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A Functorial Approach to Group C^*-Algebras

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Abstract

Let $\mathcal{A}$ be the category of unital C^*-algebras. A counterexample is given to show that the unitary group functor $\mathcal{U} : \mathcal{A} \longrightarrow \mathcal{G}rp$ (and hence the composite of the forgetful functor $\mathcal{G} : \mathcal{G}rp \longrightarrow \mathcal{S}et$ with $\mathcal{U}$) although being a faithful right adjoint, fails to be algebraic even in the commutative case. In so doing, a canonical presentation of any unital C^*-algebra as a quotient of a free product of copies (i.e., a copower in $\mathcal{A}$) of $\mathcal{C}(S^1, \mathbb{C})$ is obtained, leading to an alternative characterization of compact Hausdorff spaces, as well. Isomorphism questions are also considered.

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1 Introduction

The study of convolution or group algebras has been, and still is, an area of active research, as they are the bridge between the structure and the representation theory of groups and algebras. They are naturally linked by the various adjunctions involved- and therefore by the corresponding universal properties- in accordance with the structure considered on the objects. First of all, the full subcategory $\mathcal{G}rp$ of groups is monocoreflective in $\mathcal{M}on$, the category of monoids, the coreflector assigning to each monoid M the group $\mathcal{I}nv(M)$ of the invertible elements of it. The construction of the algebra $\mathbb{C}^{(M)}$ for a given monoid M produces a functor left adjoint to the linear structure forgetful functor from the category of unital associative algebras to that of monoids. Now the composite of the previous adjunction with this one partly utilizes the group structure.

In fact, we can lift our discussion to a more meaningful level by considering an involution. Recall that an involutive monoid M is one possessing an anti-automorphism of period two. We denote by $\mathcal{M}on^*$ the category of involutive monoids and their involution- and unit-preserving homomorphisms. Of course, every group G may be viewed as such a monoid with inherent involution its inversion. Besides, unitary elements are well-defined in any involutive monoid M and, in general, they form a proper subgroup, say $\mathcal{U}_0(M)$, of $\mathcal{I}nv(M)$. Further observe that $\mathcal{G}rp$ is a full monocoreflective subcategory of $\mathcal{M}on^*$, too, the coreflector being now the functor $\mathcal{U}_0$. On the other hand, the involution of M gives rise to an algebra involution on $\mathbb{C}^{(M)}$ (by a conjugate linearization of it) and the so resulting functor again provides a left adjoint to the forgetful functor from the category of involutive and unital associative algebras. Finally, it is well-known that there exists a bijective correspondence between the unitary representations of a given group G and the nondegenerate $*$-representations of the group C^*-algebra $\mathcal{C}^*(G)$ for it, acting on the same Hilbert space H (cf. [3] as a general reference). This situation suggests another refined adjunction, in a sense an improvement to the last composite adjunction.

Let $\mathcal{A}$ stand for the category of unital C^*-algebras and their morphisms (being automatically non-expansive). In this concern, it is known that the *closed-unit-ball functor* $O : \mathcal{A} \longrightarrow \mathcal{M}on^* \longrightarrow \mathcal{S}et$ is quasi monadic (weakly algebraic) (cf [7], [9], [10]). In this paper, we turn our attention to the study of the *unitary group functor* $\mathcal{U} : \mathcal{A} \longrightarrow \mathcal{M}on^* \longrightarrow \mathcal{G}rp$ (the composite of $\mathcal{U}_0$ with the corresponding forgetful functor) instead. It is shown that $\mathcal{U}$ is a faithful right adjoint, but fails to be algebraic even in the commutative case, as attested by a counterexample (misbehaviour of $\mathcal{U}$ on regular epimorphisms). Nevertheless, this approach leads to certain structural properties for unital C^*-algebras, by simple categorical arguments (compare with [2], Vol. 2, p. 214, Theorem 4.4.5). Specifically, the function algebra $\mathcal{C}(S^1, \mathbb{C})$ on the circle group S^1 is a regular generator (alias, separator) in $\mathcal{A}$ and (relative) $\mathcal{E}$-projective, so that any unital C^*-algebra has a canonical representation as a quotient of a free product of a certain family of copies (viz., a suitable copower) of $\mathcal{C}(S^1, \mathbb{C})$; in other words of a free group C^*-algebra. In particular, it gives evidence of an alternative characterization of compact Hausdorff spaces by utilizing the Gel'fand duality in the commutative case. Besides, aspects of the isomorphism problem are considered.

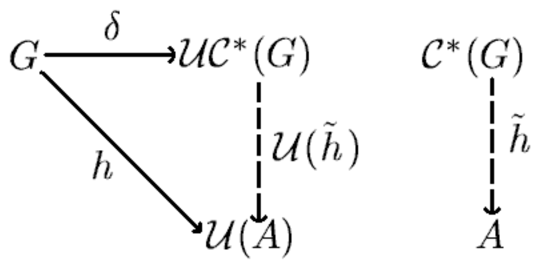
2 The adjunction refinement

Universal constructions have been playing an increasingly important role in the theory of C^*-algebras for a long time, realizing many of them in a natural and simple way. In principle, they are carried out extrinsically in terms of suitable classes of $*$-representations of the involutive algebras under consideration.

Explicitly, let I be a set of generators and R a set of "admissible" relations, in the sense that they must be realizable among bounded operators on a Hilbert space. One considers the free involutive algebra $\mathbb{C}^{(m^*(I))}$ on the set I by first forming the free involutive monoid $m^*(I)$ on I and the $*$-representations of it on Hilbert spaces corresponding to the given relations. If the supremum C^*-seminorm on $\mathbb{C}^{(m^*(I))}$ so induced exists as a finite number, then a completion of the quotient algebra of $\mathbb{C}^{(m^*(I))}$ by the nullideal N of this seminorm provides the universal C^*-algebra $\mathcal{C}^*(I, R)$ in question. Sometimes, the correct universal property is described informally and, of course, this technique is not applicable beyond C^*-algebras. Regarding the group C^*-algebra, it is constructed with the relations that each generator (i.e., group element) is a unitary of norm one (see [1] for instance).

It is more or less evident that the unitary group functor $\mathcal{U}$ is well-defined on any unital associative algebra with an involution (with or without a normed structure), whenever the morphisms are chosen to be the involution-and-unit-preserving (resp. continuous) algebras homomorphisms. Among the algebras of this type, the category $\mathcal{A}$ of unital C^*-algebras provides the most convenient setting.

Definition 2.1 *Let G be a (discrete) group. By a group C^*-algebra on G with respect to $\mathcal{U} : \mathcal{A} \longrightarrow \mathcal{G}rp$ we mean a unital C^*-algebra $\mathcal{C}^*(G)$ together with a group homomorphism $\delta : G \longrightarrow \mathcal{U}\mathcal{C}^*(G)$ such that, for every group homomorphism $h : G \longrightarrow \mathcal{U}(A)$ from G into the unitary group of any unital C^*-algebra A, there exists a unique morphism $\tilde{h} : \mathcal{C}^*(G) \longrightarrow A$ of unital C^*-algebras with $h = \mathcal{U}(\tilde{h}) \circ \delta$, as illustrated in the commutative diagram*



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By a standard argument on universal constructions, the algebras in question, if they exist, are uniquely determined within an (isometric $*-$) isomorphism. In this case, it is also true that the uniqueness requirement in the universal property of the group C^*-algebra $\mathcal{C}^*(G)$ is equivalent to the fact that the linear span of the image $\delta(G)$ must be dense in $\mathcal{C}^*(G)$. For the sake of comprehension, the rudiments of an elementary construction of group C^*-algebras are comprised in

Theorem 2.2 *For any given group G, there always exists an essentially unique group C^*-algebra $\mathcal{C}^*(G)$ on G with respect to $\mathcal{U}$.*

Proof. The complex-vector space $\mathbb{C}^{(G)}$ with the usual basis $\delta_s, s \in G$ (Kroneker delta) is converted into a unital associative algebra by means of the unique bilinear extension of the multiplication of G to all of $\mathbb{C}^{(G)}$, the convolution. Besides, the group inversion causes a natural algebra involution on $\mathbb{C}^{(G)}$ by putting $\delta_s^* := \delta_{s^{-1}}$, so that $f^*(s) = \overline{f(s^{-1})}$ for all $s \in G$ and $f \in \mathbb{C}^{(G)}$. Thus the canonical injection δ does indeed become a homomorphism from G into the unitary group of $\mathbb{C}^{(G)}$. But a completion of $\mathbb{C}^{(G)}$ with respect to the $*$-algebra norm $||f||_1 := \sum_{s\in G} |f(s)|$ is the unital and involutive Banach algebra $l^1(G)$. It consists of all complex-valued functions f on G which have a countable support and can be uniquely expressed by absolutely convergent series as $f = \sum_{s\in G} f(s)\delta_s$. Finally, the group C^*-algebra $\mathcal{C}^*(G)$ is obtained as the enveloping C^*-algebra of $l^1(G)$ ([3], p.40, section 2.7).

To verify that δ is universal from G to $\mathcal{U}$, let (A, h) be any pair with A a unital C^*-algebra and $h : G \longrightarrow \mathcal{U}(A)$ a group homomorphism. As $l^1(G)$ realizes the free Banach space on (the underlying set to) G with respect to the closed-unit-ball functor ([9], p. 30, section 3.3) and, obviously, $\mathcal{U}(A)$ is a subset of this ball of A, there is a unique linear contraction $h' : l^1(G) \longrightarrow A$ such that $h = h' \circ \delta$. Since h is a group homomorphism, h' is still multiplicative and unit-preserving. The crucial point is now that the extension h' is involution-preserving as well, if and only if, h is unitary group-valued (i.e., a unitary group morphism). Hence h' uniquely factors through the enveloping C^*-algebra $\mathcal{C}^*(G)$ of $l^1(G)$ to a unital C^*-algebras morphism $\tilde{h} : \mathcal{C}^*(G) \longrightarrow A$ with $h = \mathcal{U}(\tilde{h}) \circ \delta$. The uniqueness of such an $\tilde{h}$ is immediate from the fact that the linear span of $\delta(G)$ is dense in $\mathcal{C}^*(G)$, and this completes the proof.

Scholium. Concerning the range of the validity, it is evident that the proof is true for the category of involutive and unital associative algebras, as well. Unfortunately, this is not the case within the (super)category of involutive and unital Banach algebras and their continuous, unital and $*$-homomorphisms, because there the above extension h' need not in general be continuous.

Formally, and especially when the group G is to be stressed, we rather write δ_G for the universal morphism δ, and by abuse of notation, the same symbol is used for the composite homomorphism $G \longrightarrow l^1(G) \longrightarrow \mathcal{C}^*(G)$, the latter being also an injection. Actually, by $\mathcal{C}^* : \mathcal{G}rp \longrightarrow \mathcal{A}$ denoting the group C^*-algebra functor so resulting, for the unit δ and the counit ϵ of this adjunction, one gets

Proposition 2.3 *For every group G the component $\delta_G : G \longrightarrow \mathcal{U}\mathcal{C}^*(G)$ of the unit δ is injective; and for every unital C^*-algebra A the component $\epsilon_A : \mathcal{C}^*\mathcal{U}(A) \longrightarrow A$ of the counit ϵ is surjective.*

Proof. It is well-known that the algebra $l^1(G)$ is faithfully represented on the Hilbert space $l^2(G)$ via the left regular representation λ of G on it, induced by G acting on itself by left translation. Therefore, the respective universal morphism $l^1(G) \longrightarrow \mathcal{C}^*(G)$ is an injection, but not an isometry ([3], p.255; 13.3.6). On the other hand, the component ϵ_A of the counit is the extension of the identity automorphism of $\mathcal{U}(A)$ to the group C^*-algebra $\mathcal{C}^*\mathcal{U}(A)$, so that $\mathcal{U}(A) \subseteq Im\epsilon_A$. Since the unitary group $\mathcal{U}(A)$ spans the whole algebra A (e.g. see [4], p. 42, Proposition 15.1), ϵ_A is surjective, in fact a regular epimorphism.

Corollary 2.4 *The group C^*-algebra functor $\mathcal{C}^* : \mathcal{G}rp \longrightarrow \mathcal{A}$ is faithful left adjoint and the unitary group functor $\mathcal{U} : \mathcal{A} \longrightarrow \mathcal{G}rp$ is faithful right adjoint. Moreover, both functors (preserve and) reflect isomorphisms.*

Proof. Since epimorphisms in $\mathcal{A}$ are surjective [6], both categories $\mathcal{G}rp$ and $\mathcal{A}$ are balanced (that is, each morphism which is simultaneously a monomorphism and an epimorphism is an isomorphism). Besides, faithful functors reflect monomorphisms and epimorphisms, as well.

Remark 2.5 *(a) The adjunction isomorphism*

$$\omega_{GA} : hom_{\mathcal{A}}(\mathcal{C}^*(G), A) \longrightarrow hom_{\mathcal{G}rp}(G, \mathcal{U}(A))$$

in particular establishes a bijective correspondence between the unitary representations of the group G and the (nondegenerate) $*$-representations of its group C^*-algebra $\mathcal{C}^*(G)$ acting on the same Hilbert space H. By ([4], p. 108, Theorem 28.2 and p. 238, Theorem 56.1), this is also true of its restriction to the cyclic and irreducible representations of them, respectively.

(b) The injectivity of the universal morphism δ suggests that from the abstract point of view the theory of groups is coextensive with the theory of unitary transformation groups, rather than general transformation groups.

(c)Concerning the isomorphism problem, the meaning in non-categorical terms of (the preservation and) the reflection property is the following: A group homomorphism $h : G \longrightarrow G'$ is an isomorphism if and only if its extension $\mathcal{C}^*(h) : \mathcal{C}^*(G) \longrightarrow \mathcal{C}^*(G')$ is a (surjective isometric $*$-) isomorphism of unital C^*-algebras; dually, a unital C^*-algebra morphism $g : A \longrightarrow B$ is an (isometric $*$-) isomorphism of A onto B if and only if its restriction to the unitary groups is a bijection.

By the very definition of the character group, the adjunction isomorphism $\omega_{G\mathbb{C}} : hom_{\mathcal{A}}(\mathcal{C}^*(G), \mathbb{C}) \longrightarrow hom_{\mathcal{G}rp}(G, \mathcal{U}(\mathbb{C}))$ for an abelian group G becomes a homeomorphism with respect to the natural topologies ([3], p. 314; 18.1), so that one still recaptures the basic functional representation of the group C^*-algebra on it, as an immediate consequence of the Gel'fand duality.

Corollary 2.6 *The group C^*-algebra $\mathcal{C}^*(G)$ on an abelian group G is (isometrically $*$-) isomorphic to the function algebra $\mathcal{C}(\hat{G},\mathbb{C})$ on the character group $\hat{G}$ of G. In particular, if G is finite of order n, then $\mathcal{C}^*(G)$ is $\mathbb{C}^n$ up to an isomorphism.*

To complete the picture, further consider the forgetful functor $\mathcal{G} : \mathcal{G}rp \longrightarrow \mathcal{S}et$, right adjoint to the free group functor $\mathcal{F} : \mathcal{S}et \longrightarrow \mathcal{G}rp$. The composite $\mathcal{GU} : \mathcal{A} \longrightarrow \mathcal{S}et$ is then right adjoint to the "*free group C^*-algebra*" functor $\mathcal{C}^*\mathcal{F} : \mathcal{S}et \longrightarrow \mathcal{A}$. For an illustrative example, the free group C^*-algebra $\mathcal{C}^*(\mathbb{Z})$ on the infinite cyclic group $\mathbb{Z}$ of integers, the free (abelian) group on the one point set is, by Corollary 2.6, isomorphic to the function algebra $\mathcal{C}(S^1,\mathbb{C})$ of the circle group S^1, the unitary group of $\mathbb{C}$. It follows that $\mathcal{GU}$ is representable by $\mathcal{C}(S^1,\mathbb{C})$ and, being faithful (Corollary 2.4), $\mathcal{C}(S^1,\mathbb{C})$ is a generator in $\mathcal{A}$, in fact a regular one with the following structural property.

Theorem 2.7 *Let S^1 be the circle group, A a unital C^*-algebra and let I be a generating subset of the unitary group $\mathcal{U}(A)$ of A. Then A is (isometrically $*$-) isomorphic to a quotient algebra of the free group C^*-algebra on I, the I^{th} copower of the function algebra $\mathcal{C}(S^1,\mathbb{C})$ in $\mathcal{A}$. Furthermore, if A is commutative, then it is isomorphic to a quotient of the function algebra $\mathcal{C}((S^1)^I,\mathbb{C})$.*

Proof. A left adjoint functor preserves the existing colimits and any set I is the disjoint union of its elements. Accordingly, the I^{th} copowers ${}^I\mathcal{C}(S^1,\mathbb{C})$ of the unital C^*-algebra $\mathcal{C}(S^1,\mathbb{C})$ exist in the category $\mathcal{A}$ and provide the free group C^*-algebras on the corresponding sets I. But every group G is isomorphic to a quotient group of the free group on a generating subset I of G. Applying the left adjoint functor C^*, a canonical presentation of the group C^*-algebra $\mathcal{C}^*(G)$ as a quotient of the free group C^*-algebra ${}^I\mathcal{C}(S^1,\mathbb{C})$ on I is obtained. Now taking $G = \mathcal{U}(A)$, a composition with the presentation in Proposition 2.3 gives the desired isomorphism. In the full reflective subcategory $\mathcal{A}'$ of the commutative algebras in $\mathcal{A}$, the presentation is still valid, but now the respective copower is the I^{th} tensor power of the function algebra $\mathcal{C}(S^1,\mathbb{C})$ which, in its turn, is isomorphic to the function algebra $\mathcal{C}((S^1)^I,\mathbb{C})$ on the I^{th}-torus (also cf. [5]).

In addition to the classical embeddings, one concludes an alternative characterization of compact Hausdorff spaces (compare with [2], Vol. 1, p. 167, section 4.7; [8], p. 104; 1.61).

Corollary 2.8 *A Hausdorff space is compact if and only if it is homeomorphic to a closed subspace of a product of copies of the circle group S^1.*

Proof. That such a subspace is compact is immediate from the Tychonoff Theorem. Conversely, let $\mathcal{X}$ be the category of Compact Hausdorff spaces and X in $\mathcal{X}$. An application of the spectral functor $\Omega : \mathcal{A}' \longrightarrow \mathcal{X}^{op}$ on the quotient epimorphism $p : \mathcal{C}((S')^I, \mathbb{C}) \longrightarrow \mathcal{C}(X, \mathbb{C})$, as determined in the above Theorem, implies that the map $\Omega(p) : X \longrightarrow (S')^I$ is a continuous injection. That is, X is embedded as a closed subspace of $(S^1)^I$.

3 A counterexample

In studying faithful right adjoint functors a pertinent question is whether these are algebraic (regularly monadic). As it is already mentioned, this is for instance the case for the closed-unit-ball functor from the category of unital C^*-algebras to sets ([7], [9], [10]). The following counterexample answers this question in the negative regarding the unitary group functor $\mathcal{U}$ on $\mathcal{A}'$, and hence the composite of the forgetful functor $\mathcal{G} : \mathcal{G}rp \longrightarrow \mathcal{S}et$ with $\mathcal{U}$ ([8], p.182, Theorem 1.29), because of the misbehaviour of $\mathcal{U}$ on regular epimorphisms (being precisely the surjective morphisms in both categories). In fact, consider the inclusion $i : S^1 \longrightarrow D$ of the circle group S^1 into the closed unit disc D in the complex numbers. The C^*-algebra morphism $\mathcal{C}(i) : \mathcal{C}(D, \mathbb{C}) \longrightarrow \mathcal{C}(S^1, \mathbb{C})$ induced by functional composition is onto, by Tietze's Extension Theorem, and so a regular epimorphism (as a quotient morphism). But clearly, the unitary group of a function algebra consists of all S^1- valued functions of it. Hence, the respective restriction $\mathcal{UC}(i) : \mathcal{C}(D, S^1) \longrightarrow \mathcal{C}(S^1, S^1)$ to the unitary groups of them fails to be a surjective group morphism, as S^1 is not a retract of D (according to Brouwer's Fixed-point Theorem).

On the other hand, the free group C^*-algebras, in particular $\mathcal{C}(S^1, \mathbb{C})$, are $\mathcal{E}$-projective relative to the class $\mathcal{E}$ of all C^*-algebra (epi)morphisms q for which $\mathcal{GU}(q)$ is a surjection. As $\mathcal{GU}$ reflects (regular) epimorphisms, the counterexample also shows that $\mathcal{E}$ is a genuine subclass of the class of regular epimorphisms in $\mathcal{A}$.

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