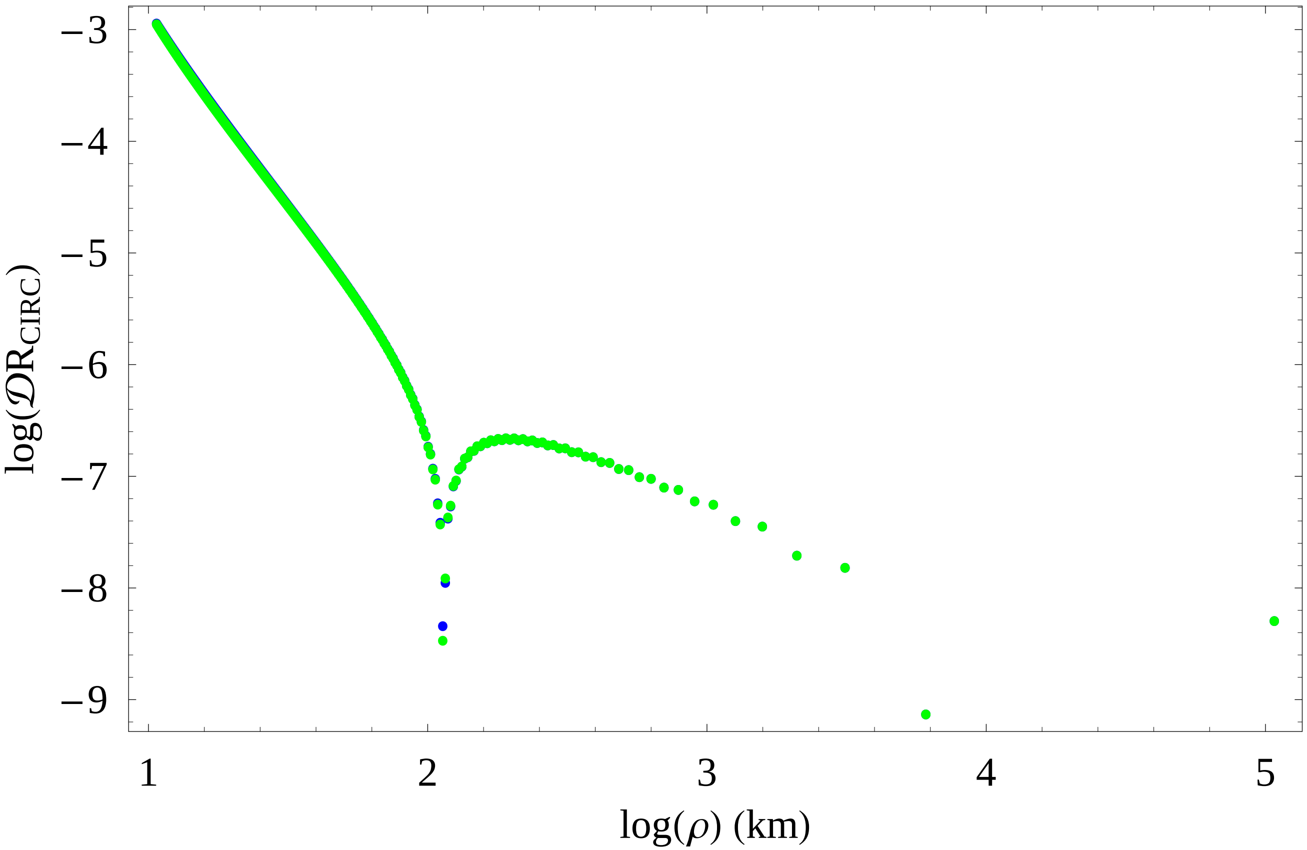
Results:



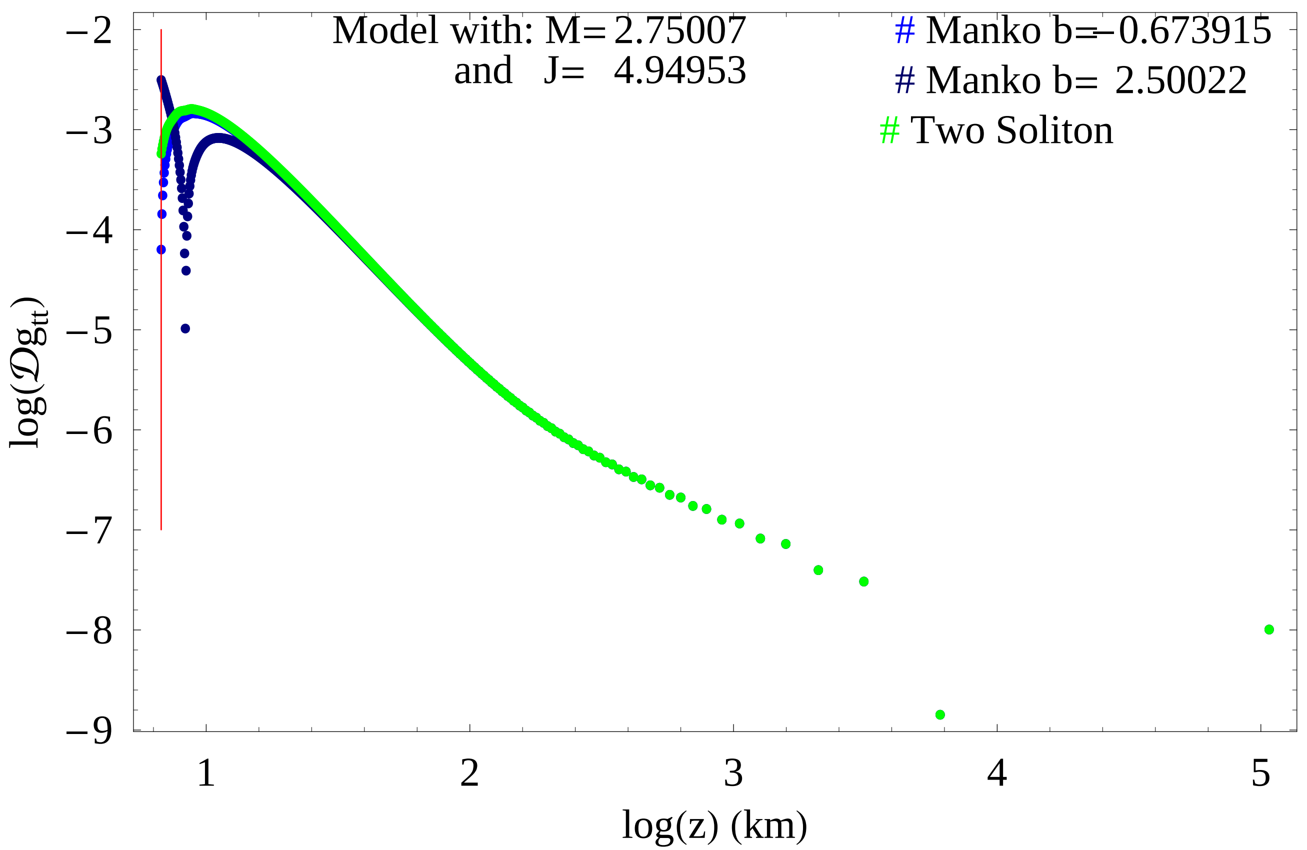
$\frac{\Delta g_{tt}}{g_{tt}}$ for the model with $j = 0.65$.



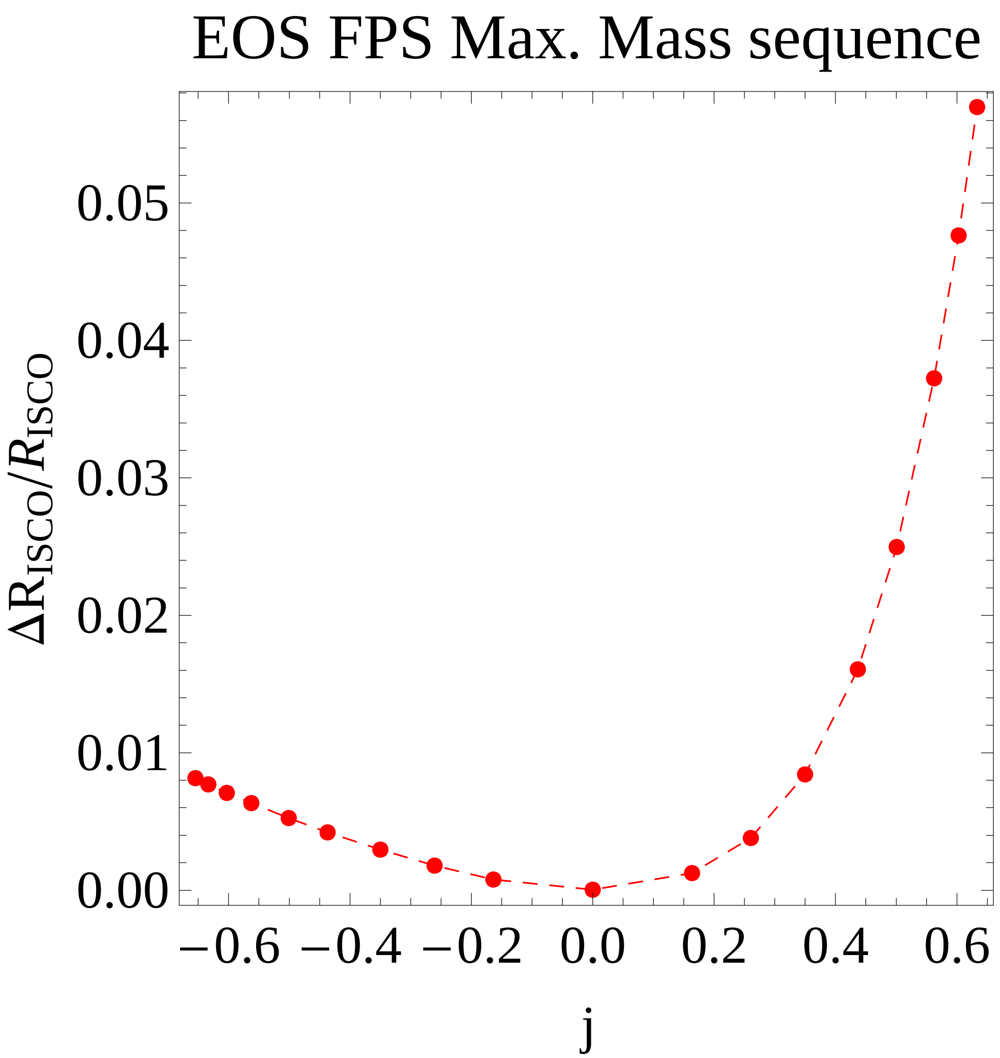
$\frac{\Delta g_{t\phi}}{g_{t\phi}}$ for the model with $j = 0.65$.



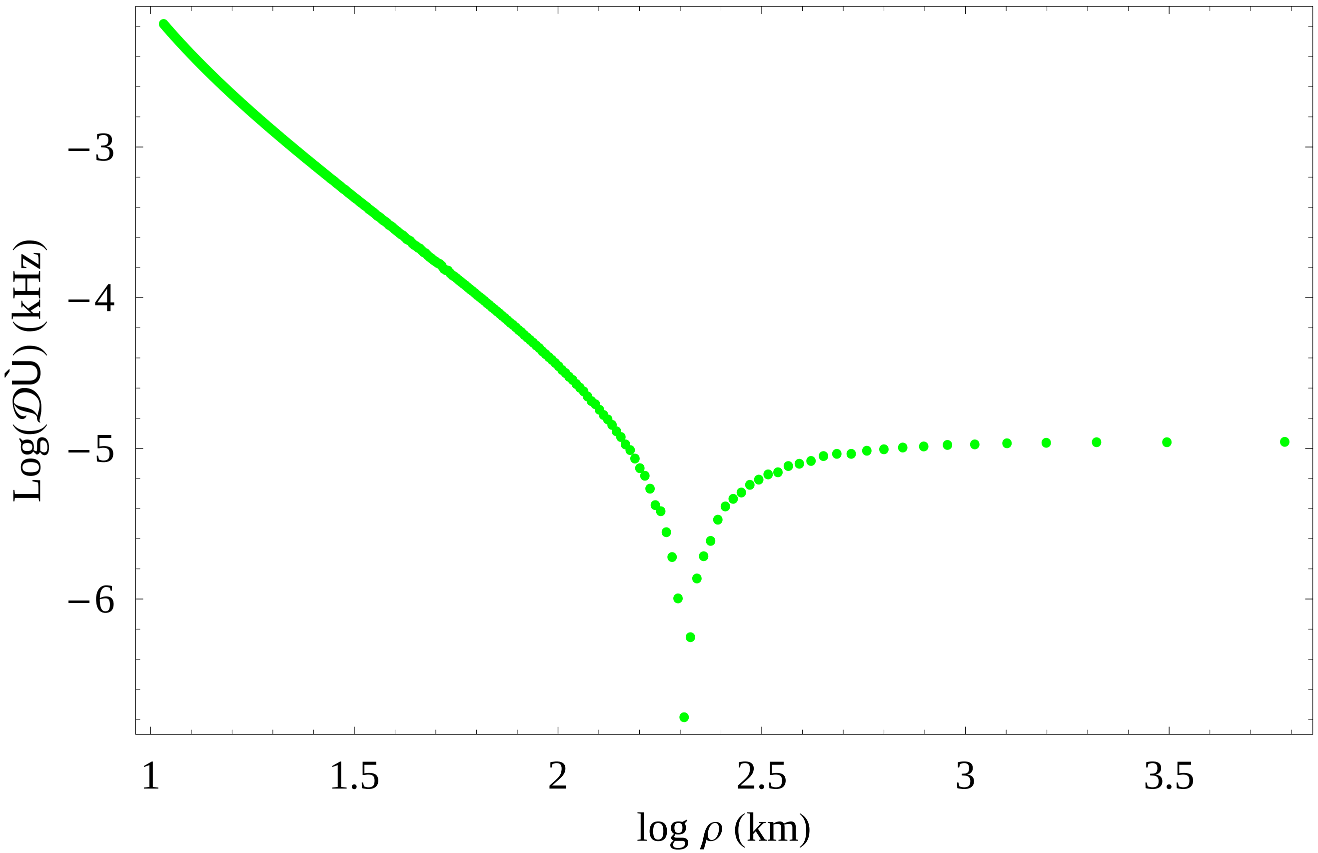
$\frac{\Delta R_{circ}}{R_{circ}}$ for the model with $j = 0.65$.



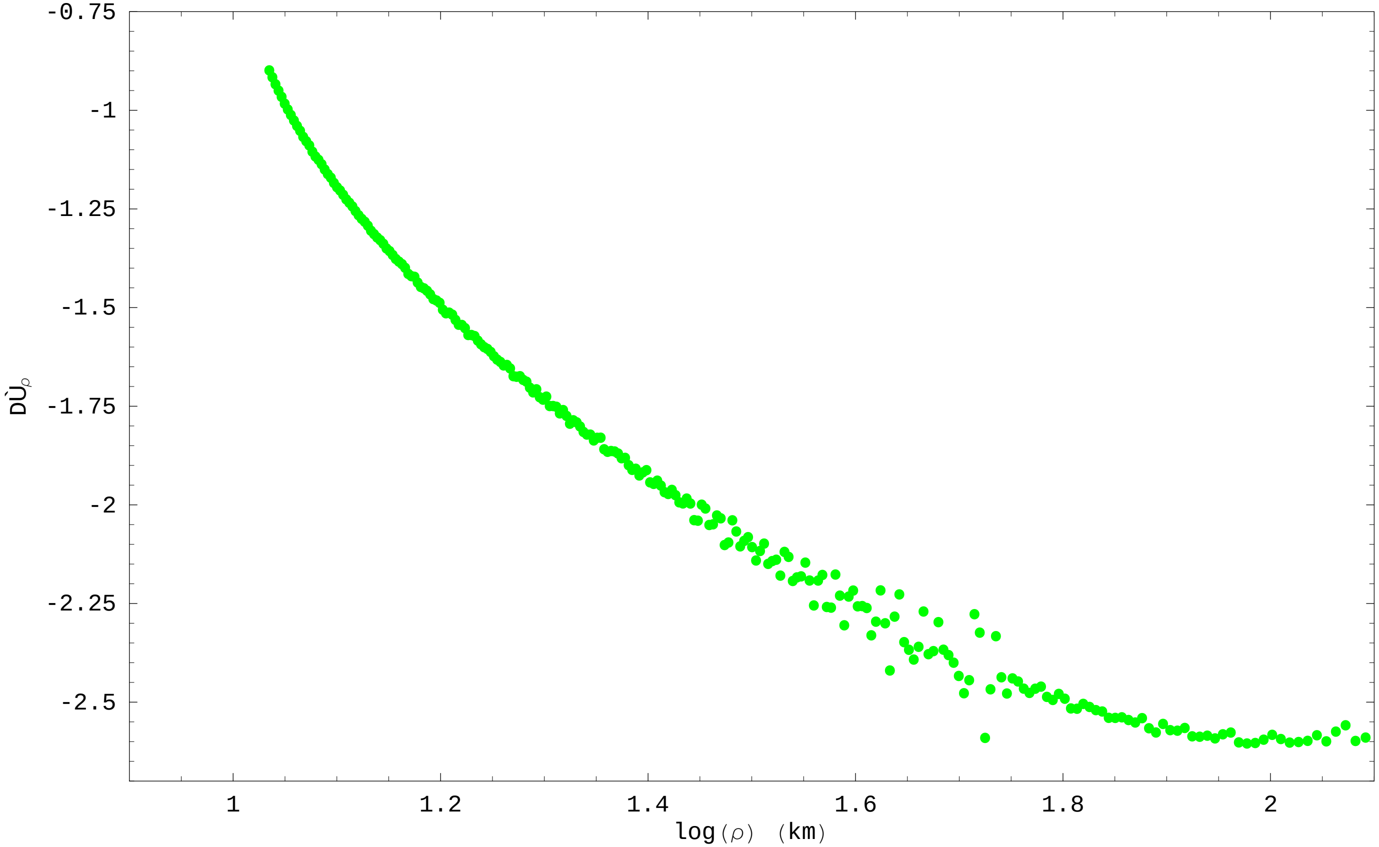
$\frac{\Delta g_{tt}}{g_{tt}}$ for the model with $j = 0.65$.



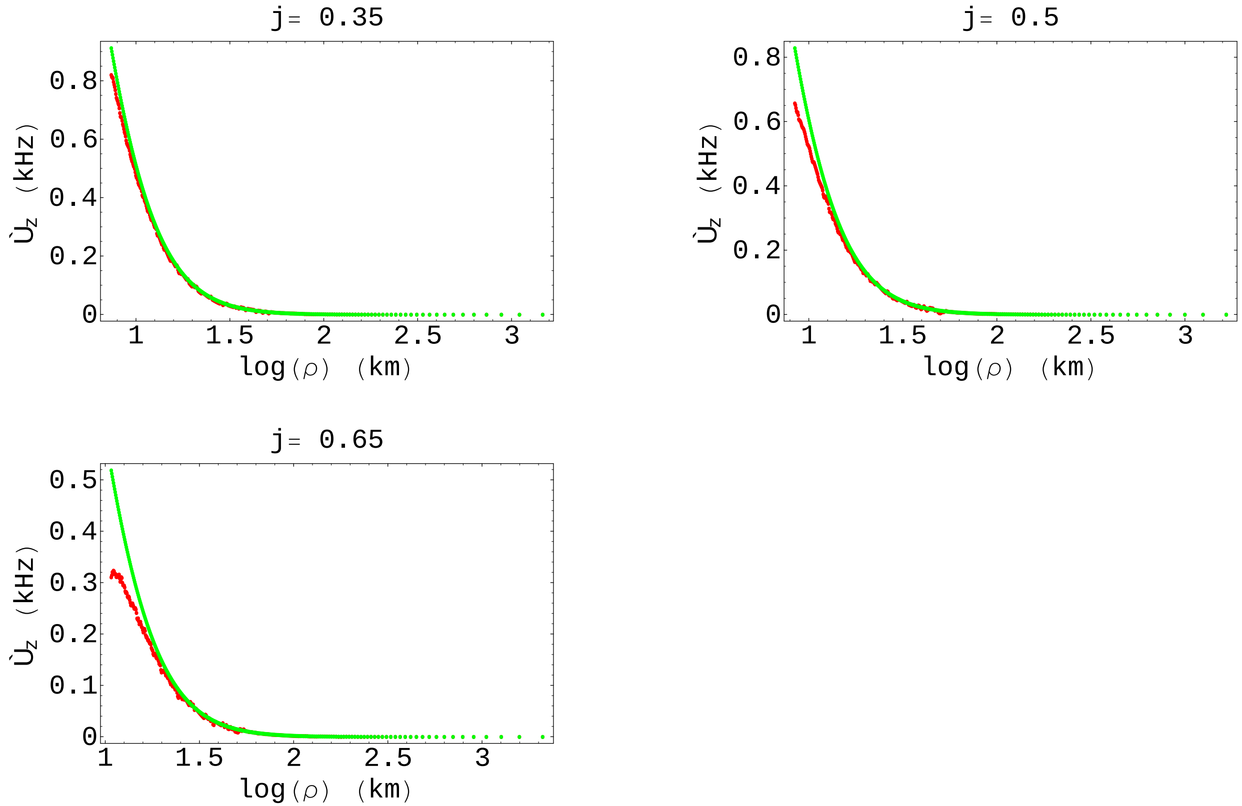
$\frac{\Delta R_{ISCO}}{R_{ISCO}}$ for co-rotating and counter-rotating circular orbits.



$\frac{\Delta\Omega}{\Omega}$ for the model with $j = 0.65$.



$\frac{\Delta\Omega_\rho}{\Omega_\rho}$ for the model with $j = 0.65$.



j= 0.35
j= 0.5
j= 0.65
$\dot{U}_z$ (kHz)
log(ρ) (km)

A few words on multipole moments.

Why Multipole moments?

- Global properties of the spacetime (the multipole structure determines the geometry),

- Relevant in gravitational wave astronomy (the observables can be related to the moments),

- Constraining the equation of state for NS (from the relation between M, S, Q),

Newtonian multipole moments:

$$\Phi(r) = \frac{\mathcal{Q}}{r} + \frac{\mathcal{Q}_a x^a}{r^3} + \frac{\mathcal{Q}_{ab} x^a x^b}{r^5} + ... \tag{7}$$

where, $\mathcal{Q}$, $\mathcal{Q}_a$, $\mathcal{Q}_{ab}$, are some integrals on the source

$$\mathcal{Q} = \int \rho d^3x,\ \mathcal{Q}_a = \int x^a \rho d^3x,\ ... \tag{8}$$

The multipole moments are generally tensorial quantities.

Definition of the moments at infinity:

$x^a \to \tilde{x}^a = r^{-2}x^a$: $\tilde{r}^2 = \tilde{x}^a\tilde{x}_a = r^{-2}$

$$\Phi(r) = \tilde{r}\left(\mathcal{Q} + \mathcal{Q}_a x^a + \mathcal{Q}_{ab}x^a x^b + ...\right) \tag{9}$$

If we define the potential at infinity $\tilde{\Phi} = \tilde{r}^{-1}\Phi$ then the moments are

$$P_{a_1...a_n} = \tilde{D}_{a_n}P_{a_1...a_{n-1}} = \tilde{D}_{a_1}...\tilde{D}_{a_n}\tilde{\Phi} \tag{10}$$

Relativistic multipole moments:

- Generalization of the Newtonian moments,

- Defined for asymptotically flat spacetimes at infinity from a "potential" by a recursive relation,

- There are two sets of moments, the Mass moments and the Rotation moments,

- For the two sets of moments we have two generating potentials, Φ_M, Φ_J, Multipole moments for stationary and axisymmetric spacetimes:

- An axisymmetric vacuum spacetime is described by the Ernst potential $\mathcal{E}$,

- The two generating potentials are given as the real (mass) an imaginary (rotation) part of the potential $\xi = \frac{1-\mathcal{E}}{1+\mathcal{E}}$,

- Because of the rotation symmetry, the moments can be reduced from tensors to scalars,

- The generating potential at infinity that produces the moments from the appropriate recursive relation is $\tilde{\xi} = \tilde{r}^{-1}\xi$

Using the multipole moments to construct a space-time.

As we have seen from the Newtonian case, the multipole moments are like the coefficients of a series expansion of the potential $\tilde{\xi}$.

Thus they can be used to construct the potential $\tilde{\xi}$

$$\tilde{\xi} = \sum_{i,j=0}^{\infty} a_{ij} \tilde{\rho}^i \tilde{z}^j \tag{11}$$

where the coefficients a_{ij} are functions of the moments.

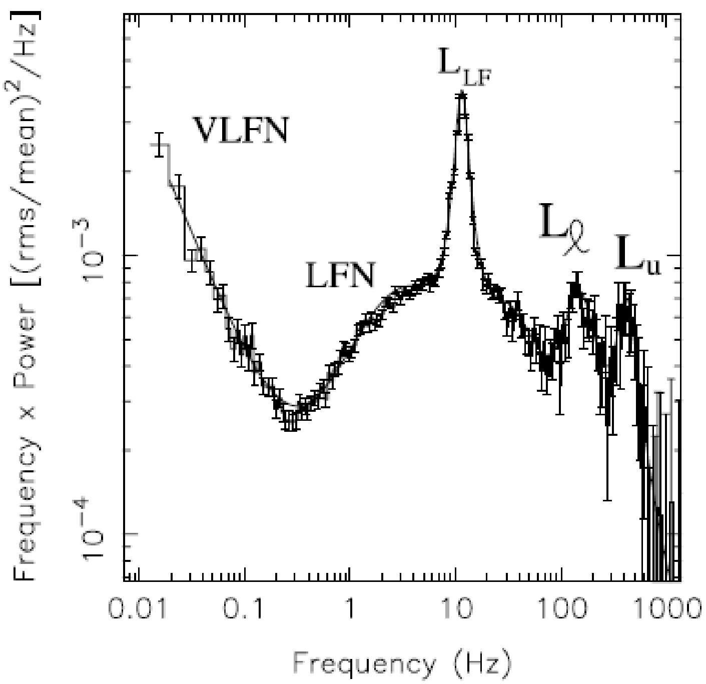
From the $\tilde{\xi}$ we can construct the Ernst potential and from the Ernst potential the metric.

Using QPOs to constrain the EOS.

Low Mass X-ray Binaries (LMXBs)

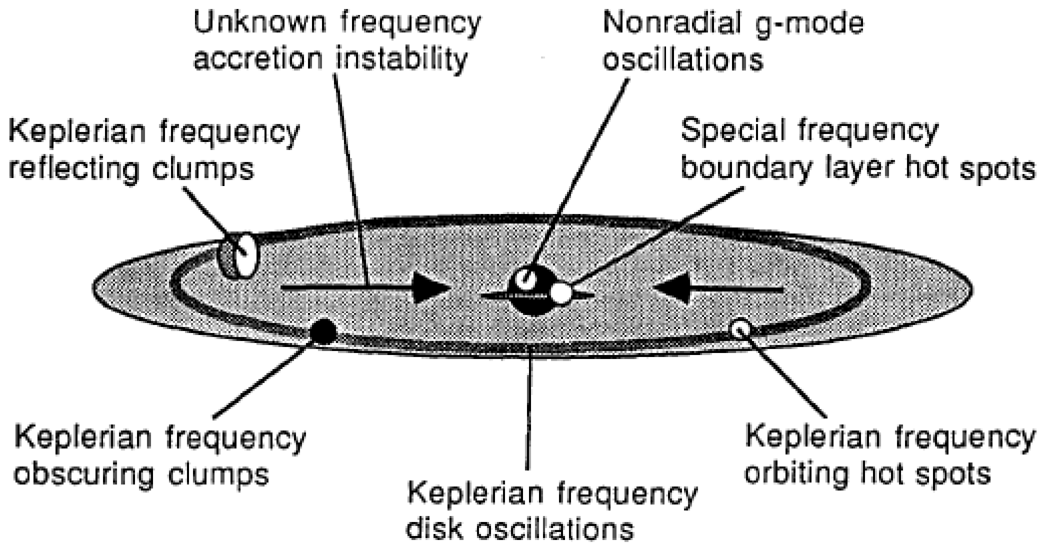
Binary systems where a compact object (Black Hole or Neutron Star) has an accretion disk with X-ray emission.

These systems have demonstrated quasi-periodic variability in their X-ray flux (QPOs).



Boutloukos et al. (2006)

There have been proposed various models for the observed variability



We will be interested in the relativistic precession model.

In the relativistic precession model, the observed frequencies could be related to:

- The orbital frequency on the equatorial plane

$$\Omega = \frac{-g_{t\phi,\rho} + \sqrt{(g_{t\phi,\rho})^2 - g_{tt,\rho} g_{\phi\phi,\rho}}}{g_{\phi\phi,\rho}},$$

- The precession frequencies of the periastron and of the orbital plane

$$\Omega_\alpha = \Omega - \left[-\frac{g^{\alpha\alpha}}{2} \left((g_{tt} + g_{t\phi}\Omega)^2 \left(\frac{g_{\phi\phi}}{\rho^2} \right)_{,\alpha\alpha} - 2(g_{tt} + g_{t\phi}\Omega)(g_{t\phi} + g_{\phi\phi}\Omega) \left(\frac{g_{t\phi}}{\rho^2} \right)_{,\alpha\alpha} \right.\right.$$
$$\left.\left. + (g_{t\phi} + g_{\phi\phi}\Omega)^2 \left(\frac{g_{tt}}{\rho^2} \right)_{,\alpha\alpha} \right) \right]^{1/2},$$

- Or the respective oscillation frequencies of the orbits (radial and vertical oscillations).

The orbital frequency and the precession frequencies can be related to the multipole moments of the spacetime.

The potential $\tilde{\xi}$ contains information about the moments. From that we can construct the metric functions,

which enter the calculation of the orbital frequency, so we can have Ω expressed as

$$\Omega = (M/\rho^3)^{1/2}(1 + \text{series in } \rho^{-1/2}). \tag{12}$$

That expression can be inverted to give

$$1/\rho = (\Omega^2/M)^{1/3}(1 + \text{series in } \Omega^{1/3}). \tag{13}$$

Since we are on the equatorial plane $z = 0$.

In the same way Ω_ρ and Ω_z can be expressed as series in $1/\rho$ with the coefficients depending on the moments.

If we replace the $1/\rho$ dependence in the precession frequencies, we get the following expressions relating the precession frequencies with the orbital frequency (Ryan, 1995)

$$\frac{\Omega_\rho}{\Omega} = 3v^2 - 4\frac{S_1}{M^2}v^3 + (\frac{9}{2} - \frac{3M_2}{2M^3})v^4 - 10\frac{S_1}{M^2}v^5 + (\frac{27}{2} - 2\frac{S_1^2}{M^4} - \frac{21M_2}{2M^3})v^6 + ...$$

$$\frac{\Omega_z}{\Omega} = 2\frac{S_1}{M^2}v^3 + \frac{3M_2}{2M^3}v^4 + (7\frac{S_1^2}{M^4} + 3\frac{M_2}{M^3})v^6 + (11\frac{S_1 M_2}{M^5} - 6\frac{S_3}{M^4})v^7 + ...$$

where $v = (M\Omega)^{1/3}$ is the orbital velocity in the newtonian limit.

Thus we have related the two precession frequencies to the orbital frequency.

In the QPO sources we observe at the same time several frequencies in the range from 0.01 up to 1000 Hz. Some sources also present the effect of twin kHz QPOs.

In the literature authors suggest that some of these frequencies could be identified with the orbital and precession frequencies (Stella and Vietri 1998,1999)

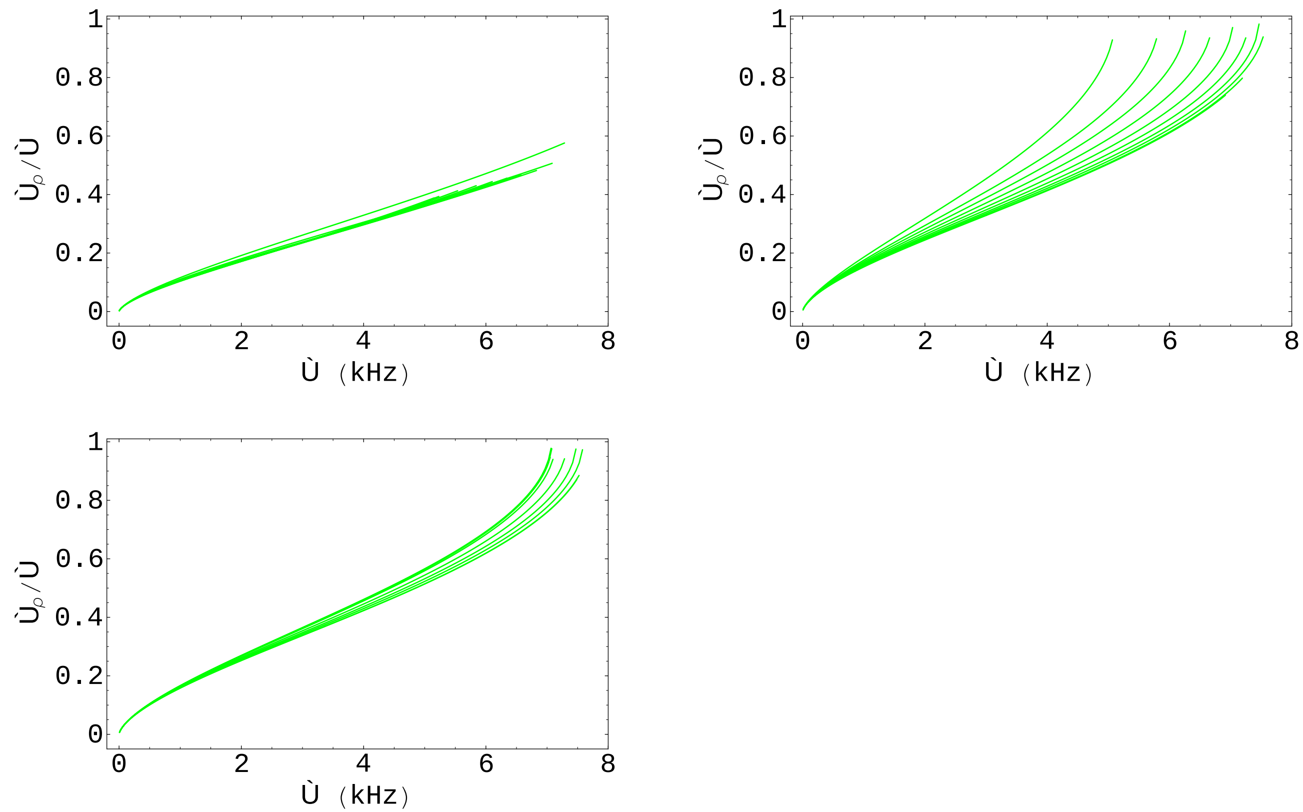
Our suggestion is that if we can identify these QPO frequencies with the orbital and precession frequencies, then from several observations from a particular source that cover various radii, we can use the previews expressions to estimate the multipole moments.

We use the Two Soliton analytical solution to produce mock frequencies and test how well the expansions

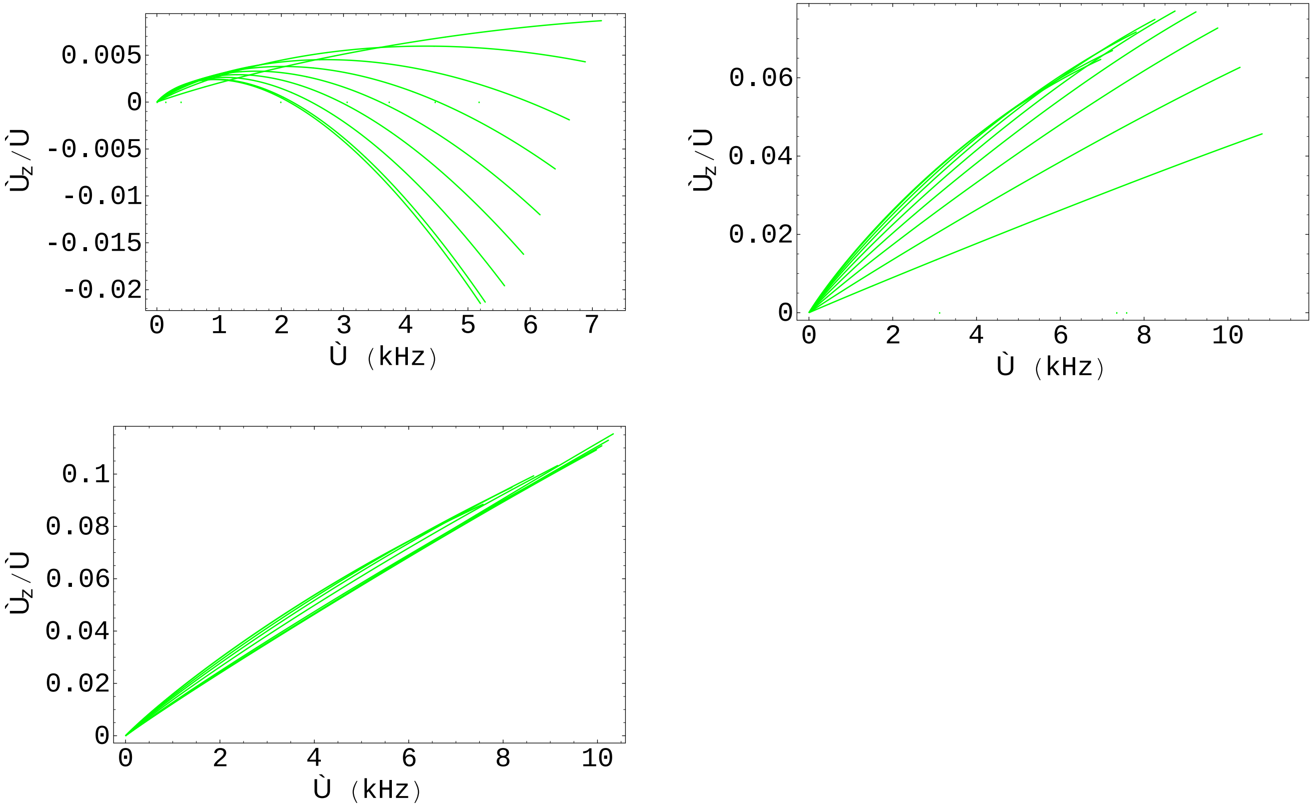
$$\frac{\Omega_\rho}{\Omega} = \sum_{n=2}^{\infty} R_n \upsilon^n$$

$$\frac{\Omega_z}{\Omega} = \sum_{n=3}^{\infty} Z_n \upsilon^n.$$

estimate the mass, the spin and the quadrupole ($M,\ S_1,\ M_2$).



$$\frac{\Omega_\rho}{\Omega} = f(\Omega) \text{ for sequences of const. rest mass.}$$



$\frac{\Omega_z}{\Omega} = f(\Omega)$ for sequences of const. rest mass.

The results of the fits are:

1st sequence of constant rest mass

ϵ_c $(10^{15} g/cm^3)$	M (km)	$j = \frac{S_1}{M^2}$	$q = \frac{M_2}{M^3}$	$s_3 = \frac{S_3}{M^4}$	$\frac{\Delta M}{M}$	$\frac{\Delta j}{j}$	$\frac{\Delta q}{q}$
0.4326	2.08	0	0	0	0.0007	-	-
0.4266	2.071	0.194	-0.307	-0.122	0.0004	0.021	0.19
0.4188	2.076	0.325	-0.833	-0.559	0.0004	0.013	0.07
0.4111	2.08	0.417	-1.345	-1.162	0.0005	0.011	0.04
0.4045	2.083	0.483	-1.775	-1.779	0.0005	0.01	0.04
0.3980	2.087	0.543	-2.209	-2.494	0.0006	0.01	0.03
0.3916	2.09	0.598	-2.639	-3.286	0.0007	0.01	0.03
0.3853	2.095	0.65	-3.076	-4.172	0.0007	0.01	0.03
0.3800	2.096	0.69	-3.442	-4.968	0.0008	0.01	0.03
0.3790	2.098	0.699	-3.516	-5.138	0.0008	0.009	0.03

2nd sequence of constant rest mass

ϵ_c $(10^{15} g/cm^3)$	M (km)	$j = \frac{S_1}{M^2}$	$q = \frac{M_2}{M^3}$	$s_3 = \frac{S_3}{M^4}$	$\frac{\Delta M}{M}$	$\frac{\Delta j}{j}$	$\frac{\Delta q}{q}$
1.4700	3.99	0	0	0	0.014	-	-
1.2010	4.01	0.178	-0.052	-0.014	0.006	0.25	9.1
1.0639	4.03	0.28	-0.133	-0.058	0.004	0.097	2.17
0.9552	4.05	0.376	-0.249	-0.148	0.002	0.046	0.73
0.8692	4.07	0.458	-0.385	-0.285	0.0015	0.024	0.29
0.8017	4.1	0.529	-0.529	-0.458	0.001	0.013	0.13
0.7495	4.12	0.588	-0.672	-0.657	0.0007	0.008	0.06
0.7101	4.14	0.635	-0.803	-0.859	0.0005	0.005	0.03
0.6729	4.16	0.682	-0.95	-1.108	0.0003	0.002	0.005
0.6600	4.168	0.7	-1.00	-1.211	0.0002	0.001	0.003

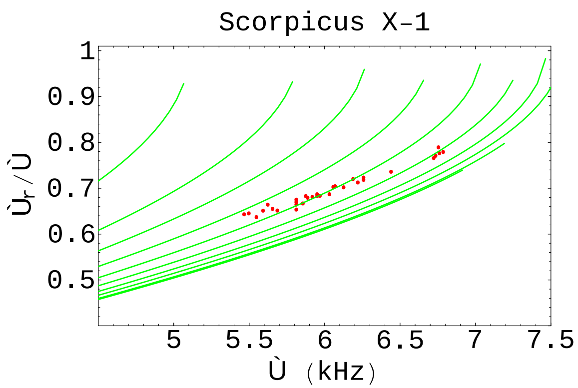
3rd sequence of constant rest mass

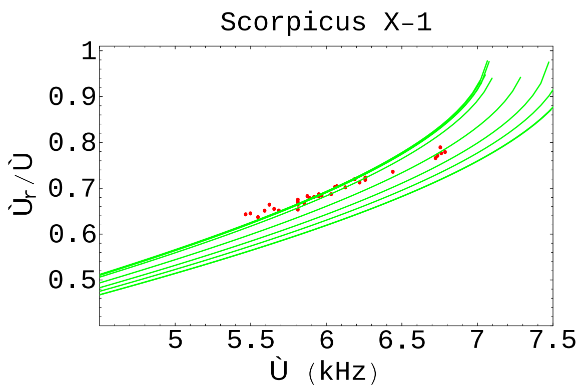
ϵ_c $(10^{15} g/cm^3)$	M (km)	$j = \frac{S_1}{M^2}$	$q = \frac{M_2}{M^3}$	$s_3 = \frac{S_3}{M^4}$	$\frac{\Delta M}{M}$	$\frac{\Delta j}{j}$	$\frac{\Delta q}{q}$
1.3847	4.321	0.478	-0.306	-0.206	0.0013	0.017	0.27
1.3199	4.321	0.479	-0.312	-0.212	0.0013	0.018	0.27
1.2500	4.324	0.484	-0.326	-0.226	0.0013	0.017	0.26
1.1992	4.326	0.489	-0.339	-0.240	0.0012	0.016	0.24
1.1160	4.333	0.505	-0.373	-0.277	0.0011	0.014	0.19
0.9665	4.355	0.556	-0.480	-0.405	0.0008	0.009	0.09
0.8781	4.377	0.602	-0.588	-0.551	0.0005	0.005	0.04
0.8172	4.396	0.641	-0.690	-0.702	0.0002	0.002	0.03
0.7604	4.419	0.685	-0.816	-0.903	0.00009	0.0003	0.02
0.7580	4.420	0.687	-0.822	-0.913	0.00006	0.0006	0.02

The fit is good as long as we stay in a frequency range where the series expansion is a good approximation and when the estimated quantities are not very small.

Except for fitting the data with the series expansions, one could use the curves $\frac{\Omega_\rho}{\Omega} = f(\Omega)$ or $\frac{\Omega_z}{\Omega} = g(\Omega)$ that can be produced with the Two-Soliton metric and try to fit them to the observed frequencies.

Various families of such curves could be produced by using the relation that the higher multipole moments (quadrupole, S_3) have with the spin parameter, the mass and the EOS (Laarakkers and Poisson, 1999).





Conclusions

- The Two-Soliton is an analytical solution in closed form that can represent the exterior of a NS well,
- Orbital and precession frequencies are observables that are not only relevant for GW astronomy,
- One could use them for some X-ray sources to estimate the first few moments (M, S, Q),
- The Two-Soliton could provide templates for these frequencies,
- The first 3 moments can be used to constrain the EOS of the NS,
- The Two-Soliton could also be used to perform dynamical analysis on orbits around a NS and identify resonances.

Thank you!!!