

# Revising the Multipole Moments of numerical spacetimes

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SFB/TR7 meeting

October 16, 2012



- A few words on multipole moments.
- Numerical spacetimes for rapidly rotating neutron stars.
- Multipole moments of numerical spacetimes.
- Consequences of correcting the multipole moments.

## Why are we interested in multipole moments?

- Global properties of the spacetime (the multipole structure determines the geometry),
- Relevant in gravitational wave astronomy (the observables can be related to the moments of the source),
- Constraining the equation of state for NS (from the relation between  $M, J, Q, \dots$ ),
- Can be used to construct analytic spacetimes for the exterior of neutron stars.

## Newtonian multipole moments:

Definition by expansion:

$$\Phi(r) = \frac{\mathcal{Q}}{r} + \frac{\mathcal{Q}_a x^a}{r^3} + \frac{\mathcal{Q}_{ab} x^a x^b}{r^5} + \dots \quad (1)$$

where,  $\mathcal{Q}$ ,  $\mathcal{Q}_a$ ,  $\mathcal{Q}_{ab}$ , are some integrals on the source

$$\mathcal{Q} = \int \rho d^3x, \quad \mathcal{Q}_a = \int x^a \rho d^3x, \quad \dots \quad (2)$$

The multipole moments are generally tensorial quantities.

Definition of the moments at infinity:

$$x^a \rightarrow \tilde{x}^a = r^{-2} x^a: \quad \tilde{r}^2 = \tilde{x}^a \tilde{x}_a = r^{-2}$$

$$\Phi(r) = \tilde{r} (\mathcal{Q} + \mathcal{Q}_a \tilde{x}^a + \mathcal{Q}_{ab} \tilde{x}^a \tilde{x}^b + \dots) \quad (3)$$

If we define the potential at infinity  $\tilde{\Phi} = \tilde{r}^{-1} \Phi$  then the moments are

$$P_{a_1 \dots a_n} = \tilde{D}_{a_n} P_{a_1 \dots a_{n-1}} = \tilde{D}_{a_1} \dots \tilde{D}_{a_n} \tilde{\Phi} \quad (4)$$

## Relativistic multipole moments:

- Generalization of the Newtonian moments,
- Defined for asymptotically flat spacetimes at infinity from a "potential" by a recursive relation,
- There are two sets of moments, the Mass moments and the Rotation moments,
- For the two sets of moments we have two generating potentials,  $\Phi_M, \Phi_J$ ,

The multipole moments for stationary and axisymmetric spacetimes can be reduced from tensors to scalars, because of the rotation symmetry.

In 1976 Butterworth and Ipser wrote the line element for a stationary and axially symmetric spacetime<sup>1</sup> (there is a timelike,  $\xi^a$ , and a spacelike,  $\eta^a$ , killing field)

$$\begin{aligned} ds^2 = & -e^{2\nu} dt^2 + r^2 \sin^2 \theta B^2 e^{-2\nu} (d\phi - \omega dt)^2 \\ & + e^{2(\zeta-\nu)} (dr^2 + r^2 d\theta^2), \end{aligned} \quad (5)$$

Asymptotic expansion of the metric functions:

$$\nu = \sum_{l=0}^{\infty} \nu_{2l}(r) P_{2l}(\mu), \quad (6)$$

$$\omega = \sum_{l=0}^{\infty} \omega_{2l}(r) P_{2l+1,\mu}(\mu), \quad (7)$$

$$B = \sum_{l=0}^{\infty} B_{2l}(r) T_{2l}^{1/2}(\mu), \quad (8)$$

$P_l$  are the Legendre polynomials,  $\mu = \cos \theta$ , and  $T_l^{1/2}$  are the Gegenbauer polynomials.

<sup>1</sup>E. M. Butterworth and J. R. Ipser, ApJ 204, 200 (1976).

From the field equations we have the asymptotic expansion:

$$\begin{aligned} \nu &= \left\{ -\frac{M}{r} + \frac{1}{3}\tilde{B}_0\frac{M}{r^3} + \frac{J^2}{r^4} + \left[ -\frac{\tilde{B}_0^2}{5} + \frac{\tilde{B}_2^2}{15} - \frac{12J^2}{5} \right] \frac{M}{r^5} + \dots \right\} \\ &\quad + \left\{ \frac{\tilde{\nu}_2}{r^3} - 2\frac{J^2}{r^4} + [\dots] \frac{1}{r^5} + \dots \right\} P_2(\mu) \\ &\quad + \left\{ \frac{\tilde{\nu}_4}{r^5} + \dots \right\} P_4(\mu) + \dots \end{aligned} \tag{9}$$

$$\begin{aligned} \omega &= \left\{ \frac{2J}{r^3} - \frac{6JM}{r^4} + \frac{6}{5} \left[ 8 - 3\frac{\tilde{B}_0}{M^2} \right] \frac{JM^2}{r^5} + (\dots) \frac{J}{r^6} + \dots \right\} P_{1,\mu}(\mu) \\ &\quad + \left\{ \frac{\tilde{\omega}_2}{r^5} + (\dots) \frac{1}{r^6} - \dots \right\} P_{3,\mu}(\mu) + \dots \end{aligned} \tag{10}$$

$$B = \left( \frac{\pi}{2} \right)^{1/2} \left( 1 + \frac{\tilde{B}_0}{r^2} \right) T_0^{1/2}(\mu) + \left( \frac{\pi}{2} \right)^{1/2} \frac{\tilde{B}_2}{r^4} T_2^{1/2}(\mu) + \dots \tag{11}$$

In 1989 Komatsu, Eriguchi, and Hechisu<sup>2</sup> wrote the same line element,

$$ds^2 = -e^{2\nu} dt^2 + r^2 \sin^2 \theta e^{2\beta} (d\phi - \omega dt)^2 + e^{2\alpha} (dr^2 + r^2 d\theta^2).$$

where,  $\nu_{\text{BI}} = \nu_{\text{KEH}} = \nu$ ,  $B_{\text{BI}} e^{-\nu} = e^{\beta_{\text{KEH}}}$ ,  $\zeta_{\text{BI}} = \nu + \alpha_{\text{KEH}}$ , and proposed a different scheme for integrating the field equations using Green's functions.

The combinations of  $\nu_{\text{KEH}}$  and  $\beta_{\text{KEH}}$

$$\gamma = \nu_{\text{KEH}} + \beta_{\text{KEH}}, \quad \rho = \nu_{\text{KEH}} - \beta_{\text{KEH}}, \quad (12)$$

along with  $\omega$  could be expressed as power series in  $1/r$ :

$$\rho = \sum_{n=0}^{\infty} \left( -2 \frac{M_{2n}}{r^{2n+1}} + \text{higher order} \right) P_{2n}(\mu), \quad (13)$$

$$\omega = \sum_{n=1}^{\infty} \left( -\frac{2}{2n-1} \frac{S_{2n-1}}{r^{2n+1}} + \text{higher order} \right) \frac{P_{2n-1}^1(\mu)}{\sin \theta}, \quad (14)$$

$$\gamma = \sum_{n=1}^{\infty} \left( \frac{D_{2n-1}}{r^{2n}} + \text{higher order} \right) \frac{\sin(2n-1)\theta}{\sin \theta}. \quad (15)$$

The coefficients  $M_{2n}$  and  $S_{2n-1}$  were identified as the mass and current-mass moments, respectively. By comparison between the above expansion and the corresponding ones of BI we see that  $M_2 = -\tilde{\nu}_2$  and  $S_3 = \frac{3}{2}\tilde{\omega}_2$ .

<sup>2</sup>H. Komatsu, Y. Eriguchi, and I. Hechisu, MNRAS **237**, 355 (1989).

In 1995 Ryan proposed a way of measuring the multipole moments of a spacetime from gravitational waves<sup>3</sup>.

Ryan's procedure expressed the energy loss per logarithmic frequency interval,  $\Delta \tilde{E}$  of an inspiralling test particle as a series expansion in the orbital frequency of the circular equatorial orbits.

The coefficients of the expansion are functions of the multipole moments.

$$\begin{aligned} \Delta \tilde{E} = & \frac{1}{3}v^2 - \frac{1}{2}v^4 + \frac{20}{9}\frac{S_1}{M^2}v^5 + \left(-\frac{27}{8} + \frac{M_2}{M^3}\right)v^6 + \frac{28}{3}\frac{S_1}{M^2}v^7 \\ & + \left(-\frac{225}{16} + \frac{80}{27}\frac{S_1^2}{M^4} + \frac{70}{9}\frac{M_2}{M^3}\right)v^8 + \left(\frac{81}{2}\frac{S_1}{M^2} + 6\frac{S_1 M_2}{M^5} - 6\frac{S_3}{M^4}\right)v^9 + \dots \end{aligned}$$

where  $v = (M\Omega)^{1/3}$  and  $S_1 = J$ .

<sup>3</sup>F. D. Ryan, Phys. Rev. D 52, 5707 (1995).

Following Ryan's procedure for the Butterworth and Iyer metric, we first express the orbital frequency

$$\Omega = \frac{d\phi}{dt} = \frac{-g_{t\phi,r} + \sqrt{(g_{t\phi,r})^2 - g_{tt,r}g_{\phi\phi,r}}}{g_{\phi\phi,r}},$$

as a power series in  $x = (M/r)^{1/2}$  and then invert it to obtain

$$\begin{aligned} x &= v + \frac{v^3}{2} + \frac{JV^4}{3} + \frac{1}{24}(13 + 4b - 6q)v^5 + \frac{JV^6}{2} \\ &\quad + \frac{(97 + 28b + 56J^2 - 144q)v^7}{144} \\ &\quad + \frac{(373J + 292bj - 330jq - 270w_2)v^8}{360} + O(v^9), \end{aligned} \quad (16)$$

where  $J = \frac{J}{M^2}$ ,  $q = \frac{\nu_2}{M^3}$ ,  $w_2 = \frac{\omega_2}{M^4}$ ,  $b = \frac{B_0}{M^2}$ .

We then expand

$$\tilde{E} = \frac{-g_{tt} - g_{t\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}, \quad (17)$$

as a power series in  $x = (M/r)^{1/2}$  and substitute the previous expression for  $x$ . Then

$$\Delta \tilde{E} = -\frac{d\tilde{E}}{d \log \Omega} = -\frac{v d\tilde{E}}{3 dv}, \quad (18)$$

will give,

$$\begin{aligned} \Delta \tilde{E} &= \frac{v^2}{3} - \frac{v^4}{2} + \frac{20jv^5}{9} - \frac{(89 + 32b + 24q)}{24}v^6 + \frac{28jv^7}{3} \\ &\quad - \frac{5(1439 + 896b - 256j^2 + 672q)}{432}v^8 \\ &\quad + \frac{((421 + 64b - 60q)j - 90w_2)v^9}{10} + O(v^{10}), \end{aligned} \quad (19)$$

By equating the coefficients of the previous power series with the corresponding ones of Ryan's expression we have<sup>4</sup>,

$$M_2^{corr} = -\tilde{\nu}_2 - \frac{4}{3}\left(\frac{1}{4} + b\right)M^3 = M_2 - \frac{4}{3}\left(\frac{1}{4} + b\right)M^3, \quad (20)$$

$$S_3^{corr} = \frac{3}{2}\tilde{\omega}_2 - \frac{12}{5}\left(\frac{1}{4} + b\right)jM^4 = S_3 - \frac{12}{5}\left(\frac{1}{4} + b\right)jM^4. \quad (21)$$

<sup>4</sup>G.P. and T. A. Apostolatos, Phys. Rev. Lett. **108** (2012), 231104.

## Quantitative difference between the corrected and the uncorrected moments.

#	$M$	$j$	$M_2$	$S_3$	$b$	$\Delta M_2(\%)$	$\Delta S_3(\%)$
4	2.087	0.414	-6.08	-10.0	-0.23	4.032	3.805
9	2.111	0.650	-15.2	-40.5	-0.20	4.008	3.716
10	2.112	0.661	-15.7	-42.7	-0.20	4.002	3.706
12	3.164	0.194	-1.68	-1.37	-0.24	21.63	31.69
15	3.231	0.485	-12.0	-26.6	-0.20	18.43	24.82
19	3.304	0.676	-27.0	-90.4	-0.17	15.86	19.77
20	3.318	0.706	-30.3	-107.	-0.16	15.40	18.88
21	3.388	0.510	-12.8	-27.8	-0.19	26.05	42.10
22	3.388	0.510	-12.9	-28.1	-0.19	25.75	41.32
28	3.458	0.659	-26.0	-82.7	-0.17	19.79	27.20
29	3.477	0.694	-30.1	-103.	-0.16	18.91	25.31
30	3.487	0.713	-32.5	-115.	-0.16	18.45	24.34

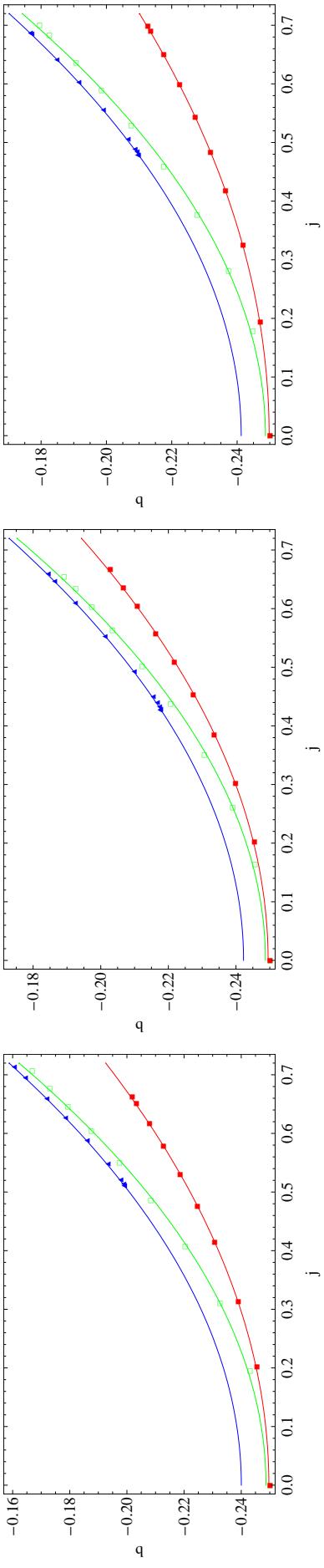
EOS AU.

#	$M$	$j$	$M_2$	$S_3$	$b$	$\Delta M_2(\%)$	$\Delta S_3(\%)$
5	2.087	0.452	-7.85	-14.6	-0.22	3.634	3.311
7	2.098	0.557	-12.0	-28.0	-0.21	3.575	3.228
9	2.106	0.636	-15.9	-42.9	-0.20	3.504	3.136
10	2.109	0.666	-17.6	-49.9	-0.20	3.469	3.091
14	2.686	0.349	-4.81	-7.43	-0.23	11.63	12.90
17	2.727	0.562	-14.6	-39.0	-0.20	9.428	9.805
19	2.744	0.633	-19.7	-60.8	-0.19	8.734	8.866
20	2.750	0.654	-21.4	-69.0	-0.18	8.543	8.607
21	2.823	0.427	-6.54	-11.7	-0.21	17.55	22.14
22	2.823	0.428	-6.69	-12.1	-0.21	16.97	21.12
29	2.882	0.647	-21.1	-67.6	-0.18	10.64	11.25
30	2.884	0.658	-22.2	-72.8	-0.18	10.42	10.94

EOS FPS.

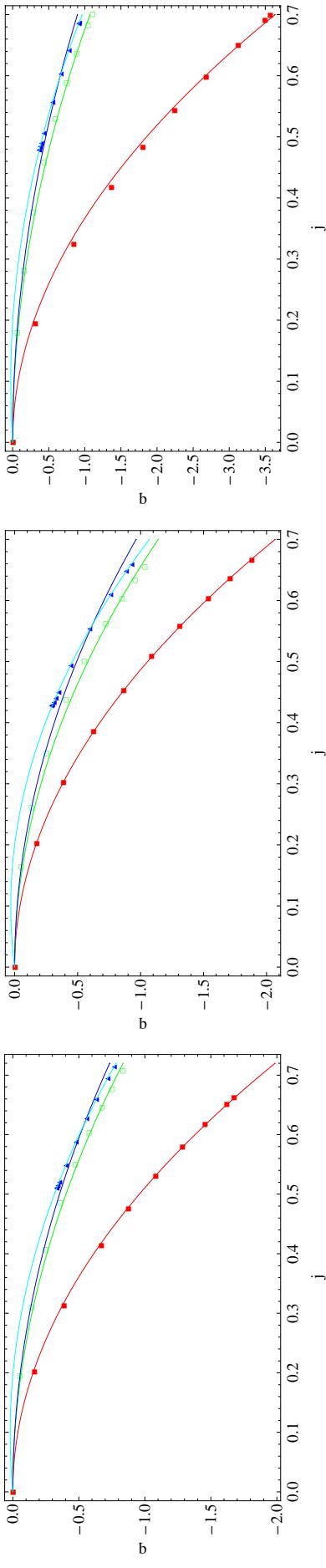
#	$M$	$j$	$M_2$	$S_3$	$b$	$\Delta M_2(\%)$	$\Delta S_3(\%)$
4	2.080	0.417	-12.2	-22.0	-0.23	1.352	1.175
7	2.090	0.598	-24.4	-63.5	-0.22	1.399	1.210
9	2.096	0.690	-32.1	-97.1	-0.21	1.417	1.220
10	2.097	0.698	-32.9	-100.	-0.21	1.420	1.222
14	4.051	0.375	-18.5	-45.4	-0.22	11.94	13.57
17	4.120	0.588	-51.8	-210.	-0.19	10.25	11.09
19	4.160	0.682	-74.9	-365.	-0.18	9.478	9.998
20	4.167	0.700	-79.8	-401.	-0.17	9.335	9.794
21	4.321	0.478	-29.0	-88.0	-0.20	17.68	22.66
22	4.321	0.479	-29.5	-90.2	-0.20	17.19	21.82
28	4.396	0.641	-66.0	-299.	-0.18	12.60	14.32
29	4.418	0.684	-78.7	-389.	-0.17	11.92	13.28
30	4.420	0.686	-79.4	-394.	-0.17	11.89	13.24

EOS L.



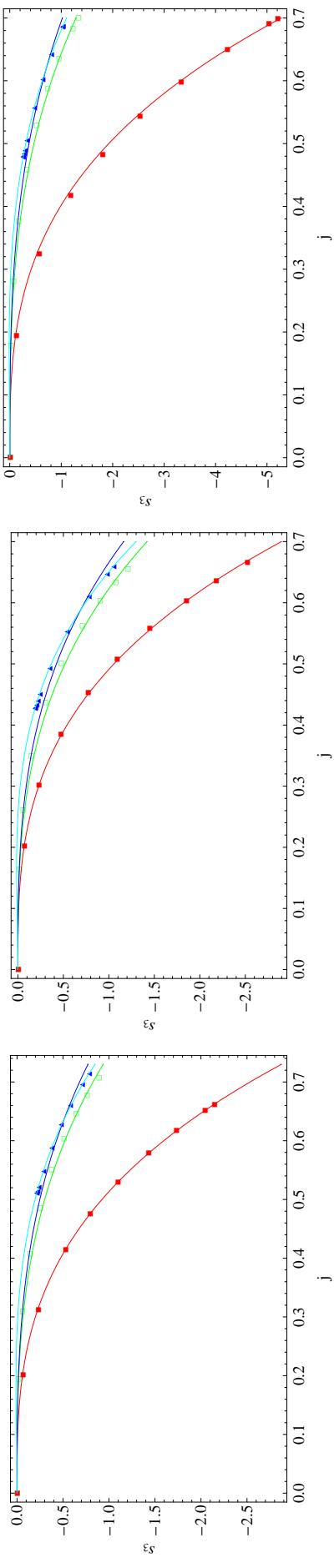
Plot of the parameter  $b$  and the corresponding fits for the three EOSs (AU on the left, FPS in the middle and L on the right). Red is the fit for sequence (i), green is the fit for sequence (ii), blue is the fit for sequence (iii).

$$b = -1/4 + a_1 j^2, \quad a_1 > 0.$$



Plot of the reduced quadrupole  $q$  and the corresponding fits for the three EOSs (AU on the left, FPS in the middle and L on the right). Red is the fit for sequence (i), green is the fit for sequence (ii), blue is the fit with one parameter for sequence (iii), and cyan is the fit with two parameters for the same sequence.

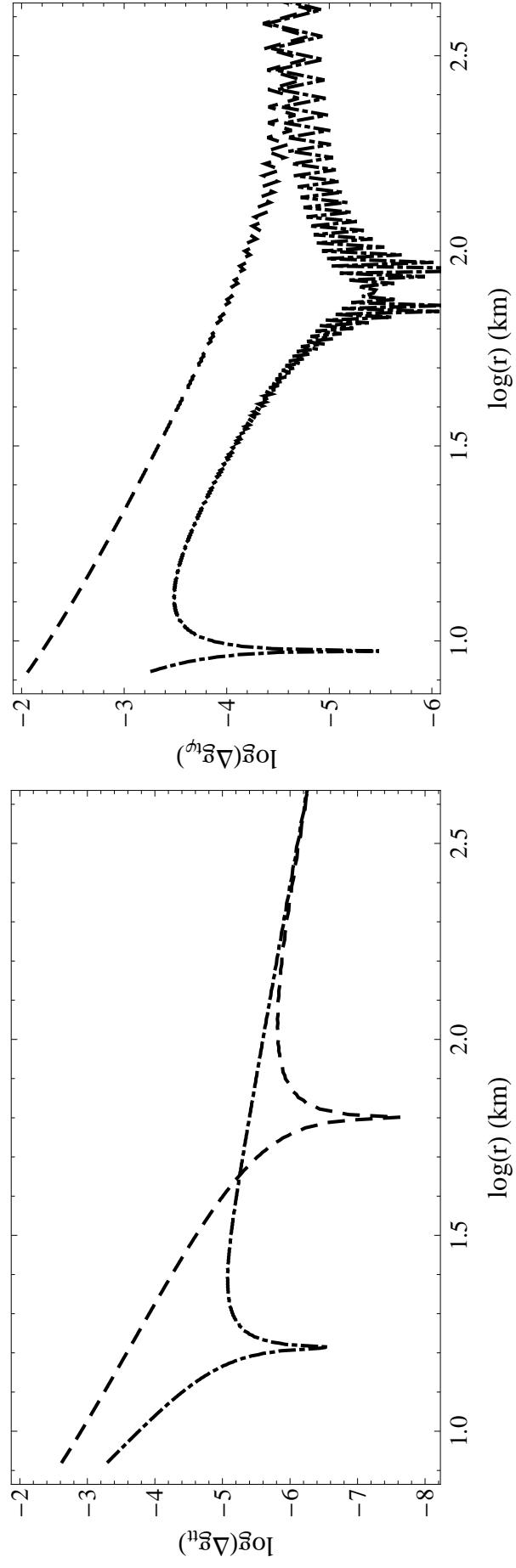
$$q = a_2 j^2, \quad a_2 < 0.$$



Plot of the reduced spin octupole  $s_3$  and the corresponding fits for the three EOSs (AU on the left, FPS in the middle and L on the right). Red is the fit for sequence (i), green is the fit for sequence (ii), blue is the fit with one parameter for sequence (iii), and cyan is the fit with two parameters for the same sequence.

$$s_3 = a_3 j^3, \quad a_3 < 0.$$

Using the three parameter Manko et al. analytic solution<sup>5</sup> to check the improvement<sup>6</sup>.



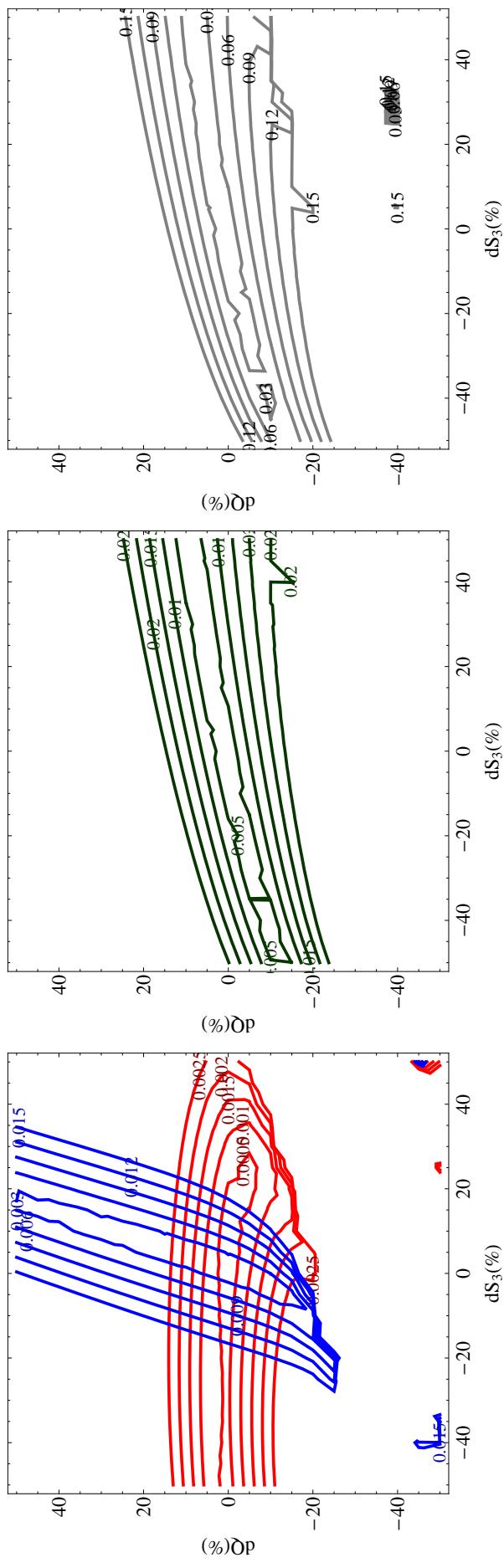
A typical log-log plot of the relative difference between the numerical and the analytic metric  $((g_{ij}^n - g_{ij}^a) / g_{ij}^n)$  for a specific numerical model before (dashed curve) and after the correction of  $M_2$  (dashed-dotted curve). The left plot is for  $g_{tt}$  and the right one for  $g_{t\phi}$ .

<sup>5</sup>V. S. Manko, E. W. Mielke, and J. D. Sanabria-Gómez, Phys. Rev. D **61**, 081501 (2000)

<sup>6</sup>G.P. and T. A. Apostolatos, Phys. Rev. Lett. **108** (2012), 231104.

Using the four parameter Two-Soliton analytic solution<sup>7</sup> to check the improvement.

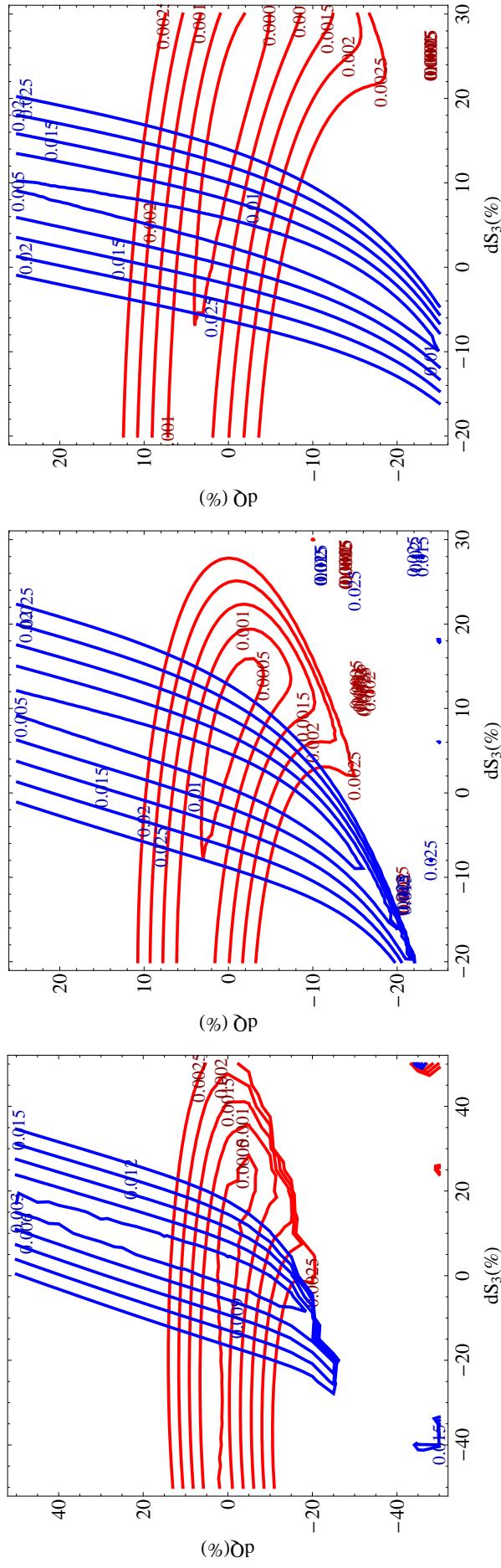
$$\sigma_{ij} = \left( \int_{R_S}^{\infty} (g_{ij}^n - g_{ij}^a)^2 dr \right)^{1/2}.$$



On the left the contour plots of the overall difference for the  $g_{tt}$  and  $g_{t\phi}$  components of the numerical metric compared to the analytic Two-Soliton metric. On the right the contour plot of the overall difference for the  $\Omega$ . In the middle the contour plot of the relative difference in  $R_{isco}$ .

<sup>7</sup>V. S. Manko, J. Martin, E. Ruiz, 1995, *J. Math. Phys.* **36** 3063

<sup>8</sup>G.P. and T. A. Apostolatos, preprint, arXiv:1209.6148 [gr-qc]



Contour plots of the overall difference for the  $g_{tt}$  and  $g_{t\phi}$  components of the numerical metric compared to the analytic Two-Soliton metric. On the left we have model #15 of EOS AU. In the middle we have model #28 of EOS AU. On the right we have model #28 of EOS L.

- The correction in the moments for the high mass neutron stars is in the range of 10-20% and can be higher for the supermassive sequences,
- Both the quadrupole and the current-octupole seem to follow the same scalings with increasing rotation parameter as the corresponding moments of the Kerr spacetime,
- The corrected moments result in an improvement to the accuracy of analytic metrics of almost an order of magnitude.