Multipole moments: from GR to scalar-tensor theory

Connection to astrophysical observables

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(Non-)Universal Properties of Neutron stars
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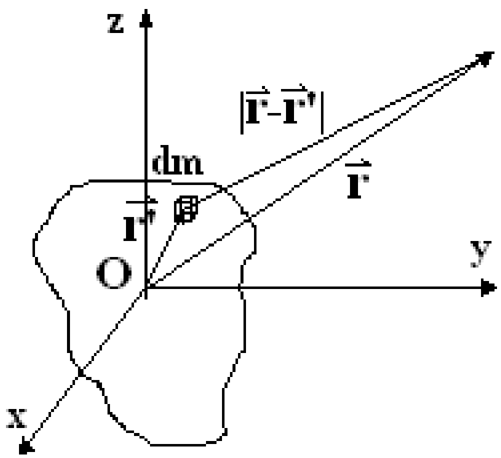
- Multipole moments
  - in Newtonian theory.
  - in General Relativity.
  - in Scalar-Tensor theory.
- Properties of neutron star moments in GR
- Astrophysical observables: Accretion discs and QPOs
- Observables and multipole moments: GR vs Scalar-Tensor theory.
- What next? (Outlook)

Why multipole moments?

- In order to explore the structure of compact objects or to test gravity, we need to probe the spacetime around them.
- Multipole moments provide a way of characterising the spacetime as well as the structure of the compact object.
- Moments can be defined in an invariant way, can be related to the source[1], and can be related to astrophysical observables.

[1] See for example Gürlebeck, Phys.Rev. D **90** 024041 (2014).



Newtonian gravity is described by potential that is the solution of a Laplace field equation in a flat space

$$\Phi(r) = G\int \frac{\rho(r')dV'}{|\vec{r}-\vec{r}'|}$$

Multipolar Expansions
(the multipoles characterise the field)
In spherical coordinates:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}} = \frac{1}{r\sqrt{1+\varepsilon^2-2\varepsilon\cos\theta'}}$$
$$= \frac{1}{r} + \frac{\cos\theta'\varepsilon}{r} + \frac{(1+3\cos 2\theta')\varepsilon^2}{4r} + \dots$$

$$\Phi(r) = G\left(\frac{1}{r}\int \rho(r')dV' + \frac{1}{r^2}\int r'\cos\theta'\rho(r')dV' + \frac{1}{r^3}\int \frac{1}{2}(3\cos^2\theta'-1)(r')^2\rho(r')dV' + \dots\right)$$

In cartesian coordinates:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = \frac{1}{r\sqrt{(x^a/r-\varepsilon^a)(x^b/r-\varepsilon^b)\delta_{ab}}}$$
$$= \frac{1}{r} + \frac{x^a\varepsilon_a}{r^2} + \frac{\frac{1}{2}(3x^ax^b-r^2\delta^{ab})\varepsilon_a\varepsilon_b}{r^3} + \dots = \frac{1}{r} + \frac{x^ax'_a}{r^3} + \frac{\frac{1}{2}(3x'_ax'_b-r'^2\delta_{ab})x^ax^b}{r^5} + \dots$$

$$\Phi(r) = G\left(\frac{1}{r}\int \rho(r')dV' + \frac{x^a}{r^3}\int x'_a\rho(r')dV' + \frac{x^ax^b}{r^5}\int \frac{1}{2}(3x'_ax'_b-r'^2\delta_{ab})\rho(r')dV' + \dots\right)$$

Newtonian multipole moments:

$$\Phi(r) = G\left(\frac{\mathcal{Q}}{r} + \frac{\mathcal{Q}_a x^a}{r^3} + \frac{\mathcal{Q}_{ab} x^a x^b}{r^5} + ...\right) \tag{1}$$

where, $\mathcal{Q}$, $\mathcal{Q}_a$, $\mathcal{Q}_{ab}$, are some integrals on the source

$$\mathcal{Q} = \int \rho(r')d^3x',\ \mathcal{Q}_a = \int x'_a \rho(r')d^3x',\ \mathcal{Q}_{ab} = \int \frac{3}{2}(x'_a x'_b - \frac{1}{3}r'^2\delta_{ab})\rho(r')d^3x'... \tag{2}$$

The multipole moments are generally tensorial quantities.

<u>Definition of the moments at infinity:</u>

$x^a \to \tilde{x}^a = r^{-2}x^a$: $\tilde{r}^2 = \tilde{x}^a\tilde{x}_a = r^{-2}$

$$\Phi(r) = \tilde{r}\left(\mathcal{Q} + \mathcal{Q}_a\tilde{x}^a + \mathcal{Q}_{ab}\tilde{x}^a\tilde{x}^b + ...\right) \tag{3}$$

If we define the potential at infinity $\tilde{\Phi} = \tilde{r}^{-1}\Phi$ then the moments are

$$P_{a_1...a_n} = \tilde{D}_{a_n}P_{a_1...a_{n-1}} = \tilde{D}_{a_1}...\tilde{D}_{a_n}\tilde{\Phi} \tag{4}$$

In General Relativity instead of a gravitational field Φ, gravity is described by a metric g_{ab}.

Relativistic multipole moments:

- Generalization of the Newtonian moments,
- Defined for asymptotically flat spacetimes at infinity from a "potential" (that is related to the metric) by a recursive relation,
  Projection formalism: We have a spacetime $(M, g_{\mu\nu})$ that admits a timelike Killing field ξ^a.
  We can use ξ_a to project the 4D space time on a 3D space. The EFE can then be written as a set of field equations for $\lambda = \xi^a \xi_a$ and $\omega_a = \epsilon_{abcd}\xi^b \nabla^c \xi^d$ on the 3D space $(S, h_{\mu\nu})$.

$$\mathcal{D}^2\lambda = \frac{1}{2}\lambda^{-1}(\mathcal{D}^m\lambda)(\mathcal{D}_m\lambda) - \lambda^{-1}\omega^m\omega_m - 2R_{mn}\xi^m\xi^n, \tag{5}$$

$$\mathcal{D}_{[a}\omega_{b]} = -\epsilon_{abmn}\xi^m R^n_p \xi^p, \tag{6}$$

$$\mathcal{D}^a\omega_a = \frac{3}{2}\lambda^{-1}\omega_m\mathcal{D}^m\lambda, \tag{7}$$

$$\mathcal{R}_{ab} = \frac{1}{2}\lambda^{-2}[\omega_a\omega_b - h_{ab}\omega^m\omega_m] + \frac{1}{2}\lambda^{-1}\mathcal{D}_a\mathcal{D}_b\lambda - \frac{1}{4}\lambda^{-2}(\mathcal{D}_a\lambda)(\mathcal{D}_b\lambda) + h_a^m h_b^n R_{mn}. \tag{8}$$

In GR, in vacuum, $R_{mn} = 0$ (EFE) and ω_a is curl-free. Thus one can define a scalar twist: $\omega_a = \mathcal{D}_a\omega$.
Then the 3D Ricci tensor and the field equations for λ, and ω can take the form:[2]

$$\tilde{\mathcal{D}}^2\lambda = \lambda^{-1}\left((\tilde{\mathcal{D}}^m\lambda)(\tilde{\mathcal{D}}_m\lambda) - (\tilde{\mathcal{D}}^m\omega)(\tilde{\mathcal{D}}_m\omega)\right), \tag{9}$$

$$\tilde{\mathcal{D}}^2\omega = 2\lambda^{-1}(\tilde{\mathcal{D}}^m\lambda)(\tilde{\mathcal{D}}_m\omega), \tag{10}$$

$$\tilde{\mathcal{R}}_{ab} = \frac{1}{2}\lambda^{-2}\left[(\tilde{\mathcal{D}}_a\lambda)(\tilde{\mathcal{D}}_b\lambda) + (\tilde{\mathcal{D}}_a\omega)(\tilde{\mathcal{D}}_b\omega)\right]. \tag{11}$$

[2] These are the expressions in the conformally related frame $\tilde{h}_{ab} = (-\lambda)h_{ab}$.

Relativistic multipole moments:

- There are two sets of moments, the Mass moments and the Rotation moments

- For the two sets of moments we have two generating potentials,
  Using the scalar quantities λ and ω, one can construct the potentials,

$$\Phi_M = \frac{1}{4}\lambda^{-1}(\lambda^2 + \omega^2 - 1) \quad \text{and} \quad \Phi_J = \frac{1}{2}\lambda^{-1}\omega, \tag{12}$$

  that satisfy the field equation

$$\mathcal{D}^a\mathcal{D}_a\Phi - (\mathcal{R}/8)\Phi = (15/8)\kappa^4\Phi. \tag{13}$$

  The properties of this equation ensure that the potentials admit a multipolar expansion.

- The moments are then given by the recursive algorithm:

$$\begin{aligned} P &= \tilde{\Phi}, \\ P_a &= \tilde{D}_a P, \\ &\vdots \\ P_{a_1 \ldots a_{s+1}} &= \mathcal{C}\left[\tilde{D}_{a_1} P_{a_2 \ldots a_{s+1}} - \frac{s(2s-1)}{2}\tilde{R}_{a_1 a_2} P_{a_3 \ldots a_{s+1}}\right], \end{aligned} \tag{14}$$

The multipole moments for stationary and axisymmetric spacetimes can be reduced from tensors to scalars, because of the rotation symmetry.

In Scalar-Tensor theory we can have the same construction as in GR but the EFE are modified (Einstein frame).

Scalar-Tensor multipole moments (for a massless scalar):[3]

- Same as in GR, but $R_{ab} = 2\partial_a\phi\partial_b\phi$ and $g^{ab}\nabla_a\nabla_b\phi = 0$.
- The field equations for λ, ω, and ϕ in the 3D space are,

$$\tilde{\mathcal{D}}^2\lambda = \lambda^{-1}\left((\tilde{\mathcal{D}}^m\lambda)(\tilde{\mathcal{D}}_m\lambda) - (\tilde{\mathcal{D}}^m\omega)(\tilde{\mathcal{D}}_m\omega)\right), \tag{15}$$
$$\tilde{\mathcal{D}}^2\omega = 2\lambda^{-1}(\tilde{\mathcal{D}}^m\lambda)(\tilde{\mathcal{D}}_m\omega), \tag{16}$$
$$\tilde{\mathcal{D}}^2\phi = 0, \tag{17}$$
$$\tilde{R}_{ab} = \frac{1}{2\lambda^2}\left[(\tilde{\mathcal{D}}_a\lambda)(\tilde{\mathcal{D}}_b\lambda) + (\tilde{\mathcal{D}}_a\omega)(\tilde{\mathcal{D}}_b\omega)\right] + 2(\tilde{\mathcal{D}}_a\phi)(\tilde{\mathcal{D}}_b\phi). \tag{18}$$

- There are 3 sets of moments, the Mass moments, the Rotation moments, and the Scalar moments.
- The generating potentials for the mass moments and the rotation moments are as in GR, Φ_M, and Φ_J, and the generating potential for the scalar moments is the scalar field itself. These fields satisfy the field equations

$$\mathcal{D}^a\mathcal{D}_a\Phi - (\mathcal{R}/8)\Phi = \kappa_1^4\Phi, \quad \text{and} \quad \mathcal{D}^a\mathcal{D}_a\phi - (\mathcal{R}/8)\phi = \kappa_2^4\phi. \tag{19}$$

The moments are again given by the same recursive relations as in GR.

[3] G P and T P Sotiriou, Phys.Rev. D **91** 044011 (2015).

Axisymmetric Scalar-Tensor multipole moments:

In the axisymmetric case, the field equations in the Einstein frame can be written in terms of an Ernst potential $\mathcal{E} = f + i\omega$, as in GR.
Therefore one can define the axisymmetric spacetime moments using the secondary Ernst potential, $\xi = \frac{1-\mathcal{E}}{1+\mathcal{E}}$ (where $\xi = \Phi_M + i\Phi_J$), while the scalar moments are given from the scalar field.

The moments will be given by the recursive algorithm,

$$\begin{aligned} P &= \tilde{\xi} \text{ for the mass and rotation or} \\ P &= \tilde{\phi} \text{ for the scalar,} \\ P_a &= \bar{D}_a P, \\ P_{a_1 \dots a_{s+1}} &= \mathcal{C}\left[\bar{D}_{a_1} P_{a_2 \dots a_{s+1}} - \frac{s}{2}(2s-1)\bar{R}_{a_1 a_2} P_{a_3 \dots a_{s+1}}\right]. \end{aligned} \tag{20}$$

Due to the rotational symmetry, the moments will now be some multiples of the symmetric trace free outer product of the axis vector and correspond to only one component of that tensor. Hence, they will be scalar quantities,

$$P_n = \frac{1}{n!} P^{(n)}_{i_1 \dots i_n} n^{i_1} \dots n^{i_n} \Big|_\Lambda = \frac{1}{n!} P^{(n)}_{2\dots 2} \Big|_\Lambda . \tag{21}$$

The first few moments (if we also assume equatorial symmetry) are:

$$P_0^g = m_0,\ P_1^g = m_1,\ P_2^g = m_2 - \frac{1}{3} m_0 w_0^2,\ P_3^g = m_3 - \frac{3}{5} m_1 w_0^2,\ \dots$$
$$P_0^\phi = w_0,\ P_1^\phi = 0,\ P_2^\phi = w_2 - \frac{1}{3} w_0 (m_0 m_0^* + w_0^2),\ P_3^\phi = 0,\ \dots$$

Neutron star multipole moments properties in GR

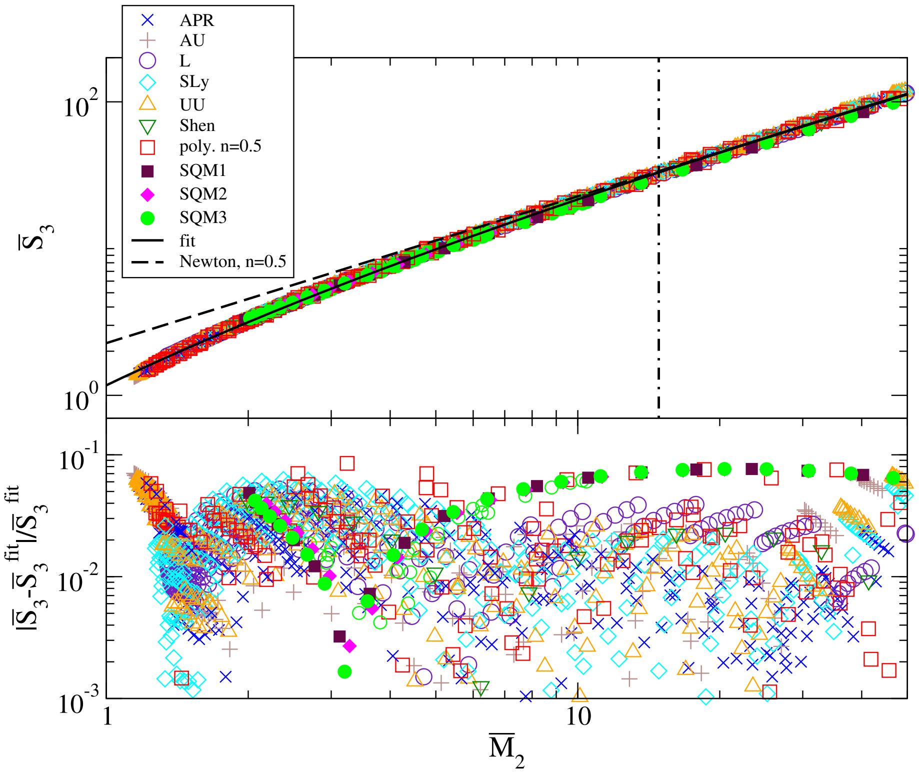
Black Hole-like behaviour of the moments[4]:

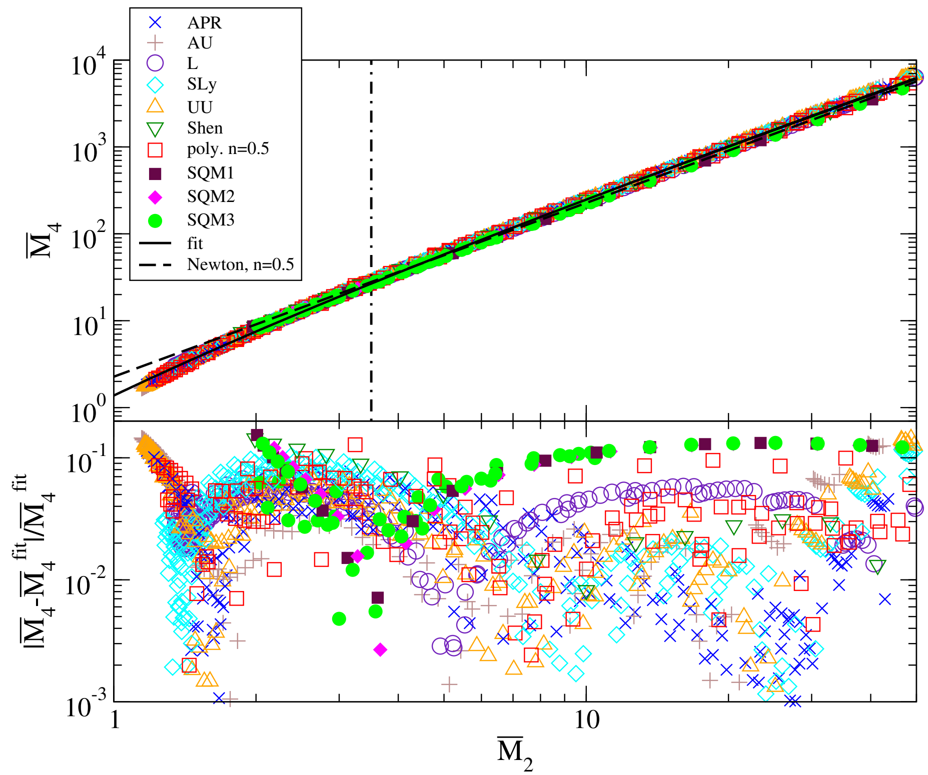
Kerr moments	Neutron star moments
$M_0 = M,$	$M_0 = M,$
$J_1 = J = jM^2,$	$J_1 = jM^2,$
$M_2 = -j^2M^3,$	$M_2 = -a(EoS,M)j^2M^3,$
$J_3 = -j^3M^4,$	$J_3 = -\beta(EoS,M)j^3M^4,$
$M_4 = j^4M^5,$	$M_4 = \gamma(EoS,M)j^4M^5,$
$\vdots$	$\vdots$
$M_{2n} = (-1)^n j^{2n}M^{2n+1},$	$M_{2n} = ?,$
$J_{2n+1} = (-1)^n j^{2n+1}M^{2n+2}$	$J_{2n+1} = ?$

[4]W.G. Laarakkers and E. Poisson, Astrophys. J. **512** 282 (1999).
G.P. and T. A. Apostolatos, Phys. Rev. Lett. **108** 231104 (2012).
K. Yagi, K. Kyutoku, G. P., N. Yunes, and T.A. Apostolatos, Phys.Rev. D **89** 124013 (2014).

Neutron star multipole moments properties in GR

EoS independent behaviour of the moments[5] (more on that in Kent's talk):





$$\bar{M}_{2n} = |M_{2n}/(j^{2n}M^{2n+1})|,\ \bar{J}_{2n+1} = |J_{2n+1}/(j^{2n+1}M^{2n+2})|$$

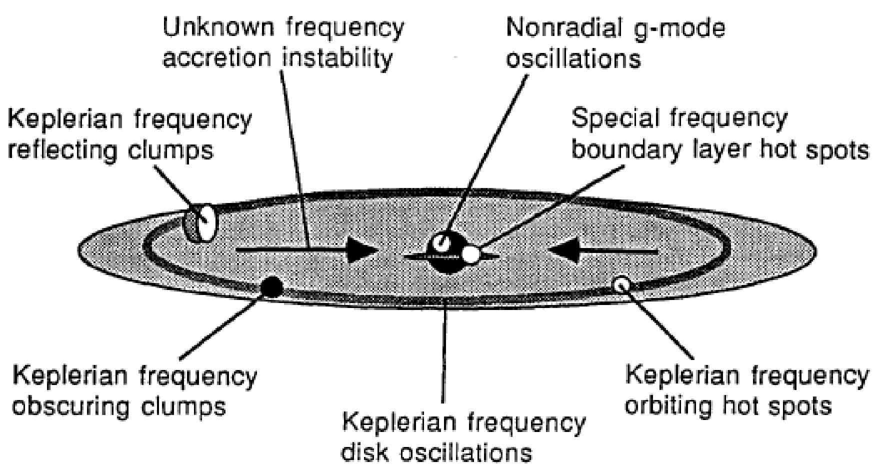
We need to explore this sort of relations in alternative theories of gravity as well. Scalar-Tensor theory would be the obvious next step.

[5] G.P. and T. A. Apostolatos, Phys.Rev.Lett. **112** 121101 (2014).
K. Yagi, K. Kyutoku, G. P., N. Yunes, and T.A. Apostolatos, Phys.Rev. D **89** 124013 (2014).

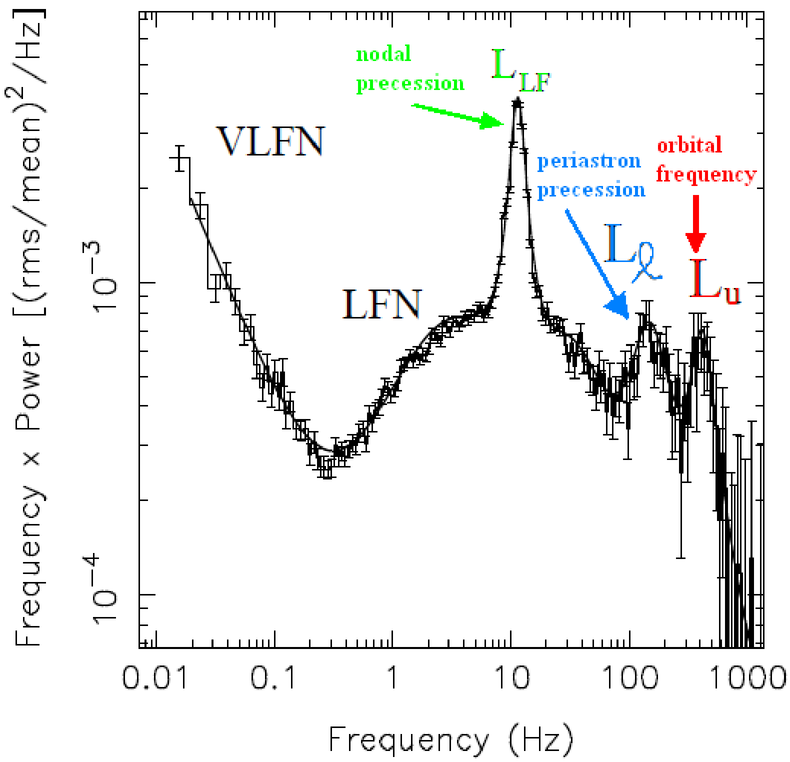
In astrophysical systems we don't observe multipole moments. We can have observables though that are related to geodesics and consequently to multipole moments.

An example of observables that can be related to orbits around neutron stars are the quasi-periodic oscillations (QPOs) of the spectrum[6] of an accretion disc.

Mechanisms for producing QPOs[7] from orbital motion



Typical X-Ray spectrum[8]



[6]Stella & Vietri, 1998, ApJ, 492, L59.

[7]F.K. Lamb, Advances in Space Research, 8 (1988) 421.

[8]Boutloukos et al., 2006, ApJ, 653, 1435-1444.

Particle motion in a spacetime with symmetries:

Symmetry in time translations is associated to an integral of motion, energy E

$$E = -p_a \xi^a = -p_t = -g_{tt} p^t - g_{t\phi} p^\phi = m \left(-g_{tt} \frac{dt}{d\tau} - g_{t\phi} \frac{d\phi}{d\tau} \right) \tag{22}$$

Symmetry in rotations is again associated to an integral of motion, angular momentum L

$$L = p_a \eta^a = p_\phi = g_{t\phi} p^t + g_{\phi\phi} p^\phi = m \left(g_{t\phi} \frac{dt}{d\tau} + g_{\phi\phi} \frac{d\phi}{d\tau} \right) \tag{23}$$

From the measure of the four-momentum, $p^a p_a = -m^2$, we have the equation,

$$-1 = g_{tt} \left(\frac{dt}{d\tau} \right)^2 + 2 g_{t\phi} \left(\frac{dt}{d\tau} \right) \left(\frac{d\phi}{d\tau} \right) + g_{\phi\phi} \left(\frac{d\phi}{d\tau} \right)^2 + g_{\rho\rho} \left(\frac{d\rho}{d\tau} \right)^2 + g_{zz} \left(\frac{dz}{d\tau} \right)^2 \tag{24}$$

Circular equatorial orbits: If we define $\Omega \equiv \frac{d\phi}{dt}$, then we have the redshift factor $\left(\frac{d\tau}{dt}\right)^2 = -g_{tt} - 2 g_{t\phi} \Omega - g_{\phi\phi} \Omega^2$,

and the energy and the angular momentum for the circular orbits take the form,

$$\tilde{E} \equiv E/m = \frac{-g_{tt} - g_{t\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}, \quad \tilde{L} \equiv L/m = \frac{g_{t\phi} + g_{\phi\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}. \tag{25}$$

From the conditions, $\frac{d\rho}{dt} = 0$, $\frac{d^2\rho}{dt^2} = 0$ and $\frac{dz}{dt} = 0$, and the equations of motion obtained assuming the Lagrangian, $L = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b$, the angular velocity can be calculated to be,

$$\Omega = \frac{-g_{t\phi,\rho} + \sqrt{(g_{t\phi,\rho})^2 - g_{tt,\rho} g_{\phi\phi,\rho}}}{g_{\phi\phi,\rho}}. \tag{26}$$

This is the orbital frequency of a particle in a circular orbit on the equatorial plane.

More general orbits:

Equation (24) can take a more general form in terms of the constants of motion,

$$-g_{\rho\rho}\left(\frac{d\rho}{d\tau}\right)^2 - g_{zz}\left(\frac{dz}{d\tau}\right)^2 = 1 - \frac{\tilde{E}^2 g_{\phi\phi} + 2\tilde{E}\tilde{L}g_{t\phi} + \tilde{L}^2 g_{tt}}{(g_{t\phi})^2 - g_{tt}g_{\phi\phi}} = V_{eff}, \tag{27}$$

With equation (27) we can study the general properties of the motion of a particle from the properties of the effective potential.

Small perturbations from circular equatorial orbits:

If we assume small deviations from the circular equatorial orbits of the form, $\rho = \rho_c + \delta\rho$ and $z = \delta z$, then we obtain the perturbed form of (27),

$$-g_{\rho\rho}\left(\frac{d(\delta\rho)}{d\tau}\right)^2 - g_{zz}\left(\frac{d(\delta z)}{d\tau}\right)^2 = \frac{1}{2}\frac{\partial^2 V_{eff}}{\partial\rho^2}(\delta\rho)^2 + \frac{1}{2}\frac{\partial^2 V_{eff}}{\partial z^2}(\delta z)^2,$$

This equation describes two harmonic oscillators with frequencies,

$$\bar{\kappa}_\rho^2 = \frac{g^{\rho\rho}}{2}\frac{\partial^2 V_{eff}}{\partial\rho^2}\bigg|_c, \quad \bar{\kappa}_z^2 = \frac{g^{zz}}{2}\frac{\partial^2 V_{eff}}{\partial z^2}\bigg|_c,$$

The differences of these frequencies (corrected with the redshift factor) from the orbital frequency, $\Omega_a = \Omega - \kappa_a$, define the precession frequencies.

The observables related to the orbits in a spacetime can be associated to its multipole moments.

The energy change per logarithmic frequency interval and the precession frequencies are related to the multipole moments,

in GR:

$$\Delta\tilde{E} = -\frac{U}{3}\frac{d\tilde{E}}{dU} = \frac{1}{3}U^2 - \frac{1}{2}U^4 + \frac{20J_1}{9M^2}U^5 + \dots$$

$$\frac{\Omega_\rho}{\Omega} = 3U^2 - 4\frac{J_1}{M^2}U^3 + \left(\frac{9}{2} - \frac{3M_2}{2M^3}\right)U^4 - 10\frac{J_1}{M^2}U^5 + \left(\frac{27}{2} - 2\frac{J_1^2}{M^4} - \frac{21M_2}{2M^3}\right)U^6 + \dots$$

$$\frac{\Omega_z}{\Omega} = 2\frac{J_1}{M^2}U^3 + \frac{3M_2}{2M^3}U^4 + \left(7\frac{J_1^2}{M^4} + 3\frac{M_2}{M^3}\right)U^6 + \left(11\frac{J_1M_2}{M^5} - 6\frac{S_3}{M^4}\right)U^7 + \dots$$

where $U = (M\Omega)^{1/3}$. The Orbital frequency gives the Keplerian mass: $\Omega = (M/r^3)^{1/2}(1 + O(r^{-1/2}))$.

in Scalar-Tensor theory:[9]

$$\Delta\tilde{E} = \frac{1}{3}U^2 + \left(\frac{2\beta_0 W_0^2}{9\bar{M}^2} - \frac{8\alpha_0 W_0}{9\bar{M}} - \frac{1}{2}\right)U^4 + \frac{20J_1}{9\bar{M}^2}U^5 + \dots$$

$$\frac{\Omega_\rho}{\Omega} = \left(3 - \frac{W_0\left(\beta_0 W_0 - 8\alpha_0\bar{M}\right)}{2\bar{M}^2}\right)U^2 - \frac{4J_1}{\bar{M}^2}U^3 + \dots$$

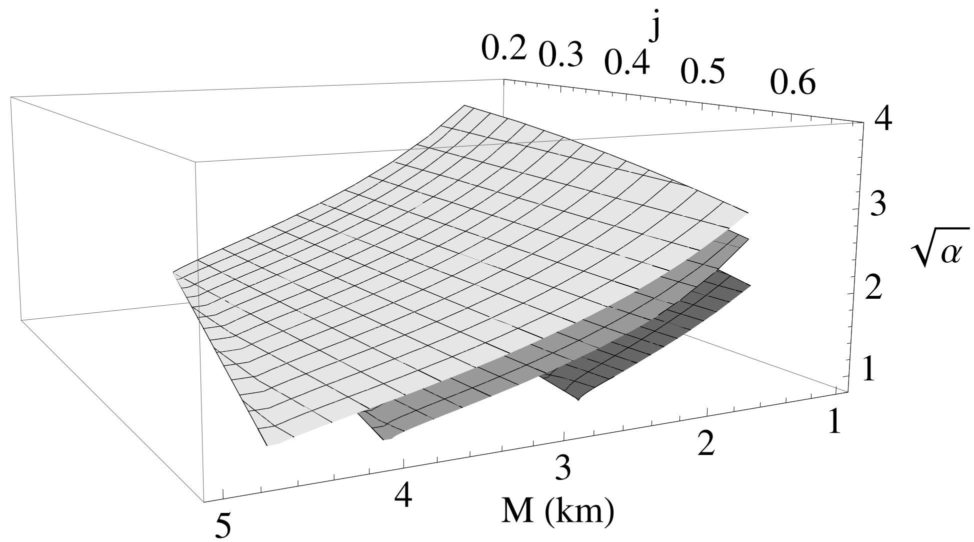
$$\frac{\Omega_z}{\Omega} = \frac{2J_1}{\bar{M}^2}U^3 + \frac{3(M_2 - \alpha_0 W_2)}{2\bar{M}^3}U^4 - \frac{2J_1W_0\left(\beta_0W_0 - \alpha_0\bar{M}\right)}{\bar{M}^4}U^5 + \dots$$

where $U = (\bar{M}\Omega)^{1/3}$. The calculations are done in the Jordan frame. Again the orbital frequency gives the Keplerian mass: $\Omega = (\bar{M}/r^3)^{1/2}(1 + O(r^{-1/2}))$, but this time the Keplerian mass is $\bar{M} = M - W_0\alpha_0$.
$W_0 \equiv P_0^\phi$ is the scalar charge, $W_2 \equiv P_2^\phi$ is the scalar quadrupole and $\alpha \equiv (d\ln A)/d\phi, \beta \equiv d\alpha/d\phi$.

These observables can distinguish between GR and Scalar-Tensor theory.

[9] G.P. and TP Sotiriou arXiv:1505.02882

Identifying the EoS: Assuming GR, a "measurement"[10] of the first 3 moments (M, J, M_2) could select an EoS[11] out of the realistic EOS candidates.



Testing GR and scalarization: If we were to "measure"[10] the coefficients of the expansions and have an independent measurement of the Keplerian mass, then we could test if the compact object is scalarized since the coefficients would be different than in GR.

[10] G.P., 2012 MNRAS, 422, 2581-2589.

[11] G.P. and T. A. Apostolatos, Phys.Rev.Lett. **112** 121101 (2014)

- Investigate the behaviour of multipole moments for scalarized neutron stars.

- Are there 3-hair multipole relations like those found in GR and is there a degeneracy between theories as the one found for I-Love-Q relations?

- Extend the definition of multipole moments to a wider class of theories, if possible...

- ... or identify other quantities that can play a similar part as the moments have.

- A more thorough astrophysical modeling of possible sources where these expressions can be applied.

Thank You.