Hyperbolic P.D.E. and Lorentzian Geometry

Demetrios Christodoulou

Physical spacetime is a 4-dimensional manifold $\mathcal{M}$ endowed with a Lorentzian metric g, relative to which it is time-oriented. In *continuum mechanics* we also have a 3-dimensional manifold $\mathcal{N}$, the *material manifold*, each point of which represents a material particle. The dynamics is described by a mapping $f : \mathcal{M} \to \mathcal{N}$ which tells us which particle is at a given event. The mapping f must satisfy the condition that the inverse image of a point $y \in \mathcal{N}$ must be a timelike curve in $\mathcal{M}$, as it is to represent the history of the particle y. Thus $df(x)$ is subject to the condition that its null space is a timelike line in $T_x\mathcal{M}$. There is then a unit future-directed timelike vector u_x in $T_x\mathcal{M}$ whose linear span is the null space of $df(x)$. The vector u_x is the *material velocity* at x. The assignment of u_x at each $x \in \mathcal{M}$ then defines a vectorfield on $\mathcal{M}$, the *material velocity*. The orthogonal complement, relative to g_x, of the null space of $df(x)$ is a spacelike hyperplane Σ_x in $T_x\mathcal{M}$, the *simultaneous space* at x. The restriction $g_x|_{\Sigma_x}$ is a positive-definite quadratic form on Σ_x. The restriction $df(x)|_{\Sigma_x}$ is then an isomorphism of Σ_x onto $T_{f(x)}\mathcal{N}$.

The material manifold $\mathcal{N}$ is in general endowed with a volume form $d\mu_\omega$, the integral of which over a domain in $\mathcal{N}$ represents the number of particles contained in the domain. In the case of *fluid mechanics* there is no other structure on $\mathcal{N}$. In the case of the *mechanics of crystalline solids* however $\mathcal{N}$ is endowed with a richer structure which we shall presently define. Let us denote by $\mathcal{X}(\mathcal{N})$ the space of C^∞ vectorfields on $\mathcal{N}$ and by ε_y the evaluation map $\mathcal{X}(\mathcal{N}) \to T_y\mathcal{N}$ taking a vectorfield to its value at $y \in \mathcal{N}$.

Definition: A *crystalline structure* on $\mathcal{N}$ is a distinguished linear subspace $\mathcal{K}$ of $\mathcal{X}(\mathcal{N})$ such that the evaluation map restricted to $\mathcal{K}$, $\varepsilon_y \ : \ \mathcal{K} \to T_y\mathcal{N}$, is an isomorphism for each $y \in \mathcal{N}$.

Each element of $\mathcal{K}$ generates a 1-parameter group of diffeomorphisms of $\mathcal{N}$. These groups represent physically the continuum limit of the groups of translations of a crystal lattice. The parametrization of the group orbits is to be thought of as proportional to the number of atoms traversed.

The *canonical form* associated to a crystalline structure is the $\mathcal{K}$ - valued 1-form ν on $\mathcal{N}$ defined by:

$$\nu(Y_y) = \varepsilon_y^{-1}(Y_y) \in \mathcal{K}$$

for each $Y_y \in T_y\mathcal{N}$ and $y \in \mathcal{N}$. The *dislocation form* is the $\mathcal{K}$ - valued 2-form λ on $\mathcal{N}$ given by:

$$\lambda = -d\nu$$

If $Y_1, Y_2 \in \mathcal{K}$ then according to the above definition $\lambda(Y_1, Y_2)$ is the following $\mathcal{K}$ - valued function on $\mathcal{N}$:

$$\lambda(Y_1, Y_2)(y) = \varepsilon_y^{-1}([Y_1, Y_2](y))$$

This is a constant function if and only if $[Y_1, Y_2] \in \mathcal{K}$. If this is the case for each pair $Y_1, Y_2 \in \mathcal{N}$ then $\mathcal{K}$ is a Lie algebra. In this case, upon choosing an identity element $e \in \mathcal{N}$, $\mathcal{N}$ becomes a Lie group so that $\mathcal{K}$ is the space of vectorfields which generate the right action of the group on itself; $\mathcal{K}$ is then, at the same time, the space of vectorfields on $\mathcal{N}$ which are invariant under left group multiplications.

The dislocation form is a concept which arises in the continuum limit when one considers a distribution of elementary dislocations in a crystal lattice. An elementary lattice dislocation has the property that if we start at an atom and move according to one group of lattice translations a certain number of atoms p, then move according to a different group of translations a number of atoms q, then according to the first $-p$ and finally according to the second $-q$, then on completing the circuit we arrive at an atom which in does not coincide with the atom from which we started, but, provided that the circuit encloses a single elementary dislocation, is arrived at in a single step corresponding to a third lattice translation. The lattice vector corresponding to this step is called *Burgers vector*. The integral of minus ν on a closed curve C in $\mathcal{N}$ represents physically the sum of the Burgers vectors of all the dislocation lines enclosed by C.

The *thermodynamic state space* is the space of local thermodynamic states of a material. For a fluid this space is $\mathbb{R}^+ \times \mathbb{R}^+$, the set of pairs (τ, σ) where $\tau \in \mathbb{R}^+$ is the *volume per particle* and $\sigma \in \mathbb{R}^+$ is the *entropy per particle*. For a crystalline solid the thermodynamic state space is $S_2^+(\mathcal{K}) \times \mathbb{R}^+$, the set of pairs (γ, σ), where $\gamma \in S_2^+(\mathcal{K})$, the space of inner products on $\mathcal{K}$, is the *thermodynamic configuration* and σ is, as above, the entropy per particle. We assume that an orientation and a volume form ω has been chosen for $\mathcal{K}$. Then for a crystalline solid the volume per particle τ is a function on $S_2^+(\mathcal{K})$ defined as follows: Each $\gamma \in S_2^+(\mathcal{K})$ defines a volume form ω_γ on $\mathcal{K}$ by the condition that if (E_1, E_2, E_3) is a positive basis for $\mathcal{K}$ which is orthonormal relative to γ then:

$$\omega_\gamma(E_1, E_2, E_3) = 1$$

It follows that there is a positive function τ on $S_2^+(\mathcal{K})$ such that:

$$\omega_\gamma = \tau(\gamma)\omega$$

The laws governing the dynamics of a given kind of material are determined by the specification of the *energy per particle* e as a function on the thermodynamic state space. Thus, in the case of fluid mechanics $e = e(\tau, \sigma)$ and the derivatives:

$$p = -\frac{\partial e}{\partial \tau} \qquad \theta = \frac{\partial e}{\partial \sigma}$$

are the *pressure* and the *temperature* respectively. In the case of the mechanics of crystalline solids $e = e(\gamma, \sigma)$ and the derivative:

$$\frac{\partial e}{\partial \gamma} = -\frac{1}{2}\pi\tau$$

defines the *stress* π, which takes values in $(S_2(\mathcal{K}))^*$. The temperature is defined as above. The *number of particles per unit volume* n and the *energy density* or *energy per unit volume* ρ are defined in terms of τ and e by:

$$n = \frac{1}{\tau} \qquad \rho = \frac{e}{\tau}$$

The equations of motion are the Euler-Lagrange equations for the mapping f, associated to the Lagrangian to be presently defined. With $x \in \mathcal{M}$, $y \in \mathcal{N}$, and $f(x) = y$, consider the set of possible values of $df(x) = v$. This is the open subset $\mathcal{V}_{(x,y)}$ of $\mathcal{L}(T_x\mathcal{M}, T_y\mathcal{N})$ defined by the condition that the null space of v is a line in $T_x\mathcal{M}$ which is timelike relative to g_x. [Here, for finite dimensional vector spaces U and V we denote by $\mathcal{L}(U, V)$ the space of linear maps of U into V.] Then L is defined on the bundle

$$\mathcal{V} = \bigcup_{(x,y)\in\mathcal{M}\times\mathcal{N}} \mathcal{V}_{(x,y)}$$

over $\mathcal{M} \times \mathcal{N}$ and assigns to each $v \in \mathcal{V}_{(x,y)}$ a top degree form on $T_x\mathcal{M}$. Since here $T_x\mathcal{M}$ is already endowed with the top degree form $d\mu_{g_x}$, the volume form of g_x, we have:

$$L(v) = L^*(v) d\mu_{g_x}$$

where L^* is simply a function on $\mathcal{V}$.

In the case of fluid mechanics this function is defined as follows. Consider Σ_x, the simultaneous space at x. Then $d\mu_{g_x}$ induces a volume form ϵ_{Σ_x} on Σ_x by:

$$\epsilon_{\Sigma_x}(X_{1,x}, X_{2,x}, X_{3,x}) = d\mu_{g_x}(u_x, X_{1,x}, X_{2,x}, X_{3,x})$$
$$: \ \forall X_{1,x}, X_{2,x}, X_{3,x} \in \Sigma_x$$

where u_x is the material velocity at x. Recall that

$$v|_{\Sigma_x} \ : \ \Sigma_x \to T_y\mathcal{N}$$

is an (orientation-preserving) isomorphism. Thus there is a positive real number $\tau(v)$ such that:

$$\epsilon_{\Sigma_x}(X_{1,x}, X_{2,x}, X_{3,x}) =$$
$$\tau(v) d\mu_{\omega_y}(v \cdot X_{1,x}, v \cdot X_{2,x}, v \cdot X_{3,x})$$
$$: \ \forall X_{1,x}, X_{2,x}, X_{3,x} \in \Sigma_x$$

where $d\mu_\omega$ is the volume form on $\mathcal{N}$. To have a pure Lagrangian description, the entropy per particle σ must be given as a positive function on $\mathcal{N}$. At each $v \in \mathcal{V}_{(x,y)}$, the pair $(\tau(v), \sigma(y)) \in \mathbb{R}^+ \times \mathbb{R}^+$ represents a *local thermodynamic state* of the fluid. The Lagrangian function L^* is then defined by:

$$L^*(v) = \rho(\tau(v), \sigma(y)) \qquad : \ \forall v \in \mathcal{V}_{(x,y)}$$

In the case of the mechanics of crystalline solids the Lagrangian function is defined as follows. For each $v \in \mathcal{V}_{(x,y)}$ we consider the isomorphism $i(v)$ of $\mathcal{K}$ onto Σ_x given by:

$$i(v) = \left(v|_{\Sigma_x}\right)^{-1} \circ \varepsilon_y$$

Then $\gamma(v) \in S_2^+(\mathcal{K})$ is defined by:

$$\gamma(v) = i^*(v) \cdot g_x|_{\Sigma_x}$$

that is:

$$\gamma(v)(Y_1, Y_2) = g_x(i(v)\cdot Y_1, i(v)\cdot Y_2) \quad : \; \forall Y_1, Y_2 \in \mathcal{K}$$

At each $v \in \mathcal{V}_{(x,y)}$, the pair $(\gamma(v), \sigma(y)) \in S_2^+(\mathcal{K}) \times \mathbb{R}^+$ represents a local thermodynamic state of the crystalline solid. The Lagrangian function L^* is then defined by:

$$L^*(v) = \rho(\gamma(v), \sigma(y)) \quad : \; \forall v \in \mathcal{V}_{(x,y)}$$

The above theories continuum mechanics fit into the general framework of Lagrangian theories of maps f of an (oriented) differentiable manifold $\mathcal{M}$ into an (oriented) differentiable manifold $\mathcal{N}$. The Lagrangian is in general defined on the bundle

$$\mathcal{V} = \bigcup_{(x,y)\in\mathcal{M}\times\mathcal{N}} \mathcal{V}_{(x,y)}$$

over $\mathcal{M} \times \mathcal{N}$, where $\mathcal{V}_{(x,y)}$ is an open subset of $\mathcal{L}(T_x\mathcal{M}, T_y\mathcal{N})$. In fact, for each $(x, y) \in \mathcal{M} \times \mathcal{N}$, L assigns to each $v \in \mathcal{V}_{(x,y)}$ an element of $\wedge_m(T_x\mathcal{M})$, $m = \dim\mathcal{M}$, that is, a top degree form on $T_x\mathcal{M}$. Hence given a map $f : \mathcal{M} \to \mathcal{N}$, the composition $L \circ df$ is an exterior differential form of top degree on $\mathcal{M}$. The *action* $\mathcal{A}(f, \Omega)$ associated to the map f and to the domain Ω with compact closure in $\mathcal{M}$, is the integral:

$$\int_{\Omega\subset\mathcal{M}} L \circ df$$

and the Euler-Lagrange equations express the requirement that, for any such domain Ω, the action $\mathcal{A}(f, \Omega)$ is stationary with respect to arbitrary variations of the map f which are compactly supported in Ω.

In the framework of continuum mechanics there is a vectorfield I on $\mathcal{M}$ associated to a map f , the *particle current*, defined by:

$$I = nu$$

where n is the number of particles per unit volume considered as a function on $\mathcal{N}$, that is:

$$n(x) = n(df(x))$$

Consider the 3-form I^* which is dual to I with respect to $d\mu_g$, that is:

$$I^* = d\mu_g(I,\ \cdot,\ \cdot,\ \cdot)$$

Then according to the above:

$$I^* = f^* d\mu_\omega$$

It follows that:

$$dI^* = 0$$

identically, which is equivalent to the *differential particle conservation law*:

$$\nabla \cdot I = 0$$

where ∇ is the covariant derivative operator associated to the metric g.

Note that in the case of the Lagrangians of continuum mechanics, for $v \in \mathcal{V}_{(x,y)}$, $L(v)$ depends on g_x only through $\tau(v)$ in the case of fluid mechanics and only through $\gamma(v)$ in the case of the mechanics of crystalline solids. The *energy-momentum-stress* tensor at x associated to a map f is the element T_x of $(S_2(T_x\mathcal{M}))^*$ defined by:

$$\frac{\partial L(df(x))}{\partial g_x} = -\frac{1}{2} T_x d\mu_{g_x}$$

The assignment of T_x to each $x \in \mathcal{M}$ then defines a symmetric 2-contravariant tensorfield T on $\mathcal{M}$, the *energy-momentum-stress* tensorfield. In the case of fluid mechanics it is given by:

$$T = \rho u \otimes u + p(g^{-1} + u \otimes u)$$

Here ρ and p are the energy density and pressure considered as functions on $\mathcal{M}$, that is:

$$\rho(x) = \rho(\tau(df(x)), s(x)) \qquad p(x) = p(\tau(df(x)), s(x))$$

where s is the entropy per particle as a function on $\mathcal{M}$, that is:

$$s(x) = \sigma(f(x))$$

In the case of the mechanics of crystalline solids T is given by:

$$T = \rho u \otimes u + S$$

where ρ is the energy density considered as a function on $\mathcal{M}$, that is:

$$\rho(x) = \rho(\gamma(df(x)), s(x))$$

and S is the *stress* tensorfield, given, at each $x \in \mathcal{M}$ by:

$$S_x(\dot{g}_x) = \pi(\gamma(df(x)), s(x)) \cdot (i^*(df(x)) \cdot \dot{g}_x|_{\Sigma_x})$$
$$: \ \forall \dot{g}_x \in S_2(T_x\mathcal{M})$$

where, as above, for any $v \in \mathcal{V}_{(x,y)}$, $i^*(v) \cdot \dot{g}_x|_{\Sigma_x} \in S_2(\mathcal{K})$ is defined by:

$$i^*(v) \cdot \dot{g}_x|_{\Sigma_x}(Y_1, Y_2) = \dot{g}_x(i(v) \cdot Y_1, i(v) \cdot Y_2)$$
$$: \ \forall Y_1, Y_2 \in \mathcal{K}$$

Under the condition that df is continuous, the Euler-Lagrange equations are equivalent to the *differential energy-momentum conservation laws*:

$$\nabla \cdot T = 0$$

This equivalence is a consequence of the invariance of the action $\mathcal{A}(f, \Omega)$ under all diffeomorphisms of $\mathcal{M}$ which coincide with the identity in the complement of a compact subset of Ω, for all domains Ω with compact closure in $\mathcal{M}$. According to the above definition of the entropy function s, the entropy function is constant along particle histories, a condition called the *adiabatic condition*. The differential energy-momentum conservation laws with the entropy function s as an unknown function on $\mathcal{M}$, in addition to the mapping $f : \mathcal{M} \to \mathcal{N}$, hold even when continuity of df is no longer assumed. Then the adiabatic condition no longer holds, we have instead a jump in s along each particle history crossing a hypersurface of discontinuity of df.

In the case of fluid mechanics we may eliminate the mapping f and consider as unknowns the functions n and s and the fluid velocity u. The energy density ρ is specified as a function of n and s. The pressure p and the temperature θ are then given according to the above by:

$$p = n\frac{\partial \rho}{\partial n} - \rho \qquad \theta = \frac{1}{n}\frac{\partial \rho}{\partial s}$$

The differential particle conservation law together with the differential energy-momentum conservation laws then constitute a first order system of partial differential equations for the unknowns n, s, and u.

The notions of ellipticity and hyperbolicity of a system of Euler-Lagrange equations refer to a given solution f_0 of the system. Any other map $f : \mathcal{M} \to \mathcal{N}$ whose graph $\overline{f} : \mathcal{M} \to \mathcal{M} \times \mathcal{N}$ lies in a suitable neighborhood of $\overline{f_0}$, the graph of f_0, can be viewed as being a perturbation of f_0. Introducing a symmetric connection A in $T\mathcal{N}$, f may obtained by exponentiating, with respect to A, a section $\dot{f}$ of $f_0^* T\mathcal{N}$, the pullback by f_0 of the tangent bundle of $\mathcal{N}$. The theory of perturbations of a given solution f_0 is thus a theory of sections of the vector bundle $\mathcal{B} = f_0^* T\mathcal{N}$ over domain manifold $\mathcal{M}$, whose fibre over $x \in \mathcal{M}$ is

$$\mathcal{B}_x = \{x\} \times T_{f_0(x)}\mathcal{N}$$

The original action of f in a domain $\Omega \subset \mathcal{M}$ then translates to an action for $\dot{f}$ in Ω. This action is the integral over Ω of the composition with $(\dot{f}, D^*\dot{f})$ of $\overset{\triangle}{L}$, the *relative Lagrangian* with respect to f_0. Here $D^*\dot{f}$ is the covariant derivative of the section $\dot{f}$ with respect to the induced connection A^* on $f_0^* T\mathcal{N}$.

With

$$\dot{\mathcal{B}} = \bigcup_{x\in\mathcal{M}} \dot{\mathcal{B}}_x, \qquad \dot{\mathcal{B}}_x = \mathcal{L}(T_x\mathcal{M}, \mathcal{B}_x)$$

$\overset{\triangle}{L}$ is defined on the bundle product

$$\mathcal{B} \times_{\mathcal{M}} \dot{\mathcal{B}} = \bigcup_{x\in\mathcal{M}} \mathcal{B}_x \times \dot{\mathcal{B}}_x$$

an element of this in the fibre over $x \in \mathcal{M}$ is a pair (Y, V), where, with $y = f_0(x)$, $Y \in T_y\mathcal{N}$ and $V \in \mathcal{L}(T_x\mathcal{M}, T_y\mathcal{N})$. Consideration of infinitesimal perturbations gives rise to $\dot{L}$, the *linearized Lagrangian* at f_0. The linearized Lagrangian is for each $x \in \mathcal{M}$ a quadratic form on $\mathcal{B}_x \times \dot{\mathcal{B}}_x$ with values in $\wedge_m(T_x\mathcal{M})$. The notions of ellipticity and hyperbolicity at x relative to f_0 refer only to the principal part of the linearized Lagrangian at f_0 and x, a quadratic form on $\dot{\mathcal{B}}_x$ with values in $\wedge_m(T_x\mathcal{M})$ given by:

$$[\dot{L}](V) = \frac{1}{2}\frac{\partial^2 L}{\partial v^2}(df_0(x))(V, V) \quad : \ \forall V \in \dot{\mathcal{B}}_x$$

Choosing a volume form ϵ on $\mathcal{M}$ [the natural choice in the physical case being $d\mu_g$] so that $L = L^*\epsilon$, we can write:

$$[\dot{L}] = \frac{1}{2}h(V,V)\epsilon$$

where

$$h = \frac{\partial^2 L^*}{\partial v^2}(df_0)$$

is a quadratic form on $\mathcal{L}(T_x\mathcal{M}, T_y\mathcal{N})$, $y = f_0(x)$. In terms of local coordinates $(x^\mu \;:\; \mu = 1, ..., m)$ on $\mathcal{M}$ and $(y^a \;:\; a = 1, ..., n)$ on $\mathcal{N}$ [dim$\mathcal{M} = m$, dim$\mathcal{N} = n$], $v \in \mathcal{V}_{(x,y)}$ takes the form:

$$v = v^a_\mu \, dx^\mu|_x \otimes \left.\frac{\partial}{\partial y^a}\right|_y$$

The coefficients (v^a_μ) of the expansion are linear coordinates on $\mathcal{V}_{(x,y)}$ and we can write:

$$\frac{1}{2}h^{\mu\nu}_{ab}V^a_\mu V^b_\nu = \frac{1}{2}\frac{\partial^2 L^*}{\partial v^a_\mu \partial v^b_\nu}(df_0(x))V^a_\mu V^b_\nu$$

The notions of ellipticity or of hyperbolicity of a system of Euler-Lagrange equations at a given solution f_0 are notions which refer to the Euler-Lagrange equations corresponding to $\dot{L}$, the linearized Lagrangian at f_0. Now two different Lagrangians give rise to the same Euler-Lagrange equations whenever their difference is a *null Lagrangian*. Thus these notions actually refer not to a given $\dot{L}$ but rather to the equivalence class obtained by adding to $\dot{L}$ an arbitrary quadratic null Lagrangian. Furthermore these notions concern only the principal part, the quadratic form on $\dot{\mathcal{B}}_x$, at each $x \in \mathcal{M}$. Now the principal part of a quadratic null Lagrangian is of the form, in terms of the ϵ - dual,

$$\frac{1}{2} n_{ab}^{\mu\nu} V_\mu^a V_\nu^b$$

where n is a quadratic form on $\dot{\mathcal{B}}_x$ with the property:

$$n_{ab}^{\mu\nu} = -n_{ab}^{\nu\mu} = -n_{ba}^{\mu\nu}$$

We call such a quadratic form *odd*.

In general, if U and V are vector spaces, then $S_2(\mathcal{L}(U,V))$, the space of quadratic forms on $\mathcal{L}(U,V) = U^* \otimes V$ decomposes into:

$$S_2(\mathcal{L}(U,V)) = S_{2+}(\mathcal{L}(U,V)) \oplus S_{2-}(\mathcal{L}(U,V))$$

where $S_{2+}(\mathcal{L}(U,V))$ is the space of *even* quadratic forms, namely those $q \in S_2(\mathcal{L}(U,V))$ which satisfy:

$$\begin{aligned} q(\alpha_1 \otimes v_1, \alpha_2 \otimes v_2) &= q(\alpha_2 \otimes v_1, \alpha_1 \otimes v_2) \\ &= q(\alpha_1 \otimes v_2, \alpha_2 \otimes v_1) \\ &: \forall \alpha_1, \alpha_2 \in U^*, \quad \forall v_1, v_2 \in V \end{aligned}$$

and $S_{2-}(\mathcal{L}(U,V))$ is the space of *odd* quadratic forms, namely those $q \in S_2(\mathcal{L}(U,V))$ which satisfy:

$$\begin{aligned} q(\alpha_1 \otimes v_1, \alpha_2 \otimes V_2) &= -q(\alpha_2 \otimes v_1, \alpha_1 \otimes v_2) \\ &= -q(\alpha_1 \otimes v_2, \alpha_2 \otimes v_1) \\ &: \forall \alpha_1, \alpha_2 \in U^*, \quad \forall v_1, v_2 \in V \end{aligned}$$

Given an arbitrary quadratic form q, we denote by q_+ and q_- its even and odd parts respectively.

In view of the above remarks, the notions of ellipticity or of hyperbolicity of a Lagrangian at $v = df_0(x) \in \mathcal{V}_{(x,y)}$, $y = f_0(x)$, should depend only on the equivalence class:

$$\{h + n \ : \ n \in S_{2-}(\mathcal{L}(T_x\mathcal{M}, T_y\mathcal{N}))\}$$

or equivalently only on h_+, the even part of h. Remark that if q is an odd quadratic form on $\mathcal{L}(U,V)$ then q vanishes on all rank 1 elements of $\mathcal{L}(U,V)$:

$$q(\alpha \otimes v, \alpha \otimes v) = 0 \quad : \ \forall \alpha \in U^*, \ \forall v \in V$$

Consequently, the following definitions comply with our requirement.

Definition: A Lagrangian L is called *regularly elliptic* at $v \in \mathcal{V}_{(x,y)}$ if $h = (\partial^2 L^* / \partial v^2)(v)$ is positive-definite on the set

$$\{\xi \otimes Y \ : \ \xi \in T_x^*\mathcal{M}, \ Y \in T_y\mathcal{N}\}$$

of all rank 1 elements of $\mathcal{L}(T_x\mathcal{M}, T_y\mathcal{N})$.

Regular ellipticity is known as the *Legendre-Hadamard* condition in the calculus of variations.

Definition: A Lagrangian L is *regularly hyperbolic* at $v \in \mathcal{V}_{(x,y)}$ if there exists a pair $(X, \xi) \in T_x\mathcal{M} \times T_x^*\mathcal{M}$ such that the restriction of $h = (\partial^2 L^* / \partial v^2)(v)$ to

$$L_\xi = \{\xi \otimes Y \ : \ Y \in T_y\mathcal{N}\}$$

is negative-definite, while the restriction of h to Σ_X^1, the set of rank 1 elements of

$$\Sigma_X = \{V \in \mathcal{L}(T_x\mathcal{M}, T_y\mathcal{N}) \ : \ V(X) = 0\}$$

is positive-definite. [Note that this implies $\xi \cdot X \neq 0$.]

Suppose that h is a regularly hyperbolic quadratic form on $\mathcal{L}(T_x\mathcal{M}, T_y\mathcal{N})$. We define $\mathcal{J}_x \subset T_x\mathcal{M}$ to be the set of all vectors $X \in T_x\mathcal{M}$ such that the restriction of h to Σ^1_X is positive-definite. Then $0 \notin \mathcal{J}_x$ and $\mathcal{J}_x$ is the disjoint union of $\mathcal{J}_x^+$ and $\mathcal{J}_x^-$, where $\mathcal{J}_x^-$ is the set of opposites of elements in $\mathcal{J}_x^+$. Similarly, we define $\mathcal{I}_x^* \subset T_x^*\mathcal{M}$ to be the set of all covectors $\xi \in T_x^*\mathcal{M}$ such that the restriction of h to L_ξ is negative-definite. Then $0 \notin \mathcal{I}_x^*$ and $\mathcal{I}_x^*$ is the disjoint union of $\mathcal{I}_x^{*+}$ and $\mathcal{I}_x^{*-}$, where $\mathcal{I}_x^{*-}$ is the set of opposites of elements in $\mathcal{I}_x^{*+}$. Once a choice of positive component $\mathcal{J}_x^+$ has been made for $\mathcal{J}_x$, the positive component $\mathcal{I}_x^{*+}$ of $\mathcal{I}_x^*$ is distinguished by

$$\xi \cdot X > 0 \quad : \ \forall (X, \xi) \in \mathcal{J}_x^+ \times \mathcal{I}_x^{*+}$$

Proposition: The sets $\mathcal{J}_x^+$, $\mathcal{J}_x^-$, $\mathcal{I}_x^{*+}$, $\mathcal{I}_x^{*-}$ are convex.

The notions of ellipticity and hyperbolicity are related as follows. Given a vectorfield X on $\mathcal{M}$ we can consider those maps of $\mathcal{M}$ into $\mathcal{N}$ which are invariant under the 1 parameter group of diffeomorphisms of $\mathcal{M}$ onto itself generated by X. This leads to a reduced Lagrangian whose associated quadratic form at $v \in \mathcal{V}_{(x,y)}$ is the restriction of the original quadratic form to Σ_X, those linear maps of $T_x\mathcal{M}$ into $T_y\mathcal{N}$ which annihilate $X(x)$. We then ask the following question: what is the set of values of X at x such that the reduced quadratic form on Σ_X is regularly elliptic? The answer is the set $\mathcal{J}_x$.

At each $\xi \in T_x^*\mathcal{M}$ we define the *characteristic form* $\chi(\xi)$, a quadratic form on $T_y\mathcal{N}$, by:

$$\chi(\xi) \cdot (Y, Y) = h(\xi \otimes Y, \xi \otimes Y) \quad : \ \forall Y \in T_y\mathcal{N}$$

In terms of local coordinates,

$$\chi_{ab}(\xi) = h_{ab}^{\mu\nu}\xi_\mu\xi_\nu$$

The *characteristic subset* $\mathcal{C}_x^*$ of $T_x^*\mathcal{M}$ is defined by:

$$\mathcal{C}_x^* = \{\xi \neq 0 \in T_x^*\mathcal{M} \ : \ \chi(\xi) \text{ is degenerate}\}$$

If a volume form on $\mathcal{N}$ is chosen we can define:

$$H(\xi) = \det\chi(\xi)$$

Then $\mathcal{C}_x^*$ is the zero level set of H on $T_x^*\mathcal{M}$. As x is an arbitrary point of $\mathcal{M}$, this defines H as a function on $T^*\mathcal{M}$. This is the *Hamiltonian function*.

The associated *canonical equations*,

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial \xi_\mu}, \qquad \frac{d\xi_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu}$$

(in local coordinates), define the *bi-characteristic flow* on the zero level set of H in $T^*\mathcal{M}$. A bi-characteristic is a path on the zero level set of H in $T^*\mathcal{M}$ which corresponds to a solution of the canonical equations. The Hamiltonian function is in fact only defined up to a transformation of the form $H \mapsto \Omega H$ where Ω is a function on $\mathcal{M}$ which nowhere vanishes. Such a transformation preserves the paths, changing only the parametrization.

Denoting by $N(\chi(\xi))$ the null space of $\chi(\xi)$ as a linear map of $T_y\mathcal{N}$ into $T^*_y\mathcal{N}$, for ξ belonging to a component of $\mathcal{C}^*_x$, $N(\chi(\xi))$ is a non-trivial subspace of $T_y\mathcal{M}$. We call this the *degrees of freedom*, or *waves*, carried by that component.

At any given point $x \in \mathcal{M}$, H is a homogeneous polynomial of degree $2n$ in ξ. Such a homogeneous polynomial is in general *irreducible*, that is it cannot be decomposed into factors of lower degree.

However in the case of fluid mechanics ($n = 3$), H is given by:

$$H = A^4 B$$

where A is linear in ξ and B quadratic:

$$A = u^\mu \xi_\mu \quad B = (h^{-1})^{\mu\nu} \xi_\mu \xi_\nu$$

Here,

$$h^{-1} = g^{-1} + \left(1 - \frac{1}{\eta^2}\right) u \otimes u$$

is a quadratic form of index 1 on $T_x^* \mathcal{M}$, at each $x \in \mathcal{M}$, the reciprocal of h, a Lorentzian metric on $\mathcal{M}$, the *acoustical metric*, given in local coordinates by:

$$h_{\mu\nu} = g_{\mu\nu} + (1 - \eta^2) u_\mu u_\nu, \quad u_\mu = g_{\mu\nu} u^\nu$$

In the above, $\eta > 0$ is the *sound speed*, defined by:

$$\eta^2 = \left(\frac{dp}{d\rho}\right)_s$$

it being assumed that the right hand side is positive.

The component of $\mathcal{C}_x^*$ corresponding to $A = 0$ carries the *vorticity waves*, while the component corresponding to $B = 0$ carries the *sound waves*. In the case of solid mechanics H is irreducible except in the case of very special energy per particle functions e.

In the case that $n = 1$, H is quadratic in ξ hence

$$\dot{x} = \frac{\partial H}{\partial \xi_\mu} \frac{\partial}{\partial x^\mu}\bigg|_x \in T_x\mathcal{M}$$

is linear in ξ. Upon substituting ξ in terms of $\dot{x}$ in H we obtain a quadratic form of index 1 in $T_x\mathcal{M}$, for each $x \in \mathcal{M}$, that is a Lorentzian metric on $\mathcal{M}$. However in the general case where H is of degree $2n$ in ξ, $\dot{x}$ is of degree $2n - 1$ in ξ and we have a generalization of the standard Lorentzian geometry.

Remark that to each non-zero vector $X \in T_x\mathcal{M}$ there corresponds a hyperplane $\Pi(X)$ in $T_x^*\mathcal{M}$:

$$\Pi(X) = \{\xi \in T_x^*\mathcal{M} \ : \ \xi \cdot X = 0\}$$

The *characteristic subset* $\mathcal{C}_x$ of $T_x\mathcal{M}$ is the set of all non-zero vectors $X \in T_x\mathcal{M}$ such that the corresponding hyperplane $\Pi(X)$ is tangent to $\mathcal{C}_x^*$.

Proposition : $\mathcal{I}_x^*$ is the interior of the innermost component of $\mathcal{C}_x^*$, the *inner characteristic core* in $T_x^*\mathcal{M}$, and $\mathcal{J}_x$ is the interior of the innermost component of $\mathcal{C}_x$, the *inner characteristic core* in $T_x\mathcal{M}$.

The *causal subset* $\overline{\mathcal{I}}_x$ of $T_x\mathcal{M}$ is defined to be the set of all $X \in T_x\mathcal{M}$ such that $\xi \cdot X \neq 0$: $\forall \xi \in \mathcal{I}^*_x$. The boundary of $\overline{\mathcal{I}}_x$ is then the set of all $X \in \overline{\mathcal{I}}_x$ such that $\exists \xi \in \partial\mathcal{I}^*_x$: $\xi \cdot X = 0$. Thus if $X \in \partial\overline{\mathcal{I}}_x$, each component of $\mathcal{I}^*_x$ lies to one side of the hyperplane $\Pi(X)$, for, $\xi \cdot X$ has one sign in each component, and $\partial\mathcal{I}^*{}_x$ intersects $\Pi(X)$ at a line through the origin. Consequently, $\Pi(X)$ is tangent to $\partial\mathcal{I}^*_x$. Hence $\partial\overline{\mathcal{I}}_x$ corresponds to that component of $\mathcal{C}_x$ which is dual to the component $\partial\mathcal{I}^*_x$ of $\mathcal{C}^*_x$. We thus have:

$$\overline{\mathcal{I}}_x \supset \mathcal{J}_x$$

We conclude with a formulation of the domain of dependence theorem. Let L be a C^∞ Lagrangian in the general theory of maps of a manifold $\mathcal{M}$ into a manifold $\mathcal{N}$. Let f_0 be a C^2 solution of the Euler-Lagrange equations corresponding to L, defined in a domain $\Omega \subset \mathcal{M}$, such that L is regularly hyperbolic at $df_0(x)$, for each $x \in \Omega$. Consider a hypersurface $\mathcal{H}$ in Ω such that for each $x \in \mathcal{H}$ the double ray $R_x(\mathcal{H})$ in $T^*_x\mathcal{M}$ defined by $T_x\mathcal{H}$ according to:

$$R_x(\mathcal{H}) = \{\xi \neq 0 \in T^*_x\mathcal{M} \mid \xi \cdot X = 0 \ : \ \forall X \in T_x\mathcal{H}\}$$

is contained in $\mathcal{I}^*_x$, as determined by the even part of $(\partial^2 L/\partial v^2)(df_0(x))$. We call such a hypersurface *spacelike* relative to L and f_0. A curve γ in Ω is called *causal* relative to L and f_0 if its tangent vector $\dot{\gamma}(t)$ at each point $\gamma(t)$ belongs to the causal subset $\overline{\mathcal{I}}_{\gamma(t)}$ of $T_{\gamma(t)}\mathcal{M}$ as determined by the even part of $(\partial^2 L/\partial v^2)(df_0(\gamma(t)))$. We then define $\mathcal{D}(\mathcal{H})$, the *domain of dependence* of $\mathcal{H}$ relative to L and f_0 to be the set of all points $x \in \Omega$ such that each causal curve γ through x, $\gamma(0) = x$, intersects $\mathcal{H}$ at a single point $\gamma(t_*)$.

Theorem: Under the above hypotheses, let f_1, a C^1 map of Ω into $\mathcal{N}$, be another solution of the Euler-Lagrange equations corresponding to L, such that

$$f_1(x) = f_0(x), \quad df_1(x) = df_0(x) \quad : \ \forall x \in \mathcal{H}$$

Then f_1 coincides with f_0 on $\mathcal{D}(\mathcal{H})$, the domain of dependence of $\mathcal{H}$ relative to L and f_0.

The above material in contained in my monograph "The Action Principle and Partial Differential Equations".