Neutron stars as matter and gravity laboratories

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CEICO seminar
Motivation
Neutron Stars, Astrophysics, and Spacetime

Neutron Star’s Structure
Non-rotating fluid configurations
Rotating fluid configurations

Neutron Star’s Multipole Moments in GR
Multipole moments and their spin dependence
Multipole moments’ universal relations (3-hair relations)
Some more universal relations (I-Love-Q)
GW170817 on neutron star structure

Neutron Stars in Scalar-Tensor theory
Scalar-Tensor theory with a massless scalar field
Moments’ universal relations in ST (preliminary results)

Astrophysical observables and NS moments
QPOs and geodesic motion frequencies
Frequencies and multipole moments
Moments and equation of state

Conclusions
Neutron stars are the results of stellar evolution. We can see them in stellar remnants. A typical example is the Crab nebula that hosts the Crab pulsar\textsuperscript{a}.

\textsuperscript{a}APOD 2006 October 26

Very often we find rapidly rotating pulsars at the end of stellar evolution. The fastest rotating known pulsar (PSR J1748-2446ad) spins at 716Hz and it is part of a binary system\textsuperscript{a}.

Low mass X-ray binaries are systems that are comprised by a compact object (NS or BH) and a regular star companion. The main source of the X-rays is the accretion disk that forms around the compact object.

\textsuperscript{a}J. W. T. Hessels et al., Science 311 1901 (2006)

Interesting astrophysics takes place around NSs that depends on the background spacetime. Matter in their interior is at very high densities, where the equation of state is unknown. NSs have strong enough gravitational fields that can test our theories of gravity.
And as we all saw on Monday the 16th, 2017, from the 17th of August 2017 we have “heard”, through gravitational waves, the inspiral and collision of a binary neutron star system.

\[ a \]

In low-mass X-ray binaries we can have observables related to geodesic motion. An example of observables related to orbits around neutron stars are the quasi-periodic oscillations (QPOs) of the spectrum\(^1\) of an accretion disc.

Mechanisms for producing QPOs\(^2\) from orbital motion

Effects: orbiting hot spots, oscillations on the disc, precessing rings or misaligned precessing discs, and so on. These could result in a modulated emission or they could be eclipsing the emission from the central object.

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Neutron stars: Fluid configurations that are in equilibrium by the action of their self-gravity and their internal forces.

**Newtonian Stars**

Hydrostatic equilibrium (spherical symmetry):
\[ \nabla P = -\rho \nabla \Phi \Rightarrow \frac{dP}{dr} = -\frac{d\Phi}{dr} \rho = -G \frac{m(r)}{r^2} \rho \]

Mass (spherical symmetry):
\[ \frac{dm}{dr} = 4\pi \rho r^2 \]

Field equations: \[ \nabla^2 \Phi = 4\pi G \rho, \]
Equation of state for the fluid: \[ P = P(\rho). \]

**Relativistic non-rotating Stars**

Instead of a gravitational field \( \Phi \), gravity is described by a metric \( g_{ab} \).

In spherical symmetry: \[ ds^2 = -e^{2\Phi} dt^2 + \left( 1 - \frac{2m(r)}{r} \right)^{-1} dr^2 + r^2 d\Omega^2. \]

Field equations: \[ G^{ab} = 8\pi G T^{ab}, \]
Equation for the metric potential \( \Phi \):
\[ \frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3 P}{r(r - 2m(r))}, \]
Definition of the Mass:
\[ \frac{dm}{dr} = 4\pi \rho r^2, \]
Hydro. equilibrium:
\[ \frac{dP}{dr} = -(\rho + P) \frac{d\Phi}{dr} = -\frac{\rho m(r)}{r^2} \left( 1 + \frac{P}{\rho} \right) \left( 1 + \frac{4\pi Pr^3}{m(r)} \right) \left( 1 - \frac{2m(r)}{r} \right)^{-1}, \]
Equation of state for the fluid: \[ P = P(\rho). \]

The spacetime outside the star is the Schwarzschild spacetime:
\[ ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega^2. \]
Realistic equations of state

Polytropic equations of state (newtonian polytropes: $M \propto R^{n-1}$)
Outline
Motivation
Neutron Star’s Structure
Neutron Star’s Multipole Moments in GR
Neutron Stars in Scalar-Tensor theory
Astrophysical observables and NS moments

Non-rotating fluid configurations
Rotating fluid configurations

Slowly Rotating Neutron Stars
\[ ds^2 = -e^\nu (1 + 2\epsilon^2 h) \, dt^2 + e^\lambda \left[ 1 + \frac{2\epsilon^2 m}{(r-2M)} \right] \, dr^2 + r^2 \left[ 1 + 2\epsilon^2 k \right] \left[ d\theta^2 + \sin^2 \theta (d\phi - \epsilon \omega \, dt)^2 \right] . \]
where \( \epsilon = \Omega/\Omega^* \) is the slow rotation small parameter with respect to \( \Omega^* = (M/R^3)^{1/2} \).


Rapidly Rotating Neutron Stars: Numerical
The line element for a stationary and axially symmetric spacetime is
\[ ds^2 = -e^{2\nu} \, dt^2 + r^2 \sin^2 \theta B^2 e^{-2\nu} (d\phi - \omega \, dt)^2 + e^{2(\zeta-\nu)} (dr^2 + r^2 d\theta^2) . \]

Komatsu, Eriguchi, and Hechisu\(^b\) proposed a scheme for integrating the field equations which is implemented by the RNS numerical code to calculate rotating neutron stars. \(^c\)


Rapidly Rotating Neutron Stars: Analytic
Using the Weyl-Papapetrou line element that describes stationary and axisymmetric vacuum spacetimes,
\[ ds^2 = -f (dt - \omega d\phi)^2 + f^{-1} \left[ e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\phi^2 \right] , \]
Ernst\(^a\) reformulated the Einstein field equations to take the form, \( (Re(\mathcal{E})) \nabla^2 \mathcal{E} = \nabla \mathcal{E} \cdot \nabla \mathcal{E} \), using the complex potential \( \mathcal{E}(\rho, z) = f(\rho, z) + i\psi(\rho, z) \), where \( f = \xi^a \xi_a \) and \( \psi \) is defined by, \( \nabla_a \psi = \epsilon_{abcd} \xi^b \nabla^c \xi^d \).

Results from numerical models:

One can use RNS to calculate models of rotating neutron stars for a given equation of state. For example we show here some models for the APR EoS:

The models with the fastest rotation have a spin parameter, $j = J/M^2$, around 0.7 and a ratio of the polar radius over the equatorial radius, $r_p/r_e$, around 0.56.

The code calculates the various physical characteristics of the NS, the metric functions on a grid, and the relativistic multipole moments, i.e., $M, S_1 \equiv J, M_2 \equiv Q, S_3 \equiv J_3$ and $M_4$. These moments characterise the NS and the spacetime around it.

### Neutron star multipole moments in GR

**Black Hole-like behaviour of the moments:**

<table>
<thead>
<tr>
<th>Kerr moments</th>
<th>Neutron star moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_0 = M$,</td>
<td>$M_0 = M$,</td>
</tr>
<tr>
<td>$J_1 \equiv J = jM^2$,</td>
<td>$J_1 = jM^2$,</td>
</tr>
<tr>
<td>$M_2 \equiv Q = -j^2 M^3$,</td>
<td>$M_2 = -a(EoS, M)j^2 M^3$,</td>
</tr>
<tr>
<td>$J_3 \equiv S_3 = -j^3 M^4$,</td>
<td>$J_3 = -\beta(EoS, M)j^3 M^4$,</td>
</tr>
<tr>
<td>$M_4 = j^4 M^5$,</td>
<td>$M_4 = \gamma(EoS, M)j^4 M^5$,</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$M_{2n} = (-1)^n j^{2n} M^{2n+1}$,</td>
<td>$M_{2n} = \ ?$,</td>
</tr>
<tr>
<td>$J_{2n+1} \equiv S_{2n+1} = (-1)^n j^{2n+1} M^{2n+2}$</td>
<td>$J_{2n+1} = \ ?$</td>
</tr>
</tbody>
</table>

where $j = J/M^2$.

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EoS independent behaviour of the moments⁵:

\[ \bar{M}_{2n} = \left| M_{2n} / (j^{2n} M^{2n+1}) \right|, \quad \bar{S}_{2n+1} = \left| S_{2n+1} / (j^{2n+1} M^{2n+2}) \right| \]

All these are properties that characterise the spacetime around neutron stars as well as the gravitational aspects of the stars themselves. Therefore we can use them to construct analytic neutron star spacetimes with only few parameters.

Neutron stars as matter and gravity laboratories

Slow rotation $Q$-Love, $I$-Love and $I - Q$ relations$^6$

Slow and rapid rotation $I - Q$ relations ($\chi \equiv j$)$^7$

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From the gravitational wave phase one can extract information for the Love numbers of each of the two stars, $\Lambda_1$ and $\Lambda_2$. In the fig. stars with larger radii are towards up and right. This gives an estimate of $R \lesssim 14 km$ for an $1.4M_\odot$ NS.\(^a\)

In the case of Scalar-Tensor theories with a massless scalar field, 

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left( \tilde{R} - 2\tilde{\nabla}^\mu \phi \tilde{\nabla}_\mu \phi \right) + S_m(g_{\mu\nu}, \psi), \]

the field equations in the Einstein frame take the form,

\[ \tilde{R}_{ab} = 2\partial_a \phi \partial_b \phi + 8\pi G \left( T_{ab} - \frac{1}{2} g_{ab} T \right), \quad \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \phi = -4\pi \alpha(\phi) T \]

These equations can be solved as in GR in order to construct neutron stars.\(^8\) On the other hand, the vacuum field equations can admit an Ernst formulation as in GR,\(^9\)

\[(Re(\mathcal{E})) \nabla^2 \mathcal{E} = \nabla \mathcal{E} \cdot \nabla \mathcal{E},\]

with the addition of a Laplace equation for the scalar field \( \nabla^2 \phi = 0. \)

One can extend the definition of multipole moments in this case as well where the moments (mass, spin, scalar) are defined in the Einstein frame but the actual physics is done in the Jordan (physical) frame, where the metric is given by the conformal transformation 

\[ g_{\mu\nu} = A^2(\phi) \tilde{g}_{\mu\nu}. \]

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Scalar-Tensor theory with a massless scalar field
Moments’ universal relations in ST (preliminary results)

Motivation

Neutron Star’s Structure
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Astrophysical observables and NS moments

ST models against GR models for APR EoS and $S_3^{ST}$, $M_4^{ST}$ vs $Q^{ST}$ relations for various EoSs.

Scalar field normalised moments plotted against the spin parameter and the quadrupole.\textsuperscript{11}

Circular equatorial orbits: If we define $\Omega \equiv \frac{d\phi}{dt}$, the energy, angular momentum and orbital frequency for the circular orbits take the form,

$$\tilde{E} \equiv \frac{\tilde{E}}{m} = \frac{-g_{tt} - g_{t\phi} \Omega}{\sqrt{-g_{tt} - 2g_{t\phi} \Omega - g_{\phi\phi} \Omega^2}},$$

$$\tilde{L} \equiv \frac{\tilde{L}}{m} = \frac{g_{t\phi} + g_{\phi\phi} \Omega}{\sqrt{-g_{tt} - 2g_{t\phi} \Omega - g_{\phi\phi} \Omega^2}},$$

$$\Omega = \frac{-g_{t\phi,\rho} + \sqrt{(g_{t\phi,\rho})^2 - g_{tt,\rho} g_{\phi\phi,\rho}}}{g_{\phi\phi,\rho}}.$$

For more general orbits: Equations of motion can take the general form,

$$-g_{\rho\rho} \left( \frac{d\rho}{d\tau} \right)^2 - g_{zz} \left( \frac{dz}{d\tau} \right)^2 = 1 - \tilde{E}^2 g_{\phi\phi} + 2\tilde{E}\tilde{L} g_{t\phi} + \tilde{L}^2 g_{tt} = V_{\text{eff}}.$$

We can study the precession properties from the properties of the effective potential.

$$-g_{\rho\rho} \left( \frac{d(\delta\rho)}{d\tau} \right)^2 - g_{zz} \left( \frac{d(\delta z)}{d\tau} \right)^2 = \frac{1}{2} \frac{\partial^2 V_{\text{eff}}}{\partial \rho^2} (\delta\rho)^2 + \frac{1}{2} \frac{\partial^2 V_{\text{eff}}}{\partial z^2} (\delta z)^2,$$

This equation describes two harmonic oscillators with epicyclic frequencies,

$$\tilde{K}_\rho^2 = \frac{g_{\rho\rho}}{2} \frac{\partial^2 V_{\text{eff}}}{\partial \rho^2} \bigg|_c,$$

$$\tilde{K}_z^2 = \frac{g_{zz}}{2} \frac{\partial^2 V_{\text{eff}}}{\partial z^2} \bigg|_c.$$

The differences of these frequencies (corrected for redshift) from the orbital frequency, $\Omega_a = \Omega - \kappa_a$, define the precession frequencies.
The energy change per logarithmic frequency interval and the precession frequencies are related to the spacetime multipole moments (Ryan, 1995),

**in GR:**
\[
\Delta \tilde{E} = - \frac{1}{3} \frac{d \tilde{E}}{dU} = \frac{1}{3} U^2 - \frac{1}{2} U^4 + \frac{20 J_1}{9 M^2} U^5 + \ldots
\]
\[
\frac{\Omega_\rho}{\Omega} = 3 U^2 - 4 \frac{J_1}{M^2} U^3 + \left( \frac{9}{2} - \frac{3M_2}{2M^3} \right) U^4 - 10 \frac{J_1}{M^2} U^5 + \left( \frac{27}{2} - 2 \frac{J_1}{M^4} - \frac{21M_2}{2M^3} \right) U^6 + \ldots
\]
\[
\frac{\Omega_z}{\Omega} = 2 \frac{J_1}{M^2} U^3 + \frac{3M_2}{2M^3} U^4 + \left( 7 \frac{J_1}{M^4} + 3 \frac{M_2}{M^3} \right) U^6 + \left( 11 \frac{J_1M_2}{M^5} - 6 \frac{S_3}{M^4} \right) U^7 + \ldots
\]

where \( U = \left( \frac{\Omega}{M \Omega} \right)^{1/3} \). The Orbital frequency gives the Keplarian mass: \( \Omega = \left( \frac{M}{r^3} \right)^{1/2} (1 + O(r^{-1/2})) \).

**in Scalar-**
\[
\Delta \tilde{E} = \frac{1}{3} U^2 + \left( \frac{2b_0 W_2}{9 M^2} - \frac{8a_0 W_0}{9 M} - \frac{1}{2} \right) U^4 + \frac{20 J_1}{9 M^2} U^5 + \ldots
\]

**Tensor theory:**
\[
\frac{\Omega_\rho}{\Omega} = \left( 3 - W_0 \left( \beta_0 W_2 - 8a_0 \bar{M} \right) \right) U^2 - \frac{4J_1}{M^2} U^3 + \ldots
\]
\[
\frac{\Omega_z}{\Omega} = 2 \frac{J_1}{M^2} U^3 + \frac{3(M_2 - a_0 W_2)}{2M^3} U^4 - \frac{2J_1 W_0}{M^4} \left( \beta_0 W_2 - a_0 \bar{M} \right) U^5 + \ldots
\]

where \( U = \left( \bar{M} \Omega \right)^{1/3} \). The calculations are done in the Jordan frame. Again the orbital frequency gives the Keplarian mass: \( \Omega = \left( \frac{\bar{M}}{r^3} \right)^{1/2} (1 + O(r^{-1/2})) \), but this time the Keplarian mass is \( \bar{M} = M - W_0 a_0 \). \( W_0 \) is the scalar charge, \( W_2 \) is the scalar quadrupole and \( a \equiv (d \ln A)/d \phi, \beta \equiv d a / d \phi \). These observables could in principle distinguish between GR and Scalar-Tensor theory.

Determining the moments $M_2, J$ (or $\alpha \equiv -\frac{M_2}{j^2 M^3}$ and $j = J/M^2$ equivalently) from QPOs for example, and having some independent knowledge of the mass of the neutron star (assuming for example that it is known from the binary system observations), one could use them to select an EoS\textsuperscript{a} out of the realistic EoS candidates (fig. shows 3 different EoSs, where each EoS traces a surface in the parameter space).

\textsuperscript{a}G.P. and T. A. Apostolatos, Phys.Rev.Lett. 112 121101 (2014)
Neutron stars exhibit some black hole-like behavior with respect to their moments structure, but the moments are different from black hole moments, so the geometry is essentially different from the geometry of Kerr black holes.

There are several NS properties that show universal behavior (EoS independent), i.e., the moments in particular.

These universal properties are present in some alternative theories of gravity as well, such as in ST theories with a massless scalar field.

The multipole moments determine the orbital dynamics and are of relevance to the study of accretion discs and quasi periodic oscillations (QPOs).

Geodesic properties such as orbital and precession frequencies could be used to measure the moments and from them the EoS.

These geodesic properties could also distinguish between different theories of gravity such as GR and Scalar-Tensor theory.

There is a lot of work to be done (in particular on the astrophysics side).

Thank you