

The Analysis of Shock
Formation
in 3-Dimensional Fluids
Part 1

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In this lecture I shall discuss the ideas of my monograph “The Formation of Shocks in 3-Dimensional Fluids”. The monograph studies the relativistic Euler equations in 3 space dimensions for a perfect fluid with an arbitrary equation of state.

The mechanics of a perfect fluid are described in the framework of special relativity by a future-directed timelike vectorfield u of unit magnitude relative to the Minkowski metric g , the *fluid 4-velocity*, and two positive functions n and s , the *number of particles per unit volume* and the *entropy per particle*. The mechanical properties of a perfect fluid are determined once we give an *equation of state*, which expresses the mass-energy density ρ as a function of n and s :

$$\rho = \rho(n, s) \tag{1}$$

According to the laws of thermodynamics, the *pressure* p and the *temperature* θ are then given by:

$$p = n \frac{\partial \rho}{\partial n} - \rho, \quad \theta = \frac{1}{n} \frac{\partial \rho}{\partial s} \quad (2)$$

The *particle current* is the vectorfield I given by:

$$I^\mu = n u^\mu \quad (3)$$

The *energy-momentum-stress tensor* is the symmetric 2-contravariant tensorfield T given by:

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p (g^{-1})^{\mu\nu} \quad (4)$$

and the *equations of motion* are the *differential conservation laws*:

$$\nabla_\mu I^\mu = 0, \quad \nabla_\nu T^{\mu\nu} = 0 \quad (5)$$

There is a substantial gain in geometric insight in working with the relativistic equations because of the spacetime geometry viewpoint of special relativity. As an example consider the equation:

$$i_u \omega = -\theta ds \quad (6)$$

Here ω is the vorticity 2-form:

$$\omega = d\beta \quad (7)$$

β being the 1-form defined by:

$$\beta_\mu = -\sqrt{\sigma} u_\mu, \quad u_\mu = g_{\mu\nu} u^\nu \quad (8)$$

with $\sqrt{\sigma}$ the relativistic *enthalpy per particle*:

$$\sqrt{\sigma} = \frac{\rho + p}{n} \quad (9)$$

In 6 i_u denotes contraction on the left by the vectorfield u . Equation 6 is equivalent to the differential energy-momentum conservation laws and is arguably the simplest explicit form of these equations.

At each point p in the spacetime manifold M , H_p , the *local simultaneous space* of the fluid at p , is the g -orthogonal complement of the linear span of u_p , the fluid velocity at p , in T_pM . The obstruction to integrability of the distribution of local simultaneous spaces is the *vorticity vector* ϖ given by:

$$\varpi^\mu = \frac{1}{2}(\epsilon^{-1})^{\mu\alpha\beta\gamma}u_\alpha\omega_{\beta\gamma} \quad (10)$$

where ϵ^{-1} is the reciprocal volume form of (M, g) , or volume form in T_p^*M at each $p \in M$.

The 1-form β plays a fundamental role in my monograph. In the irrotational case it is given by $\beta = d\phi$, where ϕ is the *wave function*. In this case, the equations of motion 5 reduce to a *nonlinear wave equation*:

$$\nabla_{\mu}(G\partial^{\mu}\phi) = 0, \quad \partial^{\mu}\phi = (g^{-1})^{\mu\nu}\partial_{\nu}\phi \quad (11)$$

where

$$G = \frac{n}{\sqrt{\sigma}} = G(\sigma), \quad \sigma = -(g^{-1})^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi \quad (12)$$

Equation 11 derives from the Lagrangian

$$L = p = L(\sigma) \quad (13)$$

the pressure as a function of the squared enthalpy.

Returning to the general case, the sound speed η is defined by:

$$\left(\frac{dp}{d\rho}\right)_s = \eta^2 \quad (14)$$

it being assumed that the left hand side is positive. The causality condition:

$$0 < \eta < 1 \quad (15)$$

is imposed, the right inequality meaning that the sound speed is less than the universal constant represented by the speed of light in vacuum.

The *acoustical metric* h is another Lorentzian metric on M such that at each $p \in M$ the simultaneous space H_p is also h -orthogonal to u_p , h agrees with g on H_p , and h assigns magnitude η to u . In terms of a formula:

$$h_{\mu\nu} = g_{\mu\nu} + (1 - \eta^2)u_\mu u_\nu, \quad u_\mu = g_{\mu\nu}u^\nu \quad (16)$$

The null cones of h are called *sound cones*. By the right inequality above, they are contained within the null cones of g , namely the light cones. What is important from the physical point of view is the *conformal geometry* induced by h on the underlying manifold. It determines the *acoustical causal structure*. That is, given any event $p \in M$ it determines $\mathcal{J}^+(p)$ the acoustical causal future of p , the set of events which are acoustically influenced by p , and $\mathcal{J}^-(p)$ the causal past of p , the set of events which acoustically influence p .

Choosing a *time function* t in Minkowski space-time, equal to the coordinate x^0 of some rectangular coordinate system, we denote by Σ_t an arbitrary level set of the function t . The Σ_t are parallel spacelike hyperplanes relative to the Minkowski metric g .

Initial data for the equations of motion 5 is given on a domain in the hyperplane Σ_0 , which may be the whole of Σ_0 . It consists in the specification of the triplet (n, s, u) on this domain. In the irrotational case, where we have the nonlinear wave equation 11, initial data consists in the specification of the pair $(\phi, \partial_t \phi)$ on such a domain. To any given initial data set there corresponds a unique *maximal classical solution* of the equations of motion 5, or of the nonlinear wave equation 11 in the irrotational case. The notion of maximal classical solution or maximal development of an initial data set is the following.

Given an initial data set, the local existence theorem asserts the existence of a *development* of this set, namely of a domain \mathcal{D} in Minkowski spacetime, whose past boundary is the domain of the initial data, and of a solution defined in \mathcal{D} and taking the given data at the past boundary, such that the following condition holds. If we consider any point $p \in \mathcal{D}$ and any curve issuing at p with the property that its tangent vector at any point q belongs to the interior or the boundary of the past component of the sound cone at q , then the curve terminates at a point of the domain of the initial data. [Drawing 1].

The local uniqueness theorem asserts that if $(\mathcal{D}_1, (n_1, s_1, u_1))$ and $(\mathcal{D}_2, (n_2, s_2, u_2))$ are two developments of the same initial data $[(\mathcal{D}_1, \phi_1)$ and (\mathcal{D}_2, ϕ_2) in the irrotational case], then (n_1, s_1, u_1) coincides with (n_2, s_2, u_2) in $\mathcal{D}_1 \cap \mathcal{D}_2$ [ϕ_1 coincides with ϕ_2 in $\mathcal{D}_1 \cap \mathcal{D}_2$ in the irrotational case]. It follows that the union of all developments of a given initial data set is itself a development, the unique *maximal development* of the initial data set.

In the monograph I consider regular initial data on Σ_0 which outside a sphere coincide with the data corresponding to a constant state. That is, outside that sphere n and s are constant and u coincides with the future-directed unit normal to Σ_0 . Under a suitable restriction on the size of the departure of the initial data from those of the constant state, I prove certain theorems which give a complete description of the maximal classical development. In particular, the theorems give a detailed description of the geometry of the boundary of the domain of the maximal classical development and a detailed analysis of the behavior of the solution at this boundary. A complete picture of shock formation in 3-dimensional fluids is thereby obtained.

I shall confine myself in this talk to the case that the initial data are irrotational hence so is the maximal classical development.

Let H be the function defined by:

$$1 - \eta^2 = \sigma H \quad (17)$$

where η is the sound speed. I denote by ℓ the value of $(dH/d\sigma)_s$ in the surrounding constant state. This constant determines the character of the shocks for small initial departures from the constant state. In particular when $\ell = 0$, no shocks form and the domain of the maximal classical solution is complete.

Consider the function $(dH/d\sigma)_s$ as a function of the thermodynamic variables p and s . Suppose that we have an equation of state such that at some value s_0 of s the function $(dH/d\sigma)_s$ vanishes everywhere along the adiabat $s = s_0$. In this case the irrotational fluid equations corresponding to the value s_0 of the entropy are equivalent to the minimal surface equation, the wave function ϕ defining a minimal graph in a Minkowski space-time of one more spatial dimension. In fact in this case the Lagrangian 13 is:

$$L = 1 - \sqrt{1 - \sigma} \quad (18)$$

and the action associated to a domain is the area of the domain minus the area of the graph over the domain.

Let O be the center of the sphere $S_{0,0}$ in Σ_0 outside which we have the constant state. Let us confine ourselves to the maximal development of the restriction of the initial data to $\Sigma_0 \setminus O$. Let u be a smooth function without critical points in $\Sigma_0 \setminus O$ such that the restriction of u to the exterior of $S_{0,0}$ is equal to minus the Euclidean distance from $S_{0,0}$. We extend u to the spacetime manifold by the condition that its level sets are outgoing null hypersurfaces relative to the acoustical metric h . Then u satisfies the h -eikonal equation:

$$(h^{-1})^{\mu\nu} \partial_\mu u \partial_\nu u = 0 \quad (19)$$

We call u an *acoustical function* and we denote by C_u an arbitrary level set of u .

Each C_u being a null hypersurface is generated by null geodesics of h . Let L be the tangent vectorfield to these geodesic generators parametrized not affinely but by t . We then define the wave fronts $S_{t,u}$ to be the surfaces of intersection $C_u \cap \Sigma_t$. Finally we define the vectorfield T to be tangential to the Σ_t and so that the flow generated by T on each Σ_t is the normal, relative to the induced on Σ_t acoustical metric \bar{h} , flow of the foliation of Σ_t by the surfaces $S_{t,u}$. This flow takes each wave front onto another wave front. [Drawing 2].

The structure introduced on the spacetime manifold by an acoustical function u , or, what is the same, the geometry of a foliation of spacetime by outgoing null hypersurfaces C_u , the level sets of u , plays a fundamental role in the problem. The most important geometric property of this foliation from the point of view of the study of shock formation is the density of the packing of its leaves C_u . One measure of this density is the *inverse spatial density* of the wave fronts, that is, the inverse density of the foliation of each spatial hyperplane Σ_t by the surfaces $S_{t,u}$. This is the function κ , given in arbitrary coordinates on Σ_t by:

$$\kappa^{-2} = (\bar{h}^{-1})^{ij} \partial_i u \partial_j u \quad (20)$$

where \bar{h}_{ij} is the induced acoustical metric on Σ_t . An equivalent definition of κ is that it is the magnitude of the vectorfield T with respect to h . [Drawing 3].

Another measure is the *inverse temporal density* of the wave fronts, the function μ , given in arbitrary coordinates in spacetime by:

$$\frac{1}{\mu} = -(h^{-1})^{\mu\nu} \partial_{\mu} t \partial_{\nu} u \quad (21)$$

The two measures are related by:

$$\mu = \alpha \kappa \quad (22)$$

where α is the inverse density, with respect to the acoustical metric, of the foliation of spacetime by the hyperplanes Σ_t . [This inverse density is of course 1 when referred to the Minkowski metric.] The function α is given in arbitrary coordinates in spacetime by:

$$\alpha^{-2} = -(h^{-1})^{\mu\nu} \partial_{\mu} t \partial_{\nu} t \quad (23)$$

It is expressed directly in terms of the 1-form $\beta = d\phi$. It turns out moreover, that it is bounded above and below by positive constants. Consequently μ and κ are equivalent measures of the density of the packing of the leaves of the foliation of spacetime by the C_u .

Shock formation is characterized by the blow up of this density or equivalently by the vanishing of κ or μ .

The maximal development being a domain in Minkowski spacetime, which by a choice of rectangular coordinates is identified with \mathcal{R}^4 , inherits the subset topology and the standard differential structure induced by the rectangular coordinates x^α . Choosing an acoustical function u we introduce *acoustical coordinates* (t, u, ϑ) , $\vartheta \in \mathcal{S}^2$, the coordinate lines corresponding to a given value of u and to constant values of ϑ being the generators of the null hypersurface C_u . The rectangular coordinates x^α are smooth functions of the acoustical coordinates (t, u, ϑ) and the Jacobian of the transformation is, up to a multiplicative factor which is bounded above and below by positive constants, the inverse temporal density function μ . The acoustical coordinates induce another differential structure on the same underlying topological manifold. However since $\mu > 0$ in the interior of the maximal development, the two differential structures coincide in the interior of the maximal development.

The main theorem of the monograph asserts that relative to the differential structure induced by the acoustical coordinates the maximal classical solution extends smoothly to the boundary of its domain. This boundary contains however a singular part B where the function μ vanishes. The rectangular coordinates themselves extend smoothly to the boundary but the Jacobian vanishes on the singular part of the boundary. The mapping from acoustical to rectangular coordinates has a continuous but not differentiable inverse at B . As a result, the two differential structures no longer coincide when the singular boundary B is included.

With respect to the standard differential structure the solution is continuous but not differentiable at B , the derivative $\hat{T}^\mu \hat{T}^\nu \partial_\mu \partial_\nu \phi$ blowing up as we approach B . Here $\hat{T} = \kappa^{-1} T$, is the vectorfield of unit magnitude with respect to h corresponding to T .

With respect to the standard differential structure, the acoustical metric h is everywhere in the closure of the domain of the maximal solution non-degenerate and continuous, but it is not differentiable at B , while with respect to the differential structure induced by the acoustical coordinates h is everywhere smooth, but it is degenerate at B .

After the proof of the main theorem, I establish a general theorem which gives sharp sufficient conditions on the initial data for the formation of a shock in the evolution. The theorem also gives a sharp upper bound on the time interval required for the onset of shock formation.

The last part of the work is concerned with the structure of the boundary of the domain of the maximal classical solution and the behavior of the solution at this boundary. The boundary of the maximal development consists of a regular part \underline{C} and a singular part B . Each component of \underline{C} is a regular incoming acoustically null hypersurface with a singular past boundary which coincides with the past boundary of an associated component of B . The union of these singular past boundaries we denote by $\partial^- B$. [Drawing 4]. Each component of B is a hypersurface which is smooth relative to both differential structures and has the intrinsic geometry of a regular null hypersurface in a regular spacetime and, like the latter, is ruled by invariant curves of vanishing arc length. [Drawing 5].

On the other hand, the extrinsic geometry of each component of B is that of an acoustically spacelike hypersurface which becomes acoustically null at its past boundary, an associated component of ∂^-B . This means that at each point $q \in B$ the past null geodesic conoid of q does not intersect B . Each component of ∂^-B is an acoustically spacelike surface which is smooth relative to both differential structures.

The main result of the last part of the work is the trichotomy theorem. This theorem shows that for each point q of the singular boundary, the intersection of the past null geodesic conoid of q with any Σ_t in the past of q splits into three parts, the parts corresponding to the outgoing and to the incoming sets of null geodesics ending at q being embedded discs with a common boundary, an embedded circle, which corresponds to the set of the remaining null geodesics ending at q . *All outgoing null geodesics ending at q have the same tangent vector at q .* This vector is then an invariant null vector associated to the singular point q . [Drawing 6].

This striking result is in fact the reason why the considerable freedom in the choice of the acoustical function does not matter in the end. For, considering the transformation from one acoustical function to another, I show that the foliations corresponding to different families of outgoing null hypersurfaces have equivalent geometric properties and degenerate in precisely the same way on the same singular boundary.

Now, the components of $\partial^- B$ are physically the surfaces where shocks begin to form. The maximal classical solution is the physical solution of the problem up to $\underline{C} \cup \partial^- B$, but not up to B . In the epilogue of the monograph the problem of the physical continuation of the solution is set up as the *shock development problem*. This is a free-boundary problem associated to each component of $\partial^- B$. In this problem it is required to construct a hypersurface of discontinuity K , the shock hypersurface, lying in the past of the associated component of B but having the same past boundary as the latter, namely the given component of $\partial^- B$, the tangent hyperplanes to K and B coinciding along $\partial^- B$.

Moreover, it is required to construct a solution of the differential conservation laws 5 in the domain in Minkowski spacetime bounded in the past by $\underline{C} \cup K$, agreeing with the maximal classical solution on $\underline{C} \cup \partial^- B$, while having jumps across K relative to the data induced on K by the maximal classical solution, jumps satisfying the jump conditions which follow from the integral form of the conservation laws. Finally K is required to be acoustically spacelike with respect to the acoustical metric induced by the maximal classical solution, and timelike with respect to the acoustical metric induced by the new solution, which holds in the future of K . The maximal classical solution thus provides the boundary conditions on $\underline{C} \cup \partial^- B$ as well as a barrier at B . [Drawing 7].

Note that a component of $\partial^- B$ is a surface which is acoustically spacelike but not necessarily spacelike relative to the Minkowski metric g . The intersection $\Sigma_t \cap K$ represents the instantaneous shock surface in the Lorentz frame defined by the time function t and the intersection $\Sigma_t \cap \partial^- B$ represents the boundary curve of the instantaneous shock surface.

Let me close with a formula for the jump in vorticity across K which shows that even though the flow before the shock may be irrotational the flow acquires vorticity immediately after. By virtue of the last condition of the shock development problem K is a time-like hypersurface relative to the Minkowski metric g . Let N be its unit normal relative to g , pointing to the future of K . Let u_0 be the fluid velocity at K induced by the past solution, u_1 be the fluid velocity at K induced by the future solution. Then by the jump conditions at each point $p \in K$ the three vectors N, u_0, u_1 lie in the same timelike plane. Let Π_p be the g -orthogonal complement of this plane.

Consider the restriction to Π_p of the differential of $[s]$, the jump in entropy across K . Let us denote this restriction by $\mathcal{A}[s]$ and the corresponding (through g) vector, element of Π_p , by $\mathcal{A}[s]^\sharp$. Then the vorticity vector at p induced by the future solution is given by:

$$\varpi_1 = \frac{\theta_1}{u_{\perp 1}} * \mathcal{A}[s]^\sharp \quad (24)$$

Here θ_1 and $u_{\perp 1}$ are, respectively, the temperature and the normal component of the fluid velocity at K induced by the future solution [$u_{\perp 0}$ and $u_{\perp 1}$ are both positive and related by $n_0 u_{\perp 0} = n_1 u_{\perp 1}$]. Also, for any $V \in \Pi_p$ we denote by $*V$ the result of rotating V counterclockwise by a right angle.