The spacetime around compact objects and astrophysical observables

George Pappas

NEB-17 RECENT DEVELOPMENTS IN GRAVITY
MYKONOS, GREECE

September 22, 2016

- Introduction - The spacetime around NSs
  - Approximate spacetimes
  - Numerical spacetimes
  - Analytic spacetimes
  - Characterising spacetimes using the multipole moments
  - NS multipole moments
- A new approximate spacetime
  - Constructing the spacetime
  - Customising the spacetime to NSs
- Comparing the proposed spacetime to NS spacetimes
  - Possible issues and good behaviour of the spacetime
  - Metric functions comparison
  - Relativistic frequencies comparison
- Astrophysical applications
- Extension to Scalar-Tensor theories of gravity

Slowly Rotating Neutron Stars

The Hartle-Thorne approximation:[1]

$$ds^2 = -e^{\bar{\nu}}\left(1+2\epsilon^2 h\right)dt^2 + e^{\lambda}\left[1+2\epsilon^2 m/(r-2M)\right]dr^2 + r^2\left[1+2\epsilon^2 k\right]\left[d\theta^2 + \sin^2\theta(d\phi - \epsilon\omega dt)^2\right].$$

where $\epsilon = \Omega/\Omega^*$ is the slow rotation small parameter with respect to $\Omega^* = (M/R^3)^{1/2}$.

Rapidly Rotating Neutron Stars: Numerical

The line element for a stationary and axially symmetric spacetime (the spacetime admits a timelike, ξ^a, and a spacelike, η^a, killing field, i.e., it has rotational symmetry and symmetry in translations in time) is [2],

$$ds^2 = -e^{2\nu}dt^2 + r^2\sin^2\theta B^2 e^{-2\nu}(d\phi - \omega dt)^2 + e^{2(\zeta-\nu)}(dr^2 + r^2 d\theta^2).$$

Komatsu, Eriguchi, and Hechisu[3] proposed a scheme for integrating the field equations using Green's functions. This scheme is implemented by the RNS numerical code to calculate rotating neutron stars [4].

Rapidly Rotating Neutron Stars: Analytic

Using the Weyl-Papapetrou line element that describes stationary and axisymmetric vacuum spacetimes,

$$ds^2 = -f\left(dt - \omega d\phi\right)^2 + f^{-1}\left[e^{2\gamma}\left(d\rho^2 + dz^2\right) + \rho^2 d\phi^2\right].$$

Ernst[5] reformulated the Einstein field equations to take the form, $(Re(\mathcal{E}))\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot\nabla\mathcal{E}$, using the complex potential $\mathcal{E}(\rho,z) = f(\rho,z) + i\psi(\rho,z)$, where $f = \xi^a\xi_a$ and ψ is defined by, $\nabla_a\psi = \varepsilon_{abcd}\,\xi^b\nabla^c\xi^d$.

[1] Hartle J. B., Thorne K. S., ApJ **153**, 807 (1968)

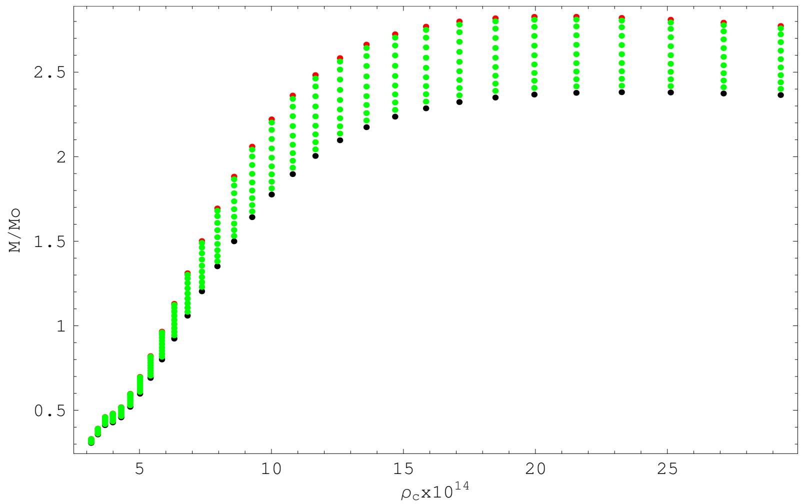
[2] E. M. Butterworth and J. R. Ipser, ApJ **204**, 200 (1976).

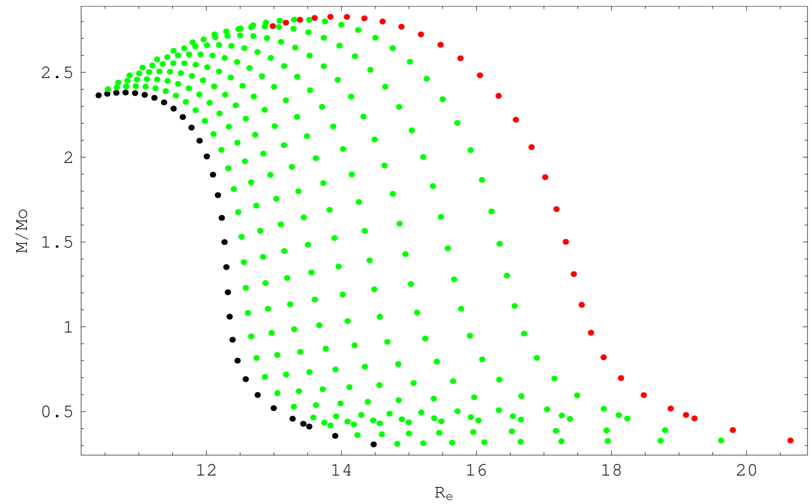
[3] H. Komatsu, Y. Eriguchi, and I. Hechisu, MNRAS **237**, 355 (1989).

[4] N. Stergioulas, J.L. Friedman, ApJ, **444**, 306 (1995).

[5] F.J. Ernst, Phys. Rev., **167**, 1175 (1968); Phys. Rev., **168**, 1415 (1968).

One can use RNS to calculate models of rotating neutron stars for a given equation of state. For example we show here some models for the APR EOS:





The models with the fastest rotation have a spin parameter, $j = J/M^2$, around 0.7 and a ratio of the polar radius over the equatorial radius, r_p/r_e, around 0.56.

The code, except from the various physical characteristics of the neutron stars, provides the metric functions in a grid on the coordinates x and μ in the whole space (for values from 0 to 1 for both variables), where $\mu = cos\theta$, $r = \frac{x r_e}{1-x}$ and r_e is a length scale.

One can also extract from the spacetime the relativistic multipole moments of the NS. In particular the RNS code can calculates the first non-zero multipole moments, i.e., M, $S_1 \equiv J$, M_2, $S_3 \equiv J_3$ and M_4[6]. These moments characterise the NS and the spacetime arounf it.

[6] G.P. and T. A. Apostolatos, Phys. Rev. Lett. **108** 231104 (2012),
K. Yagi, K. Kyutoku, G. P., N. Yunes, and T.A. Apostolatos, Phys.Rev. D **89** 124013 (2014).

Neutron star multipole moments properties in GR

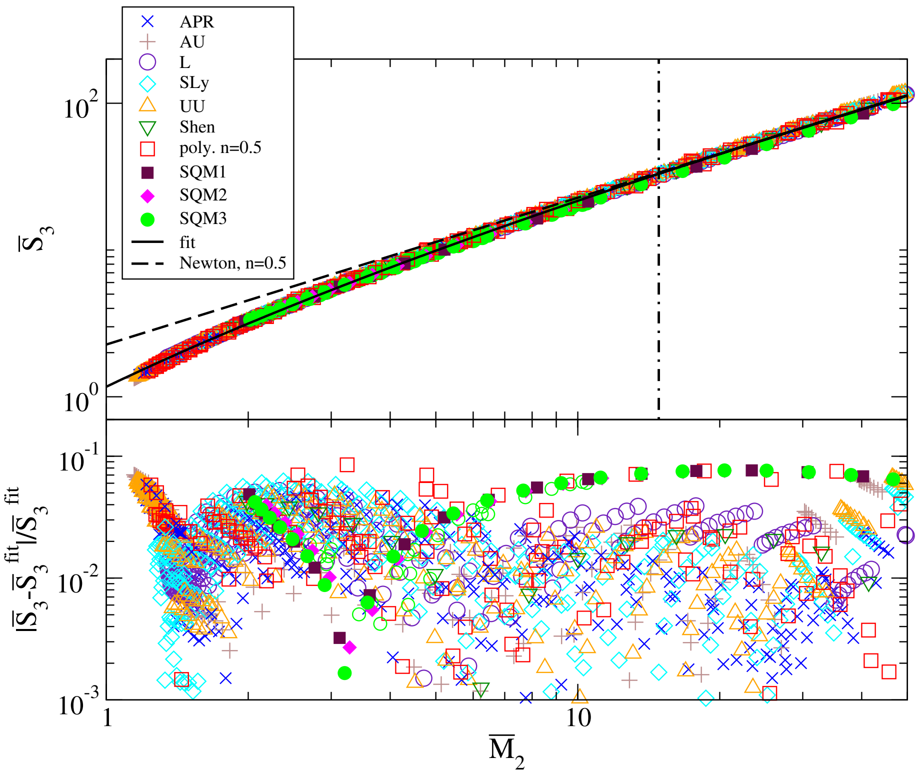
Black Hole-like behaviour of the moments[7]:

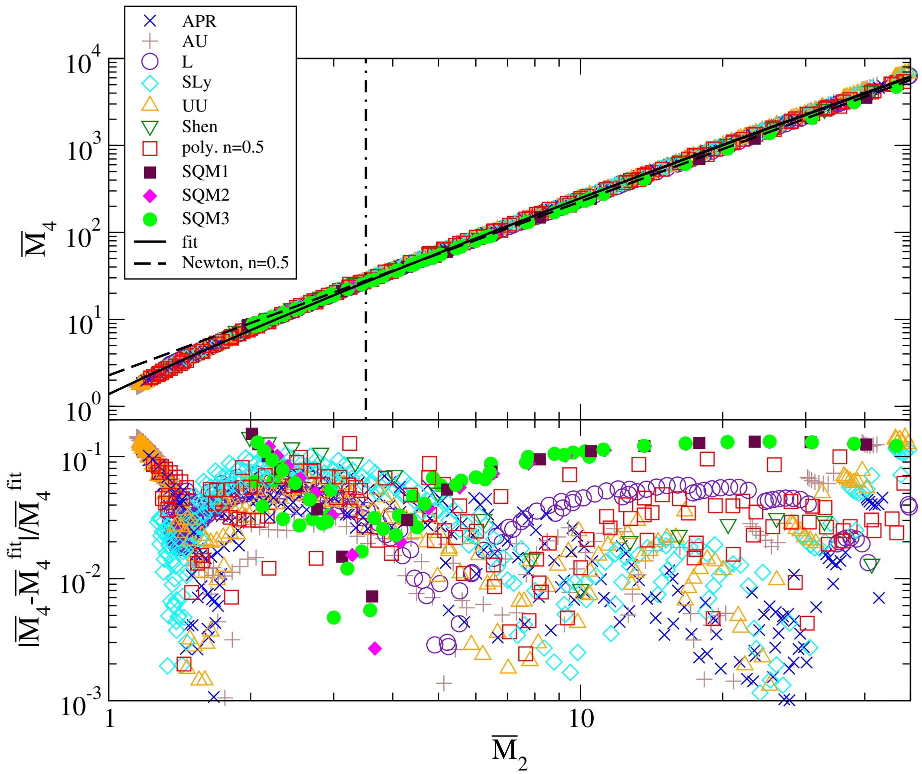
Kerr moments	Neutron star moments
$M_0 = M,$	$M_0 = M,$
$J_1 = J = jM^2,$	$J_1 = jM^2,$
$M_2 = -j^2M^3,$	$M_2 = -a(EoS, M)j^2M^3,$
$J_3 = -j^3M^4,$	$J_3 = -\beta(EoS, M)j^3M^4,$
$M_4 = j^4M^5,$	$M_4 = \gamma(EoS, M)j^4M^5,$
$\vdots$	$\vdots$
$M_{2n} = (-1)^n j^{2n} M^{2n+1},$	$M_{2n} = ?,$
$J_{2n+1} = (-1)^n j^{2n+1} M^{2n+2}$	$J_{2n+1} = ?$

[7] W.G. Laarakkers and E. Poisson, Astrophys. J. **512** 282 (1999).
G.P. and T. A. Apostolatos, Phys. Rev. Lett. **108** 231104 (2012).
K. Yagi, K. Kyutoku, G. P., N. Yunes, and T.A. Apostolatos, Phys.Rev. D **89** 124013 (2014).

Neutron star multipole moments properties in GR

EoS independent behaviour of the moments[8] :





$$\bar{M}_{2n} = |M_{2n}/(j^{2n} M^{2n+1})|,\ \bar{J}_{2n+1} = |J_{2n+1}/(j^{2n+1} M^{2n+2})|$$

All these are properties that characterize the spacetime around neutron stars as well as the gravitational aspects of the stars themselves.

[8] G.P. and T. A. Apostolatos, Phys.Rev.Lett. **112** 121101 (2014).
K. Yagi, K. Kyutoku, G. P., N. Yunes, and T.A. Apostolatos, Phys.Rev. D **89** 124013 (2014).

Two-Soliton analytic spacetime: This is a 4-parameter analytic spacetime which can be produced if one chooses the Ernst potential on the axis to have the form:

$$e(z) = \frac{(z - M - ia)(z + ib) - k}{(z + M - ia)(z + ib) - k}$$

The parameters a, b, k of the spacetime can be related to the first non-zero multipole moments through the equations,

$$J = aM, \quad M_2 = -(a^2 - k)M, \quad J_3 = -\left[a^3 - (2a - b)k\right] M,$$

where M is the mass.

One can use the multipole moments M, J, M_2, and J_3 of a numerically calculated neutron star and produce an analytic two-soliton spacetime that reproduces very accurately the numerically calculated spacetime[9]. Instead of using a specific set of values for the moments, one could reproduce any neutron star spacetime using the universal relations

$$\sqrt[3]{\bar{J}_3} = A + B_1 \left(\sqrt{\bar{M}_2}\right)^{\nu_1} + B_2 \left(\sqrt{\bar{M}_2}\right)^{\nu_2},$$

Therefore the first higher moments of a general neutron star spacetime can be expressed in terms of only three parameters, the mass M, the angular momentum J, and the quadrupole M_2,[10] having thus a universal analytic spacetime.[11]

[9] G. P., and T. A. Apostolatos, MNRAS, **429**, 3007 (2013); Other analytic spacetimes have been proposed in the past, see for eg. E. Berti, and N. Stergioulas, MNRAS, **350**, 1416 (2004)

[10] G.P. and T. A. Apostolatos, Phys.Rev.Lett. **112** 121101 (2014): $\sqrt[3]{\bar{J}_3} = -0.36 + 1.48 \left(\sqrt{\bar{M}_2}\right)^{0.65}$

[11] GP, MNRAS **454**, 4066 (2015)

The usual algorithms for constructing analytic solutions, like the Two-Soliton, result to increasingly complicated spacetimes the higher the number of parameters one introduces to characterise the spacetime.

One could try to evade this difficulty by constructing an approximate spacetime that is an approximate solution of the Ernst equation. Such a spacetime would still be clearly parameterised by the multipole moments. The ansatz that one can use to construct the solution is,

$$\mathcal{E} = \frac{1-\xi(\rho,z)}{1+\xi(\rho,z)}, \qquad \xi = \frac{1}{\sqrt{\rho^2+z^2}} \sum_{i,j=0}^{n,k} a_{ij} \left(\frac{\rho}{\rho^2+z^2}\right)^i \left(\frac{z}{\rho^2+z^2}\right)^j,$$

where the parameters a_{ij} are related to the multipole moments of the spacetime and the ξ expansion is up to the required order.

Out of this Ernst potential one has the functions, $f(\rho,z) = \frac{1}{2}(\mathcal{E}+\mathcal{E}^*)$, and $\psi(\rho,z) = \frac{1}{2i}(\mathcal{E}-\mathcal{E}^*)$.

Then from the identity $f^{-2}\nabla\psi = -\rho^{-1}\hat{n}\times\nabla\omega$, one can calculate the metric function $\omega(\rho,z)$.

Finally, the last metric function $\gamma(\rho,z)$ is calculated from the system,

$$\frac{\partial\gamma}{\partial\rho} = \frac{\rho}{4f^2}\left[\left(\frac{\partial f}{\partial\rho}\right)^2 - \left(\frac{\partial f}{\partial z}\right)^2\right] - \frac{f^2}{4\rho}\left[\left(\frac{\partial\omega}{\partial\rho}\right)^2 - \left(\frac{\partial\omega}{\partial z}\right)^2\right], \quad \frac{\partial\gamma}{\partial z} = \frac{1}{2}\left[\frac{\rho}{f^2}\frac{\partial f}{\partial\rho}\frac{\partial f}{\partial z} - \frac{f^2}{\rho}\frac{\partial\omega}{\partial\rho}\frac{\partial\omega}{\partial z}\right].$$

The approximate metric:[12]

$$
\begin{aligned}
f(\rho,z) &= 1-\frac{2M}{\sqrt{\rho^2+z^2}}+\frac{2M^2}{\rho^2+z^2}+\frac{\left(M_2-M^3\right)\rho^2-2\left(M^3+M_2\right)z^2}{\left(\rho^2+z^2\right)^{5/2}}\\
&\quad+\frac{2z^2\left(-J^2+M^4+2M_2M\right)-2MM_2\rho^2}{\left(\rho^2+z^2\right)^3}+\frac{A(\rho,z)}{28\left(\rho^2+z^2\right)^{9/2}}+\frac{B(\rho,z)}{14\left(\rho^2+z^2\right)^5},\\
\omega(\rho,z) &= -\frac{2J\rho^2}{\left(\rho^2+z^2\right)^{3/2}}-\frac{2JM\rho^2}{\left(\rho^2+z^2\right)^2}+\frac{F(\rho,z)}{\left(\rho^2+z^2\right)^{7/2}}+\frac{H(\rho,z)}{2\left(\rho^2+z^2\right)^4}+\frac{G(\rho,z)}{4\left(\rho^2+z^2\right)^{11/2}},\\
\gamma(\rho,z) &= \frac{\rho^2\left(J^2\left(\rho^2-8z^2\right)+M\left(M^3+3M_2\right)\left(\rho^2-4z^2\right)\right)}{4\left(\rho^2+z^2\right)^4}-\frac{M^2\rho^2}{2\left(\rho^2+z^2\right)^2},
\end{aligned}
$$

where,

$$
\begin{aligned}
A(\rho,z) =& \left[8\rho^2z^2\left(24J^2M+17M^2M_2+21M_4\right)+\rho^4\left(-10J^2M+7M^5+32M_2M^2-21M_4\right)\right.\\
&\left.+8z^4\left(20J^2M-7M^5-22M_2M^2-7M_4\right)\right],\\
B(\rho,z) =& \left[\rho^4\left(10J^2M^2+10M_2M^3+21M_4M+7M_2^2\right)-4z^4\left(40J^2M^2+14JS_3-7M^6-30M_2M^3\right.\right.\\
&\left.\left.-14M_4M-7M_2^2\right)-4\rho^2z^2\left(27J^2M^2-21JS_3+7M^6+48M_2M^3+42M_4M+7M_2^2\right)\right],\\
H(\rho,z) =& \left[4\rho^2z^2\left(J\left(M_2-2M^3\right)-3MS_3\right)+\rho^4\left(JM_2+3MS_3\right)\right]\\
G(\rho,z) =& \left[\rho^2\left(J^3\left(-\left(\rho^4+8z^4-12\rho^2z^2\right)\right)+JM\left(\left(M^3+2M_2\right)\rho^4-8\left(3M^3+2M_2\right)z^4\right.\right.\right.\\
&\left.\left.\left.+4\left(M^3+10M_2\right)\rho^2z^2\right)+M^2S_3\left(3\rho^4-40z^4+12\rho^2z^2\right)\right)\right]\\
F(\rho,z) =& \left[\rho^4\left(S_3-JM^2\right)-4\rho^2z^2\left(JM^2+S_3\right)\right].
\end{aligned}
$$

[12] GP in preparation: This is an approximate vacuum spacetime with $R_{ab}=0+\mathcal{O}(\bar{r}^6)$, where $\bar{r}=(\rho^2+z^2)^{-1/2}$

As with the Two-Soliton, one can use the multipole moments M, J, M_2, J_3, and additionally this time M_4, of a numerically calculated neutron star and produce an analytic two-soliton spacetime that reproduces very accurately the numerically calculated spacetime.

Again, instead of using a specific set of values for the moments, one could reproduce any neutron star spacetime using universal relations of the form

$$\sqrt[3]{\bar{J}_3} = A_1 + B_1 \left(\sqrt{\bar{M}_2}\right)^{\nu_1} + B_2 \left(\sqrt{\bar{M}_2}\right)^{\nu_2},$$

$$\sqrt[4]{\bar{M}_4} = A_2 + B_3 \left(\sqrt{\bar{M}_2}\right)^{\nu_3} + B_4 \left(\sqrt{\bar{M}_2}\right)^{\nu_4},$$

Specifically the relations one can use for neutron stars are[13]

$$\sqrt[3]{\bar{J}_3} = -0.36 + 1.48 \left(\sqrt{\bar{M}_2}\right)^{0.65},$$

$$\sqrt[4]{\bar{M}_4} = -4.749 + 0.27613 \left(\sqrt{\bar{M}_2}\right)^{1.5146} + 5.5168 \left(\sqrt{\bar{M}_2}\right)^{0.22229}$$

As before, a general neutron star spacetime is expressed in terms of only three parameters, the mass M, the angular momentum J, and the quadrupole M_2, having in this case as well a universal spacetime.[14]

[13] $\bar{M}_2 \equiv \alpha$, $\bar{J}_3 \equiv \beta$, and $\bar{M}_4 \equiv \gamma$

[14] GP in preparation

Possible issues and good behaviour of the spacetime:

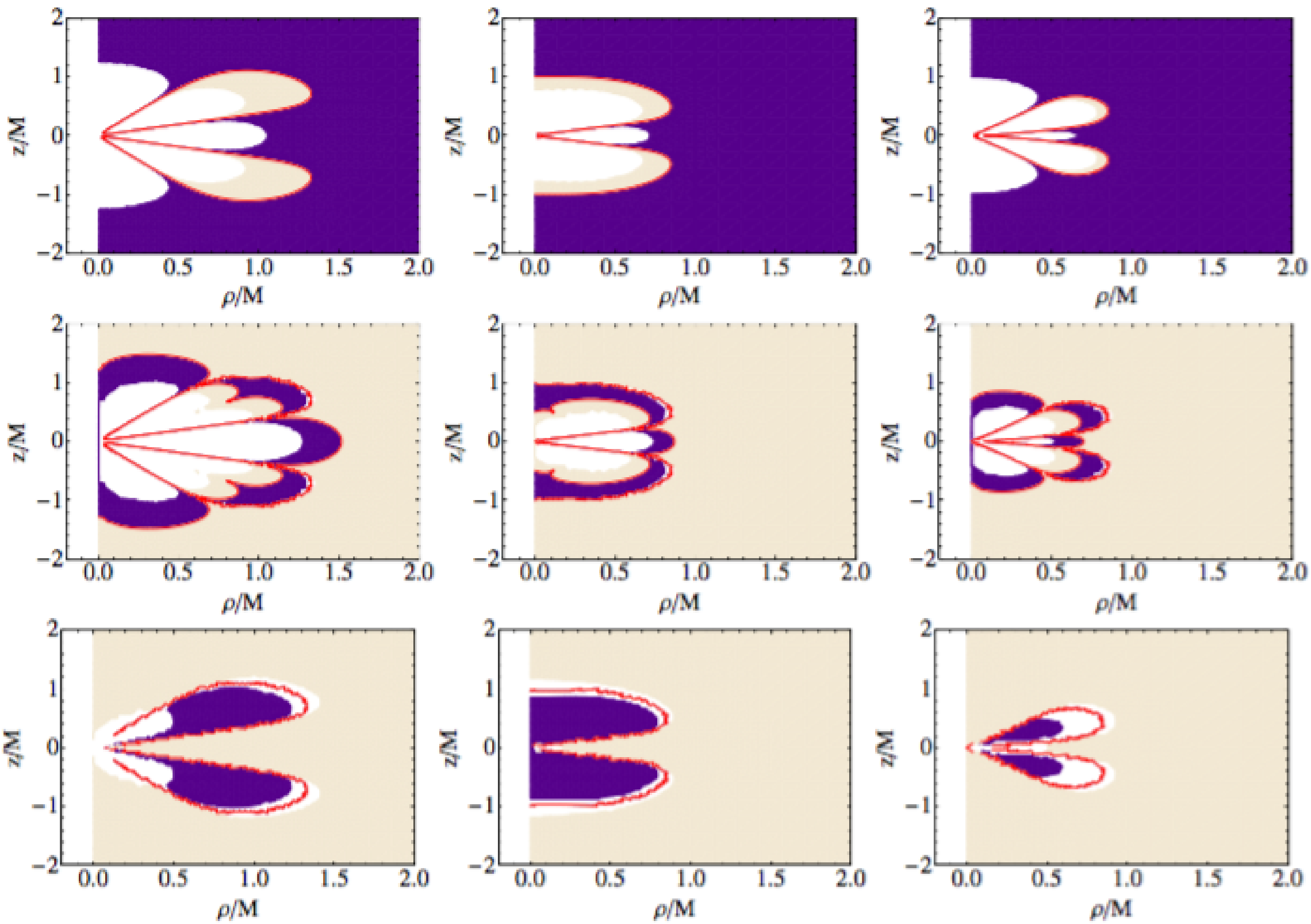


Figure 1. Typical plots of the behaviour of the metric functions g_{tt} (top row), $g_{\varphi\varphi}$ (middle row), and $g_{\rho\rho}$ (bottom row), for different values of the spin parameter j and the quadrupolar deformability α ($j = 0.5$, $\alpha = 8$ for the left column, $j = 0.25$, $\alpha = 5$ for the middle column, and $j = 0.125$, $\alpha = 3$ for the right column). The red lines indicate the locations where the metric functions are zero. The regions where the functions are negative are with dark colour, while the region where the functions are positive are with cream colour. Finally the blank regions indicate regions where the functions have singular behaviour.

Comparison of metric functions:

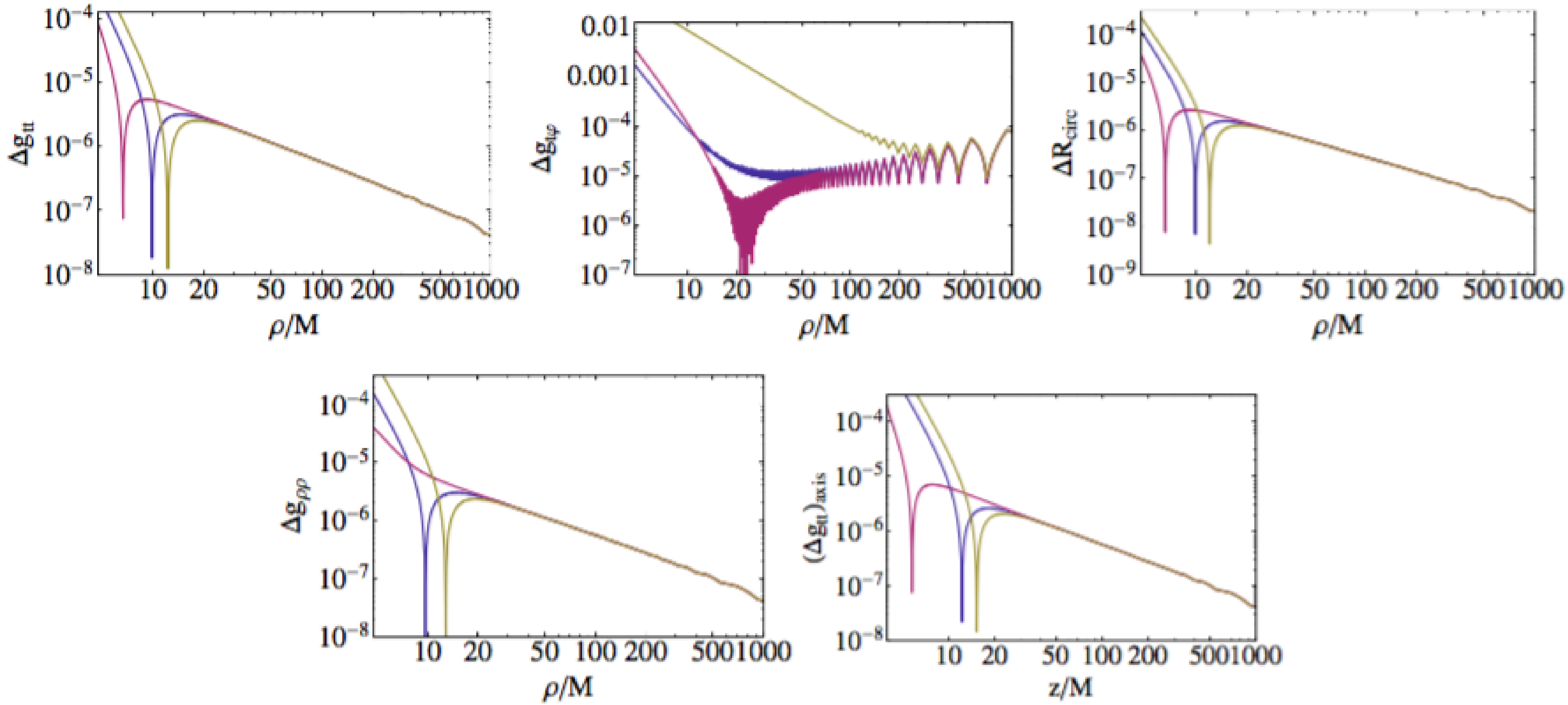


Figure 2. Typical relative difference plots for the metric functions on the equatorial plane and along the axis of symmetry. The plots are made using EoS FPS for a numerical model of $M = 1.4M_{\odot} = 2.0876$km rotating with a spin parameter of $j = 0.453$ and having $\alpha = 4.209$.The plots show three curves which correspond to the metric proposed here, the two-soliton spacetime, and the Hartle-Thorne metric.

Comparison of astrophysical observbles:

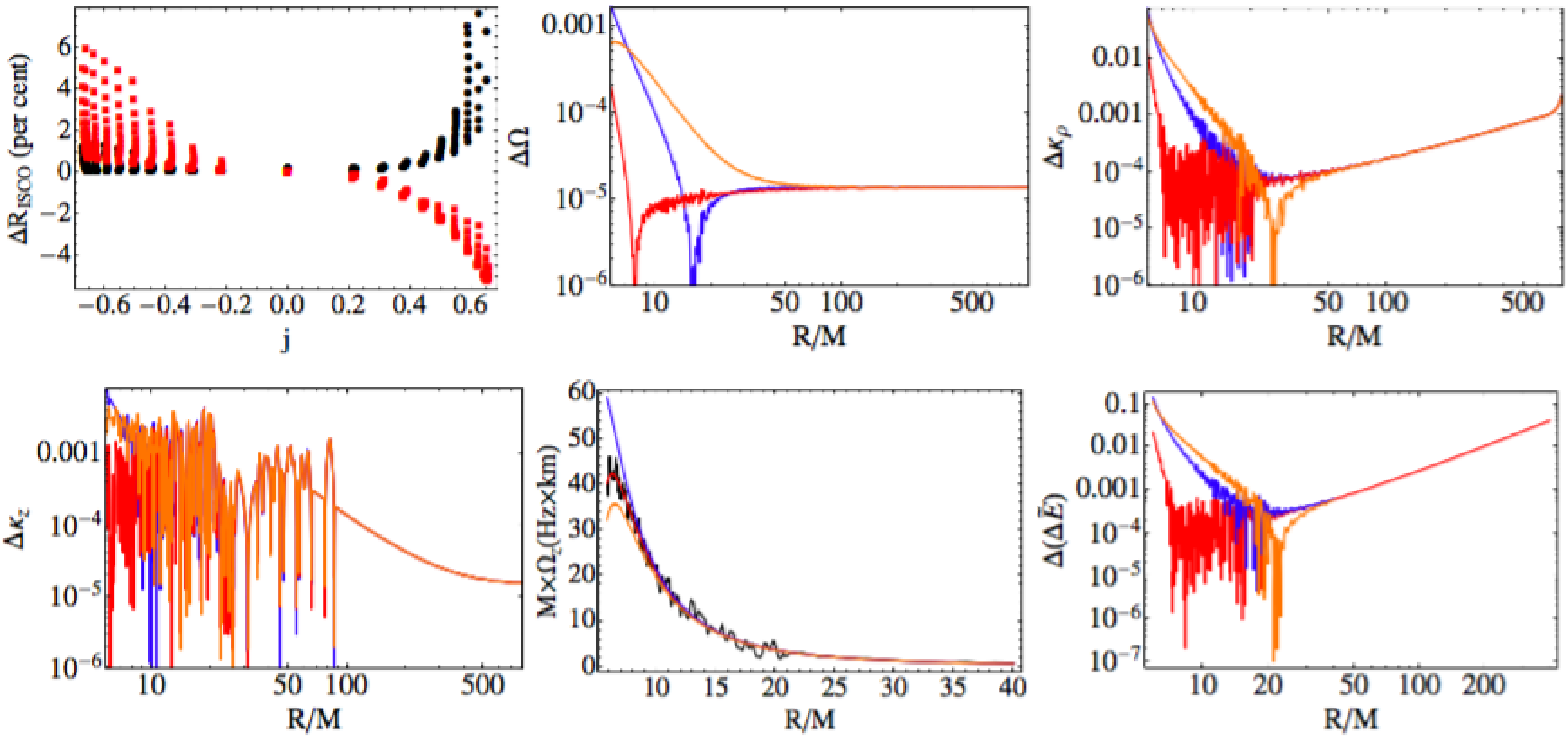
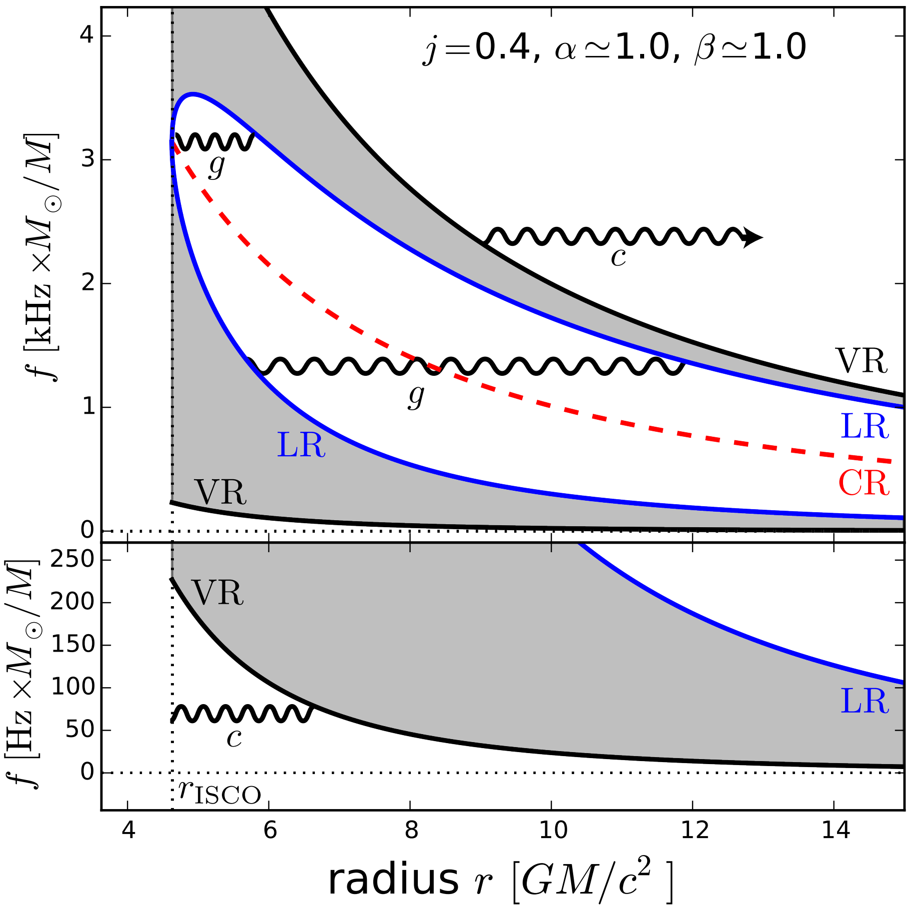


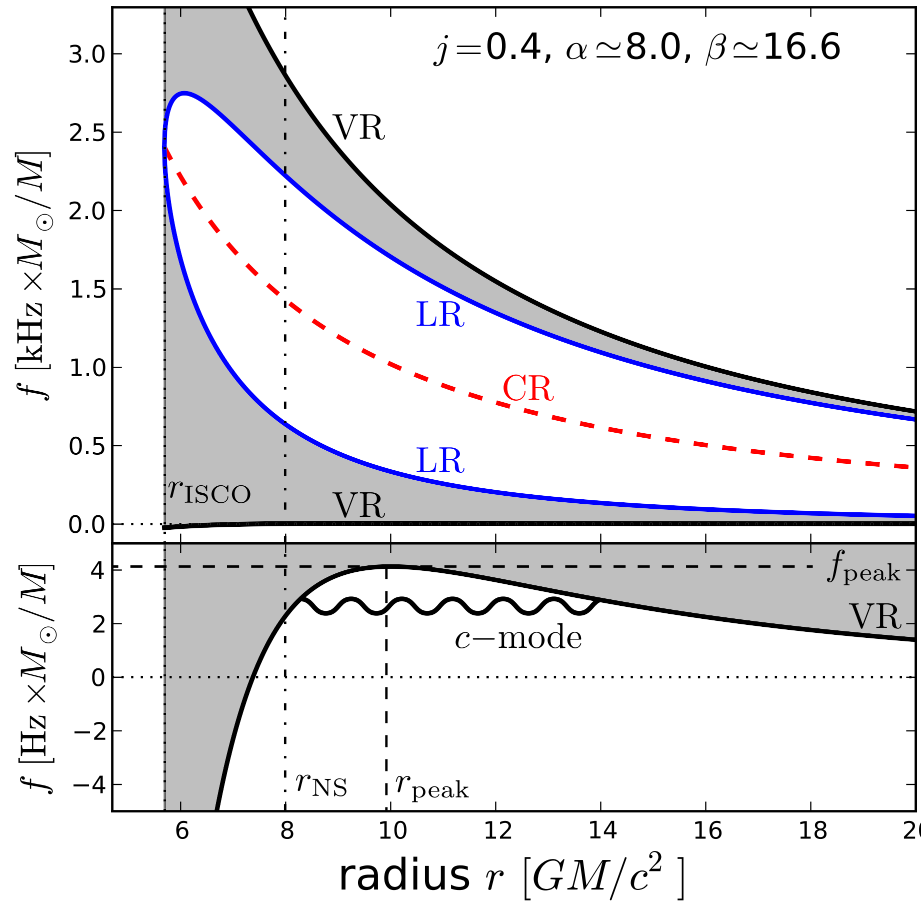
Figure 3. Typical relative difference plots for the various geodesic properties of the spacetime between the numerical spacetime and the corresponding approximate spacetime and the Hartle-Thorne spacetime. The top left plot gives the relative difference of the ISCO for the approximate metric (black circles) and the Hartle-Thorne metric (red squares). The models are constructed with the FPS EoS and we have plotted all the NS models that have an ISCO outside the surface of the star and for which the proposed metric has an ISCO (see discussion in the main text). The top middle plot shows the relative difference in the orbital frequency of circular equatorial orbits, $\Delta\Omega$, as a function of the circumferential radius over the mass, between the three analytic metrics and the numerical metric. The top right plot shows the relative difference for the radial oscillation frequency of radially perturbed orbits and the bottom left plot shows the same for the vertical oscillation frequency of slightly off-equatorial orbits, as the previous plot. The bottom middle plot shows the nodal precession frequency for the numerical and the analytical spacetimes. Finally the bottom right plot shows the relative difference of $\Delta\tilde{E}$ between the numerical and the analytic spacetimes. The frequency and $\Delta\tilde{E}$ plots are constructed using the same model as in figure 2, but the results are similar for all The EoSs.

C-modes in accretion disks around neutron stars.

D. Tsang, G.P., ApJ, 818, L11 (2016)

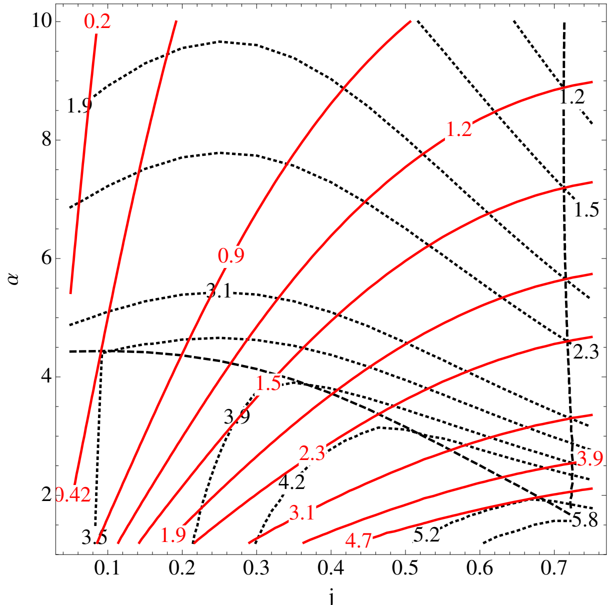


Upper: Diskoseismic propagation diagram for a Kerr black hole with spin parameter $j = 0.4$ for one-armed waves with $m = 1, n = 1$. Waves can propagate in the white regions exterior to $r_{\rm ISCO}$, and are evanescent in the shaded regions between the vertical resonances (VR) and Lindblad resonances (LR). Inertial modes (g-modes), with $m = 1, n = 1$, can become self-trapping due to the turnover of the outer Lindblad resonance, while lower frequency g-modes are quickly damped by corotation (CR). Corrugation waves can propagate at high frequencies exterior to the outer VR, and at low-frequencies interior to the inner VR. **Lower**: Enlargement of the propagation diagram at low frequencies.

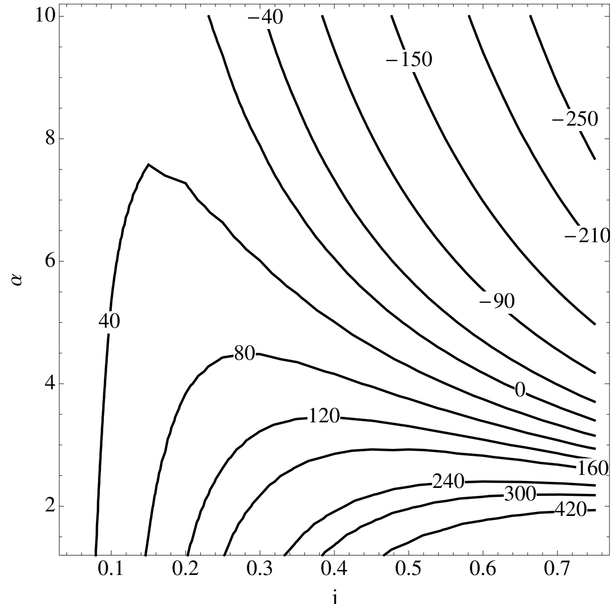


Upper: Same plot but for a neutron star spacetime with spin parameter $j = 0.4$, quadrupole rotational deformability $\alpha = 8$, and spin-octupole deformability $\beta \simeq 16.6$. Waves with frequency $f = \omega/2\pi$ can propagate in the white regions exterior to the NS radius $r_{\rm NS}$ (or wherever the disk is truncated). Wave regions are qualitatively similar to the Kerr black hole, except for the low-frequency c-mode region, where $\omega < \Omega - \Omega_\perp$. **Lower**: At low frequencies c-modes can be self-trapped due to the turnover of the Lense-Thirring frequency, $\Omega - \Omega_\perp$, at radius $r_{\rm peak}$, and frequency $f_{\rm peak}$, as a result of the spacetime quadrupole contribution.

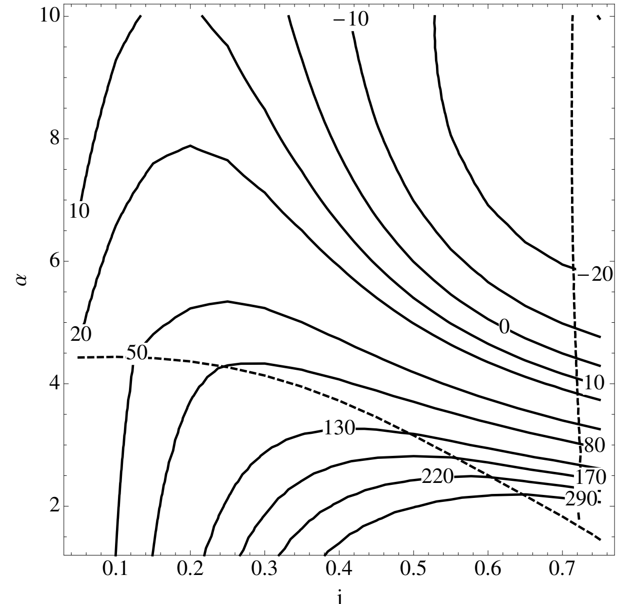
Orbital and precession frequencies: G.P. MNRAS 454, 4066 (2015)



$M \times \nu_K^{max}$ & f_{NS}

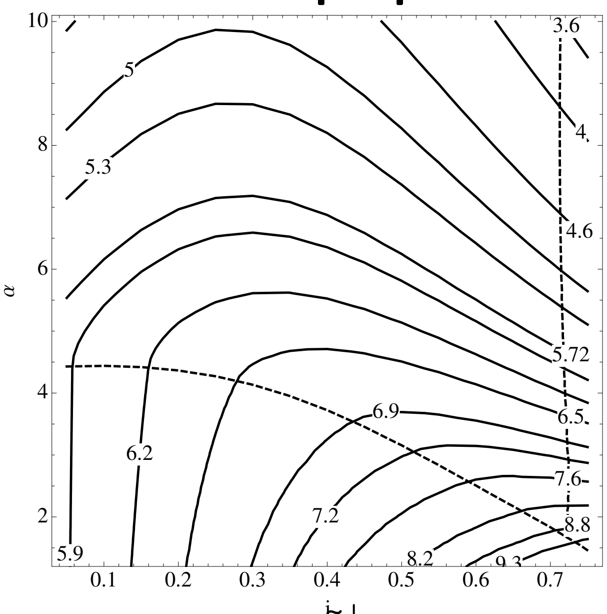


$M \times (\Omega_z/2\pi)_{ISCO}$

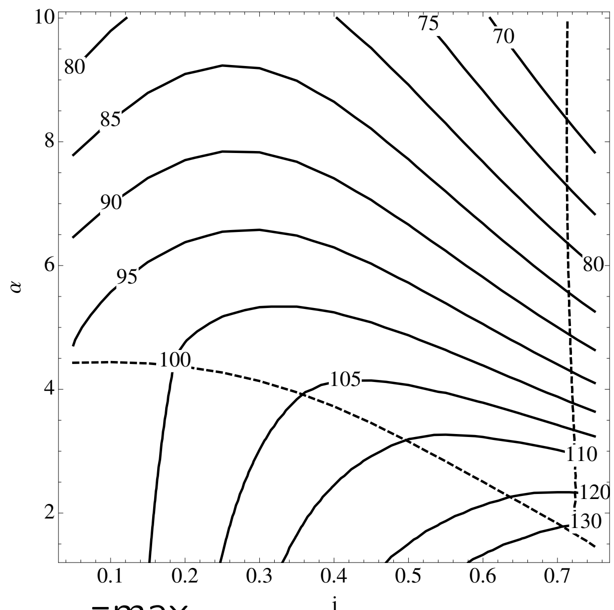


$M \times (\Omega_z/2\pi)_{inn}$

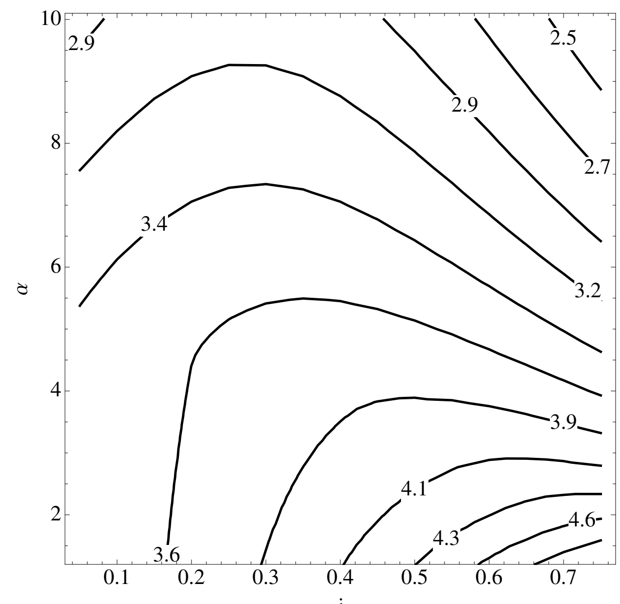
Accretion disc properties:



$\eta = 1 - \tilde{E}\big|_{\text{in}}$

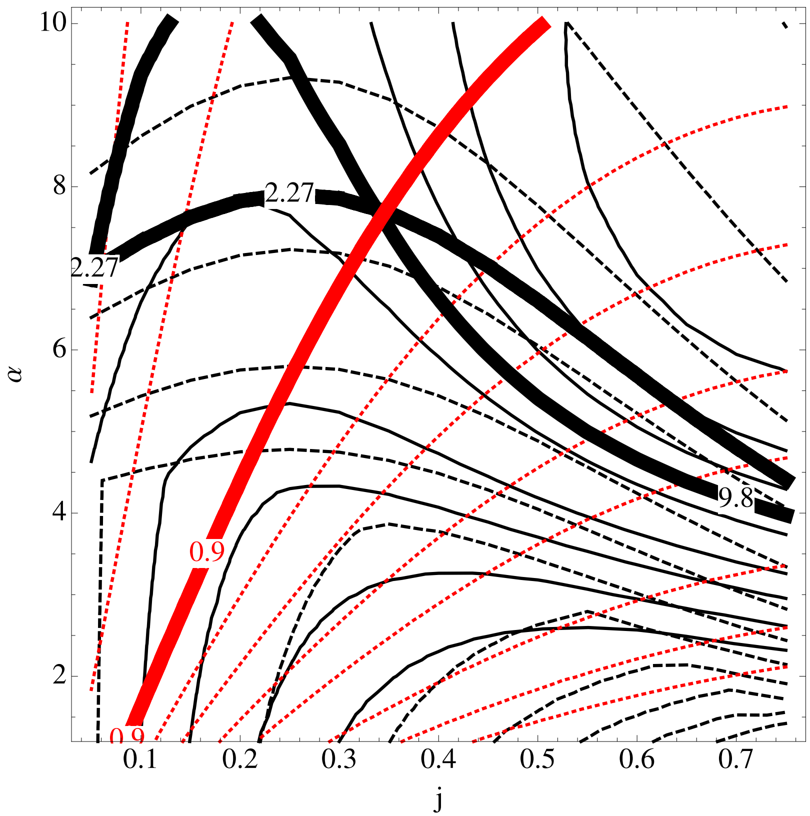


$\bar{T}_{\text{eff}}^{\text{max}}$ $(\dot{M}_0 = 10^{-12} M_\odot \text{year}^{-1})$

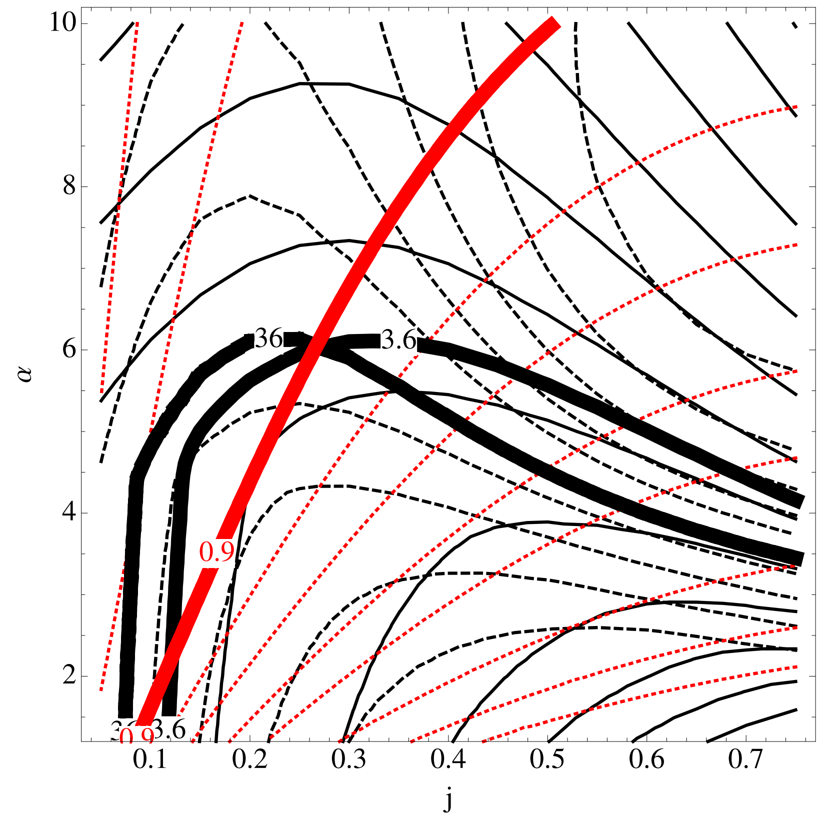


$\nu L_\nu / \dot{M}_0$

Combining the different properties to measure the moments:



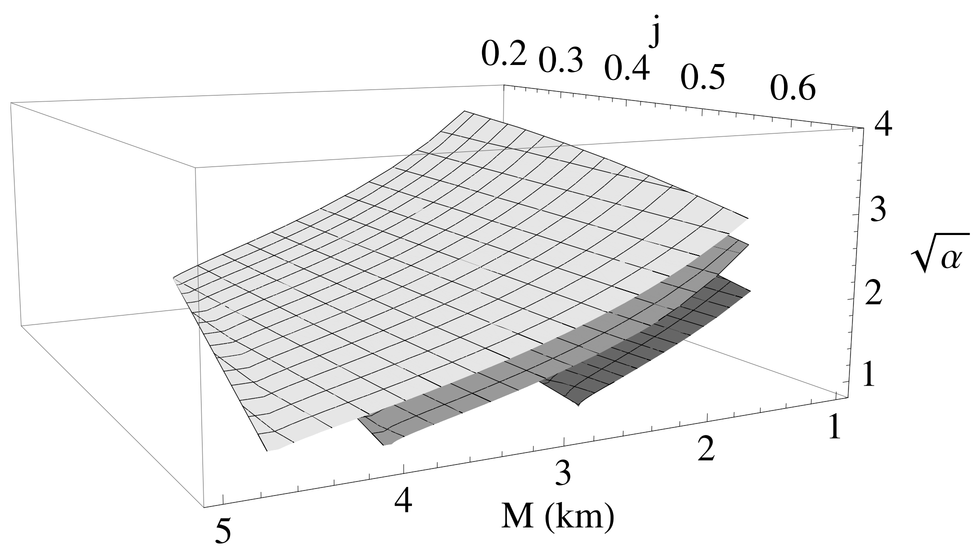
Contour plots of the orbital frequency at the orbit closest to the stellar surface (dashed black lines), the nodal precession frequency at the same orbit (solid black lines), and the rotation frequency of the star itself (dotted red lines)..



Contour plots of the maximum integrated luminosity (solid black lines), the nodal precession frequency at the orbit closest to the stellar surface (dashed black lines), and the rotation frequency of the star itself (dotted red lines).

...and constrain the EOS: Determining the parameters α and j and the independent knowledge of the mass of the neutron star (assuming for example that it is known from the binary system observations), one can evaluate the first three multipole moments.

Such a "measurement"[15] of the first 3 moments (M, J, M_2) could select an EOS[16] out of the realistic EOS candidates.



[15]G.P. MNRAS **454** 4066 (2015),
and for an alternative proposal see, G.P., 2012 MNRAS, 422, 2581-2589.

[16]G.P. and T. A. Apostolatos, Phys.Rev.Lett. **112** 121101 (2014)

In the case of Scalar-Tensor theories with a massless scalar field,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\left(\tilde{R} - 2\tilde{\nabla}^{\mu}\phi\tilde{\nabla}_{\mu}\phi\right) + S_m(g_{\mu\nu},\psi)\,,$$

the field equations in the Einstein frame take the form,

$$\tilde{R}_{ab} = 2\partial_a\phi\partial_b\phi,$$

$$\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\phi = 0$$

which can admit an Ernst formulation,[17]

$$(Re(\mathcal{E}))\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot\nabla\mathcal{E},$$

with the addition of a Laplace equation for the scalar field $\nabla^2\phi = 0$, and the γ function being given by the equations

$$\frac{\partial\gamma}{\partial\rho} = \left(\frac{\partial\gamma}{\partial\rho}\right)_{GR} + \rho\left[\left(\frac{\partial\phi}{\partial\rho}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2\right],$$

$$\frac{\partial\gamma}{\partial z} = \left(\frac{\partial\gamma}{\partial z}\right)_{GR} + 2\rho\left(\frac{\partial\phi}{\partial\rho}\right)\left(\frac{\partial\phi}{\partial z}\right).$$

One can thus extend the previous GR solution to a Scalar-Tensor solution by introducing the additional terms in γ and an appropriate scalar field. The Jordan (physical) frame metric can then be given by the conformal transformation $g_{\mu\nu} = A^2(\phi)\tilde{g}_{\mu\nu}$. This metric can be used to do astrophysics in the same way as in the GR case.[18]

[17] GP, T.P. Sotiriou, Phys. Rev. D91, 044011 (2015)

[18] GP, T.P. Sotiriou, MNRAS 454, 4066 (2015); GP in preparation

Thank You.