The spacetime around neutron stars and astrophysical observables

George Pappas

GRIT/CENTRA

Instituto Superior Técnico

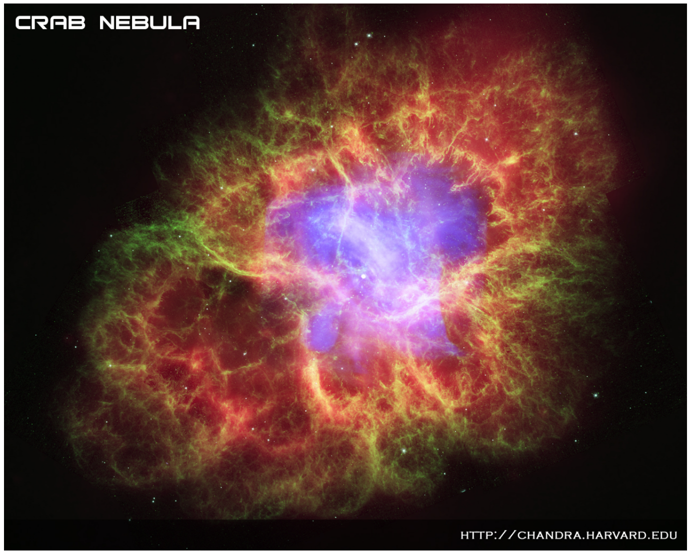
TÉCNICO LISBOA

centra
multidisciplinary centre for astrophysics

grit
gravitation in técnico

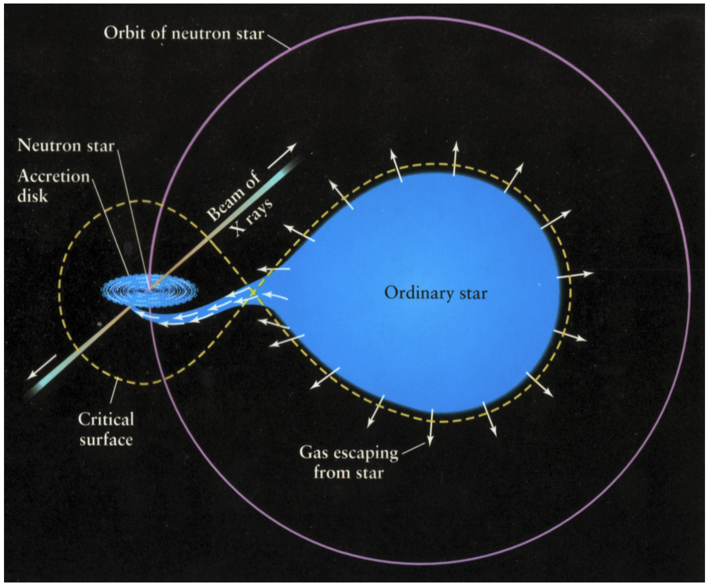
October 27, 2016

- Motivation: Why study neutron stars?
- Neutron Stars (NSs)
  - NS structure
  - Rotating NSs: approximate, numerical, and analytic spacetimes
  - NS multipole moments and NS spacetimes
- A new approximate NS spacetime
- Astrophysical applications
  - Kerr frequencies.
  - Neutron star frequencies.
  - Relation to QPOs.
  - Equation of state at supranuclear densities
- Extension to Scalar-Tensor theories of gravity
  - A spacetime in scalar-tensor theory.
  - Observables (QPOs) and multipole moments: GR vs Scalar-Tensor theory.
- Conclusions and outlook



Neutron stars are the results of stellar evolution.
We can see them in stellar remnants.
A typical example is the Crab nebula that hosts the Crab pulsar[a].

[a] APOD 2006 October 26



Very often we find rapidly rotating pulsars at the end of stellar evolution. The fastest rotating known pulsar (PSR J1748-2446ad) spins at 716Hz and it is part of a binary system[a].

Low mass X-ray binaries are systems that are comprised by a compact object (NS or BH) and a regular star companion. The main source of the X-rays is the accretion disk that forms around the compact object.

[a] J. W. T. Hessels et al., Science **311** 1901 (2006)

Interesting astrophysics takes place around NSs that depends on the background spacetime.

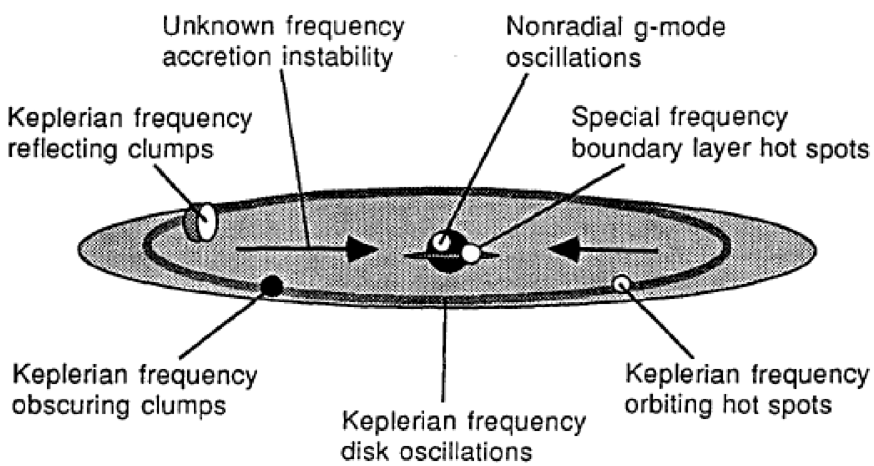
Matter in their interior is at very high densities, where the equation of state is unknown.

NSs have strong enough gravitational fields that can test our theories of gravity.

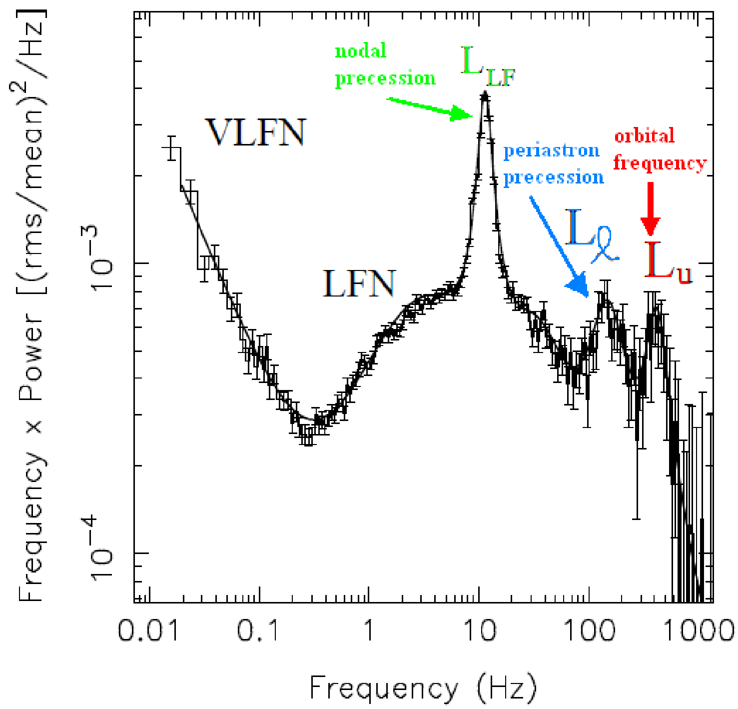
In low-mass X-ray binaries we can have observables that are related to geodesic motion.

An example of observables that can be related to orbits around neutron stars are the quasi-periodic oscillations (QPOs) of the spectrum[1] of an accretion disc.

Mechanisms for producing QPOs[2] from orbital motion



Typical X-Ray spectrum[3]



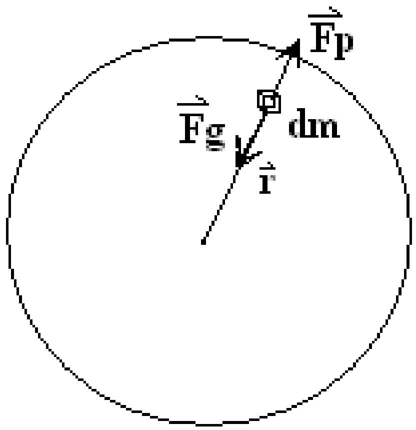
Apart from orbiting hot spots and oscillations on the disc, one could also have precessing rings or misaligned precessing discs, which either themselves have a modulated emission or they are eclipsing the emission from the central object. Furthermore, the characteristics of the emitted spectrum from the accretion provide even more observables.

[1] Stella & Vietri, 1998, ApJ, 492, L59.

[2] F.K. Lamb, Advances in Space Research, 8 (1988) 421.

[3] Boutloukos et al., 2006, ApJ, 653, 1435-1444.

Neutron stars: Fluid configurations that are in equilibrium by the action of their self-gravity and their internal forces.



Newtonian Stars

Hydrostatic equilibrium (spherical symmetry):

$$\nabla P = -\rho \nabla \Phi \Rightarrow \frac{dP}{dr} = -\frac{d\Phi}{dr}\rho = -G\frac{m(r)}{r^2}\rho$$

Mass (spherical symmetry): $\frac{dm}{dr} = 4\pi\rho r^2$

Field equations: $\nabla^2 \Phi = 4\pi G\rho$,

Equation of state for the fluid: $P = P(\rho)$.

Relativistic non-rotating Stars

Instead of a gravitational field Φ, gravity is described by a metric g_{ab}. In spherical symmetry the metric can take the form $ds^2 = -e^{2\Phi}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$.

Field equations: $G^{ab} = 8\pi G T^{ab}$,

Equation for the metric potential Φ: $\frac{d\Phi}{dr} = \frac{m(r)+4\pi r^3 P}{r(r-2m(r))}$,

Definition of the Mass: $\frac{dm}{dr} = 4\pi\rho r^2$,

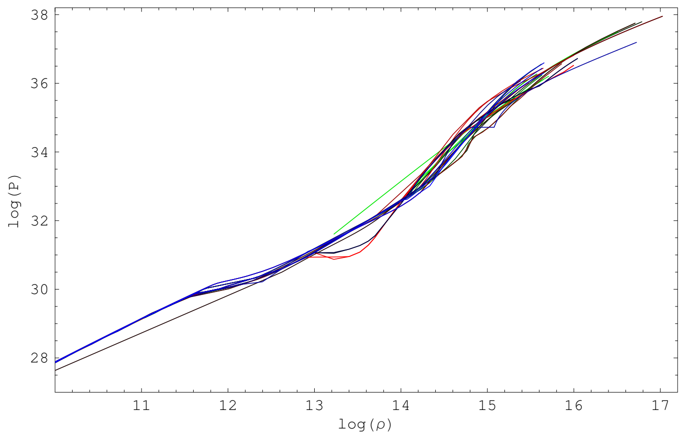
Hydro. equilibrium: $\frac{dP}{dr} = -(\rho+P)\frac{d\Phi}{dr} = -\frac{\rho m(r)}{r^2}\left(1+\frac{P}{\rho}\right)\left(1+\frac{4\pi P r^3}{m(r)}\right)\left(1-\frac{2m(r)}{r}\right)^{-1}$

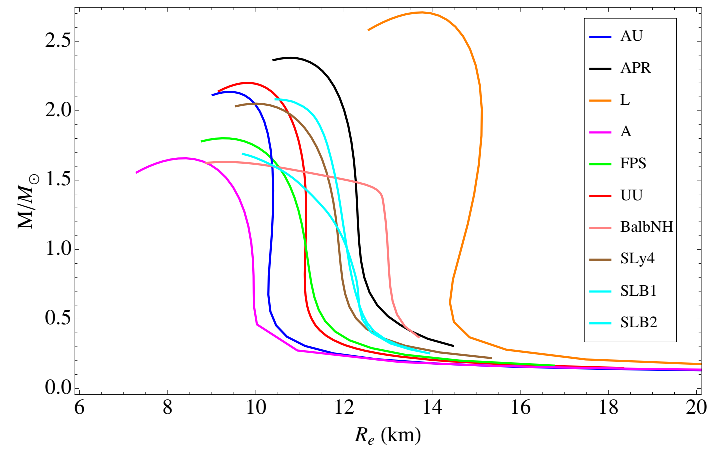
Equation of state for the fluid: $P = P(\rho)$.

The spacetime outside the star is the Schwarzschild spacetime:

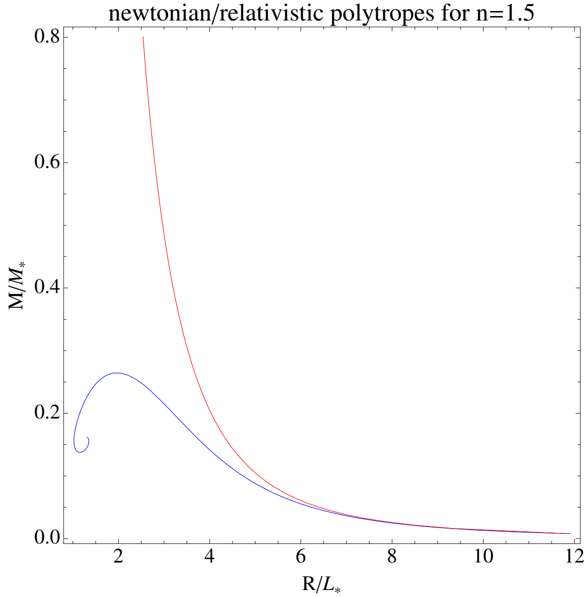
$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

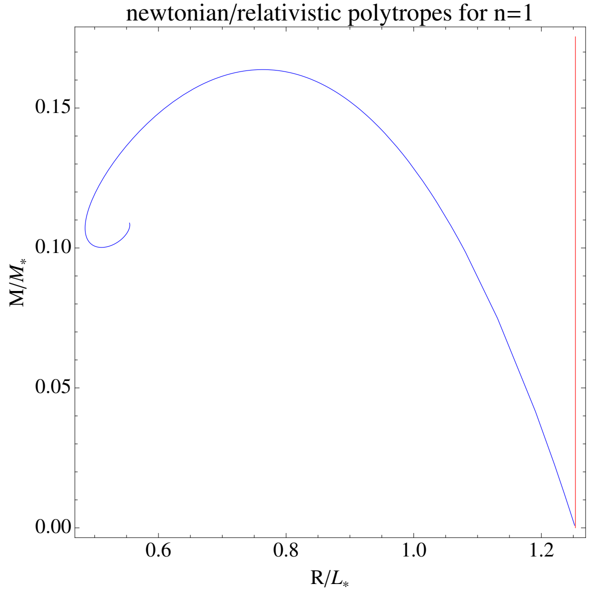
Realistic equations of state

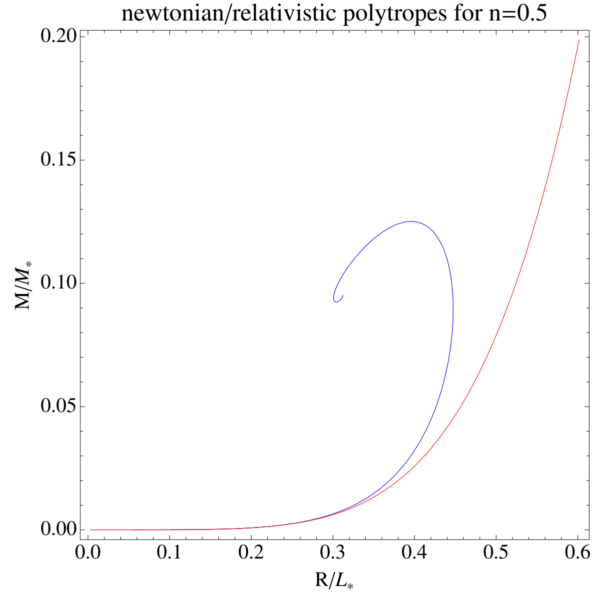




Polytropic equations of state (newtonian polytropes: $M \propto R^{\frac{n-3}{n-1}}$)







* SLB1,2 are observationally inferred EoSs

(A.W. Steiner, J.M. Lattimer, and E.F. Brown, Astrophys. J. **722** 33 (2010)).

Slowly Rotating Neutron Stars

The Hartle-Thorne approximation:[4]

$$ds^2 = -e^{\bar{\nu}}\left(1+2\epsilon^2 h\right)dt^2 + e^{\lambda}\left[1+2\epsilon^2 m/(r-2M)\right]dr^2 + r^2\left[1+2\epsilon^2 k\right]\left[d\theta^2 + \sin^2\theta(d\phi-\epsilon\omega dt)^2\right].$$

where $\epsilon = \Omega/\Omega^*$ is the slow rotation small parameter with respect to $\Omega^* = (M/R^3)^{1/2}$.

Rapidly Rotating Neutron Stars: Numerical

The line element for a stationary and axially symmetric spacetime (the spacetime admits a timelike, ξ^a, and a spacelike, η^a, killing field, i.e., it has rotational symmetry and symmetry in translations in time) is [5],

$$ds^2 = -e^{2\nu}dt^2 + r^2\sin^2\theta B^2 e^{-2\nu}(d\phi-\omega dt)^2 + e^{2(\zeta-\nu)}(dr^2 + r^2 d\theta^2).$$

Komatsu, Eriguchi, and Hechisu[6] proposed a scheme for integrating the field equations using Green's functions. This scheme is implemented by the RNS numerical code to calculate rotating neutron stars [7].

Rapidly Rotating Neutron Stars: Analytic

Using the Weyl-Papapetrou line element that describes stationary and axisymmetric vacuum spacetimes,

$$ds^2 = -f\left(dt-\omega d\phi\right)^2 + f^{-1}\left[e^{2\gamma}\left(d\rho^2+dz^2\right)+\rho^2 d\phi^2\right].$$

Ernst[8] reformulated the Einstein field equations to take the form, $(Re(\mathcal{E}))\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot\nabla\mathcal{E}$, using the complex potential $\mathcal{E}(\rho,z) = f(\rho,z) + i\psi(\rho,z)$, where $f = \xi^a\xi_a$ and ψ is defined by, $\nabla_a\psi = \varepsilon_{abcd}\,\xi^b\nabla^c\xi^d$.

[4] Hartle J. B., Thorne K. S., ApJ **153**, 807 (1968)

[5] E. M. Butterworth and J. R. Ipser, ApJ **204**, 200 (1976).

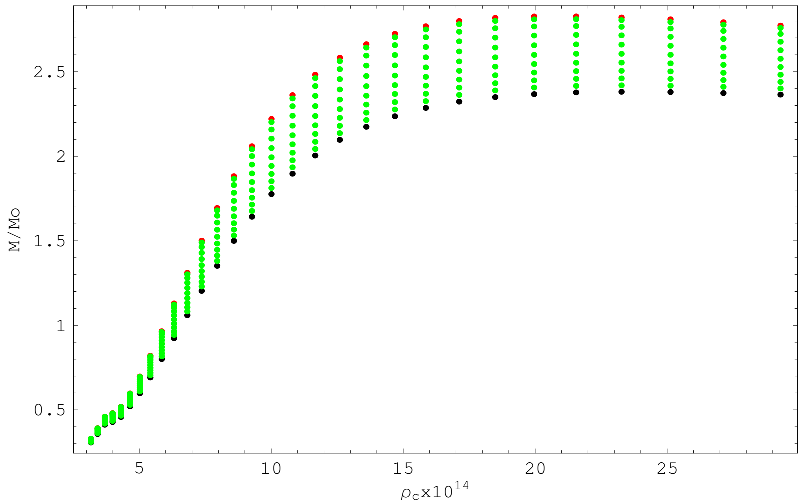
[6] H. Komatsu, Y. Eriguchi, and I. Hechisu, MNRAS **237**, 355 (1989).

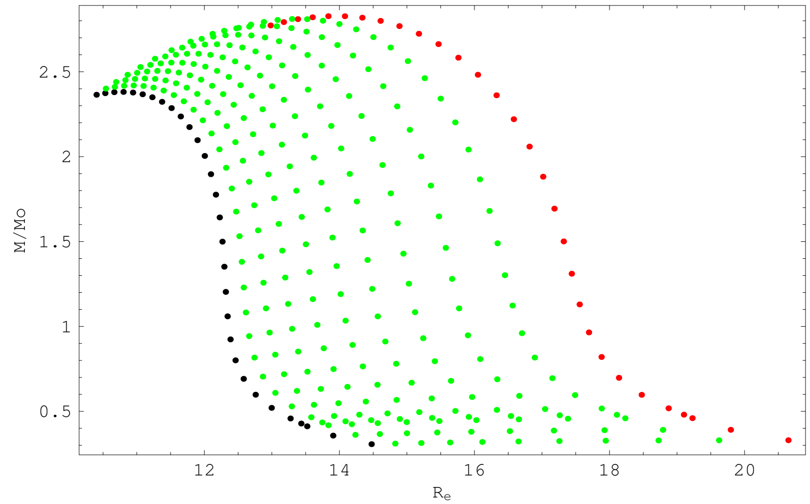
[7] N. Stergioulas, J.L. Friedman, ApJ, **444**, 306 (1995).

[8] F.J. Ernst, Phys. Rev., **167**, 1175 (1968); Phys. Rev., **168**, 1415 (1968).

Results from numerical models:

One can use RNS to calculate models of rotating neutron stars for a given equation of state. For example we show here some models for the APR EOS:





The models with the fastest rotation have a spin parameter, $j = J/M^2$, around 0.7 and a ratio of the polar radius over the equatorial radius, r_p/r_e, around 0.56.

The code, except from the various physical characteristics of the neutron stars, provides the metric functions in a grid on the coordinates x and μ in the whole space (for values from 0 to 1 for both variables), where $\mu = cos\theta$, $r = \frac{xr_e}{1-x}$ and r_e is a length scale.

One can also extract from the spacetime the relativistic multipole moments of the NS. In particular the RNS code can calculates the first non-zero multipole moments, i.e., M, $S_1 \equiv J$, M_2, $S_3 \equiv J_3$ and M_4[9]. These moments characterise the NS and the spacetime around it.

[9] G.P. and T. A. Apostolatos, Phys. Rev. Lett. **108** 231104 (2012),
K. Yagi, K. Kyutoku, G. P., N. Yunes, and T.A. Apostolatos, Phys.Rev. D **89** 124013 (2014).

Neutron star multipole moments in GR

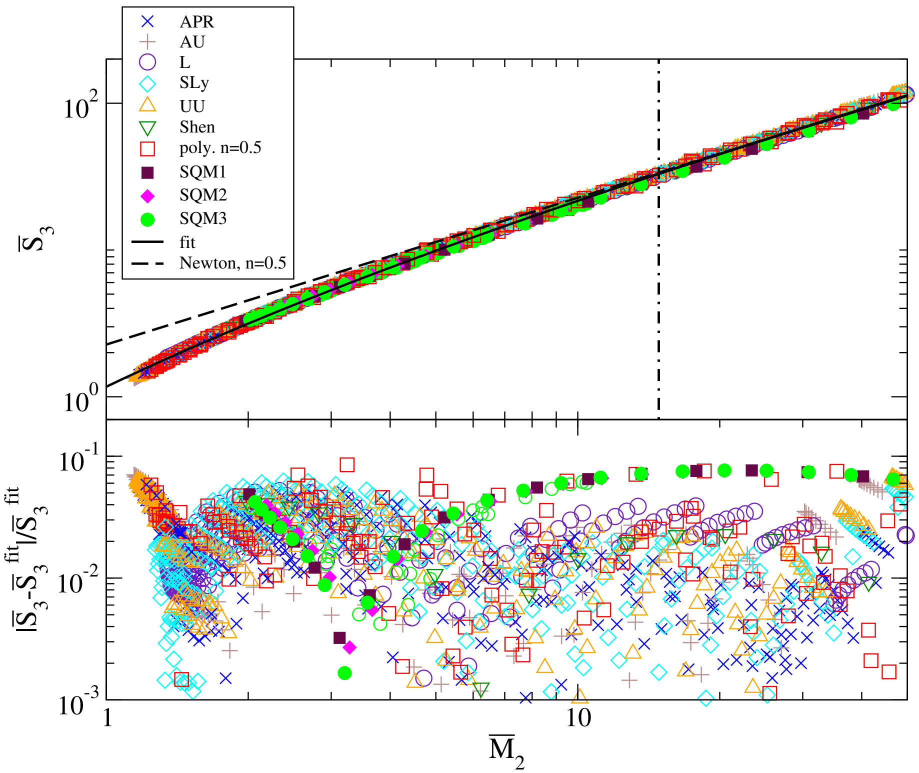
Black Hole-like behaviour of the moments[10]:

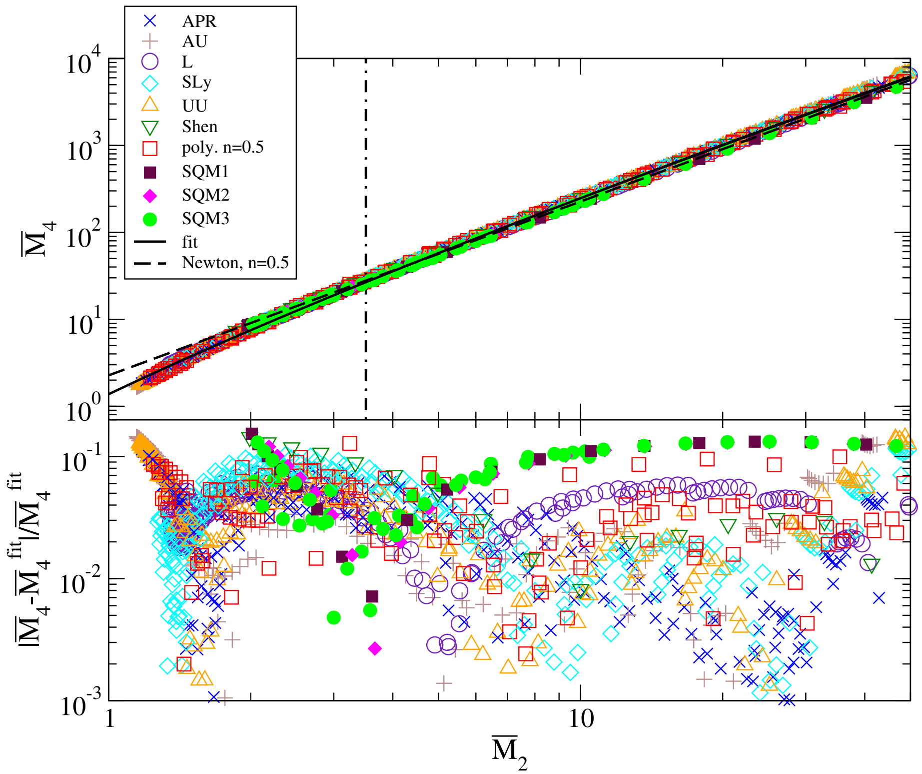
Kerr moments	Neutron star moments
$M_0 = M,$	$M_0 = M,$
$J_1 = J = jM^2,$	$J_1 = jM^2,$
$M_2 = -j^2M^3,$	$M_2 = -a(EoS, M)j^2M^3,$
$J_3 = -j^3M^4,$	$J_3 = -\beta(EoS, M)j^3M^4,$
$M_4 = j^4M^5,$	$M_4 = \gamma(EoS, M)j^4M^5,$
$\vdots$	$\vdots$
$M_{2n} = (-1)^n j^{2n} M^{2n+1},$	$M_{2n} = ?,$
$J_{2n+1} = (-1)^n j^{2n+1} M^{2n+2}$	$J_{2n+1} = ?$

[10] W.G. Laarakkers and E. Poisson, Astrophys. J. **512** 282 (1999).
G.P. and T. A. Apostolatos, Phys. Rev. Lett. **108** 231104 (2012).
K. Yagi, K. Kyutoku, G. P., N. Yunes, and T.A. Apostolatos, Phys.Rev. D **89** 124013 (2014).

Neutron star multipole moments in GR

EoS independent behaviour of the moments[11] :





$$\bar{M}_{2n} = |M_{2n}/(j^{2n}M^{2n+1})|,\ \bar{J}_{2n+1} = |J_{2n+1}/(j^{2n+1}M^{2n+2})|$$

All these are properties that characterise the spacetime around neutron stars as well as the gravitational aspects of the stars themselves.

[11] G.P. and T. A. Apostolatos, Phys.Rev.Lett. **112** 121101 (2014).
K. Yagi, K. Kyutoku, G. P., N. Yunes, and T.A. Apostolatos, Phys.Rev. D **89** 124013 (2014).

Numerical is ok but...

An analytic neutron star spacetime would be much more useful to do astrophysics (easier to implement in models, easier to parameterise, easier to solve the inverse problem)

The vacuum region of a stationary and axially symmetric space-time can be described by the Papapetrou line element[12],

$$ds^2 = -f\,(dt - w d\phi)^2 + f^{-1}\left[e^{2\gamma}\left(d\rho^2 + dz^2\right) + \rho^2 d\phi^2\right],$$

where f, w, and γ are functions of the Weyl-Papapetrou coordinates (ρ, z).

By introducing the complex potential $\mathcal{E}(\rho, z) = f(\rho, z) + i\psi(\rho, z)$ [13], the Einstein field equations take the form,

$$(Re(\mathcal{E}))\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot\nabla\mathcal{E},$$

where, $f = \xi^a\xi_a$ and ψ is defined by, $\nabla_a\psi = \varepsilon_{abcd}\,\xi^b\nabla^c\xi^d$.

An algorithm for generating solutions of the Ernst equation was developed by Sibgatullin and Manko [14]. A solution is constructed from a choice of the Ernst potential along the axis of symmetry in the form of a rational function

$$\mathcal{E}(\rho = 0, z) = e(z) = \frac{P(z)}{R(z)},$$

where $P(z), R(z)$ are polynomials of z of order n with complex coefficients in general.

[12] A. Papapetrou, Ann. Phys., **12**, 309 (1953).

[13] F.J. Ernst, Phys. Rev., **167**, 1175 (1968); Phys. Rev., **168**, 1415 (1968).

[14] V.S. Manko, N.R. Sibgatullin, Class. Quantum Grav., **10**, 1383 (1993).

An example: The two-soliton analytic spacetime.

This is a 4-parameter analytic spacetime which can be produced if one chooses the Ernst potential on the axis to have the form:

$$e(z) = \frac{(z - M - ia)(z + ib) - k}{(z + M - ia)(z + ib) - k}$$

The parameters a, b, k of the spacetime can be related to the first non-zero multipole moments through the equations,

$$J = aM, \quad M_2 = -(a^2 - k)M, \quad J_3 = -\left[a^3 - (2a - b)k\right] M,$$

where M is the mass.

One can use the multipole moments M, J, M_2, and J_3 of a numerically calculated neutron star and produce an analytic two-soliton spacetime that reproduces quite accurately the numerically calculated spacetime[15]. Instead of using a specific set of values for the moments, one could reproduce any neutron star spacetime using the universal relation

$$\sqrt[3]{\bar{J}_3} = A + B_1 \left(\sqrt{\bar{M}_2}\right)^{\nu_1} + B_2 \left(\sqrt{\bar{M}_2}\right)^{\nu_2}.$$

Using this, one can have the first higher moments of a general neutron star spacetime expressed in terms of only three parameters, the mass M, the angular momentum J, and the quadrupole M_2,[16] having thus a universal analytic spacetime.[17]

[15] G. P., and T. A. Apostolatos, MNRAS, **429**, 3007 (2013); Other analytic spacetimes have been proposed in the past, see for eg. E. Berti, and N. Stergioulas, MNRAS, **350**, 1416 (2004)

[16] G.P. and T. A. Apostolatos, Phys.Rev.Lett. **112** 121101 (2014): $\sqrt[3]{\bar{J}_3} = -0.36 + 1.48 \left(\sqrt{\bar{M}_2}\right)^{0.65}$

[17] GP, MNRAS **454**, 4066 (2015)

But, the usual algorithms for constructing analytic solutions, like the two-soliton, result to increasingly complicated spacetimes the higher the number of parameters one introduces to characterise the spacetime (why go higher?).

One could try to evade this difficulty by constructing an approximate spacetime that is an approximate solution of the Ernst equation. Such a spacetime would still be clearly parameterised by the multipole moments. The ansatz that one can use to construct the solution is,

$$\mathcal{E} = \frac{1-\xi(\rho,z)}{1+\xi(\rho,z)}, \qquad \xi = \frac{1}{\sqrt{\rho^2+z^2}} \sum_{i,j=0}^{n,k} a_{ij} \left(\frac{\rho}{\rho^2+z^2}\right)^i \left(\frac{z}{\rho^2+z^2}\right)^j,$$

where the parameters a_{ij} are related to the multipole moments of the spacetime and the ξ expansion is up to the required order.

Out of this Ernst potential one has the functions, $f(\rho,z) = \frac{1}{2}\left(\mathcal{E}+\mathcal{E}^*\right)$, and $\psi(\rho,z) = \frac{1}{2i}\left(\mathcal{E}-\mathcal{E}^*\right)$.

Then from the identity $f^{-2}\nabla\psi = -\rho^{-1}\hat{n}\times\nabla\omega$, one can calculate the metric function $\omega(\rho,z)$.

Finally, the last metric function $\gamma(\rho,z)$ is calculated from the system,

$$\frac{\partial\gamma}{\partial\rho} = \frac{\rho}{4f^2}\left[\left(\frac{\partial f}{\partial\rho}\right)^2 - \left(\frac{\partial f}{\partial z}\right)^2\right] - \frac{f^2}{4\rho}\left[\left(\frac{\partial\omega}{\partial\rho}\right)^2 - \left(\frac{\partial\omega}{\partial z}\right)^2\right], \qquad \frac{\partial\gamma}{\partial z} = \frac{1}{2}\left[\frac{\rho}{f^2}\frac{\partial f}{\partial\rho}\frac{\partial f}{\partial z} - \frac{f^2}{\rho}\frac{\partial\omega}{\partial\rho}\frac{\partial\omega}{\partial z}\right].$$

The approximate M_4-metric:[18]

$$
\begin{aligned}
f(\rho,z) &= 1-\frac{2M}{\sqrt{\rho^2+z^2}}+\frac{2M^2}{\rho^2+z^2}+\frac{\left(M_2-M^3\right)\rho^2-2\left(M^3+M_2\right)z^2}{(\rho^2+z^2)^{5/2}}\\
&\quad+\frac{2z^2\left(-J^2+M^4+2M_2M\right)-2MM_2\rho^2}{(\rho^2+z^2)^3}+\frac{A(\rho,z)}{28\,(\rho^2+z^2)^{9/2}}+\frac{B(\rho,z)}{14\,(\rho^2+z^2)^5},\\
\omega(\rho,z) &= -\frac{2J\rho^2}{(\rho^2+z^2)^{3/2}}-\frac{2JM\rho^2}{(\rho^2+z^2)^2}+\frac{F(\rho,z)}{(\rho^2+z^2)^{7/2}}+\frac{H(\rho,z)}{2\,(\rho^2+z^2)^4}+\frac{G(\rho,z)}{4\,(\rho^2+z^2)^{11/2}},\\
\gamma(\rho,z) &= \frac{\rho^2\left(J^2\left(\rho^2-8z^2\right)+M\left(M^3+3M_2\right)\left(\rho^2-4z^2\right)\right)}{4\,(\rho^2+z^2)^4}-\frac{M^2\rho^2}{2\,(\rho^2+z^2)^2},
\end{aligned}
$$

where,

$$
\begin{aligned}
A(\rho,z)=&\Big[8\rho^2z^2\left(24J^2M+17M^2M_2+21M_4\right)+\rho^4\left(-10J^2M+7M^5+32M_2M^2-21M_4\right)\\
&+8z^4\left(20J^2M-7M^5-22M_2M^2-7M_4\right)\Big],\\
B(\rho,z)=&\Big[\rho^4\left(10J^2M^2+10M_2M^3+21M_4M+7M_2^2\right)-4z^4\left(40J^2M^2+14JS_3-7M^6-30M_2M^3\right.\\
&\left.-14M_4M-7M_2^2\right)-4\rho^2z^2\left(27J^2M^2-21JS_3+7M^6+48M_2M^3+42M_4M+7M_2^2\right)\Big],\\
H(\rho,z)=&\Big[4\rho^2z^2\left(J\left(M_2-2M^3\right)-3MS_3\right)+\rho^4\left(JM_2+3MS_3\right)\Big]\\
G(\rho,z)=&\Big[\rho^2\left(J^3\left(-\left(\rho^4+8z^4-12\rho^2z^2\right)\right)+JM\left(\left(M^3+2M_2\right)\rho^4-8\left(3M^3+2M_2\right)z^4\right.\right.\\
&\left.\left.+4\left(M^3+10M_2\right)\rho^2z^2\right)+M^2S_3\left(3\rho^4-40z^4+12\rho^2z^2\right)\right)\Big]\\
F(\rho,z)=&\Big[\rho^4\left(S_3-JM^2\right)-4\rho^2z^2\left(JM^2+S_3\right)\Big].
\end{aligned}
$$

[18] GP arXiv:1610.05370: This is an approximate vacuum spacetime with $R_{ab}=0+\mathcal{O}(\bar{r}^6)$, where $\bar{r}=(\rho^2+z^2)^{-1/2}$

As with the two-soliton, one can use the multipole moments M, J, M_2, J_3, and additionally this time M_4, of a numerically calculated neutron star and produce an analytic M_4-spacetime that reproduces very accurately the numerically calculated spacetime.

Again, instead of using a specific set of values for the moments, one could reproduce any neutron star spacetime using universal relations of the form

$$\sqrt[3]{\bar{J}_3} = A_1 + B_1\left(\sqrt{\bar{M}_2}\right)^{\nu_1} + B_2\left(\sqrt{\bar{M}_2}\right)^{\nu_2},$$

$$\sqrt[4]{\bar{M}_4} = A_2 + B_3\left(\sqrt{\bar{M}_2}\right)^{\nu_3} + B_4\left(\sqrt{\bar{M}_2}\right)^{\nu_4},$$

Specifically the relations one can use for neutron stars are[19]

$$\sqrt[3]{\bar{J}_3} = -0.36 + 1.48\left(\sqrt{\bar{M}_2}\right)^{0.65},$$

$$\sqrt[4]{\bar{M}_4} = -4.749 + 0.27613\left(\sqrt{\bar{M}_2}\right)^{1.5146} + 5.5168\left(\sqrt{\bar{M}_2}\right)^{0.22229}$$

As before, a general neutron star spacetime is expressed in terms of only three parameters, the mass M, the angular momentum J, and the quadrupole M_2, having in this case as well a universal spacetime.[20]

[19] $\bar{M}_2 \equiv \alpha$, $\bar{J}_3 \equiv \beta$, and $\bar{M}_4 \equiv \gamma$

[20] GP arXiv:1610.05370

Comparison of metric functions:

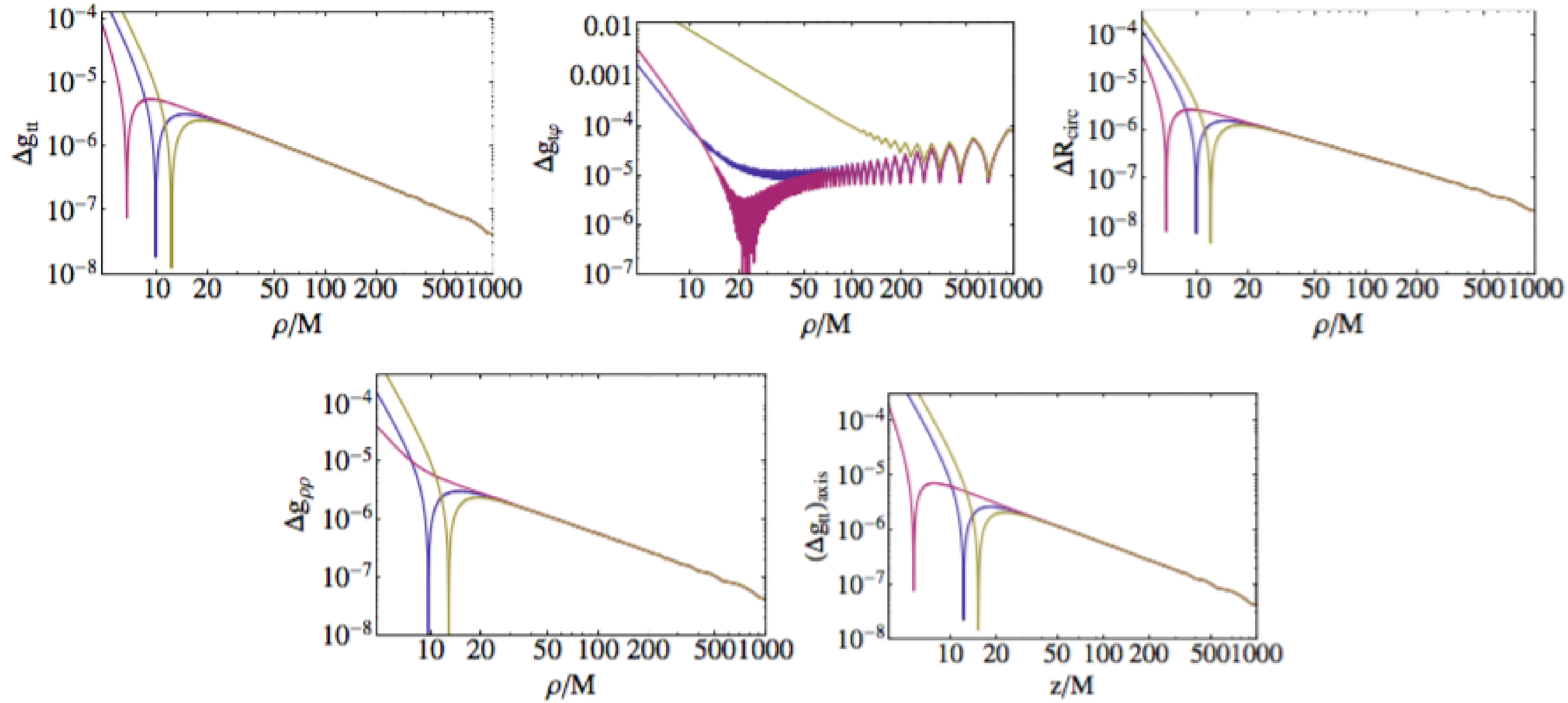


Figure 2. Typical relative difference plots for the metric functions on the equatorial plane and along the axis of symmetry. The plots are made using EoS FPS for a numerical model of $M = 1.4M_{\odot} = 2.0876$km rotating with a spin parameter of $j = 0.453$ and having $\alpha = 4.209$.The plots show three curves which correspond to the metric proposed here, the two-soliton spacetime, and the Hartle-Thorne metric.

Particle motion in a spacetime with symmetries:

Energy and angular momentum integrals of motion, $E = m\left(-g_{tt}\frac{dt}{d\tau} - g_{t\phi}\frac{d\phi}{d\tau}\right)$, $L = m\left(g_{t\phi}\frac{dt}{d\tau} + g_{\phi\phi}\frac{d\phi}{d\tau}\right)$

From the measure of the four-momentum, $p^a p_a = -m^2$, we have the equation,

$$-1 = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + 2g_{t\phi}\left(\frac{dt}{d\tau}\right)\left(\frac{d\phi}{d\tau}\right) + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 + g_{\rho\rho}\left(\frac{d\rho}{d\tau}\right)^2 + g_{zz}\left(\frac{dz}{d\tau}\right)^2 \qquad (1)$$

Circular equatorial orbits: If we define $\Omega \equiv \frac{d\phi}{dt}$, then we have the redshift factor $\left(\frac{d\tau}{dt}\right)^2 = -g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2$,

and the energy, the angular momentum and the orbital frequency for the circular orbits take the form,

$$\tilde{E} \equiv \frac{E}{m} = \frac{-g_{tt} - g_{t\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}, \quad \tilde{L} \equiv \frac{L}{m} = \frac{g_{t\phi} + g_{\phi\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}, \quad \Omega = \frac{-g_{t\phi,\rho} + \sqrt{(g_{t\phi,\rho})^2 - g_{tt,\rho}g_{\phi\phi,\rho}}}{g_{\phi\phi,\rho}} \qquad (2)$$

More general orbits: Equation (1) can take a more general form in terms of the constants of motion,

$$-g_{\rho\rho}\left(\frac{d\rho}{d\tau}\right)^2 - g_{zz}\left(\frac{dz}{d\tau}\right)^2 = 1 - \frac{\tilde{E}^2 g_{\phi\phi} + 2\tilde{E}\tilde{L}g_{t\phi} + \tilde{L}^2 g_{tt}}{(g_{t\phi})^2 - g_{tt}g_{\phi\phi}} = V_{eff}, \qquad (3)$$

With equation (3) we can study the precession properties from the properties of the effective potential.

$$-g_{\rho\rho}\left(\frac{d(\delta\rho)}{d\tau}\right)^2 - g_{zz}\left(\frac{d(\delta z)}{d\tau}\right)^2 = \frac{1}{2}\frac{\partial^2 V_{eff}}{\partial\rho^2}(\delta\rho)^2 + \frac{1}{2}\frac{\partial^2 V_{eff}}{\partial z^2}(\delta z)^2,$$

This equation describes two harmonic oscillators with frequencies, $\bar{\kappa}_\rho^2 = \frac{g^{\rho\rho}}{2}\frac{\partial^2 V_{eff}}{\partial\rho^2}\Big|_c$, $\bar{\kappa}_z^2 = \frac{g^{zz}}{2}\frac{\partial^2 V_{eff}}{\partial z^2}\Big|_c$.

The differences of these frequencies (corrected with the redshift factor) from the orbital frequency, $\Omega_a = \Omega - \kappa_a$, define the precession frequencies.

Comparison of astrophysical observbles:

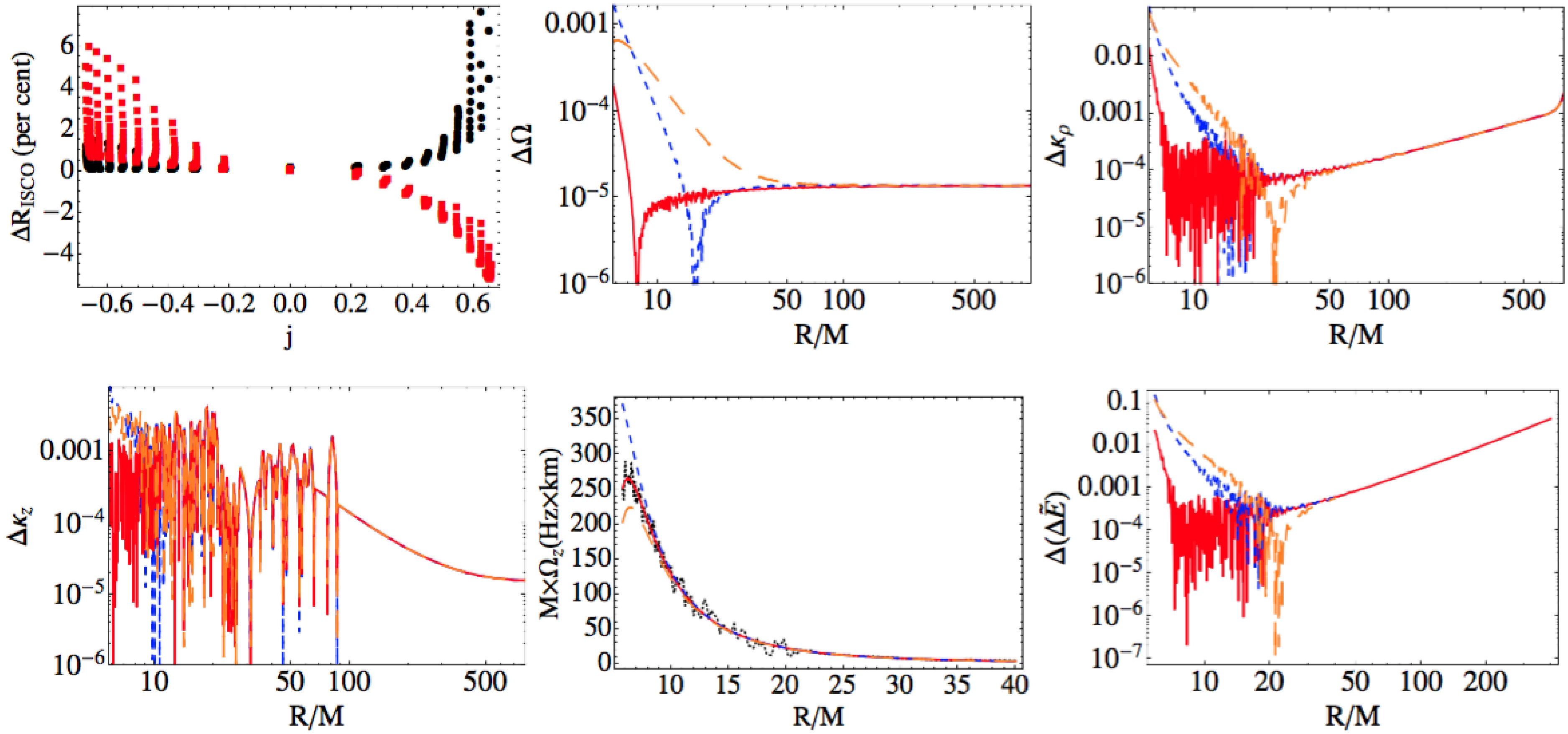


Figure 3. Typical relative difference plots for the various geodesic properties of the spacetime between the numerical spacetime and the corresponding analytic spacetime. The top left plot gives the relative difference of the ISCO for the approximate metric (black circles) and the Hartle-Thorne metric (red squares). The models are constructed with the FPS EoS and we have plotted all the NS models that have an ISCO outside the surface of the star and for which the proposed metric has an ISCO (see discussion in the main text). The top middle plot shows the relative difference in the orbital frequency of circular equatorial orbits, $\Delta\Omega$, as a function of the circumferential radius over the mass, between the three analytic metrics and the numerical metric. The top right plot shows the relative difference for the radial oscillation frequency of radially perturbed orbits and the bottom left plot shows the relative difference for the vertical oscillation frequency of slightly off-equatorial orbits. The bottom middle plot shows the nodal precession frequency $M \times \Omega_z$ for the numerical and the analytic spacetimes (we remind that $\Omega_z = 2\pi\nu_z$). Finally the bottom right plot shows the relative difference of $\Delta\tilde{E}$ between the numerical and the analytic spacetimes. The frequency and $\Delta\tilde{E}$ plots are constructed using the same model as in figure 2, but the results are similar for all the EoSs. The curves correspond to the metric proposed here (red solid curve), the two-soliton spacetime (blue dotted curve), and the Hartle-Thorne metric (orange dashed curve). The nodal precession frequency plot (bottom middle) shows also the numerical frequency (black) which follows the proposed metric curve.

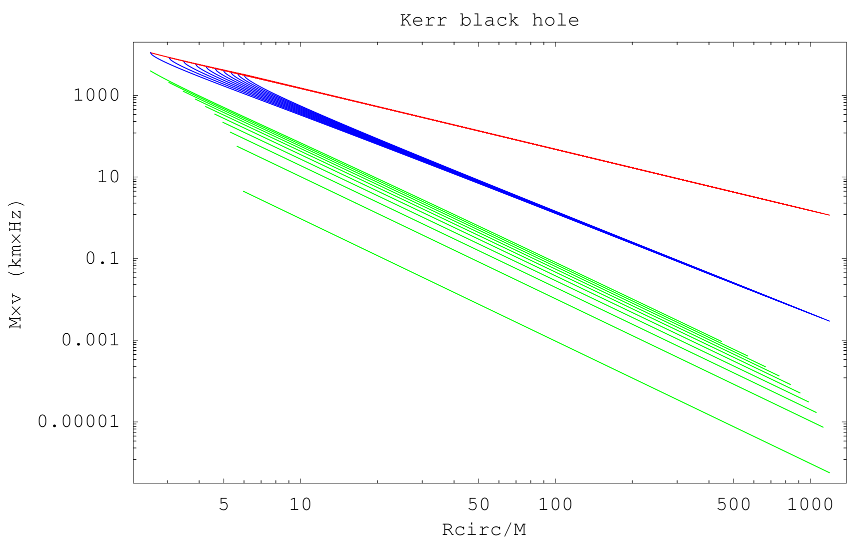
"Scaled" Frequencies for the Kerr spacetime:

Scaled Orbital frequency: $M\Omega = \frac{299790\sqrt{1/r^3}}{1+j(1/r)^{3/2}}$

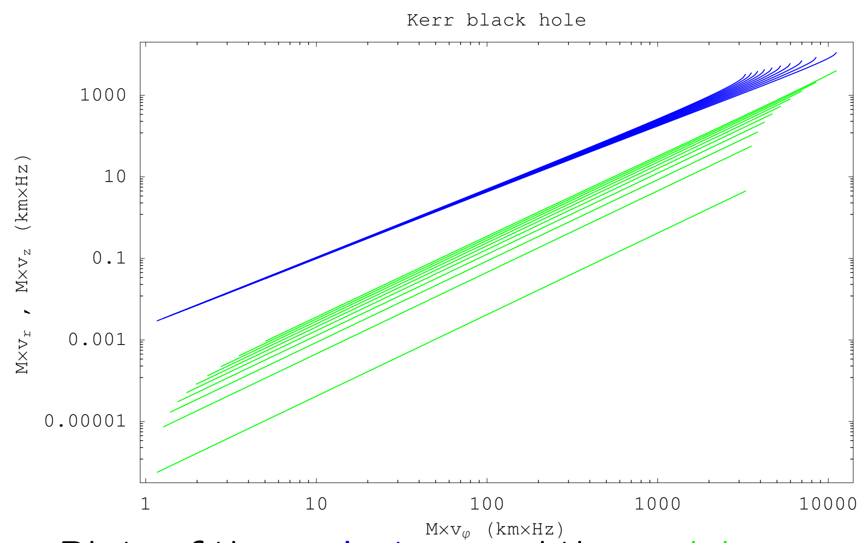
Scaled Radial frequency: $M\kappa_\rho = M\Omega\left(1 - 6(1/r) + 8j(1/r)^{3/2} - 3j^2(1/r)^2\right)^{1/2}$

Scaled Vertical frequency: $M\kappa_z = M\Omega\left(1 - 4j(1/r)^{3/2} + 3j^2(1/r)^2\right)^{1/2}$

Orbital, $M\nu_\phi$, and precession, $M\nu_a$, "scaled frequencies" for Kerr black holes for various j (0.01-0.91).



Plots of the orbital, periastron and nodal precession "scaled frequencies".

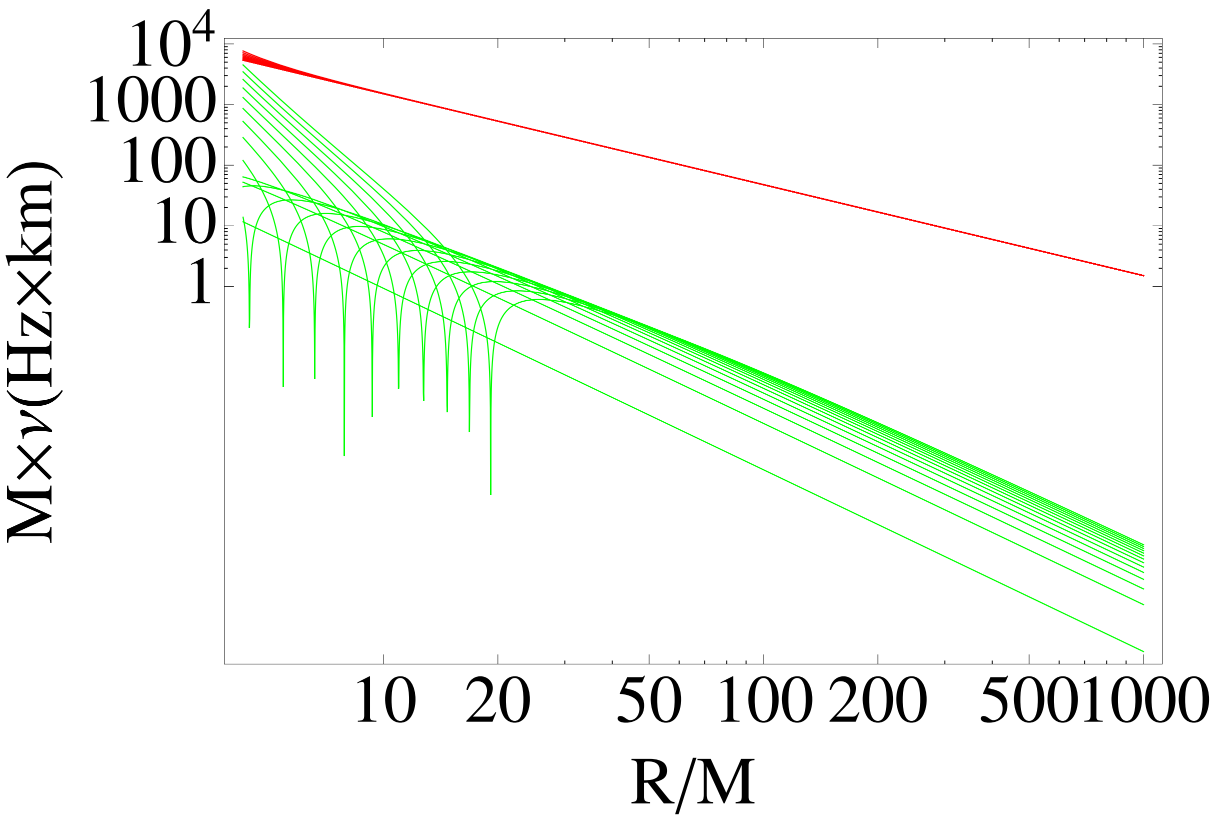


Plots of the periastron and the nodal precession "scaled frequencies" against the orbital "scaled frequency".

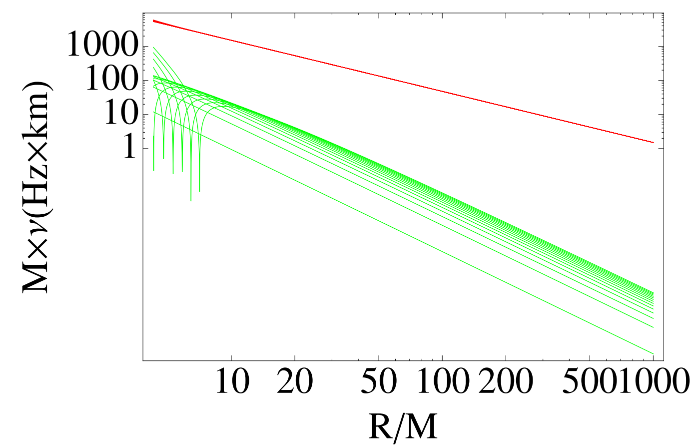
The general effect of rotation is to increase the observed frequencies (and reduce the ISCO radius; for $j \sim 1$ the horizon and ISCO radii go to $1M$).

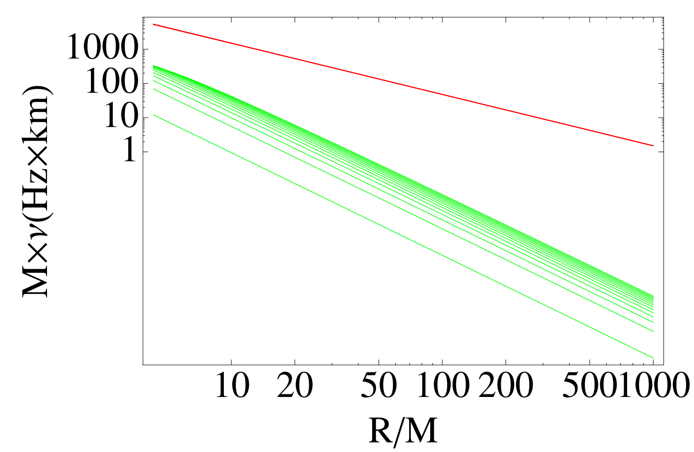
Frequencies for the spacetime around neutron stars

Orbital frequency and nodal precession frequency:
Orbital, $M\nu_\phi$, and precession, $M\nu_z$, "scaled frequencies" for neutron star models constructed with 3 different values of α for various j up to the Kepler limit(~ 0.7).



Plots of the orbital and nodal precession frequencies for different rotations for models with $\alpha = 8$. Rotation in the range, $j \sim 0.01 - 0.66$. This value of α corresponds to NSs with $M \sim 1 - 1.3 M_\odot$ depending on the EOS.



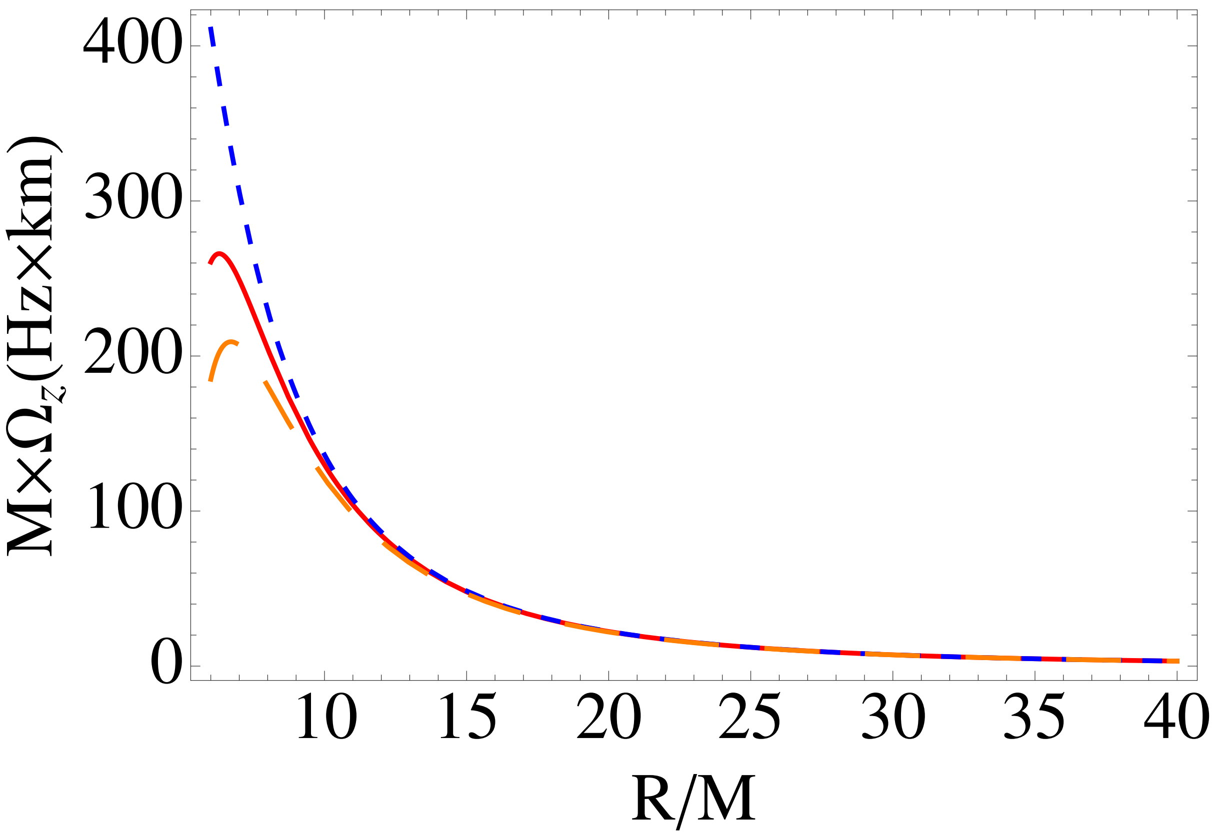


Same plots for $\alpha = 4$ (upper) and $\alpha = 2$ (lower).

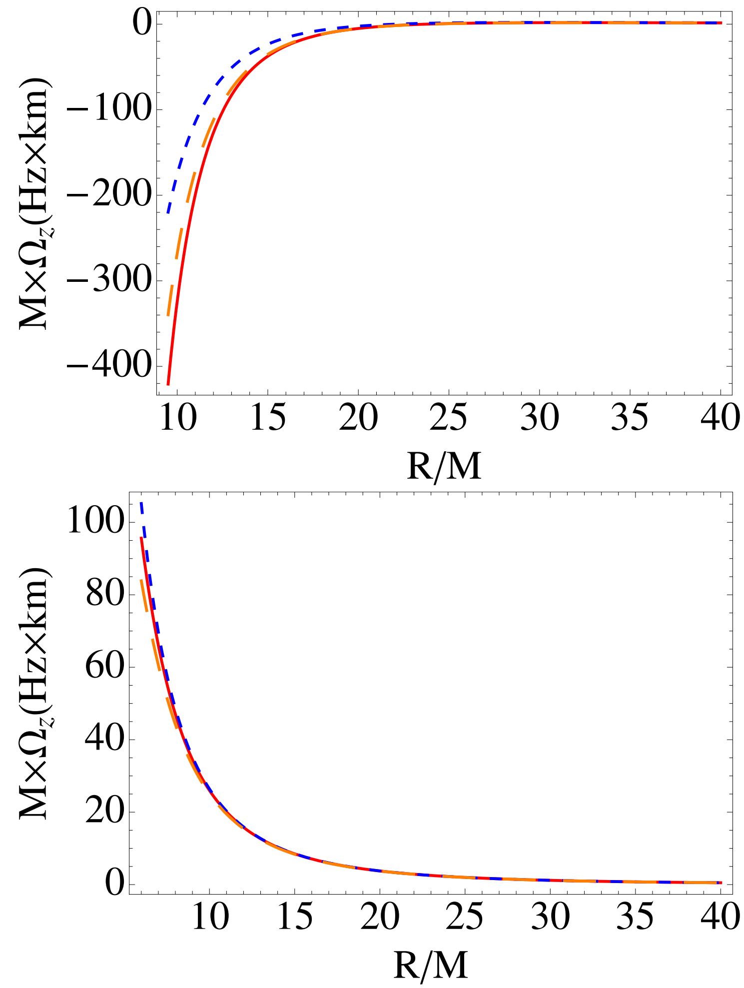
Frequencies for the spacetime around neutron stars

Nodal precession frequency and the importance of the higher moments (M_4 matters):

Nodal precession, $M\Omega_z$, "scaled frequencies" for neutron star models constructed with 3 different values of α using 3 different models for the spacetime, i.e., the two-soliton (blue), the exterior Hartle-Thorne (orange), and the M_4-metric.



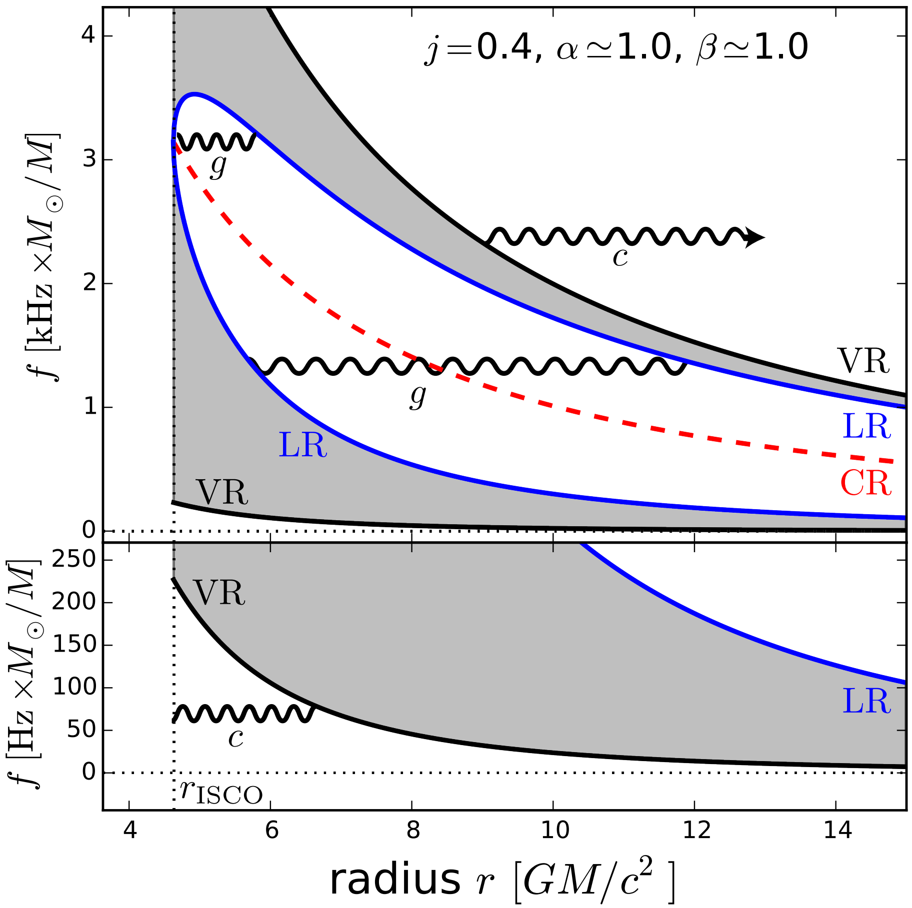
Nodal precession frequencies for $\alpha = 8$ and $j = 0.45$.



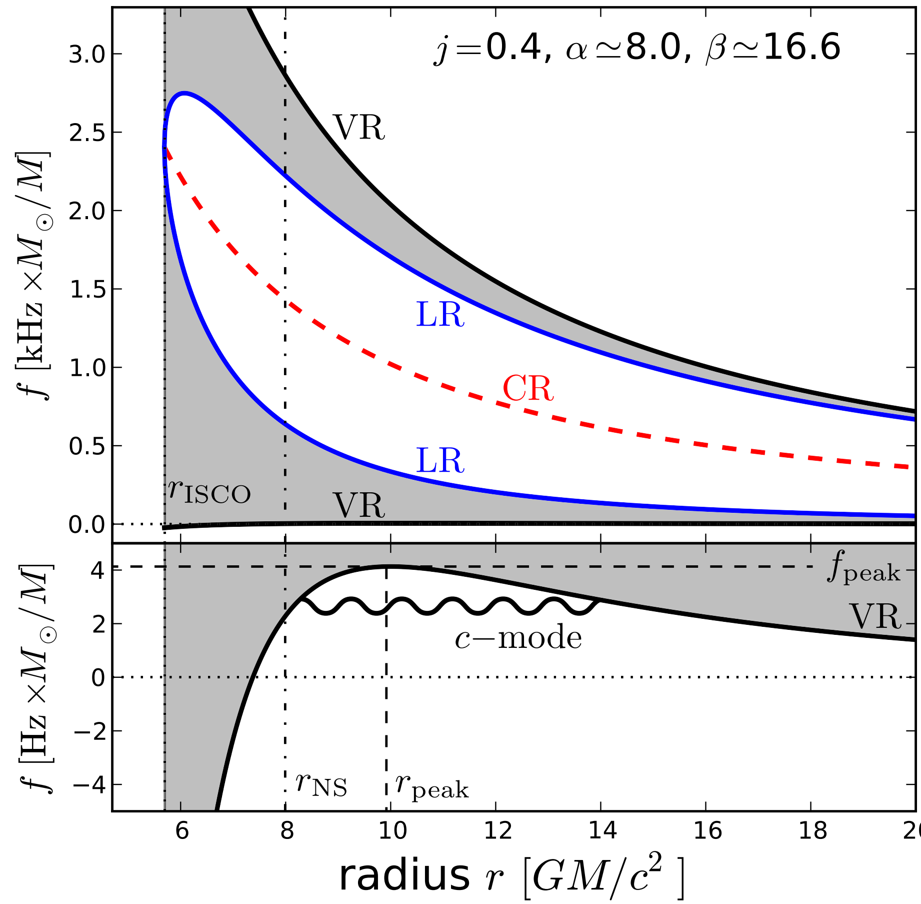
Same plots for $\alpha = 10$, $j = 0.6$ (upper) and $\alpha = 3$, $j = 0.4$ (lower).

C-modes in accretion disks around neutron stars.

D. Tsang, G.P., ApJ, 818, L11 (2016)



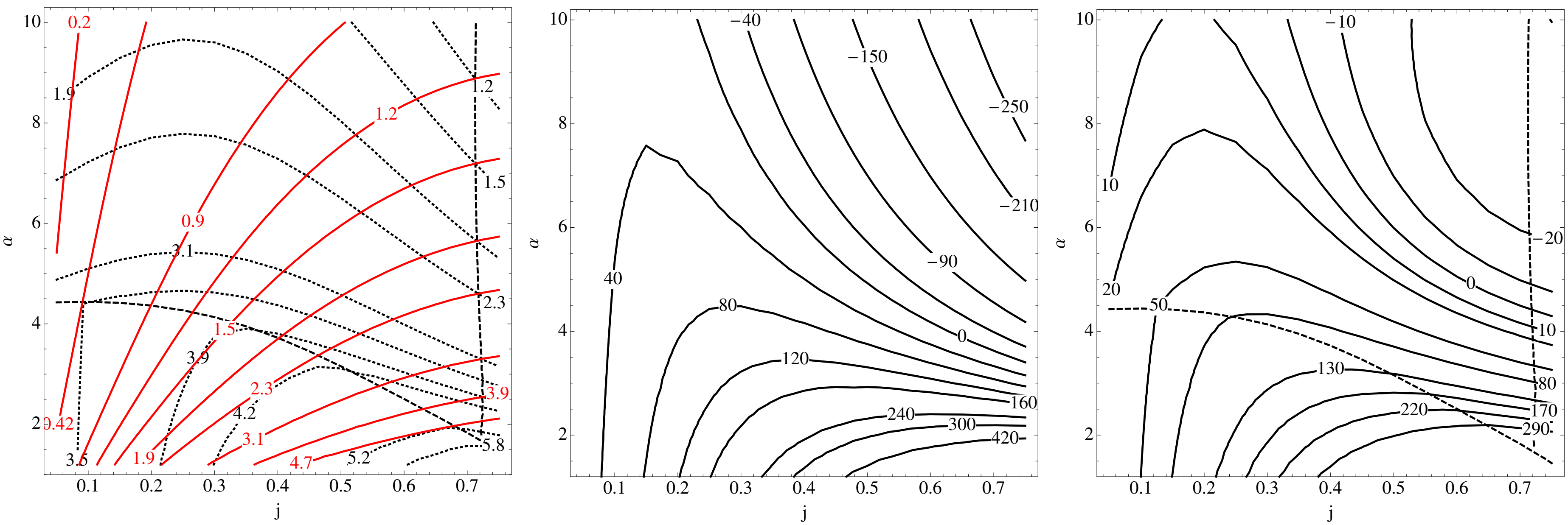
Upper: Diskoseismic propagation diagram for a Kerr black hole with spin parameter $j = 0.4$ for one-armed waves with $m = 1, n = 1$. Waves can propagate in the white regions exterior to $r_{\rm ISCO}$, and are evanescent in the shaded regions between the vertical resonances (VR) and Lindblad resonances (LR). Inertial modes (g-modes), with $m = 1, n = 1$, can become self-trapping due to the turnover of the outer Lindblad resonance, while lower frequency g-modes are quickly damped by corotation (CR). Corrugation waves can propagate at high frequencies exterior to the outer VR, and at low-frequencies interior to the inner VR. **Lower**: Enlargement of the propagation diagram at low frequencies.



Upper: Same plot but for a neutron star spacetime with spin parameter $j = 0.4$, quadrupole rotational deformability $\alpha = 8$, and spin-octupole deformability $\beta \simeq 16.6$. Waves with frequency $f = \omega/2\pi$ can propagate in the white regions exterior to the NS radius $r_{\rm NS}$ (or wherever the disk is truncated). Wave regions are qualitatively similar to the Kerr black hole, except for the low-frequency c-mode region, where $\omega < \Omega - \Omega_\perp$. **Lower**: At low frequencies c-modes can be self-trapped due to the turnover of the Lense-Thirring frequency, $\Omega - \Omega_\perp$, at radius $r_{\rm peak}$, and frequency $f_{\rm peak}$, as a result of the spacetime quadrupole contribution.

Application of the analytic spacetime (solving the inverse problem)

Orbital and precession frequencies:[21]



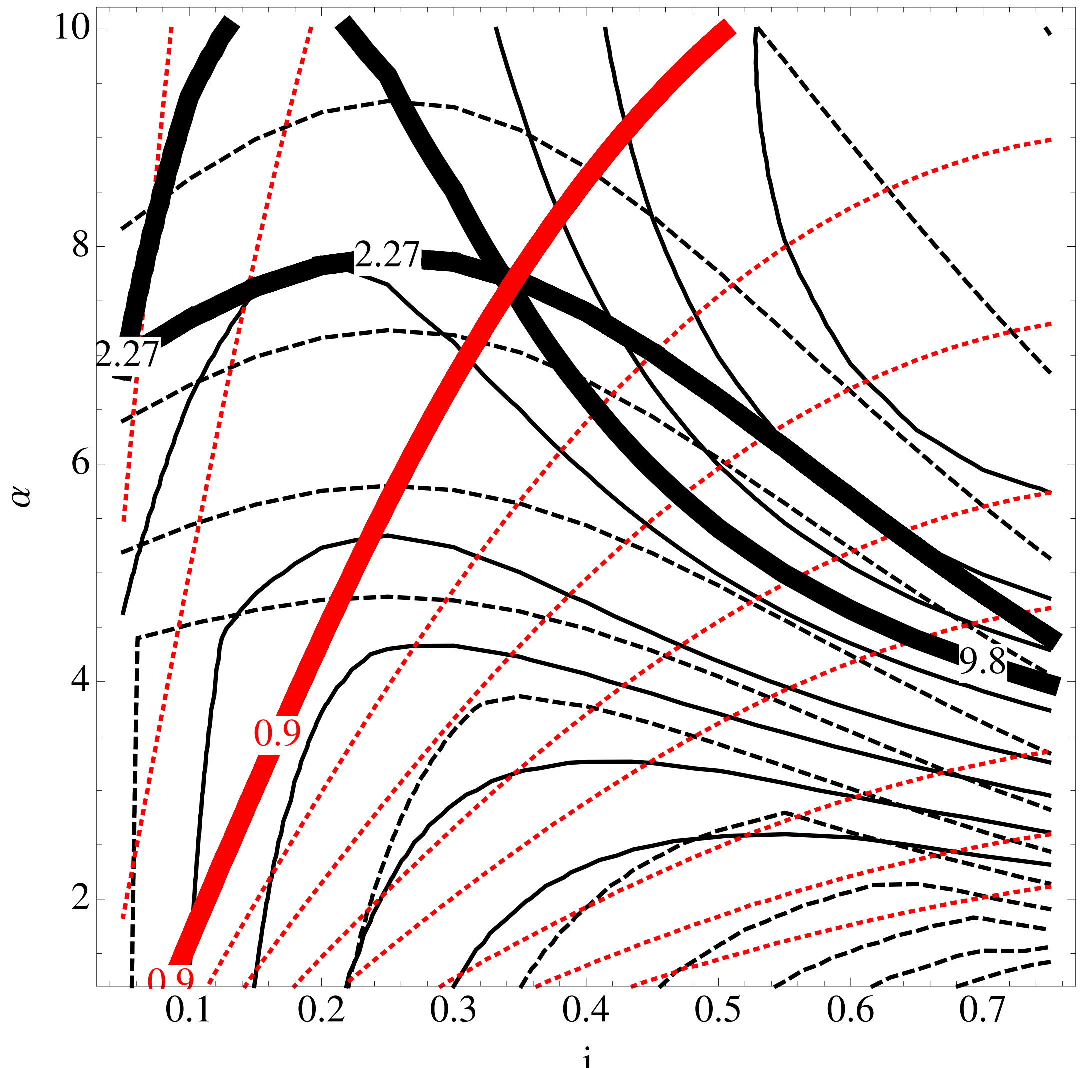
$M \times \nu_K^{max}$ & $M \times f_{NS}$ $\quad$ $M \times (\nu_z)_{ISCO}$ $\quad$ $M \times (\nu_z)_{inn}$

With a 3-parameter (Mass, angular momentum, quadrupole) NS spacetime we can have contour plots of mass reduced quantities in terms of the spin parameter $j = J/M^2$ and the quadrupolar deformability $\alpha = M_2/(j^2 M^3)$.

Here we can see some contour plots of $M \times \nu_\phi$ at the orbit closest to the stellar surface, $M \times \nu_z$ at the same orbit and at the ISCO, and $M \times f_{NS}$.

[21] G.P., 2012 MNRAS, 422, 2581-2589.

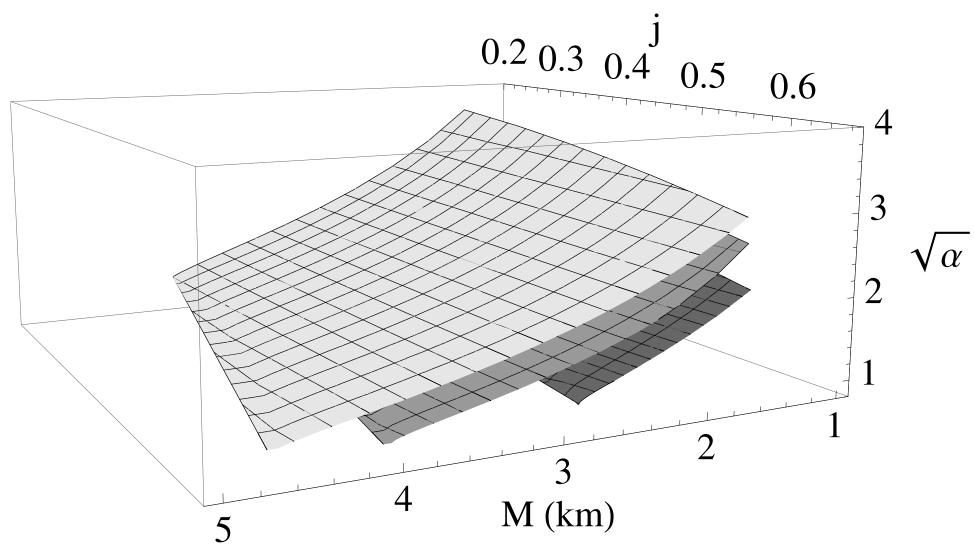
Combining the different properties to measure the moments...



Contour plots of $M \times \nu_\phi$ at the orbit closest to the stellar surface (dashed black lines), $M \times \nu_z$ at the same orbit (solid black lines), and $M \times f_{NS}$ (dotted red lines).

...and constrain the EOS:
Determining the parameters α and j and the independent knowledge of the mass of the neutron star (assuming for example that it is known from the binary system observations), one can evaluate the first three multipole moments.

Such a "measurement"[22] of the first 3 moments (M, J, M_2) could select an EOS[23] out of the realistic EOS candidates.



[22]G.P. MNRAS **454** 4066 (2015),
and for an alternative proposal see, G.P., 2012 MNRAS, 422, 2581-2589.

[23]G.P. and T. A. Apostolatos, Phys.Rev.Lett. **112** 121101 (2014)

In the case of Scalar-Tensor theories with a massless scalar field,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\left(\tilde{R} - 2\tilde{\nabla}^{\mu}\phi\tilde{\nabla}_{\mu}\phi\right) + S_m(g_{\mu\nu},\psi)\,,$$

the field equations in the Einstein frame take the form,

$$\tilde{R}_{ab} = 2\partial_a\phi\partial_b\phi,$$

$$\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\phi = 0$$

which can admit an Ernst formulation,[24]

$$(Re(\mathcal{E}))\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot\nabla\mathcal{E},$$

with the addition of a Laplace equation for the scalar field $\nabla^2\phi = 0$, and the γ function being given by the equations

$$\frac{\partial\gamma}{\partial\rho} = \left(\frac{\partial\gamma}{\partial\rho}\right)_{GR} + \rho\left[\left(\frac{\partial\phi}{\partial\rho}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2\right],$$

$$\frac{\partial\gamma}{\partial z} = \left(\frac{\partial\gamma}{\partial z}\right)_{GR} + 2\rho\left(\frac{\partial\phi}{\partial\rho}\right)\left(\frac{\partial\phi}{\partial z}\right).$$

One can thus extend the previous GR solution to a Scalar-Tensor solution by introducing the additional terms in γ and an appropriate scalar field. The Jordan (physical) frame metric can then be given by the conformal transformation $g_{\mu\nu} = A^2(\phi)\tilde{g}_{\mu\nu}$. This metric can be used to do astrophysics in the same way as in the GR case.[25]

[24] GP, T.P. Sotiriou, Phys. Rev. D91, 044011 (2015)

[25] GP, T.P. Sotiriou, MNRAS 454, 4066 (2015); GP arXiv:1610.05370

The approximate M_4-metric in scalar-tensor theory (Einstein frame):[26]

$$f(\rho,z)=1-\frac{2M}{\sqrt{\rho^2+z^2}}+\frac{2M^2}{\rho^2+z^2}+\frac{C^{ST}(\rho,z)}{3\left(\rho^2+z^2\right)^{5/2}}+\frac{D^{ST}(\rho,z)}{3\left(\rho^2+z^2\right)^{3}}+\frac{A^{ST}(\rho,z)}{420\left(\rho^2+z^2\right)^{9/2}}+\frac{B^{ST}(\rho,z)}{630\left(\rho^2+z^2\right)^{5}},$$

$$\omega(\rho,z)=-\frac{2J\rho^2}{\left(\rho^2+z^2\right)^{3/2}}-\frac{2JM\rho^2}{\left(\rho^2+z^2\right)^{2}}+\frac{F^{ST}(\rho,z)}{5\left(\rho^2+z^2\right)^{7/2}}+\frac{H^{ST}(\rho,z)}{30\left(\rho^2+z^2\right)^{4}}+\frac{G^{ST}(\rho,z)}{60\left(\rho^2+z^2\right)^{11/2}},$$

$$\gamma(\rho,z)=\frac{\rho^2}{4\left(\rho^2+z^2\right)^4}\left[\rho^2\left(J^2+M^4\right)-4z^2\left(2J^2+M^4\right)-\left(W_0\left(2M^2W_0+W_0^3+3W_2\right)+3MM_2\right)\left(4z^2-\rho^2\right)\right]$$
$$-\frac{\rho^2\left(M^2+W_0^2\right)}{2\left(\rho^2+z^2\right)^2},$$

where,

$$C^{ST}(\rho,z)=\left[\rho^2\left(3(M_2-M^3)+MW_0^2\right)-2z^2\left(3\left(M^3+M_2\right)+MW_0^2\right)\right],$$
$$D^{ST}(\rho,z)=\left[2z^2\left(M\left(3M^3+2MW_0^2+6M_2\right)-3J^2\right)-2M\rho^2\left(MW_0^2+3M_2\right)\right],$$

$$A^{ST}(\rho,z)=\left[8\rho^2z^2\left(360J^2M+91M^3W_0^2+255M^2M_2+63MW_0^4+270M_2W_0^2+90MW_2W_0+315M_4\right)\right.$$
$$-\rho^4\left(150J^2M-105M^5-154M^3W_0^2-480M_2M^2+63MW_0^4+90MW_0W_2\right.$$
$$\left.+270M_2W_0^2+315M_4\right)-8z^4\left(-300J^2M+105M^5+112M^3W_0^2+330M_2M^2+21MW_0^4\right.$$
$$\left.\left.+30MW_0W_2+90M_2W_0^2+105M_4\right)\right],$$

$$B^{ST}(\rho,z)=\left[\rho^4\left(M\left(2M\left(225J^2+84M^2W_0^2+112W_0^4+135W_2W_0\right)+945M_4\right)+30M_2\left(15M^3+34MW_0^2\right)\right.\right.$$
$$\left.+315M_2^2\right)+4z^4\left(-18\left(J\left(100JM^2+21JW_0+35S_3\right)-35MM_4\right)+150M_2\left(9M^3+5MW_0^2\right)\right.$$
$$\left.+M^2\left(315M^4+462M^2W_0^2+161W_0^4+180W_0W_2\right)+315M_2^2\right)-4\rho^2z^2\left(27\left(J\left(45JM^2-21JW_0\right.\right.\right.$$
$$\left.\left.-35S_3\right)+70MM_4\right)+30M_2\left(72M^3+61MW_0^2\right)+M^2\left(315M^4+756M^2W_0^2+413W_0^4+540W_0W_2\right)$$
$$\left.\left.+315M_2^2\right)\right],$$

[26] GP arXiv:1610.05370

The approximate M_4-metric in scalar-tensor theory (Einstein frame):

$$H^{ST}(\rho,z)=\left[\rho^2\left(M\left(-120JM^2z^2+JW_0\left(5W_0\left(\rho^2+4z^2\right)+27\left(\rho^2-4z^2\right)\right)+45S_3\left(\rho^2-4z^2\right)\right)+15JM_2\left(\rho^2+4z^2\right)\right)\right]$$

$$G^{ST}(\rho,z)=\left[\rho^2\left(15J\left(\rho^4\left(M^4-J^2\right)-8z^4\left(J^2+3M^4\right)+4\rho^2z^2\left(3J^2+M^4\right)\right)+M^2\left(JW_0\left(10W_0\left(\rho^4-8z^4+20\rho^2z^2\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+9\left(3\rho^4-40z^4+12\rho^2z^2\right)\right)+15S_3\left(3\rho^4-40z^4+12\rho^2z^2\right)\right)+30JM_2M\left(\rho^4-8z^4+20\rho^2z^2\right)\right)\right],$$

$$F^{ST}(\rho,z)=\left[\rho^2\left(-5JM^2\left(\rho^2+4z^2\right)-\left(4z^2-\rho^2\right)\left(3JW_0+5S_3\right)\right)\right].$$

and the scalar field is,

$$\phi(\rho,z)=\frac{W_0}{\sqrt{\rho^2+z^2}}\left[1-\frac{\left(M^2W_0+W_0^3+3W_2\right)\left(r^2-2z^2\right)}{6W_0\left(r^2+z^2\right)^2}\right],$$

where W_0 is the scalar charge and W_2 is the scalar quadrupole, while the mass and angular momentum moments are given by their definition in scalar-tensor theory[27] where they get contributions from the scalar field as well.

And as we have mentioned, the Jordan (physical) frame metric will be given by the conformal transformation $g_{\mu\nu}=A^2(\phi)\tilde{g}_{\mu\nu}$.

[27] GP, T.P. Sotiriou, Phys. Rev. D91, 044011 (2015)

The observables related to the orbits in a spacetime can be more immediately associated to its multipole moments.

The energy change per logarithmic frequency interval and the precession frequencies are related to the multipole moments (Ryan, 1995),

in GR:

$$\Delta\tilde{E} = -\frac{U}{3}\frac{d\tilde{E}}{dU} = \frac{1}{3}U^2 - \frac{1}{2}U^4 + \frac{20J_1}{9M^2}U^5 + \dots$$

$$\frac{\Omega_\rho}{\Omega} = 3U^2 - 4\frac{J_1}{M^2}U^3 + \left(\frac{9}{2} - \frac{3M_2}{2M^3}\right)U^4 - 10\frac{J_1}{M^2}U^5 + \left(\frac{27}{2} - 2\frac{J_1^2}{M^4} - \frac{21M_2}{2M^3}\right)U^6 + \dots$$

$$\frac{\Omega_z}{\Omega} = 2\frac{J_1}{M^2}U^3 + \frac{3M_2}{2M^3}U^4 + \left(7\frac{J_1^2}{M^4} + 3\frac{M_2}{M^3}\right)U^6 + \left(11\frac{J_1M_2}{M^5} - 6\frac{S_3}{M^4}\right)U^7 + \dots$$

where $U = (M\Omega)^{1/3}$. The Orbital frequency gives the Keplerian mass: $\Omega = (M/r^3)^{1/2}(1 + O(r^{-1/2})$.

in Scalar-Tensor theory:[28]

$$\Delta\tilde{E} = \frac{1}{3}U^2 + \left(\frac{2\beta_0W_0^2}{9\bar{M}^2} - \frac{8\alpha_0W_0}{9\bar{M}} - \frac{1}{2}\right)U^4 + \frac{20J_1}{9\bar{M}^2}U^5 + \dots$$

$$\frac{\Omega_\rho}{\Omega} = \left(3 - \frac{W_0\left(\beta_0W_0 - 8\alpha_0\bar{M}\right)}{2\bar{M}^2}\right)U^2 - \frac{4J_1}{\bar{M}^2}U^3 + \dots$$

$$\frac{\Omega_z}{\Omega} = \frac{2J_1}{\bar{M}^2}U^3 + \frac{3(M_2 - \alpha_0W_2)}{2\bar{M}^3}U^4 - \frac{2J_1W_0\left(\beta_0W_0 - \alpha_0\bar{M}\right)}{\bar{M}^4}U^5 + \dots$$

where $U = (\bar{M}\Omega)^{1/3}$. The calculations are done in the Jordan frame. Again the orbital frequency gives the Keplerian mass: $\Omega = (\bar{M}/r^3)^{1/2}(1 + O(r^{-1/2})$, but this time the Keplerian mass is $\bar{M} = M - W_0\alpha_0$.
W_0 is the scalar charge, W_2 is the scalar quadrupole and $\alpha \equiv (d\ln A)/d\phi, \beta \equiv d\alpha/d\phi$.

These observables could in principle distinguish between GR and Scalar-Tensor theory.

[28] GP, T.P. Sotiriou, MNRAS 454, 4066 (2015)

- Neutron stars exhibit some black hole-like behaviour with respect to their moments structure
- The geometry of the spacetime around rotating neutron stars is essentially different from the geometry of Kerr Black Holes.
- The difference is evident in the properties of the geodesics and especially the precession frequencies of perturbed circular equatorial orbits.
- The differences come from the fact that the quadrupole and higher order moments of a Neutron Star are larger than their BH counterparts (neutron stars are more oblate).
- An interesting result is that the nodal precession changes sign as one moves from the exterior region of the spacetime towards the innermost stable circular orbit.
- These effects related to orbital dynamics can be of relevance to the study of accretion discs and quasi periodic oscillations (QPOs) and should be taken into account in modelling the NS environment.
- Geodesic properties such as orbital and precession frequencies could distinguish between different theories of gravity such as GR and scalar-tensor theory.
- There is work to be done in the last two directions.

Thank You.