Classical Strings and Membranes in the AdS/CFT Correspondence

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This thesis is

dedicated

to my parents

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Prolegomena

This doctoral dissertation is based on the publications [1, 2, 3, 4]:

- G. Linardopoulos, *Large-Spin Expansions of Giant Magnons*, [arXiv:1502.01630]. To appear in Proceedings of Science, PoS (CORFU2014) 154

- E. Floratos, G. Linardopoulos, *Large-Spin and Large-Winding Expansions of Giant Magnons and Single Spikes*, Nucl.Phys. **B897** (2015) 229, [arXiv:1406.0796]

- E. Floratos, G. Georgiou, G. Linardopoulos, *Large-Spin Expansions of GKP Strings*, JHEP **03** (2014) 018, [arXiv:1311.5800]

- M. Axenides, E. Floratos, G. Linardopoulos, *Stringy Membranes in AdS/CFT*, JHEP **08** (2013) 089, [arXiv:1306.0220],

as well as the following talks:

- *Large-Spin Expansions of Giant Magnons.* Workshop on Quantum Fields and Strings. 14th Hellenic School and Workshops on Elementary Particle Physics & Gravity. Corfu Summer Institute, 17/09/2014, [`http://www.physics.ntua.gr/corfu2014/lectures.html`]

- *Dispersion Relation of GKP Strings.* Summer School on String Theory and Holography & Mathematica Summer School on Theoretical Physics, 6th Edition. Departamento de Física e Astronomia of Faculdade de Ciências da Universidade do Porto (FCUP), 24/07/2014, [`http://msstp.org/?q=node/294`]

- *Stringy Membranes in AdS/CFT.* "Holography 2013: Gauge/Gravity Duality and Strongly Correlated Systems". Pohang, S. Korea, June 13 – 22. APCTP, POSTECH, 15/06/2013, [`https://www.apctp.org/plan.php/holography2013/660`]

- *Rotating Strings and Membranes in AdS/CFT.* "Foundations of Quantum Mechanics and Relativistic Spacetime". Annual workshop of Working Group 3, "Quantum Theory Meets Relativity", of the COST Action MP1006, "Fundamental Problems in Quantum Physics". National & Kapodistrian University of Athens, 26/09/2012, [`http://www.pwallden.gr/foundations.asp`].

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Abstract

In its strongest version, the AdS/CFT conjecture states that $\mathcal{N}=4$, $\mathfrak{su}(N_c)$ super Yang-Mills (SYM) theory is equal to type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. It is by far the most important equation of contemporary theoretical physics, a sort of a "harmonic oscillator" for both the quantum theory of gravity and gauge theories. It goes without saying that it is imperative to fully understand its limits of validity and thoroughly investigate its implications. In particular, it would be desirable to solve the theory, i.e. to be able to compute all of its observables.

One of the most important observables of AdS/CFT is its spectrum. According to the AdS/CFT "dictionary", the spectrum of the theory comprises the energies of its string states, each of which must be equal to the scaling dimensions of its dual gauge theory operator. The full spectral problem of AdS/CFT is solved by integrability, in the sense that integrability provides the full set of algebraic equations that determine it. Integrability methods are however severely limited in the regime of long, strongly coupled operators, such as those that are dual to the Gubser-Klebanov-Polyakov (GKP) strings, giant magnons and single spike strings.

In this thesis we study classical strings and branes in the context of the AdS/CFT correspondence. Our goal is twofold: (1) develop methods for computing the $\mathrm{AdS}_5/\mathrm{CFT}_4$ spectrum in the case of long, strongly coupled operators, by using classical strings and (2) understand the role of classical membranes in AdS/CFT by investigating their stringy limits.

With regard to the first objective, we compute the classical spectra of long rotating GKP strings, giant magnons and single spikes. The conserved linear and angular momenta of these string configurations, that live either in AdS_3 or $\mathbb{R}\times\mathrm{S}^2$, are known in parametric form in terms of the strings' linear and angular velocities. We eliminate the linear and angular velocities from the expressions that give the energy of the strings, in favor of the strings' conserved charges of linear and angular momenta. This way, we find all the leading, subleading and next-to-next-to-leading terms in the dispersion relations of the aforementioned string configurations. Our results are expressed in closed forms with Lambert's W-function.

For the second objective we introduce and study "stringy membranes", a new class of membranes that live in $\mathrm{AdS}_{4/7} \times \mathrm{S}^{7/4}$ or $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ and have the same equations of motion, constraints and conserved charges with strings that live in an appropriate subset of AdS_5. Stringy membranes can be constructed whenever the target spacetime contains a compact submanifold, by identifying one of the submanifold's compact coordinates with one of the membrane worldvolume coordinates. For the stringy membranes that reproduce the pulsating and rotating GKP strings in AdS, we find that the spectrum of their transverse quadratic fluctuations displays a multiple band/gap structure governed by the Lamé equation. Conversely, string excitations are represented by a single-band/single-gap Lamé pattern. These findings confirm the picture that we have of membranes as collective excitations of some stringy counterparts.

Contents

1 Introduction

What is the greatest equation ever written? Clearly there are many choices, as a simple google search reveals. The Pythagorean theorem, Euler's equation, Maxwell's equations, Einstein's equations, Schrödinger's equation, Noether's theorem, the Callan-Symanzik equation and many others. In his 2010 TASI lectures on the gauge/gravity duality, Joseph Polchinski [5] chose the Maldacena equation

$$\mathrm{AdS} = \mathrm{CFT} \tag{1.1}$$

as his favorite equation of all time. Even if this choice seems overly enthusiastic, one thing is certain: Maldacena's original paper [6] has more than 13.000 citations,[1] while the co-founding articles of Gubser, Klebanov, Polyakov [7], Witten [8] and the early review [9], have a total of almost 20.000 citations. The Maldacena duality (1.1) certainly deserves a place in the Pantheon of the greatest equations of theoretical physics.

Equally spectacular are the implications of (1.1). One of the greatest open problems of modern theoretical physics is the unification of quantum mechanics with gravity. Formally, there's no a priori reason to expect that a theory of quantum gravity (formulated on a negatively curved spacetime such as anti-de Sitter space—AdS for short) may reduce to a very special type of a gauge theory with scale invariance (formulated on flat space of one dimension less), aka a conformal field theory (CFT). Part of the unparalleled success of AdS/CFT is owed to the fact that it became the first concrete example of the gauge/gravity duality and the holographic principle, but also a prime instance of the weak/strong coupling duality in four spacetime dimensions.

Gauge/gravity dualities provide a very special unifying framework for all the fundamental forces of nature, by regarding the gauge interactions (such as the electroweak or the strong interaction) as the alter ego of the gravitational force. With the holographic principle, our world is viewed as a hologram that encodes all the information of the higher-dimensional bulk. Weak/strong coupling dualities identify the weak-coupling regime of a theory (where perturbation theory is valid) with the strong-coupling (or non-perturbative) regime of another theory and allow us to perform calculations in a region that was inaccessible with traditional methods. The most popular form of the AdS/CFT correspondence,

$$\mathcal{N} = 4,\ \mathfrak{su}\left(N_c\right)\ \text{super Yang-Mills theory} = \text{IIB superstring theory on } \mathrm{AdS}_5 \times \mathrm{S}^5 \tag{1.2}$$

posits the equivalence of two radically different physical theories. $\mathcal{N} = 4$ super Yang-Mills theory is the most perfect of all possible gauge theories in four spacetime dimensions with the maximal allowed number of supersymmetries and conformal symmetry, which roughly means that the theory is invariant under scale transformations and finite. Type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is on the other hand a gravitational theory that is formulated on a total of ten spacetime dimensions, five of which are compactified on a 5-sphere and the rest live on 5-dimensional anti-de Sitter space.

The Maldacena conjecture seems to suggest the study of an ideal world ($\mathcal{N} = 4$ SYM) as a means to extract the properties of the real one (QCD, the theory of strong interactions). However it turns out that the AdS/CFT duality is more than just a naive toy model. In the high-temperature regime where the supersymmetry of $\mathcal{N} = 4$ SYM is explicitly broken, the theory starts resembling more and more the deconfined QCD plasma which has no chiral condensate and it is scale invariant. This form of universality lies at the heart of gauge/gravity dualities which assert that the holographic dual of QCD must be described by a modified version of the Maldacena conjecture. In this sense, the AdS/CFT duality (1.2) can be considered as the "harmonic oscillator" of modern theoretical physics.

Like the quantum harmonic oscillator, it is imperative that the AdS/CFT correspondence be studied inside out. Firstly, since AdS/CFT is still at a conjectural level that defies any reasonable attempt

[1] As of 2015.

for a proof, we have to know the limits of its validity. Many tests have been developed over the years that check the matching between the symmetries, spectra, correlators, anomalies, moduli spaces, etc. of the two implicated theories. All seem to confirm the validity of the conjecture, at least at the planar level where the rank of the gauge group becomes very large ($N_c \to \infty$).

Secondly, if we assume its validity, we have to get a full grasp of the implications of AdS/CFT. In other words we must solve the theory. Solving a theory means that we are able to compute all of its observables, e.g. spectrum, correlation functions, scattering amplitudes, expectation values of Wilson loops. A very powerful tool that has been developed in the context of AdS/CFT solvability is that of integrability. Generally speaking, a theory is integrable whenever it possesses the maximum allowed number of conservation laws that may be integrated and the theory be solved. In the case of the AdS_5/CFT_4 duality, integrability has been proven at the classical level by Bena, Polchinski and Roiban [10]. Although no formal proof of its quantum integrability currently exists, the AdS_5/CFT_4 correspondence (1.2) is thought to be quantum integrable at the planar limit ($N_c \to \infty$), where the dual string theory becomes free ($g_s \to 0$).

Integrability completely solves the spectral problem of planar AdS_5/CFT_4, in the sense that it provides the full set of algebraic equations that determines the spectrum. Integrability also provides the set of tools that can be used to solve the planar limit of AdS_5/CFT_4 theory (in the above sense, i.e. computing all of its observables). However, the integrability approach does have a number of limitations. In particular, there exist some regimes of the AdS/CFT correspondence where the solution of the above set of algebraic equations becomes impossible to obtain, either with analytic or computational means. In such cases, we have to rely on more traditional methods in order to obtain the wanted spectra. These methods involve classical strings and, in some cases, branes.

Before proceeding to the discussion of the role of classical strings and branes in the AdS/CFT correspondence, let us briefly explain why we think that the explicit computation of the planar AdS/CFT spectrum is interesting. Firstly, it seems to us that the scope of the AdS/CFT correspondence becomes somewhat limited if we do not know the exact analytic form of its spectrum. Secondly, we would like to have at our disposal tools that allow us to test the matching of the AdS/CFT spectra explicitly. Thirdly, we would like to explore the possibility of finding closed-form expressions in the AdS/CFT spectrum.

In 2002, Gubser, Klebanov and Polyakov (GKP) [11] proposed to study classical strings that rotate, spin or pulsate inside $AdS_5 \times S^5$ in order to obtain the strong coupling values of the (anomalous) scaling dimensions of certain gauge-invariant, single-trace operators that were formed by the fields of $\mathcal{N} = 4$ SYM. GKP noticed that the energy of a specific closed folded string configuration that rotates rigidly inside AdS_3, scales as the logarithm of its (large) spin, a behavior that was very reminiscent of the logarithmic scaling violations of twist QCD operators. Being able to reproduce this behavior for the anomalous scaling dimensions of scalar single-trace (twist-2) operators of $\mathcal{N} = 4$ SYM, GKP conjectured that the closed folded string that rotates rigidly in AdS_3, is the AdS/CFT dual of twist-2 operators of $\mathcal{N} = 4$ SYM and provides their anomalous scaling dimensions at strong coupling.

The GKP paradigm emphasized the benefits that accompany the study of classical strings in the context of the AdS/CFT correspondence, such as the fact that it allows the computation of the spectrum of the dual CFT at strong coupling (a regime where perturbation theory typically breaks down). Classical strings are also extensively used in the calculation of AdS/CFT correlation functions, Wilson loops and gluon scattering amplitudes. Also, the integrability properties of classical and quantum strings in planar AdS/CFT, anticipate in many respects the integrability of the whole theory. An interesting relevant question is whether the study of classical membranes within the AdS/CFT correspondence can be as beneficial as the study of classical strings.

The study of the classical dispersion relations of GKP strings was taken up seriously in the papers

[12, 3]. As we have already mentioned, GKP strings are closed strings that spin, rotate or pulsate inside the AdS_3 or $\mathbb{R} \times S^2$ submanifolds of $AdS_5 \times S^5$ and are dual to certain composite operators of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. The dispersion relations of GKP strings give the anomalous dimensions of their dual CFT operators at strong coupling. GKP strings belong to the category of "long" strings that "see" the curvature of the spacetime they live in, as opposed to "short" strings which live in an approximately flat space and the spacetime curvature only affects the subleading terms of their dispersion relations. What is more, very few results for the spectra of long strings have been obtained by using integrability methods.

Therefore one must rely on more direct methods in order to obtain the wanted spectra. Generally, the expressions for the classical conserved charges of the string energy E and the spin S/angular momentum J are known in parametric form, in terms of the angular velocity ω. The authors of the papers [12, 3] managed to invert the series that gives the conserved angular momentum of the strings in terms of their angular velocity and express the classical string energy as a function $E = E(S, J)$, using only their conserved spin/angular momentum. Only in this way can the resulting dispersion relations accommodate quantum corrections or be compared to the corresponding weak-coupling formulas, none of which is known in parametric form. It was found that the finite-size corrections to the spectra of the dual CFT operators at strong coupling can be expressed in terms of the so-called Lambert's W-function, that is defined as follows:

$$W(z)\, e^{W(z)} = z \tag{1.3}$$

and it constitutes a generalization of the logarithmic function. With Lambert's W-function, all the leading, subleading and next-to-subleading terms in the classical dispersion relations of long rotating GKP strings in AdS_3 and $\mathbb{R} \times S^2$ were computed. In [13], the W-function method was applied to the case of AdS_4/CFT_3. Moreover, the conserved energies and spins/angular momenta of long GKP strings were found to obey a number of short-long string dualities that link their conserved charges in the "short" and the "long" regime. These relations are very interesting because their quantum generalizations may allow to import the integrability results that are so rich in the regime of short strings, to the regime of long strings.

In [2], the W-function approach was upgraded to the case of giant magnons and single spikes. Giant magnons and single spikes are open single-spin strings that rotate in $\mathbb{R} \times S^2 \subset AdS_5 \times S^5$ and are dual to single-magnon and single-spinon operators of the centrally extended $\mathcal{N} = 4$ SYM. The role of giant magnons is pivotal in AdS/CFT, as they are the fundamental building blocks out of which all the states of the theory can be built. These are again "long" strings, for which very few results from integrability methods are known. Apart from their conserved energy E and angular momentum J, giant magnons and single spikes have a third conserved quantity, their linear momentum p. Also, there is a second parameter besides the angular velocity ω, namely their linear velocity v. The elimination of the parameters v and ω from the expression of the energy E, in favor of the conserved momenta J and p presents an outstanding technical challenge, as we now have to solve a much harder 3×3 system instead of a 2×2 one. The leading, subleading and next-to-subleading terms in the classical dispersion relations of both the giant magnons and single spikes have been computed in [2].

Besides the well-known example of AdS_5/CFT_4, where the bulk is 10-dimensional and hosts IIB string theory, there exists a number of AdS/CFT dualities that are formulated on an 11-dimensional bulk and host an M-theory. Just as $D = 10$ is the critical dimensionality of string spacetimes, for membranes the corresponding dimensionality increases to $D = 11$. This implies that we should perhaps replace strings with membranes as we move from the study of 10-dimensional string theory to that of 11-dimensional M-theory. The above prescription for the computation of the dual CFT spectrum via strings, should also be applicable to the case of membranes. Therefore we expect that the energy of a membrane that lives in the bulk of an 11-dimensional AdS/CFT spacetime, is equal to the scaling dimensions of an appropriately formed dual gauge theory operator.

As it turns out, branes are somewhat tricky objects to work with. The reason is that they are generally plagued with problems such as instabilities, anomalies, non-renormalizability, non-integrability, elusive quantization, non-interactivity and inexistent perturbation theory, which makes their study rather difficult. However, there seem to exist cases where many of these obstacles can be circumvented, such as matrix theory or brane theory in AdS spacetimes. The latter case is especially interesting from the point of view of the AdS/CFT correspondence. As we have already mentioned, an interesting open question is what is the role of classical membranes in AdS/CFT and to what extent can the technology that has been developed in the case of classical strings, be applied to the case of AdS/CFT branes.

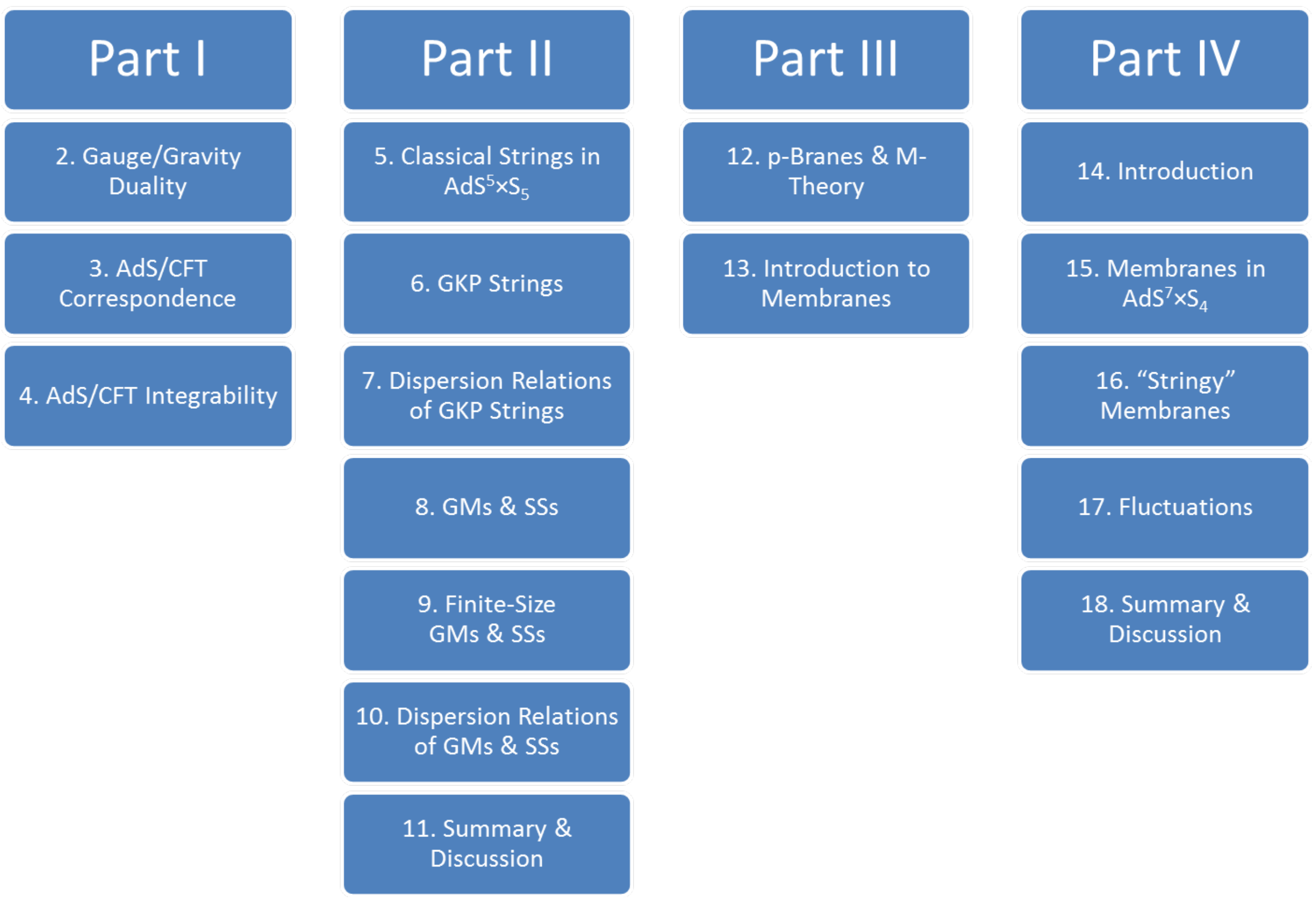
The paper [4] introduced a new class of membranes, "stringy membranes", in the context of the AdS/CFT correspondence. These are membranes that live in either $\text{AdS}_{4/7} \times \text{S}^{7/4}$ or $\text{AdS}_4 \times \text{S}^7/\mathbb{Z}_k$ and have the same equations of motion, constraints and conserved charges with strings that live in an appropriate subset of AdS_5. Stringy membranes are two-dimensional extended objects that can be constructed whenever the target spacetime contains a compact submanifold, by identifying one of the submanifold's compact coordinates with one of the membrane worldvolume coordinates. Two interesting examples of stringy membranes are the ones that fully reproduce the pulsating and rotating strings of GKP inside AdS. In the linearized approximation, the spectrum of transverse quadratic fluctuations of the stringy membranes that reproduce the rotating and pulsating strings of GKP, displays a multiple band/gap structure, governed by the Lamé equation. Conversely, string excitations are represented by a single-band/single-gap Lamé pattern. These findings confirm the picture that we have of membranes as collective excitations of some stringy counterparts.

Stringy membranes inherit all the classical characteristics of the strings that they reproduce, such as their dispersion relations and their classical integrability. Since 11-dimensional M-theory on $\text{AdS}_{4/7} \times \text{S}^{7/4}$ is dual to 3-dimensional, $\mathcal{N} = 8$ SCFT and 6-dimensional $\text{A}_{N_c-1}\,(2,0)$ SCFT respectively, both of these SCFTs are expected to contain operators that are dual to the corresponding stringy membranes. The scaling dimensions of the dual SCFT operators are expected to be equal the stringy membrane energies. This picture seems to confirm a conjecture claiming that all of the above SCFTs and $\mathcal{N} = 4$ SYM theory, possess common integrable sectors.

1.1 Overview

This doctoral dissertation is organized in four parts. Part I is a short introduction to the ideas of the gauge/gravity duality, the AdS/CFT correspondence and AdS/CFT integrability. We try to give an overview of the field and introduce all the concepts, terminology and definitions that will be used in the main body of our work.

Part II deals with classical strings that spin inside $\text{AdS}_5 \times \text{S}^5$. According to the AdS/CFT correspondence, strings in $\text{AdS}_5 \times \text{S}^5$ are dual to certain gauge-invariant operators of $\mathcal{N} = 4$ super Yang-Mills theory and provide their (anomalous) scaling dimensions at strong coupling. Our goal is to investigate the classical spectrum of these extended objects and extract closed expressions for their dispersion relations. Part II essentially follows the papers [2] and [3] and it is divided in two main sections. The first one deals with the so-called Gubser-Klebanov-Polyakov (GKP) strings, while the second one deals with giant magnons (GMs) and single spikes (SSs).



In part III we change gears and take up the study of p-branes and M-theory. We provide a brief overview of the concept of the extended object and revisit some of the most popular reasons that motivate its introduction. Before going on to discuss the action principles and the matrix models that are associated with p-branes, we discuss some of their most common problems.

In part IV we study certain classical membranes that spin inside spacetimes such as $\text{AdS}_7 \times \text{S}^4$ or $\text{AdS}_4 \times \text{S}^7/\mathbb{Z}_k$ and have a string-like behavior. Following the paper [4], these membranes are called "stringy membranes" and they can be shown to be classically equivalent to classical rotating strings. However, this equivalence disappears at the linearized level, as we also show.

Part I

Introduction to AdS/CFT

2 Gauge/Gravity Duality

The gauge/gravity duality can be formulated as follows:

$$\boxed{\Big\{\text{Quantum Gravity in } d+1 \text{ Dimensions}\Big\} = \Big\{\text{Gauge Theory in d Dimensions}\Big\}}$$

Historically, the gauge/gravity duality was put forward right after the discovery of the AdS/CFT correspondence in 1997 by Maldacena. However the basic conceptual ingredients of both had been laid down much earlier. There are 3 main theoretical indications in support of the gauge/gravity duality:

(a). The large-N_c expansion of gauge theories matches the topological expansion of string theory.

(b). The holographic principle: quantum gravity is equivalent to a QFT at the boundary of spacetime.

(c). Einstein's equations can be thought of as the RGE's of some lower dimensional QFT.

A cornucopia of gauge/gravity dualities is in existence today and the list keeps expanding. Yet, in many cases the dual theory is unknown. There's hardly any doubt that the most challenging example is quantum chromodynamics (QCD), the gravity dual of which has been called the "Holy Grail" of modern theoretical science.

Below we will try to sketch the conceptual background of gauge/gravity duality that served as a guiding principle for its first explicit realization by J. Maldacena in 1997: the AdS/CFT correspondence.

2.1 Large-N_c Expansions

In 1974 't Hooft [14] observed that the perturbative behavior of gauge theories with a large number of colors N_c is very similar to that of a string theory:

Large-N_c Expansion of Gauge Theories	$\sim$	Topological Expansion of String Theory

For a review see [15]. Here we will follow [9] for the most part. Let us begin with the Lagrangian density of a generic massive gauge (Yang-Mills) theory:

$$\mathcal{L}_{\rm YM} = {\rm Tr}\Big[d\widetilde{\Phi}_i d\widetilde{\Phi}_i + m_{ij}\widetilde{\Phi}_i\widetilde{\Phi}_j + g_{\rm YM}c_{ijk}\widetilde{\Phi}_i\widetilde{\Phi}_j\widetilde{\Phi}_k + g_{\rm YM}^2 d_{ijkl}\widetilde{\Phi}_i\widetilde{\Phi}_j\widetilde{\Phi}_k\widetilde{\Phi}_l + \dots\Big], \tag{2.1}$$

with N_c colors, assuming that the n-point vertex is proportional to the $(n-2)$th power of the coupling $g_{\rm YM}$. The gauge fields $\widetilde{\Phi}_i = \widetilde{\Phi}_i^{\mathfrak{a}} T^{\mathfrak{a}}$ are either in the adjoint or the fundamental representation of the gauge group. It is very common to rescale the gauge fields as $\widetilde{\Phi}_i \to \Phi_i/g_{\rm YM}$, so that the dependence of (2.1) on $g_{\rm YM}$ is factored out:

$$\mathcal{L}_{\rm YM} = \frac{1}{g_{\rm YM}^2}{\rm Tr}\Big[d\Phi_i d\Phi_i + m_{ij}\Phi_i\Phi_j + c_{ijk}\Phi_i\Phi_j\Phi_k + d_{ijkl}\Phi_i\Phi_j\Phi_k\Phi_l + \dots\Big]. \tag{2.2}$$

Following 't Hooft, we may also replace the coupling constant by a more convenient one for a large number of colors $N_c \to \infty$.[2] Suppose the theory (2.1) obeys the following RGE:

[2] Sending the number of colors to infinity ($N_c \to \infty$) while keeping all the other parameters fixed, is known as the 't Hooft limit.

$$\beta(x) = \mu^2 \frac{dx}{d\mu^2} = -\beta_0 x^2 - \beta_1 x^3 - \beta_2 x^4 - \beta_3 x^5 - \beta_4 x^6 - \dots, \qquad x \equiv g_{\text{YM}}^2, \tag{2.3}$$

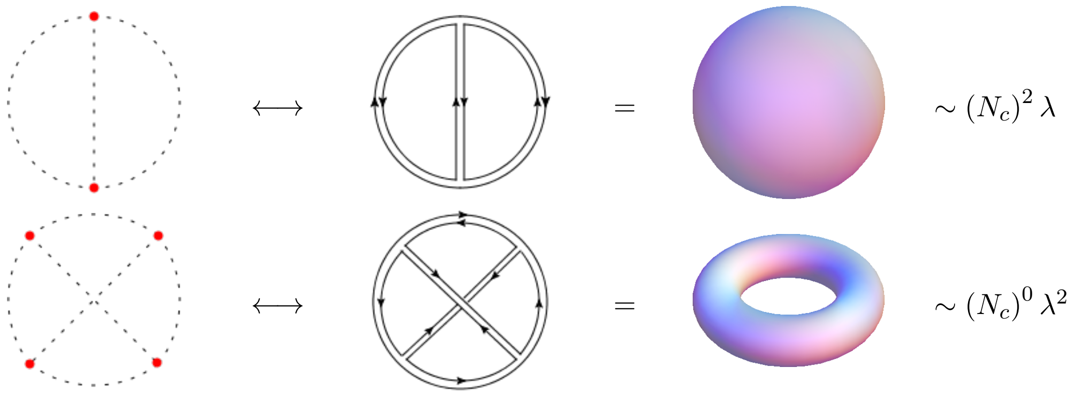
where β_n is the value of the beta function at $n+1$ loops. β_n generally depends on the number of colors N_c. We will assume that in the 't Hooft limit the n-th loop beta function scales as

$$\lim_{N_c \to \infty} \beta_n = b_n N_c^{n+1}, \tag{2.4}$$

where b_n are some numerical coefficients that are independent of N_c. Defining the 't Hooft coupling as $\lambda \equiv x N_c = g_{\text{YM}}^2 N_c$, we find that the large-N_c beta function becomes independent of N_c:

$$\beta(\lambda) = \mu^2 \frac{d\lambda}{d\mu^2} = -\lambda^2 - \sum_{n=1}^{\infty} \frac{\beta_n}{\beta_0^{n+1}} \lambda^{n+2} \longrightarrow -\lambda^2 - \sum_{n=1}^{\infty} \frac{b_n}{b_0^{n+1}} \lambda^{n+2} \quad \text{as } N_c \to \infty. \tag{2.5}$$

We now want to compute the large-N_c behavior of Feynman diagrams in theory (2.1). There is a very useful notation introduced by 't Hooft and consists in replacing the adjoint $\mathfrak{su}(N_c)$ propagators by double lines having opposite orientations. This way, the Feynman diagrams can be transformed to 2-dimensional surfaces with a varying number of handles g.



From the Lagrangian (2.2) we obtain the following large-N_c Feynman rules:

$$\begin{array}{lcl} \text{Vertices (V)} & \sim & \dfrac{1}{g_{\text{YM}}^2} = \dfrac{N_c}{\lambda} \\ \text{Propagators (E)} & \sim & g_{\text{YM}}^2 = \dfrac{\lambda}{N_c} \\ \text{Double-line loops (L)} & \sim & N_c. \end{array}$$

Therefore a disconnected Feynman diagram of the Yang-Mills theory (2.1) having V vertices, E propagators and L double-line loops will scale as follows in the 't Hooft limit:

$$\left(\frac{N_c}{\lambda}\right)^V \left(\frac{\lambda}{N_c}\right)^E (N_c)^L = (N_c)^\chi \lambda^{E-V}, \tag{2.6}$$

where χ is the Euler characteristic given by the formula $\chi = V - E + L$. Fields in the fundamental/antifundamental representation of $\mathfrak{su}(N_c)$ have single-line propagators and introduce surface boundaries (b), while the double lines of fields that transform as $\mathfrak{so}(N_c)$ or $\mathfrak{sp}(N_c)$ have the same orientation (fundamental or antifundamental) and give rise to cross-caps (c). For a surface of genus g (# handles), b boundaries (# holes) and c cross-caps (# twists), the Euler characteristic is given by

$$\chi = 2 - 2g - b - c. \tag{2.7}$$

Therefore we find that the vacuum-to-vacuum generating functional can be expanded in a double series as follows:

$$\log \mathcal{Z}_{\mathrm{YM}} = \sum_{\chi} (N_c)^{\chi} f_{\chi}(\lambda) \tag{2.8}$$

The dominant bubble diagrams in the large-N_c limit of the YM theory (2.1) are the ones with only adjoint $\mathfrak{su}(N_c)$ fields ($b = c = 0$) and no handles ($g = 0$). They are all planar diagrams.[3]

Computing connected diagrams (i.e. vacuum graphs with a number of external legs n) at large-N_c is rather straightforward. More generally, we may repeat our analysis for n-point functions of the single-trace operators

$$G_j(x_j) = \frac{1}{N_c} \operatorname{Tr}\left[\prod_i \Phi_i(x_j)\right]. \tag{2.9}$$

These are added to the Lagrangian (2.2) by coupling them to external currents g_j as

$$\mathcal{L}_G = N_c \sum_j g_j \cdot G_j(x_j) = \sum_j g_j \cdot \operatorname{Tr}\left[\prod_i \Phi_i(x_j)\right]. \tag{2.10}$$

We find that each single-trace operator $G_j(x_j)$ suppresses the correlation function by $1/N_c$, i.e.

$$\left\langle \prod_{j=1}^{n} G_j(x_j) \right\rangle \sim (N_c)^{\chi - n}, \tag{2.11}$$

so that the connected generating functional (free energy) of a generic YM theory of the form (2.1) also affords a double series expansion of the type (2.8). What is more, genus expansions like (2.8) are familiar from perturbative string theory (see e.g. [16]) where

$$\mathcal{Z}_{\text{string}} = \sum_{\chi} g_s^{-\chi} \mathcal{Z}_{\chi}, \quad \chi \equiv 2 - 2g - b - c. \tag{2.12}$$

Obviously the role of N_c in string theory is played by the inverse of the string coupling constant g_s:

$$N_c \sim \frac{1}{g_s}, \tag{2.13}$$

from which we see that the planar limit ($N_c \to \infty$) of the gauge theory corresponds to a free string theory ($g_s \to 0$). Some more implications of 't Hooft's duality are summarized in the following table.

YM Action content		String Theory
$\mathfrak{su}(N_c)$ adjoints	$b = 0$	closed
$\mathfrak{su}(N_c)$ adjoints & fundamentals	$b \neq 0$	open
$\mathfrak{su}(N_c)$ fields only	$c = 0$	orientable
$\mathfrak{so}(N_c)$ or $\mathfrak{sp}(N_c)$ fields	$c \neq 0$	non-orientable
planar	$g = b = c = 0$	free

[3] By definition, a planar graph is one that can be drawn on a plane. Equivalently no two lines may cross each other or, as we just saw, the corresponding surface cannot have any handles ($g = 0$).

The similarity between the formulae (2.8) and (2.12) implies that string theory can be thought of as some sort of large-N_c gauge theory and vice versa. However no concrete pair of a string theory and a large-N_c YM theory is known outside AdS/CFT. An additional complication is that both series (2.8) and (2.12) are divergent. String theory (in the form of dual resonance models) was originally proposed as a theory of strong interactions before being dethroned by QCD—a gauge theory. It turned out that string theory is a theory of quantum gravity after all. We may therefore ask the question if there is a systematic way to associate a non-gravitational theory (a gauge theory possibly) with a gravitational one. The answer comes from the holographic principle.

2.2 Holographic Principle

The holographic principle of 't Hooft and Susskind [17] points out that gravity/geometry can be combined with quantum mechanics/information in a non-local manner. Non-gravitational quantum field theories are generally local theories and their number of degrees of freedom is analogous to the volume of spacetime they occupy. Gravity puts severe constraints on the number of available posts by excluding a great deal of them. The number of fundamental degrees of freedom in any gravitational system is proportional to the system's area, so that the degrees of freedom may be thought to reside on an appropriate lower-dimensional surface or holographic screen. The latter hosts a non-gravitational (local) quantum field theory (QFT). Holography implies:

Quantum Gravity Theory $\cong$	Non-Gravitational Theory at the Boundary

We shall now briefly trail the steps that led to this proposal, following the very nice review of Bousso [18] to which the interested reader is referred for more details. See also [19].

Our starting point is the second law of black hole thermodynamics. It is a consequence of Hawking's black hole area theorem and Bekenstein's proposal for the entropy of black holes, both of which can be directly generalized to any number of dimensions $d+1$:[4]

$$S_{\rm BH} = \frac{kc^3}{\hbar} \cdot \frac{A}{4G_{d+1}}, \qquad dA \geq 0, \tag{2.14}$$

where G_{d+1} is the gravitational constant in $d+1$ spacetime dimensions (the value $1/4$ of the constant multiplying the area in the BH entropy formula was fixed by Hawking). The second law states:

$$dS_{\rm BH} \geq 0. \tag{2.15}$$

Bekenstein generalized the second law to include matter besides just black holes. According to the generalized second law,

$$dS_{\rm BH+matter} \geq 0. \tag{2.16}$$

The spherical entropy bound of Susskind is the condition that the generalized second law is not violated in the process of gravitational collapse of a matter system:

$$S_{\rm matter} \leq \frac{A}{4G_{d+1}}, \tag{2.17}$$

where A is the area of an asymptotically stable matter system which is either spherically symmetric or weakly gravitating.

[4] Unless otherwise noted, the convention $c = \hbar = k = 1$ will be used throughout.

The bound (2.17) seems to give rise to a holographic principle which posits that there's no gravitational theory in which the amount of information that can be stored in a region with boundary area equal to A, exceeds $A/4$ degrees of freedom or 1 bit/Planck[5] area.

The spherical entropy bound (2.17) may be upgraded to the covariant (Bousso) entropy bound, in which stability, spherical symmetry or weak gravity are not necessary attributes of the matter system. According to the covariant entropy bound, the entropy on any light-sheet of a holographic screen (light-sheets are formed by light rays emanating from the screen) is bounded from above by the area of the screen. The holographic principle then gets modified accordingly. We also know that

Holographic screens with information density of 1 bit/Planck area can be constructed at the boundary of any spacetime.

It seems natural to hypothesize that the boundary degrees of freedom are governed by some non-gravitational theory or a QFT that encodes all of the bulk dynamics. However it is not generally known how to extract or even construct the properties of the boundary QFT. One way to proceed is suggested by the holographic interpretation of the renormalization group.

2.3 Holographic Renormalization Group

It is generally accepted that the properties of physical systems depend on the scale (energy, distance, momenta) at which they are being studied. The renormalization group (RG) is the theoretical toolkit with which the response of physical systems to such changes of scale can be studied. The equations that rule the scaling behavior of systems are the renormalization group equations (RGEs). A very nice illustration of the action of the RG can be given by the Wilson-Kadanoff renormalization scheme that we will now briefly describe.[6]

Different scales generally have different degrees of freedom obeying different sets of laws. As we pass from small scales to larger ones we integrate out all the smaller degrees of freedom which become irrelevant and disappear. This is an irreversible process in which information is absorbed into a number of parameters (renormalized masses, couplings) and cannot be retrieved. As a result, the remaining degrees of freedom have completely different dynamics from the ones that we started with. Thus physics at small scales gets decoupled from the one at larger scales.

If a physical theory is renormalizable then the number of parameters that remain after the above coarse-graining is finite. If the number of parameters is infinite then the theory is non-renormalizable. The reason why we generally prefer to study renormalizable theories instead of non-renormalizable ones is that the former are much less complex than the latter. Even more manageable theories are the scale invariant or finite theories which are identical at all scales. $\mathcal{N}=4$ super Yang-Mills theory and superstrings are examples of finite theories.

It has been known since the mid-eighties that the condition for quantum scale invariance of the string sigma model gives rise to Einstein's equations, to lowest order in perturbation theory. α' corrections to Einstein's equations may be obtained by demanding that the corresponding higher-loop beta functions vanish. Let us briefly see how this comes about. Consider the string Polyakov action:

$$S_P = \frac{1}{4\pi\alpha'}\int d^2\sigma \left\{\sqrt{-\gamma}\,\gamma^{ab}G_{mn}\left(\mathbf{X}\right) + \epsilon^{ab}B_{mn}\left(\mathbf{X}\right)\right\}\partial_a\mathbf{X}^m\partial_b\mathbf{X}^n + \frac{1}{4\pi}\int d^2\sigma\sqrt{-\gamma}\,R_\gamma\Phi\left(\mathbf{X}\right), \tag{2.18}$$

where the massless states of bosonic string theory (gravitons, Kalb-Ramond fields and the dilaton) have been promoted to the background fields G_{mn}, B_{mn} and Φ. Also, γ_{ab} and ϵ_{ab} are the worldsheet

[5]As a matter of fact, the number of allowed bits cannot exceed $N = A/4\ell_p^{d-1}\log 2$ or $1/4\log 2$ bits per Planck area $\ell_p^{d-1} \equiv \hbar G_{d+1}/c^3$.

[6]For more the interested reader is referred to D. Gross' lectures in [20].

metric and the Levi-Civita symbol, R_γ is the worldsheet Ricci scalar, $\gamma = \det \gamma_{ab}$ and α' is the Regge slope. $\mathbf{X}_m$ are the spacetime and $\sigma_a = \{\tau, \sigma\}$ are the worldsheet coordinates.[7]

The expectation value of the energy-momentum tensor trace can be written as:

$$T^m_m = \left(\sqrt{-\gamma}\,\gamma^{ab}\beta^G_{mn} + \epsilon^{ab}\beta^B_{mn}\right)\partial_a\mathbf{X}^m\partial_b\mathbf{X}^n + \sqrt{-\gamma}\,R_\gamma\,\beta^\Phi \tag{2.19}$$

with the beta functionals given by [21]

$$\beta^G_{mn} = R_{mn} - \frac{1}{4}H_{mrs}H_n^{rs} + 2\nabla_m\nabla_n\Phi + O\left(\alpha'\right) \tag{2.20}$$

$$\beta^B_{mn} = \frac{1}{2}\nabla^r H_{rmn} - \left(\nabla^r\Phi\right)H_{rmn} + O\left(\alpha'\right) \tag{2.21}$$

$$\beta^\Phi = D - 26 + 3\alpha'\left\{4\left(\nabla\Phi\right)^2 - 4\nabla^2\Phi - R + \frac{H^2}{12}\right\} + O\left(\alpha'^2\right), \tag{2.22}$$

where ∇ is the spacetime covariant derivative and $H_{mrs} = 3\nabla_{[m}B_{rs]}$ the antisymmetric field strength. R_{mn} and R are the spacetime Ricci tensor and scalar. The condition for conformal invariance reads:

$$\beta^G_{mn} = \beta^B_{mn} = \beta^\Phi = 0. \tag{2.23}$$

Interestingly, the superstring action leads to exactly the same result, but in $D = 10$ dimensions instead of 26. For more see E. D'Hoker's lectures in [20]. The set of equations (2.20)–(2.22) can be derived from the following string/Jordan-frame action:

$$S_J = \frac{1}{2\kappa^2_{d+1}}\int d^{d+1}x\sqrt{-G}\,e^{-2\Phi}\left\{R_G + 4\left(\nabla\Phi\right)^2 - \frac{H^2}{12}\right\} + O\left(\alpha'\right), \quad 2\kappa^2_{d+1} \equiv 16\pi G_{d+1}. \tag{2.24}$$

We may switch to the Einstein frame by setting

$$g_{mn} = e^{-4\Phi/(d-1)}G_{mn} \tag{2.25}$$

so that the action (2.24) becomes:

$$S_E = \frac{1}{2\kappa^2_{d+1}}\int d^{d+1}x\sqrt{-g}\left\{R_g - \frac{4}{d-1}\left(\nabla\Phi\right)^2 - \frac{1}{12}e^{-8\Phi/(d-1)}H^2\right\} + O\left(\alpha'\right). \tag{2.26}$$

The action (2.26) along with its bosonic cousin

$$S = \frac{1}{2\kappa^2_{d+1}}\int d^{d+1}x\sqrt{-g}\left\{R_g - \frac{1}{2}\left(\partial\Phi\right)^2 - \frac{e^{-a(d)\Phi}}{2\left(d+1\right)!}F^2_{p+2}\right\}, \tag{2.27}$$

where F_{p+2} is a $(p+2)$-rank antisymmetric field that couples to a p-brane, are often the starting point in holographic treatments of gauge theories.

A remarkable picture has emerged in which the various background fields in the string action (metric, antisymmetric field, dilaton) play the role of coupling constants[8] and their RGEs become the supergravity equations of motion. If we treat the worldsheet scale μ as an extra (holographic) spacetime dimension, the following equivalence is obtained:

RG Flow in d-dimensional Minkowski Spacetime	$\Leftrightarrow$ Gravity in $d+1$ Dimensions

[7] Spacetime has $D = d + 1$ dimensions, so that the indices m, n, r, s take the values $0, 1, \ldots, d$. The worldsheet coordinates a, b take the values $0, 1$.

[8] To be precise, the background fields are the generating functions of coupling constants.

In the gauge/gravity duality, Einstein's (or supergravity) equations in the bulk are the RGE's and the bulk holographic coordinate is the renormalization scale of some QFT that lives on the boundary. This corresponds to an effective "geometrization" of the RG flow.

It seems that these ideas solidified in the high-energy physics community after Maldacena had published his famous paper. The first papers envisaging the possibility that gravity and supergravity equations are RGEs of some appropriate gauge theory were [22]. In a paper entitled "The wall of the Cave" [23], Polyakov tried to solve the equations (2.20)–(2.22) in certain cases, in order to gain an intuition about the dual gauge theory. Many more works followed. In the review [24] the reader may find an introduction to the subject. See also chapter 9 of the book [25]. For a recent overview see also the talk [26].

3 AdS/CFT Correspondence

The AdS/CFT correspondence [6, 7, 8] is the first explicit realization of the gauge/gravity duality and the holographic principle but also the first concrete example of a string theory that reduces to a gauge theory at large N_c. As a weak/strong coupling duality it is also an exemplary four-dimensional analog of Coleman's duality.[9] It is formulated as follows:

$$\boxed{\mathcal{N}=4,\ \mathfrak{su}\left(N_c\right)\ \text{Super Yang-Mills Theory} = \text{IIB Superstring Theory on } \mathrm{AdS}_5\times\mathrm{S}^5} \tag{3.1}$$

Nice reviews and introductions to AdS/CFT, from different perspectives and points of view, can be found in [9, 16, 24, 27, 28].

We will now present a theoretical argument that motivates the $\mathrm{AdS}_5/\mathrm{CFT}_4$ conjecture. We will take the low-energy limit of two diverse formulations of a system of N_c coinciding D3-branes, namely the *open string formulation* and the *closed string formulation*. The former is going to give rise to $\mathcal{N}=4$, $\mathfrak{su}(N_c)$ super Yang-Mills (SYM) theory and the latter to IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$. Since both theories describe the same system of D3-branes, they must be the same and (3.1) has to hold.

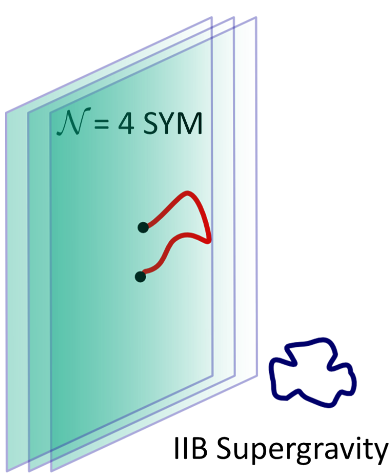
3.1 Open String Description

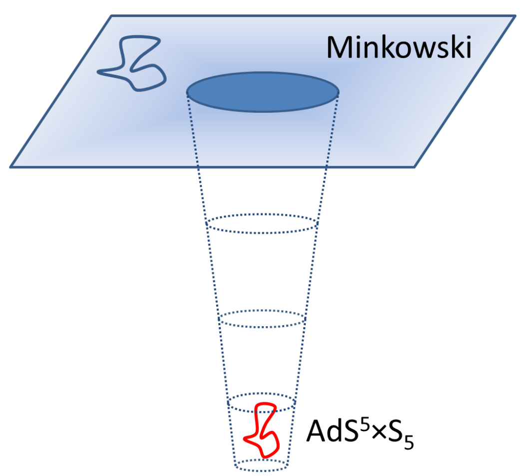
Consider N_c coinciding D3-branes in type IIB string theory. In what we call the *open string description*, the system consists of open strings with endpoints on the $(3+1)$ dimensional branes, closed strings propagating in the 10d bulk, as well as their interactions:

$$S = S_{\text{branes}} + S_{\text{bulk}} + S_{\text{interactions}}. \tag{3.2}$$

S_{branes} is just $\mathcal{N}=4$, $\mathfrak{su}(N_c)$ SYM theory in flat $3+1$ dimensions plus α' corrections, while S_{bulk} is just IIB supergravity in flat 10d plus α' corrections. As it turns out, string interactions can be switched off at low energies (a statement effectively equivalent to saying that gravity is IR-free) so that all the stringy modes decouple from each other and the action (3.2) reduces to the low-energy descriptions of non-interacting open and closed strings:

$$\left\{\begin{array}{c}\text{Open String Description}\\ \text{Low-Energy Limit}\end{array}\right\} \Rightarrow \mathcal{N}=4,\ \mathfrak{su}\left(N_c\right)\ \text{SYM} \ + \ \text{Free IIB Supergravity}. \tag{3.3}$$





[9]Maldacena's paper also seems to be breaking the citation world record. At the time of speaking it is well above 13.000 citations...

3.2 Closed String Description

In the *closed string description* the N_c D3-branes are seen as probes that source the bulk fields:

$$ds^2 = H^{-1/2}\left(-dt^2 + d\mathbf{x}_3^2\right) + H^{1/2}\left(dz^2 + z^2 d\Omega_5^2\right), \quad H(z) \equiv 1 + \left(\frac{\ell}{z}\right)^4, \quad \ell^4 = 4\pi g_s N_c \ell_s^4. \tag{3.4}$$

Far from the horizon ($z \to \infty$), the metric (3.4) reduces to 10-dimensional Minkowski spacetime. The near-horizon limit ($z \to 0$) of (3.4) is just $\mathrm{AdS}_5 \times \mathrm{S}^5$:

$$ds^2 = \frac{z^2}{\ell^2}\left(-dt^2 + d\mathbf{x}_3^2\right) + \frac{\ell^2}{z^2}\left(dz^2 + z^2 d\Omega_5^2\right) = \left\{\frac{z^2}{\ell^2}\left(-dt^2 + d\mathbf{x}_3^2\right) + \frac{\ell^2}{z^2}dz^2\right\} + \ell^2 d\Omega_5^2,$$

in the so-called horospheric/Poincaré coordinates (A.23). Taking the low-energy limit, we find that excitations living far from the horizon decouple from those in the near-horizon region and the system again becomes a sum of two systems:

$$\left\{\begin{array}{c}\text{Closed String Description}\\ \text{Low-Energy Limit}\end{array}\right\} \Rightarrow \text{IIB String Theory on } \mathrm{AdS}_5 \times \mathrm{S}^5 + \text{ Free IIB Supergravity.} \tag{3.5}$$

(3.3) and (3.5) are two descriptions of the same system of N_c coinciding D3-branes, therefore their actions must coincide. Since free IIB supergravity is a common constituent of both low-energy descriptions (3.3) and (3.5), the remaining constituents have to be identical, namely

$$\mathcal{N} = 4,\ \mathfrak{su}(N_c)\ \text{SYM} = \text{IIB Superstring Theory on } \mathrm{AdS}_5 \times \mathrm{S}^5.$$

Let us now briefly examine the two basic components of the AdS/CFT correspondence, $\mathcal{N} = 4$ SYM theory and IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. A short review of the geometry and the most common coordinate systems of anti-de Sitter space may be found in appendix A.

3.3 $\mathcal{N} = 4$ Super Yang-Mills (SYM)

$\mathcal{N} = 4$ Super Yang-Mills (SYM) theory in $d = 4$ spacetime dimensions was found in 1977 by Brink, Schwarz and Scherk and by Gliozzi, Scherk and Olive [29]. It's a theory that has the maximum allowed number of supersymmetries in $d = 4$ dimensions. Its most important property is that it is quantum conformally invariant.

The possibility that the one-loop beta function of $\mathfrak{su}(N_c)$ supersymmetric theories possessing three matter multiplets (such as the $\mathcal{N} = 4$ SYM theory) vanishes, was first considered in 1974 (prior to the discovery of $\mathcal{N} = 4$ SYM) by Ferrara and Zumino [30]. For $\mathcal{N} = 4$ SYM, the vanishing of the beta function has been confirmed up to four loops in perturbation theory [31]. Extension to all-loop orders was performed either by proving the vanishing of the axial current [32] or by going to the light-cone frame of superspace [33] or by formulating $\mathcal{N} = 4$ SYM in terms of $\mathcal{N} = 2$ superspace [34].

The perturbative finiteness of $\mathcal{N} = 4$ SYM was upgraded to non-perturbative finiteness in [35]. Therefore the theory is scale invariant. To prove superconformal invariance from scale invariance requires some more steps and the interested reader is referred to [36] for a more complete discussion.

There are various equivalent formulations of $\mathcal{N} = 4$ SYM. In one of them, the Lagrangian density can be obtained by dimensionally reducing $\mathcal{N} = 1$ SYM, from $d = 10$ to $d = 4$:

$$\mathcal{L}_{\text{SYM}} = -\frac{2}{g_{YM}^2}\mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathcal{D}_\mu\phi_i\mathcal{D}^\nu\phi_i - \frac{1}{4}\left[\phi_i,\phi_j\right]^2 + \bar{\psi}_{\mathrm{a}}\not{\mathcal{D}}\psi_{\mathrm{a}} - \frac{i}{2}\sigma_i^{\mathrm{ab}}\psi_{\mathrm{a}}\left[\phi_i,\psi_{\mathrm{b}}\right] - \frac{i}{2}\sigma_i^{\mathrm{ab}}\bar{\psi}_{\mathrm{a}}\left[\phi_i,\bar{\psi}_{\mathrm{b}}\right]\right], \tag{3.6}$$

where the definitions of the indices and the fields are ($T^{\mathfrak{a}}$ are the $\mathfrak{su}(N_c)$ generators, all in the adjoint representation):

$$A_\mu \equiv A_\mu^{\mathfrak{a}}T^{\mathfrak{a}}, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i\left[A_\mu, A_\nu\right], \quad \mu,\nu = 0,1,2,3$$

$$\phi_i \equiv \phi_i^{\mathfrak{a}} T^{\mathfrak{a}}, \quad \mathcal{D}_\mu^{\mathfrak{ab}} \equiv \delta^{\mathfrak{ab}} \partial_\mu - i\epsilon^{\mathfrak{abc}} A_\mu^{\mathfrak{c}}, \quad i = 1,2,3,4,5,6, \quad \mathfrak{a},\mathfrak{b},\mathfrak{c} = 1,2,\dots N_c^2 - 1$$

$$\psi_{\mathrm{a},\alpha} \equiv \psi_{\mathrm{a},\alpha}^{\mathfrak{a}} T^{\mathfrak{a}}, \quad \bar{\psi}_{\mathrm{a},\dot{\alpha}} \equiv \bar{\psi}_{\mathrm{a},\dot{\alpha}}^{\mathfrak{a}} T^{\mathfrak{a}}, \quad \mathrm{a},\mathrm{b} = 1,2,3,4, \quad \alpha,\dot{\alpha} = 1,2$$

$$\not{\mathcal{D}} \equiv \sigma^\mu \mathcal{D}_\mu, \quad \left(\sigma^\mu, \sigma^i\right) \equiv \text{projections of 10d Dirac matrices to 4d \& 6d respectively.}$$

The scaling dimensions of the fields appearing in $\mathcal{N} = 4$ SYM Lagrangian are:

$$[F_{\mu\nu}] = 2, \quad [A_\mu] = [\mathcal{D}_\mu] = [\phi_i] = 1, \quad [\psi_{\mathrm{a}}] = \frac{3}{2}. \tag{3.7}$$

It is customary to combine the six scalar fields ϕ_i into three complex scalars as follows:

$$\mathcal{X} \equiv \phi_1 + i\phi_2, \quad \mathcal{Y} \equiv \phi_3 + i\phi_4, \quad \mathcal{Z} \equiv \phi_5 + i\phi_6. \tag{3.8}$$

Let us also define the light-cone derivatives:

$$\mathcal{D}_+ \equiv \mathcal{D}_0 + \mathcal{D}_3, \quad \mathcal{D}_- \equiv \mathcal{D}_1 + \mathcal{D}_2. \tag{3.9}$$

For more, the reviews by Sohnius and Kovacs [37] are recommended.

3.4 IIB String Theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$

The IIB superstring action on $\mathrm{AdS}_5 \times \mathrm{S}^5$ was first written down in 1998 by Metsaev and Tseytlin [38]. It is given by the action of the Green-Schwarz superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$, which is a nonlinear sigma model (NLSM) in the coset space:

$$\frac{F}{G} = \frac{\mathfrak{psu}\,(2,2|4)}{\mathfrak{so}\,(4,1) \times \mathfrak{so}\,(5)} = \frac{\mathfrak{psu}\,(2,2|4)}{\mathfrak{sp}\,(2,2) \times \mathfrak{sp}\,(4)}. \tag{3.10}$$

Let us start from the superalgebra $\mathfrak{su}\,(2,2|4)$ which is spanned by the 8×8 matrices $\mathbb{M}$:

$$\mathbb{M} = \left(\begin{array}{c|c} \mathbb{B}_1 & \mathbb{F}_1 \\ \hline \mathbb{F}_2 & \mathbb{B}_2 \end{array}\right). \tag{3.11}$$

The 4×4 matrices $\mathbb{B}_{1,2}$ and $\mathbb{F}_{1,2}$ are respectively bosonic and fermionic. $\mathbb{M}$ have vanishing supertrace:

$$\mathrm{Str}\mathbb{M} \equiv \mathrm{Tr}\mathbb{B}_1 - \mathrm{Tr}\mathbb{B}_2 = 0. \tag{3.12}$$

$\mathfrak{psu}\,(2,2|4)$ is obtained as the quotient algebra of $\mathfrak{su}\,(2,2|4)$ over the identity element. Without going into much more details, the Lagrangian of IIB superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is given by the following expression:

$$\mathcal{L}_{\text{string}} = -\frac{1}{4\pi\alpha'} \left[\sqrt{-\gamma}\,\gamma^{ab}\,\mathrm{Str}\left(A_a^{(2)} A_b^{(2)}\right) + \kappa\epsilon^{ab}\,\mathrm{Str}\left(A_a^{(1)} A_b^{(3)}\right)\right], \tag{3.13}$$

where the first term is the kinetic and the second is a Wess-Zumino term, multiplied by the real number κ to make $\mathcal{L}_{\text{string}}$ real. Decompose the elements $\mathfrak{g}$ of the supergroup $\mathfrak{psu}\,(2,2|4)$ into a bosonic and a fermionic part as follows:

$$\mathfrak{g} = \mathfrak{g}_{\mathfrak{f}}\, \mathfrak{g}_{\mathfrak{b}}. \tag{3.14}$$

Then A is defined as:

$$A_a = \sum_{i=0}^{3} A_a^{(i)} \equiv -\mathfrak{g}^{-1}\partial_a \mathfrak{g} = -\mathfrak{g}_{\mathfrak{b}}^{-1}\mathfrak{g}_{\mathfrak{f}}^{-1}\left(\partial_a \mathfrak{g}_{\mathfrak{f}}\right)\mathfrak{g}_{\mathfrak{b}} - \mathfrak{g}_{\mathfrak{b}}^{-1}\partial_a \mathfrak{g}_{\mathfrak{b}}. \tag{3.15}$$

The decomposition of A in terms of $A^{(i)}$'s is possible because of the so-called $\mathbb{Z}_4$ grading of $\mathfrak{psu}\,(2,2|4)$. As it turns out, we may write (3.13) as follows:

$$\mathcal{L}_{\text{string}} = -\frac{1}{16\pi\alpha'}\,\mathrm{Str}\Big[\sqrt{-\gamma}\,\gamma^{ab}\left(\mathcal{B}_a + \mathcal{G}\mathcal{B}_a\mathcal{G}^{-1} + \partial_a\mathcal{G}\mathcal{G}^{-1}\right)\left(\mathcal{B}_b + \mathcal{G}\mathcal{B}_b\mathcal{G}^{-1} + \partial_b\mathcal{G}\mathcal{G}^{-1}\right) - \tag{3.16}$$

$$-2i\kappa\epsilon^{ab}\mathcal{F}_a\mathcal{G}\mathcal{F}_b^{st}\mathcal{G}^{-1}\Big], \tag{3.17}$$

where $\mathcal{B}$ and $\mathcal{F}$ are respectively the even $(0,2)$ and the odd $(1,3)$ fermionic components of $\mathfrak{g}_\mathfrak{f}^{-1}\partial_a\mathfrak{g}_\mathfrak{f}$. $\mathcal{G}$ is given by

$$\mathcal{G} = \begin{pmatrix} i\mathcal{G}_{\text{AdS}} & 0 \\ 0 & \mathcal{G}_{\text{S}} \end{pmatrix}, \qquad \mathfrak{g}_\mathfrak{f}^{-1}\partial_a\mathfrak{g}_\mathfrak{f} \equiv \mathcal{B}_a + \mathcal{F}_a \tag{3.18}$$

with

$$\mathcal{G}_{\text{AdS}} = \begin{pmatrix} 0 & -Y_{05} & Y_{12}^* & Y_{34}^* \\ Y_{05} & 0 & -Y_{34} & Y_{12} \\ -Y_{12}^* & Y_{34} & 0 & -Y_{05}^* \\ -Y_{34}^* & -Y_{12} & Y_{05}^* & 0 \end{pmatrix}, \quad \mathcal{G}_{\text{S}} = \begin{pmatrix} 0 & -X_{56} & -iX_{12}^* & -iX_{34}^* \\ X_{56} & 0 & iX_{34} & -iX_{12} \\ iX_{12}^* & -iX_{34} & 0 & -X_{56}^* \\ iX_{34}^* & iX_{12} & iX_{56}^* & 0 \end{pmatrix} \tag{3.19}$$

and the anti-de Sitter and the sphere coordinates are combined into pairs as

$$\begin{aligned} Y_{05} &= Y_0 + iY_5 & \qquad & X_{12} = X_1 + iX_2 \\ Y_{12} &= Y_1 + iY_2 & \& \qquad & X_{34} = X_3 + iX_4 \\ Y_{34} &= Y_3 + iY_4 & & X_{56} = X_5 + iX_6. \end{aligned} \tag{3.20}$$

The upshot is that the bosonic part of (3.17) ($B = F = 0$) is given by the string Polyakov action:

$$S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-\gamma}\,\gamma^{ab}\partial_a\mathbf{X}^m\partial_b\mathbf{X}^n G_{mn}(\mathbf{X}), \tag{3.21}$$

where $\mathbf{X}_m$ and G_{mn} are the $\text{AdS}_5\times\text{S}^5$ coordinates and metric tensor. The κ-fixed fermionic part is:

$$S_F = -\frac{i}{2\pi\alpha'}\int d\tau d\sigma\left(\sqrt{-\gamma}\,\gamma^{ab}\delta_{\alpha\beta} - \epsilon^{ab}s_{\alpha\beta}\right)\bar{\theta}^\alpha\rho_a\mathcal{D}_b\theta^\beta + O\left(\theta^4\right), \tag{3.22}$$

where θ are Majorana-Weyl spinors and

$$\rho_a \equiv \Gamma_\mu e_m^\mu\partial_a\mathbf{X}^m, \quad G_{mn} = e_m^\mu e_n^\nu\eta_{\mu\nu}, \quad a,b = 0,1, \quad m,n = 0,1,\ldots,9$$

$$s_{\alpha\beta} \equiv \text{diag}(1,-1), \quad \alpha,\beta = 1,2, \quad \mu,\nu = 0,1,\ldots,9$$

$$\mathcal{D}_m \equiv \partial_m + \frac{1}{4}\omega_m^{\mu\nu}\Gamma_{\mu\nu} - \frac{1}{8\cdot 5!}\Gamma^{m_1\ldots m_5}\Gamma_m F_{m_1\ldots m_5}, \quad \mathcal{D}_a \equiv \text{projection of } \mathcal{D}_m$$

$$\Gamma_\mu = e_\mu^m\Gamma_m, \text{ (10d Dirac matrices).}$$

e_m^μ is the zehnbein, $\eta_{\mu\nu}$ is the 10d Lorentz metric, $\omega_m^{\mu\nu}$ is the Lorentz connection and $F_{m_1\ldots m_5}$ is the 5-form Ramond-Ramond (RR) field. More can be found in the review [39].

3.5 Parameter Matching

When two theories are equal, their fundamental parameters are expected to be in one-to-one correspondence. Because of AdS/CFT, this must be the case for $\mathcal{N} = 4$, $\mathfrak{su}(N_c)$ SYM theory and IIB string theory on $\text{AdS}_5\times\text{S}^5$. The former depends on two fundamental parameters, the rank of the gauge group/number of colors and the 't Hooft/SYM coupling constant:

$$N_c, \quad \lambda = g_{\text{YM}}^2 N_c. \tag{3.23}$$

On the string theory side the basic parameters are the AdS_5/5-sphere radius, the fundamental string length/Regge slope and the 10-dimensional Newton's constant/Planck length:

$$\ell = R, \quad \ell_s^2 = \alpha', \quad G_{10} = \ell_p^8 = \ell_s^8 g_s^2. \tag{3.24}$$

The AdS/CFT correspondence links the fundamental parameters of the two theories as follows:

$$\boxed{\text{AdS/CFT Parameter Matching:}\quad \left(\frac{\ell_p}{\ell}\right)^4 = \frac{1}{4\pi N_c}, \qquad \left(\frac{\ell_s}{\ell}\right)^4 = \frac{1}{\lambda}}. \tag{3.25}$$

We also have for the couplings:

$$g_{\text{YM}}^2 = 4\pi g_s. \tag{3.26}$$

There are two interesting limits that one usually encounters when dealing with AdS/CFT. One is the strong-coupling limit in which, according to (3.25), the fundamental string length tends to zero and the strings are effectively point-like:[10]

$$\lambda \to \infty \quad \Leftrightarrow \quad \ell_s \to 0. \tag{3.28}$$

The second is the large-N_c/planar/'t Hooft limit which, by (3.25), corresponds to free strings:

$$N_c \to \infty \quad \Leftrightarrow \quad g_s \to 0. \tag{3.29}$$

Combining the two limits (3.28)–(3.29), we obtain the so-called classical (super)gravity approximation:

$$\boxed{\text{Classical Supergravity Approximation:}\ \ (\lambda, N_c) \to \infty \quad \Leftrightarrow \quad (\ell_s, g_s) \to 0}, \tag{3.30}$$

in which type IIB string theory reduces to classical IIB supergravity on $\text{AdS}_5 \times \text{S}^5$ that is dual to planar strongly coupled $\mathcal{N} = 4$ SYM theory.

3.6 The BMN Sector

Solving the full quantum IIB superstring sigma model on $\text{AdS}_5 \times \text{S}^5$ is an extremely difficult and so far impossible task.[11] Instead, the quantum string sigma model can be solved on a plane-wave background (see appendix C for the definition of plane-wave backgrounds and their basic properties), in which the superstring action simplifies significantly [41].

Pp-wave spacetimes are a special class of spacetimes that are α'-exact solutions of supergravity [42]. A particular type of pp-wave is the plane wave which serves as a background in certain maximally supersymmetric solutions of type IIB supergravity. Plane waves can be obtained by taking the Penrose limit of $\text{AdS}_{p+2} \times \text{S}^{q+2}$ and its orbifolds (see appendixes C.2.1–C.2.2). To take the Penrose limit, the radii of AdS_{p+2} and the $(q+2)$-sphere (ℓ and R respectively) must be sent to infinity, while their ratio must be kept fixed:

$$\ell,\ R \to \infty \quad \& \quad \frac{\ell}{R} = \text{fixed}. \tag{3.31}$$

A completely analogous limit, the Berenstein-Maldacena-Nastase (BMN) limit [43], may also be taken on the gauge theory side of the AdS/CFT correspondence, if we consider any operator of $\mathcal{N} = 4$ SYM with scaling dimension Δ and R-charge J, such that:[12]

$$N_c,\ J \to \infty \quad \& \quad \frac{N_c}{J^2} = \text{fixed}, \quad \Delta - J = \text{fixed}. \tag{3.33}$$

[10] In the opposite limit $\lambda \to 0$ the SYM theory is free, while the strings become tensionless since the string tension,

$$\lambda \to 0 \quad \Leftrightarrow \quad T = \frac{1}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi\ell^2} \to 0 \tag{3.27}$$

becomes very small [40].

[11] As it will be explained in more detail in §4.2, solving a theory basically means computing its spectrum.

[12] The scaling dimension Δ of an operator $\mathcal{O}(x)$ determines its behavior under dilations:

$$x' = \alpha x \quad \to \quad \mathcal{O}(\alpha x) = \alpha^{-\Delta}\mathcal{O}(x). \tag{3.32}$$

We're thus led to the so-called (BMN) sector of $\mathcal{N} = 4$ SYM which is dual to the Penrose-reduced IIB superstring theory on a plane-wave. The correspondence between these two limiting cases of AdS/CFT is known as the plane-wave/super Yang-Mills duality:

IIB String Theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$	$\xleftrightarrow{\mathrm{AdS}_5/\mathrm{CFT}_4}$	$\mathcal{N} = 4,\ \mathfrak{su}(N_c)$ SYM Theory
$\downarrow$ Penrose Limit		$\downarrow$ BMN Limit
IIB String Theory on plane-wave	$\xleftrightarrow{\text{plane-wave/SYM}}$	BMN Sector of $\mathcal{N} = 4$ SYM

The plane-wave/super Yang-Mills duality has been exhaustively studied (see [44] for reviews).

For later purposes it would be useful to define a very similar limit on the string theory side of AdS/CFT, that is known as the Frolov-Tseytlin (FT) limit [45]:

$$\lambda,\ J \to \infty \quad \& \quad \lambda' = \frac{\lambda}{J^2} \ll 1, \tag{3.34}$$

where J is the angular momentum of a string state of IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

3.7 Maldacena Dualities

The arguments of §3.1 and §3.2 may be repeated for other systems of branes besides the D3 system. Low-energy limits lead to decouplings analogous to (3.3)–(3.5) and give rise to a multitude of dualities between 10 or 11-dimensional theories that live in a spacetime that contains an anti-de Sitter part times a compact manifold (or a product thereof) and conformal field theories on a flat spacetime of one dimension less. The results are summarized in the following table.

Gravity Theory	**Spacetime**	**#Dim.**	**Brane System**	**Gauge Theory**	**#Dim.**
IIB String Theory	$\mathrm{AdS}_5 \times \mathrm{S}^5$	$5+5$	D3	$\mathcal{N} = 4$ SYM	$3+1$
IIB String Theory	$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{M}^4$	$3+3$	D1 + D5	$\mathcal{N} = (4,4)$ SCFT	$1+1$
IIB String Theory	$\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{M}^6$	$2+2+6$	D3	Conformal QM	$0+1$
M-Theory	$\mathrm{AdS}_7 \times \mathrm{S}^4$	$7+4$	M5	$\mathrm{A}_{N_c-1}(2,0)$ SCFT	$5+1$
M-Theory	$\mathrm{AdS}_4 \times \mathrm{S}^7$	$4+7$	M2	$\mathcal{N} = 8$ SCFT	$2+1$
M-Theory	$\mathrm{AdS}_3 \times \mathrm{S}^2 \times \mathrm{M}^6$	$3+2+6$	M5	$\mathcal{N} = (0,4)$ SCFT	$1+1$

For the manifold M, $M^4 = K3$ or T^4 and $M^6 = T^6$, $T^2 \times K3$ or CY_3. In all the cases containing a p-sphere ($p = 3,4,5,7$), there are always N_c units of p-form RR flux on S^p:

$$\int_{\mathrm{S}^p} F_p = N_c, \quad p = 3,4,5,7. \tag{3.35}$$

The near-horizon limit results in various values for the ratio $\mathfrak{k} \equiv \ell/R$ of the radius of AdS over that of the corresponding sphere. These are tabulated in the following table for each of the Maldacena dualities:

	$\mathrm{AdS}_5 \times \mathrm{S}^5$	$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{M}^4$	$\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{M}^6$	$\mathrm{AdS}_7 \times \mathrm{S}^4$	$\mathrm{AdS}_4 \times \mathrm{S}^7$	$\mathrm{AdS}_3 \times \mathrm{S}^2 \times \mathrm{M}^6$
$\mathfrak{k} =$	1	1	1	2	1/2	2

3.8 ABJM Correspondence

More recently, another group of dualities between 10 or 11-dimensional theories on AdS_4 spacetime times a compact manifold and a superconformal 3-dimensional field theory has been constructed.[13]

$$\boxed{\left\{\mathcal{N}=6,\ U(N_1)_k \times U(N_2)_{-k}\ \text{Super C-S Theory}\right\} \xrightarrow{N_{1,2}\to\infty} \left\{\text{M-Theory on } \mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k\right\}} \tag{3.36}$$

For $N_1 \neq N_2$, (3.36) is the Aharony-Bergman-Jafferis (ABJ) correspondence, while for $N_1 = N_2 = N_c$, it reduces to the Aharony-Bergman-Jafferis-Maldacena (ABJM) duality [46]. For $k = 1$ we obtain the Maldacena duality with M-theory on $\mathrm{AdS}_4 \times \mathrm{S}^7$, dual to $\mathcal{N} = 8$ SCFT. In the case of the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ gauge group, the left-hand side of (3.36) becomes the $\mathcal{N} = 8$ Bagger-Lambert-Gustavsson (BLG) theory [47]. By doubly dimensionally reducing ABJM, we're led to the following duality:

$$\boxed{\begin{array}{c}\mathcal{N}=6,\ U(N_c)_k \times U(N_c)_{-k}\ \text{Super C-S Theory} \\ k^5 \gg N_c \to \infty\ \&\ \lambda \equiv 2\pi^2 N_c/k = \text{const.}\end{array} \longleftrightarrow \text{IIA String Theory on } \mathrm{AdS}_4 \times \mathbb{CP}^3} \tag{3.37}$$

As for the flux counterparts of the above dualities, there are N_2 units of 4-form RR flux through AdS_4 in (3.36), while in (3.37) there are N_c units of 4-form RR flux through AdS_4 and k units of 2-form RR flux through $\mathbb{CP}^1 \subset \mathbb{CP}^3$.

3.9 Field/Operator Correspondence

It is often said that in conformal field theories there can be no asymptotic states/particles (consequently no traditional S-matrix) and it is operators that must assume this role.[14] The role of operators in AdS/CFT is the content of the field/operator correspondence. Let us consider the following deformation of the conformal field theory:

$$S' = S + \int d^d x\, \phi(x)\, \mathcal{O}(x), \tag{3.38}$$

where $\mathcal{O}(x)$ is a local gauge-invariant operator and $\phi(x)$ is its source. According to the field/operator correspondence, to each local gauge-invariant operator $\mathcal{O}(x)$ of the (deformed) boundary theory, there corresponds a dual bulk field $\Phi(x, y)$ such that the value of Φ at the boundary ($y \to 0$ in the conformal frame (A.23)) is the source of $\mathcal{O}(x)$:

$$\phi(x) = \Phi\Big|_{\partial \mathrm{AdS}}(x) = \lim_{y\to 0} \Phi(x, y). \tag{3.39}$$

There exists no generic algorithm which maps arbitrary boundary operators to their dual bulk fields or vice-versa. Therefore, a relatively small number of such (heuristic) identifications is known.

[13] In (3.36), super C-S stands for super Chern-Simons theory.

[14] The corresponding S-matrix goes by the name 'world sheet' S-matrix. There's also a 'space-time' S-matrix defined in terms of n-gluon amplitudes. Refer to [48] for more.

As an illustrative example of the field/operator correspondence, let us take a free scalar field in the bulk of AdS_{p+2}:

$$S_\phi = -\frac{1}{2}\int d^{p+2}x\sqrt{-g}\left(\partial_m\Phi\partial^m\Phi + m^2\Phi^2\right) \quad \& \quad ds^2 = \frac{\ell^2}{y^2}\left(-dt^2 + d\mathbf{x}_p^2 + dy^2\right). \tag{3.40}$$

If we solve the equations of motion of this field, we will find out that its behavior near the boundary of AdS $(y \to 0)$ is the following:

$$\Phi(x,y) = \underbrace{A(x)\,y^{\Delta_-}}_{\substack{\text{non-normalizable}\\ \text{term}}} + \underbrace{B(x)\,y^{\Delta_+}}_{\substack{\text{normalizable}\\ \text{term}}}, \quad y \to 0, \tag{3.41}$$

where

$$\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{d^2 + 4m^2\ell^2}\right), \qquad d = p + 1.\text{[15]} \tag{3.42}$$

For $m^2 > 0$ the behavior of (3.41) at the boundary is dominated by the first term, which blows up as $y \to 0$. Therefore

$$\phi(x) = A(x) \tag{3.43}$$

and the *non-normalizable coefficient* $A(x)$ determines the boundary Lagrangian through (3.38). We may go on and prove that Δ_+ is equal to the scaling dimension Δ of Φ's dual (scalar) operator $\mathcal{O}(x)$, which is defined as:

$$x' = \alpha x \quad \to \quad \mathcal{O}(\alpha x) = \alpha^{-\Delta}\mathcal{O}(x). \tag{3.44}$$

Now notice that since the bulk field $\Phi(x,y)$ is a scalar, it is invariant under dilatations and

$$\Phi(\alpha x, \alpha y) = \Phi(x,y) \quad \Rightarrow \quad A(\alpha x) = \alpha^{-\Delta_-}A(x) \quad \& \quad B(\alpha x) = \alpha^{-\Delta_+}B(x). \tag{3.45}$$

Therefore, (3.38)–(3.42) imply that

$$\Delta_- = d - \Delta \quad \Rightarrow \quad \Delta_+ = d - \Delta_- = \Delta. \tag{3.46}$$

As it turns out, the *normalizable coefficient* $B(x)$ can be put in 1-1 correspondence with states in the Hilbert space of the boundary theory. It can also be shown that $B(x)$ is related to the expectation value of the boundary operator $\langle\mathcal{O}(x)\rangle$. Summing up,

$$\left\{\begin{array}{c} \text{Bulk Renormalizable Modes} \longleftrightarrow \text{Boundary States} \\[1em] \text{Bulk Non-Renormalizable Modes} \longleftrightarrow \text{Boundary Lagrangian.} \end{array}\right\}$$

The previous analysis may be repeated for gauge theory operators of any spin. The following table is from reference [9]:

Field	Spin	Scaling Dimensions
Scalar	0	$\frac{1}{2}\left[d \pm \sqrt{d^2 + 4m^2\ell^2}\right]$
Spinor	1/2, 3/2	$\frac{1}{2}\left(d + 2\lvert m\rvert\ell\right)$
Vector	1	$\frac{1}{2}\left[d \pm \sqrt{(d-2)^2 + 4m^2\ell^2}\right]$
Massless Spin-2	2	d
q-form	-	$\frac{1}{2}\left[d \pm \sqrt{(d-2q)^2 + 4m^2\ell^2}\right]$.

[15] Note that for real $\Delta_\pm$, negative masses squared are allowed to a certain extent ($4m^2\ell^2 \geq -d^2$), a condition that is known as the Breitenlohner-Freedman (BF) bound.

3.10 Testing the AdS_5/CFT_4 Correspondence

We are going to finish this section with a concise discussion of the main tests of the AdS/CFT correspondence. Although our treatment will focus on the AdS_5/CFT_4 correspondence, all the tests that will be discussed can be appropriately generalized to any of the gauge/gravity dualities.

The AdS_5/CFT_4 correspondence (3.1) implies that the partition function of type IIB string theory on $AdS_5 \times S^5$ and the partition function of $\mathcal{N} = 4$, $\mathfrak{su}(N)$ super Yang-Mills (SYM) theory are equal:

$$\mathcal{Z}_{\text{string}}\left[\Phi\Big|_{\partial\text{AdS}}(x)\right] = \mathcal{Z}_{\text{CFT}}\left[\phi(x)\right], \tag{3.47}$$

where ϕ are the sources of all the gauge-invariant operators of $\mathcal{N} = 4$ SYM and Φ are their dual bulk fields. Statement (3.47) constitutes a generalization of the field/operator correspondence (3.39), according to which not only the boundary operators but every boundary observable (spectra, correlation functions, scattering amplitudes, Wilson loops, etc.) possesses a dual and equal observable in the bulk. The next level of generalization is the existence of a one-to-one mapping between the properties of type IIB superstring theory on $AdS_5 \times S^5$ and those of $\mathcal{N} = 4$ SYM. This map is colloquially known as the AdS/CFT "dictionary". Deciphering and building the dictionary of the AdS/CFT correspondence is one of the most significant problems in theoretical physics.

At the time of speaking, the official status of the AdS/CFT correspondence is "conjecture". It is not known whether a rigorous mathematical argument exists with which we can credibly prove or disprove the correspondence, neither has a theoretical algorithm of any sort been devised that, if faithfully followed, it can lead to an accepted proof or disproof of it. Remarkably, the problem of rigorously proving the AdS/CFT correspondence appears in A. Strominger's Strings 2014 list of "deep and interesting" questions that can be solved within the next 5-10 years.[16] To date, there exist three basic formulations of the AdS_5/CFT_4 correspondence:

- Weak Formulation of AdS/CFT: the correspondence is valid only for $N_c,\ \lambda \to \infty$.
- Medium Formulation of AdS/CFT: the correspondence is valid only for $N_c \to \infty$.
- Strong Formulation of AdS/CFT: the correspondence is valid for all $N_c,\ \lambda$.

However, the landscape is not 100% clear with any of them. As we will also discuss in §4.2 (dealing with AdS_5/CFT_4 integrability) there are indications that the correspondence is valid in its two weakest formulations but, in its present form, not in the strong formulation. E.g. in the tensionless limit $\lambda \to 0$, the picture is far from clear [40]. For the time being, the mainstream strategy for proving or disproving AdS/CFT, consists in computing the observables of both theories as accurately as possible and looking for agreement or disagreements. At the same time, significant effort is dedicated to completing the AdS/CFT dictionary. In the next section, a very powerful tool for computing and identifying the AdS/CFT observables will be presented: integrability.

Beyond computing/comparing and matching the observables, there exist various other tests that permit to compare the two theories of AdS/CFT, namely type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$, $\mathfrak{su}(N)$ SYM theory. In what follows we will briefly present some of these tests. Our emphasis however will be on the matching of the spectra, since this is directly related with the scope and the content of this thesis.

[16] See `http://physics.princeton.edu/strings2014/slides/Strominger.pdf`, page 13. E. Kiritsis' clue is to study the symmetries of the generalized Schwinger source functional and then try to map it in string field theory...

3.10.1 Symmetries

Type IIB superstrings are defined on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and thus they share its symmetries, namely the global bosonic isometry $\mathfrak{so}(4,2) \times \mathfrak{so}(6)$ that is extended to the AdS_5 supergroup $\mathfrak{psu}(2,2|4)$.

As we have said, $\mathcal{N}=4$, $\mathfrak{su}(N_c)$ SYM theory is conformally invariant and therefore it has the $d=4$ conformal group $\mathfrak{so}(4,2)$ as a symmetry. The Lagrangian (3.6) also has a manifest $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ R-symmetry related to the compactification from $d=10$ down to $d=4$ dimensions, under which the six scalars ϕ_i transform as vectors. Again, the $\mathfrak{so}(4,2) \times \mathfrak{so}(6)$ symmetry is extended to the super AdS_5 group $\mathfrak{psu}(2,2|4)$. All in all there are $15+15$ bosonic generators (15 conformal and 15 R-symmetries) and $16+16$ fermionic generators (16 Poincaré and 16 superconformal) in $\mathfrak{psu}(2,2|4)$:

Bosonic Generators	Fermionic Generators		
$D,\ P_\mu,\ K_\mu,\ L_{\mu\nu}$	$Q^{\mathrm{a}}_{\alpha},\ \bar{Q}^{\mathrm{a}}_{\dot\alpha}$	$\mu,\nu=0,1,2,3,$	$\mathrm{a}=1,2,3,4$
$T^{\mathfrak{a}}$	$S^{\mathrm{a}}_{\alpha},\ \bar{S}^{\mathrm{a}}_{\dot\alpha}$	$\alpha,\dot\alpha=1,2,$	$\mathfrak{a}=1,2,\ldots,15$

For later use, let us also write down the scaling dimensions of these generators:

$$[D]=[L_{\mu\nu}]=[T^{\mathfrak{a}}]=0, \quad [P_\mu]=1, \quad [K_\mu]=-1, \quad [Q]=\frac{1}{2}, \quad [S]=-\frac{1}{2}. \tag{3.48}$$

In addition to the above symmetries, both theories share a non-perturbative $\mathfrak{sl}(2,\mathbb{Z})$ symmetry or S-duality. String theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is also invariant under a certain T-duality [49]. The study of scattering amplitudes in planar $\mathcal{N}=4$ SYM theory revealed the existence of dual superconformal symmetry, a hidden symmetry that corresponds to the symmetry of $\mathcal{N}=4$ SYM Wilson loops and incorporates T-duality within $\mathcal{N}=4$ SYM. Both the dual and the ordinary superconformal symmetries combine into the Yangian symmetry, which generalizes $\mathfrak{psu}(2,2|4)$ and it is a symmetry that is typically exhibited by integrable systems.

3.10.2 Spectra

Perhaps the most important prediction of the AdS/CFT correspondence is the equality between the spectra of $\mathcal{N}=4$ SYM and type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. As we have already said, the role of particles in CFTs is played by operators, the spectrum of which is composed by their scaling (and possibly anomalous) dimensions. On the other hand we have string states and their corresponding energies. Matching the spectra of the two theories generally involves the following steps:

1. Compute the scaling dimensions Δ of all the gauge-invariant operators of $\mathcal{N}=4$ SYM.

2. Compute the energies E of IIB superstring states in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

3. Map the operators of $\mathcal{N}=4$ SYM to IIB string states in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

4. Compare the operator dimensions Δ with the dual string energies E and find agreement.

Steps 1 and 2 are less complicated in the planar limit ($N_c \to \infty$), in which IIB string theory contains only free closed string states on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Another limitation is that, except from relatively few cases, there's no general method by which to perform the state/operator mapping of step 3. That is mainly due to two reasons: (a) it is hard to quantize the string sigma model on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and (b) the spectrum of gauge-invariant operators is rather difficult to compute.

Progress in step 4 is drastically hindered by the weak/strong coupling nature of AdS/CFT. For small 't Hooft coupling ($\lambda \to 0$), $\mathcal{N} = 4$ SYM theory is weakly coupled and the spectrum may be computed perturbatively. However, the perturbative regime of IIB string theory on $\text{AdS}_5 \times \text{S}^5$ covers only the large values of 't Hooft's coupling ($\lambda \to \infty$), where the dual gauge theory is strongly coupled and far from its perturbative region. This means that we cannot directly compare the operator scaling dimensions with the energies of their dual string states, unless we find some reliable way of extrapolating our results from weak to strong coupling and vice-versa.

Let us first try to describe the spectral problem on the gauge theory side. We will consider only local gauge-invariant operators, the constituent fields of which depend on just one point in spacetime. All such operators can be divided into single and multiple-trace operators which are dual to single and multi-particle states respectively. Only the former will concern us here. In order to classify all the local gauge-invariant single-trace operators of $\mathcal{N} = 4$, $\mathfrak{su}(N_c)$ SYM, it is useful to introduce the notion of superconformal primary operators and their descendants.

Conformal primary operators are annihilated by the conformal generators K_μ, while superconformal primary operators are annihilated by the superconformal generators S_α^{a} and have the lowest dimension in a given superconformal multiplet of a unitary representation of $\mathfrak{psu}(2,2|4)$. Superconformal descendant operators are obtained by the action of the Poincaré generators on another operator of the same multiplet. Chiral superconformal primaries (aka BPS operators) are annihilated by at least one of the Poincaré supercharges and fall in short representations of the algebra. This means that BPS operators (and their descendants) are unrenormalized i.e. their scaling dimensions are protected against quantum corrections. Depending on the number of Q's that annihilate them, chiral primaries are dubbed 1/2 BPS, 1/4 BPS or 1/8 BPS. These are annihilated by half, 1/4 or 1/8 of the Poincaré supercharges respectively. A very useful corollary is that chiral primaries are composed solely out of the scalar fields of $\mathcal{N} = 4$ SYM:[17]

$$\mathcal{O}^{j_1 j_2 \dots j_n} = \text{Tr}\left[\phi^{(j_1}\phi^{j_2}\dots\phi^{j_n)}\right]. \tag{3.49}$$

All the unitary representations of $\mathfrak{psu}(2,2|4)$ have been classified by Dobrev and Petkova according to the quantum numbers of the following bosonic subgroup of $\mathfrak{so}(4,2) \times \mathfrak{so}(6)$:

$$\overbrace{\underset{\Delta}{\mathfrak{so}(1,1)} \times \underset{s_\pm}{\mathfrak{so}(1,3)}}^{\mathfrak{so}(4,2)} \times \overbrace{\underset{[r_1,r_2,r_3]}{\mathfrak{su}(4)}}^{\mathfrak{so}(6)}, \tag{3.50}$$

where $[r_1, r_2, r_3]$ are the Dynkin labels of the $\mathfrak{su}(4)$ representation. There exist four distinct series of representations, three of which are BPS (i.e. they contain a chiral primary) and one is non-BPS (without any chiral operator). All local gauge-invariant single-trace operators of $\mathcal{N} = 4$, $\mathfrak{su}(N_c)$ SYM theory are classified according to the Dobrev-Petkova scheme (3.50).

On the string theory side, a similar classification applies. String states on $\text{AdS}_5 \times \text{S}^5$ are characterized by six conserved charges: their energy E, their AdS spins $S_{1,2}$ and their S^5 spins $J_{1,2,3}$. These charges correspond to the cyclic coordinates (5.6) of the bosonic string action on $\text{AdS}_5 \times \text{S}^5$ and are in one-to-one correspondence with the operator classification (3.50) of $\mathcal{N} = 4$ SYM:

$$\overbrace{\underset{E}{\mathfrak{so}(1,1)} \times \underset{S_1,S_2}{\mathfrak{so}(1,3)}}^{\text{AdS}_5 \sim \mathfrak{so}(4,2)} \times \overbrace{\underset{J_1,J_2,J_3}{\mathfrak{su}(4)}}^{\text{S}^5 \sim \mathfrak{so}(6)}. \tag{3.51}$$

[17] In (3.49), the parentheses ( ) denote symmetrization with respect to the indices $j_1, j_2, \dots, j_n$.

In the classical supergravity limit ($N_c, \lambda \to \infty$) the spectrum of 1/2 BPS operators of $\mathcal{N} = 4$ SYM is completely matched by that of IIB supergravity compactified on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Thus BPS operators are dual to free point-like strings ($g_s, \ell_s \to 0$). For details and further references, the reader is referred to the reviews [9, 24].

The spectra have also been found to match in the BMN limit (3.33). As we have already explained, the string sigma model can be solved on plane-wave backgrounds and the energies of the free string states that are found are in complete agreement with the calculated dimensions of their dual gauge theory operators. The BMN operators are 'almost' protected—therefore 'almost' BPS—and their dual free string states are 'nearly' point-like. For more on this topic, the reader is referred to the original paper [43] and the reviews [44] of the plane-wave/SYM duality.

Beyond the BPS and BMN operators, it is integrability that comes into play, providing the tools for a complete solution of the spectral problem. We will have more to say about this in the next section. Even such a powerful technique as integrability has its shortcomings and an input from other methods is still necessary. For example there exist some regimes where the equations coming from integrability are completely intractable. What is more, the state/operator correspondence gets a bit blurred with integrability. Also, only a few methods have so far been able to provide closed formulas for the calculated spectra. In part II of this thesis we are going to describe the state of affairs in precisely these regions where integrability by itself is not enough and we will propose a possible method to proceed.

3.10.3 Correlation Functions

The computation of correlation functions in AdS/CFT essentially makes use of the fact that conformal symmetry completely determines 2 and 3-point correlation functions. For example, the correlator of two scalar (single-trace) primary operators $\mathcal{O}_{(i,j)}(x)$, having scaling dimensions Δ_i and Δ_j is given by [50]:

$$\langle \mathcal{O}_i (x_i) \mathcal{O}_j (x_j) \rangle = \frac{\delta_{ij}}{x_{ij}^{2\Delta_i}}. \tag{3.52}$$

The corresponding 3-point function is:

$$\langle \mathcal{O}_i (x_i) \mathcal{O}_j (x_j) \mathcal{O}_k (x_k) \rangle = \frac{C_{ijk}}{|x_{ij}|^{\Delta_i+\Delta_j-\Delta_k} |x_{jk}|^{\Delta_j+\Delta_k-\Delta_i} |x_{ki}|^{\Delta_k+\Delta_i-\Delta_j}}, \tag{3.53}$$

where C_{ijk} are the structure constants and Δ_k are the scaling dimension of $\mathcal{O}_k(x)$. At the planar limit ($N_c \to \infty$), the constants C_{ijk} generally admit the following weak-coupling expansion:

$$C_{ijk} = c_{ijk}^{(0)} + \lambda \cdot c_{ijk}^{(1)} + \lambda \cdot c_{ijk}^{(2)} + \ldots, \tag{3.54}$$

Higher-point correlation functions are computed from the 2 and 3-point ones by using the operator product expansion (OPE). The problem of actually computing correlation functions in the AdS/CFT correspondence therefore reduces to that of computing OPE coefficients of the form (3.54), at either weak ($\lambda \to 0$) or strong coupling ($\lambda \to \infty$).

While at weak coupling one may proceed perturbatively by computing the corresponding Feynman diagrams, the evaluation of correlation functions at strong coupling is done by using the string description through (3.47). Denoting by W the generating functional of connected gauge theory Green's functions and by S_{string} the IIB string theory action on $\mathrm{AdS}_5 \times \mathrm{S}^5$, we have:

$$e^{-S_{\text{string}}[\Phi(x,y=0)]} = \mathcal{Z}_{\text{string}} [\Phi (x, y = 0)] = \mathcal{Z}_{\text{CFT}} [\phi (x)] = e^{-W[\phi(x)]}, \tag{3.55}$$

with the boundary of AdS located at $y = 0$. The next step involves the renormalization of the Euclideanized bulk action $S^{(\text{ren})}_{\text{string}}$ and the solution of the bulk equations of motion that are obtained by minimizing it. Imposing the proper boundary conditions at $y = 0$ and substituting the solution into the renormalized action we get:

$$W\left[\phi\left(x\right)\right] \approx S^{(\text{ren})}_{\text{string}}\left[\Phi^{(\text{E})}_{\text{C}}\left(x, 0\right)\right] = S^{(\text{ren})}_{\text{sugra}}\left[\Phi^{(\text{E})}_{\text{C}}\left(x, 0\right)\right] + O\left(\alpha'\right), \tag{3.56}$$

since S_{string} is just the IIB supergravity action on $\text{AdS}_5 \times \text{S}^5$ plus α' corrections. Thus in the classical supergravity approximation (3.30),

$$W\left[\phi\left(x\right)\right] \approx S^{(\text{ren})}_{\text{sugra}}\left[\Phi^{(\text{E})}_{\text{C}}\left(x, 0\right)\right]. \tag{3.57}$$

The n-point connected correlation function between the boundary operators $\mathcal{O}\left(x\right)$ is then calculated by the formula:

$$\left\langle \mathcal{O}_1\left(x_1\right)\mathcal{O}_2\left(x_2\right)\ldots\mathcal{O}_n\left(x_n\right)\right\rangle = \left.\frac{\delta S^{(\text{ren})}_{\text{string}}\left[\Phi^{(\text{E})}_{\text{C}}\right]}{\delta\phi\left(x_1\right)\delta\phi\left(x_2\right)\ldots\delta\phi\left(x_n\right)}\right|_{\phi=0}. \tag{3.58}$$

The matching of correlation functions as calculated from both sides of the AdS/CFT correspondence has been achieved in many cases, e.g. for correlators of BPS operators. But since this is largely a subject of intense current interest, we will defer any further discussion of it and refer the interested reader to the existing literature.

3.10.4 Anomalies, Moduli Spaces, etc.

Another test of the $\text{AdS}_5/\text{CFT}_4$ correspondence is the matching of the anomalies[18] that arise when $\mathcal{N} = 4$ SYM is coupled to external gravitational or gauge fields. Deforming the gauge theory by an $\mathfrak{su}\left(4\right) \cong \mathfrak{so}\left(6\right)$ global current, an (axial) anomaly of the Adler-Bell-Jackiw type ensues, while $\mathfrak{so}\left(4, 2\right)$ currents produce a Weyl/conformal anomaly. Both anomalies may be reproduced on both sides of the duality to leading order in $1/N_c$, providing a valuable confirmation of the correspondence.

Many more compatibility tests between the two theories exist, e.g. the matching of the moduli space of $\mathcal{N} = 4$ SYM, namely $\mathcal{M} = \mathbb{R}^{6(N_c-1)}/\mathcal{S}_{N_c}$,[19] with that of string theory on $\text{AdS}_5 \times \text{S}^5$, the behavior of the two theories under deformations or finite temperature, etc. For more, a nice starting point is the review [9].

[18] An anomaly is the violation of a classical symmetry at the quantum level.
[19] $\mathcal{S}_{N_c}$ is the permutation group of N_c elements.

4 AdS/CFT Integrability

4.1 Classical & Quantum Integrability

4.1.1 Classical Integrability

We will consider classical integrability in the sense of *Liouville integrability*. Generally speaking, a classical Hamiltonian system is Liouville integrable when it possesses a maximal set of Poisson commuting invariants. In a finite, even-dimensional symplectic phase space, the coordinates q_i and the momenta p_i satisfy:

$$\{q_i, p_j\} = \delta_{ij}, \quad i, j = 1, 2, \ldots, M \in \mathbb{Z}, \tag{4.1}$$

where the Poisson bracket $\{\_, \_\}$ is defined as

$$\{f\,,\, g\} \equiv \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}. \tag{4.2}$$

The coordinates q_i and the momenta p_i also satisfy Hamilton's equations of motion:

$$\dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}. \tag{4.3}$$

We will say that an autonomous $2M$-dimensional Hamiltonian system with Hamiltonian H is completely integrable if and only if (iff) there exist M independently conserved quantities I_i such that:[20]

$$\{I_i, H\} = \{I_i, I_j\} = 0, \quad \forall\ i, j = 1, 2, \ldots, M \in \mathbb{Z}. \tag{4.4}$$

Due to a theorem of 1855 by Bour and Liouville, the solution of every classically integrable system may be obtained by finitely many algebraic operations (including inversions) and quadratures (i.e. integrations). We note however that no general criterion for deciding whether a given system is completely integrable exists. For more on the topic of classical integrability we refer the interested reader to the book by Perelomov [51].

4.1.2 Quantum Integrability

Quantum integrability comes up when we canonically quantize a classical system. Replacing all the functions of coordinates by local operators and the Poisson brackets { , } by commutators [,] so that

$$\{f\,,\, g\} \to i\hbar\left[\hat{F}\,,\,\hat{G}\right], \tag{4.5}$$

the position and momentum observables $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$ satisfy the canonical commutation relation:

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad i, j = 1, 2, \ldots, M \in \mathbb{Z}. \tag{4.6}$$

A $2M$-dimensional Hamiltonian system with Hamiltonian $\hat{H}$ is quantum integrable iff there exist M mutually commuting operators $\hat{I}_i$ such that:

$$\left[\hat{I}_i, \hat{H}\right] = \left[\hat{I}_i, \hat{I}_j\right] = 0, \quad \forall\ i, j = 1, 2, \ldots, M \in \mathbb{Z}. \tag{4.7}$$

As argued in [52], this definition of quantum integrability is incomplete because it is at odds with the classical expectation (following the Bour-Liouville theorem) that integrability implies complete solvability of a system. A more precise definition of quantum integrability for systems that support particle scattering, precludes diffractive scattering between particles [53]. This definition of quantum

[20] If only the first of these two conditions holds but not the second, the system is called integrable.

integrability is very closely related to the absence of particle production/annihilation and the factorizability of the corresponding scattering matrix that will be discussed below.

Yet another more precise definition of quantum integrability may be given in terms of a Lax pair. If two operators $\hat{A}$ and $\hat{L}$ (Lax pair) can be found so that (henceforth $\hbar = 1$)

$$\frac{d\hat{L}}{dt} = i\left[\hat{L}, \hat{H}\right] = i\left[\hat{A}, \hat{L}\right], \tag{4.8}$$

it can be proved that the matrix

$$\hat{D}\left(u\right) \equiv \det\left(u\hat{\mathbb{I}} + \hat{L}\left(t\right)\right), \tag{4.9}$$

satisfies

$$\left[\hat{D}\left(u\right), \hat{H}\right] = \left[\hat{D}\left(u\right), \hat{D}\left(u'\right)\right] = 0. \tag{4.10}$$

If in addition,

$$\hat{D}\left(u\right) = \sum_{n=0}^{\infty} \hat{Q}_n u^n. \tag{4.11}$$

$\hat{Q}_n\left(u\right)$ are local conserved charges:

$$\left[\hat{Q}_n, \hat{H}\right] = \left[\hat{Q}_m, \hat{Q}_n\right] = 0, \quad m, n = 1, 2, \ldots. \tag{4.12}$$

Hence this method offers the possibility of constructing all the conserved charges that appear in the definition (4.7).

4.2 Integrability in $\mathbf{AdS_5/CFT_4}$

Coset space sigma models have been known to be classically integrable since the work of Lüscher and Pohlmeyer [54, 55], Zakharov and Mikhailov [56] and Eichenherr and Forger [57] in the late 1970's. Classical integrability of the bosonic sector of IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ was established by Mandal, Suryanarayana and Wadia in [58]. Bena, Polchinski and Roiban [10] proved that the full kappa-invariant IIB superstring action on $\mathrm{AdS}_5 \times \mathrm{S}^5$, defined as a two-dimensional nonlinear sigma model on the coset

$$\frac{\mathfrak{psu}\left(2,2|4\right)}{\mathfrak{so}\left(4,1\right) \times \mathfrak{so}\left(5\right)}, \tag{4.13}$$

is classically integrable.

According to what has been said in the previous subsection, in order to establish integrability at the quantum level, non-diffractive scattering or equivalently the absence of particle production/annihilation and factorization of the S-matrix have to be proven. Although it is generally very difficult, and so far it has been impossible to formally prove quantum integrability for either the planar ($N_c \to \infty$) $\mathcal{N} = 4$, $\mathfrak{su}\left(N_c\right)$ SYM theory or the free ($g_s \to 0$) IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$, it is usually assumed that either theory is quantum integrable. The implications of this assumption are then investigated for agreement or possible discrepancies. There is currently a consensus that both theories are quantum integrable at the planar/free string limit. Beyond the planar level, there exist indications that integrability breaks down for finite values of N_c.

Solving a theory means that we are able to compute all of its observables: spectrum, correlation functions, scattering amplitudes, Wilson loops. In the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, integrability provides computational methods for solving the theory in the above sense. Moreover, it is claimed that the spectral problem of both planar $\mathcal{N} = 4$ SYM and free IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is fully solved by the assumption of integrability [59]. "Solving" the spectral problem is taken to mean that the full set of algebraic equations that is needed in order to determine the scaling dimensions of all the local gauge-invariant operators of planar $\mathcal{N} = 4$ SYM or the energies of all the free superstring states on $\mathrm{AdS}_5 \times \mathrm{S}^5$ as a function of the 't Hooft coupling λ, is known.

4.3 Integrability in the $\mathfrak{su}(2)$ Sector

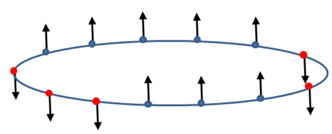
In this subsection we are going to briefly review integrability in the $\mathfrak{su}(2)$ sector of $\mathcal{N}=4$ SYM, following the very nice pedagogical reviews of Plefka [60], Dorey [61] and Minahan [62]. The $\mathfrak{su}(2)$ sector consists of the single-trace operators

$$\mathcal{O}^{(J,M)} = \mathrm{Tr}\left[\mathcal{Z}^J \mathcal{X}^M\right] + \ldots, \quad L \equiv J + M, \tag{4.14}$$

where $\mathcal{Y}$, $\mathcal{X}$, $\mathcal{Z}$ are the three complex scalar fields of $\mathcal{N}=4$ SYM, composed out of the six real scalars ϕ of the theory (3.8). As we will see below, where we will explicitly construct the exact form of the operators (4.14), the dots stand for the permutations of the fields inside the trace while each term must be multiplied by a suitable coefficient. We will also see below that this sector is dual to (closed) strings that rotate in $\mathbb{R}\times \mathrm{S}^3 \subset \mathrm{AdS}_5 \times \mathrm{S}^5$.

For reasons that will become apparent in what follows, it is very practical to regard the complex fields $\mathcal{Z}$ in (4.14) as the ground state fields (spin up) and $\mathcal{X}$ as some sort of impurities (spin down) in a spin chain. Owing to the cyclic property of traces this spin chain must be closed. Its length is L, its spin J, while M is its number of magnons. E.g. a permutation of a $(L,J,M)=(13,8,5)$ spin chain is

$$\mathrm{Tr}\left[\mathcal{Z}^5\mathcal{X}^2\mathcal{Z}^3\mathcal{X}^3\right] \quad \longleftrightarrow$$



$$= \quad |\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\rangle.$$

In order to solve the spectral problem in the $\mathfrak{su}(2)$ sector of planar $\mathcal{N}=4$ SYM we have to compute the scaling dimensions of the operators (4.14) for all the values of the coupling λ. One way to proceed is to compute their two-point function:

$$\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle = \frac{\text{const.}}{|x-y|^{2\Delta}}, \tag{4.15}$$

where Δ are the scaling dimensions of $\mathcal{O}(x)$ since by definition,

$$x' = \alpha x \quad \to \quad \mathcal{O}(\alpha x) = \alpha^{-\Delta}\mathcal{O}(x)\,. \tag{4.16}$$

Note however that the exact form of the operators $\mathcal{O}$ has not been specified yet. We only know that they have to be of the form (4.14).

The two-point function (4.15) may be evaluated in perturbation theory by the standard Feynman-diagrammatic methods. To deal with the divergences that appear from one loop on, we introduce a UV cutoff Λ and subtract the divergent parts. Effectively we define a renormalized operator

$$\mathcal{O}^A_{\text{ren}} = Z^A_B \cdot \mathcal{O}^B_{\text{bare}}, \tag{4.17}$$

that mixes all the $\mathfrak{su}(2)$ bare operators. The matrix of scaling dimensions is then given by

$$\mathbb{D} = \frac{dZ}{d\log\Lambda}\cdot Z^{-1} \tag{4.18}$$

and generates dilatations in the $\mathfrak{su}(2)$ sector

$$\mathbb{D}\cdot\mathcal{O}^{(J,M)} = \Delta\,\mathcal{O}^{(J,M)}, \tag{4.19}$$

in the sense that its eigenvectors are well-defined operators of the form (4.14) and the corresponding eigenvalues are their scaling dimensions. As we have just said these get renormalized at one loop order, so that their bare/tree-level values typically receive loop-corrections:[21]

$$\Delta = \Delta_0 + \sum_{n=1}^{\infty} \lambda^n \gamma_n. \tag{4.20}$$

The renormalized minus the bare part of the scaling dimensions is known as the anomalous dimension of the corresponding operator and is usually denoted by γ:

$$\gamma \equiv \Delta - \Delta_0 = \sum_{n=1}^{\infty} \lambda^n \gamma_n. \tag{4.21}$$

Each loop correction contributes an extra λ term, so that the n-th loop gets multiplied by λ^n. Since the scaling dimension of $\mathcal{N} = 4$ scalars is equal to one, the bare dimension of the operators (4.14) is just equal to their length L. To obtain the anomalous part, Minahan and Zarembo followed the steps that were outlined above and proved in 2002 [63] that the one-loop dilatation operator of the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM is the Hamiltonian of the Heisenberg $\text{XXX}_{1/2}$ quantum spin chain:

$$\mathbb{D} = L \cdot \mathbb{I} + \frac{\lambda}{8\pi^2} \mathbb{H} + \sum_{n=2}^{\infty} \lambda^n \mathbb{D}_n, \quad \mathbb{H} = \sum_{j=1}^{L} \left(\mathbb{I}_{j,j+1} - \mathbb{P}_{j,j+1}\right) = 2 \sum_{j=1}^{L} \left(\frac{1}{4} - \mathbf{S}_j \cdot \mathbf{S}_{j+1}\right). \tag{4.22}$$

$\boldsymbol{\sigma}$ are the standard Pauli matrices

$$\mathbf{S} \equiv \frac{\boldsymbol{\sigma}}{2} = \frac{1}{2}\left(\sigma_x, \sigma_y, \sigma_z\right), \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.23}$$

and the indices j or $j+1$ in (4.22) indicate that the corresponding matrix acts only on positions j or $j+1$ of the corresponding spin vector:

$$|\uparrow \ldots \underset{j}{\uparrow} \underset{j+1}{\downarrow} \ldots \uparrow\rangle, \quad \uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.24}$$

$\mathbb{I}_{i,j}$ and $\mathbb{P}_{i,j}$ are the spin-identity and spin-exchange operators defined respectively as

$$\left(\mathbb{I}_{i,j}\right)_{abcd} \equiv \left(\delta_{ab}\right)_i \left(\delta_{cd}\right)_j, \qquad \mathbb{P}_{i,j} \equiv \frac{1}{2}\left(\mathbb{I}_{i,j} + \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\right). \tag{4.25}$$

One may prove that their action is the following:

$$\mathbb{I}_{i,j} |\uparrow \ldots \underset{i}{\uparrow} \ldots \underset{j}{\downarrow} \ldots \uparrow\rangle = |\uparrow \ldots \underset{i}{\uparrow} \ldots \underset{j}{\downarrow} \ldots \uparrow\rangle, \tag{4.26}$$

$$\mathbb{P}_{i,j} |\uparrow \ldots \underset{i}{\uparrow} \ldots \underset{j}{\downarrow} \ldots \uparrow\rangle = |\uparrow \ldots \underset{i}{\downarrow} \ldots \underset{j}{\uparrow} \ldots \uparrow\rangle. \tag{4.27}$$

4.3.1 Coordinate Bethe Ansatz

The spin chain (4.22) may be diagonalized by the (coordinate) Bethe ansatz (BA), found by Bethe in 1931 [64]. It is convenient at first to ignore the trace condition in (4.14) and temporarily replace the corresponding closed spin chain with a periodic one having period equal to its (finite or infinite) length $L = J + M$. When we are done with the calculation we shall impose the trace condition by demanding that our results are invariant under cyclic permutations of the spin chain.

The vacuum state of the Heisenberg ferromagnet corresponds to $\mathbf{M} = 0$ magnon operators:

$$\text{Tr}\left[\mathcal{Z}^L\right] \sim |0\rangle = |\uparrow \ldots \uparrow\uparrow\uparrow \ldots \uparrow\rangle. \tag{4.28}$$

[21] Also known as curvature or α' corrections.

$\mathrm{Tr}\left[\mathcal{Z}^L\right]$ are in fact protected operators (aka chiral primary or BPS) so that their scaling dimensions are unrenormalized:

$$\Delta = J = L. \tag{4.29}$$

The BA is applied to $\mathbf{M} = 1$ magnon operators:

$$\mathrm{Tr}\left[\mathcal{Z}^J \mathcal{X}\right] \sim |x\rangle = |\uparrow \ldots \uparrow \underset{x}{\downarrow} \uparrow \ldots \uparrow\rangle. \tag{4.30}$$

These are diagonalized by the following Fourier transformation:

$$|p\rangle = \sum_{x=1}^{J+1} e^{ipx}|x\rangle \quad \longrightarrow \quad \Delta = J + 1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2} + O\left(\lambda^2\right), \tag{4.31}$$

where the scaling dimensions Δ are the eigenvalues of the dilatation operator (4.22) with corresponding eigenvector $|p\rangle$. Because of the periodicity of the spin chain, the Fourier coefficients should be periodic

$$e^{ipL} = 1. \tag{4.32}$$

$\mathbf{M} = 2$ magnon operators,

$$\mathrm{Tr}\left[\mathcal{Z}^J \mathcal{X}^2\right] \sim |x_1, x_2\rangle = |\uparrow \ldots \uparrow \underset{x_1}{\downarrow} \uparrow \ldots \uparrow \underset{x_2}{\downarrow} \uparrow \ldots \uparrow\rangle \tag{4.33}$$

are diagonalized by the eigenvectors:

$$|p_1, p_2\rangle = \sum_{x_2 > x_1 = 1}^{J+2} \left(e^{ip_1x_1 + ip_2x_2} + S_{21}e^{ip_1x_2 + ip_2x_1}\right)|x_1, x_2\rangle. \tag{4.34}$$

This wave function is the sum of two terms. The first represents two incoming magnons with momenta p_1, p_2, while in the second term the magnons have scattered by exchanging their momenta and have acquired a phase shift S_{21}. S_{21} is known as the S-matrix of the scattering process. This and the corresponding eigenvalues/scaling dimensions are found to be:

$$\Delta = J + 2 + \frac{\lambda}{2\pi^2}\sum_{j=1,2}\sin^2\frac{p_j}{2} + O\left(\lambda^2\right), \quad S_{12} = \frac{u_1 - u_2 + i}{u_1 - u_2 - i} = S_{21}^{-1}, \quad u_j = \frac{1}{2}\cot\frac{p_j}{2}, \tag{4.35}$$

where u_j is known as the rapidity of the jth magnon. The periodic b.c.'s give what is known as the Bethe ansatz equations (BAEs):

$$e^{ip_1L} = S_{12}, \quad e^{ip_2L} = S_{21}. \tag{4.36}$$

For operators with $\mathbf{M} > 2$ magnons,

$$\mathrm{Tr}\left[\mathcal{Z}^J \mathcal{X}^M\right] \sim |x_1, x_2, \ldots, x_M\rangle = |\uparrow \ldots \uparrow \underset{x_1}{\downarrow} \uparrow \ldots \uparrow \underset{x_2}{\downarrow} \uparrow \ldots \uparrow \underset{x_M}{\downarrow} \uparrow \ldots \uparrow\rangle \tag{4.37}$$

the Bethe ansatz generalizes the previous ones:

$$|p_1, p_2, \ldots, p_M\rangle = \sum_{x_M > \ldots > x_1 = 1}^{J+M}\left[\sum_{\sigma} S_{\sigma(1,2,\ldots,M)}e^{ip_jx_{\sigma_j}}\right]|x_1, x_2, \ldots, x_M\rangle, \tag{4.38}$$

where $\sigma(1, 2, \ldots, M)$ stands for a permutation of $(1, 2, \ldots, M)$ and the sum is over all permutations.[22] The corresponding eigenvalues are:

$$\Delta = J + M + \frac{\lambda}{2\pi^2}\sum_{j=1}^{M}\sin^2\frac{p_j}{2} + O\left(\lambda^2\right). \tag{4.39}$$

[22]Note also that $S_{12\ldots M} = 1$.

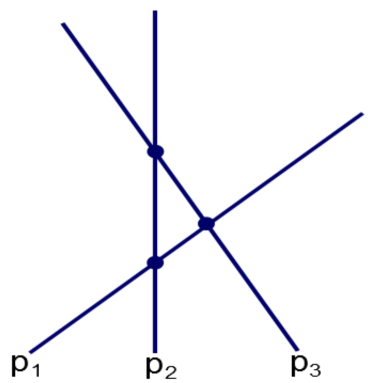
The periodicity of the spin chain leads to the following Bethe ansatz equations:

$$e^{ip_jL} = \prod_{\substack{k=1 \\ k\neq j}}^{M} S_{jk}\,,\ j = 1, 2, \ldots, M. \tag{4.40}$$

The important new feature here is the factorization of the M-magnon S-matrix into a product of 2-magnon S-matrices. This in particular implies that the system is quantum integrable. To see how this comes about, note that in 2-magnon scattering the individual particle momenta are conserved and not just their sum. The same must hold true for M-magnon scattering, whenever the S-matrix factorizes:

$$\{p_1', p_2', \ldots, p_M'\} = \{p_1, p_2, \ldots, p_M\}\,, \tag{4.41}$$

therefore we obtain M conservation laws with $0 \leq M \leq L$, meaning that the system is integrable. Integrability also has many other interesting consequences on the properties of the S-matrix, such as the Yang-Baxter relation.



We are now ready to impose the trace condition that we have suspended so far. Because of the cyclic property of traces, the Bethe eigenvectors (4.38) must be invariant under cyclic permutations of the spin chain. Thanks to factorized scattering, we only have to examine the two-magnon case. The result is that the total magnon momentum should vanish:

$$\sum_{j=1}^{M} p_j = 0. \tag{4.42}$$

Thus the only physical one-magnon states have vanishing momentum

$$|1\rangle = \sum_{k=0}^{J} \mathrm{Tr}\left[\mathcal{Z}^k \mathcal{X} \mathcal{Z}^{J-k}\right], \qquad \Delta = J+1, \qquad p = 0 \tag{4.43}$$

and are protected (BPS) operators. Two-magnon states may also be constructed by plugging $p = p_1 = -p_2$ into the BAEs (4.36):

$$|2\rangle = Le^{-i\pi n/L-1} \cdot \sum_{k=0}^{J} \cos\left[\frac{\pi n}{L-1}(2k+1)\right] \cdot \mathrm{Tr}\left[\mathcal{X}\mathcal{Z}^k \mathcal{X} \mathcal{Z}^{J-k}\right], \quad n \in \mathbb{Z}. \tag{4.44}$$

The scaling dimensions and the quantized momentum are

$$\Delta = L + \frac{\lambda}{\pi^2}\sin^2\left(\frac{\pi n}{L-1}\right) + O\left(\lambda^2\right), \quad p = \frac{2\pi n}{L-1}, \quad n \in \mathbb{Z}. \tag{4.45}$$

4.3.2 Asymptotic Bethe Ansatz

Higher-loop contributions to the dilatation operator have been explicitly calculated in perturbation theory up to four-loops:

$$\mathbb{D}_2 = \frac{1}{64\pi^4} \sum_{j=1}^{L} \left(-\mathbf{S}_j \cdot \mathbf{S}_{j+2} + 4\, \mathbf{S}_j \cdot \mathbf{S}_{j+1} - \frac{3}{4} \right) \tag{4.46}$$

$$\mathbb{D}_3 = \frac{1}{1024\pi^6} \sum_{j=1}^{L} \Big\{ -\mathbf{S}_j \cdot \mathbf{S}_{j+3} + 4 \left(\mathbf{S}_j \cdot \mathbf{S}_{j+2}\right) \left(\mathbf{S}_{j+1} \cdot \mathbf{S}_{j+3}\right) - 4 \left(\mathbf{S}_j \cdot \mathbf{S}_{j+3}\right) \left(\mathbf{S}_{j+1} \cdot \mathbf{S}_{j+2}\right) + \\ +10\, \mathbf{S}_j \cdot \mathbf{S}_{j+2} - 29\, \mathbf{S}_j \cdot \mathbf{S}_{j+1} + 5 \Big\}. \tag{4.47}$$

Obviously these become more and more complicated as the loop-order is increased, involving non-neighboring as well as higher-order interactions (e.g. $\mathbf{S}^4$ in $\mathbb{D}_3$). Details about these calculations and many original references may be found in the reviews [65].

As we have said however, integrability is thought to be an all-loop property of planar $\mathcal{N} = 4$ SYM. Based on this fundamental assumption and using the properties of the dilatation operator in the BMN limit (3.33), Beisert, Kristjansen and Staudacher [66] calculated it in two, three and four loops. With the same assumptions, the five-loop formula was also computed by Beisert, Dippel and Staudacher (BDS) in [67]. BDS's bold new proposal was that the form of the planar $\mathcal{N} = 4$ SYM dilatation operator is completely determined by integrability and BMN scaling. Based on that they provided an all-loop, asymptotic Bethe ansatz (ABA):

$$\Delta = J + M + \frac{\lambda}{8\pi^2} \sum_{j=1}^{M} E\left(p_j\right) \ , \quad E\left(p_j\right) = \frac{8\pi^2}{\lambda} \left[\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}} - 1 \right], \quad j = 1, 2, \ldots, M \tag{4.48}$$

$$e^{i p_j L} = \prod_{\substack{k=1 \\ k \neq j}}^{M} S_{jk} \ , \quad S_{jk} = \frac{u_j - u_k + i}{u_j - u_k - i} \cdot \mathcal{S}^D_{jk} \ , \quad u\left(p_j\right) = \frac{1}{2} \cot \frac{p_j}{2} \sqrt{1 + \lambda \sin^2 \frac{p_j}{2}}. \tag{4.49}$$

Asymptotic means that there's a *critical* loop order equal to the length of the spin-chain L at which the ABA ceases to hold. At this loop order the range of the spin chain interactions first exceeds the length of the chain (virtual particles start circulating around the spin chain) and the so-called wrapping corrections have to be taken into account. In fact they correspond to higher genus corrections to the dilatation operator that we neglect in the planar approximation. In the dual string theory side, the wrapping effects are due to the finite circumference of the cylindrical worldsheet.[23]

$\mathcal{S}^D_{jk}$ in equation (4.49) is known as the dressing phase/factor:

$$\mathcal{S}^D_{jk} = \sigma^2\left(p_j, p_k\right). \tag{4.50}$$

The dressing phase is introduced in order to reconcile the weak and strong coupling limits in the ABA. At weak coupling it is equal to unity up to 3-loops:

$$\sigma^2_{jk(\text{weak})} = 1 + O\left(\lambda^3\right). \tag{4.51}$$

At strong coupling it is given by the so-called Arutyunov-Frolov-Staudacher (AFS) [68] phase:

$$\sigma^2_{jk(\text{strong})} = \sigma^2_{jk(\text{AFS})}. \tag{4.52}$$

[23]For more on this issue, see footnote 37.

For $M = 2$ magnons at strong coupling ($\lambda \to \infty$) we may calculate the AFS phase exactly, finding

$$\sigma^2_{(\mathrm{AFS})}(p_1, p_2) = \exp\left\{ i\,\frac{\sqrt{\lambda}}{\pi}\left(\cos\frac{p_1}{2} - \cos\frac{p_2}{2}\right) \cdot \log\left[\frac{\sin^2(p_1 - p_2)/4}{\sin^2(p_1 + p_2)/4}\right]\right\}. \tag{4.53}$$

Part II
Spinning Strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$

5 Introduction and Motivation

As we have explained in the introduction, the AdS/CFT correspondence (3.1) implies that the spectra of $\mathcal{N} = 4$, $\mathfrak{su}(N_c)$ SYM theory and IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ should match, at least in the planar/free-string approximation. Indeed the two spectra have been found to agree in both the BPS and the BMN limits, providing substantial support to AdS/CFT. Beyond the BPS and BMN limits, planar integrability (thermodynamic BA, Y-system, quantum spectral curve) completely solves the spectral problem of AdS/CFT by producing the complete system of equations that fully determine it.

This is a very powerful verification of the planar AdS/CFT correspondence. The spectra of $\mathcal{N} = 4$ SYM and IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ are described by the same system of functional equations, meaning that they coincide. However, when it comes to actually computing the common spectrum, there are cases where the procedure turns out to be rather technically involved. Besides that, we would also like to have at our disposal tools for computing the spectra of non-integrable models and in cases where integrability is known to break down (e.g. QCD, non-planar $\mathcal{N} = 4$ SYM, p-branes).

Secondly we would like to address some of the traditional questions of AdS/CFT, like what's the AdS/CFT dictionary. From the state/operator correspondence we know that to each operator of the gauge theory there corresponds a dual IIB string state. As we have already mentioned, there are two main obstructions with the state/operator mapping, the elusive quantization of strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the technical difficulties with the computation of the SYM spectrum. Therefore no systematic procedure which assigns a gauge theory operator to every string state exists and the state/operator identification proceeds so far only heuristically.

This means that we could use our option and compute the spectra in order to conclude that a certain string state is dual to a gauge theory operator. The bonus is that we simultaneously test the AdS/CFT correspondence explicitly. As it will become apparent in what follows, the spectra must be expressed in an appropriate form to actually be of use. String energies and operator dimensions must be expressed in terms of the conserved charges and the corresponding quantum numbers. Only in this way can the energies of classical strings, valid at strong coupling λ, accommodate quantum corrections (i.e. α' or curvature corrections) and be compared to the corresponding weak-coupling results.

Finally, our approach brings closer the possibility of obtaining closed formulas for the string energies and the dual operator dimensions at strong coupling. While for the moment this appears very ambitious (even at the classical level), with sufficient ingenuity it could become more tractable. It is not at all obvious that we will always be able to transform chaotic expressions with uncorrelated random coefficients that follow a completely unpredictable and irregular pattern into an ordered and structured ensemble. Even in those happy circumstances where such an eventuality is allowed from the problem itself, it is not at all evident that it is also feasible with the computational and analytical tools that we have at our disposal.

In §3.10.2, we have described 4 steps to obtain the matching between the spectra of the two theories. As we have just argued, even before wanting to compare the AdS/CFT spectra, we have to develop techniques that permit us to explicitly compute them (in appropriate form!). The purpose of part II of this thesis is therefore twofold:

1. Compute the anomalous dimensions of $\mathcal{N} = 4$ SYM operators at strong coupling using strings.
2. If possible, find closed formulas in the dual string spectrum.

5.1 Classical Bosonic Strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$

Since we are going to heavily employ them in what follows, let us set our conventions for the study of strings right away. Consider the motion of classical closed and uncharged bosonic strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$:

$$\begin{aligned} Y_{05} &= Y_0 + iY_5 = \ell \cosh\rho \, e^{it} & & X_{12} = X_1 + iX_2 = R\cos\overline{\theta}_1 \, e^{i\overline{\phi}_1} \\ Y_{12} &= Y_1 + iY_2 = \ell \sinh\rho \cos\theta \, e^{i\phi_1} & \& \qquad & X_{34} = X_3 + iX_4 = R\sin\overline{\theta}_1 \cos\overline{\theta}_2 \, e^{i\overline{\phi}_2} \\ Y_{34} &= Y_3 + iY_4 = \ell \sinh\rho \sin\theta \, e^{i\phi_2} & & X_{56} = X_5 + iX_6 = R\sin\overline{\theta}_1 \sin\overline{\theta}_2 \, e^{i\overline{\phi}_3}, \end{aligned} \tag{5.1}$$

where Y^μ and X^i are the embedding coordinates of $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\rho \geq 0$, $t \in [0,\, 2\pi)$, $\overline{\theta}_1 \in [0,\, \pi]$, and θ, ϕ_1, ϕ_2, $\overline{\theta}_2$, $\overline{\phi}_1$, $\overline{\phi}_2$, $\overline{\phi}_3 \in [0,\, 2\pi)$. The corresponding line element is given by:[24]

$$\begin{aligned} ds^2 = \ell^2\Big[-\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \left(d\theta^2 + \cos^2\theta \, d\phi_1^2 + \sin^2\theta \, d\phi_2^2\right)\Big] + \\ + R^2\left[d\overline{\theta}_1^2 + \cos^2\overline{\theta}_1 \, d\overline{\phi}_1^2 + \sin^2\overline{\theta}_1 \, \left(d\overline{\theta}_2^2 + \cos^2\overline{\theta}_2 \, d\overline{\phi}_2^2 + \sin^2\overline{\theta_2} \, d\overline{\phi}_3^2\right)\right]. \end{aligned} \tag{5.2}$$

The Polyakov action in the conformal gauge ($\gamma_{ab} = \eta_{ab}$) reads:[25]

$$\begin{aligned} S_P = -\frac{T}{2}\int \sqrt{-\gamma}\,\gamma^{ab}\Big[G^{AdS}_{mn}(y)\partial_a y^m \partial_b y^n + G^S_{mn}(x)\partial_a x^m \partial_b x^n\Big] d\tau \, d\sigma = \\ = \frac{T}{2}\int \Big[G^{AdS}_{mn}(y)\left(\dot{y}^m \dot{y}^n - y'^{\,m} y'^{\,n}\right) + G^S_{mn}(x)\left(\dot{x}^m \dot{x}^n - x'^{\,m} x'^{\,n}\right)\Big] d\tau \, d\sigma, \end{aligned} \tag{5.3}$$

where $y^m \equiv (t,\, \rho,\, \theta,\, \phi_1,\, \phi_2)$ and $x^m \equiv \left(\overline{\theta}_1,\, \overline{\theta}_2,\, \overline{\phi}_1,\, \overline{\phi}_2,\, \overline{\phi}_3\right)$. The Virasoro constraints are:

$$T_{00} = T_{11} = \frac{1}{2}\Big[G^{AdS}_{mn}(y)\left(\dot{y}^m \dot{y}^n + y'^{\,m} y'^{\,n}\right) + G^S_{mn}(x)\left(\dot{x}^m \dot{x}^n + x'^{\,m} x'^{\,n}\right)\Big] = 0 \tag{5.4}$$

$$T_{01} = T_{10} = G^{AdS}_{mn}(y)\,\dot{y}^m y'^{\,n} + G^S_{mn}(x)\,\dot{x}^m x'^{\,n} = 0. \tag{5.5}$$

The cyclic coordinates of the action t, ϕ_1, ϕ_2, $\overline{\phi}_1$, $\overline{\phi}_2$, $\overline{\phi}_3$, give rise to the following conserved charges:

$$\begin{aligned} E &= \left|\frac{\partial L}{\partial \dot{t}}\right| = T\ell^2 \int_0^{2\pi} \dot{t}\cosh^2\rho \, d\sigma & J_1 &= \frac{\partial L}{\partial \dot{\overline{\phi}}_1} = TR^2 \int_0^{2\pi} \dot{\overline{\phi}}_1 \cos^2\overline{\theta}_1 \, d\sigma \\ S_1 &= \frac{\partial L}{\partial \dot{\phi}_1} = T\ell^2 \int_0^{2\pi} \dot{\phi}_1 \sinh^2\rho \, \cos^2\theta \, d\sigma & J_2 &= \frac{\partial L}{\partial \dot{\overline{\phi}}_2} = TR^2 \int_0^{2\pi} \dot{\overline{\phi}}_2 \sin^2\overline{\theta}_1 \cos^2\overline{\theta}_2 \, d\sigma \\ S_2 &= \frac{\partial L}{\partial \dot{\phi}_2} = T\ell^2 \int_0^{2\pi} \dot{\phi}_2 \sinh^2\rho \, \sin^2\theta \, d\sigma & J_3 &= \frac{\partial L}{\partial \dot{\overline{\phi}}_3} = TR^2 \int_0^{2\pi} \dot{\overline{\phi}}_3 \sin^2\overline{\theta}_1 \sin^2\overline{\theta}_2 \, d\sigma, \end{aligned} \tag{5.6}$$

on-shell charges consistent with the $\mathfrak{so}\,(4,2) \times \mathfrak{so}\,(6)$ global isometry of $\mathrm{AdS}_5 \times \mathrm{S}^5$.

[24] As we have seen, $R = \ell$ in $\mathrm{AdS}_5 \times \mathrm{S}^5$. However, let us keep our discussion completely general for the time being.
[25] T denotes the string tension, $T \equiv T_1 = 1/2\pi\alpha'$.

Let us also set up the bosonic string formalism in the system of embedding coordinates.[26]

$$ds^2 = \eta_{\mu\nu} dY^\mu dY^\nu + \delta_{ij} dX^i dX^j = -dY_0^2 + \sum_{i=1}^{p+1} dY_i^2 - dY_{p+2}^2 + \sum_{i=1}^{q+1} dX_i^2 \tag{5.7}$$

$$-\eta_{\mu\nu} Y^\mu Y^\nu = Y_0^2 - \sum_{i=1}^{p+1} Y_i^2 + Y_{p+2}^2 = \ell^2 \qquad \& \qquad \delta_{ij} X^i X^j = \sum_{i=1}^{q+1} dX_i^2 = R^2, \tag{5.8}$$

where $\eta_{\mu\nu} = (-, +, \ldots, +, -)$ and $\delta_{ij} = (+, +, \ldots, +, +)$. The string Polyakov action in the conformal gauge ($\gamma_{ab} = \eta_{ab}$) is given by:

$$S_P = \frac{T}{2} \int \Big[\eta_{\mu\nu} \left(\dot{Y}^\mu \dot{Y}^\nu - \acute{Y}^\mu \acute{Y}^\nu \right) + \left(\dot{X}^i \dot{X}^i - \acute{X}^i \acute{X}^i \right) + \tag{5.9}$$

$$+ \Lambda \left(\eta_{\mu\nu} Y^\mu Y^\nu + \ell^2 \right) + \widetilde{\Lambda} \left(X^i X^i - R^2 \right) \Big] d\tau d\sigma. \tag{5.10}$$

The equations of motion and the constraints (Virasoro and Lagrange) in the embedding system of coordinates are:

Equations of Motion	Virasoro Constraints	Lagrange Constraints	
$\ddot{Y}^\mu - (Y^\mu)'' = \Lambda\, Y^\mu,$	$\eta_{\mu\nu} \left(\dot{Y}^\mu \dot{Y}^\nu + \acute{Y}^\mu \acute{Y}^\nu \right) + \dot{X}^i \dot{X}^i + \acute{X}^i \acute{X}^i = 0,$	$\eta_{\mu\nu} Y^\mu Y^\nu = -\ell^2$	(5.11)
$\ddot{X}^i - \left(X^i \right)'' = \widetilde{\Lambda}\, X^i,$	$\eta_{\mu\nu} \dot{Y}^\mu \acute{Y}^\nu + \dot{X}^i \acute{X}^i = 0,$	$X^i X^i = R^2.$	(5.12)

The system (5.10) has the following 15 + 15 conservation laws,

$$S^{\mu\nu} = T \int \left(Y^\mu \dot{Y}^\nu - Y^\nu \dot{Y}^\mu \right) d\sigma, \qquad \mu, \nu = 0, 1, 2, 3, 4, 5 \tag{5.13}$$

$$J^{ij} = T \int \left(X^i \dot{X}^j - X^j \dot{X}^i \right) d\sigma, \qquad i, j = 1, 2, 3, 4, 5, 6, \tag{5.14}$$

which are compatible with the global isometry $\mathfrak{so}(4,2) \times \mathfrak{so}(6)$ of $\text{AdS}_5 \times \text{S}^5$ and the action (5.10).

A very convenient alternative way to express the action (5.10) along with its equations of motion and constraints (5.11)–(5.12) is via the worldsheet light-cone coordinates $\xi_\pm$ that are defined as follows:

$$\xi_\pm = \frac{1}{2} (\tau \pm \sigma) \qquad\qquad \partial_\pm = \partial_\tau \pm \partial_\sigma \tag{5.15}$$

[26] The following index conventions have been used so far and will also be applied in what follows. For a p-dimensional extended object (p-brane) in $D = d + 1$ dimensional spacetime, the brane coordinates, $\sigma_a = \{\tau, \sigma, \delta, \ldots\}$ are denoted by small Latin indices in the series $(a, b, c, \ldots)$. Series $(m, n, r, s, \ldots)$ denotes spacetime coordinates (taking values in the range $0, 1, 2, \ldots d$). The series of Greek letters $(\mu, \nu, \ldots)$ will generally be used in case of a metric with a Minkowski signature, while spatial parts and Euclidean metrics, e.g. the spatial part of spacetime (taking values $1, 2, \ldots d$), the spatial part of the world-sheet/volume $(\sigma, \delta, \ldots)$, scalar fields ϕ, etc. will generally use the indices of the series $(i, j, k, \ldots)$. Small Greek letter coordinates in the series ($\alpha, \beta, \ldots$, dotted or not) usually denote Weyl spinor coordinates, while Fraktur letters $(\mathfrak{a}, \mathfrak{b}, \ldots)$ label Lie group generators. As for units, $c = 1$ is used.

$$\tau = \xi_+ + \xi_- \qquad \longrightarrow \qquad \partial_\tau = \frac{1}{2}\left(\partial_+ + \partial_-\right) \tag{5.16}$$

$$\sigma = \xi_+ - \xi_- \qquad \qquad \partial_\sigma = \frac{1}{2}\left(\partial_+ - \partial_-\right). \tag{5.17}$$

The action is

$$S_P = \frac{T}{2}\int \left[\eta_{\mu\nu}\partial_+ Y^\mu \partial_- Y^\nu + \partial_+ X^i \partial_- X^i + \Lambda\left(\eta_{\mu\nu} Y^\mu Y^\nu + \ell^2\right) + \widetilde{\Lambda}\left(X^i X^i - R^2\right)\right] d\tau d\sigma, \tag{5.18}$$

while the equations of motion and the Virasoro/Lagrange constraints (5.11)–(5.12) become:

Equations of Motion	Virasoro Constraints	Lagrange Constraints	
$\partial_+\partial_- Y^\mu = \Lambda\, Y^\mu,$	$\eta_{\mu\nu}\partial_+ Y^\mu \partial_+ Y^\nu + \partial_+ X^i \partial_+ X^i = 0,$	$\eta_{\mu\nu} Y^\mu Y^\nu = -\ell^2$	(5.19)
$\partial_+\partial_- X^i = \widetilde{\Lambda}\, X^i,$	$\eta_{\mu\nu}\partial_- Y^\mu \partial_- Y^\nu + \partial_- X^i \partial_- X^i = 0,$	$X^i X^i = R^2.$	(5.20)

The formalism we have developed is very useful in proving some important reductions of the classical string sigma model.

5.2 Pohlmeyer Reduction

According to the Pohlmeyer reduction [54], the classical string sigma model in $\mathbb{R}\times \mathrm{S}^2$ can be reduced to the classical sine-Gordon (sG) equation[27] and the string sigma model in $\mathbb{R}\times\mathrm{S}^3$ is classically equivalent to the complex sine-Gordon (CsG) equation. Similar reductions [70, 71] have been carried out for the string sigma models in $\mathrm{AdS}_{2/3/4}$, which can be reduced to the Liouville, sinh-Gordon and B_2-Toda equations respectively. These reductions are summarized in the following table:

String σ-Model	Pohlmeyer Reduction	equation
$\mathbb{R}\times\mathrm{S}^2$	sine-Gordon (sG)	$\partial_+\partial_-\phi + \frac{\ell^2}{R^2}\sin\phi = 0$
$\mathbb{R}\times\mathrm{S}^3$	Complex sine-Gordon (CsG)	$\partial_+\partial_-\psi + \frac{\psi^*}{\ell^2 R^2}\frac{\partial_+\psi\partial_-\psi}{\ell^2 - \lvert\psi\rvert^2} + \frac{\psi}{R^2}\left(\ell^2 - \lvert\psi\rvert^2\right) = 0$
AdS_2	Liouville	$\partial_+\partial_- a - e^a = 0$
AdS_3	sinh-Gordon	$\partial'_+\partial'_-\hat{a} - 2\sinh\hat{a} = 0$
AdS_4	B_2-Toda	$\partial'_+\partial'_-\hat{a} - e^{\hat{a}} - e^{-\hat{a}}\cos b = 0$ $\partial'_+\partial'_- b - e^{-\hat{a}}\sin b = 0$

The Pohlmeyer fields ϕ, ψ, a and $\hat{a}$ are defined by the formulas:[28]

$$\partial_+ X^i \partial_- X^i \equiv \ell^2\cos\phi, \quad \psi \equiv \ell\sin\frac{\phi}{2}\, e^{i\chi/2}\cos\phi \tag{5.21}$$

[27] Mikhailov has shown that the two models are inequivalent at the quantum level [69].
[28] The definition of b can be found in references [70].

$$K_i\partial_\pm^2 X_i = \pm\ell^2\partial_\pm\chi\tan\frac{\phi}{2}\sin\phi, \quad K_i \equiv e_{ijkl}X_j\partial_+X_k\partial_-X_l \tag{5.22}$$

$$\eta_{\mu\nu}\partial_+Y^\mu\partial_-Y^\nu \equiv e^a, \quad \hat{a} \equiv a - \frac{1}{2}\ln\left(-u\cdot v/\ell^2\right), \tag{5.23}$$

where $u = u(\xi_+)$, $v = v(\xi_-)$. The primed coordinates $\xi'_\pm$ are given by

$$\xi'_+ \equiv \xi_+\sqrt{\frac{-u(\xi_+)}{\ell}} \quad \& \quad \xi'_- \equiv \xi_-\sqrt{\frac{v(\xi_-)}{\ell}}. \tag{5.24}$$

As shown by Bakas in 1993 [72], the complex sine-Gordon equation (CsG) can be written as a gauged Wess-Zumino-Witten (gWZW) model in the coset space $\mathfrak{su}(2)/\mathfrak{u}(1)$. As we have seen, the CsG equation is the Pohlmeyer reduction of classical strings in $\mathbb{R}\times\mathrm{S}^3$. One could then ask whether we could take advantage of the fact that the IIB superstring on $\mathrm{AdS}_5\times\mathrm{S}^5$ has the supercoset parametrization (3.10), in order to write down a gWZW model for it on some relevant coset space. This would correspond to the Pohlmeyer reduction of the classical IIB superstring sigma model on $\mathrm{AdS}_5\times\mathrm{S}^5$. The latter has been carried out along the lines we just described in [73, 74]. The gWZW model is defined upon the coset

$$\frac{G}{H} = \frac{\mathfrak{so}(4,1)\times\mathfrak{so}(5)}{\mathfrak{su}(2)^4} \tag{5.25}$$

that is deformed by an integrable potential and fermionic terms.

5.3 Neumann-Rosochatius Reduction in $\mathbb{R}\times\mathbf{S}^5$

There exists a large class of classical rotating strings in $\mathrm{AdS}_5\times\mathrm{S}^5$ that can be reduced to a certain one-dimensional integrable system that describes a particle that oscillates upon a sphere: the Neumann system [75, 76]. A subclass of the Neumann system consists of all the rigidly rotating strings in $\mathrm{AdS}_5\times\mathrm{S}^5$ and is known as the Neumann-Rosochatius (NR) system. The NR system is of course again integrable and describes a particle on a sphere subject to the potential $r^2 + r^{-2}$.

We are now going to go through a generalization of the Neumann and the Neumann-Rosochatius ansätze (set up in [77]) that will allow us to obtain two non-rigid string configurations in $\mathbb{R}\times\mathrm{S}^2$, the (infinite-size) giant magnon and the single spike. Consider the following ansatz in $\mathbb{R}\times\mathrm{S}^5$:

$$\Big\{t = \kappa\tau,\, \rho = \theta = \phi_1 = \phi_2 = 0\Big\} \times \Big\{Z_i = z_i(\xi)\, e^{i\omega_i\tau}\Big\}, \quad \xi = \alpha\sigma + \beta\tau \;\&\; z_i(\xi + 2\pi\alpha) = z_i(\xi), \tag{5.26}$$

where $Z_i = \{X_{12}, X_{34}, X_{56}\}$, $i = 1,2,3$ are the 5-sphere embedding coordinates (5.1) and $\alpha,\beta\in\mathbb{R}$. The ansatz (5.26) is the generalized Neumann ansatz. The conformal string action (5.10) becomes:

$$S_P = \frac{T}{2}\int\Big[-\ell^2\kappa^2 + \Big(\dot{Z}^i\dot{\bar{Z}}^i - \acute{Z}^i\acute{\bar{Z}}^i\Big) + \widetilde{\Lambda}\left(Z^i\bar{Z}^i - R^2\right)\Big]d\tau d\sigma. \tag{5.27}$$

The ansatz (5.26) gives rise to the following equation of motion and Largrange constraint:

$$\left(\alpha^2 - \beta^2\right)z_i'' - 2i\beta\omega_i z_i' + \omega_i^2 z_i + \widetilde{\Lambda}z_i = 0, \qquad \sum_{i=1}^3 |z_i|^2 = R^2, \tag{5.28}$$

where all the derivatives of z are w.r.t. the variable ξ. The Virasoro constraints are given by

$$\sum_{i=1}^3 2\beta\left|z_i'\right|^2 + i\,\omega_i\left(z_i\bar{z}_i' - \bar{z}_i z_i'\right) = 0 \tag{5.29}$$

$$\sum_{i=1}^{3}\left(\alpha^2+\beta^2\right)\left|z_i'\right|^2+\omega_i^2\left|z_i\right|^2+i\,\beta\,\omega_i\left(z_i\bar{z}_i'-\bar{z}_iz_i'\right)=\ell^2\kappa^2. \tag{5.30}$$

Using these constraints, the Neumann system Lagrangian and Hamiltonian density become:

$$\mathcal{L}=-\ell^2\kappa^2+\sum_{i=1}^{3}\left(\beta^2-\alpha^2\right)\left|z_i'\right|^2+i\,\beta\,\omega_i\left(z_i\bar{z}_i'-\bar{z}_iz_i'\right)+\omega_i^2\left|z_i\right|^2+\widetilde{\Lambda}\left(\left|z_i\right|^2-R^2\right) \tag{5.31}$$

$$\mathcal{H}=-\ell^2\kappa^2+\sum_{i=1}^{3}\left[\left(\alpha^2-\beta^2\right)\left|z_i'\right|^2+\omega_i^2\left|z_i\right|^2\right]=0. \tag{5.32}$$

In the Neumann-Rosochatius ansatz, z_i has the following form:

$$z_i\left(\xi\right)=r_i\left(\xi\right)e^{i\mu_i(\xi)}. \tag{5.33}$$

With (5.33), the equations of motion (5.28) become:

$$\left(\alpha^2-\beta^2\right)\left(r_i''-r_i\mu_i'^{\,2}\right)+2\beta\omega_ir_i\mu_i'+\omega_i^2r_i+\widetilde{\Lambda}r_i=0 \quad \& \quad \mu_i'=\frac{1}{\alpha^2-\beta^2}\left[\frac{C_i}{r_i^2}+\beta\omega_i\right], \tag{5.34}$$

where C_i are some real constants of integration. The (Lagrange and Virasoro) constraints of the system become:

$$\sum_{i=1}^{3}2\beta\left(r_i'^{\,2}+r_i^2\mu_i'^{\,2}\right)+2\omega_ir_i^2\mu_i'=0 \quad \& \quad \sum_{i=1}^{3}r_i^2=R^2 \tag{5.35}$$

$$\sum_{i=1}^{3}\left(\alpha^2+\beta^2\right)\left(r_i'^{\,2}+r_i^2\mu_i'^{\,2}\right)+\omega_i^2r_i^2+2\beta\omega_ir_i^2\mu_i'=\ell^2\kappa^2. \tag{5.36}$$

The NR Lagrangian and Hamiltonian are given by:

$$\mathcal{L}=\sum_{i=1}^{3}\left\{\left(\beta^2-\alpha^2\right)\left[r_i'^{\,2}+\left(\mu_i'+\frac{\beta\omega_i}{\beta^2-\alpha^2}\right)^2r_i^2\right]-\frac{\alpha^2}{\beta^2-\alpha^2}\,\omega_i^2r_i^2\right\}+\widetilde{\Lambda}\left[\sum_{i=1}^{3}r_i^2-R^2\right]-\ell^2\kappa^2 \tag{5.37}$$

$$\mathcal{H}=\sum_{i=1}^{3}\left\{\left(\alpha^2-\beta^2\right)\left(r_i'^{\,2}+r_i^2\mu_i'^{\,2}\right)+R^2\omega_i^2\right\}-\ell^2\kappa^2=0. \tag{5.38}$$

The equations (5.34)–(5.38) can be further simplified by using the value for μ_i' in (5.34):

$$\left(\alpha^2-\beta^2\right)r_i''-\frac{1}{\alpha^2-\beta^2}\frac{C_i^2}{r_i^3}+\left[\frac{\alpha^2\omega_i^2}{\alpha^2-\beta^2}+\widetilde{\Lambda}\right]r_i=0 \quad \& \quad \sum_{i=1}^{3}r_i^2=R^2. \tag{5.39}$$

The Virasoro constraints are:

$$\sum_{i=1}^{3}\left\{\left(\alpha^2-\beta^2\right)r_i'^{\,2}+\frac{1}{\alpha^2-\beta^2}\left[\frac{C_i^2}{r_i^2}+\alpha^2\omega_i^2r_i^2+2\beta C_i\omega_i\right]\right\}=\ell^2\kappa^2 \quad \& \quad \ell^2\beta\kappa^2+\sum_{i=1}^{3}C_i\omega_i=0. \tag{5.40}$$

The Lagrangian and Hamiltonian density of the system with the equations of motion (5.39) are:

$$\mathcal{L}=\sum_{i=1}^{3}\left\{\left(\beta^2-\alpha^2\right)r_i'^{\,2}-\frac{1}{\beta^2-\alpha^2}\frac{C_i^2}{r_i^2}-\frac{\alpha^2}{\beta^2-\alpha^2}\,\omega_i^2r_i^2\right\}+\widetilde{\Lambda}\left[\sum_{i=1}^{3}r_i^2-R^2\right]-\ell^2\kappa^2 \tag{5.41}$$

$$\mathcal{H}=\sum_{i=1}^{3}\left\{\left(\alpha^2-\beta^2\right)r_i'^{\,2}+\frac{1}{\alpha^2-\beta^2}\left[\frac{C_i^2}{r_i^2}+\alpha^2\omega_i^2r_i^2+2\beta C_i\omega_i\right]\right\}-\ell^2\kappa^2=0. \tag{5.42}$$

6 The Gubser-Klebanov-Polyakov (GKP) String

In 2002, Gubser, Klebanov and Polyakov (GKP) [11], proposed to study closed bosonic and uncharged strings that spin, rotate or pulsate in $\text{AdS}_5 \times \text{S}^5$, in order to obtain the (anomalous) scaling dimensions of their dual SYM operators at strong coupling, a regime inaccessible to perturbation theory from the gauge theory side. The paper of Gubser, Klebanov and Polyakov contains three prototype string ansätze for which the energy-spin relation[29] is calculated:

I. a closed string rigidly rotating in $\text{AdS}_3 \subset \text{AdS}_5 \times \text{S}^5$.

II. a closed string rigidly rotating around the pole of S^2 in $\mathbb{R} \times \text{S}^2 \subset \text{AdS}_5 \times \text{S}^5$.

III. a closed string pulsating inside $\text{AdS}_3 \subset \text{AdS}_5 \times \text{S}^5$.

Each of these string configurations is dual to a local gauge-invariant single-trace operator of $\mathcal{N} = 4$ SYM the (anomalous) scaling dimensions of which at strong coupling are equal to the energy of the closed string state.

In this section we are going to analyze the three basic string setups of GKP (I, II, III). As we have mentioned, all the GKP configurations are bosonic and uncharged so that the formalism that we set up in §5.1 is going to come very handy. As it will be explained in more detail below, long GKP strings belong to a category of configurations where integrability methods that have been developed so far are not very efficient. This means that traditional methods (namely quadratures) will have to be used in order to compute the spectra in this case. In §7 we are going to see how the predictive power of the standard spectral methods can be significantly enhanced by systemizing the computation of classical string energies. In §10 these methods are going to be applied to yet another classical string system, the giant magnon (GM).

The GKP case (I) of the AdS_3 rotating folded string is probably the most popular and has been exhaustively analyzed ever since it appeared. GKP's key observation was that the energy minus the spin of long folded strings that rotate inside AdS_3, scales like the logarithm of the spin:

$$E - S = \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}}, \quad S, \lambda \to \infty. \tag{6.1}$$

This behavior is very familiar from the study of anomalous dimensions of twist-2 Wilson operators in perturbative QCD. Indeed, GKP proceeded to reproduce this logarithmic scaling behavior by calculating the anomalous dimensions of the following twist-2, high-spin operators of $\mathcal{N} = 4$ SYM:[30]

$$\mathcal{O}_S = \text{Tr}\left[\mathcal{Z}\,\mathcal{D}_+^S\,\mathcal{Z}\right] + \ldots, \quad S \to \infty \tag{6.2}$$

where, in analogy with formula (4.14) for $\mathfrak{su}(2)$ operators, the dots in (6.2) stand for all possible distributions of the light-cone derivative $\mathcal{D}_+$ among the two fields $\mathcal{Z}$, while each term in the sum is multiplied by a suitable coefficient. Actually the calculation is almost identical to the one in QCD. However the perturbative result only scales as λ instead of $\sqrt{\lambda}$ in (6.1). GKP posited that this difference could be recompensed by the quantum corrections that the gauge theory result receives. We'll have more to say about the anomalous dimensions of twist-2 operators in QCD, $\mathcal{N} = 4$ SYM and the AdS_3 rotating GKP string in the §7.

The GKP case (II) of the $\mathbb{R} \times \text{S}^2$ rotating folded string has also been very extensively studied. In

[29] Aka dispersion relation. The term anomalous dimension will also be used interchangeably in this thesis, since the energy minus the spin of the string is equal to the anomalous dimensions of the dual gauge theory operator.

[30] Twist-J operators, $\text{Tr}\left[\mathcal{D}_+^{S_1}\mathcal{Z}\,\mathcal{D}_+^{S_2}\mathcal{Z}\ldots\mathcal{D}_+^{S_J}\,\mathcal{Z}\right]$, with $S_1 + S_2 + \ldots + S_J = S$ belong to the closed non-compact $\mathfrak{sl}(2)$ sector of $\mathcal{N} = 4$ SYM. The $\mathfrak{sl}(2)$ sector is dual to strings that rotate in $\text{AdS}_3 \times \text{S}^1$ and its dilatation operator is given by the Hamiltonian of the ferromagnetic $\text{XXX}_{-1/2}$ Heisenberg spin chain.

their original treatment [11], Gubser, Klebanov and Polyakov derived the following formula for the strong coupling value of the anomalous dimensions of the $\mathcal{N} = 4$ SYM operator that is dual to the $\mathbb{R} \times S^2$ closed and folded string (II):

$$E - J = \frac{2\sqrt{\lambda}}{\pi}, \quad J = \infty, \ \lambda \to \infty. \tag{6.3}$$

Closed folded single-spin strings rotating in $\mathbb{R} \times S^2$ can be decomposed into more elementary string theory excitations, known as giant magnons (GMs). These are open single-spin strings rotating in $S^2 \subset S^5$ that were identified in 2006 by Hofman and Maldacena [78] as the string theory duals of $\mathcal{N} = 4$ SYM magnon excitations that we saw in §4.3. The energy-spin relation of one giant magnon of angular extent $\Delta\varphi$ is:

$$E - J = \frac{\sqrt{\lambda}}{\pi} \left|\sin \frac{\Delta\varphi}{2}\right|, \quad J = \infty, \ \lambda \to \infty, \tag{6.4}$$

with $\Delta\varphi = p$ being equal to the dual magnon's momentum. Superimposing two giant magnons of maximum angular extent $\Delta\varphi = \pi$ and angular momenta $J/2$, gives the GKP formula (6.3). The GKP string (II) is therefore dual to the 2-magnon operators of $\mathcal{N} = 4$ SYM (4.33)–(4.44):

$$\mathcal{O}_J = \text{Tr}\left[\mathcal{Z}^J \mathcal{X}^2\right] + \ldots, \quad J \to \infty. \tag{6.5}$$

More on the giant magnon and its relationship with the GKP string (II) will be said in §8–§9, where GMs will be studied in detail.

The two rotating GKP string configurations (I–II) obey short-long strings dualities that connect the classical values of their conserved energies and spins in the regime of short strings to the values of these charges in the regime of long strings. These dualities are very interesting because it might be possible to upgrade them to the quantum level. Such a prospect will offer the possibility of using the energies of short strings, for which amazing results from integrability are available (Basso slope function, QSC, etc..), to compute the energies of long strings, for which the existing methods are not equally predictive as we have said.

The GKP case (III) consists of a string that pulsates inside AdS_3. The study of pulsating strings in anti-de Sitter spacetime was initiated long time ago by de Vega, Larsen and Sánchez [79] in the context of string cosmology. Gubser, Klebanov and Polyakov [11] approximated the energy of small pulsating strings in AdS_3 with the following formula:

$$E \sim \sqrt{\sqrt{\lambda} \cdot n}, \tag{6.6}$$

where n is the string excitation level. However at the time it was unclear how to extend this formula to large values of the energy E and level n, but also it was not known to which gauge theory operators the pulsating GKP string is dual. Both of these questions were answered by Minahan in [80]. The $\mathcal{N} = 4$ SYM operators that are dual to pulsating GKP strings are

$$\mathcal{O}_n = \text{Tr}\left[\mathcal{Z}\mathcal{D}_+^n \mathcal{D}_-^n \mathcal{Z}\right] + \ldots, \tag{6.7}$$

where again the dots stand for all possible permutations of the light-cone derivatives $\mathcal{D}_\pm$, sandwiched between the two complex scalars, and each term in the sum is multiplied by a suitable coefficient. The exact form of the operators $\mathcal{O}_n$ will be given in §6.3.1 where we will also quantize the pulsating GKP string à la WKB and derive a semiclassical dispersion relation that is valid for all values of the string level n. The one-loop correction to the energy of the pulsating string has been computed in [81].

This section is organized as follows. In §6.1 the rotating GKP string in AdS_3 (case I) is going to be

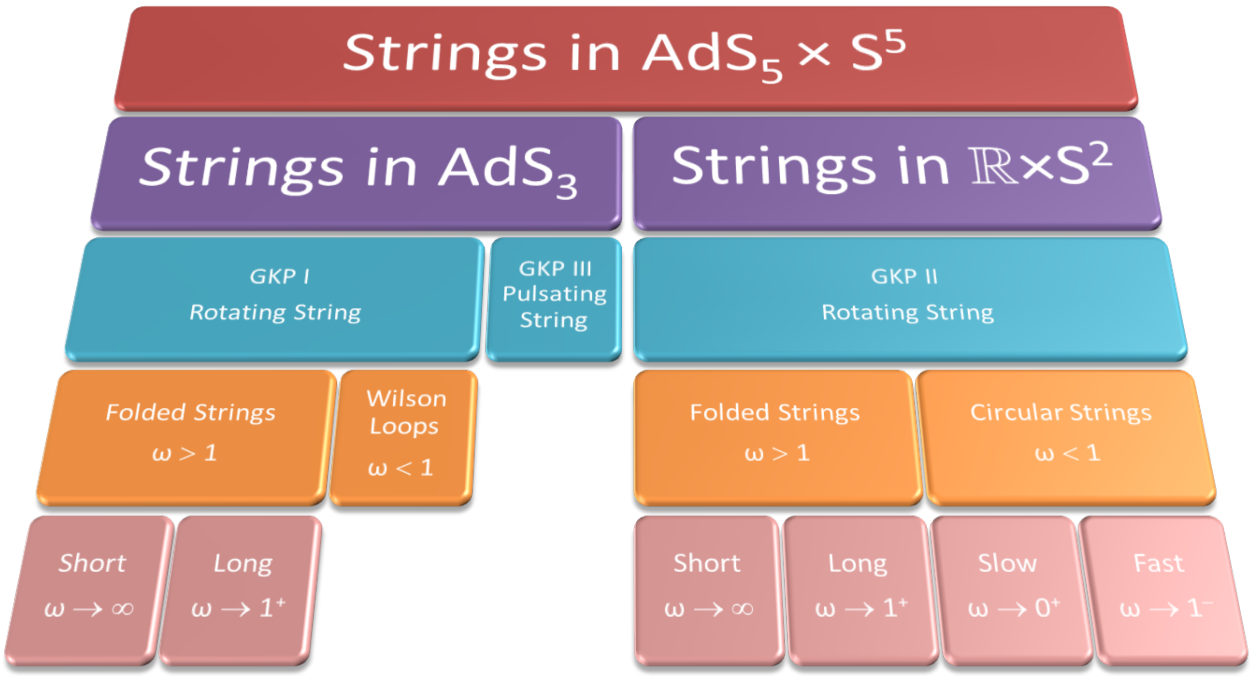


Figure 1: The GKP Strings.

presented. Depending on whether the string's angular velocity ω is greater or smaller than unity, the string either folds at the edges and rotates rigidly within AdS_3 ($\omega > 1$) or touches the AdS boundary ($\omega < 1$). In the former case ($\omega > 1$) there exist two interesting limiting cases, that of "short" strings with $\omega \to \infty$ and that of "long" strings for which $\omega \to 1^+$. Analytic expressions for the conserved string charges (namely the energy E and the spin S) of short and long folded GKP strings are given in §6.1.1 and §6.1.2. The short-long strings duality for the GKP string (I) will be derived in §6.1.3. GKP strings that touch the boundary of AdS (Wilson loops) will be omitted.

The $\mathbb{R} \times S^2$ rotating GKP string (case II) is presented in §6.2. Again there exist two main regimes depending on the value of the angular velocity ω, folded strings ($\omega > 1$), either short ($\omega \to \infty$) or long ($\omega \to 1^+$), and circular strings ($\omega < 1$) which extend along a great circle of the 2-sphere and are either "slow" ($\omega \to 0^+$) or "fast" ($\omega \to 1^-$). §6.2.1–§6.2.2 deal with the conserved charges (energy E and spin J) of the former and in §6.2.3–§6.2.4 the latter configuration is presented. Dualities between short-long and fast-slow strings are proved in §6.2.5.

This section ends with the presentation of the AdS pulsating GKP string (case III), in §6.3. The semiclassical quantization of this string is carried out in §6.3.1.

6.1 Gubser-Klebanov-Polyakov String in AdS_3

The ansatz for the GKP folded closed string (I) that rotates in $AdS_3 \subset AdS_5 \times S^5$ is:

$$\Big\{t = \kappa\tau,\ \rho = \rho(\sigma),\ \theta = \kappa\omega\tau,\ \phi_1 = \phi_2 = 0\Big\} \times \Big\{\overline{\theta}_1 = \overline{\theta}_2 = \overline{\phi}_1 = \overline{\phi}_2 = \overline{\phi}_3 = 0\Big\}. \tag{6.8}$$

In embedding space the solution is given by

$$\begin{aligned}
&Y_0 = \ell \cosh\rho(\sigma)\cos\kappa\tau, && Y_2 = Y_4 = 0, && X_1 = R = \ell \\
&Y_1 = \ell \sinh\rho(\sigma)\cos\kappa\omega\tau && && X_2 = X_3 = X_4 = X_5 = X_6 = 0 \\
&Y_3 = \ell \sinh\rho(\sigma)\sin\kappa\omega\tau && && \\
&Y_5 = \ell \cosh\rho(\sigma)\sin\kappa\tau. && &&
\end{aligned} \tag{6.9}$$

Its Polyakov action in the conformal gauge ($\gamma_{ab} = \eta_{ab}$) reads:

$$S_P = \frac{\ell^2}{4\pi\alpha'}\int\Big(-\dot{t}^2\cosh^2\rho - \rho'^{\,2} + \dot{\theta}^2\sinh^2\rho\Big)\,d\tau d\sigma = \tag{6.10}$$

$$= \frac{\ell^2}{4\pi\alpha'}\int\left(-\kappa^2\cosh^2\rho - \rho'^{\,2} + \kappa^2\omega^2\sinh^2\rho\right)d\tau d\sigma, \tag{6.11}$$

The equations of motion and the Virasoro constraints (5.4) become:

$$\rho'' + \kappa^2\left(\omega^2 - 1\right)\sinh\rho\cosh\rho = 0 \tag{6.12}$$

$$\rho'^{\,2} - \kappa^2\left(\cosh^2\rho - \omega^2\sinh^2\rho\right) = 0. \tag{6.13}$$

Both are essentially equivalent to the following equation:

$$\begin{aligned}
\frac{d\sigma}{d\rho} &= \frac{1}{\kappa\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}} = \frac{1}{\kappa\sqrt{1 - (\omega^2-1)\sinh^2\rho}} = \\
&= \frac{1}{\kappa\sqrt{(\omega^2-1)\left(q - \sinh^2\rho\right)}}\,, \qquad q \equiv \frac{1}{\omega^2 - 1}.
\end{aligned} \tag{6.14}$$

Depending on the value of the angular velocity ω, two basic cases are obtained:

(i). $\omega^2 > 1$: A folded closed rigidly rotating string with cusps at $d\sigma/d\rho\Big|_{\rho_0} = \infty$ and

$$0 \le \sinh^2\rho \le \sinh^2\rho_0 = \frac{1}{\omega^2 - 1} = q < \infty.$$

a. "Short" Strings: $\omega \to \infty$, $\rho_0 \sim 1/\omega$.

b. "Long" Strings: $\omega = 1 + 2\eta \to 1^+$, $\rho_0 \sim \ln 1/\eta \to \infty$.

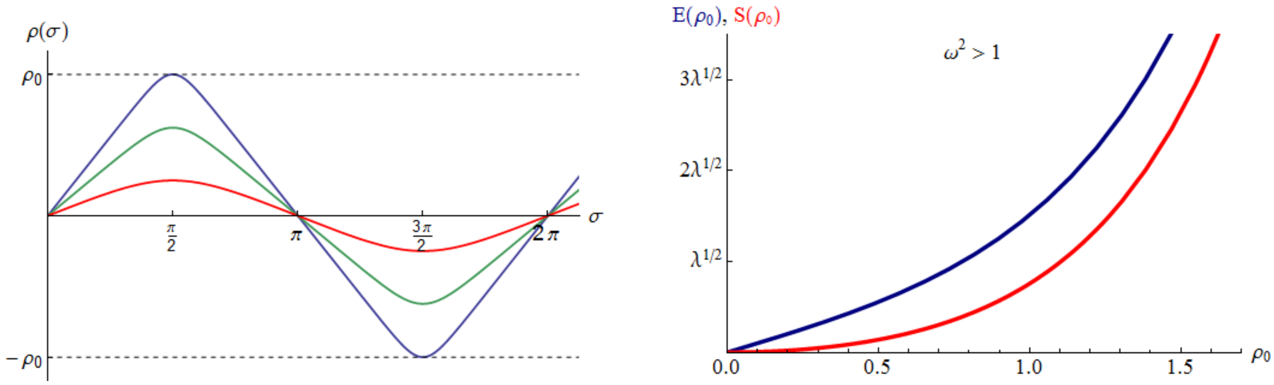


Figure 2: $\rho = \rho\,(\sigma)$ and energy/spin of the folded closed GKP string in AdS_3 (6.8) for $\omega > 1$.

(ii). $\omega^2 < 1$: Two oppositely oriented rigidly rotating Wilson loops[31] with

$$0 \leq \sinh^2 \rho \leq \sinh^2 \rho_0 = \infty.$$

The conserved charges that correspond to the two cyclic coordinates t and θ are given by the following integrals:

$$E = \left| \frac{\partial L}{\partial \dot{t}} \right| = \frac{\ell^2}{2\pi\alpha'} \int_0^{2\pi} \kappa \, \cosh^2 \rho \, d\sigma = 4 \cdot \frac{\ell^2}{2\pi\alpha'} \int_0^{\rho_0} \frac{\cosh^2 \rho \, d\rho}{\sqrt{1 - (\omega^2 - 1) \sinh^2 \rho}} \tag{6.15}$$

$$S = \frac{\partial L}{\partial \dot{\theta}} = \frac{\ell^2}{2\pi\alpha'} \int_0^{2\pi} \kappa \, \omega \, \sinh^2 \rho \, d\sigma = 4 \cdot \frac{\ell^2}{2\pi\alpha'} \int_0^{\rho_0} \frac{\omega \, \sinh^2 \rho \, d\rho}{\sqrt{1 - (\omega^2 - 1) \sinh^2 \rho}}, \tag{6.16}$$

The string has four segments that extend between $\rho = 0$ and $\rho = \rho_0$ and this accounts for the factor of 4 in front of all the ρ-integrals. One also has to calculate the length of the string

$$\sigma \cdot \kappa = \int_0^{\rho} \frac{d\rho}{\sqrt{1 - (\omega^2 - 1) \sinh^2 \rho}}, \tag{6.17}$$

where κ is a factor that fixes $\sigma\,(\rho_0) = \pi/2$. In order to calculate the integrals, we set $\omega \tanh \rho = \sin \varphi$. The results, briefly, are:

$$\sigma = \frac{1}{\kappa\,\omega} \int_0^{\varphi} \frac{d\varphi}{\left(1 - \frac{1}{\omega^2} \sin^2 \varphi\right)^{1/2}} = \frac{1}{\kappa\,\omega} \cdot \mathbb{F}\left(\varphi \middle| \frac{1}{\omega^2}\right) \Rightarrow \rho(\sigma) = \text{arctanh}\left[\frac{1}{\omega} sn\left(\kappa\omega\sigma \middle| \frac{1}{\omega^2}\right)\right], \tag{6.18}$$

where

$$\kappa = \frac{2}{\pi\omega} \mathbb{F}\left(\varphi_0 \middle| \frac{1}{\omega^2}\right) \quad \& \quad \omega \cdot \tanh \rho_0 = \sin \varphi_0$$

[31] Wilson loops touch the boundary of anti-de Sitter space at $\rho = \infty$.

$$E = \frac{2\ell^2}{\pi\alpha'\omega}\int_0^{\varphi_0}\frac{d\varphi}{\left(1-\frac{1}{\omega^2}\sin^2\varphi\right)^{3/2}} = \frac{2\ell^2}{\pi\alpha'\omega}\cdot\mathbf{\Pi}\left(\frac{1}{\omega^2},\varphi_0\,\middle|\,\frac{1}{\omega^2}\right) =$$

$$= \frac{2\ell^2}{\pi\alpha'}\left(\frac{\omega}{\omega^2-1}\cdot\mathbb{E}\left(\varphi_0\,\middle|\,\frac{1}{\omega^2}\right) - \frac{1}{2\omega\left(\omega^2-1\right)}\frac{\sin 2\varphi_0}{\sqrt{1-\frac{1}{\omega^2}\sin^2\varphi_0}}\right) \tag{6.19}$$

$$S = \frac{2\ell^2}{\pi\alpha'\omega^2}\int_0^{\varphi_0}\frac{\sin^2\varphi\, d\varphi}{\left(1-\frac{1}{\omega^2}\sin^2\varphi\right)^{3/2}} = \frac{2\ell^2}{\pi\alpha'}\left[\mathbf{\Pi}\left(\frac{1}{\omega^2},\varphi_0\middle|\frac{1}{\omega^2}\right) - \mathbb{E}\left(\varphi_0\,\middle|\,\frac{1}{\omega^2}\right)\right] =$$

$$= \frac{2\ell^2}{\pi\alpha'}\left(\frac{\omega^2}{\omega^2-1}\cdot\mathbb{E}\left(\varphi_0\,\middle|\,\frac{1}{\omega^2}\right) - \mathbb{F}\left(\varphi_0\,\middle|\,\frac{1}{\omega^2}\right) - \frac{1}{2\left(\omega^2-1\right)}\frac{\sin 2\varphi_0}{\sqrt{1-\frac{1}{\omega^2}\sin^2\varphi_0}}\right). \tag{6.20}$$

The definitions of the elliptic integrals of the first, second and third kind, $\mathbb{F}(\varphi|m)$, $\mathbb{E}(\varphi|m)$, $\mathbf{\Pi}(n,\varphi|m)$, as well as that of the Jacobian elliptic function $sn(u|m)$ are given in appendix H.

$\underline{\omega^2 > 1}$: Folded closed string (AdS_3).

For the case (i) of the closed folded string with $\omega^2 > 1$, it's $\varphi_0 = \pi/2$ so that the integrals (6.15)–(6.17) take simpler forms and can be expressed in terms of the complete elliptic integrals:

$$\rho(\sigma) = \text{arctanh}\left[\frac{1}{\omega}sn\left(\kappa\omega\sigma\,\middle|\,\frac{1}{\omega^2}\right)\right], \quad \kappa = \frac{2}{\pi\omega}\cdot\mathbb{K}\left(\frac{1}{\omega^2}\right), \quad \omega = \coth\rho_0 \tag{6.21}$$

$$E(\omega) = \frac{2\sqrt{\lambda}}{\pi}\frac{\omega}{\omega^2-1}\cdot\mathbb{E}\left(\frac{1}{\omega^2}\right) \Rightarrow \mathcal{E} \equiv \frac{\pi E}{\sqrt{\lambda}} = \frac{2\sqrt{1-x}}{x}\cdot\mathbb{E}\left(1-x\right) \tag{6.22}$$

$$S(\omega) = \frac{2\sqrt{\lambda}}{\pi}\left[\frac{\omega^2}{\omega^2-1}\cdot\mathbb{E}\left(\frac{1}{\omega^2}\right) - \mathbb{K}\left(\frac{1}{\omega^2}\right)\right] \Rightarrow \mathcal{S} \equiv \frac{\pi S}{\sqrt{\lambda}} = \frac{2}{x}\,\mathbb{E}\left(1-x\right) - 2\mathbb{K}\left(1-x\right) \tag{6.23}$$

$$\gamma \equiv \mathcal{E} - \mathcal{S} = 2\left[\frac{\sqrt{1-x}-1}{x}\cdot\mathbb{E}\left(1-x\right) + \mathbb{K}\left(1-x\right)\right], \tag{6.24}$$

where $x \equiv 1 - 1/\omega^2$ is the complementary parameter of $1/\omega^2$. In figures 2–3 we have plotted $\rho\left(\sigma\right)$ for various values of the angular velocity $\omega > 1$ and the string's energy and spin as functions of ρ_0, ω and x. Figure 4 contains the plot of the string's energy in terms of its spin, $E = E(S)$.

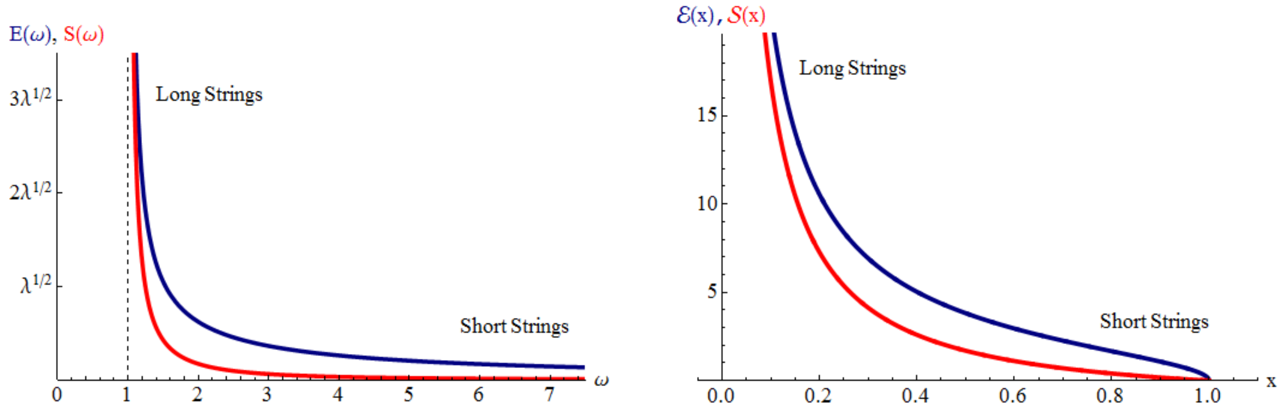


Figure 3: Energy/spin of the folded closed GKP string in AdS_3 (6.8) as functions of $\omega > 1$ and $x > 0$.

6.1.1 Short Strings in AdS_3: $\omega \to \infty$, $S \ll \sqrt{\lambda}$

In order to obtain the short-string limit, the expressions (6.22)–(6.23) may be expanded around $\omega \to \infty$ using the formulas of appendix H:

$$E = \sqrt{\lambda} \cdot \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{2n+1}{\omega^{2n+1}} = \sqrt{\lambda} \cdot \left[\frac{1}{\omega} + \frac{3}{4\,\omega^3} + \frac{45}{64\,\omega^5} + \frac{175}{256\,\omega^7} + O\left(\frac{1}{\omega^9}\right)\right] \tag{6.25}$$

$$S = \sqrt{\lambda} \cdot \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{2n}{\omega^{2n}} = \frac{\sqrt{\lambda}}{2} \cdot \left[\frac{1}{\omega^2} + \frac{9}{8\,\omega^4} + \frac{75}{64\,\omega^6} + \frac{1225}{1024\,\omega^8} + O\left(\frac{1}{\omega^{10}}\right)\right]. \tag{6.26}$$

To obtain the series (6.25) we used the identity

$$(2n+1)\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 + \sum_{k=0}^{n} \frac{1}{2k-1}\left(\frac{(2k-1)!!}{(2k)!!}\right)^2 = 0. \tag{6.27}$$

With Mathematica we may also obtain the inverse spin function $x = x(\mathcal{S})$ and the energy $\mathcal{E} = \mathcal{E}(\mathcal{S})$ in terms of the spin $\mathcal{S}$:

$$x = 1 - \frac{2\,\mathcal{S}}{\pi} + \frac{9\,\mathcal{S}^2}{2\pi^2} - \frac{87\,\mathcal{S}^3}{8\pi^3} + \frac{1.765\,\mathcal{S}^4}{64\,\pi^4} - \frac{37.071\,\mathcal{S}^5}{512\,\pi^5} + \frac{199.815\,\mathcal{S}^6}{1024\,\pi^6} - \frac{4.397.017\,\mathcal{S}^7}{8192\,\pi^7} + \ldots \tag{6.28}$$

$$\mathcal{E} = \sqrt{2} \cdot \left[\pi^{1/2}\mathcal{S}^{1/2} + \frac{3\,\mathcal{S}^{3/2}}{8\,\pi^{1/2}} - \frac{21\,\mathcal{S}^{5/2}}{128\,\pi^{3/2}} + \frac{187\,\mathcal{S}^{7/2}}{1024\,\pi^{5/2}} - \frac{9.261\,\mathcal{S}^{9/2}}{32.768\,\pi^{7/2}} + \frac{136.245\,\mathcal{S}^{11/2}}{262.144\,\pi^{9/2}} - \ldots\right]. \tag{6.29}$$

The dependence of (6.29) on the 't Hooft coupling λ can be made manifest as follows:

$$E = \left(2\sqrt{\lambda}\,S\right)^{1/2} \cdot \left[1 + \frac{3\,S}{8\sqrt{\lambda}} - \frac{21\,S^2}{128\,\lambda} + \frac{187\,S^3}{1\,024\,\lambda^{3/2}} - \frac{9\,261\,S^4}{32\,768\,\lambda^2} + \frac{136\,245\,S^5}{262\,144\,\lambda^{5/2}} - O\left(\frac{S^6}{\lambda^3}\right)\right]. \tag{6.30}$$

Quantum corrections to the short AdS_3 string have been calculated up to 1-loop in [82]:

$$E_{qc} = \left(2\sqrt{\lambda}\,S\right)^{1/2}\cdot\left[\left(\mathrm{a}_{00}+\frac{\mathrm{a}_{01}}{\sqrt{\lambda}}+\ldots\right)+\left(\mathrm{a}_{10}+\frac{\mathrm{a}_{11}}{\sqrt{\lambda}}+\ldots\right)\frac{S}{\sqrt{\lambda}}+\left(\mathrm{a}_{20}+\frac{\mathrm{a}_{21}}{\sqrt{\lambda}}+\ldots\right)\frac{S^2}{\lambda}+\ldots\right],$$

with the first few coefficients being,

$$\mathrm{a}_{00}=1\,,\qquad \mathrm{a}_{01}=3-4\ln 2\,,\qquad \mathrm{a}_{10}=\frac{3}{8}\,,\qquad \mathrm{a}_{11}=-\frac{1219}{576}+\frac{3}{2}\ln 2+\frac{3}{4}\zeta(3).$$

More recent results have been obtained by Basso [83] as well as with the $P\mu$ system, but we don't have time to report them here.

6.1.2 Long Strings in AdS_3: $\omega\to 1^+$, $S\gg\sqrt{\lambda}$

Going to the opposite regime $\omega\to 1^+$ ($S\gg\lambda$), let us summarize the results of [12]:

$$\begin{aligned}E=&\,\frac{2\sqrt{\lambda}}{\pi\omega}\cdot\left\{\frac{\omega^2}{\omega^2-1}+\frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)\Gamma(n+3/2)}{n!(n+1)!}\Big[2\psi(n+1)-2\psi(n+1/2)-\right.\\&\left.-\ln(1-1/\omega^2)-\frac{1}{(n+1)\,(2n+1)}\Big]\cdot(1-1/\omega^2)^n\right\}=\\=&\,\frac{2\sqrt{\lambda}}{\pi\omega}\cdot\left\{\frac{\omega^2}{\omega^2-1}-\frac{1}{4}\left[\ln\left(1-1/\omega^2\right)-4\ln 2+1\right]-\frac{3}{32}\left(1-1/\omega^2\right)\Big[\ln\left(1-1/\omega^2\right)-\right.\\&\left.-4\ln 2+\frac{13}{6}\Big]-\frac{15}{256}\left(1-1/\omega^2\right)^2\left[\ln\left(1-1/\omega^2\right)-4\ln 2+\frac{12}{5}\right]+\ldots\right\}\end{aligned}\tag{6.31}$$

$$\begin{aligned}S=&\,\frac{2\sqrt{\lambda}}{\pi}\cdot\left\{\frac{\omega^2}{\omega^2-1}-\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(\Gamma(n+1/2))^2}{n!(n+1)!}\Big[2\psi(n+1)-2\psi(n+1/2)-\ln(1-1/\omega^2)+\right.\\&\left.+\frac{1}{n+1}\Big]\cdot(1-1/\omega^2)^n\right\}=\\=&\,\frac{2\sqrt{\lambda}}{\pi}\cdot\left\{\frac{\omega^2}{\omega^2-1}+\frac{1}{4}\left[\ln\left(1-1/\omega^2\right)-4\ln 2-1\right]+\frac{1}{32}\left(1-1/\omega^2\right)\Big[\ln\left(1-1/\omega^2\right)-\right.\\&\left.-4\ln 2+\frac{3}{2}\Big]+\frac{3}{256}\left(1-1/\omega^2\right)^2\left[\ln\left(1-1/\omega^2\right)-4\ln 2+2\right]+\ldots\right\}.\end{aligned}\tag{6.32}$$

The two series may also be expressed via the complementary parameter $x\equiv 1-1/\omega^2\to 0^+$:

$$\begin{aligned}\mathcal{E}\equiv\frac{\pi\,E}{\sqrt{\lambda}}=&\,2\sqrt{1-x}\cdot\left\{\frac{1}{x}+\sum_{n=0}^{\infty}x^n\left(d_n\ln x+h_n\right)\right\}=\\&=\frac{2}{x}-2\sum_{n=0}^{\infty}x^n\cdot\left\{\frac{(2n-1)!!}{(2n+2)!!}+\sum_{k=0}^{n}\frac{(2k-3)!!}{(2k)!!}\left(d_{n-k}\ln x+h_{n-k}\right)\right\}\end{aligned}\tag{6.33}$$

$$\mathcal{S}\equiv\frac{\pi\,S}{\sqrt{\lambda}}=\frac{2}{x}+2\sum_{n=0}^{\infty}x^n\left(c_n\ln x+b_n\right).\tag{6.34}$$

The coefficients of the series (6.33) and (6.34) are given by:[32]

$$
\begin{aligned}
d_n &= -\frac{1}{4}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \cdot \frac{2n+1}{n+1} \\
h_n &= -d_n \cdot \left[4\ln 2 + 2\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{2}{2k-1}\right) + \frac{1}{n+1} - \frac{2}{2n+1}\right] \\
c_n &= \frac{1}{4}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \cdot \frac{1}{n+1} = -\frac{d_n}{2n+1} \\
b_n &= -c_n \cdot \left[4\ln 2 + 2\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{2}{2k-1}\right) + \frac{1}{n+1}\right],
\end{aligned} \tag{6.35}
$$

where $n = 0\,,1\,,2\,,\ldots$. Explicitly, the first few of them are:

$$
\begin{aligned}
d_0 &= -\frac{1}{4}\,, & d_1 &= -\frac{3}{32}\,, & d_2 &= -\frac{15}{256} \\
h_0 &= \ln 2 - \frac{1}{4}\,, & h_1 &= \frac{3}{8}\ln 2 - \frac{13}{64}\,, & h_2 &= \frac{15}{64}\ln 2 - \frac{9}{64} \\
c_0 &= \frac{1}{4}\,, & c_1 &= \frac{1}{32}\,, & c_2 &= \frac{3}{256} \\
b_0 &= -\ln 2 - \frac{1}{4}\,, & b_1 &= -\frac{1}{8}\ln 2 + \frac{3}{64}\,, & b_2 &= -\frac{3}{64}\ln 2 + \frac{3}{128}.
\end{aligned} \tag{6.36}
$$

6.1.3 Short-Long Strings Duality

Following Georgiou and Savvidy [12], we will now derive a formula that links the conserved energy and spin of "short" strings ($\omega \to \infty$) with the energy and spin of "long" strings ($\omega \to 1^+$). Take Legendre's relation between complete elliptic integrals of the first and second kind (see e.g. [84, 85]):

$$
\mathbb{E}(k)\mathbb{K}(k') + \mathbb{K}(k)\mathbb{E}(k') - \mathbb{K}(k)\mathbb{K}(k') = \frac{\pi}{2}, \tag{6.37}
$$

where the arguments $k = 1/\omega^2$ and $k' = x = 1/\omega'^{\,2}$ satisfy $k + k' = 1$. Solve (6.22)–(6.23) for $\mathbb{E}(k)$ and $\mathbb{K}(k)$ and substitute their values in (6.37). We obtain the following relation between classical folded short and long strings that spin in AdS_3:

$$
\frac{1}{\omega}ES' + \frac{1}{\omega'}E'S - SS' = \frac{2\lambda}{\pi}, \quad \lambda \to \infty. \tag{6.38}
$$

There's an alternative expression of (6.38) in terms of the anomalous dimensions $\gamma \equiv \mathcal{E} - \mathcal{S}$:

$$
\frac{1}{\omega}\gamma\,\mathcal{S}' + \frac{1}{\omega'}\gamma'\,\mathcal{S} + \left(\frac{1}{\omega} + \frac{1}{\omega'} - 1\right)\mathcal{S}\mathcal{S}' = 2\pi, \quad \mathcal{E} \equiv \frac{\pi\,E}{\sqrt{\lambda}}, \quad \mathcal{S} \equiv \frac{\pi\,S}{\sqrt{\lambda}}. \tag{6.39}
$$

It is not known whether short-long string dualities similar to (6.38)–(6.39) can also be formulated at the quantum level. A short-long strings duality will also be found in the GKP case (II) below. Even more short-long dualities will be constructed in appendix E.

[32] In double factorial notation, it's $0!! = 1$, $(-1)!! = 1$, $(-3)!! = -1$.

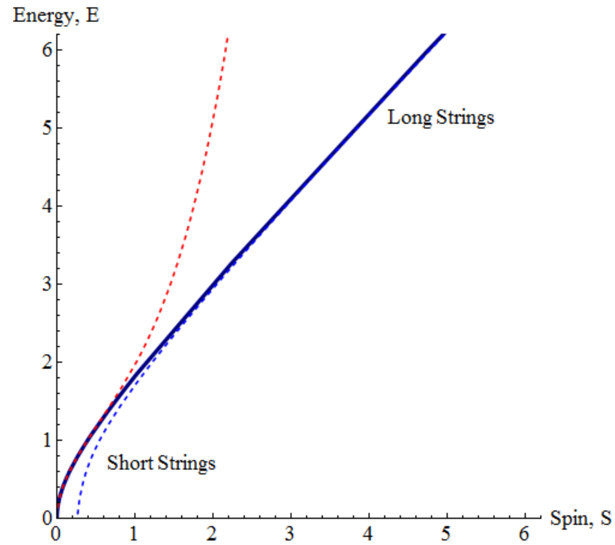


Figure 4: Energy versus spin of the folded closed AdS$_3$ string (6.8) for $\omega^2 > 1$. The red dashed line is the plot of the first 4 terms of the "short" approximation (6.30), while the blue dashed line corresponds to the string's leading "long" approximation (determined by the coefficients (7.104)–(7.105)).

6.2 Gubser-Klebanov-Polyakov String in $\mathbb{R} \times \mathrm{S}^2$

The GKP folded closed string (II) has its center at the pole of S^2 and rotates around it:

$$\Big\{t = \kappa\tau,\ \rho = \theta = \phi_1 = \phi_2 = 0\Big\} \times \Big\{\bar{\theta}_1 = \bar{\theta}_1(\sigma),\ \bar{\theta}_2 = \kappa\omega\tau,\ \bar{\phi}_1 = \bar{\phi}_2 = \bar{\phi}_3 = 0\Big\}. \tag{6.40}$$

In embedding coordinates the ansatz reads:

$$\begin{aligned}
&Y_0 = \ell\cos\kappa\tau\,, \quad Y_5 = \ell\sin\kappa\tau\,, && X_1 = \ell\cos\bar{\theta}_1(\sigma)\,, && X_2 = X_4 = X_6 = 0\\
&Y_1 = Y_2 = Y_3 = Y_4 = 0 && X_3 = \ell\sin\bar{\theta}_1(\sigma)\cos\kappa\omega\tau &&\\
& && X_5 = \ell\sin\bar{\theta}_1(\sigma)\sin\kappa\omega\tau. &&
\end{aligned} \tag{6.41}$$

In the conformal gauge ($\gamma_{ab} = \eta_{ab}$) the string has the following Polyakov action:

$$S_P = \frac{\ell^2}{4\pi\alpha'}\int\left(-\dot{t}^2 + \dot{\bar{\theta}}_2^2\sin^2\bar{\theta}_1 - \bar{\theta}_1'^2\right)d\tau d\sigma = \frac{\ell^2}{4\pi\alpha'}\int\left(-\kappa^2 + \kappa^2\omega^2\sin^2\bar{\theta}_1 - \bar{\theta}_1'^2\right)d\tau d\sigma. \tag{6.42}$$

This action gives rise to the following equations of motion and Virasoro constraints (5.4):

$$\bar{\theta}_1'' + \kappa^2\omega^2\sin\bar{\theta}_1\cos\bar{\theta}_1 = 0 \tag{6.43}$$

$$\bar{\theta}_1'^{\,2} - \kappa^2\left(1 - \omega^2\sin^2\bar{\theta}_1\right) = 0, \tag{6.44}$$

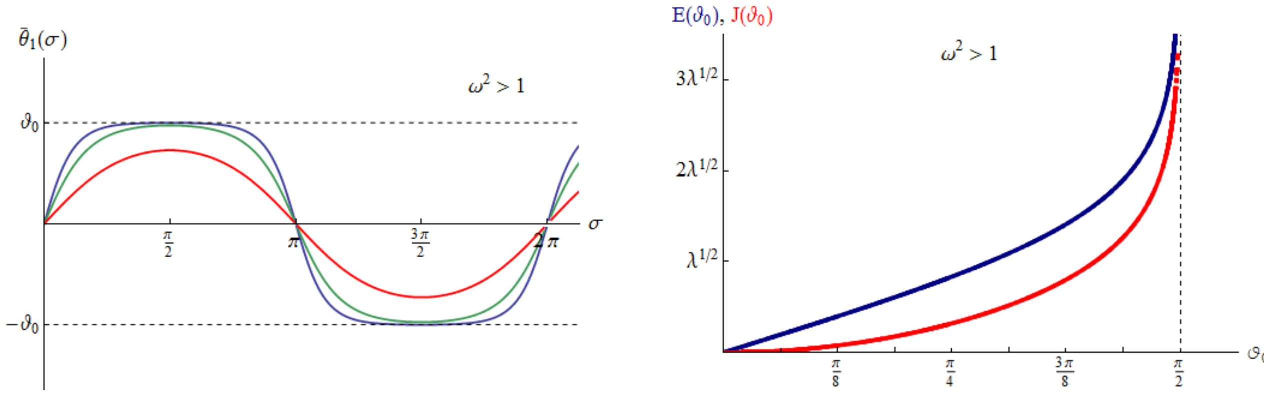


Figure 5: $\overline{\theta} = \theta_1(\sigma)$ and energy/spin of the closed folded GKP string in $\mathbb{R} \times \mathrm{S}^2$ (6.40) for $\omega^2 > 1$.

both of which are equivalent to:

$$\frac{d\sigma}{d\overline{\theta}_1} = \frac{1}{\kappa\sqrt{1 - \omega^2 \sin^2 \overline{\theta}_1}}. \tag{6.45}$$

Depending on the value of $\omega \neq 1$, the following cases are obtained:

(i). $\omega^2 > 1$: A folded closed string rigidly rotating around the pole of S^2,

$$\overline{\theta}_1 \in [0\,, \arcsin 1/\omega = \vartheta_0].$$

a. "Short" Strings: $\omega \to \infty$, $\vartheta_0 \sim 1/\omega$.

b. "Long" Strings: $\omega = 1 + 2\eta \to 1^+$, $\vartheta_0 \to \pi/2$.

(ii). $\omega^2 < 1$: A circular string stretched along an S^2 polar great circle, rotating rigidly around the poles,[33]

$$\overline{\theta}_1 \in [0\,, \pi/2 = \vartheta_0].$$

The integrals for the string length and conserved charges are given by:

$$\sigma\left(\overline{\theta}_1\right) = \int_0^{\overline{\theta}_1} \frac{d\overline{\theta}_1}{\kappa\sqrt{1 - \omega^2 \sin^2 \overline{\theta}_1}} = \frac{1}{\kappa} \cdot \mathbb{F}\left(\overline{\theta}_1 \middle| \omega^2\right) \tag{6.46}$$

$$E = \left|\frac{\partial L}{\partial \dot{t}}\right| = \frac{\ell^2}{2\pi\alpha'} \int_0^{2\pi} \kappa\, d\sigma = \frac{\kappa\,\ell^2}{\alpha'} = 4 \cdot \frac{\ell^2}{2\pi\alpha'} \int_0^{\vartheta_0} \frac{d\overline{\theta}_1}{\sqrt{1 - \omega^2 \sin^2 \overline{\theta}_1}} = \frac{2\ell^2}{\pi\alpha'} \cdot \mathbb{F}\left(\vartheta_0 \middle| \omega^2\right) \tag{6.47}$$

$$J = \frac{\partial L}{\partial \dot{\overline{\theta}}_2} = \frac{\ell^2}{2\pi\alpha'} \int_0^{2\pi} \kappa\,\omega\, \sin^2 \overline{\theta}_1\, d\sigma = 4 \cdot \frac{\ell^2}{2\pi\alpha'} \int_0^{\vartheta_0} \frac{\omega\, \sin^2 \overline{\theta}_1\, d\overline{\theta}_1}{\sqrt{1 - \omega^2 \sin^2 \overline{\theta}_1}} = \frac{2\ell^2}{\pi\alpha'\omega} \left(\mathbb{F}\left(\vartheta_0 \middle| \omega^2\right) - \mathbb{E}\left(\vartheta_0 \middle| \omega^2\right)\right). \tag{6.48}$$

[33]In this case the string has no cusp at ϑ_0 ($d\sigma/d\overline{\theta}_1|_{\vartheta_0} \neq \infty$). Its four pieces unfold to form a great circle.

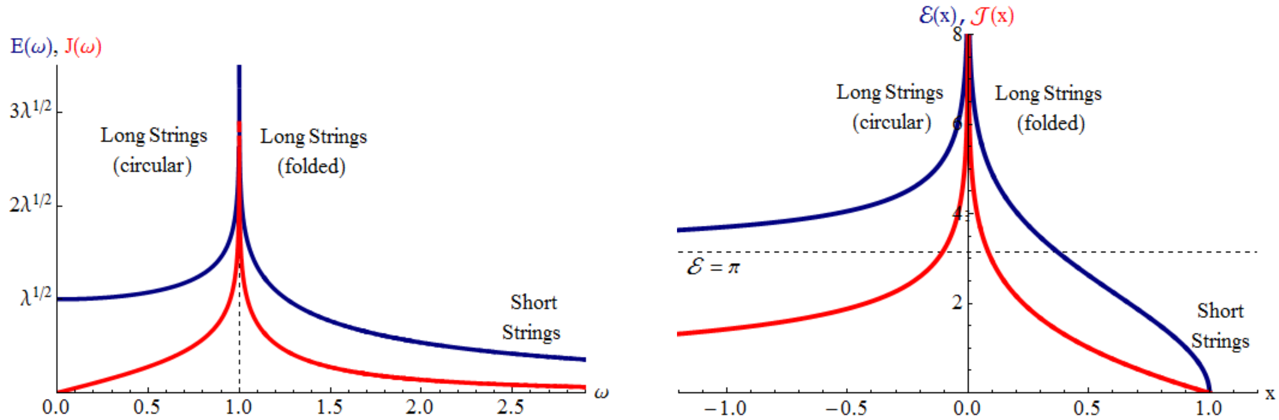


Figure 6: Energy/spin of the closed folded/circular string in $\mathbb{R}\times \mathrm{S}^2$ (6.40) as function of ω and x.

These integrals all diverge for $\omega = 1$. Setting $\sigma(\vartheta_0) = \pi/2$ yields:

$$\bar{\theta}_1(\sigma) = am\left[\kappa\,\sigma\,\middle|\,\omega^2\right] \quad , \quad \kappa = \frac{2}{\pi}\cdot\mathbb{F}\left(\vartheta_0|\omega^2\right). \tag{6.49}$$

From (6.47)–(6.49) the following results are obtained:

$\underline{\omega^2 < 1}$: Circular string ($\mathbb{R}\times\mathrm{S}^2$).

$$\bar{\theta}_1(\sigma) = am\left[\kappa\,\sigma\,\middle|\,\omega^2\right] \quad , \quad \kappa = \frac{2}{\pi}\cdot\mathbb{K}\left(\omega^2\right) \tag{6.50}$$

$$E(\omega) = \frac{2\sqrt{\lambda}}{\pi}\cdot\mathbb{K}\left(\omega^2\right) \Rightarrow \mathcal{E} \equiv \frac{\pi\,E}{\sqrt{\lambda}} = 2\,\mathbb{K}\left(1-\widetilde{x}\right) \tag{6.51}$$

$$J(\omega) = \frac{2\sqrt{\lambda}}{\pi\,\omega}\cdot\Big[\mathbb{K}\left(\omega^2\right) - \mathbb{E}\left(\omega^2\right)\Big] \Rightarrow \mathcal{J} \equiv \frac{\pi\,J}{\sqrt{\lambda}} = \frac{2}{\sqrt{1-\widetilde{x}}}\cdot\Big[\mathbb{K}\left(1-\widetilde{x}\right) - \mathbb{E}\left(1-\widetilde{x}\right)\Big]. \tag{6.52}$$

$$\gamma \equiv \mathcal{E} - \mathcal{J} = \frac{2}{\sqrt{1-\widetilde{x}}}\left[\left(\sqrt{1-\widetilde{x}}-1\right)\mathbb{K}\left(1-\widetilde{x}\right) + \mathbb{E}\left(1-\widetilde{x}\right)\right], \tag{6.53}$$

$\underline{\omega^2 > 1}$: Folded closed string ($\mathbb{R}\times\mathrm{S}^2$).

$$\bar{\theta}_1(\sigma) = am\left[\kappa\,\sigma\,\middle|\,\omega^2\right], \quad \kappa = \frac{2}{\pi\,\omega}\cdot\mathbb{K}\left(\frac{1}{\omega^2}\right), \quad \omega = \csc\vartheta_0 \tag{6.54}$$

$$E(\omega) = \frac{2\sqrt{\lambda}}{\pi\,\omega}\cdot\mathbb{K}\left(\frac{1}{\omega^2}\right) \Rightarrow \mathcal{E} \equiv \frac{\pi\,E}{\sqrt{\lambda}} = 2\sqrt{1-x}\cdot\mathbb{K}\left(1-x\right) \tag{6.55}$$

$$J(\omega) = \frac{2\sqrt{\lambda}}{\pi}\cdot\left[\mathbb{K}\left(\frac{1}{\omega^2}\right) - \mathbb{E}\left(\frac{1}{\omega^2}\right)\right] \Rightarrow \mathcal{J} \equiv \frac{\pi\,J}{\sqrt{\lambda}} = 2\left[\mathbb{K}\left(1-x\right) - \mathbb{E}\left(1-x\right)\right] \tag{6.56}$$

$$\gamma \equiv \mathcal{E} - \mathcal{J} = 2\left[\left(\sqrt{1-x}-1\right)\cdot\mathbb{K}\left(1-x\right) + \mathbb{E}\left(1-x\right)\right], \tag{6.57}$$

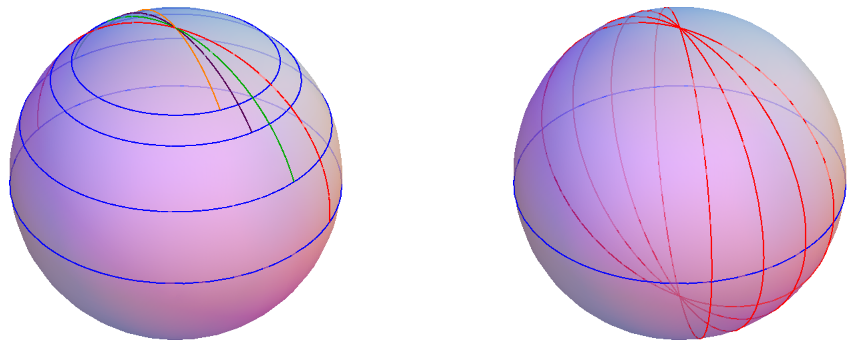


Figure 7: Plots of the $\mathbb{R}\times\mathrm{S}^2$ GKP string (II). On the left, various snapshots of the closed folded string on the sphere have been plotted, for four different values of the angular velocity $\omega > 1$ (each with a different color). This string rotates rigidly around its fixed polar point. On the right, we have plotted four snapshots of a circular string ($\omega < 1$), which rotates rigidly around the polar axis.

where $x \equiv 1 - 1/\omega^2$ and $\widetilde{x} \equiv 1 - \omega^2$ are the complementary parameters of $1/\omega^2$ and ω^2 respectively. In figures 5–6, $\overline{\theta}_1 = \overline{\theta}_1(\sigma)$ has been plotted for various values of $\omega > 1$, while the energy/spin of the $\mathbb{R}\times\mathrm{S}^2$ string have been plotted in terms of the variables ϑ_0, ω and x. In figure 7, both the folded (left) and the circular (right) GKP string has been plotted on a 2-sphere.

For $\omega > 1$, there are two interesting regimes where we would like to obtain $E = E(J)$ and the anomalous dimensions $\gamma = \gamma(J)$, the short-string limit $\omega \to \infty$ and the long-string limit $\omega \to 1^+$.

6.2.1 Short Folded Strings in $\mathbb{R}\times\mathrm{S}^2$: $\omega \to \infty$, $J \ll \sqrt{\lambda}$

The expansions of the energy and spin of short $\mathbb{R}\times\mathrm{S}^2$ strings ($\omega \to \infty$) in terms of the angular frequency ω are given by (cf. appendix H):

$$E = \sqrt{\lambda}\cdot\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\frac{1}{\omega^{2n+1}} = \sqrt{\lambda}\cdot\left[\frac{1}{\omega} + \frac{1}{4\,\omega^3} + \frac{9}{64\,\omega^5} + \frac{25}{256\,\omega^7} + O\left(\frac{1}{\omega^9}\right)\right] \tag{6.58}$$

$$J = \sqrt{\lambda}\cdot\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\frac{2n}{2n-1}\frac{1}{\omega^{2n}} = \frac{\sqrt{\lambda}}{2}\cdot\left[\frac{1}{\omega^2} + \frac{3}{8\,\omega^4} + \frac{15}{64\,\omega^6} + O\left(\frac{1}{\omega^8}\right)\right]. \tag{6.59}$$

We may invert the series (6.59) with e.g. Mathematica and then plug the obtained inverse spin function $x = x(\mathcal{J})$ into the expression for the energy (6.58). This yields $\mathcal{E} = \mathcal{E}(\mathcal{J})$:

$$x = 1 - \frac{2\mathcal{J}}{\pi} + \frac{3\mathcal{J}^2}{2\pi^2} - \frac{3\mathcal{J}^3}{8\pi^3} - \frac{5\mathcal{J}^4}{64\pi^4} + \frac{9\mathcal{J}^5}{512\pi^5} + \frac{21\mathcal{J}^6}{1024\pi^6} + \frac{35\mathcal{J}^7}{8192\pi^7} - \frac{459\mathcal{J}^8}{131.072\pi^8} - \ldots \tag{6.60}$$

$$\mathcal{E} = \sqrt{2}\cdot\left[\pi^{1/2}\mathcal{J}^{1/2} + \frac{\mathcal{J}^{3/2}}{8\,\pi^{1/2}} + \frac{3\,\mathcal{J}^{5/2}}{128\,\pi^{3/2}} + \frac{\mathcal{J}^{7/2}}{1024\,\pi^{5/2}} - \frac{61\,\mathcal{J}^{9/2}}{32.768\,\pi^{7/2}} - \frac{201\,\mathcal{J}^{11/2}}{262.144\,\pi^{9/2}} + \ldots\right]. \tag{6.61}$$

The short string coefficients of the energy of closed folded $\mathbb{R}\times\mathrm{S}^2$ strings (6.58) differ from ones of the AdS_3 strings (6.25) by a factor of $(2n+1)$, $n = 0, 1, \ldots$ Also, the coefficients of the angular momentum J in (6.59) differ from the ones of the spin S in (6.26) by $1/(2n-1)$. This is due to the fact that (6.22) and (6.23) may be obtained from (6.55), (6.56) by differentiation/integration. (6.61) may also be written as follows:

$$E = \left(2\sqrt{\lambda}\, J\right)^{1/2} \cdot \left[1 + \frac{J}{8\sqrt{\lambda}} + \frac{3J^2}{128\,\lambda} + \frac{J^3}{1\,024\,\lambda^{3/2}} - \frac{61 J^4}{32\,768\,\lambda^2} - \frac{201 J^5}{262\,144\,\lambda^{5/2}} + O\left(\frac{J^6}{\lambda^3}\right)\right]. \quad (6.62)$$

6.2.2 Long Folded Strings in $\mathbb{R} \times \mathrm{S}^2$: $\omega \to 1^+$, $J \gg \sqrt{\lambda}$

The energy and the spin of long $\mathbb{R} \times \mathrm{S}^2$ strings ($\omega \to 1^+$) become (using the formulas of appendix H):

$$E = \frac{\sqrt{\lambda}}{\pi^2 \omega} \cdot \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+1/2)}{n!}\right)^2 \left[2\psi(n+1) - 2\psi(n+1/2) - \ln\left(1 - 1/\omega^2\right)\right] \cdot \left(1 - 1/\omega^2\right)^n =$$

$$= \frac{\sqrt{\lambda}}{\pi\omega} \cdot \left\{\left[4\ln 2 - \ln\left(1 - 1/\omega^2\right)\right] + \frac{1}{4}\left(1 - 1/\omega^2\right)\left[4\ln 2 - 2 - \ln\left(1 - 1/\omega^2\right)\right] + \ldots\right\} \quad (6.63)$$

$$J = \frac{\sqrt{\lambda}}{\pi} \cdot \left\{4\ln 2 - 2 - \ln\left(1 - 1/\omega^2\right) - \frac{1}{2\pi}\sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)\,\Gamma(n+3/2)}{((n+1)!)^2}\left[2\,\psi(n+1) - \right.\right.$$

$$\left.\left. -2\,\psi(n+1/2) - \ln\left(1 - 1/\omega^2\right) + \frac{2n}{(n+1)(2n+1)}\right] \cdot \left(1 - 1/\omega^2\right)^{n+1}\right\} =$$

$$= \frac{\sqrt{\lambda}}{\pi} \cdot \left\{\left[4\ln 2 - 2 - \ln\left(1 - 1/\omega^2\right)\right] - \frac{1}{4}\left(1 - 1/\omega^2\right)\left[4\ln 2 - \ln\left(1 - 1/\omega^2\right)\right] + \ldots\right\}. \quad (6.64)$$

Using the complementary parameter $x \equiv 1 - 1/\omega^2 \to 0^+$ we may write the above series as follows:

$$\mathcal{E} \equiv \frac{\pi E}{\sqrt{\lambda}} = 2\sqrt{1-x} \cdot \sum_{n=0}^{\infty} x^n \left(d_n \ln x + h_n\right) = -2\sum_{n=0}^{\infty} x^n \cdot \sum_{k=0}^{n} \frac{(2k-3)!!}{(2k)!!}\left(d_{n-k}\ln x + h_{n-k}\right) \quad (6.65)$$

$$\mathcal{J} \equiv \frac{\pi J}{\sqrt{\lambda}} = 2\sum_{n=0}^{\infty} x^n \left(c_n \ln x + b_n\right). \quad (6.66)$$

The coefficients of (6.65) and (6.66) are given by:

$$d_n = -\frac{1}{2}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2$$

$$h_n = -d_n \cdot \left[4\ln 2 + 2\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{2}{2k-1}\right)\right]$$

$$c_n = \frac{1}{2}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \cdot \frac{1}{2n-1} = -\frac{d_n}{2n-1}$$

$$b_n = -c_n \cdot \left[4\ln 2 + 2\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{2}{2k-1}\right) + \frac{2}{2n-1}\right], \quad (6.67)$$

for $n = 0\,,1\,,2\,,\ldots$ The first few of them are:

$$d_0 = -\frac{1}{2}\,, \qquad d_1 = -\frac{1}{8}\,, \qquad d_2 = -\frac{9}{128}$$

$$h_0 = 2\ln 2\,, \qquad h_1 = \frac{1}{2}\ln 2 - \frac{1}{4}\,, \qquad h_2 = \frac{9}{32}\ln 2 - \frac{21}{128}$$

$$c_0 = -\frac{1}{2}\,, \qquad c_1 = \frac{1}{8}\,, \qquad c_2 = \frac{3}{128}$$

$$b_0 = 2\ln 2 - 1\,, \qquad b_1 = -\frac{1}{2}\ln 2\,, \qquad b_2 = -\frac{3}{32}\ln 2 + \frac{5}{128}. \tag{6.68}$$

6.2.3 Slow Circular Strings in $\mathbb{R}\times \mathbf{S}^2$: $\omega \to 0^+$, $J \ll \lambda$

Despite the fact that the GKP strings on the sphere for which $\omega < 1$ (circular strings) are unstable,[34] they're very similar to the GKP strings with $\omega > 1$ (folded strings) that were studied in §6.2.1–§6.2.2. In this subsection and the next we will obtain the expressions for $E = E\,(J)$ for slow (small J) and fast (large J) circular strings in $\mathbb{R}\times \mathrm{S}^2$. In the case of slow circular strings ($\omega \to 0^+$) the expansions of the energy (6.51) and the spin (6.52) in terms of the angular frequency ω, are given by:

$$E = \sqrt{\lambda}\cdot\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\omega^{2n} = \sqrt{\lambda}\cdot\left[1+\frac{\omega^2}{4}+\frac{9\,\omega^4}{64}+\frac{25\,\omega^6}{256}+\frac{1225\,\omega^8}{16\,384}+O\left(\omega^{10}\right)\right] \tag{6.69}$$

$$J = \sqrt{\lambda}\cdot\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\frac{2n}{2n-1}\cdot\omega^{2n-1} = \frac{\sqrt{\lambda}}{2}\cdot\left[\omega+\frac{3\,\omega^3}{8}+\frac{15\,\omega^5}{64}+\frac{175\,\omega^7}{1024}+O\left(\omega^{9}\right)\right]. \tag{6.70}$$

Inverting the series (6.70) and then plugging the inverse spin function $\widetilde{x} \equiv 1-\omega^2 = \widetilde{x}\,(\mathcal{J})$ into the expression for the energy (6.69), we are lead to $\mathcal{E} = \mathcal{E}\,(\mathcal{J})$:

$$\widetilde{x} = 1 - \frac{4\,\mathcal{J}^2}{\pi^2}+\frac{12\,\mathcal{J}^4}{\pi^4}-\frac{33\,\mathcal{J}^6}{\pi^6}+\frac{175\,\mathcal{J}^8}{2\pi^8}-\frac{1821\,\mathcal{J}^{10}}{8\pi^{10}}+\frac{4683\,\mathcal{J}^{12}}{8\pi^{12}}-\ldots \tag{6.71}$$

$$\mathcal{E} = \pi + \frac{\mathcal{J}^2}{\pi}-\frac{3\,\mathcal{J}^4}{4\pi^3}+\frac{\mathcal{J}^6}{\pi^5}-\frac{103\,\mathcal{J}^8}{64\pi^7}+\frac{183\,\mathcal{J}^{10}}{64\pi^9}-\frac{1383\,\mathcal{J}^{12}}{256\pi^{11}}+\frac{2725\,\mathcal{J}^{14}}{256\pi^{13}}-\ldots \tag{6.72}$$

The latter may also be written as follows:

$$E = \sqrt{\lambda}\cdot\left[1+\frac{J^2}{\lambda}-\frac{3\,J^4}{4\,\lambda^2}+\frac{J^6}{\lambda^3}-\frac{103\,J^8}{64\,\lambda^4}+\frac{183\,J^{10}}{64\,\lambda^5}-\frac{1383\,J^{12}}{256\,\lambda^6}+\frac{2725\,J^{14}}{256\,\lambda^7}-O\left(\frac{J^{16}}{\lambda^8}\right)\right]. \tag{6.73}$$

[34] As GKP put it, $\mathbb{R}\times\mathrm{S}^2$ strings with $\omega < 1$ are unstable towards "slipping off the side" of S^2. In the section for giant magnons we will see that the GKP strings on the sphere can be formed by two giant magnons (having maximum momentum) which are stable in their "elementary" region and unstable in their "doubled" region. The circular GKP strings are unstable because they are formed by two "doubled" GMs.

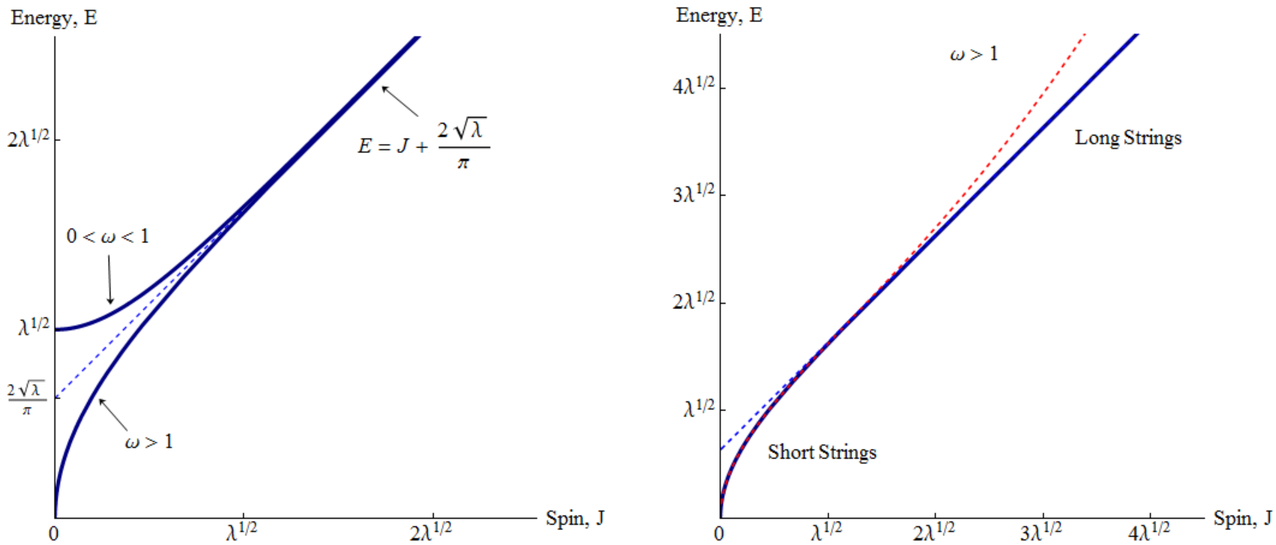


Figure 8: Energy versus the angular momentum of the closed folded/circular $\mathbb{R}\times S^2$ string (6.40). In the right graph ($\omega > 1$), the red dashed line plots the first 4 terms of the "short" approximation (6.62) and the blue dashed line corresponds to the first two terms of the "long" approximation (G.3).

6.2.4 Fast Circular Strings in $\mathbb{R}\times \mathrm{S}^2$: $\omega \to 1^-$, $J \gg \lambda$

The case $\omega \to 1^-$ of fast circular strings on the sphere should be treated similarly to the $\omega \to 1^+$ case:

$$\mathcal{E} \equiv \frac{\pi E}{\sqrt{\lambda}} = 2\sum_{n=0}^{\infty} \widetilde{x}^n \left(d_n \ln \widetilde{x} + h_n\right) \tag{6.74}$$

$$\mathcal{J} \equiv \frac{\pi J}{\sqrt{\lambda}} = \frac{2}{\sqrt{1-\widetilde{x}}} \cdot \sum_{n=0}^{\infty} \widetilde{x}^n \left(c_n \ln \widetilde{x} + b_n\right) = 2\sum_{n=0}^{\infty} \widetilde{x}^n \cdot \sum_{k=0}^{n} \frac{(2k-1)!!}{(2k)!!} \left(c_{n-k} \ln \widetilde{x} + b_{n-k}\right), \tag{6.75}$$

where the complementary parameter is $\widetilde{x} \equiv 1-\omega^2 \to 0^-$, b_n and the coefficients c_n, d_n, h_n are the same as those defined in (6.67)–(6.68).

6.2.5 Short-Long Strings Duality

As we have already mentioned, a short-long strings duality (more precisely a fast/slow folded strings duality) may also be formulated in the case of $\mathbb{R}\times S^2$ strings. If we solve (6.55)–(6.56) for $\mathbb{E}(k)$ and $\mathbb{K}(k)$ and plug them into Legendre's relation (6.37), we're led to the following duality relation between classical folded short and long strings in $\mathbb{R}\times S^2$:

$$\omega\,\omega'\,EE' - \omega\,EJ' - \omega'\,E'J = \frac{2\lambda}{\pi}, \qquad \omega > 1, \quad \lambda \to \infty, \tag{6.76}$$

where the arguments of the elliptic functions are $k = 1/\omega^2$ and $k' = x = 1/\omega'^{\,2}$ respectively and satisfy $k + k' = 1$. Since large values of $\omega' \to \infty$ ("short/slow" strings) correspond to values of $\omega \to 1^+$ near unity ("long/fast" strings), (6.76) provides a map between the corresponding energies and spins. (6.76) is completely analogous to the short-long duality (6.38), found for closed folded strings spinning inside AdS_3 [12]. It is a classical duality between strings that rotate in $\mathbb{R}\times S^2$ but again, it would be interesting to investigate whether it can be generalized to the quantum level. We can also write (6.76) in terms of $\gamma \equiv \mathcal{E} - \mathcal{J}$. A completely analogous relation may be formulated for fast and slow circular strings, using (6.51)–(6.52). The result is:

$$E E' - \omega' \, E J' - \omega \, E' J = \frac{2\lambda}{\pi}, \qquad \omega < 1, \quad \lambda \to \infty, \tag{6.77}$$

where $\widetilde{k} = \omega^2$, $\widetilde{k}' = \widetilde{x} = \omega'^2$ and $\widetilde{k} + \widetilde{k}' = 1$. More about short-long string dualities can be found in appendix E.

6.3 Pulsating Gubser-Klebanov-Polyakov String

Let us complete our brief overview of the GKP bosonic string configurations by presenting setup (III). This consists of a closed folded string that pulsates inside the AdS_3 part of $AdS_5 \times S^5$ and is given by the following ansatz:

$$\Big\{ t = t\,(\tau)\,,\, \rho = \rho(\tau),\, \theta = 0,\, \phi_1 = w\sigma,\, \phi_2 = 0 \Big\} \times \Big\{ \bar{\theta}_1 = \bar{\theta}_2 = \bar{\phi}_1 = \bar{\phi}_2 = \bar{\phi}_3 = 0 \Big\}. \tag{6.78}$$

In embedding space the above ansatz is expressed as follows:

$$\begin{aligned} &Y_0 = \ell \cosh \rho(\tau) \cos t(\tau), \qquad Y_3 = Y_4 = 0, \qquad X_1 = R = \ell \\ &Y_1 = \ell \sinh \rho(\tau) \cos w\sigma \qquad\qquad\qquad\qquad X_2 = X_3 = X_4 = X_5 = X_6 = 0 \\ &Y_2 = \ell \sinh \rho(\tau) \sin w\sigma \\ &Y_5 = \ell \cosh \rho(\tau) \sin t(\tau). \end{aligned} \tag{6.79}$$

The string Polyakov action (in the conformal gauge $\gamma_{ab} = \eta_{ab}$) takes the following form, if we also perform the integral with respect to the σ variable ($\sigma \in [0\,, 2\pi)$):[35]

$$S_P = \frac{\ell^2}{4\pi\alpha'} \int \left(-\dot{t}^2 \cosh^2\rho + \dot{\rho}^2 - \phi_1'^2 \sinh^2\rho\right) d\tau d\sigma = \tag{6.80}$$

$$= \frac{\ell^2}{2\alpha'} \int \left(-\dot{t}^2 \cosh^2\rho + \dot{\rho}^2 - w^2 \sinh^2\rho\right) d\tau. \tag{6.81}$$

The equations of motion and the Virasoro constraints then become:

$$\ddot{t} \cosh^2\rho + 2\,\dot{t}\,\dot{\rho}\, \cosh\rho \sinh\rho = 0 \tag{6.82}$$

$$\ddot{\rho} + \sinh\rho \cosh\rho \left(\dot{t}^2 + w^2\right) = 0 \tag{6.83}$$

$$\dot{\rho}^2 - \dot{t}^2 \cosh^2\rho + w^2 \sinh^2\rho = 0. \tag{6.84}$$

The following equations are obtained:

$$\frac{d\tau}{d\rho} = \frac{\cosh\rho}{w\sqrt{e^2 - \sinh^2\rho \cosh^2\rho}}\,, \qquad e = \frac{\dot{t}}{w} \cdot \cosh^2\rho\,(\tau) \equiv \sinh\rho_0 \cosh\rho_0 = \text{const.}, \tag{6.85}$$

[35]Because the conformal gauge $\gamma_{ab} = \eta_{ab}$ is incompatible with the static time gauge $t = \tau$ (the t-equation of motion (6.82) is not satisfied), we are obliged to use $t = t\,(\tau)$.

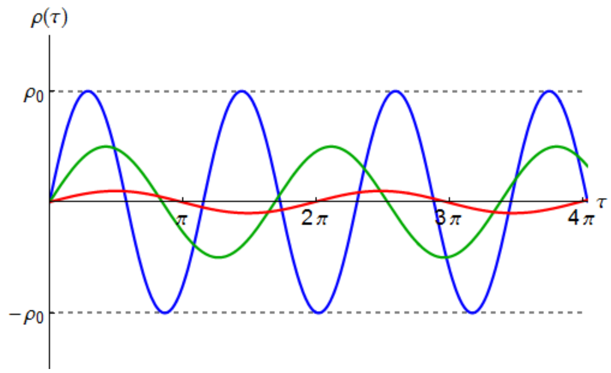


Figure 9: $\rho = \rho(\tau)$ of the pulsating GKP string in AdS_3, given by (6.78).

from which we infer that $\rho < \rho_0$. The conserved energy as well as the string's length are given by:

$$E = \left|\frac{\partial L}{\partial \dot{t}}\right| = \frac{\ell^2}{2\pi\alpha'}\int_0^{2\pi} \dot{t}\cosh^2\rho\, d\sigma = \frac{\ell^2}{\alpha'}\cdot \dot{t}\,\cosh^2\rho = \frac{w\,e\,\ell^2}{\alpha'} = w\sqrt{\lambda}\,e \tag{6.86}$$

$$\tau(\rho) = \int_0^\rho \frac{\cosh\rho\, d\rho}{w\sqrt{e^2 - \cosh^2\rho\sinh^2\rho}} = \int_0^{\sinh\rho} \frac{dx}{w\sqrt{e^2 - x^2 - x^4}}. \tag{6.87}$$

Performing the length integral we obtain $\tau(\rho)$ and, by inversion, $\rho(\tau)$:

$$\tau(\rho) = \frac{\mathbb{F}\left[\arcsin\left(\frac{\sinh\rho}{\sinh\rho_0}\right)\middle| - \tanh^2\rho_0\right]}{w\cosh\rho_0} \Leftrightarrow \tag{6.88}$$

$$\rho(\tau) = \left|\operatorname{arcsinh}\left[\sinh\rho_0\cdot sn\left(w\tau\cosh\rho_0\middle| - \tanh^2\rho_0\right)\right]\right|. \tag{6.89}$$

This is an oscillatory time-periodic solution that we have plotted for various ρ_0's in figure 9.

6.3.1 Semiclassical Quantization

Following [79, 80], we are now going to semiclassically quantize the pulsating GKP string in AdS_3. To facilitate the process, let us switch from global variables to $\tanh\rho = \sin\xi$, with $\xi \in [0, \pi/2]$. Polyakov's action (6.81) becomes:

$$S_P = \frac{\ell^2}{2\alpha'}\int\left(-\dot{t}^2\cosh^2\rho + \dot{\rho}^2 - w^2\sinh^2\rho\right)d\tau = \frac{\ell^2}{2\alpha'}\int \sec^2\xi\left(-\dot{t}^2 + \dot{\xi}^2 - w^2\sin^2\xi\right)d\tau. \tag{6.90}$$

The Hamiltonian density is equal to zero, according to the constraint (6.84):

$$\mathcal{H} = \pi_t\dot{t} + \pi_\xi\dot{\xi} - \mathcal{L} = \frac{\ell^2}{2\alpha'}\left(-\dot{t}^2\cosh^2\rho + \dot{\rho}^2 + w^2\sinh^2\rho\right) = \frac{\ell^2}{2\alpha'}\sec^2\xi\left(-\dot{t}^2 + \dot{\xi}^2 + w^2\sin^2\xi\right) = 0.$$

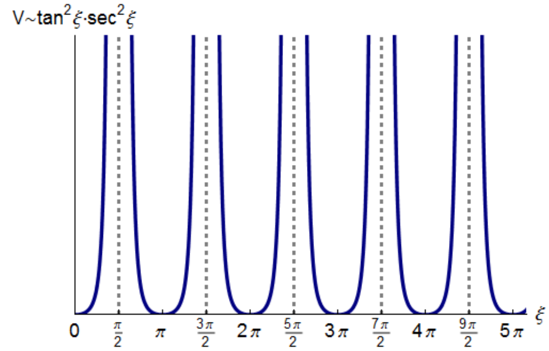


Figure 10: Plot of the effective potential $V = w^2\lambda\, \tan^2\xi\, \sec^2\xi$.

In terms of the canonical variables,

$$\pi_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = -\frac{\ell^2}{\alpha'}\,\dot{t}\,\sec^2\xi \qquad \& \qquad \pi_\xi = \frac{\partial \mathcal{L}}{\partial \dot{\xi}} = \frac{\ell^2}{\alpha'}\,\dot{\xi}\,\sec^2\xi, \tag{6.91}$$

the Hamiltonian density can be written as:

$$\mathcal{H} = \frac{\alpha'}{2\ell^2}\cos^2\xi\,\left(-\pi_t^2 + \pi_\xi^2\right) + \frac{w^2\ell^2}{2\alpha'}\tan^2\xi = 0 \tag{6.92}$$

or equivalently,

$$H^2 = \pi_t^2 = \pi_\xi^2 + \left(\frac{w\,\ell^2}{\alpha'}\right)^2 \tan^2\xi\,\sec^2\xi = \pi_\xi^2 + w^2\lambda\,\tan^2\xi\,\sec^2\xi. \tag{6.93}$$

(6.93) is the Hamiltonian of a Klein-Gordon particle inside the periodic potential (plotted in figure 10) $V = w^2\lambda\,\tan^2\xi\,\sec^2\xi$. This potential typically implies a band structure, however we are going to consider only its classical region here. Start by first-quantizing (6.93) ($\pi_\mu \to i\hbar\partial_\mu$) as follows:

$$-\hbar^2\,\partial_t^2\Psi\,(t,\xi) = -\hbar^2\,\partial_\xi^2\Psi\,(t,\xi) + w^2\lambda\,\tan^2\xi\,\sec^2\xi\cdot\Psi\,(t,\xi) \Rightarrow$$
$$\Rightarrow -\hbar^2\,\psi''\,(\xi) = \left(E^2 - w^2\lambda\,\tan^2\xi\,\sec^2\xi\right)\cdot\psi\,(\xi)\;, \quad \Psi\,(t,\xi) = e^{-i\,E\,t/\hbar}\cdot\psi\,(\xi)\,. \tag{6.94}$$

The allowed energies and wave functions may be found approximately by the WKB method (see e.g. [86]) for $\psi\,(0) = \pm 1$:

$$\int_0^{\xi_0}\sqrt{E_n^2 - w^2\lambda\,\tan^2\xi\,\sec^2\xi}\cdot d\xi = \hbar\left(n + \frac{1}{4}\right)\pi + O\left(\hbar^2\right)\;, \quad n = 0\,,1\,,2\,,\ldots \tag{6.95}$$

$$\psi_n\,(\xi) = (-1)^n\left[1 - \frac{w^2\lambda}{E_n^2}\tan^2\xi\,\sec^2\xi\right]^{-\frac{1}{4}}\cdot\cos\left(\frac{1}{\hbar}\int_0^{\xi}\sqrt{E_n^2 - w^2\lambda\,\tan^2\xi'\,\sec^2\xi'}\cdot d\xi'\right), \tag{6.96}$$

in the physical optics approximation. The integral may be performed and the result is:

$$w\sqrt{\lambda}\cosh\rho_0\cdot\Big\{\sinh^2\rho_0\,\mathbf{\Pi}(-\sinh^2\rho_0;-\tanh^2\rho_0)+\mathbb{K}(-\tanh^2\rho_0)+\mathbb{E}(-\tanh^2\rho_0)\Big\}=$$
$$=\hbar\left(n+\frac{1}{4}\right)\pi+O\left(\hbar^2\right), \quad (6.97)$$

where from the definition (6.85) of $e=\sinh\rho_0\cosh\rho_0$, the single classical turning point $\tanh\rho_0=\sin\xi_0$ satisfies

$$\sinh\rho_0=\left[\frac{1}{2}\left(-1+\sqrt{1+4e^2}\right)\right]^{1/2} \ \& \ \cosh\rho_0=\left[\frac{1}{2}\left(1+\sqrt{1+4e^2}\right)\right]^{1/2}, \ e\equiv\frac{E\alpha'}{w\ell^2}=\frac{E}{w\sqrt{\lambda}}. \quad (6.98)$$

If we expand around $E=\infty$, we retrieve the result of [80]:

$$\hbar\left(2n+\frac{1}{2}\right)\pi+O\left(\hbar^2\right)=\pi E-\frac{2\,(2\pi)^{3/2}}{\Gamma\left(\frac{1}{4}\right)^2}\cdot w^{1/2}\lambda^{1/4}E^{1/2}+\frac{\Gamma\left(\frac{1}{4}\right)^2}{12\,(2\pi)^{1/2}}\cdot w^{3/2}\lambda^{3/4}E^{-1/2}-$$
$$-\frac{3\pi^{3/2}}{20\sqrt{2}\,\Gamma\left(\frac{1}{4}\right)^2}\cdot w^{5/2}\lambda^{5/4}E^{-3/2}+O\left(w^{7/2}\lambda^{7/4}E^{-5/2}\right). \quad (6.99)$$

We may revert (6.99) for large values of n, obtaining the following double series of n and λ:[36]

$$E=2\,n+\frac{8\sqrt{w\pi}}{\Gamma\left(\frac{1}{4}\right)^2}\cdot\lambda^{1/4}n^{1/2}+\left[\frac{1}{2}+\frac{16w\pi}{\Gamma\left(\frac{1}{4}\right)^4}\cdot\lambda^{1/2}\right]+\left[\frac{\sqrt{w\pi}}{\Gamma\left(\frac{1}{4}\right)^2}\cdot\lambda^{1/4}+\left(\frac{16\pi^{\frac{3}{2}}}{\Gamma\left(\frac{1}{4}\right)^6}-\frac{\Gamma\left(\frac{1}{4}\right)^2}{24\pi^{\frac{3}{2}}}\right)w^{3/4}\lambda^{3/4}\right]\cdot n^{-1/2}$$
$$+O\left(n^{-3/2}\right). \quad (6.100)$$

We can also expand around $E=0$:

$$\hbar\left(2n+\frac{1}{2}\right)\pi+O\left(\hbar^2\right)=\frac{E^2}{4\,w\,\lambda^{1/2}}-\frac{5\,E^4}{32\,w^3\,\lambda^{3/2}}+\frac{63\,E^6}{256\,w^5\,\lambda^{5/2}}-\frac{2145\,E^8}{4096\,w^7\,\lambda^{7/2}}+O\left(w^{-9}\,\lambda^{-9/2}\,E^{10}\right) \quad (6.101)$$

Upon inverting for small n we obtain ($\hbar\to 1$):

$$E=2\left(\sqrt{\lambda}\,w\right)^{1/2}\left(n+\frac{1}{4}\right)^{1/2}+\frac{5\left(n+\frac{1}{4}\right)^{3/2}}{2\left(\sqrt{\lambda}\,w\right)^{1/2}}-\frac{77\left(n+\frac{1}{4}\right)^{5/2}}{16\left(\sqrt{\lambda}\,w\right)^{3/2}}+\frac{1365\left(n+\frac{1}{4}\right)^{7/2}}{64\left(\sqrt{\lambda}\,w\right)^{5/2}}+$$
$$+O\left(w^{-7/2}\,\lambda^{-7/4}\,n^{9/2}\right), \quad (6.102)$$

[36] We set $\hbar=1$.

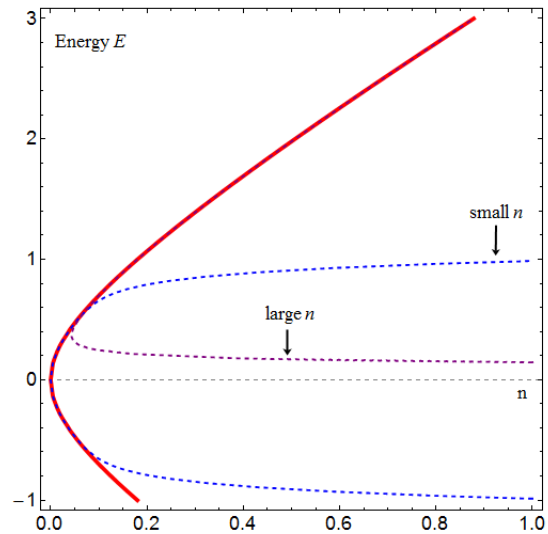


Figure 11: Implicit plot of the energy E versus the level n of the AdS_3 pulsating string, according to (6.97). The blue dashed line corresponds to the "small" n approximation (6.100) and the purple dashed line to the approximation (6.102) for "large" values of n.

which obviously agrees with the GKP formula (6.6). Leading behaviors $\sim \lambda^{1/4} n^{1/2}$ are characteristic of the small-spin limit in which the spacetime is approximately flat. Compare with the other small-spin limits (6.30)–(6.62) as well as with the string energies in flat spacetimes (appendix D).

According to [80], the operators that correspond to the above string states are of the following form:

$$\frac{n!}{\sqrt{(2n)!}} \sum_{\text{perms}} \text{Tr}\left[\mathcal{Z}\mathcal{D}_+^{n_1^+}\mathcal{D}_-^{n_1^-}\ldots\mathcal{D}_+^{n_k^+}\mathcal{D}_-^{n_k^-}\mathcal{Z}\right] \cdot \exp\left(\text{i}\varphi\left(n_1^\pm, \ldots, n_k^\pm\right)\right) \tag{6.103}$$

and the phase is given by

$$\varphi\left(n_1^\pm, \ldots, n_k^\pm\right) = -\frac{2\pi w}{n} \sum_{i\leq j}^{k} n_i^+ n_j^- \quad \& \quad \sum_{i=1}^{k} n_i^\pm = 2n. \tag{6.104}$$

7 Dispersion Relations of GKP Strings

Having presented the basics of GKP strings, we are now going to derive the classical energy-spin relation (aka dispersion relation or anomalous dimensions) of the two rigidly rotating configurations I (6.8) and II (6.40), for large values of the spin. As we have argued above, long rotating strings belong to an exceptional class of configurations where integrability methods are not as impressive as they are at weak coupling, or when the strings and the operators are short. Therefore, until it is known precisely how integrability works in the regime of long operators and strings, more traditional methods (namely quadratures) have to be used in order to be able to calculate the corresponding spectra. One such method was put forward in paper [3], following an earlier attempt by Georgiou and Savvidy [12].

The objective of this section is the calculation of the classical dispersion relation of GKP strings (I) and (II) by using the method of [3]. Since we will be working exclusively on the string theory side of the planar AdS/CFT correspondence, our results will be valid for large values of the 't Hooft coupling constant $\lambda \to \infty$ and for $N_c = \infty$. In this limit, all $1/N_c$ corrections are suppressed. In addition, we shall only be concerned with classical GKP strings, i.e. we are not going to consider any quantum corrections (α' corrections) that these strings generally receive. Formally this also means that $\lambda = \infty$.

As we have said, we are going to deal exclusively with long strings, i.e. strings that have large (yet not infinite) values of conserved charges, $\mathcal{E}, \mathcal{J}, \mathcal{S} \to \infty$. This is one of the few remaining cases where integrability cannot yet offer much help. The results for the string spectra that we shall obtain by using the method of [3] have not been obtained by any other (integrability) method, e.g. Lüscher corrections, the algebraic curve, the TBA, the Y-system or the quantum spectral curve (QSC). We also feel necessary to emphasize that these results are semi-analytical, so that it is impossible to obtain them by using a computer.

There are many reasons why we need to know the dispersion relation of long rigidly rotating GKP strings. First, although integrability tells us that the planar spectra of $\mathcal{N} = 4$ SYM theory and IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ must match because they are described by the same system of algebraic equations, we want to explicitly verify this matching in all possible regimes. Secondly, we want to be able to actually calculate the spectra in order to address some traditional questions of AdS/CFT, but also because we would like to improve the way integrability works in certain limits and possibly even go beyond integrability. Thirdly, we want to investigate the possibility of describing the string and gauge theory spectra by means of closed formulae that are ideally valid for all values of the 't Hooft coupling λ.

Long GKP strings that rotate in AdS_3 (case I) and $\mathbb{R} \times \mathrm{S}^2$ (case II) are respectively dual to the following (long) operators of $\mathcal{N} = 4$ SYM:

$$\mathcal{O}_S = Tr\left[\mathcal{D}_+^m \mathcal{Z} \mathcal{D}_+^{S-m} \mathcal{Z}\right] + \ldots \quad \& \quad \mathcal{O}_J = \mathrm{Tr}\left[\mathcal{X} \mathcal{Z}^m \mathcal{X} \mathcal{Z}^{J-m}\right] + \ldots, \quad S, J \to \infty, \tag{7.1}$$

where $N_c, \lambda = \infty$ and the dots stand for permutations of the fields inside the trace, multiplied by an appropriate coefficient. Twist-2 ($\mathcal{O}_S$) and 2-magnon ($\mathcal{O}_J$) operators belong respectively to the $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$ sectors of $\mathcal{N} = 4$ SYM. At one loop the dilatation operator of these two sectors coincides with the Hamiltonian of the Heisenberg ferromagnetic $\mathrm{XXX}_{\pm 1/2}$ spin chain. Neither of the operators (7.1) is BPS so that the corresponding scaling dimensions contain an anomalous part.

We have already mentioned that "wrapping corrections" appear on the gauge theory side of AdS/CFT when the loop-order becomes greater than the length of the SYM operator. Increasing the number of loops theoretically takes us closer to strong coupling where the string description becomes valid. Tree-level on the string theory side (classical strings) corresponds to ∞ gauge theory loops and, as long as the SYM operator has not an infinite length (or "size"), its dispersion relation is expected to receive wrapping corrections.[37] Conversely, increasing the number of loops from the string theory side by adding $\alpha' \sim \sqrt{\lambda}$ (quantum) corrections to the classical result, moves us towards $\mathcal{N}=4$ SYM.

Let us start from twist-2 operators and the GKP case (I) of the AdS_3 rotating string. From the QCD point of view, twist operators of high spin S play a very important role in deep inelastic scattering (DIS), where their anomalous scaling dimensions are responsible for the (logarithmic) violation of Bjorken scaling. The anomalous dimensions of twist-2 operators have been calculated in perturbative QCD[38] at one-loop [88], two-loops [89] and three loops [90]. The QCD results can be used to compute the corresponding anomalous dimensions in perturbative $\mathcal{N}=1,2,4$ SYM theories. At weak 't Hooft coupling λ, the anomalous dimensions of twist-2, high-spin S operators $\text{Tr}\big[\mathcal{Z}\,\mathcal{D}_+^S\,\mathcal{Z}\big]$ of $\mathcal{N}=4$ SYM, have been calculated at one-loop [91], two-loops [92] and, using the property of transcedentality, to three-loops [93]. We end up with the following logarithmic behavior that is also known as Sudakov scaling:

$$\gamma\,(S\,,g) = \Delta - (S+2) = f(g)\ln S + \ldots\;, \qquad g = \frac{\sqrt{\lambda}}{4\pi} \to 0, \tag{7.2}$$

where $f(g)$ is the cusp anomalous dimension or the universal scaling function of $\mathcal{N}=4$ SYM. It can be computed from the Beisert-Eden-Staudacher (BES) equation at weak [94, 48] and strong coupling [95, 96]. The strong-coupling result agrees with the explicit 2-loop calculation from the string theory side [45, 97, 98].

The general structure of the large-spin expansion of the anomalous dimensions of twist-2 operators $\mathcal{O}_S$ of $\mathcal{N}=4$ SYM theory is identical at weak and strong coupling [99]:

$$E - S = f\ln(S/\sqrt{\lambda}) + \sum_{n=1}^{\infty} f_{(nn)}\frac{\ln^n(S/\sqrt{\lambda})}{S^n} + \sum_{n=2}^{\infty} f_{(nn-1)}\frac{\ln^{n-1}(S/\sqrt{\lambda})}{S^n} + \ldots + \sum_{n=0}^{\infty}\frac{f_n}{S^n}, \tag{7.3}$$

albeit with a different set of coefficients $f_{(nk)}(\sqrt{\lambda})$ in each case:

$$f_{(nk)} = \sum_{m}^{\infty}\tilde{\mathrm{f}}_{nkm}\lambda^m\,, \quad \text{(weak coupling)} \quad \& \quad f_{(nk)} = \sum_{m}^{\infty}\frac{\mathrm{f}_{nkm}}{\left(\sqrt{\lambda}\right)^m}\,, \quad \text{(strong coupling)}. \tag{7.4}$$

Various other methods can be applied to the computation of the anomalous dimensions (7.2) of twist-2 operators (6.2) in perturbative $\mathcal{N}=4$ SYM theory. By analytically solving the Baxter equation, three-loop [100] and four-loop [101] expressions have been obtained. By computing wrapping corrections after three-loops, the anomalous dimensions to four and five-loops have been computed in [102, 103].

At strong coupling λ, all $f_{(nk)}$ can theoretically be obtained from the thermodynamic Bethe ansatz (TBA) [104]. In [99, 105], the coefficients f_0, f_1, $f_{(11)}$, of (7.3) were calculated at one loop using string perturbation theory. It is rather straightforward to compute the first few classical coefficients of

[37] Wrapping corrections are known as "finite-size" corrections because they first appear at the critical loop order L, where L is the size of the system and they vanish for infinite system sizes, $L=\infty$. On the gauge theory side, the size of the system is equal to the length of the spin chain, typically determined by its bare scaling dimension Δ_0, the spins S and J, or the number of magnons M. On the string theory side, the size of the system is determined by the circumference $2\pi r$ of the cylindrical worldsheet, $(\tau,\sigma)\in(-\infty,+\infty)\times[-r,+r]$. It can be shown (see e.g. equation (8.20)) that there exists a parametrization of the periodic spatial worldsheet coordinate $\sigma\,(-r)=\sigma\,(+r)$, such that the conserved string energy $\mathcal{E}\propto r$. Therefore, whenever the energy $\mathcal{E}$ is (in)finite, so is the worldsheet circumference $2\pi r$. Since it is almost always the case that the string's energy $\mathcal{E}$ is an increasing function of the conserved charges $\mathcal{S}$ and $\mathcal{J}$, these can also serve as a measure of the system's size, which will be infinite whenever either of them is infinite and finite whenever both of them are finite.

[38] For further references along with a concise historical perspective, see [87].

(7.3) at strong coupling with a symbolic computations program, e.g. Mathematica (see appendix G.1). However, computer methods are generally limited by the available computer power. No one has ever managed to calculate all the coefficients of (7.3) analytically. In [12] Georgiou and Savvidy succeeded in calculating all the classical leading ($f_{(nn)}$) and subleading ($f_{(nn-1)}$) terms at strong coupling, by introducing an iterative method that can potentially generate them all. In [3] all the classical next-to-subleading coefficients ($f_{(nn-2)}$) were computed by using the Lambert W-function representation of (7.3). In §7.2, we are going to revisit this derivation, giving more details and intermediate results.

Let us now briefly summarize the classical results. First express (7.3) in the following form:

$$\mathcal{E}-\mathcal{S}=\rho_c\,\ln\mathcal{S}+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\rho_{(nk)}\frac{\ln^k\mathcal{S}}{\mathcal{S}^n}=\rho_c\,\ln\mathcal{S}+\rho_0+\sum_{n=1}^{\infty}\rho_{(nn)}\frac{\ln^n\mathcal{S}}{\mathcal{S}^n}+\sum_{n=2}^{\infty}\rho_{(nn-1)}\frac{\ln^{n-1}\mathcal{S}}{\mathcal{S}^n}+$$
$$+\sum_{n=3}^{\infty}\rho_{(nn-2)}\frac{\ln^{n-2}\mathcal{S}}{\mathcal{S}^n}+\ldots+\frac{\rho_1}{\mathcal{S}}+\frac{\rho_2}{\mathcal{S}^2}+\frac{\rho_3}{\mathcal{S}^3}+\ldots,\qquad \mathcal{S},\lambda\to\infty,\tag{7.5}$$

where $\mathcal{E}=\pi E/\sqrt{\lambda}$, $\mathcal{S}=\pi S/\sqrt{\lambda}$. We find the following coefficients:

$$\rho_c=1\quad,\quad\rho_0=3\ln 2-1\quad,\quad\rho_1=\frac{1}{2}\left(3\ln 2-1\right)\quad,\quad\rho_2=-\frac{9\ln^2 2}{8}+\frac{27\ln 2}{16}-\frac{5}{16}.\tag{7.6}$$

We also find,

$$\rho_{(mm)}=\frac{(-1)^{m+1}}{2^m}\frac{1}{m},\tag{7.7}$$

$$\rho_{(m+1,m)}=\frac{(-1)^{m+1}}{2^{m+1}}\Big[H_m+\frac{m}{4}+\,1-3\ln 2\Big]\tag{7.8}$$

$$\rho_{(m+2,m)}=\frac{(-1)^{m+1}}{2^{m+3}}\cdot(m+1)\cdot\Bigg\{H_{m+1}^2-H_{m+1}^{(2)}+\frac{1}{2}\left(m-12\ln 2+5\right)\cdot H_{m+1}+\frac{m\left(m-1\right)}{24}-$$
$$-\frac{3}{2}\left(m+5\right)\ln 2+9\ln^2 2\Bigg\}.\tag{7.9}$$

As we have said, the series $\rho_{(mm)}$ and $\rho_{(m+1,m)}$ were derived for the first time in [12]. The next-to-next-to-leading coefficients $\rho_{(m+2,m)}$ were derived in [3].

2-magnon operators are dual to the $\mathbb{R}\times\mathrm{S}^2$ rotating string, GKP case (II). The GKP strings on the 2-sphere are directly related to giant magnons (GMs) which are open single-spin strings that rotate in $\mathbb{R}\times\mathrm{S}^2$. The GMs are the string theory duals of magnon excitations that belong to the $\mathfrak{su}(2)$ sector of $\mathcal{N}=4$ SYM, encountered in §4.3. The GKP string on the sphere is formed by the superposition of two giant magnons of maximum angular extent $\Delta\varphi=\pi$ and angular momenta $J/2$ each. Therefore GKP strings in $\mathbb{R}\times\mathrm{S}^2$ are dual to 2-magnon operators having maximum momentum $p=\pi$.

The anomalous dimensions of the 2-magnon operators $\mathcal{O}_J$ are given by the asymptotic Bethe ansatz (4.48) up to $J+1$ loops:

$$\Delta-J=2\sqrt{1+\frac{\lambda}{\pi^2}}.\tag{7.10}$$

We may obtain its weak and strong coupling limits as follows:

$$\Delta - J = 2 + \frac{\lambda}{\pi^2} - \frac{\lambda^2}{4\pi^4} + \frac{\lambda^3}{8\pi^6} - \ldots, \qquad \lambda \to 0 \quad \text{(weak coupling)} \tag{7.11}$$

$$\Delta - J = \frac{2\sqrt{\lambda}}{\pi} + 0 + \frac{\pi}{\sqrt{\lambda}} - \frac{\pi^3}{4\lambda^{3/2}} + \ldots, \qquad J, \lambda \to \infty \quad \text{(strong coupling)}. \tag{7.12}$$

Each extra term on the r.h.s. of equations (7.11)–(7.12) corresponds to a quantum (α' or curvature) correction at an increasing loop-order. We see that the bare weak-coupling dimensions $\Delta_0 = J + 2$ get corrected by powers of λ, while the strong coupling result of GKP (6.3) gets corrected by powers of $1/\sqrt{\lambda}$.

On the other hand, wrapping corrections first appear in equation (7.11) at $J+2$ loops. Unless the system's size is infinite ($J = \infty$, in which case there are no wrapping corrections), the wrapping corrections are present in the strong coupling expansion (7.12) even at the tree level (which corresponds to ∞ loop-order from the gauge theory viewpoint). The classical and quantum finite-size corrections that the GKP dispersion relation (6.3) receives at strong coupling and large but finite angular momentum J, have the form of exponentially suppressed terms. The classical part of these finite-size corrections has the following structure:

$$\mathcal{E} - \mathcal{J} = 2 + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \widetilde{\mathcal{A}}_{nm} \mathcal{J}^{n-m-1} \, e^{-n(\mathcal{J}+2)} + O\left(\frac{1}{\sqrt{\lambda}}\right), \quad \lambda, \mathcal{J} \to \infty, \tag{7.13}$$

where $\mathcal{E} \equiv \pi E/\sqrt{\lambda}$ and $\mathcal{J} \equiv \pi J/\sqrt{\lambda}$. It is assumed that the coefficients of all the negative powers of $\mathcal{J}$ in (7.13) are zero (e.g. $\mathcal{A}_{11} = \mathcal{A}_{12} = \ldots = 0$). Many classical terms may be obtained by a direct Mathematica computation (cf. appendix G.1). As it will be shown in §7.1 below, the anomalous dimensions (7.13) can be written in terms of Lambert's W-function $W\left(\pm 4\mathcal{J}e^{-\mathcal{J}-2}\right)$ and take the form

$$\mathcal{E} - \mathcal{J} = 2 - \frac{1}{\mathcal{J}}\left(2W + W^2\right) - \frac{1}{2\mathcal{J}^2}\left(W^2 + W^3\right) - \frac{1}{16\mathcal{J}^3} \frac{W^3\left(11\,W^2 + 26\,W + 16\right)}{1 + W} + \ldots \tag{7.14}$$

where Lambert's W-function is defined by the implicit relation (for more, see appendix I):

$$W(z)\, e^{W(z)} = z \Leftrightarrow W\left(z\, e^z\right) = z. \tag{7.15}$$

The plus sign in the argument of Lambert's W-function in equation (7.14) corresponds to the closed and folded case ($\omega > 1$), while the minus sign corresponds to the case of circular strings ($\omega < 1$). Expanding Lambert's W-function, the second, third and fourth term in (7.14) provide the leading ($\widetilde{\mathcal{A}}_{n0}$), subleading ($\widetilde{\mathcal{A}}_{n1}$) and next-to-subleading ($\widetilde{\mathcal{A}}_{n2}$) terms of (7.13):

- leading terms: $-\dfrac{1}{\mathcal{J}}\left(2W + W^2\right) = \displaystyle\sum_{n=1}^{\infty} \widetilde{\mathcal{A}}_{n0} \mathcal{J}^{n-1} e^{-n(\mathcal{J}+2)}$
- subleading terms: $-\dfrac{1}{2\mathcal{J}^2}\left(W^2 + W^3\right) = \displaystyle\sum_{n=2}^{\infty} \widetilde{\mathcal{A}}_{n1} \mathcal{J}^{n-2} e^{-n(\mathcal{J}+2)}$
- next-to-subleading terms: $-\dfrac{1}{16\mathcal{J}^3} \dfrac{W^3\left(11\,W^2 + 26\,W + 16\right)}{1 + W} = \displaystyle\sum_{n=3}^{\infty} \widetilde{\mathcal{A}}_{n2} \mathcal{J}^{n-3} e^{-n(\mathcal{J}+2)}$.

Precise expressions for these series may be written down, see equations (7.45), (7.58) and (7.63). We may also argue that all the terms of (7.13) (N^k-subleading terms) can be written in terms of Lambert's W-function.

Before proceeding to analytically derive the energy-spin relations (7.5) and (7.14), let us briefly sketch how to obtain them. In the case of equation (7.14) for long folded strings in $\mathbb{R}\times\mathrm{S}^2$ ($\omega > 1$), our starting point is the 2×2 system of equations (6.65)–(6.66):

$$\mathcal{E} = d\,(x)\ln x + h\,(x) \tag{7.16}$$

$$\mathcal{J} = c\,(x)\ln x + b\,(x)\,, \tag{7.17}$$

where $x \equiv 1 - 1/\omega^2$ is the complementary parameter of the angular velocity ω and $d\,(x)$, $h\,(x)$, $c\,(x)$, $b\,(x)$ are the power series that appear in (6.65)–(6.66), with coefficients d_n, h_n, c_n and b_n respectively, given in (6.67). All we do is to use the Lagrange-Bürmann inversion formula to invert the equation (7.17) for the inverse spin function $x = x\,(\mathcal{J})$ and then plug it back into (7.16) to obtain the anomalous dimensions $\gamma \equiv \mathcal{E} - \mathcal{J} = \gamma\,(\mathcal{J})$ in terms of Lambert's W-function. This leads to equation (7.14) for the leading, subleading and next-to-subleading coefficients of the dispersion relation of long folded strings in $\mathbb{R}\times\mathrm{S}^2$. The procedure is rather involved technically and that is why its implementation spans the following five sections §7.1.1–§7.1.5.

Once the formalism is set however, it is rather straightforward to reapply it. In §7.1.6 the above algorithm is repeated for fast circular strings in $\mathbb{R}\times\mathrm{S}^2$ ($\omega < 1$) by using the series (6.74)–(6.75) to write down the system (7.16)–(7.17). In §7.2, long folded strings in AdS_3 ($\omega > 1$) are taken up by using the equations (6.33)–(6.34) and the coefficients (6.35) in order to solve the system (7.16)–(7.17). The W-function representation of the dispersion relation of long GKP strings in AdS_3 can then be used to extract the coefficients (7.6)–(7.9) of equation (7.5).

The expressions (7.5)–(7.14), not only give the classical string energies to a remarkable depth, they also provide closed and neatly re-organized formulas for the string spectra. This reorganization sheds light on the structure of the large-spin expansions of the anomalous dimensions of GKP strings and their dual operators, but it could also affect the way that we view the corresponding weak-coupling, small-spin and quantum expansions. Integrability methods might also benefit from better-recognizable structures in the string spectra.

7.1 Gubser-Klebanov-Polyakov String in $\mathbb{R}\times\mathrm{S}^2$

7.1.1 Inverse Spin Function

Let us now see how to invert the J-series of equation (6.66) in terms of $x = x\,(\mathcal{J})$. Begin by solving (6.66) for $\ln x$:

$$\mathcal{J} = 2\sum_{n=0}^{\infty} x^n\Big(c_n \ln x + b_n\Big) \Rightarrow \ln x = \frac{\mathcal{J}/2 - \sum_{n=0}^{\infty} b_n x^n}{\sum_{n=0}^{\infty} c_n x^n} \Rightarrow$$

$$\Rightarrow \ln x = \left[\frac{\mathcal{J}/2 - b_0}{c_0} - \sum_{n=1}^{\infty}\frac{b_n}{c_0}\,x^n\right]\cdot\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{c_k}{c_0}\,x^k\right)^n. \tag{7.18}$$

Subsequently, we perform the products between the series and exponentiate the resulting equation:

$$x = x_0\cdot\exp\Bigg\{-\left[c_1\,\frac{\mathcal{J}}{2} + b_1\,c_0 - b_0 c_1\right]\frac{x}{c_0^2} +$$

$$+\sum_{n=2}^{\infty}\left[\left(\frac{\mathcal{J}}{2}-b_0\right)\frac{\mathbf{P}_n^{(-1)}}{n!}-b_0-\sum_{k=0}^{n-2}b_{n-k-1}\frac{\mathbf{P}_{k+1}^{(-1)}}{(k+1)!}\right]\frac{x^n}{c_0}\Bigg\}, \tag{7.19}$$

where

$$x_0 \equiv \exp\left[\frac{\mathcal{J}/2-b_0}{c_0}\right] = 16\, e^{-\mathcal{J}-2} \tag{7.20}$$

solves (7.18) to lowest order in x and $\mathbf{P}_n^{(r)}$ are known as potential polynomials (defined in appendix J.2). Suppose that we want to solve the following equation:

$$x = x_0 \cdot \exp\left[\sum_{n=1}^{\infty} \mathrm{a}_n\, x^n\right] = x_0 \cdot \exp\left(\mathrm{a}_1\, x + \mathrm{a}_2\, x^2 + \mathrm{a}_3\, x^3 + \ldots\right) \tag{7.21}$$

in terms of x (having computed the a_n's from (7.19)). A possible way to do this is to try to revert the series (7.19) with respect to x by using the Lagrange inversion theorem [85, 106]. As it turns out, the function to be inverted has a very convenient form that significantly simplifies the computation of its inverse. This fact was discovered by J.-L. Lagrange and H. H. Bürmann [107] and the following formula (applied here to the exponential function) is known as the Lagrange-Bürmann inversion formula:

$$x = \sum_{n=1}^{\infty}\frac{x_0^n}{n!}\cdot\left\{\frac{d^{n-1}}{dz^{n-1}}\exp\left[\sum_{m=1}^{\infty} n\,\mathrm{a}_m\, z^m\right]\right\}_{z=0}. \tag{7.22}$$

In order to evaluate the n-th derivative of the exponential of a power series, we use the exponential formula:

$$\exp\left[\sum_{m=1}^{\infty} n\,\mathrm{a}_m\, z^m\right] = \sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{B}_k\left(n\cdot \mathrm{a}_1, 2\,n\cdot \mathrm{a}_2, \ldots, k!\, n\cdot \mathrm{a}_k\right)\, z^k, \tag{7.23}$$

where $\mathbf{B}_n\left(x_1, x_2, \ldots, x_n\right)$ are the (exponential) complete Bell polynomials, defined in appendix J.1. We find,

$$\left\{\frac{d^{n-1}}{dz^{n-1}}\exp\left[\sum_{m=1}^{\infty} n\,\mathrm{a}_m\, z^m\right]\right\}_{z=0} = \mathbf{B}_{n-1}\left(n\cdot \mathrm{a}_1, 2\,n\cdot \mathrm{a}_2, \ldots, (n-1)!\, n\cdot \mathrm{a}_{n-1}\right) =$$

$$= n!\cdot\sum_{k=0}^{n-1}\frac{n^{k-1}}{k!}\widehat{\mathbf{B}}_{n-1,k}\left(\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_{n-1}\right) = \sum_{k, j_i=0}^{n-1} n^k \binom{n-1}{j_1\,,\; j_2\,,\; \ldots\,,\; j_{n-1}}\mathrm{a}_1^{j_1}\mathrm{a}_2^{j_2}\ldots\mathrm{a}_{n-1}^{j_{n-1}}, \tag{7.24}$$

where $\widehat{\mathbf{B}}_{n,k}\left(x_1, x_2, \ldots, x_n\right)$ are the (ordinary) partial Bell polynomials (see appendix J.1) and

$$j_1 + j_2 + \ldots + j_{n-1} = k \quad \& \quad j_1 + 2\,j_2 + \ldots + (n-1)\,j_{n-1} = n-1. \tag{7.25}$$

The parameter x then becomes:

$$x = \sum_{n=1}^{\infty} x_0^n \cdot \sum_{k=0}^{n-1} \frac{n^{k-1}}{k!} \, \widehat{\mathbf{B}}_{n-1,k}(\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_{n-1}) =$$

$$= \sum_{n=1}^{\infty} x_0^n \cdot \sum_{k, j_i=0}^{n-1} \frac{n^k}{n!} \binom{n-1}{j_1\,,\; j_2\,,\; \ldots\,,\; j_{n-1}} \mathrm{a}_1^{j_1} \mathrm{a}_2^{j_2} \ldots \mathrm{a}_{n-1}^{j_{n-1}}. \tag{7.26}$$

Now notice that (7.19) implies that the a_i's can only depend linearly on $\mathcal{J}$, so that the inverse spin function $x = x(\mathcal{J})$ has to be of the following form:

$$x = \sum_{n=1}^{\infty} x_0^n \cdot \sum_{k=0}^{n-1} a_{nk} J^k, \tag{7.27}$$

where the coefficients a_{nk} do not depend on J. To see why (7.27) must be true, consider the following two constraints on the values of j:

$$\left.\begin{array}{r} j_1 + j_2 + \ldots + j_{n-1} = k \\ j_1 + 2\, j_2 + \ldots + (n-1)\, j_{n-1} = n-1 \end{array}\right\} \Rightarrow k + j_2 + \ldots + (n-2)\, j_{n-1} = n-1, \tag{7.28}$$

i.e. the maximum power of J in (7.27) is $n-1$. Another conclusion that is implied by these two constraints is that all the leading in $\mathcal{J}$ contributions to x are determined by the leading in $\mathcal{J}$ terms of a_1, all the subleading in $\mathcal{J}$ contributions to x are controlled by a_1 and the leading in $\mathcal{J}$ terms of a_2, etc., i.e. all the coefficients of $x(\mathcal{J})$ up to $x_0^n \, \mathcal{J}^{n-m}$ are controlled by $\mathrm{a}_1, \ldots \mathrm{a}_{m-1}$ and the leading term of a_m. This is better understood if one notices from $k + j_2 + \ldots + (n-2)\, j_{n-1} = n-1$ that when some j_m in (7.26) is $j_m \neq 0$ (minimum value 1), $k = j_m + \ldots + j_{n-1}$ is at most $n-1-(m-1) = n-m$. This conclusion concerning the number of terms that fully determine $x(\mathcal{J})$ agrees with what we expect from equation (7.21).

7.1.2 Anomalous Dimensions

Having $x(\mathcal{J})$ at our disposal, it is possible to express the anomalous scaling dimensions $\gamma = \mathcal{E} - \mathcal{J}$ of the $\mathbb{R} \times \mathrm{S}^2$ closed folded string as a function of $\mathcal{J}$:

$$\mathcal{E} - \mathcal{J} = 2 \sum_{n=0}^{\infty} x^n \left(f_n \, \ln x + g_n\right) = 2 \sum_{n=0}^{\infty} x^n \left[A_n + f_n \, \ln \frac{x}{x_0}\right], \quad \mathcal{E} \equiv \frac{\pi E}{\sqrt{\lambda}}, \quad \mathcal{J} \equiv \frac{\pi J}{\sqrt{\lambda}}, \tag{7.29}$$

where,

$$f_n \equiv -c_n - \sum_{k=0}^{n} \frac{(2k-3)!!}{(2k)!!} \cdot d_{n-k}\,, \quad g_n \equiv -b_n - \sum_{k=0}^{n} \frac{(2k-3)!!}{(2k)!!} \cdot h_{n-k}\,, \quad n = 0, 1, 2, \ldots \tag{7.30}$$

The first few of the coefficients f_n and g_n are:

$$f_0 = 0, \qquad f_1 = 0, \qquad f_2 = \frac{1}{32}, \qquad f_3 = \frac{3}{128}$$

$$g_0 = 1, \qquad g_1 = -\frac{1}{4}, \qquad g_2 = -\frac{1}{8}\ln 2 - \frac{5}{64}, \qquad g_3 = -\frac{3}{32}\ln 2 - \frac{7}{256}. \tag{7.31}$$

The coefficients A_n are defined as:

$$A_n \equiv g_n + f_n \ln x_0 = g_n + 2 f_n \left(2\ln 2 - \frac{\mathcal{J}}{2} - 1\right) \tag{7.32}$$

and the first few A's are:

$$A_0 = 1\,, \quad A_1 = -\frac{1}{4}\,, \quad A_2 = -\frac{1}{64}\left(2\mathcal{J} + 9\right)\,, \quad A_3 = -\frac{1}{256}\left(6\mathcal{J} + 19\right). \tag{7.33}$$

For large spin $\mathcal{J}$, we may invert the series (7.18)–(7.19) and obtain $x = x\left(\mathcal{J}\right)$ by using Mathematica. Then the inverse spin function $x\left(\mathcal{J}\right)$ can be inserted into equation (7.29) and give the energy-spin relation of the GKP string (II), or equivalently the anomalous dimensions of the $\mathcal{N} = 4$ SYM operators $\text{Tr}\left[\mathcal{X}\mathcal{Z}^m\,\mathcal{X}\,\mathcal{Z}^{J-m}\right]$ as a function the (large) R-charge $\mathcal{J}$. The results of such a computation for the inverse spin function $x = x\left(\mathcal{J}\right)$ and the anomalous dimensions $\gamma = \gamma\left(\mathcal{J}\right)$ can be found in equations (G.2)–(G.3) of appendix G. Both series contain the following kinds of terms:

$$\begin{aligned} &\text{Leading terms (L): } \mathcal{J}^{n-1}\,\left(e^{-\mathcal{J}-2}\right)^n \\ &\text{Next-to-leading/Subleading terms (NL): } \mathcal{J}^{n-2}\,\left(e^{-\mathcal{J}-2}\right)^n \\ &\text{NNL terms: } \mathcal{J}^{n-3}\,\left(e^{-\mathcal{J}-2}\right)^n \\ &\qquad\qquad\vdots \end{aligned} \tag{7.34}$$

Using the fact that the series (6.65), (6.66) and (7.29) have the same structure, we may prove the following corollary. To obtain $\mathcal{E} - \mathcal{J}$ up to a given subleading order, we have to know the inverse spin function $x\left(\mathcal{J}\right)$, that has to be inserted into (7.29) in order to give $\mathcal{E} = \mathcal{E}\left(\mathcal{J}\right)$, up to no more than the same order. Using the equations (7.21) and (7.27), we obtain:

$$\ln\frac{x}{x_0} = \sum_{k=1}^{\infty} \mathrm{a}_k\, x^k = \mathrm{a}_1\, x + \mathrm{a}_2\, x^2 + \mathrm{a}_3\, x^3 + \ldots \tag{7.35}$$

$$x = \sum_{n=1}^{\infty} x_0^n \cdot \sum_{k=0}^{n-1} a_{nk}\mathcal{J}^k = \sum_{n=1}^{\infty} \mathcal{J}^{n-1}\, x_0^n \cdot \sum_{k=0}^{n-1} \frac{\widetilde{a}_{nk}}{\mathcal{J}^k} = \frac{1}{\mathcal{J}} \sum_{n=1}^{\infty} \mathcal{J}^n\, x_0^n \cdot \sum_{k=0}^{n-1} \frac{\widetilde{a}_{nk}}{\mathcal{J}^k}, \tag{7.36}$$

where $a_{nm} = \widetilde{a}_{n(n-k-1)}$ are constants and a_n are linear functions of $\mathcal{J}$. The last equation follows from (7.27) after some reshuffling. The energy-spin relation (7.29) is then written as follows:

$$\mathcal{E} - \mathcal{J} = 2\sum_{n=0}^{\infty} x^n \left(f_n\, \ln x + g_n\right) = 2\sum_{n=0}^{\infty} x^n \left[A_n + f_n\, \ln\frac{x}{x_0}\right] = 2\sum_{n=0}^{\infty} x^n \left[A_n + \sum_{k=1}^{\infty} f_n\, \mathrm{a}_k\, x^k\right]. \tag{7.37}$$

All the leading terms of x^n are of the order $1/\mathcal{J}^n$ (observe the form of the expansion (7.36) for x, which is nothing more than the equation (G.2) written symbolically) and they multiply either A_n or $f_{n-k} \cdot \mathrm{a}_k$ in the expression (7.37) for $\mathcal{E} - \mathcal{J}$, both of which are linear in $\mathcal{J}$. Thus the r-th subleading term of $\mathcal{E} - \mathcal{J}$ (which is of the order $1/\mathcal{J}^r$) cannot receive contributions from its x^{r+2} terms (for which $\mathcal{J}/\mathcal{J}^{r+2} \sim \mathcal{J}^{r+1}$). Therefore, in order to get precisely the first r-subleading orders of $\mathcal{E} - \mathcal{J}$ ($r = 1$ leading, $r = 2$ subleading, etc.), no more than the first $r + 1$ powers of x must be retained in (7.37). Additionally, the last power of x to be kept in (7.37) (namely x^{r+1}) should not be multiplied by terms which are independent of $\mathcal{J}$.

We can then see why we need precisely n subleading terms in the x-expansion in order to calculate $\mathcal{E} - \mathcal{J}$ up to the n-th subleading order. Keep less powers inside x and $x \cdot A_1 = -x/4$ will miss some of the subleading terms. Terms deeper than $1/\mathcal{J}^n$ into x cannot contribute, since there exist no powers of $\mathcal{J}$ in the expression for $\mathcal{E} - \mathcal{J}$ that can potentially lift them up to the wanted power. All of these observations will become clearer below.

7.1.3 Leading Terms

Let us now see how the above can be applied to the computation of the anomalous dimensions to leading order in $\mathcal{J}$. We will compute the coefficients of the following series:

$$E - J\Big|_{(\mathrm{L})} = \sum_{n=1}^{\infty} \widetilde{\mathcal{A}}_{n0}\, \mathcal{J}^{n-1} \left(e^{-\mathcal{J}-2}\right)^n . \tag{7.38}$$

As we have explained, we only need to find the leading terms of x, i.e. the terms of the following series:

$$x_{(\mathrm{L})} = \sum_{n=1}^{\infty} \alpha_n\, \mathcal{J}^{n-1} \left(e^{-\mathcal{J}-2}\right)^n . \tag{7.39}$$

The leading term of x is in turn determined if on the r.h.s. of (7.18) we keep all the terms that multiply $x^0 = 1$ and just the leading in $\mathcal{J}$ terms that multiply $x^1 = x$. (7.18) then becomes:

$$\ln x_{(\mathrm{L})} = \frac{\mathcal{J}/2 - b_0}{c_0} - \frac{c_1}{c_0^2}\frac{\mathcal{J}}{2} \cdot x_{(\mathrm{L})} \Rightarrow x_0 = x_{(\mathrm{L})}\, \exp\left[\frac{c_1}{c_0^2}\frac{\mathcal{J}}{2} \cdot x_{(\mathrm{L})}\right] = x_{(\mathrm{L})}\, e^{\mathcal{J} \cdot x_{(\mathrm{L})}/4}, \tag{7.40}$$

where $x_0 = 16\, e^{-\mathcal{J}-2}$. This is equation (7.21) for the leading terms of x. We may solve it either by the inversion method that was described in the previous section, or we can calculate the following tetration:

$$x_{(\mathrm{L})} = x_0\, e^{-x_0 \mathcal{J}/4 \cdot e^{-x_0 \mathcal{J}/4 \cdot e^{\cdots}}} = x_0 \cdot {}^{\infty}\!\left(e^{-x_0 \mathcal{J}/4}\right) . \tag{7.41}$$

There's a neat formula for the infinite exponential appearing in (7.41) involving the Lambert W-function (for the definition and the properties of the W-function, the reader is referred to appendix I)

$${}^{\infty}(e^z) = \frac{W(-z)}{-z} = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}\, z^{n-1}, \tag{7.42}$$

in its principal branch W_0.[39] We therefore find

$$x_{(\mathrm{L})}= \frac{4}{\mathcal{J}}\, W\left(4\,\mathcal{J}\, e^{-\mathcal{J}-2}\right) = \sum_{n=1}^{\infty} \alpha_n\, \mathcal{J}^{n-1}\, \left(e^{-\mathcal{J}-2}\right)^n\,, \tag{7.43}$$

where we have defined:

$$\alpha_n \equiv (-1)^{n+1}\, 2^{2n+2} \cdot \frac{n^{n-1}}{n!}\,. \tag{7.44}$$

To obtain the energy-spin relation to leading order in $\mathcal{J}$, we have to insert formula (7.43) for $x_{(\mathrm{L})}$ into (7.29) and keep only the leading terms. The result is:

$$\begin{aligned} E - J\Big|_{(\mathrm{L})} &= \frac{2\sqrt{\lambda}}{\pi}\left\{1 + g_1\, x_{(\mathrm{L})} - f_2\, \mathcal{J} x^2_{(\ell)}\right\} = \frac{2\sqrt{\lambda}}{\pi}\left\{1 - \frac{x_{(\mathrm{L})}}{4} - \frac{\mathcal{J} x^2_{(\mathrm{L})}}{32}\right\} = \\ &= \frac{2\sqrt{\lambda}}{\pi}\left\{1 - \frac{1}{2\mathcal{J}}\left[2\cdot W\left(4\,\mathcal{J}\, e^{-\mathcal{J}-2}\right) + W^2\left(4\,\mathcal{J}\, e^{-\mathcal{J}-2}\right)\right]\right\} = \\ &= \frac{2\sqrt{\lambda}}{\pi}\left\{1 - \frac{1}{16}\sum_{n=1}^{\infty}\left[4\,\alpha_n + \sum_{k=1}^{n-1}\alpha_k\,\alpha_{n-k}\right]\cdot \mathcal{J}^{n-1}\left(e^{-\mathcal{J}-2}\right)^n\right\}. \end{aligned} \tag{7.45}$$

These are all the leading terms $\widetilde{\mathcal{A}}_{n0}$ of (7.13).

7.1.4 Next-to-Leading Terms

To calculate the subleading coefficients of the anomalous dimensions

$$E - J\Big|_{(\mathrm{NL})} = \sum_{n=2}^{\infty} \widetilde{\mathcal{A}}_{n1}\, \mathcal{J}^{n-2}\, \left(e^{-\mathcal{J}-2}\right)^n\,, \tag{7.46}$$

we need the leading and subleading terms of x in (7.26):

$$x_{(\mathrm{NL})} = \sum_{n=2}^{\infty} \beta_n\, \mathcal{J}^{n-2}\, \left(e^{-\mathcal{J}-2}\right)^n\,. \tag{7.47}$$

This means that we only have to keep all the terms that multiply $x^{0,1}$ on the r.h.s. of (7.18), and only the leading in $\mathcal{J}$ terms that multiply x^2. Equation (7.18), precise up to next-to-leading/subleading (NL) order becomes:

$$\ln x_{(\mathrm{L+NL+\ldots})} = \frac{\mathcal{J}/2 - b_0}{c_0} - \frac{\mathcal{J} c_1/2 + b_1 c_0 - b_0 c_1}{c_0^2}\cdot x_{(\mathrm{L+NL+\ldots})} + \frac{c_1^2 - c_0 c_2}{c_0^3}\,\frac{\mathcal{J}}{2}\cdot x^2_{(\mathrm{L+NL+\ldots})} \Rightarrow$$

[39] We must choose the principal branch W_0 so that x has the correct behavior, $x \to 0^+$ as $\mathcal{J} \to +\infty$. Conversely, in the W_{-1} branch, $x \to -4$. More, in appendix I.

$$\Rightarrow x_{(\mathrm{L+NL+}\ldots)} = x_0 \cdot \exp\left[-\frac{\mathcal{J}+2}{4} \cdot x_{(\mathrm{L+NL+}\ldots)} - \frac{7\mathcal{J}}{64} \cdot x^2_{(\mathrm{L+NL+}\ldots)}\right]. \tag{7.48}$$

To solve this equation, we first invert it by means of the Lagrange-Bürmann formula. Writing,

$$x_0 = x_{(\mathrm{L+NL+}\ldots)} \cdot \exp\left[\frac{\mathcal{J}+2}{4} \cdot x_{(\mathrm{L+NL+}\ldots)} + \frac{7\mathcal{J}}{64} \cdot x^2_{(\mathrm{L+NL+}\ldots)}\right], \tag{7.49}$$

we find the inverse as in equation (7.26). Explicitly

$$x_{(\mathrm{L+NL+}\ldots)} = \sum_{n=1}^{\infty} \frac{d^{n-1}}{dx^{n-1}} \left\{\exp\left[-\frac{\mathcal{J}+2}{4} \cdot n\, x - \frac{7\mathcal{J}}{64} \cdot n\, x^2\right]\right\}\Bigg|_{x=0} \cdot \frac{x_0^n}{n!}. \tag{7.50}$$

Noting that

$$\exp\left(\alpha\, x + \beta\, x^2\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \frac{d^n}{dz^n} \left\{\exp\left(\alpha\, z + \beta\, z^2\right)\right\}\Bigg|_{z=0} = \sum_{n=0}^{\infty} \frac{\left(\alpha\, x + \beta\, x^2\right)^n}{n!} =$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{\substack{k,\, j_1 = 0 \\ n = k + j_1 \\ 0 \le j_1 \le k}}^{n} \frac{(k+j_1)!}{(k-j_1)!\, j_1!}\, \alpha^{k-j_1} \beta^{j_1} \Rightarrow$$

$$\Rightarrow \frac{d^n}{dz^n} \left\{\exp\left(\alpha\, z + \beta\, z^2\right)\right\}\Bigg|_{z=0} = \sum_{\substack{k,\, j_1 = 0 \\ n = k + j_1 \\ 0 \le j_1 \le k}}^{n} \frac{(k+j_1)!}{(k-j_1)!\, j_1!}\, \alpha^{k-j_1} \beta^{j_1}, \tag{7.51}$$

we find:

$$x_{(\mathrm{L+NL+}\ldots)} = \sum_{n=1}^{\infty} \frac{x_0^n}{n!} \sum_{\substack{k,\, j_1 = 0 \\ n-1 = k + j_1 \\ 0 \le j_1 \le k}}^{n-1} (-1)^k\, n^k\, \frac{(n-1)!}{(k-j_1)!\, j_1!} \cdot \left(\frac{\mathcal{J}+2}{4}\right)^{k-j_1} \left(\frac{7\mathcal{J}}{64}\right)^{j_1}. \tag{7.52}$$

The next step is to select and keep only the leading (L) and next-to-leading (NL) terms. Begin by expanding the binomial in powers of $\mathcal{J}$:

$$\left(\frac{\mathcal{J}+2}{4}\right)^{k-j_1} \cdot \left(\frac{7\mathcal{J}}{64}\right)^{j_1} = \left(\frac{1}{2}\right)^{k-j_1} \left(\frac{7\mathcal{J}}{64}\right)^{j_1} \cdot \sum_{m=0}^{k-j_1} \binom{k-j_1}{m} \left(\frac{\mathcal{J}}{2}\right)^{m} =$$

$$= \frac{7^{j_1}}{2^{k+4j_1}} \cdot \sum_{m=0}^{k-j_1} \binom{k-j_1}{m} \left(\frac{\mathcal{J}}{2}\right)^{m+j_1} = \frac{7^{j_1}}{2^{k+4j_1}} \cdot \left(\left(\frac{\mathcal{J}}{2}\right)^{j_1} + \ldots + (k-j_1)\left(\frac{\mathcal{J}}{2}\right)^{k-1} + \left(\frac{\mathcal{J}}{2}\right)^{k}\right).$$

The leading terms $\mathcal{J}^{n-1}\, x_0^n$ correspond to $k = n-1$, $m = j_1 = 0$ and give rise to the leading power series (7.43)–(7.44) of the previous section. The next-to-leading/subleading terms $\mathcal{J}^{n-2}\, x_0^n$ correspond to the sum of the terms with either $k = n-1$, $j_1 = 0$ and $m = 1$ or $k = n-2$, $j_1 = 1$ and $m = 0$. We find:

$$x_{(\mathrm{NL})} = \sum_{n=1}^{\infty} \frac{x_0^n}{n!} \cdot \left\{ (-1)^{n-1}\, n^{n-1}\, \frac{(n-1)}{2^{2n-3}} + (-1)^{n-2}\, n^{n-2}\, \frac{7\,(n-1)\,(n-2)}{2^{2n}} \right\} \cdot \mathcal{J}^{n-2}. \tag{7.53}$$

The leading and the next-to-leading terms of x are given by:

$$x_{(\mathrm{L+NL})} = \sum_{n=1}^{\infty} \left(\alpha_n\, \mathcal{J}^{n-1} + \beta_n\, \mathcal{J}^{n-2}\right) \cdot \left(e^{-\mathcal{J}-2}\right)^n \tag{7.54}$$

where the α's are defined in (7.44) and

$$\beta_n \equiv (-1)^{n+1}\, 2^{2n} \cdot \frac{n^{n-2}}{n!} \cdot (n-1)\,(n+14)\,. \tag{7.55}$$

We can express the series (7.54) with the aid of Lambert's W-function, by using the formulas (I.8)–(I.13) of appendix I:

$$x_{(\mathrm{L+NL})} = \sum_{n=1}^{\infty} \left(\alpha_n\, \mathcal{J}^{n-1} + \beta_n\, \mathcal{J}^{n-2}\right) \cdot \left(e^{-\mathcal{J}-2}\right)^n = \frac{4}{\mathcal{J}}\, W - \frac{1}{\mathcal{J}^2}\, \frac{W^2\,(7\,W+8)}{1+W}, \tag{7.56}$$

where the argument of the W-function is $4\,\mathcal{J}\, e^{-\mathcal{J}-2}$. To obtain the leading and the next-to-leading coefficients of the dispersion relation, insert (7.56) into (7.29) and keep only the terms of the leading and the next-to-leading/subleading order:

$$\begin{aligned} E - J\Big|_{(\mathrm{L+NL})} = \frac{2\sqrt{\lambda}}{\pi} \Bigg\{ & 1 + A_1\left(x_{(\mathrm{L})} + x_{(\mathrm{NL})}\right) - f_2\, \mathcal{J}\, x_{(\mathrm{L})}^2 - 2 f_2\, \mathcal{J}\, x_{(\mathrm{L})} \cdot x_{(\mathrm{NL})} + \\ & + \left(g_2 + 2\,(2\ln 2 - 1)\, f_2\right) x_{(\mathrm{L})}^2 - \left(\frac{c_1 f_2}{2 c_0^2} + f_3\right) \mathcal{J}\, x_{(\mathrm{L})}^3 \Bigg\} = \\ = \frac{2\sqrt{\lambda}}{\pi} \Bigg\{ & 1 - \frac{x_{(\mathrm{L})}}{4} - \frac{x_{(\mathrm{NL})}}{4} - \frac{\mathcal{J}}{32}\, x_{(\mathrm{L})}^2 - \frac{\mathcal{J}}{16}\, x_{(\mathrm{L})} \cdot x_{(\mathrm{NL})} - \frac{9}{64}\, x_{(\mathrm{L})}^2 - \frac{\mathcal{J}}{32}\, x_{(\mathrm{L})}^3 \Bigg\}. \end{aligned} \tag{7.57}$$

From this expression we can read the next-to-leading/subleading coefficients (the leading ones were given in (7.45)):

$$E - J\Big|_{(\mathrm{NL})} = -\frac{2\sqrt{\lambda}}{\pi} \left\{ \frac{x_{(\mathrm{NL})}}{4} + \frac{\mathcal{J}}{16}\, x_{(\mathrm{L})} \cdot x_{(\mathrm{NL})} + \frac{9}{64} x_{(\mathrm{L})}^2 + \frac{\mathcal{J}}{32} x_{(\mathrm{L})}^3 \right\} = -\frac{2\sqrt{\lambda}}{\pi}\, \frac{1}{4\mathcal{J}^2}\left(W^2 + W^3\right) \Rightarrow$$

$$\Rightarrow E - J\Big|_{(\mathrm{NL})} = -\frac{\sqrt{\lambda}}{32\,\pi} \sum_{n=1}^{\infty} \Bigg\{ 16\,\beta_n + \sum_{k=1}^{n-1} \alpha_k \left[9\,\alpha_{n-k} + 8\,\beta_{n-k} \right] +$$

$$+4\sum_{k,m=1}^{n-2}\alpha_k\,\alpha_m\,\alpha_{n-k-m}\Bigg\}\cdot\mathcal{J}^{n-2}\left(e^{-\mathcal{J}-2}\right)^n. \tag{7.58}$$

7.1.5 NNL Terms

Likewise, we can go on and compute higher-order terms in the long-string expansion of $E-J$. Equation (7.26) gives,

$$x_{(\mathrm{L+NL+NNL+}\dots)}=\sum_{n=1}^{\infty}\frac{x_0^n}{n!}\cdot\sum_{k,j=0}^{n-1}\frac{(-1)^k\,n^k\,(n-1)!}{(k-j_1-j_2)!\,j_1!\,j_2!}\left(\frac{\mathcal{J}+2}{4}\right)^{k-j_1-j_2}\left(\frac{7\mathcal{J}+9}{64}\right)^{j_1}\left(\frac{15\mathcal{J}}{256}\right)^{j_2}, \tag{7.59}$$

with $n-1=k+j_1+2j_2$ and $0\le j_1+j_2\le k$. We need to keep only the leading (L), subleading (NL) and next-to-subleading (NNL) terms, which can be used to write the resulting power series in terms of Lambert's W-function by using the formulas (I.8)–(I.13) of appendix I. We find:

$$x_{(\mathrm{L+NL+NNL})}=\sum_{n=1}^{\infty}\left(\alpha_n\,\mathcal{J}^{n-1}+\beta_n\,\mathcal{J}^{n-2}+\gamma_n\,\mathcal{J}^{n-3}\right)\cdot\left(e^{-\mathcal{J}-2}\right)^n \tag{7.60}$$

where the γ_n's are defined as (the α's and β's are defined in (7.44)–(7.55)):

$$\gamma_n\equiv(-1)^{n+1}\,2^{3n-6}\cdot\frac{n^{n-3}}{n!}\cdot(n-1)\,(n-2)\left(n^2+41n+228\right). \tag{7.61}$$

The inverse spin function $x=x\left(\mathcal{J}\right)$ (up to NNL order) is then given by

$$x_{(\mathrm{L+NL+NNL})}=\frac{4}{\mathcal{J}}\,W-\frac{1}{\mathcal{J}^2}\frac{W^2\,(7\,W+8)}{1+W}+\frac{1}{8\,\mathcal{J}^3}\frac{W^3\left(76W^3+269W^2+312W+120\right)}{(1+W)^3}, \tag{7.62}$$

where the arguments of the W-functions are $W\left(4\,\mathcal{J}\,e^{-\mathcal{J}-2}\right)$. We insert (7.62) into (7.29), keeping only up to next-to-subleading terms. Then the next-to-subleading (NNL) coefficients of $E-J$ are:

$$E-J\Big|_{\mathrm{NNL}}=-\frac{2\sqrt{\lambda}}{\pi}\Bigg\{\frac{x_{(\mathrm{NNL})}}{4}+\frac{9}{32}\,x_{(\mathrm{L})}\cdot x_{(\mathrm{NL})}+\frac{\mathcal{J}}{32}\,x^2_{(\mathrm{NL})}+\frac{\mathcal{J}}{16}\,x_{(\mathrm{L})}\cdot x_{(\mathrm{NNL})}+\frac{23}{256}\,x^3_{(\mathrm{L})}+$$

$$+\frac{3\,\mathcal{J}}{32}\,x^2_{(\mathrm{L})}\cdot x_{(\mathrm{NL})}+\frac{111\,\mathcal{J}}{4096}\,x^4_{(\mathrm{L})}\Bigg\}=-\frac{2\sqrt{\lambda}}{\pi}\,\frac{1}{32\mathcal{J}^3}\,\frac{W^3\left(11\,W^2+26\,W+16\right)}{1+W}\Rightarrow$$

$$\Rightarrow E-J\Big|_{\mathrm{NNL}}=-\frac{\sqrt{\lambda}}{128\,\pi}\sum_{n=1}^{\infty}\Bigg\{64\,\gamma_n+8\sum_{k=1}^{n-1}\Big[9\,\alpha_k\,\beta_{n-k}+2\,\beta_k\,\beta_{n-k}+4\,\alpha_k\,\gamma_{n-k}\Big]+\sum_{k,m=1}^{n-2}\alpha_k\,\alpha_m\cdot$$

$$\cdot\Big[23\,\alpha_{n-k-m}+48\,\beta_{n-k-m}\Big]+\frac{111}{8}\sum_{k,m,s=1}^{n-3}\alpha_k\,\alpha_m\alpha_s\,\alpha_{n-k-m-s}\Bigg\}\cdot\mathcal{J}^{n-3}\left(e^{-\mathcal{J}-2}\right)^n. \tag{7.63}$$

Our final results for the leading, next-to-leading and next-to-next-to-leading terms of the inverse spin function and the anomalous dimensions of the long 2-magnon operators $\text{Tr}\left[\mathcal{X}\mathcal{Z}^m\mathcal{X}\mathcal{Z}^{J-m}\right]$ of $\mathcal{N}=4$ SYM theory at strong 't Hooft coupling are:

$$x = \frac{4\,W}{\mathcal{J}} - \frac{1}{\mathcal{J}^2}\frac{W^2\,(7\,W+8)}{1+W} + \frac{1}{8\,\mathcal{J}^3}\frac{W^3\left(76W^3+269W^2+312W+120\right)}{(1+W)^3} + \ldots \tag{7.64}$$

$$\mathcal{E}-\mathcal{J} = 2 - \frac{1}{\mathcal{J}}\left(2W+W^2\right) - \frac{1}{2\,\mathcal{J}^2}\left(W^2+W^3\right) - \frac{1}{16\,\mathcal{J}^3}\frac{W^3\left(11\,W^2+26\,W+16\right)}{1+W} + \ldots, \tag{7.65}$$

where $\mathcal{E} \equiv \pi E/\sqrt{\lambda}$ and $\mathcal{J} \equiv \pi J/\sqrt{\lambda}$. If the series (7.64) and (7.65) are expanded around $\mathcal{J} \to \infty$ by using Taylor's formula (I.3), it is found that they completely agree with the ones calculated with Mathematica (G.2)–(G.3). Formally there's no obstruction in going deeper and deeper in (7.13) and obtaining all the terms of the classical long string expansion. It seems that the Lambert W-functions will keep appearing to all subleading orders of x and will therefore determine all the orders of $\mathcal{E}-\mathcal{J}$ as well. To see how this may come about, just note that the equation (7.26) will generally contain a term of the form $n^n/n!$ that multiplies some Laurent polynomial of n that originates from the multinomial coefficient and the expansion of the a_i's in powers of $\mathcal{J}$. The formulas (I.8)–(I.13) of appendix I may then be used to express the resulting power series in terms of Lambert's function.

We can compare, if we wish, the equations (7.64) and (7.65) for the inverse spin function and the anomalous dimensions of long strings (i.e. $E \gg \sqrt{\lambda}$) with the corresponding expressions for short strings (6.60)–(6.61) i.e. those for which $J \ll \sqrt{\lambda}$. See also figure 27 in appendix G.1 for the plots of (7.64) and (7.65).

7.1.6 Fast Circular Strings on S^2: $\omega \to 1^-$, $J \gg \lambda$

Fast circular strings on the sphere (having $\omega \to 1^-$) can be treated similarly to long folded strings (for which $\omega \to 1^+$). Let us briefly obtain the corresponding expressions for this case too. Our starting point is the series of anomalous dimensions:

$$\mathcal{E}-\mathcal{J} = 2\sum_{n=0}^{\infty}\widetilde{x}^n\left(f_n\,\ln\widetilde{x}+g_n\right) = 2\sum_{n=0}^{\infty}\widetilde{x}^n\left[A_n + f_n\,\ln\frac{\widetilde{x}}{x_0}\right], \tag{7.66}$$

$$f_n \equiv d_n - \sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}\cdot c_{n-k}\,, \quad g_n \equiv h_n - \sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}\cdot b_{n-k} \tag{7.67}$$

where the complementary parameter is $\widetilde{x} \equiv 1-\omega^2 \to 0^-$ and d_n, h_n, c_n, b_n are defined in (6.67)–(6.68). The A_n's are given by

$$A_n \equiv g_n + f_n \ln x_0 = g_n + f_n\,\left(4\ln 2 - \mathcal{J} - 2\right), \quad n = 0\,,1\,,2\,,\ldots \tag{7.68}$$

and x_0's defined in (7.20). The result is:

$$\widetilde{x} = -\frac{4\,W}{\mathcal{J}} - \frac{1}{\mathcal{J}^2}\frac{W^2\,(9\,W+8)}{1+W} - \frac{1}{8\,\mathcal{J}^3}\frac{W^3\left(140W^3+397W^2+376W+120\right)}{(1+W)^3} + \ldots \tag{7.69}$$

$$\mathcal{E}-\mathcal{J} = 2 - \frac{1}{\mathcal{J}}\left(2W+W^2\right) - \frac{1}{2\,\mathcal{J}^2}\left(W^2+W^3\right) - \frac{1}{16\,\mathcal{J}^3}\frac{W^3\left(11\,W^2+26\,W+16\right)}{1+W} + \ldots, \tag{7.70}$$

where the argument of the W-function is $W\left(-4\,\mathcal{J}\,e^{-\mathcal{J}-2}\right)$. Expanding the series (7.69)–(7.70) around $\mathcal{J}\to\infty$ with the aid of the series (I.3) of appendix I, we find complete agreement with the corresponding large-spin expansions (G.4)–(G.5) that were obtained with Mathematica. Although the inverse spin functions $x\left(\mathcal{J}\right)$ and $\widetilde{x}\left(\mathcal{J}\right)$ are completely different for long folded and fast circular strings in $\mathbb{R}\times\mathrm{S}^2$ (cf. (7.64), (7.69)), the corresponding expressions for the anomalous dimensions in terms of the Lambert W-function coincide (cf. (7.65), (7.70)). Because the arguments of the W-functions have opposite signs in these two cases, the formulas of the anomalous dimensions $\gamma=\gamma\left(\mathcal{J}\right)$ will have a periodic sign difference (cf. (G.3), (G.5)). Apparently, this sign flip seems to be associated with the transition from the stable case of long folded strings, to the instability of fast circular strings.[40]

7.2 Gubser-Klebanov-Polyakov String in AdS_3

7.2.1 Inverse Spin Function

In this subsection we are going to calculate the inverse spin function of the closed folded GKP string in AdS_3. Again, we will first have to revert the series (6.34) for $x=x\left(\mathcal{S}\right)$. Solving (6.34) for $\ln x$, we get

$$\mathcal{S}=\frac{2}{x}+2\sum_{n=0}^{\infty}x^n\left(c_n\ln x+b_n\right)\Rightarrow\ln x=\frac{-1/x+\mathcal{S}/2-\sum_{n=0}^{\infty}b_nx^n}{\sum_{n=0}^{\infty}c_nx^n}\Rightarrow$$

$$\Rightarrow\ln x=\left[-\frac{1}{c_0\,x}+\frac{\mathcal{S}/2-b_0}{c_0}-\sum_{n=1}^{\infty}\frac{b_n}{c_0}\,x^n\right]\cdot\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{c_k}{c_0}\,x^k\right)^n. \tag{7.71}$$

(7.71) is then equivalent to an equation of the following form (cf. equation (7.21)):

$$x=x_0\cdot\exp\left[\frac{\mathrm{a}_0}{x}+\sum_{n=1}^{\infty}\mathrm{a}_n\,x^n\right]=x_0\cdot\exp\left(\frac{\mathrm{a}_0}{x}+\mathrm{a}_1\,x+\mathrm{a}_2\,x^2+\mathrm{a}_3\,x^3+\ldots\right), \tag{7.72}$$

where the a_n's depend linearly on $\mathcal{S}$ ($\mathrm{a}_0=-c_0^{-1}=-4$) and x_0 is defined as:

$$x_0\equiv\exp\left[\frac{\mathcal{S}/2-b_0}{c_0}+\frac{c_1}{c_0^2}\right]=16\,e^{2\mathcal{S}+3/2}. \tag{7.73}$$

An important remark should be made at this point. Although equation (7.72) for the inverse spin function of a closed folded string that rotates inside AdS_3 is very similar to the inverse spin function (7.21) of closed (folded or circular) strings in $\mathbb{R}\times\mathrm{S}^2$, it has two significant differences: it contains a $1/x$ term, and x_0 is an increasing function of the spin $\mathcal{S}$. This means that we cannot solve the equation (7.72) by using the algorithm of section (7.1.1), but we have to apply a slightly varied method. Consider x^* that is defined as

$$x^*=x_0\cdot e^{\mathrm{a}_0/x^*}\Rightarrow x^*=\frac{\mathrm{a}_0}{W\left(\mathrm{a}_0/x_0\right)}=x_0\cdot e^{W\left(\mathrm{a}_0/x_0\right)} \tag{7.74}$$

[40]The author kindly thanks professor I. Bakas for this remark.

and $W(z)$ is the Lambert W-function (see appendix I). In effect x^* solves equation (7.72) to lowest order.[41] Setting

$$x = x^* \cdot e^{\mathfrak{u}} \tag{7.75}$$

with $\mathfrak{u} \to 0$ and plugging it into equation (7.72), we obtain with the aid of (7.74):

$$\mathfrak{u} - \frac{\mathrm{a}_0}{x^*} \sum_{k=1}^{\infty} (-1)^k \frac{\mathfrak{u}^k}{k!} - \sum_{n=1}^{\infty} \mathrm{a}_n \, (x^*)^n \, e^{n\mathfrak{u}} = 0. \tag{7.76}$$

We may revert the series (7.76) for v by using standard series reversion. Expanding the exponential in (7.76), we get

$$\left(1 + \frac{\mathrm{a}_0}{x^*} - \sum_{k=1}^{\infty} k \, \mathrm{a}_k \, (x^*)^k\right) \mathfrak{u} - \sum_{n=2}^{\infty} \left[(-1)^n \frac{\mathrm{a}_0}{x^*} + \sum_{k=1}^{\infty} k^n \, \mathrm{a}_k \, (x^*)^k\right] \frac{\mathfrak{u}^n}{n!} = \sum_{n=1}^{\infty} \mathrm{a}_n \, (x^*)^n . \tag{7.77}$$

The inverse series is a power series in x^*

$$\mathfrak{u} = \sum_{n=1}^{\infty} C_n \, (x^*) \cdot \left(\sum_{m=1}^{\infty} \mathrm{a}_m \, (x^*)^m\right)^n , \tag{7.78}$$

with $C_n\,(x^*)$ satisfying (for series reversion, see [85])

$$\begin{aligned} C_1 \cdot \left(1 + \frac{\mathrm{a}_0}{x^*} - \sum_{k=1}^{\infty} k \, \mathrm{a}_k \, (x^*)^k\right) &= 1 \\ C_2 \cdot \left(1 + \frac{\mathrm{a}_0}{x^*} - \sum_{k=1}^{\infty} k \, \mathrm{a}_k \, (x^*)^k\right)^3 &= \frac{\mathrm{a}_0}{x^*} + \sum_{k=1}^{\infty} k^2 \, \mathrm{a}_k \, (x^*)^k \\ &\vdots \end{aligned} \tag{7.79}$$

Practically we may obtain the inverse series by using Mathematica. We find:

$$\mathfrak{u} = \frac{\mathrm{a}_1}{\mathrm{a}_0} (x^*)^2 + \left[\frac{\mathrm{a}_2}{\mathrm{a}_0} - \frac{\mathrm{a}_1}{\mathrm{a}_0^2}\right] (x^*)^3 + \left[\frac{\mathrm{a}_1}{\mathrm{a}_0^3} + \frac{3\,\mathrm{a}_1^2 - 2\,\mathrm{a}_2}{2\,\mathrm{a}_0^2} + \frac{\mathrm{a}_3}{\mathrm{a}_0}\right] (x^*)^4 + \ldots \tag{7.80}$$

x in equation (7.75) is then given by

$$x = x^* + \frac{\mathrm{a}_1}{\mathrm{a}_0} (x^*)^3 + \left[\frac{\mathrm{a}_2}{\mathrm{a}_0} - \frac{\mathrm{a}_1}{\mathrm{a}_0^2}\right] (x^*)^4 + \left[\frac{\mathrm{a}_3}{\mathrm{a}_0} + \frac{2\,\mathrm{a}_1^2 - \mathrm{a}_2}{\mathrm{a}_0^2} + \frac{\mathrm{a}_1}{\mathrm{a}_0^3}\right] (x^*)^5 + \ldots \tag{7.81}$$

[41] Since we solve (7.71) in the region where $\mathcal{S} \to +\infty$, $x \to 0^+$ and $\mathrm{a}_0 < 0$, $x_0 \to +\infty$, the W_{-1} branch of Lambert's function must be chosen. See also the comment below the equation (7.97).

while $1/x$ is given by

$$\frac{1}{x} = \frac{1}{x^*} - \frac{\mathrm{a}_1}{\mathrm{a}_0}x^* - \left[\frac{\mathrm{a}_2}{\mathrm{a}_0} - \frac{\mathrm{a}_1}{\mathrm{a}_0^2}\right](x^*)^2 - \left[\frac{\mathrm{a}_3}{\mathrm{a}_0} + \frac{\mathrm{a}_1^2 - \mathrm{a}_2}{\mathrm{a}_0^2} + \frac{\mathrm{a}_1}{\mathrm{a}_0^3}\right](x^*)^3 + \ldots \tag{7.82}$$

We may now obtain all the coefficients a_n by expanding (7.71). When we then plug them into (7.81)–(7.82), we get the following series for x and $1/x$:

$$x = x^* + \left(\frac{\mathcal{S}}{16} + \frac{3}{64}\right)(x^*)^3 + \left(\frac{\mathcal{S}}{32} + \frac{23}{1024}\right)(x^*)^4 + \left(\frac{\mathcal{S}^2}{128} + \frac{55\mathcal{S}}{2048} + \frac{349}{24.576}\right)(x^*)^5 + \ldots \tag{7.83}$$

$$\frac{1}{x} = \frac{1}{x^*} - \left(\frac{\mathcal{S}}{16} + \frac{3}{64}\right)x^* - \left(\frac{\mathcal{S}}{32} + \frac{23}{1024}\right)(x^*)^2 - \left(\frac{\mathcal{S}^2}{256} + \frac{43\mathcal{S}}{2048} + \frac{295}{24.576}\right)(x^*)^3 + \ldots \tag{7.84}$$

7.2.2 Anomalous Dimensions

The series of the anomalous scaling dimensions $\gamma = \mathcal{E} - \mathcal{S}$ of the closed folded AdS_3 string is given in terms of $x = x(\mathcal{S})$ by the following expression:

$$\mathcal{E} - \mathcal{S} = 2\sum_{n=0}^{\infty} x^n \left(f_n \ln x + g_n\right) = \sum_{n=0}^{\infty} x^n \left[A_n + f_n \ln\frac{x}{x_0}\right], \quad \mathcal{E} \equiv \frac{\pi E}{\sqrt{\lambda}}, \quad \mathcal{S} \equiv \frac{\pi S}{\sqrt{\lambda}}, \tag{7.85}$$

where,

$$x_0 \equiv \exp\left[\frac{\mathcal{S}/2 - b_0}{c_0} + \frac{c_1}{c_0^2}\right] = 16\, e^{2\mathcal{S}+3/2} \tag{7.86}$$

and also

$$f_n \equiv -c_n - \sum_{k=0}^{n} \frac{(2k-3)!!}{(2k)!!} \cdot d_{n-k}$$

$$g_n \equiv -b_n - \frac{(2n-1)!!}{(2n+2)!!} - \sum_{k=0}^{n} \frac{(2k-3)!!}{(2k)!!} \cdot h_{n-k}\,, \quad n = 0, 1, 2, \ldots \tag{7.87}$$

has been defined. The first few of the coefficients f_n and g_n are:

$$f_0 = -\frac{1}{2}\,, \qquad f_1 = 0\,, \qquad f_2 = \frac{1}{128}\,, \qquad f_3 = \frac{1}{128}$$

$$g_0 = 2\ln 2 - \frac{1}{2}\,, \quad g_1 = -\frac{1}{4}\,, \quad g_2 = -\frac{1}{32}\ln 2 - \frac{3}{32}\,, \quad g_3 = -\frac{1}{32}\ln 2 - \frac{37}{768}. \tag{7.88}$$

The coefficients A_n are given by:

$$A_n \equiv g_n + f_n \ln x_0 = g_n + f_n\left(4\ln 2 + 2\mathcal{S} + \frac{3}{2}\right) \tag{7.89}$$

and the first three of them are:

$$A_0 = -\mathcal{S} - \frac{5}{4}\,, \quad A_1 = -\frac{1}{4}\,, \quad A_2 = \frac{1}{256}\left(4\mathcal{S} - 21\right)\,, \quad A_3 = \frac{1}{192}\left(3\mathcal{S} - 7\right). \tag{7.90}$$

It will also be useful to obtain the anomalous dimensions $\gamma = \mathcal{E} - \mathcal{S}$ in terms of x^*. First insert (7.72) into (7.85):

$$\mathcal{E} - \mathcal{S} = \frac{2\mathrm{a}_0\, f_0}{x} + 2A_0 + 2\sum_{n=1}^{\infty} x^n \left[A_n + \mathrm{a}_0\, f_{n+1} + \sum_{k=0}^{n-1} f_{n-k-1}\, \mathrm{a}_{k+1}\right]. \tag{7.91}$$

Plugging the series (7.83) and (7.84) into (7.91), we're led to the following result:

$$\mathcal{E} - \mathcal{S} = \frac{4}{x^*} - \left(2\mathcal{S} + \frac{5}{2}\right) - \frac{9\,x^*}{16} - \left(\frac{\mathcal{S}}{32} + \frac{35}{128}\right)(x^*)^2 - \left(\frac{5\,\mathcal{S}}{128} + \frac{2213}{12.288}\right)(x^*)^3 -$$
$$- \left(\frac{\mathcal{S}^2}{512} + \frac{361\,\mathcal{S}}{8192} + \frac{6665}{49.152}\right)(x^*)^4 - \left(\frac{19\,\mathcal{S}^2}{4096} + \frac{1579\,\mathcal{S}}{32.768} + \frac{433.501}{3.932.160}\right)(x^*)^5 + \ldots \tag{7.92}$$

For large spin $\mathcal{S}$, the series (7.71) may be reverted and $x = x\left(\mathcal{S}\right)$ can be obtained with Mathematica. The inverse spin function $x\left(\mathcal{S}\right)$ can be plugged into equation (7.85) and give the energy-spin relation of the GKP string case (I) and the anomalous dimensions of the $\mathcal{N} = 4$ SYM operators $\mathrm{Tr}\left[\mathcal{Z}\,\mathcal{D}_+^S\,\mathcal{Z}\right]$, as a function the (large) R-charge $\mathcal{S}$. The results of such a computation for the inverse spin function $x = x\left(\mathcal{S}\right)$ and the anomalous dimensions $\gamma = \gamma\left(\mathcal{S}\right)$, can be found in equations (G.6)–(G.7) of appendix G. Both series generally contain the following kinds of terms ($n = 0, 1, 2, \ldots$):

Leading terms (L): $\dfrac{\ln^n \mathcal{S}}{\mathcal{S}^n}$

Next-to-leading/Subleading terms (NL): $\dfrac{\ln^n \mathcal{S}}{\mathcal{S}^{n+1}}$

NNL terms: $\dfrac{\ln^n \mathcal{S}}{\mathcal{S}^{n+2}}$

$\vdots$ (7.93)

As it will turn out, the expansion of the inverse spin function $x = x\left(\mathcal{S}\right)$ does not contain any of the leading terms $\ln^n \mathcal{S}/\mathcal{S}^n$, whereas the expansion of the anomalous dimensions $\gamma = \mathcal{E} - \mathcal{S}$ contains them all. The large-spin expansion of the energy-spin relation then takes the following form (7.5):

$$\mathcal{E} - \mathcal{S} = \rho_c\, \ln\mathcal{S} + \sum_{n=0}^{\infty}\sum_{k=0}^{n} \rho_{(nk)} \frac{\ln^k \mathcal{S}}{\mathcal{S}^n} = \rho_c\, \ln\mathcal{S} + \rho_0 + \sum_{n=1}^{\infty} \rho_{(nn)} \frac{\ln^n \mathcal{S}}{\mathcal{S}^n} + \sum_{n=2}^{\infty} \rho_{(nn-1)} \frac{\ln^{n-1} \mathcal{S}}{\mathcal{S}^n} +$$
$$+ \sum_{n=3}^{\infty} \rho_{(nn-2)} \frac{\ln^{n-2} \mathcal{S}}{\mathcal{S}^n} + \ldots + \frac{\rho_1}{\mathcal{S}} + \frac{\rho_2}{\mathcal{S}^2} + \frac{\rho_3}{\mathcal{S}^3} + \ldots \tag{7.94}$$

It is interesting to also note also the presence of the unusual super-leading term $f\ \ln\mathcal{S}$. We may write:

$$E-S= f\ \ln\left(S/\sqrt{\lambda}\right)+\sum_{n=0}^{\infty}\sum_{k=0}^{n}f_{(nk)}\,\frac{\ln^{k}\left(S/\sqrt{\lambda}\right)}{S^{n}}=f\ \ln\left(S/\sqrt{\lambda}\right)+f_0+\sum_{n=1}^{\infty}f_{(nn)}\frac{\ln^{n}\left(S/\sqrt{\lambda}\right)}{S^{n}}+$$
$$+\sum_{n=2}^{\infty}f_{(nn-1)}\frac{\ln^{n-1}\left(S/\sqrt{\lambda}\right)}{S^{n}}+\sum_{n=3}^{\infty}f_{(nn-2)}\frac{\ln^{n-2}\left(S/\sqrt{\lambda}\right)}{S^{n}}+\ldots+\frac{f_1}{S}+\frac{f_2}{S^2}+\frac{f_3}{S^3}+\ldots \tag{7.95}$$

where

$$\rho_c=\frac{\pi\,f}{\sqrt{\lambda}},\quad \rho_0=\frac{\pi}{\sqrt{\lambda}}\left(f_0-f\ln\pi\right),\quad \rho_1=\left(\frac{\pi}{\sqrt{\lambda}}\right)^2\left(f_1-f_{11}\ln\pi\right)$$
$$\rho_2=\left(\frac{\pi}{\sqrt{\lambda}}\right)^3\left(f_2-f_{21}\ln\pi+f_{22}\ln^2\pi\right)$$
$$\rho_3=\left(\frac{\pi}{\sqrt{\lambda}}\right)^4\left(f_3-f_{31}\ln\pi+f_{32}\ln^2\pi-f_{33}\ln^3\pi\right)$$
$$\rho_{(nn)}=\left(\frac{\pi}{\sqrt{\lambda}}\right)^{n+1}\cdot f_{(nn)}\,,\quad \rho_{(nn-1)}=\left(\frac{\pi}{\sqrt{\lambda}}\right)^{n+1}\left(f_{(nn-1)}-n\,f_{(nn)}\ln\pi\right)$$
$$\rho_{(nn-2)}=\left(\frac{\pi}{\sqrt{\lambda}}\right)^{n+1}\left(f_{(nn-2)}-(n-1)\,f_{(nn-1)}\ln\pi+\frac{n\,(n-1)}{2}f_{(nn)}\ln^2\pi\right). \tag{7.96}$$

7.2.3 Leading Terms

To calculate the leading in $\mathcal{S}$ terms of the series (7.85), we must use the formula (7.74):

$$x^*=x_0\cdot e^{W_{-1}(\mathrm{a}_0/x_0)}=\frac{\mathrm{a}_0}{W_{-1}\left(\mathrm{a}_0/x_0\right)}=\frac{-4}{W_{-1}\left[-\frac{1}{4}\,e^{-2\mathcal{S}-3/2}\right]}, \tag{7.97}$$

where we have chosen the W_{-1} branch of Lambert's function because we should have $x^*\to 0^+$, as $\mathcal{S}\to+\infty$ and $W_{-1}\to-\infty$ (conversely $W_0\to 0^-$ for $\mathcal{S}\to+\infty$, making x^* blow up as $x^*\to+\infty$. See figure 31). To leading order, the inverse spin function $x=x\left(\mathcal{S}\right)$ is obtained by using Taylor's expansion (I.4) in the W_{-1} branch. For $1/x^*$ we obtain,

$$\frac{1}{x^*}=-\frac{1}{4}\left\{\ln\left|\frac{\mathrm{a}_0}{x_0}\right|-\ln\ln\left|\frac{\mathrm{a}_0}{x_0}\right|+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^n}{m!}\left[\begin{array}{c}n+m\\ n+1\end{array}\right]\frac{\left(\ln\ln|\mathrm{a}_0/x_0|\right)^m}{\left(\ln|\mathrm{a}_0/x_0|\right)^{n+m}}\right\}=$$
$$=\frac{\mathcal{S}}{2}+\frac{\ln 2}{2}+\frac{3}{8}+\frac{1}{4}\ln\left[2\mathcal{S}+2\ln 2+\frac{3}{2}\right]-\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^m}{4\,m!}\left[\begin{array}{c}n+m\\ n+1\end{array}\right]\frac{\left(\ln\left[2\mathcal{S}+2\ln 2+3/2\right]\right)^m}{\left(2\mathcal{S}+2\ln 2+3/2\right)^{n+m}}=$$
$$=\frac{\mathcal{S}}{2}+\frac{\ln\mathcal{S}}{4}+\frac{3}{4}\ln 2+\frac{3}{8}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n}\left(\frac{\ln 2+3/4}{\mathcal{S}}\right)^n-\sum_{n,q=0}^{\infty}\sum_{m=1}^{\infty}\sum_{p=0}^{m}\frac{(-1)^m}{2^{n+m+2}\,m!}\left[\begin{array}{c}n+m\\ n+1\end{array}\right]\cdot$$
$$\cdot\left(\begin{array}{c}-n-m\\ q\end{array}\right)\left(\begin{array}{c}m\\ p\end{array}\right)\frac{\ln^p\mathcal{S}}{\mathcal{S}^{n+m}}\left(\ln 2-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{\ln 2+3/4}{\mathcal{S}}\right)^k\right)^{m-p}\left(\frac{\ln 2+3/4}{\mathcal{S}}\right)^q, \tag{7.98}$$

where the unsigned Stirling numbers of the first kind $\left[\begin{array}{c} n+m \\ n+1 \end{array}\right]$ are defined in appendix I, equation (I.5) and, for large $\mathcal{S}$, the following identity has been used:

$$\ln\left[2\mathcal{S}+2\ln 2+\frac{3}{2}\right]=\ln\mathcal{S}+\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{\ln 2+3/4}{\mathcal{S}}\right)^{n}. \tag{7.99}$$

To get x^{*}, let us expand the reciprocal of (7.98). The result is,

$$x^{*}=\frac{2}{\mathcal{S}}\cdot\left\{1+\frac{\ln\mathcal{S}}{2\,\mathcal{S}}+(2\ln 2+1)\,\frac{3}{4\mathcal{S}}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n}\frac{(\ln 2+3/4)^{n}}{\mathcal{S}^{n+1}}-\sum_{n,q=0}^{\infty}\sum_{m=1}^{\infty}\sum_{p=0}^{m}\frac{(-1)^{m}}{2^{n+m+1}\,m!}\cdot\right.$$
$$\cdot\left[\begin{array}{c} n+m \\ n+1 \end{array}\right]\left(\begin{array}{c} -n-m \\ q \end{array}\right)\left(\begin{array}{c} m \\ p \end{array}\right)\frac{\ln^{p}\mathcal{S}}{\mathcal{S}^{n+m+1}}\left(\ln 2-\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}\left(\frac{\ln 2+3/4}{\mathcal{S}}\right)^{k}\right)^{m-p}\cdot$$
$$\left.\cdot\left(\frac{\ln 2+3/4}{\mathcal{S}}\right)^{q}\right\}^{-1}\Rightarrow \tag{7.100}$$

$$x^{*}=\frac{2}{\mathcal{S}}-\left[\ln\mathcal{S}+\left(3\ln 2+\frac{3}{2}\right)\right]\frac{1}{\mathcal{S}^{2}}+\left[\frac{\ln^{2}\mathcal{S}}{2}+(1+3\ln 2)\ln\mathcal{S}+\left(\frac{9\ln^{2}2}{2}+3\ln 2+\frac{3}{8}\right)\right]\frac{1}{\mathcal{S}^{3}}-$$
$$-\Big[\frac{\ln^{3}\mathcal{S}}{4}+\left(\frac{1}{2}+\frac{9\ln 2}{4}\right)\ln^{2}\mathcal{S}+\left(\frac{27\ln^{2}2}{4}+3\ln 2+\frac{1}{16}\right)\ln\mathcal{S}+\Big(\frac{27\ln^{3}2}{4}+\frac{9\ln^{2}2}{2}+\frac{3\ln 2}{16}-$$
$$-\frac{3}{16}\Big)\Big]\frac{1}{\mathcal{S}^{4}}+\Big[\frac{\ln^{4}\mathcal{S}}{8}+\left(\frac{3\ln 2}{2}+\frac{5}{24}\right)\ln^{3}\mathcal{S}+\left(\frac{27\ln^{2}2}{4}+\frac{15\ln 2}{8}-\frac{3}{16}\right)\ln^{2}\mathcal{S}+\Big(\frac{27\ln^{3}2}{2}+$$
$$+\frac{45\ln^{2}2}{8}-\frac{9\ln 2}{8}-\frac{13}{32}\Big)\ln\mathcal{S}+\Big(\frac{81\ln^{4}2}{8}+\frac{45\ln^{3}2}{8}-\frac{27\ln^{2}2}{16}-\frac{39\ln 2}{32}-\frac{15}{128}\Big)\Big]\frac{1}{\mathcal{S}^{5}}+\ldots \tag{7.101}$$

Now note that the series (7.98) for $1/x^{*}$ contains every kind of small terms (that is terms $\to 0$ as $\mathcal{S}\to\infty$): leading terms $\ln^{n}\mathcal{S}/\mathcal{S}^{n}$, subleading terms $\ln^{n}\mathcal{S}/\mathcal{S}^{n+1}$, next-to-subleading terms $\ln^{n}\mathcal{S}/\mathcal{S}^{n+2}$, etc. up to $1/\mathcal{S}^{n}$ terms, with $n=1,\,2,\,3\,\ldots$ On the other hand, the series (7.100)–(7.101) for x^{*} do not contain any of the leading terms. Likewise $(x^{*})^{2}$ will not contain any leading and subleading terms, $(x^{*})^{3}$ no leading, subleading and next-to-subleading terms, etc. To calculate $E-S$ to leading order in $\mathcal{S}$, we need only the first two terms of (7.92), namely

$$\mathcal{E}-\mathcal{S}\Big|_{\mathrm{L}+\ldots}=\frac{4}{x^{*}}-\left(2\,\mathcal{S}+\frac{5}{2}\right), \tag{7.102}$$

because the other terms of (7.92) contribute to NL order onwards. We get

$$\mathcal{E}-\mathcal{S}\Big|_{\mathrm{L}+\ldots}=\ln\mathcal{S}+(3\ln 2-1)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{\ln 2+3/4}{\mathcal{S}}\right)^{n}-\sum_{n,q=0}^{\infty}\sum_{m=1}^{\infty}\sum_{p=0}^{m}\frac{(-1)^{m}}{2^{n+m}}\cdot$$

$$\cdot \frac{1}{m!} \left[\begin{array}{c} n+m \\ n+1 \end{array} \right] \left(\begin{array}{c} -n-m \\ q \end{array} \right) \left(\begin{array}{c} m \\ p \end{array} \right) \frac{\ln^p \mathcal{S}}{\mathcal{S}^{n+m}} \left(\frac{\ln 2 + 3/4}{\mathcal{S}} \right)^q \left[\ln 2 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \cdot \right.$$

$$\left. \cdot \left(\frac{\ln 2 + 3/4}{\mathcal{S}} \right)^k \right]^{m-p}. \tag{7.103}$$

For $p = m$ and $n = q = 0$ we may read off all the coefficients of the leading terms:

$$\rho_{(mm)} = -\frac{(-1)^m}{2^m \, m!} \cdot \left[\begin{array}{c} m \\ 1 \end{array} \right] \left(\begin{array}{c} -m \\ 0 \end{array} \right) \left(\begin{array}{c} m \\ m \end{array} \right) = \frac{(-1)^{m+1}}{2^m} \frac{1}{m}. \tag{7.104}$$

As we have already said, our result agrees with those of [12] (we have used $\left[{m \atop 1} \right] = (m-1)!$). Also,

$$\rho_c = 1 \quad \& \quad \rho_0 = 3 \ln 2 - 1. \tag{7.105}$$

7.2.4 Next-to-Leading Terms

For the subleading coefficients we need the following terms of (7.92),

$$\mathcal{E} - \mathcal{S} \Big|_{\text{L+NL+...}} = \frac{4}{x^*} - \left(2\,\mathcal{S} + \frac{5}{2} \right) - \frac{9\,x^*}{16} - \frac{\mathcal{S}}{32} \,(x^*)^2 \,. \tag{7.106}$$

Reading off all the terms that give contributions to subleading order from equations (7.98)–(7.101), we get

$$\rho_{(m+1,m)} = \frac{(-1)^{m+1}}{2^{m+1}} \left[H_m + \frac{m}{4} + 1 - 3 \ln 2 \right], \tag{7.107}$$

which also agrees with the results of [12] (here we have used $\left[{m+1 \atop 2} \right] = m!\, H_m$). We may also confirm the value of the coefficient ρ_1 in equation (7.94):

$$\rho_1 = \frac{1}{2} \,(3 \ln 2 - 1)\,. \tag{7.108}$$

7.2.5 NNL Terms

Going to the next-to-next-to-leading (NNL) order, the contributing terms of (7.92) are:

$$\mathcal{E} - \mathcal{S} \Big|_{\text{L+NL+NNL+...}} = \frac{4}{x^*} - \left(2\,\mathcal{S} + \frac{5}{2} \right) - \frac{9\,x^*}{16} - \left(\frac{\mathcal{S}}{32} + \frac{35}{128} \right) (x^*)^2 - \frac{5\,\mathcal{S}}{128} \,(x^*)^3 -$$

$$- \frac{\mathcal{S}^2}{512} \,(x^*)^4 + \ldots \tag{7.109}$$

We can read off all the NNL terms (with the aid of the property (I.6) of Stirling numbers):

$$\rho_{(m+2,m)} = \frac{(-1)^{m+1}}{2^{m+3}} \cdot (m+1) \cdot \Bigg\{ H_{m+1}^2 - H_{m+1}^{(2)} + \frac{1}{2}(m - 12\ln 2 + 5) \cdot H_{m+1} + \frac{m(m-1)}{24} - \\ - \frac{3}{2}(m+5)\ln 2 + 9\ln^2 2 \Bigg\}. \tag{7.110}$$

The coefficient ρ_2 of (7.94) is:

$$\rho_2 = -\frac{9\ln^2 2}{8} + \frac{27\ln 2}{16} - \frac{5}{16}. \tag{7.111}$$

These results agree with those found in [99, 108] for the first few terms of the series (7.95). For more, refer to appendix G and formula (G.7).

Let us finish this paragraph by writing down the W-function expressions for the inverse spin function $x = x(\mathcal{J})$ and the anomalous dimensions $\gamma = \gamma(\mathcal{J})$. Plugging x^* from (7.97) into equations (7.83) and (7.92) we obtain:

$$x = -\frac{4}{W_{-1}} - \frac{4\mathcal{S}+3}{(W_{-1})^3} + \left[8\mathcal{S} + \frac{23}{4}\right]\frac{1}{(W_{-1})^4} - \left[8\,S^2 + \frac{55}{2}\,S + \frac{349}{24}\right]\frac{1}{(W_{-1})^5} + \Bigg[38\,S^2 + \\ + \frac{711\,S}{8} + \frac{3745}{96}\Bigg]\frac{1}{(W_{-1})^6} - \left[20\,S^3 + 176\,S^2 + \frac{4765\,S}{16} + \frac{26.659}{240}\right]\frac{1}{(W_{-1})^7} + \\ + \left[\frac{466\,S^3}{3} + \frac{6077\,S^2}{8} + \frac{48.955\,S}{48} + \frac{2.543.083}{7680}\right]\frac{1}{(W_{-1})^8} - \ldots, \tag{7.112}$$

$$\mathcal{E} - \mathcal{S} = -W_{-1} - \left(2\mathcal{S} + \frac{5}{2}\right) + \frac{9}{4\,W_{-1}} - \left[\frac{\mathcal{S}}{2} + \frac{35}{8}\right]\frac{1}{(W_{-1})^2} + \left[\frac{5\,\mathcal{S}}{2} + \frac{2213}{192}\right]\frac{1}{(W_{-1})^3} - \\ - \left[\frac{\mathcal{S}^2}{2} + \frac{361\,\mathcal{S}}{32} + \frac{6665}{192}\right]\frac{1}{(W_{-1})^4} + \left[\frac{19\,\mathcal{S}^2}{4} + \frac{1579\,\mathcal{S}}{32} + \frac{433.501}{3840}\right]\frac{1}{(W_{-1})^5} - \\ - \left[\frac{5\,\mathcal{S}^3}{6} + \frac{259\,\mathcal{S}^2}{8} + \frac{81.799\,\mathcal{S}}{384} + \frac{2.963.887}{7680}\right]\frac{1}{(W_{-1})^6} + \Bigg[\frac{34\,\mathcal{S}^3}{3} + \frac{3069\,\mathcal{S}^2}{16} + \\ + \frac{175.481\,\mathcal{S}}{192} + \frac{2.350.780.111}{1.720.320}\Bigg]\frac{1}{(W_{-1})^7} - \ldots, \tag{7.113}$$

where $\mathcal{E} \equiv \pi E/\sqrt{\lambda}$, $\mathcal{S} \equiv \pi S/\sqrt{\lambda}$ and the argument of the W-functions is $W_{-1}\left(-e^{-2\mathcal{S}-3/2}/4\right)$. Again, the terms of both series (7.112) and (7.113) can be arranged in decreasing order of importance, e.g. according to (7.102) the first two terms of (7.113) are the leading coefficients, (7.106) implies that the first four terms contain all the next-to-leading coefficients, etc. We can check that the formulae (7.112)–(7.113) are correct, by first expanding them around $\mathcal{S} \to +\infty$ with the aid of the expansion (I.4) of the W-function in its W_{-1} branch, and then by comparing them with the series (G.6)–(G.7) that have been obtained with Mathematica.

7.3 Reciprocity

The fact that twist (aka quasipartonic) operators[42] can be classified according to the representations of the collinear subgroup $\mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{so}(4,2)$, which are labeled by the conformal spin s

$$s \equiv \frac{\pi}{2\sqrt{\lambda}}(S+\Delta) = \mathcal{S} + \frac{1}{2}\gamma(\mathcal{S}), \tag{7.114}$$

where

$$\gamma \equiv \mathcal{E} - \mathcal{S}, \qquad \mathcal{E} \equiv \frac{\pi E}{\sqrt{\lambda}}, \qquad \mathcal{S} \equiv \frac{\pi S}{\sqrt{\lambda}}, \tag{7.115}$$

implies that the anomalous dimensions $\gamma(\mathcal{S})$ must be a function of the conformal spin s:

$$\gamma(\mathcal{S}) = P(s) = P\left(\mathcal{S} + \frac{1}{2}\gamma(\mathcal{S})\right). \tag{7.116}$$

Equivalently we may write,

$$P(\mathcal{S}) = \gamma\left(\mathcal{S} - \frac{1}{2}P(\mathcal{S})\right). \tag{7.117}$$

This equation may be inverted by using the Lagrange-Bürmann formula (7.22), as follows [109]:

$$P(\mathcal{S}) = \sum_{k=1}^{\infty} \frac{1}{k!}\left(-\frac{1}{2}\partial_{\mathcal{S}}\right)^{k-1} \gamma^k(\mathcal{S}). \tag{7.118}$$

The property of reciprocity or parity-invariance of the anomalous dimensions $\gamma(\mathcal{S})$ is expressed through the condition that, for large values of the spin $\mathcal{S}$, the inverse anomalous dimension $P(\mathcal{S})$ contains only even negative powers of the quadratic Casimir operator of the collinear group, $C^2 = s_0(s_0-1)$:[43]

$$P(\mathcal{S}) = \sum_{k=0}^{\infty} \frac{c_k \ln C}{C^{2k}}. \tag{7.119}$$

Calculating $P(\mathcal{S})$ by plugging the formula (G.7) into (7.118), we find that $P(\mathcal{S})$ is "simple" in that it contains no leading logarithms of $\mathcal{S}$ (as opposed to $\gamma(\mathcal{S})$ in (G.7)) and that it also has the property of reciprocity [99, 105, 108], because it contains only even negative powers of $\mathcal{S}$ and it is thus symmetric under $C = \mathcal{S} \to -\mathcal{S}$:

$$P(\mathcal{S}) = \ln \mathcal{S} + \Big[3\ln 2 - 1\Big] + \left[\frac{\ln \mathcal{S}}{16} + \left(\frac{3\ln 2}{16} + \frac{1}{16}\right)\right]\frac{1}{\mathcal{S}^2} - \left[\frac{\ln^2 \mathcal{S}}{128} + \left(\frac{3\ln 2}{64} + \frac{7}{1024}\right)\ln \mathcal{S} + \right.$$
$$\left. + \left(\frac{9\ln^2 2}{128} + \frac{21\ln 2}{1024} + \frac{5}{2048}\right)\right]\frac{1}{\mathcal{S}^4} + O\left(\frac{1}{\mathcal{S}^6}\right). \tag{7.120}$$

Repeating our analysis by plugging the formula (7.94) into (7.118) and then demanding that all the leading coefficients, as well as those that multiply the odd negative powers of $1/\mathcal{S}$ must vanish, we get a system of equations between the coefficients ρ. Solving this system of equations, we obtain the following set of formulas that are known as the MVV relations:[44]

$$\rho_1 = \frac{1}{2}\rho_c\,\rho_0, \quad \rho_{11} = \frac{\rho_c^2}{2}, \quad \rho_{22} = -\frac{\rho_c^3}{8}, \quad \rho_{33} = \frac{\rho_c^4}{24}, \quad \rho_{44} = -\frac{\rho_c^5}{64}, \quad \rho_{55} = \frac{\rho_c^6}{160} \tag{7.121}$$

[42] Namely operators composed out of scalars, gauginos or vector fields and their light-cone derivatives.

[43] We write $s_0 \equiv \pi(S+\Delta_0)/2\sqrt{\lambda}$ for the bare conformal spin.

[44] The Gribov-Lipatov reciprocity [110] leads to certain relations between the coefficients of the large-spin expansion of anomalous dimensions, known as Moch-Vermaseren-Vogt (MVV) relations [90]. For more on the concept of reciprocity in the context of QCD, see [109, 111, 112]. For other twist operators of $\mathcal{N}=4$ SYM, see [113]. For reciprocity in the context of $\mathcal{N}=6$ super Chern-Simons (ABJM) theory, see [114].

$$\rho_{32} + \rho_c\,\rho_{21} = \frac{\rho_c^4}{16} - \frac{1}{8}\,\rho_c^3\,\rho_0, \quad \rho_c\rho_2 + \rho_{21}\,(\rho_0 - \rho_c) + \rho_{31} = -\frac{\rho_c^4}{8} + \frac{3}{8}\,\rho_c^3\,\rho_0 - \frac{1}{4}\,\rho_c^2\,\rho_0^2 \tag{7.122}$$

$$\rho_2\left(\rho_0 - \frac{\rho_c}{2}\right) - \frac{1}{2}\,\rho_0\,\rho_{21} + \rho_3 = -\frac{1}{8}\,\rho_c^3\,\rho_0 + \frac{1}{4}\,\rho_c^2\,\rho_0^2 - \frac{1}{12}\,\rho_c\,\rho_0^3 \tag{7.123}$$

$$\rho_c^3\,\rho_2 + \rho_{21}\left(3\rho_c^2\,\rho_0 - \frac{37\,\rho_c^3}{12}\right) - 2\,\rho_c\,\rho_{42} + 2\,\rho_{43}\,(\rho_c - \rho_0) - \rho_{53} = \frac{95}{96}\,\rho_c^5\,\rho_0 - \frac{5}{8}\,\rho_c^4\,\rho_0^2 - \frac{67\rho_c^6}{192} \tag{7.124}$$

$$\vdots$$

We may also want to test the idea of *inheritance*, i.e. to check whether low perturbative orders may influence or even control higher orders in string perturbation theory. The idea of inheritance originally came up in a QCD context (for a review see [112]), where it was observed that the lower orders in perturbation theory are able to transmit their structure to higher orders. One may check that the quantum corrections that were calculated in [99] verify the first two of the above MVV relations up to one loop. For the remaining MVV formulae, inheritance is expected (and has indeed been shown) to break down. This is because of the wrapping effects that begin to set in after the critical loop-order and are also present at strong coupling. In fact, wrapping effects are responsible for the breaking down of both the reciprocity and simplicity in the large-spin expansion of the anomalous dimensions of twist-2 operators of $\mathcal{N} = 4$ SYM at strong coupling [108].

For the GKP strings in $\mathbb{R} \times \mathrm{S}^2$, there exists a transformation that allows the corresponding dispersion relation to be expressed in a form that resembles the dispersion relation of the closed folded GKP string in AdS_3 and permits comparisons between the two. The transformation

$$\mathcal{S} \equiv \frac{1}{16}\,e^{\mathcal{J}+2} \Leftrightarrow \mathcal{J} = \ln\mathcal{S} + 4\ln 2 - 2, \tag{7.125}$$

leads to the following set of terms in the energy-spin relation of closed folded strings in $\mathbb{R} \times \mathrm{S}^2$:

$$\begin{aligned} &\text{Leading terms (L): } \frac{\ln^n\mathcal{S}}{\mathcal{S}^{n+1}} \\ &\text{Next-to-leading/Subleading terms (NL): } \frac{\ln^n\mathcal{S}}{\mathcal{S}^{n+2}} \\ &\text{NNL terms: } \frac{\ln^n\mathcal{S}}{\mathcal{S}^{n+3}} \\ &\qquad\vdots \end{aligned} \tag{7.126}$$

In contradistinction with the AdS case, all the $\ln^n\mathcal{S}/\mathcal{S}^n$ terms are absent from the large-spin expansion of the anomalous dimensions:

$$\gamma = 2 - \frac{1}{2\mathcal{S}} + \left[\frac{\ln\mathcal{S}}{16} + \left(\frac{\ln 2}{4} - \frac{5}{32}\right)\right]\frac{1}{\mathcal{S}^2} - \left[\frac{\ln^2\mathcal{S}}{64} + \left(\frac{\ln 2}{8} - \frac{9}{128}\right)\ln\mathcal{S} + \left(\frac{\ln^2 2}{4} - \frac{9\ln 2}{32} + \right.\right.$$
$$\left.\left. + \frac{3}{32}\right)\right]\frac{1}{\mathcal{S}^3} + \left[\frac{\ln^3\mathcal{S}}{192} + \left(\frac{\ln 2}{16} - \frac{17}{512}\right)\ln^2\mathcal{S} + \left(\frac{\ln^2 2}{4} - \frac{17\ln 2}{64} + \frac{163}{2048}\right)\ln\mathcal{S} + \left(\frac{\ln^3 2}{3} - \right.\right.$$
$$\left.\left. - \frac{17\ln^2 2}{32} + \frac{163\ln 2}{512} - \frac{1735}{24.576}\right)\right]\frac{1}{\mathcal{S}^4} - O\left(\frac{1}{\mathcal{S}^5}\right). \tag{7.127}$$

Having expressed the anomalous dimensions $\gamma = \mathcal{E} - \mathcal{J}$ in terms of the variable $\mathcal{S} = e^{\mathcal{J}+2}/16$, it is tempting to try to pose and answer questions that normally arise in the case of twist operators. If we calculate $P(\mathcal{S})$ by plugging the formula (7.127) into (7.118), we find that $P(\mathcal{S})$ is simple in that it contains only subleading logarithms of $\mathcal{S}$ (not simpler than $\gamma(\mathcal{S})$ however because (7.127) does not contain leading logarithms too), and it has no reciprocity/parity invariance, since both the even and the odd negative powers of $\mathcal{S}$ appear (therefore there's no symmetry under $C = \mathcal{S} \to -\mathcal{S}$):

$$P(\mathcal{S}) = 2 - \frac{1}{2\mathcal{S}} + \left[\frac{\ln\mathcal{S}}{16} + \left(\frac{\ln 2}{4} - \frac{21}{32}\right)\right]\frac{1}{\mathcal{S}^2} - \left[\frac{\ln^2\mathcal{S}}{64} + \left(\frac{\ln 2}{8} - \frac{25}{128}\right)\ln\mathcal{S} + \left(\frac{\ln^2 2}{4} - \frac{25\ln 2}{32} + \right.\right.$$
$$\left.\left. + \frac{27}{32}\right)\right]\frac{1}{\mathcal{S}^3} + \left[\frac{\ln^3\mathcal{S}}{192} + \left(\frac{\ln 2}{16} - \frac{41}{512}\right)\ln^2\mathcal{S} + \left(\frac{\ln^2 2}{4} - \frac{41\ln 2}{64} + \frac{947}{2048}\right)\ln\mathcal{S} + \left(\frac{\ln^3 2}{3} - \right.\right.$$
$$\left.\left. - \frac{41\ln^2 2}{32} + \frac{947\ln 2}{512} - \frac{13.255}{24.576}\right)\right]\frac{1}{\mathcal{S}^4} + O\left(\frac{1}{\mathcal{S}^5}\right). \tag{7.128}$$

8 Infinite-Size Giant Magnons and Single Spikes

In §4.3 we saw how the concept of the magnon emerged in the $\mathfrak{su}(2)$ sector of planar $\mathcal{N} = 4$ SYM, when we considered all the gauge-invariant single trace operators of the sector and the spectrum of their scaling dimensions. The coordinate Bethe ansatz (4.38) was then used to determine the scaling dimensions of all the operators at one-loop order in α' perturbation theory. With the asymptotic Bethe ansatz (ABA) of Beisert, Dippel and Staudacher (BDS) (4.48)–(4.49), an all-loop prediction for the magnon energies became possible for infinite operator sizes. For the $M = 1$ magnon states

$$\mathcal{O}_M = \sum_{m=1}^{J+1} e^{imp} \left|\mathcal{Z}^{m-1}\mathcal{X}\mathcal{Z}^{J-m+1}\right\rangle, \quad p \in \mathbb{R} \tag{8.1}$$

$$\tag{8.2}$$

of infinite size ($J = \infty$),[45] the ABA dictates:

$$\Delta - J = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}, \quad J = \infty, \text{ all } \lambda. \tag{8.3}$$

(8.3) is a non-perturbative prediction for the one-magnon spectrum at infinite size that is valid to all loop orders, in weak and strong coupling. Its weak and strong coupling limits are:

$$\Delta - J = 1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2} - \frac{\lambda^2}{8\pi^4}\sin^4\frac{p}{2} + \frac{\lambda^3}{16\pi^6}\sin^6\frac{p}{2} - \ldots, \qquad \lambda \to 0 \text{ (weak coupling)} \tag{8.4}$$

$$\Delta - J = \frac{\sqrt{\lambda}}{\pi}\sin\frac{p}{2} + 0 + \frac{\pi}{2\sqrt{\lambda}}\csc\frac{p}{2} - \frac{\pi^3}{8\lambda^{3/2}}\csc^3\frac{p}{2} + \ldots, \qquad \lambda \to \infty \text{ (strong coupling)}. \tag{8.5}$$

Strictly speaking, the one-magnon operators (8.1) are not physical states of $\mathcal{N} = 4$ SYM since as we have explained, the cyclicity of the trace (4.42) implies that their total momentum p must vanish. As Beisert has shown in [115], in order to accommodate states (8.1) having a non-vanishing momentum $p \neq 0$ in $\mathcal{N} = 4$ SYM theory, the corresponding symmetry algebra $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2) \subset \mathfrak{psu}(2,2|4)$ must be extended with two central charges. To obtain meaningful gauge theory states, either the single-magnon momentum must vanish (4.43), or zero-momentum operators of two or more magnons must be formed (4.44).

The string theory duals of the $\mathcal{N} = 4$ SYM magnon excitations at infinite size ($J = \infty$) are the giant magnons (GMs) that were found in 2006 by Hofman and Maldacena (HM) [78]. Giant magnons are open single-spin strings that rotate rigidly in $\mathbb{R} \times \mathrm{S}^2 \subset \mathrm{AdS}_5 \times \mathrm{S}^5$. They are the elementary excitations of type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ that serve as a sort of a fundamental building block of all the closed string states and multi-soliton solutions of the theory. Despite the fact that both the conserved energy and the spin of infinite-size GMs diverge ($E, J = \infty$) their difference remains finite so that the energy-spin relation of a single giant magnon of momentum/angular extent $p = \Delta\varphi$ is:

$$E - J = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{\Delta\varphi}{2}\right|, \quad J = \infty, \ \lambda \to \infty. \tag{8.6}$$

[45] The following convention shall be employed throughout the text: $E, J, p = \infty/v, \omega = 1$ will denote infinite size (as obtained by computing the limit $\lim_{J,p\to\infty/v,\omega\to 1}$) and $E, J, p \to \infty/v, \omega \to 1$ will denote large but still finite size (i.e. before considering the limit $\lim_{J,p\to\infty/v,\omega\to 1}$).

Open strings are absent from the spectrum of a type IIB string theory however and we are faced with the same problem that we had with the gauge theory magnons of non-vanishing momentum. As we did for magnons, in order to accommodate GMs within string theory, the symmetry algebra of the corresponding string sigma model (3.13) has to be centrally extended with two additional charges. Physical string states can only be formed by two or more GMs with vanishing total momentum.

The $\mathbb{R}\times\mathrm{S}^2$ GKP string (II) that he have extensively studied in §6.2–§7.1 is such a closed string state with vanishing total momentum. Infinite-size GKP strings in $\mathbb{R}\times\mathrm{S}^2$ can be formed by the superposition of two HM giant magnons that have maximum angular extent $\Delta\varphi=\pi$ and angular momentum equal to $J/2$ each. They are dual to 2-magnon operators Tr $\left[\mathcal{Z}^J\mathcal{X}^2\right]$ and their dispersion relation at infinite-size is

$$E-J=\frac{2\sqrt{\lambda}}{\pi}, \quad J=\infty,\ \lambda\to\infty. \tag{8.7}$$

A second class of solutions in $\mathbb{R}\times\mathrm{S}^2$ is that of single spikes (SSs), single-spin strings with a spike, that rotate on the 2-sphere just like giant magnons do [116, 117]. Technically, single spikes are very closely related to giant magnons. In the conformal gauge, the single spike ansatz follows from the HM giant magnon by interchanging the world-sheet coordinates on the 2-sphere, i.e. $\tau\leftrightarrow\sigma$ while leaving the temporal spacetime coordinate intact, $t=\tau$. The paper [118] claims that the $\tau\leftrightarrow\sigma$ transform carries us from large-spin strings in $\mathbb{R}\times\mathrm{S}^2$ to large-winding ones, and from the holomorphic sector of $\mathcal{N}=4$ SYM to its non-holomorphic sector.

The conserved charges of momentum and energy of infinite-size/momentum single spikes diverge as $E,p=\infty$, while their difference remains finite. Their dispersion relation is:

$$E-T\Delta\varphi=\frac{\sqrt{\lambda}}{\pi}\arcsin\left(\frac{\pi J}{\sqrt{\lambda}}\right), \quad p=\infty,\ \lambda\to\infty, \tag{8.8}$$

where $\Delta\varphi=p$ is the momentum/angular extent of the single spike. The giant magnon dispersion relation (8.6) may be obtained from (8.8) by making the transformation $\pi E/\sqrt{\lambda}-\Delta\varphi/2\mapsto p/2$ and $J\mapsto E-J$.

We won't have much to say about the operators that are dual to single spikes. The interested reader may refer to the papers [119, 120, 118, 121] for more information. As we saw, giant magnons are dual to one-magnon states of the $\mathfrak{su}(2)$ sector of $\mathcal{N}=4$ SYM, the elementary excitations above the ferromagnetic ground state $\mathrm{Tr}\mathcal{Z}^J$ of the Heisenberg $\mathrm{XXX}_{1/2}$ spin chain. As we will see in more detail below, the string theory dual of the BPS operators $\mathrm{Tr}\mathcal{Z}^J$ is a point-like string that rotates around the equator of the 2-sphere in $\mathbb{R}\times\mathrm{S}^2$ with infinite angular momentum J. A very analogous statement can be made for single spikes: single spikes in $\mathbb{R}\times\mathrm{S}^2$ are elementary excitations above the anti-ferromagnetic ground state $\mathrm{Tr}\mathbb{S}^{L/2}+\ldots$ of an $\mathfrak{so}(6)$ spin chain of $\mathcal{N}=4$ SYM, where $\mathbb{S}$ are the $\mathcal{N}=4$ SYM operators $\mathbb{S}\sim\mathcal{X}\overline{\mathcal{X}}+\mathcal{Y}\overline{\mathcal{Y}}+\mathcal{Z}\overline{\mathcal{Z}}$ and $L\equiv J+M$. The anti-ferromagnetic vacuum is in turn dual to the "hoop" string, a string at rest that is wound around the equator of S^2. The $\tau\leftrightarrow\sigma$ transform may again be used to translate between the two solutions.

As opposed to the ferromagnetic ground state $\mathrm{Tr}\mathcal{Z}^J$, that has the minimal number of magnons and occupies the "bottom" of the $\mathcal{N}=4$ SYM spectrum, the operators $\mathrm{Tr}\mathbb{S}^{L/2}$ are near the "top" of the spectrum with a number of magnons that is comparable to the operator's length. The strings that are dual to the (anti)ferromagnetic ground state (the hoop string and the point-like string respectively) are expected to be (un)stable. Unstable single spikes and hoop strings may be stabilized in many ways, e.g. by adding extra angular momenta [122].

The (in)stability of giant magnons (stable) and single spikes (unstable) may also be inferred by the (in)stability properties of their Pohlmeyer images. As we saw in §5.2, any string configuration on $\mathbb{R}\times\mathrm{S}^2$ can be mapped to a solution of the sine-Gordon equation. It will turn out that HM giant magnons are dual to the stable sine-Gordon soliton and infinite-momentum single spikes are dual to an unstable solution of the sG equation.

The Pohlmeyer image of giant magnons and single spikes can also be used to compute their S-matrices. Because of the factorizability of the $\mathfrak{su}(2)$ sector (4.40), we only need to consider 2-particle scattering. For giant magnons, the S-matrix that is computed from the Pohlmeyer reduction coincides with the strong coupling limit of the magnon S-matrix (4.49)–(4.52) that we encountered in §4.3.2. This constitutes further evidence that magnons and giant magnons are AdS/CFT duals [78]. The scattering of infinite-momentum single spikes was studied in [123] and the corresponding phase-shift was found to be equal (up to non-logarithmic terms) to the one for magnons and giant magnons. Okamura [121] explained this result by regarding single-spike scattering as factorized scattering between infinitely many giant magnons. In the paper [2] it was shown how this result may follow from the Pohlmeyer reduction.

Let us now set up the formalism that will allow us to study giant magnons and single spikes. Consider the following general ansatz of a string in $\mathbb{R}\times\mathrm{S}^2\subset\mathrm{AdS}_5\times\mathrm{S}^5$:

$$\Big\{t=t\left(\tau,\sigma\right),\,\rho=\theta=\phi_1=\phi_2=0\Big\}\times\Big\{\overline{\theta}_1=\theta\left(\tau,\sigma\right),\,\overline{\phi}_1=\phi\left(\tau,\sigma\right),\,\overline{\theta}_2=\overline{\phi}_2=\overline{\phi}_3=0\Big\}, \tag{8.9}$$

and the change of variables

$$z\left(\tau,\sigma\right)=R\,\sin\theta\left(\tau,\sigma\right), \tag{8.10}$$

so that $z\in[-R,R]$ and $\phi\in[0,2\pi)$. The embedding coordinates (5.1) of the string become:

$$Y_{05}=Y_0+i\,Y_5=R\,e^{i\,t(\tau,\sigma)} \quad \& \quad X_{12}=X_1+iX_2=\sqrt{R^2-z^2\left(\tau,\sigma\right)}\cdot e^{i\,\phi(\tau,\sigma)} \tag{8.11}$$

$$Y_{12}=Y_{34}=0 \qquad\qquad X_{34}=X_3=z\left(\tau,\sigma\right),\quad X_4=X_{56}=0. \tag{8.12}$$

The conformal gauge ($\gamma_{ab}=\eta_{ab}$) string Polyakov action is given by

$$S_P=\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\Bigg\{-\left(\dot{t}^2-t'^2\right)+\frac{\dot{z}^2-z'^{\,2}}{R^2-z^2}+\frac{1}{R^2}\left(R^2-z^2\right)\left(\dot{\phi}^2-\phi'^{\,2}\right)\Bigg\}. \tag{8.13}$$

In the static gauge $t=\tau$, the following set of Virasoro constraints (5.11)–(5.12) is obtained:

$$\dot{X}^i\dot{X}^i+\acute{X}^i\acute{X}^i=\frac{R^2}{R^2-z^2}\left(\dot{z}^2+z'^{\,2}\right)+\left(R^2-z^2\right)\left(\dot{\phi}^2+\phi'^{\,2}\right)=R^2 \tag{8.14}$$

$$\dot{X}^i\acute{X}^i=\frac{R^2\,\dot{z}z'}{R^2-z^2}+\left(R^2-z^2\right)\dot{\phi}\phi'=0. \tag{8.15}$$

According to what we have said in §5.2, the classical string sigma model in $\mathbb{R}\times\mathrm{S}^2$ can be mapped to the sine-Gordon equation by the Pohlmeyer reduction. Defining the Pohlmeyer field ψ as

$$\dot{X}^i\dot{X}^i-\acute{X}^i\acute{X}^i=\frac{R^2}{R^2-z^2}\left(\dot{z}^2-z'^{\,2}\right)+\left(R^2-z^2\right)\left(\dot{\phi}^2-\phi'^{\,2}\right)=R^2\cos 2\psi, \tag{8.16}$$

then ψ must satisfy the following sine-Gordon (sG) equation:

$$\ddot{\psi} - \psi'' + \frac{1}{2}\sin 2\psi = 0. \tag{8.17}$$

We also impose the following boundary conditions to the string's motion in $\mathbb{R}\times \mathrm{S}^2$:

$$p \equiv \Delta\phi = \Delta\varphi = \varphi(r,\tau) - \varphi(-r,\tau)\,, \quad \Delta z = z(r,\tau) - z(-r,\tau) = 0, \tag{8.18}$$

where p is the conserved momentum and $\pm r$ are the open string's world-sheet endpoints, i.e. $\sigma \in [-r, r]$. The conserved charges of the string are given by:

$$p \equiv \Delta\phi = \Delta\varphi = \int_{-r}^{+r} \varphi'\, d\sigma \tag{8.19}$$

$$E = \left|\frac{\partial L}{\partial \dot{t}}\right| = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{+r} \dot{t}\, d\sigma = \frac{r\sqrt{\lambda}}{\pi} \tag{8.20}$$

$$J = \frac{\partial L}{\partial \dot{\phi}} = \frac{\sqrt{\lambda}}{2\pi R^2}\int_{-r}^{+r} \left(R^2 - z^2\right)\dot{\phi}\, d\sigma. \tag{8.21}$$

This section is organized as follows. In §8.1 we present the infinite-size or Hofman-Maldacena giant magnons. In §8.2 the $(\mathbb{R}\times \mathrm{S}^2)$ infinite-size/momentum/winding single spike string is presented. §8.3 deals with the scattering and the bound states of infinite-size giant magnons and single spikes.

8.1 The Hofman-Maldacena (HM) Giant Magnon

To obtain the Hofman-Maldacena (or infinite-size) giant magnon, we set

$$\bar{\theta}_1 = \theta(\sigma - v\tau) \quad \& \quad \bar{\phi}_1 = \tau + \varphi(\sigma - v\tau) \tag{8.22}$$

in the ansatz (8.9), getting (in the static gauge $t = \tau$):

$$\Big\{t = \tau,\, \rho = \theta = \phi_1 = \phi_2 = 0\Big\} \times \Big\{\bar{\theta}_1 = \theta(\sigma - v\tau)\,,\, \bar{\phi}_1 = \tau + \varphi(\sigma - v\tau)\,,\, \bar{\theta}_2 = \bar{\phi}_2 = \bar{\phi}_3 = 0\Big\}, \tag{8.23}$$

where v is the giant magnon's velocity. Plugging,

$$z = z(\sigma - v\tau) \quad \& \quad \phi = \tau + \varphi(\sigma - v\tau) \tag{8.24}$$

into the Virasoro constraints (8.14)–(8.15), we obtain:

$$\varphi' = \frac{v}{1 - v^2}\cdot\frac{z^2}{R^2 - z^2}\,, \quad v \neq 1 \tag{8.25}$$

$$z'^{\,2} = \frac{z^2\left(\zeta_v^2 - z^2\right)}{R^2\left(1 - v^2\right)^2}\,, \quad \zeta_v^2 \equiv R^2\left(1 - v^2\right). \tag{8.26}$$

The equations (8.25)–(8.26) have the following solution:

$$z(\tau,\sigma) = R\sin\theta(\tau,\sigma) = \frac{R}{\gamma}\,\mathrm{sech}\Big[\gamma(\sigma - v\tau)\Big], \quad \gamma \equiv \frac{1}{\sqrt{1 - v^2}} \tag{8.27}$$

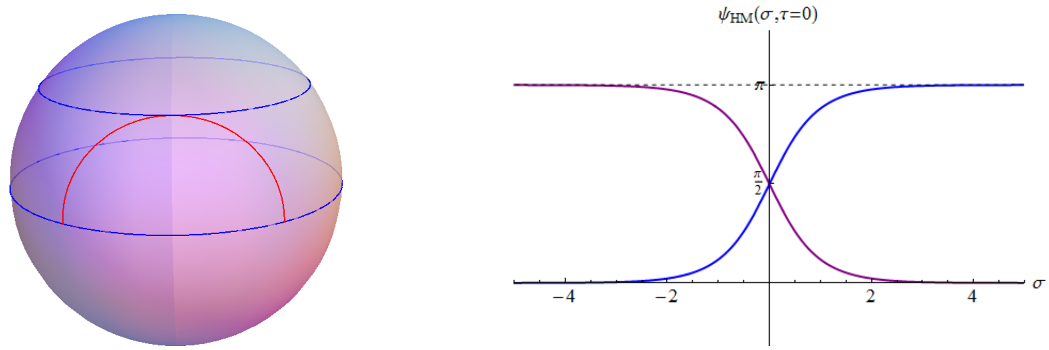


Figure 12: A $v = 0.8$ Hofman-Maldacena giant magnon (left) and its Pohlmeyer images (8.31) (right).

$$\phi(\tau,\sigma) = \tau + \arctan\left[\frac{1}{\gamma\, v}\tanh\gamma\,(\sigma - v\tau)\right]. \tag{8.28}$$

We have plotted a HM giant magnon (8.27)–(8.28) with $v = 0.8$ on the left of figure 12. Its two edges touch the equator and move at the speed of light, while the string rotates rigidly around the 2-sphere.

As we have already said, while both the conserved energy and the angular momentum (8.20)–(8.21) of the giant magnon diverge, their difference remains finite:

$$\left.\begin{aligned} & p = 2\arcsin\sqrt{1-v^2} \Rightarrow v = \cos p/2 \\ & \mathcal{E} \equiv \pi E/\sqrt{\lambda} = \sqrt{1-v^2}\cdot\mathbb{K}(1) = \infty \\ & \mathcal{J} \equiv \pi J/\sqrt{\lambda} = \sqrt{1-v^2}\cdot\Big[\mathbb{K}(1) - 1\Big] = \infty \end{aligned}\right\} \Rightarrow \mathcal{E} - \mathcal{J} = \sqrt{1-v^2} = \sin\frac{p}{2}. \tag{8.29}$$

The fact that the tree-level dispersion relation of infinite-size giant magnons (8.29) is identical to the strong coupling limit (8.5) of the corresponding dispersion relation of the $\mathcal{N} = 4$ SYM single-magnon operators (8.1), implies that the magnons and the giant magnons must be AdS/CFT duals. Below we shall provide further evidence for this duality, by calculating the S-matrix of giant magnons and showing that it coincides with the strong-coupling limit of the magnon S-matrix. In order to be able to do so, we will need to know what the Pohlmeyer image of the giant magnon is.

To determine which of the sine-Gordon solutions corresponds to the Pohlmeyer reduction of the HM giant magnon (8.27)–(8.28), we insert the ansatz (8.23) and the two Virasoro constraints (8.25)–(8.26) into (8.16). We find

$$\sin^2\psi = \frac{z^2}{\zeta_v^2} = \frac{z^2}{R^2(1-v^2)}, \tag{8.30}$$

which has the following solution:

$$\psi(\tau,\sigma) = 2\arctan e^{\pm\gamma(\sigma - v\tau)} = \arcsin\operatorname{sech}\left[\gamma\,(\sigma - v\tau)\right] \tag{8.31}$$

and corresponds to the kink/antikink solution of sG (see e.g. [124]). (8.31) has been plotted on the right-hand side of figure 12. In the singular case $v = 1$, the two constraints (8.25)–(8.26) become

$$z = 0 \quad \& \quad \varphi'\left(1 - \varphi'\right) = 0, \tag{8.32}$$

leading to the following two solutions:

$$z = 0 \quad \& \quad \phi = \tau + c \quad \text{or} \quad \phi = \sigma + c. \tag{8.33}$$

The first is a point-like string that rotates at the equator of the 2-sphere and it is dual to the BPS operator $\text{Tr}\mathcal{Z}^J$ of the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM. The second is the "hoop" string, a stationary string that is wound around the equator of the 2-sphere and it is dual to the $\mathcal{N} = 4$ SYM operators $\text{Tr}\mathbb{S}^{L/2}$, where $\mathbb{S}$ are the operators $\mathbb{S} \sim \mathcal{X}\overline{\mathcal{X}} + \mathcal{Y}\overline{\mathcal{Y}} + \mathcal{Z}\overline{\mathcal{Z}}$. It is rather straightforward to compute the conserved charges and the dispersion relations of both the point-like and the hoop string:

$$\mathcal{E} = \mathcal{J}, \; p = 0 \quad (\text{Point-Like String}) \quad \& \quad \mathcal{E} = \frac{p}{2}, \; \mathcal{J} = 0 \quad (\text{Hoop String}). \tag{8.34}$$

The corresponding Pohlmeyer reductions are also straightforward to obtain from (8.16)–(8.33) and they are given by $\psi = 0$ for the point-like string and $\psi = \pi/2$ for the hoop string.

8.2 Infinite-Momentum Single Spikes

Infinite-momentum/winding single spikes are obtained for

$$\overline{\theta}_1 = \theta\,(\sigma - \omega\tau) \quad \& \quad \overline{\phi}_1 = \omega\tau + \varphi\,(\sigma - \omega\tau) \tag{8.35}$$

in (8.9), which leads to (static gauge, $t = \tau$):

$$\Big\{t = \tau, \, \rho = \theta = \phi_1 = \phi_2 = 0\Big\} \times \Big\{\overline{\theta}_1 = \theta\,(\sigma - \omega\tau)\,, \overline{\phi}_1 = \omega\tau + \varphi\,(\sigma - \omega\tau)\,, \overline{\theta}_2 = \overline{\phi}_2 = \overline{\phi}_3 = 0\Big\} \tag{8.36}$$

where ω is the angular velocity of the single spike. If we insert

$$z = z\,(\sigma - \omega\tau) \quad \& \quad \phi = \omega\tau + \varphi\,(\sigma - \omega\tau) \tag{8.37}$$

into the constraint equations (8.14)–(8.15), we get:

$$\varphi' = \frac{\omega^2}{1-\omega^2} \cdot \frac{z^2 - \zeta_\omega^2}{R^2 - z^2}\,, \quad \omega \neq 1 \tag{8.38}$$

$$z'^{\,2} = \frac{\omega^2}{R^2\left(1-\omega^2\right)^2} \cdot z^2\left(\zeta_\omega^2 - z^2\right)\,, \quad \zeta_\omega^2 \equiv R^2\left[1 - \frac{1}{\omega^2}\right]. \tag{8.39}$$

For $\omega = 1$, the equations (8.32)–(8.33) for the point-like or the hoop string are obtained. For $\omega \neq 1$, (8.38)–(8.39) have the following solution:

$$z\,(\tau,\sigma) = R\sin\theta\,(\tau,\sigma) = R\sqrt{1 - \frac{1}{\omega^2}} \cdot \text{sech}\left(\frac{\sigma - \omega\tau}{\sqrt{\omega^2 - 1}}\right) \tag{8.40}$$

$$\phi\,(\tau,\sigma) = \sigma - \arctan\left[\sqrt{\omega^2 - 1}\tanh\left(\frac{\sigma - \omega\tau}{\sqrt{\omega^2 - 1}}\right)\right]. \tag{8.41}$$

An infinite-momentum single spike (8.40)–(8.41) with $\omega = 1.6$ has been plotted on the left of figure 13. The spiky string is wound around the equator of the 2-sphere while it rotates rigidly around it.

Infinite-momentum single spikes have their conserved linear momentum and energy (8.19)–(8.20)

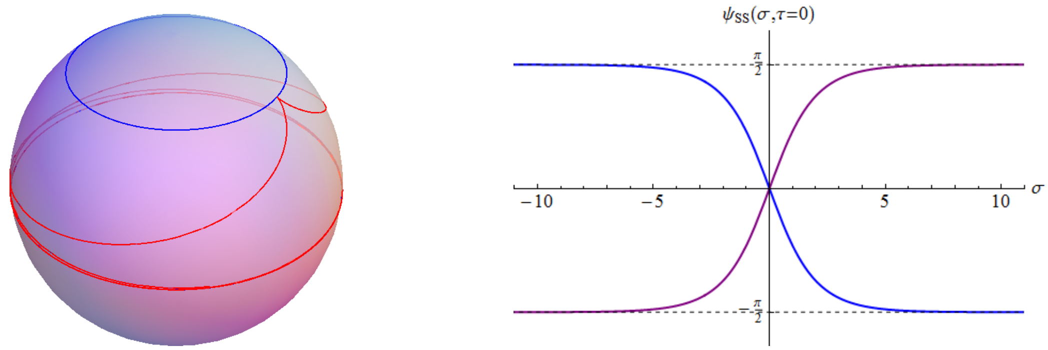


Figure 13: An infinite-size single spike with $\omega = 1.6$ (left) and its Pohlmeyer images (8.44) (right).

diverge while their difference remains finite ($\mathcal{E} \equiv \pi E/\sqrt{\lambda}$, $\mathcal{J} \equiv \pi J/\sqrt{\lambda}$):

$$\left.\begin{aligned} &p = 2\left[\sqrt{\omega^2-1}\cdot\mathbb{K}(1) - \arcsin\sqrt{1-1/\omega^2}\right] = \infty \\ &\mathcal{E} = \sqrt{\omega^2-1}\cdot\mathbb{K}(1) = \infty \\ &\mathcal{J} = \sqrt{1-1/\omega^2} \leq 1 \end{aligned}\right\} \Rightarrow \mathcal{E} - \frac{p}{2} = \arcsin\sqrt{1-\frac{1}{\omega^2}} = \arcsin\mathcal{J}. \quad (8.42)$$

To study the scattering between single spikes we need to know their Pohlmeyer reduction. If we insert the ansatz (8.36) and the Virasoro constraints (8.38)–(8.39) into (8.16), we will find that

$$\sin^2\psi = 1 - \frac{z^2}{\zeta_\omega^2}. \quad (8.43)$$

Therefore the Pohlmeyer reduction of the single spike is:

$$\psi(\tau,\sigma) = \frac{\pi}{2} - 2\arctan e^{\pm(\sigma-\omega\tau)/\sqrt{\omega^2-1}} = \arcsin\tanh\left[\frac{\sigma-\omega\tau}{\sqrt{\omega^2-1}}\right] \quad (8.44)$$

and it corresponds to an unstable solution of the sine-Gordon equation. The plot of (8.44) for an infinite-size single spike with $\omega = 1.6$, can be found on the right of figure 13.

As we have already mentioned in the introduction of this section, there exists a transformation that allows to transform between infinite-size giant magnons and single spikes. The $\tau \leftrightarrow \sigma$ symmetry or "2D duality" [118],

$$\tau \leftrightarrow \sigma\,,\; v \leftrightarrow \frac{1}{\omega}\,,\; \psi \leftrightarrow \left[\frac{\pi}{2} - \psi\right] \quad \Leftrightarrow \quad \text{Giant Magnons} \leftrightarrow \text{Single Spikes}. \quad (8.45)$$

transforms the GM solutions (8.27)–(8.28) and their Pohlmeyer reduction (8.31) to the SS ones, (8.40)–(8.41) and (8.44). The $\tau \leftrightarrow \sigma$ transform can also be used to transform the point-like string to the hoop string (8.33). As we shall see, the 2D duality also applies to giant magnons and single spikes of finite sizes.

In order to transform between the dispersion relations of GMs and SSs, the following transform should be applied:

$$\mathcal{E} - \frac{p}{2} \mapsto \frac{p}{2} \quad \& \quad \mathcal{J} \mapsto \mathcal{E} - \mathcal{J} \quad \Leftrightarrow \quad \text{Single Spikes} \mapsto \text{Giant Magnons}. \quad (8.46)$$

The transform (8.46) maps the energy-momentum relation (8.42) to (8.29). (8.46) obviously also works for the dispersion relations of the point-like and the hoop string (8.34).

8.3 Bound States & Scattering

8.3.1 Scattering

The sine-Gordon images of (infinite-size) giant magnons and single spikes can be used to calculate their S-matrices and study their bound states. Let us start by considering giant magnon scattering.

Following Hofman and Maldacena [78], we may study the scattering of the Pohlmeyer images of GMs which are the kink/antikink (soliton/antisoliton) solutions of the sine-Gordon equation (8.31). It is slightly more convenient to consider the soliton-antisoliton solution, although the same result can be found from any of the remaining 2-soliton scattering solutions of sG, namely the soliton-soliton or the antisoliton-antisoliton. The kink-antikink solution of the sG equation (8.17) is (see e.g. [124]):

$$\tan\frac{\psi_{\text{s-a}}}{2} = \frac{\sinh(v\gamma\tau)}{v\cosh\gamma\sigma} = \frac{1}{v}\cdot\frac{e^{v\gamma\tau}-e^{-v\gamma\tau}}{e^{\gamma\sigma}+e^{-\gamma\sigma}}. \tag{8.47}$$

The solution (8.47) describes two solitons that are initially at $\sigma = \pm\infty$ when $\tau = -\infty$, then they start approaching each other, they interact and they end up at the opposite side $\sigma = \mp\infty$, when $\tau = +\infty$. The Pohlmeyer reduction $\psi_{\text{s-a}}$, as well as the corresponding energy density $dE_{\text{s-a}}/d\sigma$ of giant magnon scattering with $v = 0.5$, have been plotted on the left graphs of figures 14–15 respectively.

We now want to obtain the time delay that the two sG solitons experience in their center of mass (CM) frame, as they pass through each other. By comparing the values of (8.47) at $\sigma = \pm\infty$, $\tau = \pm\infty$, we find

$$\Delta T_{\text{cm}} = \frac{2\sqrt{1-v^2}}{v}\cdot\ln v, \tag{8.48}$$

where $0 \leq v \leq 1$ is the soliton's velocity in the CM frame. The equation (8.48) may be transformed to a reference frame where the two solitons have arbitrary speeds v_1 and v_2. The result is:

$$\Delta T_{12} = \frac{2}{v_1\gamma_1}\cdot\ln v = \tan\frac{p_1}{2}\cdot\ln\left[\frac{1-\cos\frac{p_1-p_2}{2}}{1-\cos\frac{p_1+p_2}{2}}\right], \qquad \begin{array}{l} v_i = \tan\hat{\theta}_i = \cos p_i/2 \\ \gamma_i = \cosh\hat{\theta}_i = \csc p_i/2 \end{array}, \tag{8.49}$$

where $\hat{\theta}_i$ is the rapidity of the soliton $i = 1, 2$ and $v = \tanh(\hat{\theta}_1 - \hat{\theta}_2/2)$. The prescription for the calculation of phase-shifts from time delays in quantum field theory has been laid down by Jackiw and Woo in [125]:

$$\Delta T_{12} = \frac{\partial\delta_{12}}{\partial\epsilon_1}, \tag{8.50}$$

where $\epsilon_1 = \sin p/2$ is the energy of one of the solitons. Inserting (8.49) in (8.50), we find:

$$\delta_{12} = \frac{\sqrt{\lambda}}{\pi}\left\{\left(\cos\frac{p_2}{2}-\cos\frac{p_1}{2}\right)\ln\left[\frac{1-\cos\frac{p_1-p_2}{2}}{1-\cos\frac{p_1+p_2}{2}}\right] - p_1\sin\frac{p_1}{2}\right\}. \tag{8.51}$$

The last term in (8.51) depends on the worldsheet gauge that we have chosen and the definition of the spatial variable σ. Had we chosen a gauge with $dJ/d\sigma = \text{const.}$, instead of one with $dE/d\sigma = \text{const.}$, we could have arranged for the last term in (8.51) to drop out and the phase-shift to become, for $\sin p_{1,2}/2 > 0$:

$$\delta_{12} = \delta(p_1, p_2) = -\frac{\sqrt{\lambda}}{\pi}\left(\cos\frac{p_1}{2}-\cos\frac{p_2}{2}\right)\ln\left[\frac{\sin^2\frac{p_1-p_2}{4}}{\sin^2\frac{p_1+p_2}{4}}\right]. \tag{8.52}$$

It can be shown that the phase-shift (8.52) is equal to the strong coupling value of the 2-magnon phase-shift of the dressing phase (4.50),

$$\sigma^2_{12(\text{AFS})} = e^{i\delta_{12}} \tag{8.53}$$

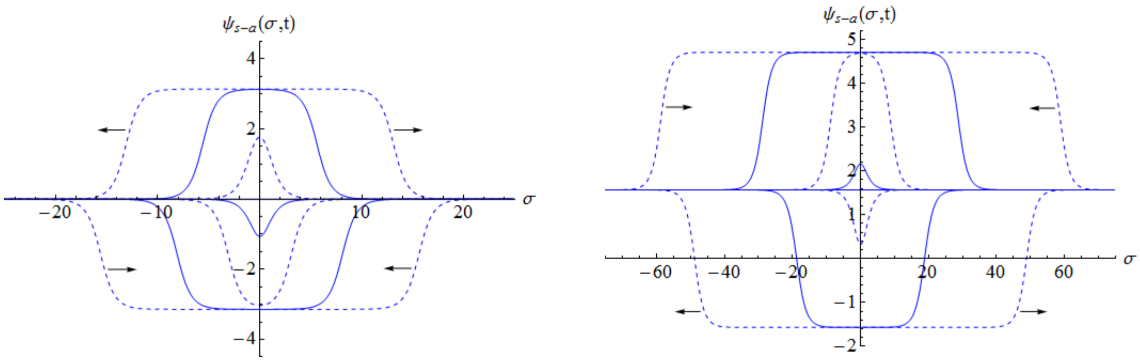


Figure 14: Sine-Gordon solution of soliton-antisoliton scattering between two giant magnons with $v = 0.5$ (left) and two single spikes with $\omega = 2$ (right).

aka AFS phase (4.52). We won't have time to present the derivation of the formula (8.52) from the AFS phase here, so we refer the reader to the paper [68] for details.

Another way to compute the giant magnon phase-shift (8.51)–(8.52) is by using the specific solution of the string sigma model in $\mathbb{R}\times\mathrm{S}^2$ that describes the scattering between two giant magnons. This was carried out in [126], where the 2-giant magnon scattering solution was constructed by the so-called dressing method, that starts from the $\mathbb{R}\times\mathrm{S}^2$ point-like string (8.33) and successively builds more complicated string solutions in $\mathbb{R}\times\mathrm{S}^2$. In a similar manner, one may "dress" the hoop string (8.33) and obtain the scattering solution between two single spikes, from which the single spike phase-shift may be calculated. This has been done in [123] and the result is:

$$\delta\left(q_1,q_2\right) = -\frac{\sqrt{\lambda}}{\pi}\left\{\left(\cos\frac{q_1}{2}-\cos\frac{q_2}{2}\right)\ln\left[\frac{\sin^2\frac{q_1-q_2}{4}}{\sin^2\frac{q_1+q_2}{4}}\right]-q_1\sin\frac{q_1}{2}\right\}, \tag{8.54}$$

where q is defined as $\mathcal{J} = \left(1-1/\omega^2\right)^{1/2} \equiv \sin q/2$, while ω and $\mathcal{J}$ are the spike's angular velocity and conserved angular momentum respectively. For $p \leftrightarrow q$ (8.54) obviously agrees with the giant magnon phase-shift (8.52) that we computed above, up to the non-logarithmic term $q\sin q/2$ which comes with the opposite sign. A qualitative explanation for the coincidence of the logarithmic terms of (8.51)–(8.54) has been given by Okamura in [121], where single spike scattering was regarded as factorized scattering between infinitely many giant magnons.

A simpler derivation of the single spike phase-shift was given in [2] by using the $\tau\leftrightarrow\sigma$ transform (8.45).[46] Roughly speaking, the $\tau\leftrightarrow\sigma$ transform can be used to transform the sG solutions that correspond to GMs, into solutions of the sG equation that correspond to single spikes. Then, the single spike phase-shift can be calculated from the Pohlmeyer image of the single spike scattering solution à la Hofman-Maldacena. We note that both the logarithmic and the non-logarithmic terms of the phase-shift formula of [2] agree with the GM phase-shift (8.51).

Let us see how the recipe of [2] works. This time it is more convenient to start from the soliton-soliton scattering solution of the sG equation:

$$\tan\frac{\psi_{\text{s-s}}}{2} = \frac{v\,\sinh\gamma\sigma}{\cosh v\gamma\tau}. \tag{8.55}$$

This solution of the sG equation has topological charge[47] $Q = +2$ and it is the Pohlmeyer reduction of two giant magnons that scatter in their center of mass frame. When it is $\tau\leftrightarrow\sigma$ transformed according

[46] The authors of [2] followed a suggestion that appeared in footnote 2 of reference [123].

[47] The topological charge Q of a sG solution is defined as $Q = 1/\pi\int_{-\infty}^{+\infty}\partial_\sigma\psi\,d\sigma$.

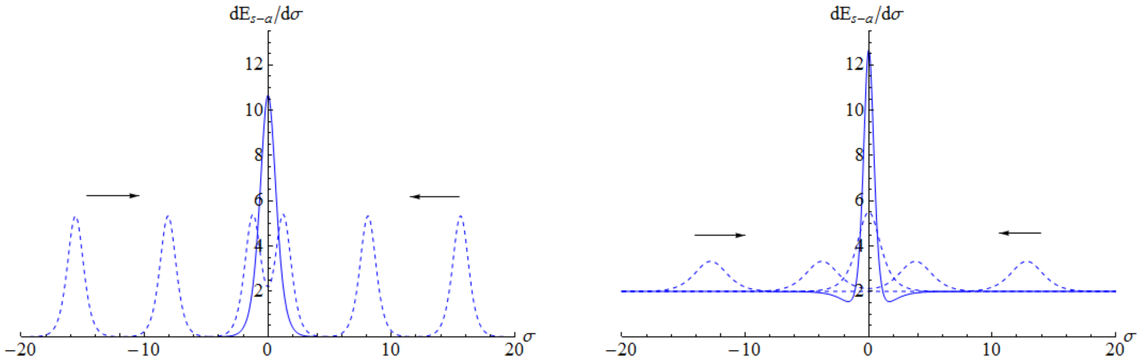


Figure 15: Energy density of soliton-antisoliton scattering between two giant magnons with $v = 0.5$ (left) and two single spikes with $\omega = 2$ (right).

to (8.45), the transformed solution

$$\tan\frac{\psi_{\text{s-a}}}{2} = \frac{\omega\cosh\sigma/\sqrt{\omega^2-1} - \sinh\omega\tau/\sqrt{\omega^2-1}}{\omega\cosh\sigma/\sqrt{\omega^2-1} + \sinh\omega\tau/\sqrt{\omega^2-1}}, \tag{8.56}$$

satisfies the sine-Gordon equation (8.17) and has a topological charge of $Q = 0$, which means that it corresponds to the scattering between two solutions that carry opposite topological charges. The equation (8.56) is the Pohlmeyer reduction of a string solution that describes the scattering of two single spikes in their center of mass frame. On the right-hand side of figures 14–15, we have plotted the sG wavefunction (8.56) and its energy density that correspond to the scattering between two single spikes with $\omega = 2$. Likewise we may obtain the solutions of the sG equation that describe the scattering between single spikes of the same topological charge (and total charge $Q = \pm 2$).

Now that we have obtained the Pohlmeyer reduction of single spike scattering (8.56), we may go on and calculate the phase-shift à la Hofman and Maldacena. The result is the same if the $Q = \pm 2$ solutions of sG are used instead. In a reference frame where the velocities of the two solutions are $v_1 = 1/\omega_1$ and $v_2 = 1/\omega_2$, the time delay is found to be:

$$\Delta T_{12} = \frac{1}{\gamma_1}\ln v = \sin\frac{q_1}{2}\cdot\ln\left[\frac{1-\cos\frac{q_1-q_2}{2}}{1-\cos\frac{q_1+q_2}{2}}\right], \quad v = \tanh\left[\frac{\hat{\theta}_1-\hat{\theta}_2}{2}\right], \tag{8.57}$$

where $\cosh\hat{\theta}_i \equiv \gamma_i = (1-v_i^2)^{-1/2} = \csc q_i/2$ for $i = 1, 2$. The single spike phase shift for $\sin q_i/2 > 0$ is recovered by means of the formula,

$$\Delta T_{12} = \frac{\partial\delta_{12}}{\partial\varepsilon_1}, \quad \varepsilon_i \equiv \mathcal{E}_i - \frac{p_i}{2} = \arcsin\mathcal{J}_i = \frac{q_i}{2}, \quad i = 1, 2. \tag{8.58}$$

We find:

$$\delta(q_1, q_2) = \frac{\sqrt{\lambda}}{\pi}\left\{\left(\cos\frac{q_2}{2} - \cos\frac{q_1}{2}\right)\ln\left[\frac{\sin^2\frac{q_1-q_2}{4}}{\sin^2\frac{q_1+q_2}{4}}\right] - q_1\sin\frac{q_1}{2}\right\}. \tag{8.59}$$

8.3.2 Bound States

We close this section with two examples of the $\sigma \leftrightarrow \tau$ transform with single spike bound states. It is obvious that any N-soliton solution of the sG equation can be $\sigma \leftrightarrow \tau$ transformed and give rise to some new single spike solutions. An example is the breather (Q = 0) solution,

$$\tan\frac{\psi_b}{2} = \frac{\sin a\gamma_a\tau}{a\cosh\gamma_a\sigma}, \quad \gamma_a \equiv \frac{1}{\sqrt{1+a^2}} \tag{8.60}$$

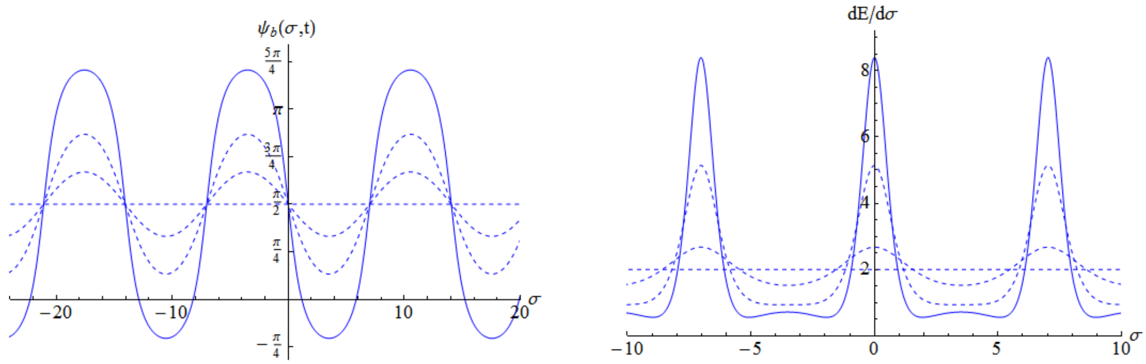


Figure 16: Sine-Gordon wavefunction (left) and energy density (right) of the breather solution for single spikes (8.61), with $\omega = 2$.

which takes the following form under the $\tau \leftrightarrow \sigma$ transform:

$$\tan\frac{\psi_b}{2} = \frac{\cosh\omega\gamma_\omega\tau - \omega\sin\gamma_\omega\sigma}{\cosh\omega\gamma_\omega\tau + \omega\sin\gamma_\omega\sigma}. \tag{8.61}$$

(8.61) satisfies the sine-Gordon equation (8.17). For $\omega = 2$ its wavefunction has been plotted on the left graph of figure 16 while the right graph of the same figure contains a plot of the energy density. Initially, the solution is constant at $\psi = \pi/2$, then between times $\tau = -\tau_0$ and $\tau = 0$ its amplitude and energy start growing until they become the wiggly periodic lines of figure 16, with extrema at $\sigma = k\pi/2\gamma_\omega$. Afterwards, both wavefunctions start decreasing again and attain their constant initial value $\psi = \pi/2$, at $\tau = \tau_0$.

Another stable solution of sG with 3 solitons is the "wobble", which contains a breather and a kink (or antikink) [127, 128]:

$$\tan\frac{\psi_w}{2} = \frac{\frac{\sqrt{1-a^2}}{a}\sin a\tau + \frac{e^\sigma}{2}\left(e^{-\sqrt{1-a^2}\sigma} + r_{\rm a}^2 e^{\sqrt{1-a^2}\sigma}\right)}{\cosh\left(\sqrt{1-a^2}\,\sigma\right) + \frac{\sqrt{1-a^2}}{a}\,r_{\rm a}\,e^\sigma\sin a\tau}\,, \quad r_{\rm a} \equiv \frac{1-\sqrt{1-a^2}}{1+\sqrt{1-a^2}}. \tag{8.62}$$

With the $\tau \leftrightarrow \sigma$ transform (8.62) becomes:

$$\tan\frac{\psi_w}{2} = \frac{\sqrt{\omega^2-1}\,(r_\omega e^\tau - 1)\sin\frac{\sigma}{\omega} + \frac{1}{2}\left[(1-e^\tau)\,e^{-\frac{\sqrt{\omega^2-1}}{\omega}\cdot\tau} + \left(1 - r_\omega^2\,e^\tau\right)e^{\frac{\sqrt{\omega^2-1}}{\omega}\cdot\tau}\right]}{\sqrt{\omega^2-1}\,(r_\omega e^\tau + 1)\sin\frac{\sigma}{\omega} + \frac{1}{2}\left[(1+e^\tau)\,e^{-\frac{\sqrt{\omega^2-1}}{\omega}\cdot\tau} + (1 + r_\omega^2\,e^\tau)\,e^{\frac{\sqrt{\omega^2-1}}{\omega}\cdot\tau}\right]}, \tag{8.63}$$

where

$$r_\omega \equiv \frac{\omega - \sqrt{\omega^2-1}}{\omega + \sqrt{\omega^2-1}}. \tag{8.64}$$

The solution (8.63) also exhibits the "flare"-like behavior of the breather (8.61).

9 Finite-Size Giant Magnons and Single Spikes

The finite-size generalizations of giant magnons and single spikes can be obtained by inserting,

$$\theta = \theta\left(\sigma - v\omega\tau\right) , \quad \varphi \equiv \phi - \omega\,\tau = \varphi\left(\sigma - v\omega\tau\right) \tag{9.1}$$

into the ansatz (8.9), so that in the static gauge $t = \tau$ (8.9) becomes:

$$\Big\{t = \tau,\, \rho = \theta = \phi_1 = \phi_2 = 0\Big\} \times \Big\{\bar{\theta}_1 = \theta\left(\sigma - v\omega\tau\right), \bar{\phi}_1 = \omega\,\tau + \varphi\left(\sigma - v\omega\tau\right), \bar{\theta}_2 = \bar{\phi}_2 = \bar{\phi}_3 = 0\Big\} \tag{9.2}$$

Finite-size giant magnons and single spikes are open strings in $\mathbb{R} \times \mathrm{S}^2$ that rotate with angular velocity ω and at the same time they translate with phase velocity $v_p = v \cdot \omega$. If we plug,

$$z = z\left(\sigma - v\omega\tau\right) , \quad \varphi \equiv \phi - \omega\,\tau = \varphi\left(\sigma - v\omega\tau\right) \tag{9.3}$$

into the constraint equations (8.14)–(8.15) and the Pohlmeyer reduction (8.16), we obtain:

$$\varphi' = \frac{v\,\omega^2}{1 - v^2\omega^2} \cdot \frac{z^2 - \zeta_\omega^2}{R^2 - z^2}, \quad \zeta_\omega^2 \equiv R^2\left[1 - \frac{1}{\omega^2}\right], \quad v \cdot \omega \neq 1 \tag{9.4}$$

$$z'^{\,2} = \frac{\omega^2}{R^2\left(1 - v^2\omega^2\right)^2} \cdot \left(z^2 - \zeta_\omega^2\right)\left(\zeta_v^2 - z^2\right), \quad \zeta_v^2 \equiv R^2\left(1 - v^2\right) \tag{9.5}$$

$$\sin^2\psi = \frac{z^2 - \zeta_\omega^2}{\zeta_v^2 - \zeta_\omega^2} \quad \text{(Pohlmeyer reduction).} \tag{9.6}$$

For $v \cdot \omega = 1$ we're led to the trivial solution $z = \zeta_v = \zeta_\omega$. This solution is only possible if $z = 0$ and $v = \omega = 1$, which is just the point-like string and its $\sigma \leftrightarrow \tau$ dual hoop string (8.33). Combining the equations (9.4) and (9.5), we obtain:

$$\frac{dz}{d\varphi} = \frac{R^2 - z^2}{R\,v\,\omega}\sqrt{\frac{\zeta_v^2 - z^2}{z^2 - \zeta_\omega^2}}. \tag{9.7}$$

It is relatively simple to retrieve the infinite-size limits of giant magnons and single spikes from the finite-size ansatz (9.2) and equations (9.4)–(9.6). The Hofman-Maldacena (or infinite-size) giant magnon (8.25)–(8.26) and its Pohlmeyer reduction (8.30) are retrieved for $\omega = 1$ and $|v| \leq 1$, while for $v = 1$ and $\omega \geq 1$ we recover the infinite-size (or infinite momentum/winding) single spikes (8.38)–(8.39) and their Pohlmeyer reduction (8.43).

Depending on the relative values of the open string's linear and angular velocities v and ω, there exist four main regimes of solutions of the constraints (9.4)–(9.5) and the Pohlmeyer reduction (9.6):

1. Giant magnon, elementary region: $0 \leq |v| < 1/\omega \leq 1$.

2. Giant magnon, doubled region: $0 \leq |v| \leq 1 \leq 1/\omega$.

3. Single spike, elementary region: $0 \leq 1/\omega < |v| \leq 1$.

4. Single spike, doubled region: $0 \leq 1/\omega \leq 1 \leq |v|$.

See also table 1. The choice of the names "elementary" and "doubled" will become clear below, where each the above regions will be studied in more detail. In §10, we will examine the classical dispersion relations of these solutions.

	$\omega \leq 1$	$\omega \geq 1$	
$v\omega \leq 1$	GM Doubled (9.2)	GM Elementary (9.1)	–
$v\omega \geq 1$	–	SS Elementary (9.3)	SS Doubled (9.4)
	$v \leq 1$		$v \geq 1$

Table 1: Elementary and doubled regions of giant magnons and single spikes.

9.1 Giant Magnon: Elementary Region, $0 \leq |v| < 1/\omega \leq 1$

In this case the open string is an arc in $\mathbb{R} \times \mathrm{S}^2$ (giant magnon) that extends between the parallels ζ_ω and ζ_v:

$$0 \leq \zeta_\omega^2 = z_{\text{min}}^2 \leq z^2 \leq z_{\text{max}}^2 = \zeta_v^2 \leq R^2. \tag{9.8}$$

The conserved momentum/angular extent of finite-size giant magnons in the elementary region is:

$$p \equiv \Delta\phi = \Delta\varphi = \int_{-r}^{+r} \varphi' \, d\sigma = \frac{2}{\sqrt{1 - v^2}} \left[\frac{1}{v\omega} \mathbf{\Pi}\left(\left[1 - \frac{1}{v^2}\right]\eta; \eta\right) - v\omega \, \mathbb{K}(\eta)\right], \tag{9.9}$$

where

$$\eta \equiv 1 - \frac{z_{\text{min}}^2}{z_{\text{max}}^2} = \frac{1 - v^2\omega^2}{\omega^2 (1 - v^2)} \Leftrightarrow \omega = \frac{1}{\sqrt{\eta + v^2 (1 - \eta)}}. \tag{9.10}$$

The conserved charges of the energy and the angular momentum are found to be:

$$E = \frac{\sqrt{\lambda}}{2\pi} \int_{-r}^{+r} \dot{t} \, d\sigma = \frac{r\sqrt{\lambda}}{\pi} = \frac{\sqrt{\lambda}}{\pi\omega} \cdot \frac{1 - v^2\omega^2}{\sqrt{1 - v^2}} \mathbb{K}(\eta) \,, \quad r = \frac{1 - v^2\omega^2}{\omega\sqrt{1 - v^2}} \mathbb{K}(\eta) \tag{9.11}$$

$$J = \frac{\sqrt{\lambda}}{2\pi R^2} \int_{-r}^{+r} \left(R^2 - z^2\right) \dot{\phi} \, d\sigma = \frac{\sqrt{\lambda}}{\pi} \cdot \sqrt{1 - v^2} \Big(\mathbb{K}(\eta) - \mathbb{E}(\eta)\Big). \tag{9.12}$$

As we have said, the infinite-size (Hofman-Maldacena) giant magnon can be recovered in the limit $\omega = 1$ and $J = \infty$. To obtain the finite-size version of the $\mathbb{R} \times \mathrm{S}^2$ closed folded GKP string that we studied in §6.2, two elementary giant magnons with velocities $v = 0$, maximum momentum $p = \pi$ and angular momenta $J/2$ must be superposed. The Virasoro constraints (9.4)–(9.5) of giant magnons in the elementary region, have the following solutions:

$$z(\tau, \sigma) = R\sqrt{1 - v^2} \cdot \text{dn}\left(\frac{\sigma - v\omega\tau}{\omega\,\eta\sqrt{1 - v^2}}, \eta\right) \,, \quad n \cdot r \leq \sigma - v\omega\tau \leq (n + 1) \cdot r \tag{9.13}$$

$$\varphi(z) = \frac{(-1)^n}{\sqrt{1 - v^2}} \left\{\frac{1}{v\omega} \mathbf{\Pi}\left(\left[1 - \frac{1}{v^2}\right]\eta, \arcsin\left[\frac{1}{\sqrt{\eta}}\sqrt{1 - \frac{z^2}{z_{\text{max}}^2}}\right] \Big| \eta\right) - \right.$$

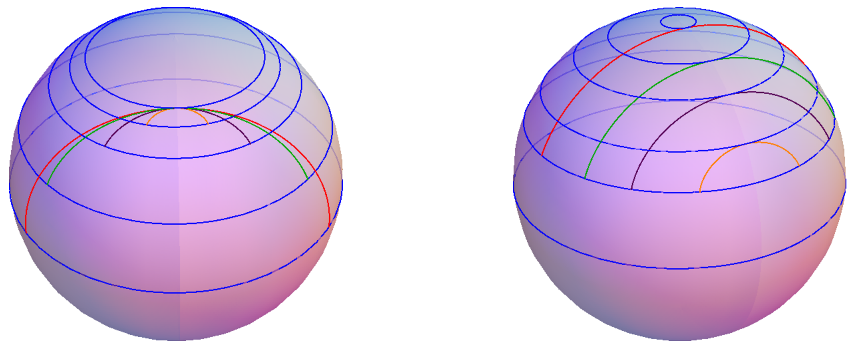


Figure 17: Plots of finite-size giant magnons with $\omega > 1$ (elementary region), for $v =$ const. (left) and $\omega =$ const. (right).

$$-v\omega\,\mathbb{F}\left(\arcsin\left[\frac{1}{\sqrt{\eta}}\sqrt{1-\frac{z^2}{z_{\text{max}}^2}}\right],\eta\right)\Bigg\} + \left\lfloor\frac{n+1}{2}\right\rfloor\cdot p\,, \quad z_{\text{min}} \leq z \leq z_{\text{max}}, \quad (9.14)$$

where $\lfloor y \rfloor$ is the floor function of y. In figure 17 we have drawn various snapshots of elementary giant magnons, by plotting (9.14) upon a sphere for various values of the velocities v and ω, and for $-r \leq \sigma \leq r$, $\tau = 0$. With Mathematica we may also animate the elementary giant magnon and verify that it performs a worm-like movement around the 2-sphere, as described by Arutyunov, Frolov and Zamaklar in [129]. The elementary giant magnon corresponds to a single-spin helical string of type (i), according to the terminology of Okamura and Suzuki [130].

By solving the equation (9.6), we obtain the following Pohlmeyer reduction of finite-size giant magnons in the elementary region:

$$\psi(\tau,\sigma) = \frac{\pi}{2} + \text{am}\left(\frac{\sigma - v\omega\tau}{\omega\,\eta\,\sqrt{1-v^2}},\eta\right). \quad (9.15)$$

This solution describes a quasi-periodic series of sine-Gordon kinks that is also known as kink chain/train (see also [131]). The period of the kink chain/train is given by

$$\psi(\tau,\sigma) = \psi(\sigma + L,\tau) + n\pi\,, \quad L = 2\sqrt{\eta\left(1-v^2\omega^2\right)}\cdot\mathbb{K}(\eta)\,, \quad n = 0,\pm1,\pm2,\ldots \quad (9.16)$$

Since each period of the kink train contains exactly one soliton (a kink), the parameter region $0 \leq |v| < 1/\omega \leq 1$ has been dubbed "elementary" by Klose and McLoughlin in [131]. The 2-d plot of (9.15) for $v = 0.1$ and $\omega = 1.01$, in terms of the worldsheet variables σ and τ can be found in the leftmost graph of figure 22. The stability properties of the sine-Gordon solution (9.15) have been studied in [132], according to which (9.15) corresponds to a linearly stable subluminal rotational wave (with $v \cdot \omega < 1$).

9.2 Giant Magnon: Doubled Region, $0 \leq |v| \leq 1 \leq 1/\omega$

In this case the open string is an arc on the 2-sphere that touches the equator and is bound above by the parallel ζ_v:

$$\zeta_\omega^2 = -z_{\text{min}}^2 \leq 0 \leq z^2 \leq z_{\text{max}}^2 = \zeta_v^2 \leq R^2. \quad (9.17)$$

In the doubled region, the finite-size giant magnon's conserved momentum/angular extent is found to be

$$p \equiv \Delta\phi = \Delta\varphi = \int_{-r}^{+r}\varphi'\,d\sigma = \frac{2\omega}{\sqrt{1-v^2\omega^2}}\left[\frac{1}{v\omega}\,\mathbf{\Pi}\left(1-\frac{1}{v^2};\frac{1}{\eta}\right) - v\omega\,\mathbb{K}\left(\frac{1}{\eta}\right)\right], \quad (9.18)$$

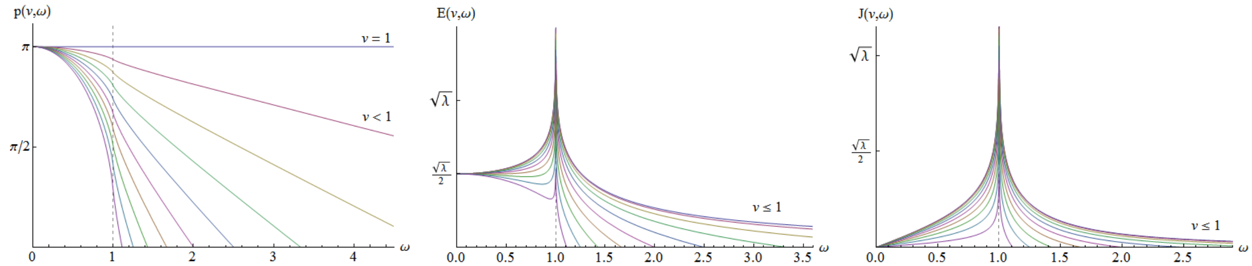


Figure 18: Momentum, energy, spin of finite-size giant magnons in terms of the angular velocity ω.

where

$$\eta \equiv 1 + \frac{z_{\text{min}}^2}{z_{\text{max}}^2} = \frac{1 - v^2\omega^2}{\omega^2\left(1 - v^2\right)} \Leftrightarrow \omega = \frac{1}{\sqrt{\eta + v^2\left(1 - \eta\right)}}. \tag{9.19}$$

The conserved energy and angular momentum of the giant magnon in the doubled region are given by:

$$E = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{+r} \dot{t}\, d\sigma = \frac{r\sqrt{\lambda}}{\pi} = \frac{\sqrt{\lambda}}{\pi}\cdot\sqrt{1 - v^2\omega^2}\,\mathbb{K}\left(\frac{1}{\eta}\right), \quad r = \sqrt{1 - v^2\omega^2}\,\mathbb{K}\left(\frac{1}{\eta}\right) \tag{9.20}$$

$$J = \frac{\sqrt{\lambda}}{2\pi R^2}\int_{-r}^{+r}\left(R^2 - z^2\right)\dot{\phi}\, d\sigma = \frac{\sqrt{\lambda}}{\pi}\cdot\frac{\sqrt{1 - v^2\omega^2}}{\omega}\left[\mathbb{K}\left(\frac{1}{\eta}\right) - \mathbb{E}\left(\frac{1}{\eta}\right)\right]. \tag{9.21}$$

In the doubled region, the Virasoro constraints (9.4)–(9.5) have the following solution:

$$z\left(\tau,\sigma\right) = R\sqrt{1 - v^2}\cdot\text{cn}\left(\frac{\sigma - v\omega\tau}{\sqrt{1 - v^2\omega^2}}, \frac{1}{\eta}\right), \quad 2\,n\cdot r \leq \sigma - v\omega\tau \leq 2\left(n + 1\right)\cdot r \tag{9.22}$$

$$\varphi\left(z\right) = \frac{\left(-1\right)^n\omega}{\sqrt{1 - v^2\omega^2}}\left\{\frac{1}{v\omega}\,\mathbf{\Pi}\left(1 - \frac{1}{v^2}, \arccos\left[\frac{z}{z_{\text{max}}}\right]\middle|\,\frac{1}{\eta}\right) - \right.$$
$$\left. - v\omega\,\mathbb{F}\left(\arccos\left[\frac{z}{z_{\text{max}}}\right], \frac{1}{\eta}\right)\right\} + 2\left\lfloor\frac{n + 1}{2}\right\rfloor\cdot p\,, \quad -z_{\text{max}} \leq z \leq z_{\text{max}}. \tag{9.23}$$

The HM giant magnon (8.27)–(8.28) can be retrieved from (9.22)–(9.23) in the limit $\omega = 1$. The circular GKP string in $\mathbb{R}\times\text{S}^2$ that we studied in §6.2 is formed by two doubled giant magnons with velocities $v = 0$, maximum momentum $p = \pi$ and angular momenta $J/2$. Drawings of the doubled region giant magnons, for various values of their velocities v and ω can be found in figure 19. The motion of giant magnons in the doubled region is a combination of rotation and translation: the GM is initially tangent to the parallel $z = z_{\text{max}}$ of the northern hemisphere, then it starts gradually moving towards the parallel $z = -z_{\text{max}}$ of the southern hemisphere, before it moves again towards its initial position. Then the motion repeats. Doubled region giant magnons have also been classified by Okamura and Suzuki [130] as single-spin helical strings of type (ii). Figure 18 contains the plots of the momentum, the energy and the spin of both the elementary ($\omega \geq 1$) and the doubled ($\omega \leq 1$) giant magnons in terms of their angular velocities ω and various values of their linear velocities v.

The Pohlmeyer reduction (9.6) of the $\mathbb{R}\times\text{S}^2$ string (9.22)–(9.23) is a periodic series of sine-Gordon

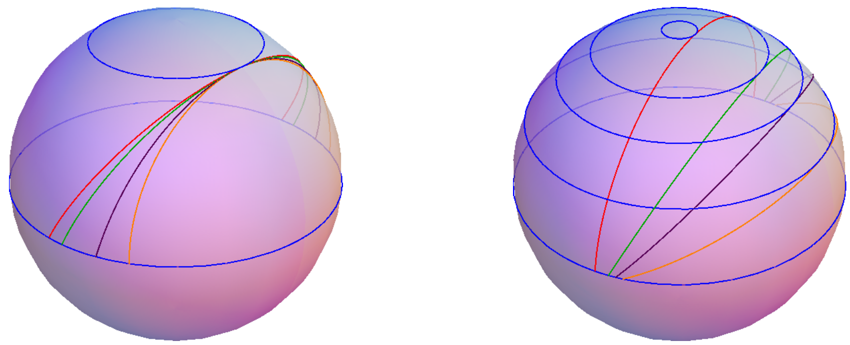


Figure 19: Plots of finite-size giant magnons with $\omega < 1$ (doubled region), for $v = $ const. (left) and $\omega = $ const. (right).

kinks and antikinks that is known as kink-antikink chain/train:

$$\psi(\tau,\sigma) = \arccos\left[\frac{1}{\sqrt{\eta}}\,\mathrm{sn}\left(\frac{\sigma - v\omega\tau}{\sqrt{1-v^2\omega^2}}, \frac{1}{\eta}\right)\right]. \tag{9.24}$$

(9.24) has been plotted in the second graph of figure 22 for $v = 0.4$ and $\omega = 0.3$. The half-period of the kink-antikink train is

$$\psi(\tau,\sigma) = -\psi(\sigma + L,\tau) + n\pi\,, \quad L = 2\sqrt{1-v^2\omega^2}\cdot\mathbb{K}\left(\frac{1}{\eta}\right)\,, \quad n = 0, \pm1, \pm2, \ldots \tag{9.25}$$

Each period L of the kink-antikink train contains exactly two solitons (one kink and one antikink), that is the reason that the parameter region $0 \leq |v| \leq 1 \leq 1/\omega$ has been called "doubled" by Klose and McLoughlin in [131], convention that we also follow here. According to [132], the sG solution (9.24) is a spectrally unstable subluminal ($v \cdot \omega < 1$) librational wave.

9.3 Single Spike: Elementary Region, $0 \leq 1/\omega < |v| \leq 1$

In this case the string extends between the parallels ζ_v and ζ_ω, but it is multiply wound around the 2-sphere and it has a spike instead of being arc-shaped:

$$0 \leq \zeta_v^2 = z_{\text{min}}^2 \leq z^2 \leq z_{\text{max}}^2 = \zeta_\omega^2 \leq R^2. \tag{9.26}$$

The conserved momentum of the finite-size single spike in its elementary region is found to be:

$$p \equiv \Delta\phi = \Delta\varphi = \int_{-r}^{+r} \varphi'\, d\sigma = \frac{2v\omega}{\sqrt{1-1/\omega^2}}\Big[\mathbb{K}(\eta) - \mathbf{\Pi}\left(1 - v^2\omega^2;\eta\right)\Big], \tag{9.27}$$

where

$$\eta \equiv 1 - \frac{z_{\text{min}}^2}{z_{\text{max}}^2} = \frac{v^2\omega^2 - 1}{\omega^2 - 1} \Leftrightarrow \omega = \sqrt{\frac{1-\eta}{v^2 - \eta}}. \tag{9.28}$$

The conserved charges of energy and angular momentum of single spikes in their elementary region are:

$$E = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{+r} \dot{t}\, d\sigma = \frac{r\sqrt{\lambda}}{\pi} = \frac{\sqrt{\lambda}}{\pi}\cdot\frac{v^2\omega^2 - 1}{\sqrt{\omega^2 - 1}}\,\mathbb{K}(\eta)\,, \quad r = \frac{v^2\omega^2 - 1}{\sqrt{\omega^2 - 1}}\,\mathbb{K}(\eta) \tag{9.29}$$

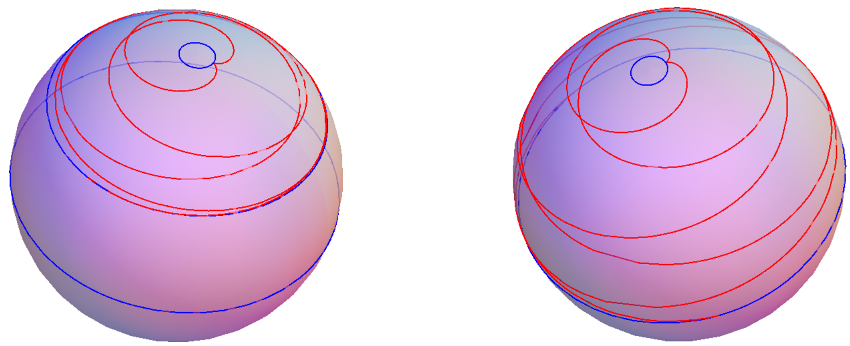


Figure 20: Plots of finite-size single spikes ($v \cdot \omega > 1$) in the elementary (left) and the doubled region (right).

$$J = \frac{\sqrt{\lambda}}{2\pi R^2}\int_{-r}^{+r}\left(R^2 - z^2\right)\dot{\phi}\, d\sigma = \frac{\sqrt{\lambda}}{\pi}\cdot\sqrt{1-\frac{1}{\omega^2}}\left[\mathbb{E}\left(\eta\right) - \frac{1-v^2}{1-1/\omega^2}\,\mathbb{K}\left(\eta\right)\right]. \tag{9.30}$$

The constraint equations (9.4)–(9.5) admit the following solutions:

$$z\left(\tau,\sigma\right) = R\sqrt{1-\frac{1}{\omega^2}}\cdot\text{dn}\left(\frac{\sigma - v\omega\tau}{\eta\sqrt{\omega^2-1}},\eta\right) \tag{9.31}$$

$$\varphi\left(z\right) = \frac{\left(-1\right)^n\, v\omega}{\sqrt{1-1/\omega^2}}\left\{\mathbb{F}\left(\arcsin\left[\frac{1}{\sqrt{\eta}}\sqrt{1-\frac{z^2}{z_{\text{max}}^2}}\right],\eta\right) - \right.$$
$$\left. -\mathbf{\Pi}\left(1 - v^2\omega^2, \arcsin\left[\frac{1}{\sqrt{\eta}}\sqrt{1-\frac{z^2}{z_{\text{max}}^2}}\right]\Bigg|\,\eta\right)\right\} + \left\lfloor\frac{n+1}{2}\right\rfloor\cdot p\,, \quad z_{\text{min}} \leq z \leq z_{\text{max}}. \tag{9.32}$$

By plotting the equation (9.32) upon a sphere with Mathematica, we may obtain drawings of elementary region single spikes—e.g. the one on the left of figure 20. The motion of elementary region single spikes is very reminiscent of the motion of elementary region giant magnons that has been described in §9.1. As we have already mentioned, for $v = 1$ our finite-momentum/winding solution approaches the infinite-size single spike that we've studied in §8.2.

The Pohlmeyer reduction of the solution (9.31)–(9.32) is given by the following wavefunction:

$$\psi\left(\tau,\sigma\right) = \text{am}\left(\frac{\sigma - v\omega\tau}{\eta\,\sqrt{\omega^2-1}},\eta\right). \tag{9.33}$$

Once more, (9.33) is a kink chain/train, very similar to the kink chain/train (9.15) that corresponds to the Pohlmeyer reduction of giant magnons. The chain contains exactly one kink per period, that is why we call this parameter region "elementary". (9.33) has been plotted for $v = 0.9$ and $\omega = 2$ in figure 22. The period of the kink train (9.33) is

$$\psi\left(\tau,\sigma\right) = \psi\left(\sigma + L,\tau\right) + n\pi\,, \quad L = 2\sqrt{\eta\left(v^2\omega^2-1\right)}\cdot\mathbb{K}\left(\eta\right)\,, \quad n = 0,\pm1,\pm2,\ldots \tag{9.34}$$

According to [132], (9.33) corresponds to a spectrally unstable superluminal ($v\cdot\omega > 1$) rotational wave.

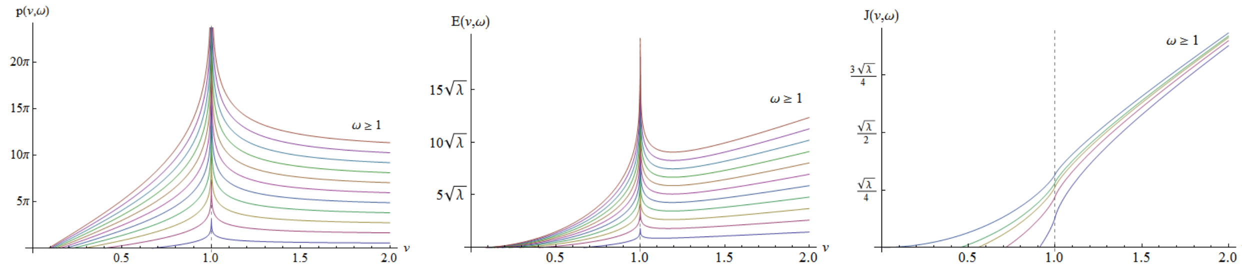


Figure 21: Momentum, energy, spin of finite-size single spikes in terms of the linear velocity v.

9.4 Single Spike: Doubled Region, $0 \leq 1/\omega \leq 1 \leq |v|$

For $0 \leq 1/\omega \leq 1 \leq |v|$, the solution (9.2) describes a spiky open string that is multiply wound around the 2-sphere and extends between the equator and the parallel ζ_ω:

$$\zeta_v^2 = -z_{\text{min}}^2 \leq 0 \leq z^2 \leq z_{\text{max}}^2 = \zeta_\omega^2 \leq R^2. \tag{9.35}$$

The conserved momentum of the single spike in the doubled region is:

$$p \equiv \Delta\phi = \Delta\varphi = \int_{-r}^{+r} \varphi' \, d\sigma = \frac{2v\omega^2}{\sqrt{v^2\omega^2 - 1}} \left[\mathbb{K}\left(\frac{1}{\eta}\right) - \mathbf{\Pi}\left(1 - \omega^2; \frac{1}{\eta}\right)\right], \tag{9.36}$$

where

$$\eta \equiv 1 + \frac{z_{\text{min}}^2}{z_{\text{max}}^2} = \frac{v^2\omega^2 - 1}{\omega^2 - 1} \Leftrightarrow \omega = \sqrt{\frac{1-\eta}{v^2 - \eta}}. \tag{9.37}$$

The conserved energy and angular momentum of doubled region single spikes are:

$$E = \frac{\sqrt{\lambda}}{2\pi} \int_{-r}^{+r} \dot{t} \, d\sigma = \frac{r\sqrt{\lambda}}{\pi} = \frac{\sqrt{\lambda}}{\pi} \cdot \sqrt{v^2\omega^2 - 1} \, \mathbb{K}\left(\frac{1}{\eta}\right), \quad r = \sqrt{v^2\omega^2 - 1} \, \mathbb{K}\left(\frac{1}{\eta}\right) \tag{9.38}$$

$$J = \frac{\sqrt{\lambda}}{2\pi R^2} \int_{-r}^{+r} \left(R^2 - z^2\right) \dot{\phi} \, d\sigma = \frac{\sqrt{\lambda}}{\pi} \cdot \frac{\sqrt{v^2\omega^2 - 1}}{\omega} \, \mathbb{E}\left(\frac{1}{\eta}\right), \tag{9.39}$$

In this case, the Virasoro constraints (9.4)–(9.5) are solved by:

$$z(\tau, \sigma) = R\sqrt{1 - \frac{1}{\omega^2}} \cdot \text{cn}\left(\frac{\sigma - v\omega\tau}{\sqrt{v^2\omega^2 - 1}}, \frac{1}{\eta}\right) \tag{9.40}$$

$$\varphi(z) = \frac{(-1)^n \, v\omega^2}{\sqrt{v^2\omega^2 - 1}} \left\{\mathbb{F}\left(\arccos\left[\frac{z}{z_{\text{max}}}\right], \frac{1}{\eta}\right) - \mathbf{\Pi}\left(1 - \omega^2, \arccos\left[\frac{z}{z_{\text{max}}}\right] \Bigg| \frac{1}{\eta}\right)\right\} + 2\left\lfloor\frac{n+1}{2}\right\rfloor \cdot p,$$

$$-z_{\text{max}} \leq z \leq z_{\text{max}}. \tag{9.41}$$

See the right drawing of figure 20 for a plot of the doubled region single spike. The string starts

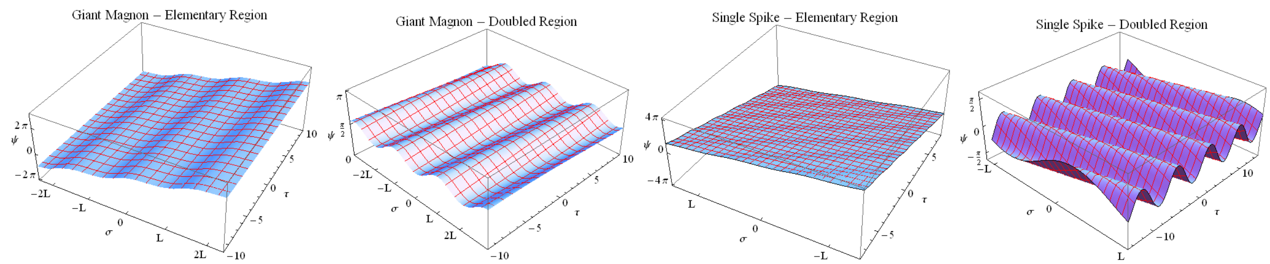


Figure 22: Pohlmeyer reductions of giant magnons and single spikes. The Pohlmeyer reduction of elementary giant magnons (9.15) (first plot) is plotted for $v = 0.1$ and $\omega = 1.01$. The wavefunction (9.24) of doubled giant magnons (second plot) is plotted for $v = 0.4$ and $\omega = 0.3$. The Pohlmeyer reduction of elementary single spikes (9.33) (third plot) has $v = 0.9$ and $\omega = 2$. The sG wavefunction of doubled single spikes (9.42) (fourth plot) has $v = 1.4$ and $\omega = 3$.

unwinding from the north pole and gradually winds around the south pole. Then the motion is reversed and repeated. For $v = 1$ our finite-momentum/winding solution (9.40)–(9.41) approaches the infinite-momentum/winding single spike (8.40)–(8.41) that was studied in §8.2. The plots of the momentum, the energy and the spin of both the elementary ($v \leq 1$) and the doubled region ($v \geq 1$) single spikes as functions of their linear velocities v and for various values of their angular velocities ω, can be found in figure 21.

The Pohlmeyer reduction is again a kink-antikink chain/train, similar to the kink-antikink train of the doubled region giant magnons (9.24):

$$\psi(\tau, \sigma) = \arcsin\left[\frac{1}{\sqrt{\eta}} \operatorname{sn}\left(\frac{\sigma - v\omega\tau}{\sqrt{v^2\omega^2 - 1}}, \frac{1}{\eta}\right)\right]. \tag{9.42}$$

Each period of the train (9.42) contains exactly two solitons, that is why the parameter region $0 \leq 1/\omega \leq 1 \leq |v|$ is called "doubled", in accordance with what has been said before. The quasi-periodic solution of the sG equation (9.42) has been plotted for $v = 1.4$ and $\omega = 3$ in figure 22. The half-period of the kink-antikink chain is:

$$\psi(\tau, \sigma) = -\psi(\sigma + L, \tau) + n\pi\,, \quad L = 2\sqrt{v^2\omega^2 - 1} \cdot \mathbb{K}\left(\frac{1}{\eta}\right), \quad n = 0, \pm 1, \pm 2, \ldots \tag{9.43}$$

(9.42) corresponds to a (spectrally) unstable superluminal ($v \cdot \omega > 1$) librational wave [132].

9.5 Symmetries

Before closing this section and pass to the computation of the dispersion relations of giant magnons and single spikes, let us say a few things about symmetries. The $\tau \leftrightarrow \sigma$ symmetry or "2D duality" (8.45) that was used to transform between giant magnons and single spikes of infinite size, is also applicable at finite-size:

$$\tau \leftrightarrow \sigma\,, \; v \leftrightarrow \frac{1}{\omega}\,, \; \psi \leftrightarrow \left[\frac{\pi}{2} - \psi\right] \quad \Leftrightarrow \quad \text{Giant Magnons} \leftrightarrow \text{Single Spikes}. \tag{9.44}$$

(9.44) maps elementary region giant magnons to elementary region single spikes and doubled region giant magnons to doubled region single spikes. The 2D duality (9.44) acts on the ansätze (with the exception of the temporal coordinate $t = \tau$ which is unaffected), the parameter regions of velocities v and ω, the solutions (z and ϕ) and the Pohlmeyer reductions ψ of GMs and SSs. The conserved charges p, J, E are not correctly transformed by the $\tau \leftrightarrow \sigma$ transform.

There exists a second transformation between the various parameter regions of giant magnons and single spikes (summarized in table 1) that is worth discussing. The substitution $\eta \leftrightarrow -\eta$ can be used to relate the elementary regions of giant magnons and single spikes, firstly by transforming between the solution and the Pohlmeyer reduction z, φ, ψ of elementary GMs and SSs and secondly by flipping the signs of the corresponding conserved charges p, J, E.

The elementary regions of giant magnons and single spikes can also be related to the respective doubled regions by the transformation $\eta \leftrightarrow 1/\eta$. Again, while the solutions z, φ, ψ are taken from the elementary to the doubled region of giant magnons or single spikes, the corresponding conserved charges p, J, E are not transformed correctly under this transformation. On the other hand, it is not known how to relate the doubled regions of giant magnons and single spikes by a similar transformation. None of the transformations that we have discussed is known to affect the dispersion relations of giant magnons and single spikes that we are going to study below.

10 Dispersion Relations of Giant Magnons and Single Spikes

In this section we are going to study the classical dispersion relations of finite-size giant magnons and single spikes in both their elementary and doubled regions. Giant magnons are the AdS/CFT duals of 1-magnon operators that appear in the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM. They are bosonic single spin open strings that rotate in $\mathbb{R} \times \mathrm{S}^2 \subset \mathrm{AdS}_5 \times \mathrm{S}^5$, the classical energy of which is equal to the scaling dimensions of 1-magnon operators of $\mathcal{N} = 4$ SYM theory at strong coupling. The S-matrix of giant magnons (as computed from their Pohlmeyer reduction) agrees with the magnon S-matrix at strong coupling (given by the AFS phase), allowing us to identify them as their AdS/CFT duals.

As we have already explained, magnons and giant magnons cannot be part of the AdS/CFT spectrum. The former have non-vanishing momentum that violates the cyclicity of the trace condition and the latter are open strings which cannot belong to a type IIB string theory. However (giant) magnons are an indispensable tool in the study of the AdS/CFT spectrum because they are the fundamental building blocks out of which all the states in the theory may be built. This is in complete analogy with the sine-Gordon equation, where it is known that all of its solutions can be built out of only a small number of fundamental excitations. As a matter of fact, the solitons of the sine-Gordon equation are the Pohlmeyer duals of giant magnons.

The reason we are forced to study the dispersion relation of giant magnons is that the corresponding gauge theory prescription is valid only asymptotically. Indeed, the asymptotic Bethe ansatz (ABA) that we have seen in §4.3.2 ceases to hold when the loop order becomes equal to the length of the operator under study. For infinite system sizes the ABA stays alive and kicking up to infinite loops, i.e. all the way up to strong coupling where the string description takes over. We will see below that all the evidence that we have from the string theory side agrees with the ABA at infinite size. Beyond the critical loop order at finite size we must calculate wrapping corrections to the magnon anomalous dimensions from the weakly coupled gauge theory side, and classical or quantum (that is α' or curvature) corrections from the string theory side where the gauge theory coupling is strong.

Correcting the spectrum from either side of AdS/CFT moves us towards the other side, i.e. by including gauge theory corrections to the operator scaling dimensions we approach the string theory result and by adding α' (or loop) corrections to the string energies we approach the gauge theory result. In other words tree level gauge theory is equivalent to considering infinite string theory loops, and tree level string theory corresponds to infinite gauge theory loops. The two descriptions ought to meet somewhere in the middle of the AdS/CFT spectrum.

Based on what we have said above, for operators that have large yet not infinite sizes $J \to \infty$, the ABA will only start receiving wrapping corrections after the large but finite critical loop-order $L \sim J \to \infty$. But then the coupling will almost be strong and the string theory description will be just above the tree or classical level. This is precisely the regime that interests us at finite-size. It should be clear that since string theory is just above the tree level and gauge theory well-above the critical loop order, wrapping corrections will generally be present in the string theory spectrum, even at the classical level. These classical and quantum corrections to the ABA (8.3) are known as finite-size corrections and as we will see, they have the form of exponentially suppressed terms.

In our treatment, single spikes are viewed as an analytic continuation of giant magnons. Single spikes are single spin strings in $\mathbb{R} \times \mathrm{S}^2$ that wind many times around the 2-sphere and have a spike in their center. As we have seen in section (9.5), single spikes can be simply obtained from giant magnons by a $\sigma \leftrightarrow \tau$ transform and the transformation $\eta \leftrightarrow -\eta$. The dispersion relations of giant magnons and single spikes can also be related by an appropriate change of variables. Generally speaking, it is to be expected that what we have said above for giant magnons should also be applicable to single spikes as well.

Before we begin our investigation of the giant magnon/single spike dispersion relation, let us briefly restate our arguments about why we think that the explicit calculation of the planar AdS/CFT spectrum is interesting. First and foremost, it seems to us that the scope of AdS/CFT becomes very limited if we do not know how to compute its full spectrum. Secondly, in most cases where we can explicitly calculate the AdS/CFT spectrum we may also thoroughly and unambiguously verify its matching on the two sides of the correspondence. Matching the spectra means that we can also complete the dictionary of AdS/CFT by mapping each and every operator of the planar $\mathcal{N}=4$ SYM to its dual free string state in $AdS_5 \times S^5$. Thirdly, with the full analytic spectrum of AdS/CFT at our disposal, it is very intriguing to search for closed-form expressions at weak and strong coupling.

As in the case of GKP strings, the method for computing the AdS/CFT spectrum in the case of giant magnons and single spikes does not depend on integrability. Besides, we are focused on a regime where integrability-based methods (e.g. the thermodynamic Bethe ansatz (TBA), the Y-system or the quantum spectral curve) have not yet managed to produce any spectacular results. All the computations of the paper [2] that we are going to review below have not been obtained before with any other method, neither are they derivable by means of a computer. Developing a spectral method that does not take into account integrability has the disadvantage of possibly being more complicated than needed since it ignores a very important simplifying assumption (namely that the system is integrable) but it also has the advantage of being applicable whenever integrability itself becomes more involved than needed or it is simply absent (e.g. in non-planar AdS/CFT, QCD, p-branes). We are therefore offered the chance to compute spectra in more generic frameworks. We shall also see that we can make a lot of progress towards finding closed formulas in the AdS/CFT spectrum.

Consider once again the $M=1$ magnon states of $\mathcal{N}=4$ SYM:

$$\mathcal{O}_M = \sum_{m=1}^{J+1} e^{imp} \left|\mathcal{Z}^{m-1}\mathcal{X}\mathcal{Z}^{J-m+1}\right\rangle, \quad p \in \mathbb{R}, \quad \lambda, \mathcal{J} \to \infty. \tag{10.1}$$

The general form of the dispersion relation of finite-size magnon states (10.1) at strong coupling, or equivalently finite-size giant magnons is:

$$\epsilon\left(p\right) = \epsilon_\infty + \underbrace{\sqrt{\lambda}\,\delta\epsilon_{\text{cl}} + \delta\epsilon_{\text{1-loop}} + \frac{1}{\sqrt{\lambda}}\delta\epsilon_{\text{2-loop}} + \ldots,}_{\text{finite-size corrections}} \qquad J, \lambda \to \infty, \tag{10.2}$$

where $\epsilon\left(p\right) \equiv E - J$ and ϵ_∞ is the all-loop 1-magnon formula of BDS (8.3):

$$\lim_{J\to\infty} \epsilon\left(p\right) = \epsilon_\infty \equiv \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} = \frac{\sqrt{\lambda}}{\pi}\sin\frac{p}{2} + 0 + \frac{\pi}{2\sqrt{\lambda}}\csc\frac{p}{2} - \frac{\pi^3}{8\lambda^{3/2}}\csc^3\frac{p}{2} + \ldots, \quad \lambda \to \infty \tag{10.3}$$

At finite-size, ϵ_∞ receives classical and quantum corrections $\delta\epsilon_{\text{cl}}$ and $\delta\epsilon_{n\text{-loop}}$. By generalizing the Hofman-Maldacena ansatz (8.23) to finite-size, Arutyunov, Frolov and Zamaklar (AFZ) [129] derived the first few terms of the classical finite-size expansion $\delta\epsilon_{\text{cl}}$:

$$\delta\epsilon_{\text{cl}} = -\frac{4}{\pi}\sin\frac{p}{2}\left\{\sin^2\frac{p}{2}\,e^{-\mathcal{L}} + \left[8\mathcal{J}^2\cos^2\frac{p}{2} + 4\sin\frac{p}{2}\left(3\cos p + 2\right)\mathcal{J} + \right.\right.$$
$$\left.\left. + \sin^2\frac{p}{2}\left(6\cos p + 7\right)\right]e^{-2\mathcal{L}} + \ldots\right\}, \quad \mathcal{J} \equiv \frac{\pi J}{\sqrt{\lambda}}, \quad \mathcal{L} \equiv 2\mathcal{J}\csc\frac{p}{2} + 2. \tag{10.4}$$

It has been proven by Astolfi, Forini, Grignani and Semenoff in [133] that the spectrum of finite-size giant magnons in the uniform light-cone gauge is completely independent of the gauge parameter. Many more terms in (10.4) can be computed with Mathematica—see appendixes F.3–G.2. The general structure of the classical finite-size corrections $\delta\epsilon_{\rm cl}$ is the following:

$$\delta\epsilon_{\rm cl}= \frac{1}{\pi}\cdot\sum_{n=1}^{\infty}\sum_{m=0}^{2n-2}\mathcal{A}_{nm}(p)\,\mathcal{J}^{2n-m-2}e^{-2n\left(\mathcal{J}\csc\frac{p}{2}+1\right)}=$$
$$=\frac{1}{\pi}\cdot\sum_{m=0}^{\infty}\mathcal{J}^{-m-2}\left\{\sum_{n=\lfloor\frac{m}{2}+1\rfloor}^{\infty}\mathcal{A}_{nm}(p)\,\mathcal{J}^{2n}e^{-2n\left(\mathcal{J}\csc\frac{p}{2}+1\right)}\right\}, \qquad (10.5)$$

where all the coefficients of the negative powers of $\mathcal{J}$ are zero (e.g. $\mathcal{A}_{11}=\mathcal{A}_{12}=\ldots=0$, etc.). The AFZ formula (10.4) contains the terms $\mathcal{A}_{10}$, $\mathcal{A}_{20}$, $\mathcal{A}_{21}$, $\mathcal{A}_{22}$ of (10.5). Klose and McLoughlin [131] have obtained the terms $\mathcal{A}_{10}$–$\mathcal{A}_{60}$:

$$\delta\epsilon_{\rm cl}=-\frac{4}{\pi}\sin^3\frac{p}{2}\,e^{-\mathcal{L}}\Bigg[1+2\,\mathcal{L}^2\cos^2\frac{p}{2}\,e^{-\mathcal{L}}+8\,\mathcal{L}^4\cos^4\frac{p}{2}\,e^{-2\mathcal{L}}+\frac{128}{3}\mathcal{L}^6\cos^6\frac{p}{2}\,e^{-3\mathcal{L}}+$$
$$+\frac{800}{3}\mathcal{L}^8\cos^8\frac{p}{2}\,e^{-4\mathcal{L}}+\frac{9216}{5}\mathcal{L}^{10}\cos^{10}\frac{p}{2}\,e^{-5\mathcal{L}}+\ldots\Bigg], \qquad (10.6)$$

The leading term $\mathcal{A}_{10}$ of (10.4)–(10.5) has also been obtained by the algebraic curve method in [134] and by the Lüscher-Klassen-Melzer (LKM) formulae [135] at strong coupling in [136, 137, 138].

In [2] all the coefficients $\mathcal{A}_{n0}$, $\mathcal{A}_{n1}$, $\mathcal{A}_{n2}$ of (10.5) have been computed. In §10.1–§10.4 we are going to revisit this paper. Let us first summarize the result. To leading order, the classical part $\delta\epsilon_{\rm cl}$ of the dispersion relation of giant magnons and the anomalous scaling dimensions of the operators (10.1) at strong coupling, in both their elementary and doubled regions can be expressed in terms of Lambert's W-function as follows:

$$\delta\epsilon_{\rm cl}=\frac{1}{4\pi\mathcal{J}^2}\tan^2\frac{p}{2}\sin^3\frac{p}{2}\left[W+\frac{W^2}{2}\right]-\frac{1}{16\pi\mathcal{J}^3}\tan^4\frac{p}{2}\sin^2\frac{p}{2}\left[(3\cos p+2)\,W^2+\frac{1}{6}(5\cos p+11)\,W^3\right]$$
$$-\frac{1}{512\pi\mathcal{J}^4}\tan^6\frac{p}{2}\sin\frac{p}{2}\Bigg\{(7\cos p-3)^2\,\frac{W^2}{1+W}-\frac{1}{2}(25\cos 2p-188\cos p-13)\,W^2-\frac{1}{2}(47\cos 2p$$
$$+196\cos p-19)W^3-\frac{1}{3}(13\cos 2p+90\cos p+137)\,W^4\Bigg\}+\ldots, \qquad (10.7)$$

where the argument of Lambert's W-function is $W\left(\pm 16\mathcal{J}^2\cot^2(p/2)\,e^{-\mathcal{L}}\right)$, in the principal branch and $\mathcal{L}\equiv 2\mathcal{J}\csc p/2+2$. The minus sign inside the argument of W refers to the elementary region of giant magnons ($0\le|v|<1/\omega\le 1$) and the plus sign is for the doubled region ($0\le|v|\le 1\le 1/\omega$). The leading, subleading and next-to-subleading coefficients of (10.5) ($\mathcal{A}_{n0}$, $\mathcal{A}_{n1}$, $\mathcal{A}_{n2}$) can be found if we expand Lambert's W-function in (10.7) around $\mathcal{J}\to\infty$, by using Taylor's expansion (I.3). The result is:

- leading terms: $\displaystyle\sum_{n=1}^{\infty}\mathcal{A}_{n0}(p)\;\mathcal{J}^{2n-2}\,e^{-n\,\mathcal{L}}=\frac{1}{4\mathcal{J}^2}\tan^2\frac{p}{2}\sin^3\frac{p}{2}\left[W+\frac{W^2}{2}\right],$

- next-to-leading terms: $\sum_{n=2}^{\infty} \mathcal{A}_{n1}(p)\, \mathcal{J}^{2n-3}\, e^{-n\,\mathcal{L}} = -\frac{1}{16\mathcal{J}^3} \tan^4\frac{p}{2} \sin^2\frac{p}{2}\left[(3\cos p+2)\,W^2 + \frac{1}{6}(5\cos p+11)\,W^3\right],$

- next-to-next-to-leading terms: $\sum_{n=2}^{\infty} \mathcal{A}_{n2}(p)\, \mathcal{J}^{2n-4}\, e^{-n\,\mathcal{L}} = -\frac{1}{512\mathcal{J}^4} \tan^6\frac{p}{2} \sin\frac{p}{2}\Big\{ (7\cos p-3)^2\,\frac{W^2}{1+W} -$

$$-\frac{1}{2}(25\cos 2p - 188\cos p - 13)\,W^2 - \frac{1}{2}(47\cos 2p + 196\cos p - 19)\,W^3 - \frac{1}{3}(13\cos 2p + 90\cos p + 137)\,W^4\Big\},$$

The coefficients $\mathcal{A}_{n0}$, $\mathcal{A}_{n1}$, $\mathcal{A}_{n2}$ agree completely with the AFZ results (10.4), the Klose-McLoughlin formula (10.6), as well as the formulae (G.12)–(G.13) that were computed with Mathematica.

From what we have said so far it should be clear that the ABA formula of BDS (10.3) is confirmed at strong coupling by the classical (tree) level formula of Hofman and Maldacena (8.6). By perturbing the IIB string sigma model in $\mathbb{R}\times\mathrm{S}^2$, it has been shown in [139] that the infinite-volume one-loop shift vanishes:

$$\delta\epsilon_{\text{1-loop}} = 0, \quad J=0,\ \lambda\to\infty, \tag{10.8}$$

which also agrees with the BDS formula (10.3) at one-loop order. At finite volume, the calculation of α' corrections can be accomplished either via the algebraic curve method [138] or by computing the Lüscher F and μ-terms [137]. The general form of the one-loop shift at finite volume is:

$$\delta\epsilon_{\text{1-loop}} = a_{1,0}\, e^{-2D} + \sum_{\substack{n=0\\m=1}}^{\infty} a_{n,m} e^{-2nD-m\mathcal{L}}, \quad D \equiv \mathcal{J} + \sin\frac{p}{2}. \tag{10.9}$$

The calculation of the terms $a_{n,0}$ and $a_{1,m}$ of (10.9) proceeds along the lines of the papers [138, 140]. The leading term $a_{1,0}$ is given by:

$$a_{1,0} = \frac{1}{\sqrt{D}}\frac{8\sin^2 p/4}{(\sin p/2-1)}\left[1 - \frac{7+4\sin p - 4\cos p + \sin p/2}{16\,(\sin p/2-1)}\cdot\frac{1}{D} + O\left(\frac{1}{D^2}\right)\right]. \tag{10.10}$$

The leading finite-size term in the dispersion relation of single spikes has been computed in [141]:

$$E - T\Delta\varphi = \frac{\sqrt{\lambda}}{\pi}\left[\frac{q}{2} + 4\sin^2\frac{q}{2}\tan\frac{q}{2}\cdot e^{-(q+\Delta\varphi)\cdot\cot\frac{q}{2}}\right], \quad q \equiv 2\arcsin\left(\frac{\pi J}{\sqrt{\lambda}}\right), \quad \Delta\phi, \lambda\to\infty. \tag{10.11}$$

In appendix G.2 many more terms of (10.11) have been computed with Mathematica. The code can be found in appendix F.3. The structure of the classical finite-size corrections of the single spike dispersion relation at finite volume is very similar to the one for giant magnons (10.5), however the roles of $\Delta\phi = p$ and $\mathcal{J}$ have been interchanged:

$$\left.\mathcal{E} - \frac{p}{2}\right|_{\text{clas}} = \frac{q}{2} + \sum_{n=1}^{\infty}\sum_{m=0}^{2n-2} \hat{\mathcal{A}}_{nm}(q)\, p^{2n-m-2} e^{-n(q+p)\cot\frac{q}{2}}, \tag{10.12}$$

where again all the negative powers of the momentum p are absent from (10.12) (e.g. $\hat{\mathcal{A}}_{11} = \hat{\mathcal{A}}_{12} = \ldots = 0$, etc.). All the coefficients $\hat{\mathcal{A}}_{n0}$, $\hat{\mathcal{A}}_{n1}$, $\hat{\mathcal{A}}_{n2}$ of (10.12) have been computed in the reference [2].

We will review the paper [2] in §10.1–§10.4 below. For the moment let us first state the results for single spikes. The leading, subleading and next-to-subleading coefficients ($\hat{\mathcal{A}}_{n0}$, $\hat{\mathcal{A}}_{n1}$, $\hat{\mathcal{A}}_{n2}$) of (10.12), in the classical part of the dispersion relation of elementary single spikes ($0 \leq 1/\omega < |v| \leq 1$), can be expressed in terms of Lambert's W-function as follows:

- leading terms: $\sum_{n=1}^{\infty} \hat{\mathcal{A}}_{n0}(q)\, p^{2n-2}\, e^{-n\,\mathcal{R}} = -\frac{1}{p^2}\sin^4\frac{q}{2}\,\tan\frac{q}{2}\left[W + \frac{W^2}{2}\right].$

- next-to-leading terms: $\sum_{n=2}^{\infty} \hat{\mathcal{A}}_{n1}(q)\, p^{2n-3}\, e^{-n\,\mathcal{R}} = \frac{1}{p^3}\sin^6\frac{q}{2}\left\{\left[\left(\sec^2\frac{q}{2} + 2q\csc q - \frac{1}{2}\right)\right]W^2 + \left[5 + 3\sec^2\frac{q}{2}\right]\frac{W^3}{6}\right\}.$

- next-to-next-to-leading terms: $\sum_{n=2}^{\infty} \hat{\mathcal{A}}_{n2}(q)\, p^{2n-4}\, e^{-n\,\mathcal{R}} = \frac{1}{64\,p^4}\sin^4\frac{q}{2}\tan^3\frac{q}{2}\Bigg\{2\left(5 + 7\cos q - 8q\cot\frac{q}{2}\right)^2\frac{W^2}{1+W} - \Big(96\cdot$

$\cdot q^2\cot^2\frac{q}{2} - 52q\csc^4\frac{q}{2}\sin^3 q + 45\cos 2q + 148\cos q + 79\Big)W^2 - \Big(16q\,(11 + 5\cos q)\cot\frac{q}{2} - 37\cos 2q - 172\cos q - 79\Big)W^3$

$- (11\cos 2q + 64\cos q + 85)\,W^4\Bigg\},$

with the argument of Lambert's function equal to $W\left(4p^2\csc^2(q/2)\,e^{-\mathcal{R}}\right)$, in the principal branch W_0, $\mathcal{R} \equiv (p+q)\cot q/2$ and $\sin q/2 \equiv \mathcal{J}$. In the doubled region of single spikes ($0 \leq 1/\omega \leq 1 \leq |v|$) the argument of Lambert's function becomes $W\left(-4p^2\csc^2(q/2)\,e^{-\mathcal{R}}\right)$, and the corresponding coefficients of the leading and the subleading series $\hat{\mathcal{A}}_{n0}$, $\hat{\mathcal{A}}_{n1}$ are the same as in the elementary region. The next-to-next-to-leading series $\hat{\mathcal{A}}_{n2}$ in the doubled region is given by:

- next-to-next-to-leading terms: $\sum_{n=2}^{\infty} \hat{\mathcal{A}}_{n2}(q)\, p^{2n-4}\, e^{-n\,\mathcal{R}} = \frac{1}{64\,p^4}\sin^4\frac{q}{2}\tan^3\frac{q}{2}\Bigg\{2\left(5 + 7\cos q - 8q\cot\frac{q}{2}\right)^2\frac{W^2}{1+W} - \Big(96\cdot$

$\cdot q^2\cot^2\frac{q}{2} - 52q\csc^4\frac{q}{2}\sin^3 q + 45\cos 2q + 276\cos q - 256\csc^2\frac{q}{2} + 463\Big)W^2 - \Big(16q(11 + 5\cos q)\cot\frac{q}{2} - 37\cos 2q -$

$-172\cos q - 79\Big)W^3 - (11\cos 2q + 64\cos q + 85)\,W^4\Bigg\}.$

The terms in red are absent from the corresponding formula in the elementary region. The coefficients $\hat{\mathcal{A}}_{n0}$, $\hat{\mathcal{A}}_{n1}$, $\hat{\mathcal{A}}_{n2}$ can be computed with the Taylor expansion (I.3) of Lambert's W-function. They are in complete agreement with the Ahn-Bozhilov formula (10.11) and the results (G.14)–(G.15) of appendix G.2 that were computed with Mathematica.

Let us also overview the method by which the classical coefficients in the elementary and the doubled regions of giant magnons and single spikes are obtained. In contrast to GKP strings where our starting point was the 2×2 system of equations (7.16)–(7.17), in the case of giant magnons/single spikes we start with a 3×3 system:

$$\mathcal{E} = d\,(a, x)\ln x + h\,(a, x) \tag{10.13}$$

$$\mathcal{J} = c\,(a, x)\ln x + b\,(a, x) \tag{10.14}$$

$$p = f\,(a, x)\ln x + g\,(a, x)\,, \tag{10.15}$$

where for elementary giant magnons it's $x = 1 - \eta$, η is defined in equation (9.10) and $v \equiv \cos a$. $d\,(a,x)$, $h\,(a,x)$, $c\,(a,x)$, $b\,(a,x)$, $f\,(a,x)$, $g\,(a,x)$ are the coefficients of the series (9.11), (9.12) and (9.9), when these are expressed in terms of the variables x and a. The system (10.13)–(10.15) can be solved as follows. First the logarithm is eliminated from the equations (10.14)–(10.15), leading to an expression $p = p(\mathcal{J}, a, x)$ of the linear momentum in terms of the conserved angular momentum $\mathcal{J}$ and the parameters a and x. Then $p(\mathcal{J}, a, x)$ is expanded in a double series w.r.t. the variables a and x and it is inverted for $a = a(x, p, \mathcal{J})$. $a(x, p, \mathcal{J})$ is plugged into the equations (10.13)–(10.14) and a system like (7.16)–(7.17) is obtained:

$$\mathcal{E} = d(x, p, \mathcal{J}) \ln x + h(x, p, \mathcal{J}) \tag{10.16}$$

$$\mathcal{J} = c(x, p, \mathcal{J}) \ln x + b(x, p, \mathcal{J}). \tag{10.17}$$

The method of §7 may now be applied in order to eliminate the variable x from the system (10.16)–(10.17) and derive the giant magnon dispersion relation $\gamma \equiv \mathcal{E} - \mathcal{J} = \gamma(p, \mathcal{J})$ in terms of the momenta p and $\mathcal{J}$.

The algorithm is exactly the same for giant magnons in the doubled region, except that $\widetilde{x} = 1 - 1/\eta$ and η is defined from equation (9.19), while $d(a, \widetilde{x})$, $h(a, \widetilde{x})$, $c(a, \widetilde{x})$, $b(a, \widetilde{x})$, $f(a, \widetilde{x})$, $g(a, \widetilde{x})$ are taken from the series (9.20), (9.21) and (9.18).

To treat large-momentum single spikes we must set $a \equiv \arccos 1/\omega$ and eliminate the logarithm from the equations (10.14)–(10.15). This leads to an expression $\mathcal{J} = \mathcal{J}(a, x, p)$ for the angular momentum which is then inverted for $a = a(x, p, \mathcal{J})$ and inserted into the equations (10.13), (10.15). The resulting 2×2 system

$$\mathcal{E} = d(x, p, \mathcal{J}) \ln x + h(x, p, \mathcal{J}) \tag{10.18}$$

$$p = f(x, p, \mathcal{J}) \ln x + g(x, p, \mathcal{J}), \tag{10.19}$$

can be solved like the corresponding system for the GKP strings (7.16)–(7.17) in §7. For single spikes in the elementary region, $x = 1 - \eta$, where η is defined in equation (9.28) and $1/\omega \equiv \cos a$. The coefficients $d(a, x)$, $h(a, x)$, $c(a, x)$, $b(a, x)$, $f(a, x)$, $g(a, x)$ are defined from the series (9.29), (9.30) and (9.27). Single spikes in the doubled region have $\widetilde{x} = 1 - 1/\eta$ and η is defined in equation (9.37). The coefficients $d(a, \widetilde{x})$, $h(a, \widetilde{x})$, $c(a, \widetilde{x})$, $b(a, \widetilde{x})$, $f(a, \widetilde{x})$, $g(a, \widetilde{x})$ are defined from the series (9.38), (9.39) and (9.36).

This section is organized as follows. In §10.1 we are going to implement the above algorithm in the case of elementary giant magnons ($0 \le |v| < 1/\omega \le 1$) and in §10.2 it shall be applied to the doubled giant magnons ($0 \le |v| \le 1 \le 1/\omega$). In §10.3–§10.4, the elementary ($0 \le 1/\omega < |v| \le 1$) and the doubled ($0 \le 1/\omega \le 1 \le |v|$) single spikes will be studied.

10.1 Giant Magnon, Elementary Region: $0 \le |v| < 1/\omega \le 1$

Let us begin with the elementary giant magnons for which,

$$0 \le |v| \le 1/\omega \le 1. \tag{10.20}$$

As we have said, the elementary region giant magnons are arc-shaped open strings in $\mathbb{R} \times \mathrm{S}^2$ that extend between the parallels ζ_ω and ζ_v:

$$0 \le R^2 \left[1 - \frac{1}{\omega^2}\right] \equiv \zeta_\omega = z^2_{\text{min}} \le z^2 \le z^2_{\text{max}} = \zeta_v \equiv R^2 \left(1 - v^2\right) \le R^2. \tag{10.21}$$

If we define the variable x as

$$x \equiv 1 - \eta = \frac{z^2_{\text{min}}}{z^2_{\text{max}}} = \frac{\omega^2 - 1}{\omega^2 (1 - v^2)}, \tag{10.22}$$

the following system of equations is obtained:

$$\mathcal{E} \equiv \frac{\pi\, E}{\sqrt{\lambda}} = \frac{\sqrt{1-v^2}}{\sqrt{1-x\,(1-v^2)}}\,(1-x)\cdot\mathbb{K}\,(1-x) \tag{10.23}$$

$$\mathcal{J} \equiv \frac{\pi\, J}{\sqrt{\lambda}} = \sqrt{1-v^2}\,\big(\mathbb{K}\,(1-x)-\mathbb{E}\,(1-x)\big) \tag{10.24}$$

$$\gamma = \mathcal{E}-\mathcal{J} = \sqrt{1-v^2}\,\left\{\mathbb{E}\,(1-x)-\left(1-\frac{1-x}{\sqrt{1-x\,(1-v^2)}}\right)\mathbb{K}\,(1-x)\right\} \tag{10.25}$$

$$p = \frac{1}{v}\frac{1}{\sqrt{1-x\,(1-v^2)}\cdot\mathbb{K}\,(x)}\left\{\pi v\sqrt{1-x\,(1-v^2)}\cdot\mathbb{F}\left(\arcsin\sqrt{1-v^2},x\right)+2\,(1-x)\,\sqrt{1-v^2}\cdot\right.$$

$$\left.\cdot\left[\mathbb{K}\,(x)-\mathbf{\Pi}\left(\frac{x\,v^2}{1-x\,(1-v^2)};x\right)\right]\cdot\mathbb{K}\,(1-x)\right\}. \tag{10.26}$$

(10.26) is derived from the momentum of giant magnons (9.9) and the addition formula (H.14) of the complete elliptic integrals of the third kind. Let us now see how the algorithm that we described in the previous section can be used in order to obtain the dispersion relation of elementary giant magnons $\mathcal{E} = \mathcal{E}\,(p,\mathcal{J})$, for large but finite angular momentum $J\to\infty$ and $x\to 0^+$.

10.1.1 Inverse Momentum

We first have to express the velocity v of giant magnons in terms of the momenta p and $\mathcal{J}$. The formulas (10.23)–(10.26) have a logarithmic singularity at $x\to 0^+$ which they inherit from the following two elliptic functions:

$$\mathbb{K}\,(1-x) = \sum_{n=0}^{\infty} x^n\,(d_n\ln x + h_n) \tag{10.27}$$

$$\mathbb{K}\,(1-x)-\mathbb{E}\,(1-x) = \sum_{n=0}^{\infty} x^n\,(c_n\ln x + b_n)\,. \tag{10.28}$$

The coefficients of the series (10.27) and (10.28) are the following:

$$d_n = -\frac{1}{2}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2, \qquad h_n = -4\,d_n\cdot(\ln 2+H_n-H_{2n})$$

$$c_n = -\frac{d_n}{2n-1}\,, \qquad b_n = -4\,c_n\cdot\left[\ln 2+H_n-H_{2n}+\frac{1}{2\,(2n-1)}\right], \tag{10.29}$$

where $n = 0\,,1\,,2\,,\ldots$ Eliminating the logarithms from the equations (10.24), (10.26), we are led to

$$p = \frac{\pi\cdot\mathbb{F}\,(a,x)}{\mathbb{K}\,(x)}+\frac{2\,(1-x)\tan a}{\mathbb{K}\,(x)\,\sqrt{1-x\sin^2 a}}\cdot\left[\mathbb{K}\,(x)-\mathbf{\Pi}\left(\frac{x\,\cos^2 a}{1-x\sin^2 a};x\right)\right]\cdot\left\{\sum_{n=0}^{\infty}h_n x^n+\frac{\sum_{n=0}^{\infty}d_n x^n}{\sum_{n=0}^{\infty}c_n x^n}\cdot\right.$$

$$\cdot \left(\mathcal{J} \csc a - \sum_{n=0}^{\infty} b_n x^n \right) \Bigg\}, \quad (10.30)$$

where $v = \cos a$ ($\arccos 1/\omega \leq a \leq \pi/2$). The equation (10.30), that gives $p = p(\mathcal{J}, a, x)$, can be expanded in a double series around $x = 0$ and $a = p/2$ and then it can be inverted for the variable a with Mathematica. See appendix G.2, equation (G.10). Then we may plug $a(x, p, \mathcal{J})$ into the equations (10.24)–(10.25) and apply the method that we used in the case of GKP strings in order to invert the equation (10.24) by computing the inverse spin function $x = x(p, \mathcal{J})$. If we insert the $x(p, \mathcal{J})$ that we found into the anomalous dimensions formula (10.25), we will obtain the dispersion relation of elementary giant magnons in terms of the W-function.

10.1.2 Inverse Spin Function

As we have said, the velocity $v = \cos a(x, p, \mathcal{J})$ that we have found in the previous subsection must be inserted into equation (10.24) that gives the spin of the GM and the resulting angular momentum series $\mathcal{J} = \mathcal{J}(x, p)$ must be inverted for the inverse spin function $x = x(p, \mathcal{J})$. Then by plugging $x(p, \mathcal{J})$ into $\gamma = \gamma(x, p)$ that is given by equation (10.25), we find $\gamma = \gamma(p, \mathcal{J})$. Let us first solve the equation (10.24) for $\ln x$:

$$\mathcal{J} = \sin a(x, p, \mathcal{J}) \cdot \sum_{n=0}^{\infty} x^n \left(c_n \ln x + b_n \right) \Rightarrow \ln x = \left[\frac{\mathcal{J} \csc a - b_0}{c_0} - \sum_{n=1}^{\infty} \frac{b_n}{c_0} x^n \right] \cdot \sum_{n=0}^{\infty} \left(- \sum_{k=1}^{\infty} \frac{c_k}{c_0} x^k \right)^n . \quad (10.31)$$

(10.31) may be written as a series of the following form (cf. (7.21)–(7.72)):

$$x = x_0 \cdot \exp\left[\sum_{n=1}^{\infty} \mathrm{a}_n \, x^n \right] = x_0 \cdot \exp\left(\mathrm{a}_1 \, x + \mathrm{a}_2 \, x^2 + \mathrm{a}_3 \, x^3 + \ldots \right), \quad (10.32)$$

where the coefficients $\mathrm{a}_n = \mathrm{a}_n(p, \mathcal{J})$ can be computed from (10.31). We have also defined:

$$x_0 \equiv \exp\left[\frac{\mathcal{J} \csc \frac{p}{2} - b_0}{c_0} \right] = 16 \, e^{-2\mathcal{J} \csc \frac{p}{2} - 2} \quad (10.33)$$

which solves (10.31) to lowest order in the variable x. We can use the Lagrange-Bürmann formula (7.22) to invert the series (10.32). We find:

$$x = \sum_{n=1}^{\infty} x_0^n \cdot \sum_{k, j_i = 0}^{n-1} \frac{n^k}{n!} \binom{n-1}{j_1 \, , \, j_2 \, , \, \ldots \, , \, j_{n-1}} \mathrm{a}_1^{j_1} \mathrm{a}_2^{j_2} \ldots \mathrm{a}_{n-1}^{j_{n-1}}, \quad (10.34)$$

where

$$j_1 + j_2 + \ldots + j_{n-1} = k \quad \& \quad j_1 + 2\, j_2 + \ldots + (n-1)\, j_{n-1} = n - 1.$$

By expanding (10.31) we may prove that the a_n's have the following form:

$$\mathrm{a}_n = \sum_{m=0}^{n+1} \mathrm{a}_{nm} \mathcal{J}^m, \quad (10.35)$$

where the a_{nm} are known functions of the momentum/angular extent p. If we insert (10.35) into (10.34) and use the identities

$$\left. \begin{array}{r} j_1 + j_2 + \ldots + j_{n-1} = k \\ j_1 + 2\, j_2 + \ldots + (n-1)\, j_{n-1} = n - 1 \end{array} \right\} \Rightarrow k + j_2 + \ldots + (n-2)\, j_{n-1} = n - 1, \quad (10.36)$$

we may also show that the inverse spin function series $x = x(p, \mathcal{J})$ has the following general form:

$$x = \sum_{n=1}^{\infty} x_0^n \cdot \sum_{m=0}^{2n-2} \widetilde{a}_{nm} \mathcal{J}^m, \tag{10.37}$$

where the $\widetilde{a}_{nm}$ depend on the momentum p. The $\widetilde{a}_{nm}$'s are computed in terms of the a_{nm}'s in equation (10.35), by inserting (10.35) into (10.34). The result should coincide with equation (G.11), where x has been computed with Mathematica. It can be proven that the leading in $\mathcal{J}$ contributions to x (i.e. the terms $\widetilde{a}_{n,2n-2}$) are determined by a_{12}, the next-to-leading in $\mathcal{J}$ contributions to x (terms $\widetilde{a}_{n,2n-3}$) are determined by a_1 and a_{23}, and so on up to the term $\widetilde{a}_{nn}$. In other words, all the coefficients of $x(\mathcal{J})$ up to $x_0^n \mathcal{J}^{2n-2-\mathfrak{m}}$ $(0 \leq \mathfrak{m} \leq n-2)$ are determined by $\mathrm{a}_1, \ldots, \mathrm{a}_{\mathfrak{m}}$, and $\mathrm{a}_{\mathfrak{m}+1,\mathfrak{m}+2}$. The next-to-leading terms $\widetilde{a}_{n0}, \ldots, \widetilde{a}_{n,n-1}$ (multiplying $x_0^n \mathcal{J}^m$ for $0 \leq m \leq n-1$) are determined from the coefficients $\mathrm{a}_1, \ldots, \mathrm{a}_{n-2}$ and $\mathrm{a}_{n-1,m}$. To prove these statements, the formula (10.35) must be inserted into the equation (10.34). We find:

$$x = \sum_{n=1}^{\infty} \frac{x_0^n}{n!} \cdot \Bigg\{ n^{n-1} \mathrm{a}_1^{n-1} + (n-1)(n-2)\, n^{n-2} \mathrm{a}_1^{n-3} \mathrm{a}_2 + (n-1)(n-2)(n-3)\, n^{n-3} \Big[\mathrm{a}_1^{n-4} \mathrm{a}_3 + \\ + \frac{1}{2}(n-4)\, \mathrm{a}_1^{n-5} \mathrm{a}_2^2 \Big] + \ldots \Bigg\}. \tag{10.38}$$

In order to evaluate the inverse spin function x, we must calculate the coefficients a_1, a_2, a_3 from the equation (10.31) and insert them into the equation (10.38). Here we will keep only the leading, subleading and next-to-subleading terms and ignore all the higher-order contributions. Then we must transform the resulting series into Lambert's functions by using the formulae (I.8)–(I.13) of appendix I. The final result for the inverse spin function $x = x(p, \mathcal{J})$ is:

$$x = -\frac{1}{\mathcal{J}^2} \tan^2 \frac{p}{2} \cdot W + \frac{1}{8\mathcal{J}^3} \tan^3 \frac{p}{2} \sec \frac{p}{2} \cdot \left[\frac{7\cos p - 3}{1+W} - (\cos p - 5) \right] \cdot W^2 - \frac{1}{64\mathcal{J}^4} \tan^4 \frac{p}{2} \sec^2 \frac{p}{2} \cdot \\ \cdot \Bigg\{ \frac{1}{2}(7\cos p - 3)^2 \frac{W}{(1+W)^3} - \frac{1}{6}(241 \cos 2p - 924 \cos p + 731) \frac{W}{1+W} - \frac{1}{3}(335 \cos p - 463) \cdot \\ \sin^2 \frac{p}{2} \cdot W - \frac{1}{12}(41 \cos 2p - 1284 \cos p + 667) W^2 - \frac{1}{3}(\cos 2p + 36 \cos p - 85) W^3 \Bigg\} + \ldots \tag{10.39}$$

The arguments of Lambert's W-functions in (10.39) are $W\left(-16\mathcal{J}^2 \cot^2(p/2)\, e^{-2\mathcal{J} \csc p/2 - 2}\right)$ in the principal branch W_0. If we use the Taylor expansion of the W-function in the W_0 branch (I.3) to expand the formula (10.39) for $\mathcal{J} \to \infty$, we recover the leading, subleading and next-to-subleading terms of the inverse spin function. These agree with the inverse spin function (G.11) that has been computed in appendix G.2 with Mathematica. Let us also define:

$$x_{(\mathrm{L})} = -\frac{1}{\mathcal{J}^2} \tan^2 \frac{p}{2} \cdot W \tag{10.40}$$

$$x_{(\mathrm{NL})} = \frac{1}{8\mathcal{J}^3} \tan^3 \frac{p}{2} \sec \frac{p}{2} \cdot \left[\frac{7\cos p - 3}{1+W} - (\cos p - 5) \right] \cdot W^2 \tag{10.41}$$

$$x_{(\mathrm{NNL})} = -\frac{1}{64\mathcal{J}^4} \tan^4 \frac{p}{2} \sec^2 \frac{p}{2} \cdot \Bigg\{ \frac{1}{2}(7\cos p - 3)^2 \frac{W}{(1+W)^3} - \frac{1}{6}(241 \cos 2p - 924 \cos p + 731) \frac{W}{1+W} -$$

$$-\frac{1}{3}(335\cos p-463)\sin^2\frac{p}{2}\cdot W-\frac{1}{12}(41\cos 2p-1284\cos p+667)\,W^2-$$

$$-\frac{1}{3}(\cos 2p+36\cos p-85)\,W^3\Bigg\}. \tag{10.42}$$

10.1.3 Dispersion Relation

To compute the classical part of the dispersion relation of finite-size giant magnons, we must insert the inverse spin function $x=x(p,\mathcal{J})$ in (10.39) (derived in the previous section) into $\gamma=\mathcal{E}-\mathcal{J}$, in equation (10.25). We first expand (10.25) around $x\to 0^+$ by using the series (10.27)–(10.28):

$$\mathcal{E}-\mathcal{J}=\sum_{n=0}^{\infty}x^n\left(f_n\,\ln x+g_n\right), \tag{10.43}$$

where the coefficients f_n and g_n are functions of x, p and $\mathcal{J}$. They are defined as:

$$f_n\equiv\sin a\left[\frac{1-x}{\sqrt{1-x\sin^2 a}}\,d_n-c_n\right]\;,\quad g_n\equiv\sin a\left[\frac{1-x}{\sqrt{1-x\sin^2 a}}\,h_n-b_n\right]\;,\quad n=0,1,2,\ldots \tag{10.44}$$

Next we substitute the computed value of $\sin a\,(x,p,\mathcal{J})$ (as given by equation (G.10) in appendix G.2) into (10.44), and replace $\ln x/x_0$ by its equal in equation (10.32). The dispersion relation (10.43) is then written as follows:

$$\mathcal{E}-\mathcal{J}=\sum_{n=0}^{\infty}x^n\left(f_n\,\ln x+g_n\right)=\sum_{n=0}^{\infty}x^n\left[A_n+f_n\,\ln\frac{x}{x_0}\right]=A_0+\sum_{n=1}^{\infty}x^n\left[A_n+\sum_{k=1}^{n}f_{n-k}\cdot \mathrm{a}_k\right], \tag{10.45}$$

where now f_n and g_n are functions of only the momentum p and the spin $\mathcal{J}$. The A_n's are given by

$$A_n\equiv g_n+f_n\ln x_0=g_n+2f_n\left(2\ln 2-\mathcal{J}\csc\frac{p}{2}-1\right). \tag{10.46}$$

Generally, A_n and f_n have following form:

$$A_n=\sum_{m=0}^{n}A_{nm}\mathcal{J}^m\quad\&\quad f_n=\sum_{m=0}^{n-1}f_{nm}\mathcal{J}^m, \tag{10.47}$$

where A_{nm} and f_{nm} are known functions of the momentum p. We can now write down all the terms of the expansion (10.45) that contribute to the anomalous dimensions up to next-to-next-to-leading (NNL) order. In (10.45) we make the replacements (10.35), (10.47) and $x=x_{(\mathrm{L})}+x_{(\mathrm{NL})}+x_{(\mathrm{NNL})}+\ldots$, getting:

$$\begin{aligned}\mathcal{E}-\mathcal{J}=&A_0+\left\{A_1x_{(\mathrm{L})}+(A_{22}+f_1\mathrm{a}_{12})\,\mathcal{J}^2x_{(\mathrm{L})}^2\right\}+\left\{A_1x_{(\mathrm{NL})}+(A_{21}+f_1\mathrm{a}_{11})\,\mathcal{J}x_{(\mathrm{L})}^2+\right.\\&+2\,(A_{22}+f_1\mathrm{a}_{12})\,\mathcal{J}^2x_{(\mathrm{L})}x_{(\mathrm{NL})}+\left.(A_{33}+f_1\mathrm{a}_{23}+f_{21}\mathrm{a}_{12})\,\mathcal{J}^3x_{(\mathrm{L})}^3\right\}+\left\{A_1x_{(\mathrm{NNL})}+\right.\\&+(A_{20}+f_1\mathrm{a}_{10})\,x_{(\mathrm{L})}^2+2\,(A_{21}+f_1\mathrm{a}_{11})\,\mathcal{J}x_{(\mathrm{L})}x_{(\mathrm{NL})}+(A_{22}+f_1\mathrm{a}_{12})\,\mathcal{J}^2\left(x_{(\mathrm{NL})}^2+2x_{(\mathrm{L})}x_{(\mathrm{NNL})}\right)+\\&+(A_{32}+f_1\mathrm{a}_{22}+f_{21}\mathrm{a}_{11}+f_{20}\mathrm{a}_{12})\,\mathcal{J}^2x_{(\mathrm{L})}^3+3\,(A_{33}+f_1\mathrm{a}_{23}+f_{21}\mathrm{a}_{12})\,\mathcal{J}^3x_{(\mathrm{L})}^2x_{(\mathrm{NL})}+\end{aligned}$$

$$+\left(A_{44}+f_1\mathrm{a}_{34}+f_{21}\mathrm{a}_{23}+f_{32}\mathrm{a}_{12}\right)\mathcal{J}^4 x_{(\mathrm{L})}^4\Big\},^{48} \tag{10.48}$$

Inserting (10.40)–(10.42) into this formula and performing the calculus, we obtain the following NNLO energy-spin relation of elementary region giant magnons:

$$\mathcal{E}-\mathcal{J}=\sin\frac{p}{2}+\frac{1}{4\mathcal{J}^2}\tan^2\frac{p}{2}\sin^3\frac{p}{2}\left[W+\frac{W^2}{2}\right]-\frac{1}{16\mathcal{J}^3}\tan^4\frac{p}{2}\sin^2\frac{p}{2}\Big[\left(3\cos p+2\right)W^2+$$
$$+\frac{1}{6}\left(5\cos p+11\right)W^3\Big]-\frac{1}{512\mathcal{J}^4}\tan^6\frac{p}{2}\sin\frac{p}{2}\Big\{\left(7\cos p-3\right)^2\frac{W^2}{1+W}-$$
$$-\frac{1}{2}\left(25\cos 2p-188\cos p-13\right)W^2-\frac{1}{2}\left(47\cos 2p+196\cos p-19\right)W^3-$$
$$-\frac{1}{3}\left(13\cos 2p+90\cos p+137\right)W^4\Big\}+\ldots, \tag{10.49}$$

where the arguments of the W-functions are again $W\left(-16\mathcal{J}^2\cot^2\left(p/2\right)e^{-2\mathcal{J}\csc p/2-2}\right)$ in the principal branch W_0. If we expand (10.49) around $\mathcal{J}\to\infty$, we recover the leading, subleading and next-to-subleading terms of the elementary giant magnon dispersion relation. These agree with the large-spin expansion (G.12) of the anomalous dimensions that were evaluated in appendix G.2 with Mathematica. Our results also agree with the GM finite-size corrections of Arutyunov, Frolov and Zamaklar (10.4) and Klose and McLoughlin (10.6). For $p=\pi$, (10.49) becomes:

$$\mathcal{E}-\mathcal{J}=1-4e^{-2\mathcal{J}-2}+4\left(4\mathcal{J}-1\right)e^{-4\mathcal{J}-4}-128\mathcal{J}^2e^{-6\mathcal{J}-6}. \tag{10.50}$$

Superposing two such GMs with angular momenta equal to $\mathcal{J}/2$, we retrieve the first few terms in the dispersion relation of long and folded GKP strings in $\mathbb{R}\times\mathrm{S}^2$, equation (G.3).

10.2 Giant Magnon, Doubled Region: $0\leq|v|\leq 1\leq 1/\omega$

As we have said, we can follow the exact same algorithm that we followed in the previous section to derive the classical part of the dispersion relation of giant magnons in the doubled region. The only difference is that the variable $\widetilde{x}=1-1/\eta$ is used instead of x, with η defined in equation (9.19) and with the string's conserved charges given by equations (9.18), (9.20), (9.21). We find:

$$\mathcal{E}-\mathcal{J}=\sin\frac{p}{2}+\frac{1}{4\mathcal{J}^2}\tan^2\frac{p}{2}\sin^3\frac{p}{2}\left[W+\frac{W^2}{2}\right]-\frac{1}{16\mathcal{J}^3}\tan^4\frac{p}{2}\sin^2\frac{p}{2}\Big[\left(3\cos p+2\right)W^2+$$
$$+\frac{1}{6}\left(5\cos p+11\right)W^3\Big]-\frac{1}{512\mathcal{J}^4}\tan^6\frac{p}{2}\sin\frac{p}{2}\Big\{\left(7\cos p-3\right)^2\frac{W^2}{1+W}-$$
$$-\frac{1}{2}\left(25\cos 2p-188\cos p-13\right)W^2-\frac{1}{2}\left(47\cos 2p+196\cos p-19\right)W^3-$$
$$-\frac{1}{3}\left(13\cos 2p+90\cos p+137\right)W^4\Big\}+\ldots, \tag{10.51}$$

[48] We also use $A_1=A_{10}$.

where the argument of Lambert's W-function has the opposite sign than before, i.e. it's given by $W\left(16\mathcal{J}^2\cot^2(p/2)\,e^{-2\mathcal{J}\csc p/2-2}\right)$ in the principal branch W_0. We notice that the W-dependence of (10.51) at NNLO is identical with (10.49), despite the fact that the inverse spin function $\widetilde{x}=\widetilde{x}(p,\mathcal{J})$ is not given by (10.39). If we expand (10.51) for $\mathcal{J}\to\infty$ we recover the Mathematica result (G.13) up to NNLO. For $p=\pi$, (10.51) becomes:

$$\mathcal{E}-\mathcal{J}=1+4e^{-2\mathcal{J}-2}+4\left(4\mathcal{J}-1\right)e^{-4\mathcal{J}-4}+128\mathcal{J}^2\,e^{-6\mathcal{J}-6}. \tag{10.52}$$

Superposing two doubled GMs (10.52) with angular momenta equal to $\mathcal{J}/2$, we get the first few terms in the dispersion relation of long circular GKP strings in $\mathbb{R}\times\mathrm{S}^2$, equation (G.5).

10.3 Single Spike, Elementary Region: $0\leq 1/\omega<|v|\leq 1$

For single spikes in the elementary region, the procedure for deriving the classical dispersion relation up to NNLO is slightly different. We must set $a\equiv\arccos 1/\omega$ and eliminate the logarithm from the equations (9.27) and (9.30). Also $x=1-\eta$, where η is defined by (9.28). The expression $\mathcal{J}=\mathcal{J}(a,x,p)$ that we obtain is inverted for $a=a(x,p,\mathcal{J})$ and it is inserted into the equations (9.27), (9.29). The variable x is then eliminated from the resulting 2×2 system that contains the momentum $p=p(x,\mathcal{J})$ and the energy $\mathcal{E}=\mathcal{E}(x,\mathcal{J})$. The result is:

$$\begin{aligned}\mathcal{E}-\frac{p}{2}=&\frac{q}{2}-\frac{1}{p^2}\sin^4\frac{q}{2}\tan\frac{q}{2}\left[W+\frac{W^2}{2}\right]+\frac{1}{p^3}\sin^6\frac{q}{2}\Bigg\{\left[\left(\sec^2\frac{q}{2}+2q\csc q-\frac{1}{2}\right)\right]W^2+\left[5+3\sec^2\frac{q}{2}\right]\\&\cdot\frac{W^3}{6}\Bigg\}+\frac{1}{64\,p^4}\sin^4\frac{q}{2}\tan^3\frac{q}{2}\Bigg\{2\left(5+7\cos q-8q\cot\frac{q}{2}\right)^2\frac{W^2}{1+W}-\Big(96q^2\cot^2\frac{q}{2}-52q\csc^4\frac{q}{2}\cdot\\&\cdot\sin^3 q+45\cos 2q+148\cos q+79\Big)W^2-\Big(16q\left(11+5\cos q\right)\cot\frac{q}{2}-37\cos 2q-172\cos q-\\&-79\Big)W^3-\left(11\cos 2q+64\cos q+85\right)W^4\Bigg\}+\ldots\end{aligned} \tag{10.53}$$

The arguments of the Lambert W-functions are $W\left(\pm 4p^2\csc^2(q/2)\,e^{-(p+q)\cdot\cot\frac{q}{2}}\right)$ in the principal branch W_0. We have defined $\mathcal{J}\equiv\sin q/2$. The minus sign in the argument of W corresponds to the elementary region of single spikes and the plus sign to the doubled region.

10.4 Single Spike, Doubled Region: $0\leq 1/\omega\leq 1\leq|v|$

To obtain the dispersion relation in the doubled region of single spikes, we set $\widetilde{x}=1-1/\eta$ with η defined in equation (9.37). Then we follow the same algorithm that we followed in the case of the elementary region single spikes for the conserved charges (9.36), (9.38), (9.39). We find:

$$\begin{aligned}\mathcal{E}-\frac{p}{2}=&\frac{q}{2}-\frac{1}{p^2}\sin^4\frac{q}{2}\tan\frac{q}{2}\left[W+\frac{W^2}{2}\right]+\frac{1}{p^3}\sin^6\frac{q}{2}\Bigg\{\left[\left(\sec^2\frac{q}{2}+2q\csc q-\frac{1}{2}\right)\right]W^2+\left[5+3\sec^2\frac{q}{2}\right]\\&\cdot\frac{W^3}{6}\Bigg\}+\frac{1}{64\,p^4}\sin^4\frac{q}{2}\tan^3\frac{q}{2}\Bigg\{2\left(5+7\cos q-8q\cot\frac{q}{2}\right)^2\frac{W^2}{1+W}-\Big(96q^2\cot^2\frac{q}{2}-52q\csc^4\frac{q}{2}\cdot\end{aligned}$$

$$\cdot \sin^3 q + 45\cos 2q + 276\cos q - 256\csc^2\frac{q}{2} + 463\Big)W^2 - \Big(16q\,(11 + 5\cos q)\cot\frac{q}{2} - 37\cos 2q -$$

$$-172\cos q - 79\Big)W^3 - (11\cos 2q + 64\cos q + 85)\,W^4\Bigg\} + \ldots \tag{10.54}$$

In contrast to the dispersion relation of giant magnons which have the same W-dependence in their elementary and doubled regions, the dispersion relation of single spikes in the elementary region is not the same with the one in the doubled region. We have marked the terms which differ between the formulas (10.53)–(10.54) with red color. Both anomalous dimensions converge to the infinite-momentum/winding dispersion relation (8.8) for $p = \infty$. We can check that both expressions (10.53)–(10.54) are correct, if we expand them for large momentum/winding $p \to \infty$. We recover all the LO, NLO and NNLO terms of formulae (G.14)–(G.15) that were obtained with Mathematica. The leading finite-size correction of (10.53) agrees with the Ahn-Bozhilov formula (10.11).

11 Part II Summary and Discussion

In part II of this thesis (§5–§10) we studied free spinning strings in $AdS_5 \times S^5$. Because of the AdS/CFT correspondence (3.1) free string states in $AdS_5 \times S^5$ are dual to local operators of the planar $\mathcal{N} = 4$ SYM theory. We may use this duality in order to compute the spectrum of the gauge theory at strong coupling where the strings are effectively weakly coupled in α'. Our focus was put on two fundamental string configurations which we studied in detail at infinite and finite-size: the Gubser-Klebanov-Polyakov (GKP) strings and giant magnons (GMs)/single spikes (SSs). Our goal was to compute the anomalous dimensions of the $\mathcal{N} = 4$ SYM operators that are dual to the above configurations and investigate the possibility of expressing them in closed forms.

Even though the full classical expressions for each of the system's charges at strong coupling are known in parametric form as functions of the dual string's velocities v and ω, the anomalous dimensions have to be expressed solely in terms of the conserved charges. Only in this way they can accommodate quantum corrections and they can be compared to the corresponding weak-coupling formulas, neither of which is known in parametric form.

11.1 GKP Strings

The GKP strings were introduced in §6. These consist of the following setups in AdS_3 and $\mathbb{R} \times S^2$:

I. a closed string rigidly rotating in $AdS_3 \subset AdS_5 \times S^5$.

II. a closed string rigidly rotating around the pole of S^2 in $\mathbb{R} \times S^2 \subset AdS_5 \times S^5$.

III. a closed string pulsating inside $AdS_3 \subset AdS_5 \times S^5$.

Each of these string configurations was studied in detail. The GKP strings I and II can be either folded or not folded and short/slow or long/fast. They obey classical short-long and slow-fast strings dualities that connect the values of their conserved charges in the corresponding regimes. Solutions with energy E and spins S or J can be related to solutions with energy E' and spins S' or J' via the equations (6.38), (6.76)–(6.77). Not all the charges have to belong to the same GKP configuration (see appendix E). All the short-long and fast-slow dualities are purely classical ($\lambda = \infty$) but it would be interesting to promote them to the quantum level or to find their analogues at weak coupling.

The dispersion relation of pulsating strings in AdS_3 (GKP strings III) was found by the WKB method. These strings are dual to the following operators of $\mathcal{N} = 4$ SYM:

$$\mathcal{O}_n = \text{Tr}\left[\mathcal{Z}\mathcal{D}_+^n\mathcal{D}_-^n\mathcal{Z}\right] + \ldots, \quad \lambda \to \infty, \tag{11.1}$$

where $\mathcal{Z}$ is a complex scalar field of $\mathcal{N} = 4$ SYM (3.8) and $\mathcal{D}_\pm$ are the light-cone derivatives (3.9).

In §7 we computed the classical part of the finite-size corrections to the large-spin dispersion relations of twist-2 and 2-magnon operators of $\mathcal{N} = 4$ super Yang-Mills theory at strong coupling:

$$\mathcal{O}_S = Tr\left[\mathcal{D}_+^m\mathcal{Z}\mathcal{D}_+^{S-m}\mathcal{Z}\right] + \ldots \quad \& \quad \mathcal{O}_J = \text{Tr}\left[\mathcal{X}\mathcal{Z}^m\mathcal{X}\mathcal{Z}^{J-m}\right] + \ldots, \quad \lambda, S, J \to \infty, \tag{11.2}$$

where $\mathcal{X}$ is another complex scalar of $\mathcal{N} = 4$ SYM (3.8). The twist-2 and the 2-magnon operators are dual to semiclassical single-spin strings that rotate in AdS_3 and $\mathbb{R} \times S^2$ respectively, dubbed above GKP strings I and II.

Following the paper [3], we have used the Lagrange-Bürmann inversion formula to invert certain functions of the elliptic integrals that are related to the conserved spins of the long GKP strings I and II. Next, we expressed the corresponding dispersion relations and the anomalous dimensions of their dual $\mathcal{N}=4$ SYM operators (11.2) in terms of Lambert's W-function. This way, not only we succeeded in predicting infinitely many and previously unbeknown terms in the dispersion relations of the GKP strings, but we also obtained compact, almost closed-form expressions for the corresponding spectra.

Inverting the elliptic integrals and the Jacobian elliptic functions w.r.t. the parameter m, constitutes an active field of research in computational mathematics.[49] It seems that the presence of a logarithmic singularity at $m=1$ (see the corresponding Taylor series of the elliptic integrals in appendix H) obstructed any progress in calculating these inverses. The authors of the paper [3] noticed that the equation (7.21) can be inverted by the Lagrange-Bürmann formula and the result can be expressed with Lambert's W-function. For AdS_3 the process had to be modified slightly because of the term $1/x$ on the r.h.s. of (7.72). It is this $1/x$ term that leads to consider the W_{-1} branch of the W-function instead of the W_0 branch and to logarithmic rather than exponential corrections in the inverse spin function and the anomalous dimensions.

It would be interesting to generalize the W-function expressions (7.64)–(7.65), (7.69)–(7.70) and (7.112)–(7.113) to all the subleading orders by means of a closed formula, a recursive process or an algorithm. It seems that Lambert's W-functions will keep appearing to all the subsequent orders. Further, we could study the effect of changing branches in Lambert's W-function. Going from the W_0 branch of the W-function to the W_{-1} branch and vice-versa, implies that the inverse spin function x either blows up (i.e. $x\to\pm\infty$) or exhibits a behavior that is different from $x\to 0$. In the case of GKP strings in $\mathbb{R}\times S^2$ we saw that if we flip the sign in the argument of Lambert's function (cf. (7.65)–(7.70)) we go from folded and stable ($\omega>1$) to circular and unstable ($\omega<1$) GKP strings and vice-versa. Perhaps this relationship could be generalized to a more profound symmetry. In other words, the Lambert W-function formalism could help reveal the symmetries (e.g. the near conjugate symmetry $W_k(\overline{z})=\overline{W}_{-k}(z)$) that are hidden inside the large-spin expansions of strings in $AdS_5\times S^5$.

All of our expressions for long/fast strings can be easily verified with Mathematica. See appendix G. For short/slow strings, the elliptic integrals do not have a logarithmic singularity for $m<1$ and the expressions for $\mathcal{E}=\mathcal{E}(\mathcal{J})$ and $\mathcal{E}=\mathcal{E}(\mathcal{S})$ can be obtained with Mathematica by simple series reversion. It is not completely impossible that the short/slow series (6.30), (6.62), (6.73) could also afford W-function parametrizations. It would be interesting to have compact forms for the short/slow series. This would facilitate the comparison between the dispersion relations of short/slow spinning strings in curved spacetimes and those of closed strings that rotate inside a flat spacetime. Strings in flat spacetimes are briefly examined in appendix D.

It would be worth investigating whether the quantum finite-size corrections at strong coupling or the weak-coupling analogues of the anomalous dimensions of long twist-2 and 2-magnon operators of $\mathcal{N}=4$ SYM, can also be expressed via the W-function. The anomalous dimensions of long twist operators of QCD (responsible for scaling violations in DIS) could also afford a W-function parametrization at strong coupling. Already, the 3-loop running coupling constant of QCD is known to have a similar W-function parametrization (see appendix I).

Another setup where the W-function formalism is expected to apply, is the solution of Einstein's equations in thermal backgrounds and dilaton geometries (see e.g. [143]). The reason for this is quite subtle and it is related to what we have said in §2.3 about the holographic renormalization group. In the context of holography, Einstein's equations are the RG equations of a certain QFT that lives on the boundary of spacetime. The solutions of one and two-loop RG equations however, have already been shown to be expressible in terms of Lambert's W-function (see appendix I and references therein). In fact, we can rigorously prove that the solution of RG equations up to any loop-order can be written in terms of Lambert's W-function. That this is possible can be seen from equation (7.72), which is nothing more than the antiderivative of a generic RG-equation $\beta(x)=\mu^2 dx/d\mu^2=-x^2\sum\beta_n x^n$. In [13],

[49] See e.g. [142].

W-function expressions were found for the dispersion relation of strings rotating inside $\text{AdS}_4 \times \mathbb{CP}^3$.

11.2 Giant Magnons & Single Spikes

Giant magnons and single spikes of infinite and finite sizes were presented in §8–§9. Giant magnons are open single-spin strings in $\mathbb{R} \times \text{S}^2$ that perform a wave-like motion around the 2-sphere. Single spikes are single-spin strings in $\mathbb{R} \times \text{S}^2$ that are wound around the equator of the 2-sphere and rotate around it. Depending on the relative values of their angular and linear velocities, the giant magnons and the single spikes can be either elementary or doubled. We may study the scattering of classical GMs and SSs by using their Pohlmeyer images in the sine-Gordon equation. The giant magnons are dual to the (anti)kink solitons of sG while the single spikes correspond to certain unstable solutions of sG. There exists a transformation, namely the $\sigma \leftrightarrow \tau$ transform, that allows to transform between giant magnons and single spikes and their Pohlmeyer reductions.

The scattering matrix of infinite-size giant magnons that is computed by means of their Pohlmeyer reduction agrees with the strong-coupling limit of the magnon S-matrix. The S-matrix of infinite-size single spikes is equal to the one for giant magnons up to non-logarithmic terms. It is very tempting to ask if scattering between giant magnons and single spikes is possible in the infinite-size limit.[50] However, the sG solutions that correspond to the single spike and the giant magnon seem to belong to different sectors of the theory, which forbids the existence of scattering solutions with GMs and SSs as asymptotic states. The "dressing" method also fails to provide such GM-SS scattering solutions, as does the generalization of the sG solutions to solutions of the complex sine-Gordon equation. One could also try to form solutions of the sG equation or the string sigma model by using the picture of single spikes as superposition of an infinite number of giant magnons.

In §10 we computed the classical part of the finite-size corrections to the dispersion relations of large-spin giant magnons and large-momentum/winding single spikes. The former are dual to the single-magnon operators of $\mathcal{N} = 4$ SYM:

$$\mathcal{O}_M = \sum_{m=1}^{J+1} e^{imp} \left|\mathcal{Z}^{m-1}\mathcal{X}\mathcal{Z}^{J-m+1}\right\rangle, \quad p \in \mathbb{R}, \quad \lambda, J \to \infty \tag{11.3}$$

at strong coupling. Single spikes are dual to (single) spinon operators:

$$\mathcal{O}_{\mathbb{S}} \sim \sum_{m=0}^{(L-1)/2} \left|\mathbb{S}^m \, \mathcal{X} \, \mathbb{S}^{(L-1)/2-m}\right\rangle, \quad J \in \mathbb{R} \quad \lambda, p \to \infty \tag{11.4}$$

at strong coupling, with $\mathbb{S} \sim \mathcal{X}\overline{\mathcal{X}} + \mathcal{Y}\overline{\mathcal{Y}} + \mathcal{Z}\overline{\mathcal{Z}}$.

We have computed all the leading ($\mathcal{A}_{n0}$, $\hat{\mathcal{A}}_{n0}$), next-to-leading ($\mathcal{A}_{n1}$, $\hat{\mathcal{A}}_{n1}$) and next-to-next-to-leading ($\mathcal{A}_{n2}$, $\hat{\mathcal{A}}_{n2}$) terms of the classical finite-size corrections to the dispersion relations of giant magnons (10.5) and single spikes (10.12), in both their elementary and doubled regions. As in the case of GKP strings, the corresponding dispersion relations have been expressed in terms of Lambert's W-function. It is not known whether there's a similar role for the W-function at weak coupling too. Since the above results for the dispersion relation of giant magnons and single spikes have not been obtained by any other method, they can be used as a check for the correct inclusion of classical wrapping effects at strong coupling by other integrability methods such as Lüscher corrections, the thermodynamic Bethe ansatz (TBA)/Y-system and the quantum spectral curve (QSC). Furthermore, since the quantum finite-size corrections to the GM dispersion relation are only known to lowest order in λ, the classical results could elucidate the structure of the quantum expansion and possibly suggest

[50]This can be seen as another way of asking whether scattering between a ferromagnetic and an anti-ferromagnetic magnon is possible. Such magnonic experiments do not seem as impossible as they were in the past.

more efficient ways to quantize this system.

The formulas (10.39)–(10.49) and (10.53)–(10.54) could be generalized to all the subleading orders by means of general formulas, a recursive process or an algorithm. The Lambert functions will keep appearing to all the subsequent orders, in complete analogy with the case of GKP strings. The quantum finite-size corrections to the dispersion relation of giant magnons and single spikes may also be expressible in terms of Lambert's W-function.

The expressions for the inverse spin function $x = x(p, \mathcal{J})$ and the anomalous dimensions $\gamma = \gamma(p, \mathcal{J})$ of both giant magnons and single spikes can be easily verified with Mathematica and the formulas of appendix G.2. As we have said, GKP strings in $\mathbb{R} \times \mathrm{S}^2$ are formed by the superposition of two giant magnons with maximum momenta $p = \pi$ and angular momenta $J/2$. With these substitutions the magnon dispersion relation (G.12) reduces to the dispersion relation of the GKP string II (G.3). However the structures of these two dispersion relations are somewhat different and the terms that are leading, subleading, etc. in one are not the same as the terms that are leading, subleading, etc. in the dispersion relation of the other. Therefore, two GMs with maximum momentum $p = \pi$ and spin $J/2$ only give (10.50) in lieu of the corresponding terms of (G.3).

Let us end this section by anticipating some further applications of the W-function formalism. The form of the finite-size corrections to the dispersion relation of GMs in γ-deformed backgrounds[51] [144] is very reminiscent of the ones appearing in undeformed backgrounds (10.4):

$$E - J = \frac{\sqrt{\lambda}}{\pi} \sin\frac{p}{2}\left\{1 - 4\sin^2\frac{p}{2}\cos\Xi\, e^{-2-2\pi J/\sqrt{\lambda}\sin\frac{p}{2}} + \ldots\right\}, \quad \Xi \equiv \frac{2\pi\,(n_2 - \beta\, J)}{2^{3/2}\cos^3 p/4}, \tag{11.5}$$

where $n_2 \in \mathbb{Z}$ is the string winding number and β is the real deformation parameter that satisfies $|n_2 - \beta\, J| \leq 1/2$ [145]. Finite-size effects are also very interesting in the cases of dyonic giant magnons [146] and the giant magnons of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence (3.37) [147]. Similar remarks apply to the generalizations of single spikes in the ABJM theory and the γ-deformed backgrounds, but also to the anti-de Sitter analogues of giant magnons, namely the spiky Kruczenski strings [148] (the GKP strings in AdS_3 can be thought of as 2-spike Kruczenski strings).

The computation of correlation functions at strong coupling could also be made with the W-function methods that were developed in part II of this thesis. Finally, as we are going to see in the following two parts, it is sometimes possible for higher-dimensional extended objects, such as p-branes and membranes, to share many of the neat characteristics of strings.[52] It is natural then to expect that the Lambert W-function formalism will be applicable to these cases as well. Finite-size effects for p-branes, e.g. for M2-branes in $\mathrm{AdS}_4 \times \mathrm{S}^7$ [151] could also be studied in the same spirit.

[51] Aka real Lunin-Maldacena backgrounds.

[52] See e.g. the papers [4, 149]. Magnon-like dispersion relations for membranes that rotate in $\mathrm{AdS}_4 \times \mathrm{S}^7$ have been found in [150].

Part III

Rudiments of p-Branes & M-Theory

12 Generalities

It is customary to dissociate the development of the theory of one-dimensional extended objects (strings) from that of higher-dimensional ones (branes), despite the fact that their itineraries and aims were always inextricably intertwined. Thus although the beginnings of string theory is placed in 1943 with Heisenberg's S-matrix program [152], the official kickoff for the study of branes takes place with Dirac's 1962 theory of electrons [153].[53] In the detailed historical account of Duff [157], the development of brane theory is divided into four main periods:

1962–1986: Bosonic p-branes.

1986–1995: Super p-branes.

1995– M-theory.

2000– Brane world cosmology.

Before going any further, let us first clarify the term "branes" [158]. Generally we should distinguish between Dp-branes, which are p-dimensional extended objects that host the endpoints of open strings and are generally studied within (10-dimensional) string theory and Mp-branes which are p-dimensional extended objects within (11-dimensional) M-theory.

We will now attempt to give a short introduction to p-branes from an M-theory perspective. We will focus mainly on classical bosonic M2-branes. Broad, extended reviews of the subject can be found in the references [159, 160, 161, 162, 163, 164].

12.1 Uses of Extended Objects

Let us start by going through the main motivations for the theory of p-branes:[54]

1. Description of elementary particles.
2. Study of the strong interaction.
3. Generalization of superstrings and superparticles.
4. The membrane paradigm of black holes.
5. M-theory.
6. AdS/CFT correspondence.
7. Brane world cosmology.

We will now mention a few things about each of these motivations.

[53] According to [154, 155], the first paper envisaging the possibility of non-local field theory and consequently of extended objects such as membranes, was written in 1950 by Yukawa [156].

[54] Related brief historical remarks can also be found in the references [165, 166, 167].

12.1.1 Description of Elementary Particles

The idea that the elementary particles are not point-like but have small finite sizes, has its origins in the concept of electromagnetic mass that was introduced by J. J. Thomson in 1881 and was later developed by Heaviside, Searle, Abraham, Lorentz and others. Closely related notions are the 4/3 factor problem, the Poincaré stresses and the electron self-force (or bootstrap force), wonderfully presented in chapter 28 of Feynman's second volume of *Lectures in Physics* [168]. In his inaugurating 1962 paper of brane theory [153], Dirac treated the electron as a charged quantized membrane, the first excited state of which was the muon.[55]

12.1.2 Study of the Strong Interaction

We mentioned in §2.1 that string theory was initially proposed as a theory of strong interactions before being overthrown by QCD in the early seventies. There two main points of contact between string theory and QCD: (a) the bosonic string interpretation of the Veneziano formula by Nambu, Nielsen and Susskind in 1970 and (b) the explanation of the almost linear Regge trajectories of hadrons by Goddard, Goldstone, Rebbi and Thorn in 1973 with bosonic string theory.

The 1974-75 bag models of the MIT, SLAC and Budapest groups [170] replaced the stringy descriptions of strong interactions. They essentially modelled the hadrons as quark bubbles/bags which contain quarks and gluons. The bag itself is a 2-dimensional dynamic membrane which confines the partons inside a space that QCD applies. The bag models successfully predict the spectrum and some other properties of hadrons while they provide a very intuitive picture of quark confinement.

	D = 2	3	4	5	6	7	8	9	10	11
p = 0	✓ (red)	✓ (red) ✓ (blue)	✓ (blue)	✓ (red)	✓ (blue)			✓ (red)	✓ (blue)	
1		✓ (red) ✓ (blue)	✓ (red) ✓ (blue)		✓ (red) ✓ (blue)				✓ (red) ✓ (blue)	
2		✓ (blue)	✓ (red) ✓ (blue)	✓ (red)	✓ (blue)	✓ (red)			✓ (blue)	✓ (red)
3			✓ (blue)		✓ (red) ✓ (blue)		✓ (red) ✓ (blue)		✓ (blue)	
4					✓ (blue)			✓ (red)	✓ (blue)	
5					✓ (blue)	✓ (blue)			✓ (red) ✓ (blue)	✓ (blue)
6									✓ (blue)	
7									✓ (blue)	
8									✓ (blue)	
9									✓ (blue)	

12.1.3 Generalization of Superstrings and Superparticles

A rather obvious motivation for developing the theory of extended objects was the need to generalize strings and point-particles to higher-dimensional objects [165] and superstrings to supermembranes [171]. It seems that higher-dimensional extensions of superstrings favor the Green-Schwarz formulation with manifest spacetime supersymmetry. As it turns out [172], supersymmetric Mp-branes can exist only in $D \leq 11$ spacetime dimensions. The chart of the Mp-brane dimensionalities that can be supersymmetrized in each spacetime dimension is known as the *brane scan* [160]. With double

[55] From the experimental point of view however, the recent upper bound for the electron radius $R_e < 10^{-18}$ m [169] is about 3 orders of magnitude smaller than the classical electron radius $r_e = e^2/4\pi\epsilon_0 m_e c^2 \sim 2.817 \times 10^{-15}$ m.

dimensional reduction (DDR), any Mp-brane configuration may be reduced to a lower-dimensional extended object in one spacetime dimension less. In the above table, the red checkmarks denote the existence of scalar supermultiplets (containing scalars and spinors) while the blue checkmarks stand for higher spin supermultiplets. Obviously, DDR is possible only in the scalar case.

12.1.4 The Membrane Paradigm of Black Holes

We saw back in §2.2, where we briefly introduced the Holographic Principle, that its main motivation was the area law and the Bekenstein-Hawking formula (2.14) for the entropy of black holes. The Hawking area law is also the motivation for what is known as the black hole "membrane paradigm" [173].

According to the membrane paradigm, the black hole horizon behaves as a fluid under small disturbances. It can effectively be replaced by a "stretched horizon", an M2-brane that is made from a 2-dimensional viscous charged and electrically conducting fluid that obeys the Navier-Stokes equations.

The membrane paradigm fits very nicely into the existing network of physical ideas and it can be used in conjunction with the other entries to give rise to interesting new concepts. For example, when it is combined with the representation of elementary particles and hadrons by 2-dimensional surfaces §12.1.1–§12.1.2, it gives rise to the black hole description of elementary particles and hadrons [174]. This set of ideas goes back to the work Einstein, Infeld and Hoffmann on the black hole electron.

The holographic analogue of the membrane paradigm is known as the fluid/gravity correspondence. Just as the area law of black holes was generalized to the Holographic Principle that applies to any spacetime, the black hole membrane paradigm can be extended to the fluid/gravity duality. Largely inspired by the AdS/CFT correspondence (introduced in §3), the fluid/gravity duality asserts that the boundary data of all asymptotically AdS spacetimes are governed by the Navier-Stokes equations in the hydrodynamic limit.

Before going too far with the analogies, we should note that the "membrane" in the membrane paradigm is a very thin 2-d sheet of fluid that surrounds the black hole and obeys the Navier-Stokes equations, instead of a 2-dimensional extended object that has a dynamics of its own.[56] However the depiction of the fluid in terms of a membrane, has certain advantages when it comes to discuss the position of the black hole microstates or the location of the black hole degrees of freedom that are going to be quantized [166].

12.1.5 M-Theory

The unifying framework of the five superstring theories is known as M-theory:

$$\left.\begin{array}{c}\text{Type-I}\\ \text{Type-IIA}\\ \text{Type-IIB}\\ \text{Heterotic } \mathfrak{so}\,(32)\\ \text{Heterotic } \mathrm{E}_8\times\mathrm{E}_8\end{array}\right\}\ \text{M-Theory.}$$

The "M" in "M-theory" stands for "magic, mystery and matrix" according to Witten [176] and "membrane" according to Duff [157]. Common lore also attributes "M" to "mother" because of the "maternal" role of M-theory in the web of the five fundamental string theories [177]. The relevance

[56] Interestingly, Bordemann and Hoppe [175] have shown how to reduce the membrane equations of motion to those of an inviscid fluid.

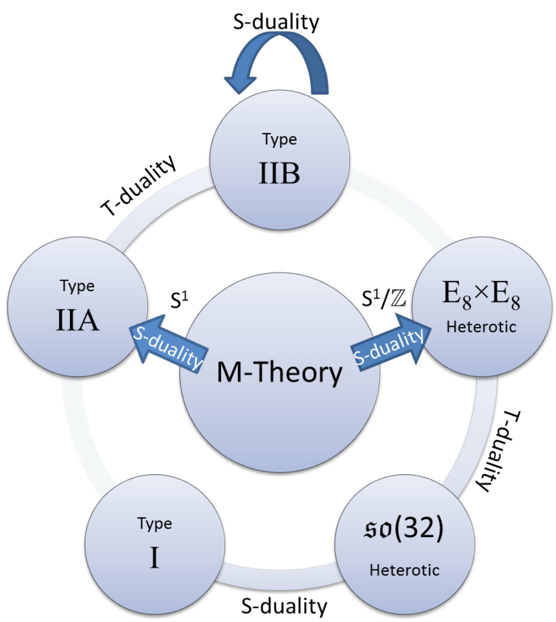
of the terms "matrix" and "membrane" will be elucidated below. Recommended reviews of M-theory are [178, 179]. More popular accounts can be found in [176]. There are three major milestones of the M-theory hypothesis:

• In 1988, Duff, Howe, Inami, Stelle [180] proved that the double dimensional reduction of the 11-dimensional supermembrane, yields the IIA superstring.

• In 1995, Townsend [181] argued that when 11-dimensional supergravity is compactified on S^1, IIA supergravity is obtained. IIA supergravity is the low-energy limit of IIA string theory. According to Townsend, the above paper of Duff, Howe, Inami and Stelle then implied that IIA string theory is just a compactified 11-dimensional supermembrane theory.

$$\text{IIA Superstring Theory (10d)} \xrightleftharpoons[\text{compactify on } S^1]{g_s \to \infty} \text{M-Theory (11d)} \xrightarrow{\text{low-energy}} \mathbf{1}_{11} \text{ Supergravity}$$

• A few months later, Witten [182] provided further supporting evidence to the M-theory conjecture. To the existing (since the eighties) set of T-dualities between IIA and IIB theory and $E_8 \times E_8$ and $\mathfrak{so}(32)$ heterotic strings, Witten added a whole new family of weak/strong coupling dualities (aka S-dualities). These transformed IIB theory to itself (self-duality), type I theory to $\mathfrak{so}(32)$ and types IIA and $E_8 \times E_8$ theories to some unknown 11-dimensional theory, M-theory.



A flurry of research activity followed Witten's groundbreaking announcements. The picture that emerged was that the five 10-dimensional string theories were interconnected by the various dualities in such a way, that they looked like the nuts and bolts of a broader 11-dimensional theory. However, nothing more was known about this 11-dimensional theory (not to mention a Lagrangian) apart from the fact that it was intrinsically non-perturbative. In addition, the 11-dimensional brane scan indicated that M2-branes were the fundamental entities to be considered in M-theory. The M-theory advent is now officially part of the second superstring revolution (1994–2000).

12.1.6 AdS/CFT Correspondence

The AdS/CFT correspondence (3.1) was introduced in §3. Besides the most popular case of AdS_5/CFT_4 there exist many more dualities in 10 and 11 dimensions. See §3.7.

12.1.7 Brane World Cosmology

One of the earliest attempts to model the universe as a brane living inside a higher-dimensional spacetime was Rubakov and Shaposhnikov's 1983 paper [183]. Brane world cosmology literally exploded after Randall and Sundrum published their famous papers [184] in 1999.

12.2 Towards M(embrane) Theory

Believing that a theory of higher-dimensional extended objects is necessary and useful and welcome is of course a completely different issue than actually developing such a theory. Very early on, it was realized that the theories of higher-dimensional bosonic/supersymmetric objects such as M2-branes are plagued with a series of diehard issues:

1. Instabilities.
2. Anomalies.
3. Ghosts.
4. Non-Renormalizability.
5. Integrability & Solvability.
6. Quantization.
7. Membrane Interactions & Perturbation Theory.

Every cloud has a silver lining however and the general discussion about all of these issues helped refine a list of a few basic ingredients of good quantum theories of supermembranes:

I. Discrete State Spectrum.

II. Massless States.

III. Mass Gap.

Let us now briefly examine each of these issues 1–7.

12.2.1 Membrane Instabilities

Membrane instabilities are perhaps the most popular objection against membranes. They were discovered as early as 1978, sixteen years after Dirac's monumental 1962 paper that initiated all the activity in the field. In sum, although Dirac's electron model was cleared from causality problems (such as classical runaway solutions, well-known in the case of the Abraham-Lorentz model), it was found to suffer from the so-called quadrupole instabilities [185]. Unfortunately, this was not just an unhappy coincidence but membrane instabilities had come to stay: branes are prone to developing spikes (or "hair") and stringy tubes [186].[57] The reasoning is very simple. Because of their extended nature, the energy of branes that is proportional to their spatial surface, remains constant if we suddenly decide to stretch any one of its sides while at the same time we shrink any of the remaining ones, in such a way that their product (equal to their energy) is the same [161].

Many discussions ensued from the above no-go situation and M(embrane) theory became stalled for years. Today it is known that the problem of membrane instabilities is cured by quantization, it returns because of supersymmetry in the form of continuous membrane spectra but it finds a natural explanation within the "matrix theory conjecture" [163]. Outside matrix quantum mechanics the

[57] Townsend [181] compares membrane spikes to a "fakir's bed of nails"...

problem still persists for the majority of spacetimes. As Nicolai and Helling mention in [161], one could consider curing this problem by adding higher curvature corrections to the membrane action. Alas these would be in conflict with the possible counterterms that would render the theory renormalizable. On the other hand, there exists a number of special membrane backgrounds (such as $\mathbb{R}^9 \times \mathrm{S}^1 \times \mathrm{S}^1$ [180] and $\mathrm{AdS}_4 \times \mathrm{M}^7$ [187]) that are known to be instability-free.

12.2.2 Anomalies

In string theory, (conformal) anomalies are linked to the notion of critical spacetime dimensions D at which they happen to vanish. It will be seen below that due to the presence of a constant cosmological term in the membrane action, the corresponding worldvolume is not conformally invariant and the conformal symmetry cannot be possibly considered as a valid anomalous symmetry candidate. On the other hand it has been shown that diffeomorphism anomalies vanish only in the case of $p = 2$ dimensional branes that live in $D = 11$ spacetime dimensions [188]. All the other allowed dimensionalities of the *brane scan* as well as all the bosonic p-branes, have been shown to suffer from incurable anomalies.[58]

12.2.3 Ghosts

Kinetic terms with the wrong sign are known as ghosts. Ghost instabilities are a commonplace in both the string and brane actions because of the negative sign in front of the kinetic term that corresponds to the temporal coordinate. Classically, ghosts are removed by gauge-fixing that trades manifest Lorentz-invariance with ghost freedom. At the quantum level, we have seen that all anomalies (including possible ghost contributions) are cancelled only for the dimensionalities $p = 1, 2$ and $D = 10, 11$.

12.2.4 Non-Renormalizability

Perhaps the most vexing problem of higher-dimensional extended objects is their being notoriously non-renormalizable. All nonlinear sigma models (NLSM's) in $p \geq 2$ dimensions are non-renormalizable by power-counting.[59] One of the earliest ideas to tackle this problem [180, 190] invoked asymptotic safety and insisted that brane theories depend on a finite number of parameters after all, despite being evidently non-renormalizable. Indeed membrane non-renormalizability has been explicitly demonstrated in the bosonic case [191], however this result does not seem to constrain neither the supermembrane nor the case of curved AdS backgrounds [192], for which it is hoped that renormalizable examples could be found. Non-renormalizability implies in many respects the end of the membrane adventure, since the infinity of curvature counterterms that would have to be added to the membrane action, would be at odds with any quantized version of the theory [161]. Matrix theory on the other hand is perfectly renormalizable for any finite matrix dimensionality N and it remains so, as long as N does not become infinite ($N \to \infty$). More recent attempts deal with the problem of M2-brane non-renormalizability in some appropriately defined large-N limit of the NLSM [193].

12.2.5 Integrability

A slightly undervalued motivation for the development of the theory of relativistic M2-branes, has been the striking resemblance of their dynamics to Yang-Mills theories. Owing to the work of Goldstone and Hoppe [194], it has been known that regularized spherical bosonic membranes are equivalent to $\mathfrak{su}(\infty)$ classical Yang-Mills theory. This is mainly due to the fact that the group of area-preserving diffeomorphisms is a (residual) symmetry of (gauge-fixed) bosonic membranes and it can be approximated by $\mathfrak{su}(N)$ in the case of spherical membranes [195]. Here N is the matrix dimensionality in the

[58] The anomaly-free superstring and supermembrane ($p = 1, 2,\ D = 10, 11$) compose what is known as the O-series of the *brane scan*.

[59] See e.g. [189].

regularized description of membranes. More will be said in the following section. The previous theorem can be generalized to the area-preserving diffeomorphisms of arbitrary-genus membranes, which should also reduce to $\mathfrak{su}(\infty)$ Yang-Mills theory in the large-N limit [196]. Yet another result stressing the deep analogies between the dynamics of membranes and Yang-Mills theories is the existence of self-dual closed bosonic membranes [197].

The association of membranes and Yang-Mills theories that we just sketched, is largely responsible for a very popular rumour (mostly circulating during the 1990's) that membranes cannot be integrable and therefore "nothing can be solved".[60] Now it is known that general Yang-Mills theories cannot possibly be integrable [198], except maybe in certain special occasions [199] such as the large-N_c limit or self-dual Yang-Mills. That branes can in fact be integrable has been advocated by Bordemann and Hoppe in a series of publications [175, 200], where M2-branes were shown to possess a Lax pair and thus an infinity of conservation laws. In curved AdS/CFT backgrounds, the classical integrability of certain configurations of M2-branes has been recently investigated by Bozhilov [201]. Membrane integrability probably has an important role to play when it comes to its quantization without the use of matrix models.

12.2.6 Quantization

Not surprisingly, all the p-brane problems that we have mentioned so far, obstruct any attempt to consistently quantize the theory of supermembranes. Brane instabilities lead to infinitely degenerate path integral contributions that contain, apart from the original p-brane configuration, all of its spiky versions. Anomalies and ghosts seem to prevent almost all p-brane dimensionalities except M2-branes in 11 spacetime dimensions, while non-renormalizability is an emphatic no-go condition. Non-integrability and non-solvability present the unsettling technical difficulty of wanting to quantize a theory without knowing how to solve it exactly.

Apparently there are two main escape routes. They have already been delineated in the above discussion. One is matrix theory which miraculously resurrects quantum membranes. We will briefly discuss matrix theory in the following section. A second alternative is to consider M2-branes in curved 11-dimensional AdS/CFT backgrounds, where supermembranes are stable, renormalizable and also possibly integrable.

Such considerations have been put forward in [202]. Supermembranes in $\mathrm{AdS}_4 \times \mathrm{S}^7$ are stable and they possess a discrete spectrum (I) and an infinity of massless states (II) [187]. The "membrane at the end of the universe" [203] is a static configuration that sits at the boundary of $\mathrm{AdS}_4 \subset \mathrm{AdS}_4 \times \mathrm{S}^7$ (taken to be $\mathrm{S}^2 \times \mathrm{S}^1$) and it is equivalent to a renormalizable $\mathfrak{osp}\,(8|4)$ supersingleton theory with $\mathcal{N} = 8$ superconformal symmetry [192]. The quantization of $\mathfrak{osp}\,(8|4)$ supersingleton theory may be consistently carried out, giving rise to an infinite tower of massless higher-spin (HS) fields (familiar from Vasiliev's HS theory) that are directly mapped to the $\mathrm{AdS}_4 \times \mathrm{S}^7$ supermembrane spectrum [204].

An analogous framework should apply to $\mathrm{AdS}_4 \times \mathrm{S}^7$'s sibling background, $\mathrm{AdS}_7 \times \mathrm{S}^4$ [190]. The HS symmetry that is associated with $\mathrm{AdS}_7 \times \mathrm{S}^4$ is given in terms of the 7-dimensional $hs\,(8^*|4)$ gauge theory with $\mathcal{N} = 2$ supersymmetry, consistent with our expectations from the Maldacena dualities in §3.7 [205].

12.2.7 Membrane Interactions & Perturbation Theory.

M-theory is defined as the strong coupling limit of string theory, in which $g_s \to \infty$. Accordingly, (super-)membranes are inherently non-perturbative objects. It is not generally known how to set up membrane interactions in the absence of a coupling constant. Conversely, matrix theory treats membrane interactions in a very efficient way [163].

[60] Quoting [175].

13 Introduction to Membranes

13.1 Bosonic Membranes

13.1.1 Dirac-Nambu-Goto Action

The Dirac-Nambu-Goto (DNG) [153, 165] action for a bosonic M2-brane (membrane) in $D = d + 1$ spacetime dimensions is:

$$S_{DNG} = -T_2 \int d^3\sigma \sqrt{-h}, \quad T_2 \equiv \frac{1}{(2\pi)^2 \ell_p^3}, \tag{13.1}$$

where ℓ_p is the Planck length of D-dimensional spacetime and $\sigma_a = \{\tau, \sigma_1, \sigma_2\} = \{\tau,\ \sigma,\ \delta\}$ are the membrane/worldvolume coordinates.[61] On the other hand, $g_{mn}(X)$ is the spacetime metric and h_{ab} is its induced (pull-back) metric on the membrane:

$$h_{ab} \equiv \partial_a X^m \partial_b X^n \, g_{mn}(X)\,, \quad h \equiv \det h_{ab}, \tag{13.2}$$

where X_m are the spacetime coordinates.

Equations of Motion

If we vary the action w.r.t. the spacetime coordinates X_m, we obtain the following equations of motion [206]:

$$\widehat{\delta}_X S_{DNG} = 0\,^{62} \Rightarrow \frac{1}{\sqrt{-h}}\,\partial_a \left(\sqrt{-h}\, h^{ab} \partial_b X^m\right) + h^{ab} \partial_a X^n \partial_b X^p \, \Gamma^m_{np}(X) = 0, \tag{13.3}$$

where $\Gamma^m_{np}(X)$ are the second-kind Christoffel symbols of the spacetime metric $g_{mn}(X)$.

Symmetries

Just as in the case of strings, the action (13.1) inherits the (global) symmetries of the spacetime metric $g_{mn}(X)$ and in addition it possesses (local) reparametrization/diffeomorphism invariance:

$$X^{m\prime}\left(\tau', \sigma_1', \sigma_2'\right) = X^m(\tau, \sigma_1, \sigma_2) \;\longrightarrow\; \widehat{\delta} X^m = \xi_a \, \partial^a X^m \quad \text{(infinitesimal)}, \tag{13.4}$$

where $\sigma_a' = \sigma_a + \xi_a(\tau, \sigma_1, \sigma_2)$ and ξ_a are some infinitesimal vectors.

13.1.2 Polyakov Action

The (Howe-Tucker-) Polyakov [207] action for a bosonic M2-brane in $D = d+1$ spacetime dimensions is:

$$S_P = -\frac{T_2}{2} \int d^3\sigma \sqrt{-\gamma} \left(\gamma^{ab} h_{ab} - 1\right), \tag{13.5}$$

[61] See footnote 26 for a summary of the index conventions that we use.

[62] In the present part, all the variations will be denoted by $\widehat{\delta}$ so as to avoid confusion with the membrane worldvolume coordinate $\sigma_2 = \delta$.

with the same conventions as above. The auxiliary metric γ_{ab} is known as the membrane/worldvolume metric.

Equations of Motion

Variation of the action functional w.r.t. the membrane metric γ_{ab}, yields the following equation of motion:

$$\widehat{\delta}_\gamma S_P = 0 \Rightarrow \gamma_{ab} = h_{ab}. \tag{13.6}$$

Substituting this equation back into the action (13.5), we see that the DNG and the Polyakov actions are equivalent on-shell, so that upon varying (13.5) w.r.t. the spacetime coordinates X^m, the equations of motion (13.3) are retrieved.

Symmetries

Again, (13.5) inherits the (global) symmetries of the spacetime metric $g_{mn}(X)$ and possesses (local) reparametrization/diffeomorphism invariance:

$$X^{m\prime}\left(\tau', \sigma', \delta'\right) = X^m\left(\tau, \sigma, \delta\right) \longrightarrow \widehat{\delta} X^m = \xi_a\, \partial^a X^m \quad \text{(infinitesimal)} \tag{13.7}$$

$$\gamma'_{ab}\left(\tau', \sigma', \delta'\right) = \partial_a \sigma^c\, \partial_b \sigma^d\, \gamma_{cd}\left(\tau, \sigma, \delta\right) \longrightarrow \widehat{\delta}\gamma_{ab} = \nabla_a \xi_b + \nabla_b \xi_a, \tag{13.8}$$

where again $\sigma'_a = \sigma_a + \xi_a\left(\tau, \sigma_1, \sigma_2\right)$, ξ_a are infinitesimal vectors and ∇_a stands for the worldvolume covariant derivative.

Unlike strings, the membrane Polyakov action is no longer Weyl-invariant due to the cosmological term in the action (13.5) that is proportional to 1. The energy-momentum tensor is equal to zero and it is conserved on-shell:

$$T_{ab} \equiv \frac{-1}{T_2\sqrt{-\gamma}} \frac{\widehat{\delta} S_P}{\widehat{\delta}\gamma^{ab}} = \frac{1}{2}\left[h_{ab} - \frac{1}{2}\left(h^c_c - 1\right)\gamma_{ab}\right]: \quad \nabla^a T_{ab} = 0. \tag{13.9}$$

Wess-Zumino Term

Generally speaking, the bosonic M2-branes couple to the $D = d + 1$ dimensional action (2.27) via the following Wess-Zumino (WZ) flux term [208]:

$$S_{\mathrm{WZ}} = -6\, T_2 \int d^3\sigma\, \dot{X}^m\, \partial_\sigma X^n\, \partial_\delta X^p\, A_{mnp}(X), \tag{13.10}$$

where $p = 2$. The antisymmetric 3-form field $A_{mnp}(X)$ is defined as:

$$F_4 \equiv dA_3 \quad \Leftrightarrow \quad F_{mnpq} = 3\partial_{[m} A_{npq]}, \tag{13.11}$$

with F_4 being the 4-form field in action (2.27). The membrane equations of motion are modified accordingly in order to accommodate (13.10). In $D = 11$ spacetime dimensions, the action (2.27) reduces to the bosonic part of 11-dimensional supergravity.

13.1.3 Gauge-Fixing

Given an action that is invariant under reparametrizations of the 3 membrane coordinates σ_a, a total of 3 degrees of freedom can be gauged away by appropriately selecting the membrane coordinates. Since $\gamma_{ab} = h_{ab}$ is a 3×3 symmetric matrix having 6 degrees of freedom, the gauge-fixing procedure leaves 3 degrees of freedom supplemented by 3 constraints [194, 209].

An especially convenient gauge choice is the following:

$$\gamma_{00} = h_{00} = -\frac{4}{\nu^2} \cdot \det h_{ij} = -\frac{2}{\nu^2} \left\{X^i, X^j\right\}^2 \qquad \gamma_{0i} = h_{0i} = 0, \qquad i, j = 1, 2, \tag{13.12}$$

where ν is a real constant that facilitates the passage to the matrix description that is going to be presented later. The Polyakov action (13.5) becomes:

$$S_P = \frac{\nu\, T_2}{4} \int d^3\sigma \left(g_{mn} \dot{X}^m \dot{X}^n - \frac{2}{\nu^2}\, g_{mn} g_{pq} \left\{X^m, X^p\right\} \left\{X^n, X^q\right\} \right). \tag{13.13}$$

The Poisson bracket $\{\_, \_\}$ is defined as in (4.2):

$$\{f\,,\, g\} \equiv \epsilon^{ij} \partial_i f \partial_j g = \partial_\sigma f\ \partial_\delta g - \partial_\delta f\ \partial_\sigma g. \tag{13.14}$$

For the conjugate momenta $\pi^m = \nu\, T_2\, \dot{X}^m/2$, the corresponding Hamiltonian is:

$$H = \frac{\nu\, T_2}{4} \int d^2\sigma \left(g_{mn} \dot{X}^m \dot{X}^n + \frac{2}{\nu^2}\, g_{mn} g_{pq} \left\{X^m, X^p\right\} \left\{X^n, X^q\right\} \right). \tag{13.15}$$

Variation of (13.13) w.r.t. X^m gives the following equations of motion:

$$\ddot{X}^m + \Gamma^m_{nr} \dot{X}^n \dot{X}^r - \frac{4}{\nu^2} \Bigg\{ g_{pq} \Gamma^m_{nr} \left\{X^n, X^p\right\} \left\{X^r, X^q\right\} + g_{nr} \left\{\left\{X^m, X^n\right\}, X^r\right\} - \\ - 2\Gamma_{nrp} \left\{X^m, X^r\right\} \left\{X^n, X^p\right\} \Bigg\} = 0. \tag{13.16}$$

The constraints are:

$$\gamma_{00} = -\frac{4}{\nu^2} \cdot \det h_{ij} \Rightarrow\ g_{mn} \dot{X}^m \dot{X}^n + \frac{2}{\nu^2}\, g_{mn} g_{pq} \left\{X^m, X^p\right\} \left\{X^n, X^q\right\} = 0 \tag{13.17}$$

$$\gamma_{0i} = 0 \Rightarrow\ g_{mn} \dot{X}^m \partial_i X^n = \left\{ g_{mn}\, \dot{X}^m, X^n \right\} = 0. \tag{13.18}$$

Although the procedure described above completely fixes the gauge, there will occasionally be times when the membrane coordinates σ_a will not be uniquely specified. If the membrane time τ happens to be such a coordinate, we can set it equal to some function of the spacetime variables $\tau = \tau\left(X^m\right)$, without affecting the gauge choice or decreasing the allowed degrees of freedom. There exist two popular gauge choices, namely the static gauge $\tau = X^0$ and the light-cone gauge $\tau \propto X^0 + X^i$ (where X^i is any spatial spacetime coordinate).

In order to decide whether a certain worldvolume gauge is compatible with some time gauge (or more generally a specific membrane configuration), one has to check the equations of motion and the gauge constraints for inconsistencies. For example, the static gauge is clearly incompatible with the ansatz (16.10) (and (6.78) in the case of strings), while the light-cone gauge is inconsistent with most of the ansätze that are studied in this thesis.

13.1.4 Membranes in Flat Spacetimes

In flat spacetimes $g_{mn} \mapsto \eta_{\mu\nu}$, the gauge-fixed Polyakov action (13.12)–(13.13) becomes:

$$S_P = \frac{\nu T_2}{4} \int d^3\sigma \left(\dot{X}^\mu \dot{X}_\mu - \frac{2}{\nu^2} \{X^\mu, X^\nu\} \{X_\mu, X_\nu\}\right) \tag{13.19}$$

while the corresponding Hamiltonian is:

$$H = \frac{\nu T_2}{4} \int d^2\sigma \left(\dot{X}^\mu \dot{X}_\mu + \frac{2}{\nu^2} \{X^\mu, X^\nu\} \{X_\mu, X_\nu\}\right). \tag{13.20}$$

The Christoffel symbols vanish in flat spacetimes so that the equations of motion (13.16) and constraints (13.17)–(13.18) become:

$$\ddot{X}^\mu - \frac{4}{\nu^2} \{\{X^\mu, X^\nu\}, X_\nu\} = 0 \tag{13.21}$$

$$\dot{X}^\mu \dot{X}_\mu + \frac{2}{\nu^2} \{X^\mu, X^\nu\} \{X_\mu, X_\nu\} = 0 \quad \& \quad \dot{X}^\mu \partial_i X_\mu = \left\{\dot{X}^\mu, X_\mu\right\} = 0. \tag{13.22}$$

There exists a neat way to express the flat spacetime Lagrangian and equations of motion of bosonic membranes in $D = d + 1$ dimensions. Defining the spacetime light-cone coordinates $X^\pm$ as:

$$X^\pm \equiv \frac{1}{\sqrt{2}} \left(X^0 \pm X^d\right) \tag{13.23}$$

and choosing the light-cone gauge

$$X^+ = \tau, \tag{13.24}$$

the above equations of motion and gauge constraints become:

$$\ddot{X}^- - \frac{4}{\nu^2} \left\{\{X^-, X^j\}, X^j\right\} = 0 \quad \& \quad \ddot{X}^j - \frac{4}{\nu^2} \left\{\left\{X^j, X^k\right\}, X^k\right\} = 0 \tag{13.25}$$

$$\dot{X}^- = \frac{1}{2} \dot{X}^j \dot{X}^j + \frac{1}{\nu^2} \left\{X^j, X^k\right\} \left\{X^j, X^k\right\} \quad \& \quad \partial_i X^- = \dot{X}^j \partial_i X^j \Leftrightarrow \left\{\dot{X}^j, X^j\right\} = 0, \tag{13.26}$$

where exceptionally $i = 1, 2$ and $j, k = 1, 2, \ldots, d - 1$. The membrane Hamiltonian is given by

$$H = \frac{\nu T_2}{4} \int d^2\sigma \left(\dot{X}^j \dot{X}^j + \frac{2}{\nu^2} \left\{X^j, X^k\right\} \left\{X^j X^k\right\}\right). \tag{13.27}$$

and the total momentum in the direction X^+ is

$$p^+ = \int_0^{2\pi} \pi^+ d^2\sigma = \int_0^{2\pi} \frac{\nu T_2}{2} \cdot \dot{X}^+ d^2\sigma = \frac{\nu}{2\ell_p^3}. \tag{13.28}$$

13.2 Bosonic p-Branes

The Dirac-Nambu-Goto and Polyakov actions of M2-branes (membranes) that we saw in the previous section, may be directly generalized to higher-dimensional extended bosonic objects that are known as Mp-branes. The DNG action of a bosonic Mp-brane that lives in $D = d + 1$ spacetime dimensions $(d \geq p)$ is [160, 210]:

$$S_{DNG} = -T_p \int d\tau d^p\sigma \sqrt{-h}, \tag{13.29}$$

where T_p is the tension of the Mp-brane and $\sigma_a = \{\tau, \sigma_1, \sigma_2, \ldots, \sigma_p\}$ are the brane worldvolume coordinates. If $g_{mn}(X)$ is the spacetime metric and X_m are the corresponding coordinates, then the induced metric on the brane h_{ab} is given by:

$$h_{ab} \equiv \partial_a X^m \partial_b X^n \, g_{mn}(X), \quad h \equiv \det h_{ab} = \frac{1}{(p+1)!} \{X^{m_1}, X^{m_2}, \ldots, X^{m_{p+1}}\}^2, \tag{13.30}$$

where the classical Nambu bracket $\{\_, \_, \ldots, \_\}$ is defined as:[63]

$$\{f_1, f_2, \ldots, f_n\} \equiv \epsilon^{i_1 i_2 \ldots i_n} \partial_{i_1} f_1 \, \partial_{i_2} f_2 \, \ldots \, \partial_{i_n} f_n. \tag{13.31}$$

The Polyakov action for bosonic Mp-branes is:

$$S_P = -\frac{T_p}{2} \int d\tau d^p\sigma \sqrt{-\gamma} \left[\gamma^{ab} h_{ab} - (p-1)\right], \tag{13.32}$$

where γ_{ab} is the worldvolume (auxiliary) metric of the p-brane. The equations of motion and the symmetries of each of the p-brane actions, are identical to those that were found above for M2-branes, i.e. (13.3)–(13.6) and local reparametrization/diffeomorphism invariance. P-branes also couple to the action (2.27) via a Wess-Zumino (WZ) flux term:

$$S_{\mathrm{WZ}} = -\frac{T_p}{(p+1)!} \int d\tau d^p\sigma \, \epsilon^{i_1 i_2 \ldots i_{p+1}} \partial_{i_1} X^{m_1} \, \partial_{i_2} X^{m_2} \, \ldots \, \partial_{i_{p+1}} X^{m_{p+1}} \, A_{m_1 m_2 \ldots m_{p+1}}(X), \tag{13.33}$$

where again,

$$F_{p+2} \equiv dA_{p+1} \quad \Leftrightarrow \quad F_{m_1 m_2 \ldots m_{p+1}} = (p+1) \, \partial_{[m_1} A_{m_2 m_3 \ldots m_{p+2}]}. \tag{13.34}$$

The gauge may again be fixed as before (13.12) by taking advantage of the diffeomorphism invariance of the Polyakov action (13.32) and reduce the initial $(p+1)(p+2)/2$ degrees of freedom that are present in the worldvolume metric $\gamma_{ab} = h_{ab}$, to $p(p+1)/2$ degrees of freedom. In a flat Minkowski spacetime, the (uncharged) p-brane equations of motion and gauge constraints become:

$$\ddot{X}^\mu - \frac{4}{\nu^2 (p-1)!} \left\{\{X^\mu, X^{\mu_1}, \ldots, X^{\mu_{p-1}}\}, X_{\mu_1}, \ldots, X_{\mu_{p-1}}\right\} = 0 \tag{13.35}$$

$$\dot{X}^\mu \dot{X}_\mu + \frac{4}{\nu^2 p!} \{X^{\mu_1}, \ldots, X^{\mu_p}\}^2 = 0 \qquad \& \qquad \dot{X}^\mu \partial_i X_\mu = 0 \tag{13.36}$$

[63] For more, see [211] and references therein.

and the corresponding Hamiltonian is:

$$H = \frac{\nu T_2}{4} \int d^2\sigma \left(\dot{X}^\mu \dot{X}_\mu + \frac{4}{\nu^2 p!} \left\{ X^{\mu_1}, \ldots, X^{\mu_p} \right\}^2 \right). \tag{13.37}$$

13.2.1 Volume-Preserving Diffeomorphisms

There exists an interesting property of the classical Nambu bracket that permits to identify the group of area preserving diffeomorphisms with a residual symmetry of the gauge-fixed p-brane. We have implicitly used this property while fixing the gauge in (13.12)–(13.30). Generally speaking, the classical Nambu bracket obeys the following identity:

$$\det \left(\partial_a X^\mu \partial_b X_\mu \right) = \frac{1}{p!} \left\{ X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_p} \right\}^2 \tag{13.38}$$

so that the spatial worldvolume of the Mp-brane is given by:

$$\int_\Sigma d^p\sigma \sqrt{\det \left(\partial_a X^\mu \partial_b X_\mu \right)} = \int_\Sigma d^p\sigma \sqrt{\frac{1}{p!} \cdot \left\{ X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_p} \right\}^2}. \tag{13.39}$$

We see that the flat bosonic p-brane Hamiltonian (13.37) is invariant under time-independent transformations that preserve the spatial worldvolume (13.39). These are known as volume-preserving diffeomorphisms, SDiff(Σ). The statement affords appropriate generalizations to curved spacetimes and supersymmetric branes.

M2-branes are invariant under area-preserving diffeomorphisms. As we have mentioned in §12.2.5, SDiff$\left(\mathrm{S}^2\right)$ can be approximated by $\mathfrak{su}\,(N \to \infty)$, where N is the matrix dimensionality of the regularized membrane. Similar statements can be made for higher-genus membranes.

13.3 Supermembranes

As we have said in §12.1.3 the higher-dimensional generalizations of superstrings (i.e. super p-branes) favor the Green-Schwarz (GS) formulation, where the spacetime supersymmetry is manifest rather than the worldvolume supersymmetry. We saw back in §3.4 that the GS formalism was also useful in writing down the Lagrangian of IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

In this short section we are going to skim through the GS formulation of the 11-dimensional supermembrane. More can be found in the original article [171], as well as the reviews [159, 160]. The generalization to higher spacetime and worldvolume dimensions is rather straightforward if only a little problematic, as we have extensively discussed in §12.2.

Let Z^M encode the bosonic/fermionic coordinates of the curved target superspace:

$$Z^M = \left(X^m, \theta_\alpha \right), \qquad M = (m, \alpha), \quad m = 0, \ldots, 11, \quad \alpha = 1, \ldots, 32 \tag{13.40}$$

and Π_a^A be the pull-back of the corresponding supervielbein E_M^A:

$$\Pi_a^A = \partial_a Z^M E_M^A, \qquad A = (\mu, \dot{\alpha}), \quad \mu = 0, \ldots, 11,, \quad \dot{\alpha} = 1, \ldots, 32, \quad a = 0, 1, 2. \tag{13.41}$$

Then the action of the 11-dimensional supermembrane is given by

$$S = -T_2 \int d^3\sigma \left\{ \sqrt{-h} + \epsilon^{abc} \Pi_a^A \Pi_b^B \Pi_c^C B_{CBA} \right\}, \tag{13.42}$$

where the three-form field B_{CBA} couples to 11-dimensional supergravity, appropriately formulated in superspace [212], and

$$h_{ab} \equiv \gamma^{ab}\Pi_a^{\mu}\Pi_b^{\nu}\eta_{\mu\nu}, \qquad h \equiv \det h_{ab}. \tag{13.43}$$

Alternatively, the supermembrane action can be expressed via the auxiliary metric γ_{ab} as follows:

$$S = -\frac{T_2}{2}\int d^3\sigma \left\{ \sqrt{-\gamma}\left[\gamma^{ab}h_{ab} - 1\right] + 2\epsilon^{abc}\Pi_a^A\Pi_b^B\Pi_c^C B_{CBA} \right\}. \tag{13.44}$$

The actions (13.42)–(13.44) are invariant under 3-dimensional reparametrizations/diffeomorphisms of the superspace coordinates Z^M as well as local fermionic κ-transformations. As functions of the background fields E_M^A and B_{CBA}, (13.42)–(13.44) are also invariant under 11-dimensional gauge transformations. As in the case of bosonic membranes, (13.42)–(13.44) may be varied w.r.t. the superspace coordinates Z^M and the worldvolume metric γ_{ab}, giving rise to the supermembrane analogs of the bosonic equations of motion (13.3)–(13.6).

To obtain the supermembrane action in flat 11-dimensional spacetime, we must set:

$$E_m^{\mu} = \delta_m^{\mu}, \qquad E_m^{\alpha} = 0 \tag{13.45}$$

$$E_{\beta}^{\alpha} = \delta_{\beta}^{\alpha}, \qquad E_{\alpha}^{\mu} = -i\left(\Gamma^{\mu}\right)_{\alpha\beta}\theta^{\beta} \tag{13.46}$$

$$B_{mn\alpha} = -\frac{i}{6}\left(\Gamma_{mn}\theta\right)_{\alpha}, \qquad B_{m\alpha\beta} = -\frac{1}{6}\left(\Gamma_{mn}\theta\right)_{(\alpha}\left(\Gamma^{n}\theta\right)_{\beta)} \tag{13.47}$$

$$B_{\alpha\beta\gamma} = -\frac{i}{6}\left(\Gamma_{\mu\nu}\theta\right)_{(\alpha}\left(\Gamma^{\mu}\theta\right)_{\beta}\left(\Gamma^{\nu}\theta\right)_{\gamma)}, \qquad B_{mnr} = 0, \tag{13.48}$$

where θ_α are 32-component Majorana spinors and $E_{\dot\alpha}^{\alpha} = \delta_{\dot\alpha}^{\alpha}$. The flat spacetime action then becomes,

$$S = -\frac{T_2}{2}\int d^3\sigma \Bigg\{ \sqrt{-\gamma}\Big[\gamma^{ab}\Pi_a^{\mu}\Pi_{b\mu} - 1\Big] + \\ + i\epsilon^{abc}\left(\bar\theta\Gamma_{mn}\partial_a\theta\right)\left[\Pi_b^m\Pi_c^n + i\Pi_b^m\left(\bar\theta\Gamma^n\partial_c\theta\right) - \frac{1}{3}\left(\bar\theta\Gamma^m\partial_b\theta\right)\left(\bar\theta\Gamma^n\partial_c\theta\right)\right]\Bigg\}, \tag{13.49}$$

where

$$\Pi_a^m = \partial_a X^m - i\bar\theta\Gamma^m\partial_a\theta. \tag{13.50}$$

In addition to local reparametrization and κ-invariance, (13.49) is invariant under the super-Poincaré transformations. As before, we may go on to fix the gauge as in (13.12) and obtain the corresponding equations of motion and constraints in the standard way. The fermionic κ-symmetry of the supermembrane is fixed as follows:

$$\Gamma^{+}\theta = 0, \tag{13.51}$$

where the 11-dimensional (32×32) light-cone gamma matrices are defined as:

$$\Gamma^{+} \equiv \frac{1}{\sqrt{2}}\left(\Gamma^0 + \Gamma^{10}\right) = \begin{pmatrix} 0 & 0 \\ \sqrt{2}i & 0 \end{pmatrix} \tag{13.52}$$

$$\Gamma^{-} \equiv \frac{1}{\sqrt{2}}\left(-\Gamma^0 + \Gamma^{10}\right) = \begin{pmatrix} 0 & \sqrt{2}i \\ 0 & 0 \end{pmatrix} \tag{13.53}$$

$$\Gamma^{i} \equiv \begin{pmatrix} \gamma_i & 0 \\ 0 & -\gamma_i \end{pmatrix}, \quad i = 1, \ldots, 9, \tag{13.54}$$

where γ_i are 16×16 Euclidean gamma matrices ($\{\gamma_i, \gamma_j\} = 2\delta_{ij}$). If we further fix the light-cone gauge as in (13.24) the corresponding supermembrane Hamiltonian gets significantly simplified. The result is:

$$H = \frac{\nu T_2}{4} \int d^2\sigma \left[\dot{X}^j \dot{X}^j + \frac{2}{\nu^2}\left\{X^j, X^k\right\}\left\{X^j X^k\right\} - \frac{4}{\nu}\,\theta^T \gamma_i \left\{X^i, \theta\right\}\right], \tag{13.55}$$

where θ stand for 16-component Majorana spinors.

In §3.4 we formulated the IIB superstring action on $\mathrm{AdS}_5 \times \mathrm{S}^5$ à la Metsaev and Tseytlin, that is by writing it as a nonlinear sigma model (NLSM) in the super-coset space (3.10). With the full GS supermembrane formalism at our disposal, we may similarly proceed and set up the supermembrane Lagrangians in the curved 11-dimensional AdS/CFT backgrounds that are part of exact solutions of 11-dimensional supergravity, namely $\mathrm{AdS}_{4,7} \times \mathrm{S}^{7,4}$. The corresponding super-coset spaces are:

$$\frac{F_1}{G_1} = \frac{\mathfrak{osp}\,(8|4)}{\mathfrak{so}\,(3,1) \times \mathfrak{so}\,(7)} \qquad \& \qquad \frac{F_2}{G_2} = \frac{\mathfrak{osp}\,(6,2|4)}{\mathfrak{so}\,(6,1) \times \mathfrak{so}\,(4)}. \tag{13.56}$$

More about the supermembrane action in these backgrounds can be found in the papers [213].

13.3.1 11-Dimensional Supergravity

Let us now briefly examine the theory that couples to the 11-dimensional supermembrane, that is 11-dimensional supergravity. More can be found in the original Cremmer-Julia-Scherk (CJS) paper [214] as well as in many textbooks, e.g. [215].

The Lagrangian of 11-dimensional supergravity (1_{11} sugra for short) is built out of three fields, the graviton/elfbein e^{μ}_{m}, the Majorana gravitino ψ_m and the antisymmetric 3-form field A_{mnp} (13.11):

$$\mathcal{L} = -\frac{e}{2\kappa_{11}^2} R - \frac{ie}{2}\overline{\psi}_m \Gamma^{mnr} D_n \psi_r - \frac{e}{48} F_{mnrs} F^{mnrs} + \frac{\sqrt{2}\, e\, \kappa_{11}}{384}\left(\overline{\psi}_m \Gamma^{mnrspq}\psi_n + 12\overline{\psi}^r \Gamma^{pq}\psi^s\right)\Big(F_{rspq} + \\ + \hat{F}_{rspq}\Big) + \frac{\sqrt{2}\,\kappa_{11}}{144^2}\,\epsilon^{m_1 \ldots m_{11}} F_{m_1 \ldots m_4} F_{m_5 \ldots m_8} A_{m_n \ldots m_{11}}\ , \tag{13.57}$$

where we have followed the conventions of the CJS paper according to which $\eta_{\mu\nu} = (+, -, -, \ldots, -)$,[64]

$$K_{m\rho\sigma} \equiv \frac{i\kappa_{11}^2}{8}\left[-\overline{\psi}_p \Gamma_{m\rho\sigma}{}^{pq}\psi_q + \left(\overline{\psi}_m \Gamma_\sigma \psi_\rho - \overline{\psi}_m \Gamma_\rho \psi_\sigma + \overline{\psi}_\sigma \Gamma_m \psi_\rho\right)\right] \tag{13.58}$$

[64] With the exception of the present paragraph, the metric we have been using in this thesis is a "mostly plus" metric. However we are a bit sloppy with our notation in this section, since our main purpose is to briefly present the original formulation of 11-dimensional supergravity without going into too many details.

$$\omega_{m\rho\sigma} \equiv \omega^0_{m\rho\sigma} + K_{m\rho\sigma} \quad \& \quad \hat{\omega}_{m\rho\sigma} \equiv \omega_{m\rho\sigma} + \frac{i\kappa_{11}^2}{8}\,\overline{\psi}_p \Gamma_{m\rho\sigma}{}^{pq}\psi_q \tag{13.59}$$

$$D_n \equiv \partial_n + \frac{1}{8}\left(\omega_{n\rho\sigma} + \hat{\omega}_{n\rho\sigma}\right)\Gamma^{\rho\sigma} \tag{13.60}$$

$$F_{mnrs} \equiv 4\partial_{[m}A_{nrs]} \quad \& \quad \hat{F}_{mnrs} \equiv F_{mnrs} - \frac{3\kappa_{11}}{\sqrt{2}}\,\psi_{[m}\Gamma_{nr}\psi_{s]}, \tag{13.61}$$

$\kappa_{11}^2 \equiv 8\pi G_{11}$, e is the elfbein determinant $e \equiv \det e^\mu_m = \sqrt{-g}$ and $\omega^0_{m\rho\sigma}$ is the Christoffel connection. Care should be taken as to distinguish between the Christoffel symbols of §13.1 and the 32×32 Dirac gamma matrices in $D = 11$ dimensions that appear in the present section:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}, \qquad \Gamma^m = e^m_\mu \Gamma^\mu. \tag{13.62}$$

Note also that $\{\ ,\ \}$ stands for the anti-commutator and not the Poisson bracket (13.14) of §13.1. The bosonic part of the action (13.57) is obtained by setting the gravitino field ψ_m equal to zero:

$$\mathcal{L}_B = -\frac{e}{2\kappa_{11}^2}R - \frac{e}{48}F_{mnrs}F^{mnrs} + \frac{\sqrt{2}\kappa_{11}}{144^2}\,\epsilon^{m_1\ldots m_{11}}F_{m_1\ldots m_4}F_{m_5\ldots m_8}A_{m_n\ldots m_{11}}. \tag{13.63}$$

13.4 M(atrix) Theory

13.4.1 Matrix-Regularized Membranes

2-dimensional spherical surfaces may be regularized by an ingenious method that was devised by Goldstone and Hoppe in 1982 [194]. Consider a 2-dimensional unit sphere,

$$x^2 + y^2 + z^2 = 1 \tag{13.64}$$

in spherical coordinates:

$$x_1 = \cos\varphi\sin\vartheta, \quad x_2 = \sin\varphi\sin\vartheta, \quad x_3 = \cos\vartheta. \tag{13.65}$$

For the worldvolume variables $\sigma_1 = \varphi$, $\sigma_2 = \cos\vartheta$, the Poisson brackets of x_i's satisfy,

$$\{x_i, x_j\} = e_{ijk}x_k, \tag{13.66}$$

which is very reminiscent of the $\mathfrak{su}(2)$ algebra. One is then tempted to make the following replacements:

$$x_i \mapsto \frac{2}{N}\mathbf{J}_i, \qquad \{\ ,\ \} \mapsto -\frac{iN}{2}[\ ,\ ], \qquad \frac{1}{4\pi}\int d^2\sigma\ldots, \mapsto \frac{1}{N}\mathrm{Tr}[\ldots] \tag{13.67}$$

where the $N \times N$ matrices $\mathbf{J}_i$ furnish a representation of $\mathfrak{su}(2)$ with spin equal to $(N-1)/2$:

$$[\mathbf{J}_i, \mathbf{J}_j] = ie_{ijk}\mathbf{J}_k. \tag{13.68}$$

The replacement rules (13.67) suggest a way to regularize gauge-fixed spherical bosonic membranes in flat backgrounds that are described by the system (13.25)–(13.27). Spatial spacetime coordinates X^i are upgraded to $N \times N$ matrices $\mathbf{X}^i$, the Poisson brackets are replaced by commutators and integrals by traces as follows:

$$x_i \mapsto \mathbf{X}_i, \qquad \{\ ,\ \} \mapsto -\frac{iN}{2}[\ ,\ ], \qquad \frac{1}{4\pi}\int d^2\sigma\ldots, \mapsto \frac{1}{N}\mathrm{Tr}[\ldots]. \tag{13.69}$$

With (13.69), the Hamiltonian (13.27) becomes:

$$H = \frac{1}{2\pi\ell_p^3} \cdot \mathrm{Tr}\left(\frac{1}{2}\dot{\mathbf{X}}^i\dot{\mathbf{X}}^i - \frac{1}{4}\left[\mathbf{X}^i, \mathbf{X}^j\right]\left[\mathbf{X}^i, \mathbf{X}^j\right]\right), \tag{13.70}$$

where ν in (13.27) has been set equal to the matrix dimensionality N. The regularized (spatial) equations of motion and constraints (13.25)–(13.26) become:

$$\ddot{\mathbf{X}}^i + \left[\left[\mathbf{X}^i, \mathbf{X}^j\right], \mathbf{X}^k\right] = 0 \quad \& \quad \left[\dot{\mathbf{X}}^i, \mathbf{X}^j\right] = 0. \tag{13.71}$$

To go supersymmetric, rules (13.69) are applied to the susy Hamiltonian (13.55). The result is known as Matrix Theory [216]:

$$\boxed{H_0 = \frac{1}{2\pi\ell_p^3} \cdot \mathrm{Tr}\left(\frac{1}{2}\dot{\mathbf{X}}^i\dot{\mathbf{X}}^i - \frac{1}{4}\left[\mathbf{X}^i, \mathbf{X}^j\right]\left[\mathbf{X}^i, \mathbf{X}^j\right] + \theta^T\gamma_i\left[\mathbf{X}^i, \theta\right]\right)}, \tag{13.72}$$

which although it has been derived only for spherical membranes,[65] it can be directly generalized to supersymmetric membranes of arbitrary topologies with the replacement rule (13.69).

We have already mentioned in §12.2.1 that the theory of classical (bosonic and supersymmetric) membranes suffers from incurable instabilities that apparently hinder all reasonable attempts to quantize the theory. Matrix theory however appears to cure the problem in the case of bosonic membranes (13.70). The reasoning is this: flat directions ("spikes") are eventually disfavored because they give rise to a large effective confining potential that stabilizes the system.

On the other hand, super-matrix theory (13.72) resurrects instabilities in the form of continuous supermembrane spectra [186]. The fermionic contribution to the zero-point energy of the membrane is exactly the opposite of the bosonic one, so that the supermembrane spectrum is no longer discrete (I) and cannot be possibly associated to particles. As we will see below, the BFSS conjecture [218] provides a very satisfactory explanation for this fact, which ceases to be problematic.

13.4.2 The Matrix Theory Conjecture

The matrix theory conjecture (aka the BFSS conjecture) [218], strengthened the intimate connection between the theory of quantum supermembranes and M-theory that was first established by Townsend during the M-theory revolution. Banks, Fischler, Shenker and Susskind (BFSS) observed that the low-energy Hamiltonian of N, type IIA D0-branes is precisely equivalent to the matrix theory (13.72). Since IIA string theory arises from the compactification of M-theory on S^1 and the latter becomes non-relativistic in the infinite-momentum frame (IMF), BFSS concluded that:

$$\left.\begin{array}{r} \#N \to \infty, \ \text{(non-relativistic) type IIA D0-branes} \\ \mathfrak{su}(\infty) \ \text{susy QM (13.72)} \end{array}\right\} = \begin{array}{c} \text{M-theory compactified} \\ \text{on the IMF.} \end{array} \tag{13.73}$$

With the matrix theory conjecture, the problem of continuous supermembrane spectra is resolved because supermembranes are treated not as elementary objects but as composite ones that are made up from gravitons = D0-branes. The energy of a system that contains two or more gravitons can

[65] The case of the 2-torus can be studied along the lines of the 2-sphere. More can be found in [217].

take any value whatsoever and super-matrix theory with $N \geq 2$ contains multi-particle states with continuous spectra.

The finite-N version of the matrix theory conjecture was put forward in 1997 by Susskind [219]. Instead of M-theory on the IMF, finite-N matrix theory (13.72) is equivalent to a sector of M-theory where the retarded time x^- has been periodically identified. This is generally known as discrete light-cone quantization (DLCQ):

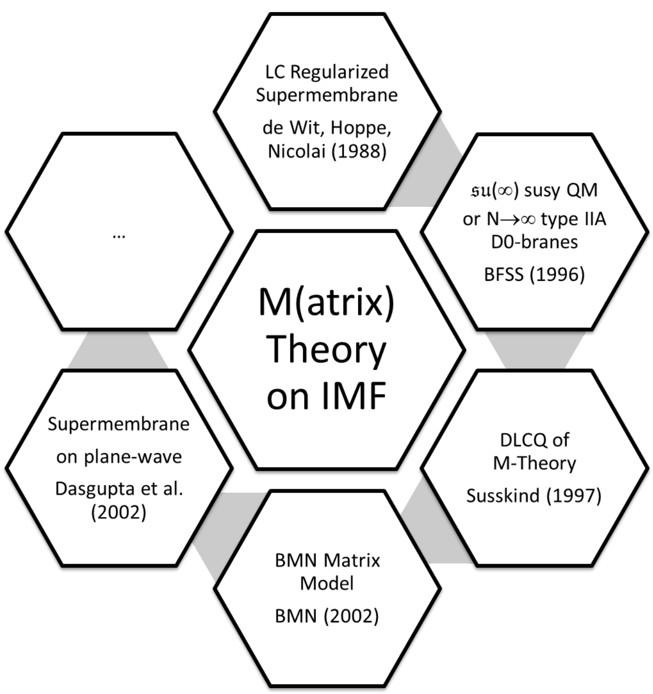
$$\left.\begin{array}{r} \#N \text{ (low-energy) type IIA D0-branes} \\ \mathfrak{su}\,(N) \text{ susy QM (13.72)} \end{array}\right\} = \begin{array}{l} \text{DLCQ Sector of M-theory with} \\ \#N \text{ units of compact momentum.} \end{array} \qquad (13.74)$$

In sum, matrix theory is an inherently many-body theory that provides a second-quantized model of M-theory in flat 11-dimensional target space. The gravitational interactions arise in matrix theory through the inclusion of quantum effects.

13.4.3 Matrix Theory in Curved Backgrounds

The matrix theory Hamiltonian (13.72) and the corresponding matrix theory conjectures (13.73)–(13.74) have been formulated and are only valid in flat 11-dimensional backgrounds. It is natural to want to set up matrix models of M-theory on curved 11-dimensional backgrounds and especially plane-wave backgrounds and $\mathrm{AdS}_{4,7} \times \mathrm{S}^{7,4}$ spacetimes that are directly related to exact solutions of 11-dimensional supergravity.

A DLCQ description of 6-dimensional $\mathrm{A}_{N-1}\,(2,0)$ SCFT theory that, as we saw in §3.7, is the holographic dual of M-theory on $\mathrm{AdS}_7 \times \mathrm{S}^4$ has been proposed in [220]. It is based on quantum mechanics on some appropriately defined large-instanton moduli space. Matrix theory on weakly curved backgrounds has been studied by Taylor and Van Raamsdonk [221].



In 2002, Berenstein, Maldacena and Nastase (BMN) [43] proposed a matrix model description of the DLCQ of M-theory on the following (homogeneous) plane-wave background (C.5):

$$ds^2 = -2dudv - \left[\sum_{i=1}^{3} \frac{\mu^2}{9} x^i x^i - \sum_{j=4}^{9} \frac{\mu^2}{36} x^j x^j \right] du^2 + \sum_{i=1}^{9} dx^i dx^i. \tag{13.75}$$

The DLCQ of M-theory on the homogeneous plane-wave (13.75) is given by the following Hamiltonian:

$$H = H_0 + \frac{1}{2} \cdot \mathrm{Tr} \left(\sum_{i=1}^{3} \frac{\mu^2}{9} \mathbf{X}_i^2 + \sum_{j=4}^{9} \frac{\mu^2}{36} \mathbf{X}_j^2 - \frac{i\mu}{8} \theta^T \gamma_{123} \theta - \frac{2i\mu}{3} \epsilon^{ijk} \mathbf{X}_i \mathbf{X}_j \mathbf{X}_k \right). \tag{13.76}$$

For simplicity, we have omitted the overall Planck length factor. The authors of [222] have showed that the BMN matrix model (13.76) can be derived either by regularizing the supermembrane on the plane-wave background (13.75), or from the dynamics of type IIA D0-branes.

Part IV

Rotating Membranes

14 Introduction and Motivation

Having introduced supersymmetric branes and presented the main ideas about them, we are ready to zoom on the AdS/CFT correspondence and ask whether the study of branes as individual entities can elucidate certain aspects of the duality and contribute to the redaction of its "dictionary". However, one has to realize that the procedure of turning from particles and strings to p-branes can occasionally be quite murky. Instabilities, anomalies, non-renormalizability, non-integrability, elusive quantization, non-interactivity and nonexistent perturbation theory, are some of the issues that have always plagued Mp-branes, as saw in §12.2. These issues are very likely to persist in AdS/CFT. Therefore, a down-to-earth strategy would suggest investing only in those traits of branes that stand greater chances of fitting in a self-consistent framework.

In this spirit, we have chosen to focus on the study of the stringy properties of classical M2-branes that live in the 11-dimensional spacetimes that are relevant to the AdS/CFT correspondence. As we have explained in §12.2, M2-branes in curved 11-dimensional backgrounds such as $\text{AdS}_m \times \text{S}^n$ are relatively immune to most of the usual "maladies" of Mp-brane virology. Even in the worst-case scenario where 11-dimensional anti-de Sitter M2-branes turn out to be unreliable to work with, we have chosen to invest in perhaps their most reliable aspect, that is their stringy behavior. The purpose of part IV of this thesis is therefore twofold:

1. Elucidate the role of individual Mp-branes in AdS/CFT.

2. Study the stringy aspects of M2-branes in the context of the AdS/CFT correspondence.

In part II of the thesis, we studied extensively the three basic string setups of GKP and explained in detail all of their virtues with regard to the AdS/CFT correspondence. The GKP strings convey important information about the scaling of the dual gauge theory states at strong coupling, which is impossible to obtain by other (i.e. perturbative) means. In the present part, we will prove an interesting property of anti-de Sitter membranes, namely that they are capable of encoding all the dynamics of anti-de Sitter GKP strings. In fact, AdS membranes are capable of reproducing not only the anti-de Sitter GKP strings, but any classical string configuration in $\text{AdS}_5 \subset \text{AdS}_5 \times \text{S}^5$.

"Stringy membranes" are literally membrane configurations in a stringy disguise. They are defined in spacetimes with a compact submanifold, such as all the backgrounds that are related to the AdS/CFT correspondence (listed in §3.7). Their essential property is that they are wrapped around one of the compact dimensions of the background and they reproduce the action, equations of motion and conserved charges of a string that lives in the non-compact part of spacetime. There are two interesting consequences of this construction. Firstly, at the level of classical quadratic fluctuations of stringy membranes in $\text{AdS}_m \times \text{S}^n$ there seems to exist an infinite set of purely membrane modes, in addition to the set of purely stringy ones. Secondly, just as the $\text{AdS}_5/\text{CFT}_4$ parameter matching affords to strings in the bulk of AdS an effective string tension that is equal to the square root of the 't Hooft coupling $\sqrt{\lambda}$, stringy membranes are similarly endowed with an effective tension that is equal to $\sqrt{\lambda'} = R\sqrt{\lambda}/g_s\,\ell_s$.[66]

One family of stringy membranes with the above properties can be obtained by embedding the bosonic (conformal) string Polyakov action in AdS_5 into the Polyakov action of membranes in

[66] R denotes the radius of the compact dimension, g_s is the string coupling constant and ℓ_s is the fundamental string length.

$AdS_7 \times S^4$. It can then be shown that the action, equations of motion and Virasoro constraints of every string solution in AdS_5 can be reproduced by a properly constructed membrane of $AdS_7 \times S^4$. Similarly, every string configuration in $AdS_4 \subset AdS_5$ can be reproduced by a stringy membrane of $AdS_4 \times S^7/\mathbb{Z}_k$. As an illustration of the properties of stringy membranes, we may write down membrane ansätze that reproduce the dynamics of the two anti-de Sitter GKP configurations, namely the rotating GKP string (I) that we examined in §6.1 and the pulsating GKP string (III) that we examined in §6.3. To further investigate the true relationship between the stringy membranes and the string solutions that they reproduce, we may analyze the spectrum of quadratic fluctuations around the corresponding stringy membranes. For the two stringy membranes that reproduce the GKP strings, we find that a decoupled subset of fluctuations that is transverse to the direction of the stringy membrane admits a Lamé multi-band/multi-gap structure, which uniquely characterises their membrane nature. On the other hand, string excitations are represented by a single-band/single-gap pattern. These findings confirm the picture that we have of membranes as collective excitations of some stringy counterparts.

Thus we see that the study of classical membranes à la GKP begins to pay off. More will be said in the discussion section §18, but let us give a brief overview here. Because of the way we constructed them, we have all the reasons to expect that stringy membranes will correspond to certain string-like operators of the dual SCFTs. For example, the GKP configuration (I), given by (6.8) is dual to twist-2 operators (6.2). The stringy membrane that reproduces the GKP string (I) should also be dual to twist-2 SCFT operators like (6.2). The state-operator correspondence is also likely to be applicable here and the energies of stringy membranes are expected to equal the scaling dimensions of the stringy gauge theory operators. Without stringy membranes we had no way to find SCFT operators that are dual to M-theory states, especially in theories like the 6-dimensional $A_{N_c-1}\,(2,0)$ SCFT, about which very few things are known. The stringy limit of AdS/CFT membranes teaches us that stringy membranes are mapped to the gauge theory operators that are dual to the strings they reproduce and that the scaling dimensions of the latter are equal to the energies of the stringy membranes.

A second lesson that we draw from stringy membranes is that M-theory in backgrounds such as $AdS_{4,7} \times S^{7,4}$, most probably has certain classically integrable "stringy" sectors, where all the technology and methods from the integrable $AdS_5 \times S^5$ string paradigm can be imported. The reason for this is simple: stringy membranes have the same action and equations of motion with bosonic strings in $AdS_5 \subset AdS_5 \times S^5$, which are known to be classically integrable [58] (see also §5.2). Therefore they too are expected to be classically integrable. Stringy membranes also seem to confirm a conjecture that was put forward some time ago [201], that the various AdS/CFT dualities possess common integrable sectors. As this conjecture awaits for a rigorous proof (probably coming from integrability), stringy membranes further suggest that the "family" of theories with common integrable sectors could actually be bigger and contain, apart from the AdS/CFT group, other theories like QCD and ABJM.

Part IV is organized as follows. §15 is a brief introduction to classical bosonic membranes in $AdS_7 \times S^4$. In §16 we present stringy membranes. We examine two principal ansätze of stringy membranes in $AdS_7 \times S^4$ that fully reproduce the action and equations of motion of the GKP spinning string configurations in AdS: (I) the AdS_3 closed & folded GKP string (6.8) and (III) the AdS_3 pulsating GKP string (6.78). Subsequently, we prove (along the general lines of the paper [223]) that the on-shell action, equations of motion and conserved charges of bosonic strings that live in $AdS_5 \subset AdS_5 \times S^5$ can be reproduced by appropriate membrane ansätze in $AdS_7 \times S^4$. Analogous statements are formulated for bosonic strings in $AdS_4 \times S^7/\mathbb{Z}_k$, in §16.2–§16.3. In §17 we examine the stability of the two stringy membranes that correspond to the GKP strings (I) and (III). A summary and a discussion of stringy membranes can be found in §18.

15 Spinning Membranes in $\text{AdS}_7 \times \text{S}^4$

In this section we will briefly consider classical and uncharged (the WZ term (13.10) is absent) bosonic membranes in $\text{AdS}_7 \times \text{S}^4$,[67]

$$
\begin{aligned}
&Y_{07} = Y_0 + iY_7 = 2\cosh\rho\, e^{it} && X_{12} = X_1 + iX_2 = \cos\overline{\theta}_1\, e^{i\overline{\phi}_1} \\
&Y_{12} = Y_1 + iY_2 = 2\sinh\rho\cos\theta_1\, e^{i\phi_1} \qquad \& && X_{34} = X_3 + iX_4 = \sin\overline{\theta}_1\cos\overline{\theta}_2\, e^{i\overline{\phi}_2} \\
&Y_{34} = Y_3 + iY_4 = 2\sinh\rho\sin\theta_1\cos\theta_2\, e^{i\phi_2} && X_5 = \sin\overline{\theta}_1\sin\overline{\theta}_2 \\
&Y_{56} = Y_5 + iY_6 = 2\sinh\rho\sin\theta_1\sin\theta_2\, e^{i\phi_3},
\end{aligned}
\tag{15.1}
$$

where Y^μ and X^i are the embedding coordinates of $\text{AdS}_7 \times \text{S}^4$ and $\rho \geq 0$, $t \in [0,\, 2\pi)$,[68] $\theta_1,\, \overline{\theta}_1 \in [0,\, \pi]$, and $\theta_2,\, \phi_1,\, \phi_2,\, \phi_3,\, \overline{\theta}_2,\, \overline{\phi}_1,\, \overline{\phi}_2 \in [0,\, 2\pi)$. The $\text{AdS}_7 \times \text{S}^4$ line element is given by:

$$
\begin{aligned}
ds^2 = {}& G^{\text{AdS}}_{mn}(y)dy^m dy^n + G^{\text{S}}_{mn}(x)dx^m dx^n = \\
={}&4\Big[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\,\Big(d\theta_1^2 + \cos^2\theta_1\, d\phi_1^2 + \sin^2\theta_1\left(d\theta_2^2 + \cos^2\theta_2 d\phi_2^2 + \right.\\
&\left.\left. + \sin^2\theta_2 d\phi_3^2\right)\Big)\Big] + \Big[d\overline{\theta}_1^2 + \cos^2\overline{\theta}_1\, d\overline{\phi}_1^2 + \sin^2\overline{\theta}_1\,\Big(d\overline{\theta}_2^2 + \cos^2\overline{\theta}_2\, d\overline{\phi}_2^2\Big)\Big],
\end{aligned}
\tag{15.2}
$$

where $y^m \equiv (t,\, \rho,\, \theta_1,\, \theta_2,\, \phi_1,\, \phi_2,\, \phi_3)$ and $x^m \equiv \left(\overline{\theta}_1,\, \overline{\theta}_2,\, \overline{\phi}_1,\, \overline{\phi}_2\right)$. With the gauge choice (13.12), the membrane Polyakov action (13.13) becomes (for $\nu = 2$) in $\text{AdS}_7 \times \text{S}^4$ (15.2):

$$
\begin{aligned}
S_P = \frac{T_2}{2}\int\Big[&G^{\text{AdS}}_{mn}(y)\dot{y}^m\dot{y}^n + G^{\text{S}}_{mn}(x)\dot{x}^m\dot{x}^n - \frac{1}{2}G^{\text{AdS}}_{mn}(y)G^{\text{AdS}}_{pq}(y)\{y^m, y^p\}\{y^n, y^q\} - \\
&-\frac{1}{2}G^{\text{S}}_{mn}(x)G^{\text{S}}_{pq}(x)\{x^m, x^p\}\{x^n, x^q\} - G^{\text{AdS}}_{mn}(y)G^{\text{S}}_{pq}(x)\{y^m, x^p\}\{y^n, x^q\}\Big]d\tau\, d\sigma\, d\delta.
\end{aligned}
\tag{15.3}
$$

The constraints (13.17)–(13.18) that follow from the gauge-fixing (13.12) become ($i, j = 1, 2,\ \nu = 2$):

$$
\begin{aligned}
\gamma_{00} =- \det h_{ij} \Rightarrow {}& G^{\text{AdS}}_{mn}(y)\dot{y}^m\dot{y}^n + G^{\text{S}}_{mn}(x)\dot{x}^m\dot{x}^n + \frac{1}{2}G^{\text{AdS}}_{mn}(y)G^{\text{AdS}}_{pq}(y)\{y^m, y^p\}\{y^n, y^q\} + \\
&+\frac{1}{2}G^{\text{S}}_{mn}(x)G^{\text{S}}_{pq}(x)\{x^m, x^p\}\{x^n, x^q\} + G^{\text{AdS}}_{mn}(y)G^{\text{S}}_{pq}(x)\{y^m, x^p\}\{y^n, x^q\} = 0
\end{aligned}
\tag{15.4}
$$

$$
\gamma_{0i} = G^{\text{AdS}}_{mn}(y)\,\dot{y}^m\partial_i y^n + G^{\text{S}}_{mn}(x)\dot{x}^m\partial_i x^n = 0 \Rightarrow \Big\{G^{\text{AdS}}_{mn}(y)\,\dot{y}^m, y^n\Big\} + \Big\{G^{\text{S}}_{mn}(x)\,\dot{x}^m, x^n\Big\} = 0. \tag{15.5}
$$

The action (15.3) and the constraints (15.4)–(15.5) are invariant under the global isometry group of $\text{AdS}_7 \times \text{S}^4$, that is $\mathfrak{so}(6,2) \times \mathfrak{so}(5)$. The following 28 + 10 Noether charges (spins and angular

[67] In $\text{AdS}_7 \times \text{S}^4$ it's $\mathfrak{k} = \ell/R = 2$, as we saw in §3.7. Then, for $R = 1 \Leftrightarrow \ell = 2$. R and ℓ may be restored in all of the formulas of part IV, by setting $\delta \mapsto \delta/R$ and $\delta \in [0,\, 2\pi R)$.

[68] To avoid time periodicity (a typical feature of anti-de Sitter spacetime) we must consider the universal covering space of AdS, in which $t \in \mathbb{R}$.

momenta) are conserved on-shell:

$$S^{\mu\nu} = T_2 \int_0^{2\pi} \left(Y^\mu \dot{Y}^\nu - Y^\nu \dot{Y}^\mu\right) d\sigma d\delta, \qquad \mu\,, \nu = 0, 1, \ldots, 7 \tag{15.6}$$

$$J^{ij} = T_2 \int_0^{2\pi} \left(X^i \dot{X}^j - X^j \dot{X}^i\right) d\sigma d\delta, \qquad i, j = 1, 2, \ldots, 5. \tag{15.7}$$

Certain charges among (15.6)–(15.7) correspond to the cyclic coordinates of the action (15.3), namely t, ϕ_1, ϕ_2, ϕ_3, $\overline{\phi}_1$, $\overline{\phi}_2$. The expressions for the cyclic charges can be directly read off from (15.3), by using the corresponding line element (15.2):

$$E = \left|\frac{\partial L}{\partial \dot{t}}\right| = 4\,T_2 \int_0^{2\pi} \dot{t} \cosh^2\rho \, d\sigma d\delta = S^{07} \tag{15.8}$$

$$S_1 = \frac{\partial L}{\partial \dot{\phi}_1} = 4\,T_2 \int_0^{2\pi} \dot{\phi}_1 \sinh^2\rho\, \cos^2\theta_1 \, d\sigma d\delta = S^{12} \tag{15.9}$$

$$S_2 = \frac{\partial L}{\partial \dot{\phi}_2} = 4\,T_2 \int_0^{2\pi} \dot{\phi}_2 \sinh^2\rho\, \sin^2\theta_1 \cos^2\theta_2 \, d\sigma d\delta = S^{34} \tag{15.10}$$

$$S_3 = \frac{\partial L}{\partial \dot{\phi}_3} = 4\,T_2 \int_0^{2\pi} \dot{\phi}_3 \sinh^2\rho \sin^2\theta_1 \sin^2\theta_2 \, d\sigma d\delta = S^{56} \tag{15.11}$$

$$J_1 = \frac{\partial L}{\partial \dot{\overline{\phi}}_1} = 4\,T_2 \int_0^{2\pi} \dot{\overline{\phi}}_1 \cos^2\overline{\theta}_1 \, d\sigma d\delta = J^{12} \tag{15.12}$$

$$J_2 = \frac{\partial L}{\partial \dot{\overline{\phi}}_2} = 4\,T_2 \int_0^{2\pi} \dot{\overline{\phi}}_2 \sin^2\overline{\theta}_1 \cos^2\overline{\theta}_2 \, d\sigma d\delta = J^{34}. \tag{15.13}$$

In (15.8)–(15.13), L stands for the membrane Lagrangian that is defined by the formula $S_P = \int L\, d\tau$.

16 Spinning Membranes as Spinning Strings

16.1 Stringy Membranes in $\text{AdS}_7 \times \text{S}^4$

The purpose of this section is to investigate the stringy behavior of classical membranes in $\text{AdS}_7 \times \text{S}^4$. It will be shown that the GKP folded closed string (I) that rotates in $\text{AdS}_3 \subset \text{AdS}_5 \times \text{S}^5$,[69] has the same action and equations of motion with a specific membrane soliton that spins in $\text{AdS}_3 \subset \text{AdS}_7 \times \text{S}^4$. A similar result will be shown to hold for the pulsating closed folded string (III) of GKP.[70] The two results will then be generalized to all the string solitons that live in $\text{AdS}_5 \subset \text{AdS}_5 \times \text{S}^5$, for which it will be shown that there is always a membrane soliton in $\text{AdS}_7 \times \text{S}^4$ with the same action and equations of motion.

Consider the following ansatz for a membrane that rotates in $\text{AdS}_3 \times \text{S}^1 \subset \text{AdS}_7 \times \text{S}^4$:

$$\Big\{t = \kappa\tau,\ \rho = \rho(\sigma),\ \phi_1 = \kappa\omega\tau,\ \phi_2 = \phi_3 = \theta_1 = \theta_2 \ = 0\Big\} \times \Big\{\overline{\phi}_1 = \delta,\ \overline{\theta}_1 = \overline{\theta}_2 = \overline{\phi}_2 = 0\Big\}. \tag{16.1}$$

[69] The AdS_3 folded closed string (I) of GKP [11] was studied in §6.1.

[70] The AdS_3 pulsating closed folded string (III) of GKP [11] was studied in §6.3.

In embedding coordinates ($R = 1$, $\ell = 2$), the ansatz (16.1) reads:

$$\begin{aligned}
Y_0 &= 2\cosh\rho(\sigma)\cos\kappa\tau \ , \quad Y_3 = Y_4 = Y_5 = Y_6 = 0\,, & X_1 &= \cos\delta \\
Y_1 &= 2\sinh\rho(\sigma)\cos\kappa\omega\tau & X_2 &= \sin\delta \\
Y_2 &= 2\sinh\rho(\sigma)\sin\kappa\omega\tau & X_3 &= X_4 = X_5 = 0 \\
Y_7 &= 2\cosh\rho(\sigma)\sin\kappa\tau. & &
\end{aligned} \tag{16.2}$$

Polyakov's action (15.3) and constraint equation (15.4) become:[71]

$$S_P = 2\,T_2 \int \left(-\dot{t}^2\cosh^2\rho + \dot{\phi}_1^2\sinh^2\rho\,\cos^2\theta_1 - \cos^2\overline{\theta}_1\,\rho'^2\,\overline{\phi}_1'^2\,\left\{\sigma,\delta\right\}^2\right) d\tau d\sigma d\delta = \tag{16.4}$$

$$= \frac{2\,T_1}{\ell_s g_s}\int \left(-\kappa^2\cosh^2\rho + \kappa^2\omega^2\sinh^2\rho - \rho'^{\,2}\right) d\tau d\sigma \tag{16.5}$$

$$\rho'^{\,2} - \kappa^2\left(\cosh^2\rho - \omega^2\sinh^2\rho\right) = 0 \qquad \text{(constraint)}. \tag{16.6}$$

Now compare the action (16.5) and the corresponding gauge constraint (16.6), with the on-shell action (6.11) and the Virasoro constraint (6.13) of the GKP string (I). They are identical! With the exception of the factor $\cos^2\overline{\theta}_1\,\overline{\phi}_1'^2$, the off-shell action (16.4) is also identical to the off-shell stringy action (6.10). To prove the equivalence of the systems (6.10)–(6.13) and (16.4)–(16.6), just note that the action (16.4) has only ρ with a non-vanishing equation of motion:

$$\rho'' + \kappa^2\left(\omega^2 - 1\right)\sinh\rho\cosh\rho = 0, \tag{16.7}$$

Equation (16.7) is the same as the stringy one, equation (6.12). At the same time, all the conserved charges of the membrane action (16.4) are identical to the ($\omega^2 > 1$) stringy ones (6.22)–(6.23):

$$E(\omega) = \frac{16\,T_1}{g_s\ell_s}\cdot\frac{\omega}{\omega^2-1}\,\mathbb{E}\left(\frac{1}{\omega^2}\right) \tag{16.8}$$

$$S(\omega) = \frac{16\,T_1}{g_s\ell_s}\cdot\left(\frac{\omega^2}{\omega^2-1}\,\mathbb{E}\left(\frac{1}{\omega^2}\right) - \mathbb{K}\left(\frac{1}{\omega^2}\right)\right) = S_1. \tag{16.9}$$

We have therefore proven that the membrane (16.1) is dynamically equivalent to the AdS_3 closed folded GKP string (I) that is given by the ansatz (6.8).

[71] In $D = 11$ spacetime dimensions, we may express the 10-dimensional string coupling constant g_s in terms of the Planck length ℓ_{11} and the string fundamental length ℓ_s, by dimensionally reducing 11-dimensional supergravity to $D = 10$ spacetime dimensions:

$$g_s = \left(\frac{R_c}{\ell_{11}}\right)^{3/2}, \qquad \ell_s^2 = \frac{\ell_{11}^3}{R_c} \quad\longrightarrow\quad g_s = \left(\frac{\ell_{11}}{\ell_s}\right)^3, \tag{16.3}$$

where R_c is the radius of compactification. The 11-dimensional membrane tension becomes $T_2 = \left[(2\pi)^2 g_s \ell_s^3\right]^{-1}$ [16].

For the oscillating AdS_3 GKP string (III), we can also find a dynamically equivalent membrane in $AdS_7 \times S^4$. Consider the following ansatz of a pulsating membrane in $AdS_7 \times S^4$:

$$\Big\{t = t\,(\tau)\,,\; \rho = \rho\,(\tau)\,,\; \theta_1 = \frac{\pi}{2},\; \theta_2 = \sigma,\; \phi_1 = \phi_2 = \phi_3 = 0\Big\} \times \Big\{\overline{\phi}_1 = \delta,\; \overline{\theta}_1 = \overline{\theta}_2 = \overline{\phi}_2 = 0\Big\}. \quad (16.10)$$

In embedding coordinates, the ansatz (16.10) reads:

$$\begin{aligned}
&Y_0 = 2\cosh\rho(\tau)\cos t\,(\tau)\,, \quad Y_1 = Y_2 = Y_4 = Y_6 = 0\,, && X_1 = \cos\delta \\
&Y_3 = 2\sinh\rho(\tau)\cos\sigma && X_2 = \sin\delta \\
&Y_5 = 2\sinh\rho(\tau)\sin\sigma && X_3 = X_4 = X_5 = 0 \\
&Y_7 = 2\cosh\rho(\tau)\sin t\,(\tau)\,.
\end{aligned} \quad (16.11)$$

The off-shell and on-shell Polyakov action of the pulsating membrane configuration (16.10), along with the corresponding constraint equation are given by:

$$S_P = 2\,T_2 \int \Big(-\dot{t}^2\cosh^2\rho + \dot{\rho}^2 - \sinh^2\rho\,\sin^2\theta_1\,\cos^2\overline{\theta}_1\,\theta_2'^2\,\overline{\phi}_1'^2\,\{\sigma,\delta\}^2\Big)\,d\tau d\sigma d\delta = \quad (16.12)$$

$$= \frac{2\,T_1}{\ell_s g_s}\int \left(-\dot{t}^2\cosh^2\rho + \dot{\rho}^2 - \sinh^2\rho\right) d\tau d\sigma \quad (16.13)$$

$$\dot{\rho}^2 - \dot{t}^2\cosh^2\rho + \sinh^2\rho = 0 \qquad \text{(constraint)}. \quad (16.14)$$

The on-shell membrane Polyakov action (16.13) and its constraint (16.14), are identical to the stringy ones that are given by equations (6.81)–(6.84) for $w = 1$. Therefore the pulsating membrane (16.10) is dynamically equivalent to the AdS_3 pulsating GKP string (6.78). The t and ρ equations of motion of the membrane (16.10) are also identical to the stringy ones, (6.82)–(6.83) (with $w = 1$):

$$\ddot{t}\cosh^2\rho + 2\,\dot{t}\,\dot{\rho}\,\cosh\rho\sinh\rho = 0 \quad (16.15)$$

$$\ddot{\rho} + \sinh\rho\cosh\rho\left(\dot{t}^2 + 1\right) = 0. \quad (16.16)$$

As promised, we may generalize the two previous examples to any[72] string soliton that lives in $AdS_5 \subset AdS_5 \times S^5$ and has no dynamical parts in S^5.[73] Thus we are going to prove the following proposition:

■ **16.1.1.** *Every classical pure AdS_5 string soliton has an equivalent $AdS_7 \times S^4$ membrane soliton (and not vice versa).*

[72]This statement does not include all the ansätze that are incompatible with the choice of the conformal gauge ($\gamma_{ab} = \eta_{ab}$) in the string Polyakov action (5.3). An interesting generalization of our proposal would include all the AdS_5 Polyakov string configurations independently of the gauge choice, or equivalently all the AdS_5 Nambu-Goto string ansätze.

[73]For convenience, we are going to dub all the $\mathfrak{sl}(2)$ string solitons that have no S^5 counterparts, "pure".

Proof: Consider the membrane Polyakov action (15.3) and the corresponding constraint equations (15.4)–(15.5) in the gauge (13.12):

$$S_2 = \frac{T_2}{2}\int\Big[G^{\text{AdS}}_{mn}(y)\dot{y}^m\dot{y}^n + G^{\text{S}}_{mn}(x)\dot{x}^m\dot{x}^n - \frac{1}{2}G^{\text{AdS}}_{mn}(y)G^{\text{AdS}}_{pq}(y)\{y^m,y^p\}\{y^n,y^q\} - \\ -\frac{1}{2}G^{\text{S}}_{mn}(x)G^{\text{S}}_{pq}(x)\{x^m,x^p\}\{x^n,x^q\} - G^{\text{AdS}}_{mn}(y)G^{\text{S}}_{pq}(x)\{y^m,x^p\}\{y^n,x^q\}\Big]d\tau\, d\sigma\, d\delta \tag{16.17}$$

$$G^{\text{AdS}}_{mn}(y)\dot{y}^m\dot{y}^n + G^{\text{S}}_{mn}(x)\dot{x}^m\dot{x}^n + \frac{1}{2}G^{\text{AdS}}_{mn}(y)G^{\text{AdS}}_{pq}(y)\{y^m,y^p\}\{y^n,y^q\} + \\ +\frac{1}{2}G^{\text{S}}_{mn}(x)G^{\text{S}}_{pq}(x)\{x^m,x^p\}\{x^n,x^q\} + G^{\text{AdS}}_{mn}(y)G^{\text{S}}_{pq}(x)\{y^m,x^p\}\{y^n,x^q\} = 0 \tag{16.18}$$

$$G^{\text{AdS}}_{mn}(y)\,\dot{y}^m\partial_i y^n + G^{\text{S}}_{mn}(x)\,\dot{x}^m\partial_i x^n = \Big\{G^{\text{AdS}}_{mn}(y)\,\dot{y}^m, y^n\Big\} + \Big\{G^{\text{S}}_{mn}(x)\,\dot{x}^m, x^n\Big\} = 0, \tag{16.19}$$

where as before $y^m \equiv (t,\,\rho,\,\theta_1,\,\theta_2,\,\phi_1,\,\phi_2,\,\phi_3)$ and $x^m \equiv \left(\overline{\theta}_1,\,\overline{\theta}_2,\,\overline{\phi}_1,\,\overline{\phi}_2\right)$ and $G_{mn}\,(y,\,x)$ are the components of (15.2). Let σ denote the spatial string worldsheet coordinate:

$$y^m = y^m\,(\tau\,,\sigma) \quad \& \quad x^m = x^m\,(\tau\,,\delta)\,, \tag{16.20}$$

then the above action and constraints become:

$$S_2 = \frac{T_2}{2}\int\Big[G^{\text{AdS}}_{mn}(y)\dot{y}^m\dot{y}^n + G^{\text{S}}_{mn}(x)\dot{x}^m\dot{x}^n - G^{\text{AdS}}_{mn}(y)G^{\text{S}}_{pq}(x)y'^{\,m}y'^{\,n}x'^{\,p}x'^{\,q}\Big]d\tau\, d\sigma\, d\delta \tag{16.21}$$

$$G^{\text{AdS}}_{mn}(y)\dot{y}^m\dot{y}^n + G^{\text{S}}_{mn}(x)\dot{x}^m\dot{x}^n + G^{\text{AdS}}_{mn}(y)G^{\text{S}}_{pq}(x)y'^{\,m}y'^{\,n}x'^{\,p}x'^{\,q} = 0 \tag{16.22}$$

$$G^{\text{AdS}}_{mn}(y)\,\dot{y}^m y'^{\,n} = G^{\text{S}}_{mn}(x)\,\dot{x}^m x'^{\,n} = 0. \tag{16.23}$$

We now set $x^3 = \overline{\phi}_1 = \delta$ for the coordinate of the 4-sphere that corresponds to $G^{\text{S}}_{33} = \cos^2\overline{\theta}_1$. Then,

$$S_2 = \frac{T_2}{2}\int\Big[G^{\text{AdS}}_{mn}(y)\Big(\dot{y}^m\dot{y}^n - \cos^2\overline{\theta}_1\,\overline{\phi}_1{}'^2\,y'^{\,m}y'^{\,n}\Big) + G^{\text{S}}_{mn\neq3}(x)\dot{x}^m\dot{x}^n - \\ -G^{\text{AdS}}_{mn}(y)G^{\text{S}}_{pq\neq3}(x)y'^{\,m}y'^{\,n}x'^{\,p}x'^{\,q}\Big]d\tau\, d\sigma\, d\delta \tag{16.24}$$

$$G^{\text{AdS}}_{mn}(y)\left(\dot{y}^m\dot{y}^n + \cos^2\overline{\theta}_1\,y'^{\,m}y'^{\,n}\right) + G^{\text{S}}_{mn\neq3}(x)\dot{x}^m\dot{x}^n + \\ +G^{\text{AdS}}_{mn}(y)G^{\text{S}}_{pq\neq3}(x)y'^{\,m}y'^{\,n}x'^{\,p}x'^{\,q} = 0 \tag{16.25}$$

$$G^{\text{AdS}}_{mn}(y)\,\dot{y}^m y'^{\,n} = G^{\text{S}}_{mn\neq3}(x)\,\dot{x}^m x'^{\,n} = 0. \tag{16.26}$$

The proposition 16.1.1 follows upon setting $x^{m\neq3} = 0$, $y^{m>5} = 0$ and performing the δ-integration:

$$S_2 = \frac{T_2}{2}\int G^{\text{AdS}}_{mn\leq5}(y^{p\leq5})\left(\dot{y}^m\dot{y}^n - \cos^2\overline{\theta}_1\,\overline{\phi}_1{}'^2\,y'^{\,m}y'^{\,n}\right)d\tau\, d\sigma = \tag{16.27}$$

$$= \frac{T_1}{2g_s\ell_s}\int G^{\text{AdS}}_{mn\leq 5}(y^{p\leq 5})\left(\dot{y}^m\dot{y}^n - y'^{\,m}y'^{\,n}\right)d\tau\, d\sigma = \frac{S_1}{g_s\ell_s} \tag{16.28}$$

$$G^{\text{AdS}}_{mn\leq 5}(y^{p\leq 5})\left(\dot{y}^m\dot{y}^n + y'^{\,m}y'^{\,n}\right) = G^{\text{AdS}}_{mn\leq 5}(y^{p\leq 5})\,\dot{y}^m y'^{\,n} = 0, \tag{16.29}$$

which is a just the action and the Virasoro constraints of a classical string in AdS_5. To see this, compare (16.28)–(16.29) with the corresponding string Polyakov action and Virasoro constraints in $\text{AdS}_5 \times \text{S}^5$ (in the conformal gauge, $\gamma_{ab} = \eta_{ab}$), (5.3)–(5.5):

$$S_1 = \frac{T_1}{2}\int \Big[G^{\text{AdS}}_{mn}(y)\left(\dot{y}^m\dot{y}^n - y'^{\,m}y'^{\,n}\right) + G^{\text{S}}_{mn}(x)\left(\dot{x}^m\dot{x}^n - x'^{\,m}x'^{\,n}\right)\Big]d\tau\, d\sigma \tag{16.30}$$

$$T_{00} = T_{11} = \frac{1}{2}\Big[G^{\text{AdS}}_{mn}(y)\left(\dot{y}^m\dot{y}^n + y'^{\,m}y'^{\,n}\right) + G^{\text{S}}_{mn}(x)\left(\dot{x}^m\dot{x}^n + x'^{\,m}x'^{\,n}\right)\Big] = 0 \tag{16.31}$$

$$T_{01} = T_{10} = G^{\text{AdS}}_{mn}(y)\,\dot{y}^m y'^{\,n} + G^{\text{S}}_{mn}(x)\,\dot{x}^m x'^{\,n} = 0. \tag{16.32}$$

The $\overline{\theta}_1$ and $\overline{\phi}_1$ equations of motion in (16.27) are trivially satisfied and the remaining equations of motion of (16.27) are identical to the equations of motion that are obtained by varying the string action (16.30). Therefore the two systems are dynamically equivalent.

The not vice versa part in 16.1.1 follows from the fact that we may construct many inequivalent $\text{AdS}_7 \times \text{S}^4$ membrane configurations with dependence on both σ and δ, that are impossible to obtain from the action of a classical bosonic string in $\text{AdS}_5 \times \text{S}^5$. □

16.2 Stringy Membranes in $\text{AdS}_4 \times \text{S}^7$

The proposition 16.1.1 may be modified in order to apply in $\text{AdS}_4 \times \text{S}^7$. If we assume that the string's target space coordinates depend on both world-sheet coordinates $\{\tau\,,\sigma\}$,

$$y^m = \left(t = t\,(\tau\,,\sigma)\,,\, \rho = \rho\,(\tau\,,\sigma)\,,\, \theta = \theta\,(\tau\,,\sigma)\,,\, \phi_1 = \phi_1\,(\tau\,,\sigma)\,,\, \phi_2 = \phi_2\,(\tau\,,\sigma)\right), \tag{16.33}$$

then the proposition 16.1.1 can only apply to a subset of all possible AdS_5 classical string configurations, namely strings in $\text{AdS}_4 \subset \text{AdS}_5$. These are the only ones that can be obtained from a membrane in $\text{AdS}_4 \times \text{S}^7$. For example both stringy membranes (16.1)–(16.10) that we encountered above and give rise to GKP strings in $\text{AdS}_3 \subset \text{AdS}_4 \subset \text{AdS}_5$, are such membranes since they live in $\text{AdS}_4 \subset \text{AdS}_4 \times \text{S}^7$. More generically,

■ **16.2.1.** *Every classical pure string soliton of $AdS_4 \subset AdS_5$*[74] *has an equivalent $AdS_4 \times S^7$ membrane soliton (and not vice versa).*

If we drop the condition of full dependence of the string's target-space coordinates on both world-sheet coordinates $\{\tau\,,\sigma\}$ as in (16.33), it should be possible to apply the above method and obtain (i) $\text{AdS}_{4/7} \times \text{S}^{7/4}$ stringy membranes that are equivalent to certain special string configurations that live in $\text{AdS}_5 \times \text{S}^5$ and (ii) $\text{AdS}_4 \times \text{S}^7$ stringy membranes that are equivalent to strings that live in AdS_5.

[74] By writing $\text{AdS}_4 \subset \text{AdS}_5$, we mean that one of the two azimuthal angles of the 3-sphere of AdS_5 has been set equal to zero.

16.3 Stringy Membranes in $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$

Stringy membranes are also meaningful in orbifolded spacetimes such as $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$. For $k = 1$, this is just the $\mathrm{AdS}_4 \times \mathrm{S}^7$ spacetime that we saw above. As we have discussed in §3.8, geometries like $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ provide the gravitational backgrounds of the ABJM correspondence (3.37).

In the context of the ABJM theory, the question has been posed whether a logarithmic type behavior is possible for the anomalous dimensions of any state in the theory. As we saw back in §6, the logarithmic behavior of anomalous dimensions is possible for twist-2 operators of $\mathcal{N} = 4$ SYM theory and their dual closed and folded GKP strings in AdS_3 (I). Based on what we have said, we can answer the above question in an affirmative way by using the properties of stringy membranes. Consider the metric of $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ [203]:

$$\begin{aligned} ds^2 &= G^{\mathrm{AdS}}_{mn}(y)dy^m dy^n + G^{\mathrm{S}/\mathbb{Z}}_{mn}(x)dx^m dx^n = \\ &= \ell^2\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho \cdot d\Omega_2^2\right) + R^2 d\overline{\Omega}^2_{7/\mathbb{Z}_k} \end{aligned} \tag{16.34}$$

$$d\overline{\Omega}^2_{7/\mathbb{Z}_k} = \left(\frac{d\overline{y}}{k} + \widetilde{A}\right)^2 + ds^2_{\mathbb{CP}^3}\,, \tag{16.35}$$

$$\widetilde{A} \equiv \frac{1}{2}\left(\cos^2\overline{\xi} - \sin^2\overline{\xi}\right)d\overline{\psi} + \frac{1}{2}\cos^2\overline{\xi}\,\cos\overline{\theta}_1\,d\overline{\phi}_1 + \frac{1}{2}\sin^2\overline{\xi}\,\cos\overline{\theta}_2\,d\overline{\phi}_2$$

$$\begin{aligned} ds^2_{\mathbb{CP}^3} &= d\overline{\xi}^2 + \cos^2\overline{\xi}\,\sin^2\overline{\xi}\left(d\overline{\psi} + \frac{1}{2}\cos\overline{\theta}_1\,d\overline{\phi}_1 - \frac{1}{2}\cos\overline{\theta}_2\,d\overline{\phi}_2\right)^2 + \\ &\quad + \frac{1}{4}\cos^2\overline{\xi}\left(d\overline{\theta}_1^2 + \sin^2\overline{\theta}_1\,d\overline{\phi}_1^2\right) + \frac{1}{4}\sin^2\overline{\xi}\left(d\overline{\theta}_2^2 + \sin^2\overline{\theta}_2\,d\overline{\phi}_2^2\right). \end{aligned} \tag{16.36}$$

The membrane configurations (16.1)–(16.10) may be easily obtained from (16.34)–(16.36). In the ansätze (16.1)–(16.10), we only have to assign $\overline{y} = k\delta$ (also, for $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ it's $\mathfrak{k} = \ell/R = 1/2$) and set the six remaining angles in S^7 equal to zero. Logarithmic behavior will then be possible for the ABJM states that are dual to the $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ avatar of the stringy membrane (16.1) that fully captures the dynamics and properties of the closed and folded GKP strings (I). More generally, we may formulate the following proposition:

■ **16.3.1.** *Every classical pure string soliton of $AdS_4 \subset AdS_5$ has an equivalent $AdS_4 \times S^7/\mathbb{Z}_k$ membrane soliton (and not vice versa).*

Statements like 16.3.1 are to be expected, since it is known that type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ can be obtained from the supermembrane action on $\mathrm{AdS}_4 \times \mathrm{S}^7$ by double dimensional reduction [224].

With this we conclude our presentation of stringy membranes in anti-de Sitter spacetimes. In the following section we are going to study the stability properties of the two stringy membrane configurations that we examined above, namely (16.1)–(16.10).

17 Membrane Fluctuations

The subject of the present section is the stability of stringy membranes. Generally, we would expect stringy membranes to be unstable, because of their δ-component that is wound around a great circle of $S^{4/7}$. The δ-component is prone to collapsing towards either pole of the corresponding hypersphere, leading the total system towards a more stable configuration with lower energy. Indeed, this has been known to be true for classical bosonic strings that are wound around a great circle of a 2-sphere and have no other dynamics [225]. On the other hand stringy membranes share a common Lagrangian, equations of motion and gauge constraints with their equivalent strings, so that we would expect them to inherit many of their stabilities/instabilities. If we suppose for the sake of the argument that the string configuration that is dual to a stringy membrane is unstable, there exist many ways that can render it stable, e.g. by adding more angular momenta [225, 226], stable AdS components [76, 227], pulsating parts [228], orientifold projections [229] and flux terms [230]. Even those strings that are known to be unstable have been extensively studied and have proven very useful in the context of the AdS/CFT correspondence [76, 225], since the instabilities may sometimes be quite insignificant from the point of view of the dual gauge theory [231]. One possible way to explain this state of affairs is that unstable solutions are often easily extendable to stable configurations, while at the same time they maintain their wanted gauge theory properties. The stability analysis of strings and membranes in anti-de Sitter spacetimes has not been developed satisfactorily (even at the level of numerics), mainly because of its difficulty [232]. Having at our disposal results about the stability of AdS strings and membranes, would enable us to draw very useful conclusions about the stability of our stringy membrane configurations.

On the other hand, we should not forget that stringy membranes are membranes and not strings. This property can sometimes enhance the stability of the resulting system. Whereas a simple stringy membrane that is wound around a 2-sphere has zero surface tension and it is expected to be stable, a similarly wound string around the 2-sphere cannot be stable as we saw above. Since stringy membranes are meant to reproduce the behavior of classical strings in AdS_5, it is important to be able to make concrete statements about their advantages/disadvantages in the domain of stability. Work on membrane fluctuations in various backgrounds can be found in [233].

The main result of the present section will be that the fluctuations of stringy membranes are governed by the Lamé equation. Let us briefly review the main applications of Lamé equations, before we embark on the stability analysis of stringy membranes. The Lamé equation arises when we separate the variables of the Laplace equation in the ellipsoidal coordinate system [234]. It belongs to the class of "quasi-exactly solvable" (QES) systems [235], so-called because their solutions can be determined by algebraic means in certain cases [236, 237]. Because the stabilities and instabilities of Lamé systems are organized in multiple bands and gaps, the range of their physical applications is quite extended. Among their most interesting applications are: (a) they provide an alternative to the Kronig-Penney model of electrons in one-dimensional crystals [238, 236]; (b) they seem to govern explosive particle production (preheating) due to parametric resonance in post-inflationary universe [239]; (c) they come up in sphaleron fluctuations of the ϕ^4 and 1+1 dimensional abelian Higgs models [240]; (d) they are intimately connected to the spectral curve of $\mathfrak{su}(2)$ BPS monopoles [241]; (e) they arise very often in supersymmetric quantum mechanics [242], etc. [243, 244]. The Lamé equation repeatedly appears in all the studies of string fluctuations in anti-de Sitter spacetimes [81, 105, 228]. As we will see below, the fluctuations of stringy membranes give rise to a much richer Lamé band/gap structure.

We will mainly work in the embedding coordinate system of $AdS_{p+2} \times S^q$ for which,

$$ds^2 = \eta_{\mu\nu} dY^\mu dY^\nu + \delta_{ij} dX^i dX^j = -dY_0^2 + \sum_{i=1}^{p+1} dY_i^2 - dY_{p+2}^2 + \sum_{i=1}^{q+1} dX_i^2 \tag{17.1}$$

$$-\eta_{\mu\nu} Y^\mu Y^\nu = Y_0^2 - \sum_{i=1}^{p+1} Y_i^2 + Y_{p+2}^2 = \ell^2 \quad , \quad \delta_{ij} X^i X^j = \sum_{i=1}^{q+1} X_i^2 = R^2, \tag{17.2}$$

where $\eta_{\mu\nu} = (-,+,+,\ldots,+,-)$, $\delta_{ij} = (+,+,\ldots,+)$, $\mu,\nu = 0,1,\ldots,p+2$ and $i,j = 1,2,\ldots,q+1$. The constraints (17.2) are taken into account by including two Lagrange multipliers Λ, $\widetilde{\Lambda}$ in the gauge-fixed according to (13.12) membrane Polyakov action (13.13) (with $\nu = 2$):

$$S_P = \frac{T_2}{2}\int d^3\sigma \Bigg[\dot{Y}^\mu\dot{Y}_\mu + \dot{X}^i\dot{X}^i - \frac{1}{2}\{Y^\mu,Y^\nu\}\{Y_\mu,Y_\nu\} - \frac{1}{2}\{X^i,X^j\}\{X^i,X^j\} - \\ -\{Y^\mu,X^i\}\{Y_\mu,X^i\} + \Lambda\left(Y^\mu Y_\mu + \ell^2\right) + \widetilde{\Lambda}\left(X^i X^i - R^2\right)\Bigg]. \tag{17.3}$$

If we vary the action (17.3), we obtain the following equations of motion:

$$\ddot{Y}^\mu = \left\{\{Y^\mu,Y^\nu\},Y_\nu\right\} + \left\{\{Y^\mu,X^i\},X^i\right\} + \Lambda\, Y^\mu \tag{17.4}$$

$$\ddot{X}^i = \left\{\{X^i,X^j\},X^j\right\} + \left\{\{X^i,Y^\mu\},Y_\mu\right\} + \widetilde{\Lambda}\, X^i. \tag{17.5}$$

The Lagrange constraints are:

$$Y^\mu Y_\mu = -\ell^2, \qquad X^i X^i = R^2, \tag{17.6}$$

while the two constraints that follow from the gauge-fixing (13.12) are given by:

$$\dot{Y}^\mu\partial_\sigma Y_\mu + \dot{X}^i\partial_\sigma X^i = \dot{Y}^\mu\partial_\delta Y_\mu + \dot{X}^i\partial_\delta X^i = 0 \tag{17.7}$$

$$\dot{Y}^\mu\dot{Y}_\mu + \dot{X}^i\dot{X}^i + \frac{1}{2}\{Y^\mu,Y^\nu\}\{Y_\mu,Y_\nu\} + \frac{1}{2}\{X^i,X^j\}\{X^i,X^j\} + \{Y^\mu,X^i\}\{Y_\mu,X^i\} = 0. \tag{17.8}$$

Because of the constraint (17.8), the membrane Hamiltonian is identically equal to zero:

$$H = \frac{T_2}{2}\int d^2\sigma \Bigg[\dot{Y}^\mu\dot{Y}_\mu + \dot{X}^i\dot{X}^i + \frac{1}{2}\{Y^\mu,Y^\nu\}\{Y_\mu,Y_\nu\} + \frac{1}{2}\{X^i,X^j\}\{X^i,X^j\} + \\ +\{Y^\mu,X^i\}\{Y_\mu,X^i\} - \Lambda\left(Y^\mu Y_\mu + \ell^2\right) - \widetilde{\Lambda}\left(X^i X^i - R^2\right)\Bigg] = 0. \tag{17.9}$$

Let us consider the following perturbations:[75]

$$Y^\mu = Y_0^\mu + \delta Y^\mu \quad , \quad X^i = X_0^i + \delta X^i \quad , \quad \Lambda = \Lambda_0 + \delta\Lambda \quad , \quad \widetilde{\Lambda} = \widetilde{\Lambda}_0 + \delta\widetilde{\Lambda}, \tag{17.10}$$

where $\left\{Y_0, X_0, \Lambda_0, \widetilde{\Lambda}_0\right\}$ is a solution of the equations of motion (17.4)–(17.5) and the constraints

[75] The reader should be careful in order not to confuse the world-volume coordinate $\delta \equiv \sigma_2$, with the variational δ's that appear in δS_P, δX, δY, $\delta\Lambda$, $\delta\widetilde{\Lambda}$ and denote the perturbations of S_P, X, Y, Λ and $\widetilde{\Lambda}$.

(17.6)–(17.8). The quadratic fluctuation action is given by:

$$\delta S_P = \frac{T_2}{2}\int d^3\sigma \Big[\delta\dot{Y}^\mu\,\delta\dot{Y}_\mu + \delta\dot{X}^i\,\delta\dot{X}^i - \{Y_0^\mu, Y_0^\nu\}\{\delta Y_\mu, \delta Y_\nu\} - \{\delta Y^\mu, Y_0^\nu\}\{\delta Y_\mu, Y_{0\,\nu}\} -$$

$$-\{\delta Y^\mu, Y_0^\nu\}\{Y_{0\,\mu}, \delta Y_\nu\} - \{X_0^i, X_0^j\}\{\delta X^i, \delta X^j\} - \{\delta X^i, X_0^j\}\{\delta X^i, X_0^j\} -$$

$$-\{\delta X^i, X_0^j\}\{X_0^i, \delta X^j\} - 2\{Y_0^\mu, X_0^i\}\{\delta Y_\mu, \delta X^i\} - \{\delta Y^\mu, X_0^i\}\{\delta Y_\mu, X_0^i\} -$$

$$-2\{\delta Y^\mu, X_0^i\}\{Y_{0\,\mu}, \delta X^i\} - \{Y_0^\mu, \delta X^i\}\{Y_{0\,\mu}, \delta X^i\} + 2\,Y_0^\mu\,\delta Y_\mu\,\delta\Lambda +$$

$$+2\,X_0^i\,\delta X^i\,\delta\widetilde{\Lambda}\Big]. \tag{17.11}$$

To lowest order, the fluctuations obey the following equations:

$$\delta\ddot{Y}^\mu = \{\{Y_0^\mu, Y_0^\nu\}, \delta Y_\nu\} + \{\{\delta Y^\mu, Y_0^\nu\}, Y_{0\,\nu}\} + \{\{Y_0^\mu, \delta Y^\nu\}, Y_{0\,\nu}\} + \big\{\{Y_0^\mu, X_0^i\}, \delta X^i\big\} +$$

$$+\big\{\{\delta Y^\mu, X_0^i\}, X_0^i\big\} + \big\{\{Y_0^\mu, \delta X^i\}, X^i\big\} + \Lambda_0\delta Y^\mu + Y_0^\mu\,\delta\Lambda \tag{17.12}$$

$$\delta\ddot{X}^i = \Big\{\Big\{X_0^i, X_0^j\Big\}, \delta X_j\Big\} + \Big\{\Big\{\delta X^i, X_0^j\Big\}, X_0^j\Big\} + \Big\{\{X_0^i, \delta X^j\}, X_0^j\Big\} + \{\{X_0^i, Y_0^\mu\}, \delta Y_\mu\} +$$

$$+\big\{\{\delta X^i, Y_0^\mu\}, Y_{0\,\mu}\big\} + \big\{\{X_0^i, \delta Y^\mu\}, Y_{0\,\mu}\big\} + \widetilde{\Lambda}_0\delta X^i + X_0^i\,\delta\widetilde{\Lambda} \tag{17.13}$$

and the constraints:

$$Y_0^\mu\,\delta Y_\mu = X_0^i\,\delta X^i = 0 \quad , \quad \dot{Y}_0^\mu\,\partial_\sigma\delta Y_\mu + \delta\dot{Y}^\mu\,\partial_\sigma Y_{0\,\mu} + \dot{X}_0^i\,\partial_\sigma\delta X^i + \delta\dot{X}^i\,\partial_\sigma X_0^i = 0$$

$$\dot{Y}_0^\mu\,\partial_\delta\delta Y_\mu + \delta\dot{Y}^\mu\,\partial_\delta Y_{0\,\mu} + \dot{X}_0^i\,\partial_\delta\delta X^i + \delta\dot{X}^i\,\partial_\delta X_0^i = 0 \tag{17.14}$$

$$\dot{Y}_0^\mu\,\delta\dot{Y}_\mu + \dot{X}_0^i\,\delta\dot{X}^i + \{Y_0^\mu, Y_0^\nu\}\{\delta Y_\mu, Y_{0\,\nu}\} + \{X_0^i, X_0^j\}\{\delta X^i, X_0^j\} + \{Y_0^\mu, X_0^i\}\{\delta Y_\mu, X_0^i\} +$$

$$+\{Y_0^\mu, X_0^i\}\{Y_{0\,\mu}, \delta X^i\} = 0. \tag{17.15}$$

Stringy membranes in $\mathrm{AdS}_{p+2}\times\mathrm{S}^q$ have:

$$Y_0^\mu = Y_0^\mu\,(\tau, \sigma) \tag{17.16}$$

$$X_0^i = (\cos\delta\,,\,\sin\delta\,,\,0\,,\,\ldots\,,\,0) \;\longrightarrow\; X_0^i X_0^i = 1 \tag{17.17}$$

$$X_0^{i\,\prime} = (-\sin\delta\,,\,\cos\delta\,,\,0\,,\,\ldots\,,\,0) \;\longrightarrow\; X_0^{i\,\prime} X_0^{i\,\prime} = 1 \tag{17.18}$$

$$X_0^{i\,\prime\prime} = -\left(\cos\delta\,,\,\sin\delta\,,\,0\,,\,\ldots\,,\,0\right) = -X_0^i \;\longrightarrow\; X^{i\,\prime\prime}X^{i\,\prime\prime} = 1. \tag{17.19}$$

Plugging (17.16)–(17.19) into the equations of motion and constraints of the solutions (17.4)–(17.8) and the equations of motion and constraints of the fluctuations (17.12)–(17.15), we obtain the following system of equations (setting $R = 1$):

$$\ddot{Y}_0^\mu = Y_0^{\mu\,\prime\prime} + \Lambda_0\,Y_0^\mu \quad , \quad Y_0^{\mu\,\prime}Y_{0\,\mu}' = -\dot{Y}_0^\mu\dot{Y}_{0\,\mu} = \widetilde{\Lambda}_0 = -\ell^2/2\,\Lambda_0 \tag{17.20}$$

$$Y_0^\mu Y_{0\,\mu} = -\ell^2 \quad , \quad \dot{Y}_0^\mu Y_{0\,\mu}' = 0, \tag{17.21}$$

fluctuation equations,

$$\begin{aligned}\delta\ddot{Y}^\mu =& \partial_\sigma^2\delta Y^\mu + \widetilde{\Lambda}_0\,\partial_\delta^2\delta Y^\mu - \left(X_0^{i\,\prime\prime}\,\partial_\sigma\delta X^i - X_0^{i\,\prime}\,\partial_{\sigma,\delta}^2\delta X^i + Y_0^{\nu\,\prime}\,\partial_\delta^2\delta Y_\nu\right)Y_0^{\mu\,\prime} + \\ &+2\left(X_0^{i\,\prime}\,\partial_\delta\delta X^i\right)Y_0^{\mu\,\prime\prime} + \Lambda_0\,\delta Y^\mu + Y_0^\mu\,\delta\Lambda\end{aligned} \tag{17.22}$$

$$\begin{aligned}\delta\ddot{X}^i =& \partial_\sigma^2\delta X^i + \widetilde{\Lambda}_0\,\partial_\delta^2\delta X^i - \left(X_0^{j\,\prime}\,\partial_\sigma^2\delta X^j + Y_0^{\mu\,\prime\prime}\,\partial_\delta\delta Y_\mu - Y_0^{\mu\,\prime}\,\partial_{\sigma,\delta}^2\delta Y_\mu\right)X_0^{i\,\prime} + \\ &+2\left(Y_0^{\mu\,\prime}\,\partial_\sigma\delta Y_\mu\right)X_0^{i\,\prime\prime} + \widetilde{\Lambda}_0\,\delta X^i + X_0^i\,\delta\widetilde{\Lambda}\end{aligned} \tag{17.23}$$

and constraints:

$$Y_0^\mu\,\delta Y_\mu = X_0^i\,\delta X^i = 0\,,\; \dot{Y}_0^\mu\,\partial_\sigma\delta Y_\mu + \delta\dot{Y}^\mu\,Y_{0\,\mu}' = \dot{Y}_0^\mu\,\partial_\delta\delta Y_\mu + \delta\dot{X}^i\,X_0^{i\,\prime} = 0 \tag{17.24}$$

$$\dot{Y}_0^\mu\,\delta\dot{Y}_\mu + Y_0^{\mu\,\prime}\,\partial_\sigma\delta Y_\mu + \widetilde{\Lambda}_0\,\left(X_0^{i\,\prime}\,\partial_\delta\delta X^i\right) = 0. \tag{17.25}$$

It is interesting to note that, although the equations of motion (17.20)–(17.21) do not explicitly depend on the world-volume coordinate $\delta = \sigma_2$ (they are string equations), the fluctuation equations (17.22)–(17.25) depend explicitly on the world-volume coordinate δ through the S^4 coordinates $X^i\,(\delta)$ and their derivatives. Since no coordinate transformation that eliminates the δ-dependence from the fluctuation equations (17.22)–(17.25) exists, we conclude that stringy membranes are equivalent to strings only up to leading order.

In what follows, only the fluctuations along the directions that are transverse to the membrane will be examined, i.e. the fluctuations for which $Y_0^\mu = X_0^i = 0$. These fluctuations are easier to study, since they decouple from the ones that lie along the parallel directions of the stringy membrane, as can be seen from (17.22)–(17.25). The fluctuation equations then become:

$$\delta\ddot{Y}^\mu = \partial_\sigma^2\delta Y^\mu + \widetilde{\Lambda}_0\,\partial_\delta^2\delta Y^\mu + \Lambda_0\,\delta Y^\mu \tag{17.26}$$

$$\delta\ddot{X}^i = \partial_\sigma^2\delta X^i + \widetilde{\Lambda}_0\,\partial_\delta^2\delta X^i + \widetilde{\Lambda}_0\,\delta X^i. \tag{17.27}$$

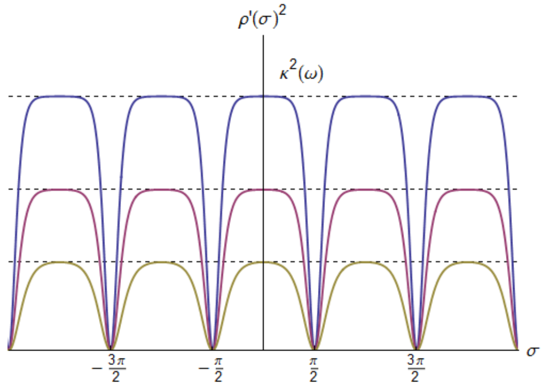


Figure 23: Plot of Lamé's potential (17.33) of the stringy membrane (16.1)–(17.31).

17.1 Rotating Stringy Membranes

To study the transverse fluctuations of rotating stringy membranes we set:

$$\delta Y^{\mu} = \sum_{r,m} e^{ir\tau + im\delta}\, \widetilde{y}^{\mu}_{r,m}(\sigma)\;, \qquad \delta X^{i} = \sum_{r,m} e^{ir\tau + im\delta}\, \widetilde{x}^{i}_{r,m}(\sigma)\;, \qquad m \in \mathbb{Z}. \tag{17.28}$$

If we plug (17.28) into (17.26)–(17.27), then the corresponding equations along the transverse directions $Y_0^{\mu} = X_0^{i} = 0$ take the following form (for simplicity, we omit the dependencies of $\widetilde{y}^{\mu}_{r,m}(\sigma)$ and $\widetilde{x}^{i}_{r,m}(\sigma)$ on r, m and σ):

$$\left(\widetilde{y}^{\mu}\right)'' + \left(r^2 - m^2\widetilde{\Lambda}_0 + \Lambda_0\right)\widetilde{y}^{\mu} = 0 \tag{17.29}$$

$$\left(\widetilde{x}^{i}\right)'' + \left(r^2 - m^2\widetilde{\Lambda}_0 + \widetilde{\Lambda}_0\right)\widetilde{x}^{i} = 0. \tag{17.30}$$

Now consider the $\text{AdS}_7 \times \text{S}^4$ stringy membranes (16.1) for which ($\ell = 2$),[76]

$$Y_0^{\mu} = 2\left(\cosh\rho(\sigma)\cos\kappa\tau\,,\, \sinh\rho(\sigma)\cos\kappa\omega\tau\,,\, \sinh\rho(\sigma)\sin\kappa\omega\tau\,,\, 0\,,\, 0\,,\, 0\,,\, 0\,,\, \cosh\rho(\sigma)\sin\kappa\tau\right). \tag{17.31}$$

The Lagrange multipliers Λ_0 and $\widetilde{\Lambda}_0$, for the stringy membrane (17.31) are given by:

$$\Lambda_0 = -2\rho'^2 \quad \& \quad \widetilde{\Lambda}_0 = 4\rho'^2, \tag{17.32}$$

where $\rho'(\sigma)^2$ is a σ-periodic, even function[77] (plotted for various ω's in figure 23) that is given by:

$$\rho'^2 = \kappa^2\left(\cosh^2\rho - \omega^2\sinh^2\rho\right) = \kappa^2 \cdot sn^2\left[\kappa\omega\left(\sigma + \frac{\pi}{2}\right)\Big|\frac{1}{\omega^2}\right] \tag{17.33}$$

$$\omega\cdot\kappa(\omega) = \frac{2}{\pi}\cdot\mathbb{K}\left(\frac{1}{\omega^2}\right) \quad , \quad \omega^2 > 1.$$

The fluctuation equations (17.29)–(17.30) along the transverse directions $Y^{\mu} = X^{i} = 0$, can be shown

[76]It is straightforward to extend the results of this section to stringy membranes in $\text{AdS}_4 \times \text{S}^7/\mathbb{Z}^k$. See table 2.

[77]Note that for large ω's, we can make the approximation $\rho'^2 = \kappa^2 \cdot cd^2\left[\kappa\omega\sigma\big|1/\omega^2\right] \sim \kappa^2\cos^2\sigma$.

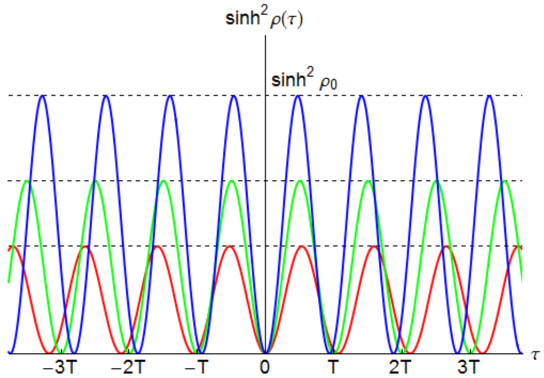


Figure 24: Plot of the Lamé potential (17.39) of the stringy membrane (16.10)–(17.38).

to obey the Jacobi form of Lamé's equation,

$$\frac{d^2 z}{du^2} + \left[h - \nu\left(\nu+1\right) k^2 sn^2\left(u|k^2\right)\right] z = 0, \tag{17.34}$$

provided that we set:

$$z = \widetilde{y}^{\mu}(\sigma)\ ,\ u = \kappa\omega\left(\sigma + \frac{\pi}{2}\right)\ ,\ h = \left(\frac{r}{\kappa\omega}\right)^2\ ,\ \nu\left(\nu+1\right) = 2\left(2\,m^2+1\right)\ ,\ k = \frac{1}{\omega}$$
$$z = \widetilde{x}^{i}(\sigma)\ ,\ u = \kappa\omega\left(\sigma + \frac{\pi}{2}\right)\ ,\ h = \left(\frac{r}{\kappa\omega}\right)^2\ ,\ \nu\left(\nu+1\right) = 4\left(m^2-1\right)\ ,\ k = \frac{1}{\omega}.$$

17.2 Pulsating Stringy Membranes

To study the transverse fluctuations of pulsating stringy membranes we set:

$$\delta Y^{\mu} = \sum_{m,n} e^{in\sigma + im\delta}\ \widetilde{y}^{\mu}_{m,n}\left(\tau\right)\ , \qquad \delta X^{i} = \sum_{m,n} e^{in\sigma + im\delta}\ \widetilde{x}^{i}_{m,n}\left(\tau\right)\ , \qquad m \in \mathbb{Z}, \tag{17.35}$$

then the equations for the transverse fluctuations (17.26)–(17.27), take the following form (again we have omitted the dependencies of $\widetilde{y}^{\mu}_{n,m}\left(\tau\right)$ and $\widetilde{x}^{i}_{n,m}\left(\tau\right)$ on n, m and τ):

$$\ddot{\widetilde{y}}^{\mu} + \left(n^2 + m^2\widetilde{\Lambda}_0 - \Lambda_0\right)\widetilde{y}^{\mu} = 0 \tag{17.36}$$

$$\ddot{\widetilde{x}}^{i} + \left(n^2 + m^2\widetilde{\Lambda}_0 - \widetilde{\Lambda}_0\right)\widetilde{x}^{i} = 0. \tag{17.37}$$

Consider the $\mathrm{AdS}_7 \times \mathrm{S}^4$ pulsating configuration (16.10) ($\ell = 2$):

$$Y^{\mu}_0 = 2\left(\cosh\rho(\tau)\cos t\left(\tau\right)\ ,\ 0\ ,\ 0\ ,\ \sinh\rho(\tau)\cos\sigma\ ,\ 0\ ,\ \sinh\rho(\tau)\sin\sigma\ ,\ 0\ ,\ \cosh\rho(\tau)\sin t\left(\tau\right)\right). \tag{17.38}$$

If we solve the equations of motion (17.20)–(17.21), the following Lamé potential is obtained:

$$\sinh^2\rho\left(\tau\right) = \sinh^2\rho_0 \cdot sn^2\left[\tau\cdot\cosh\rho_0\,\Big|\, -\tanh^2\rho_0\right], \tag{17.39}$$

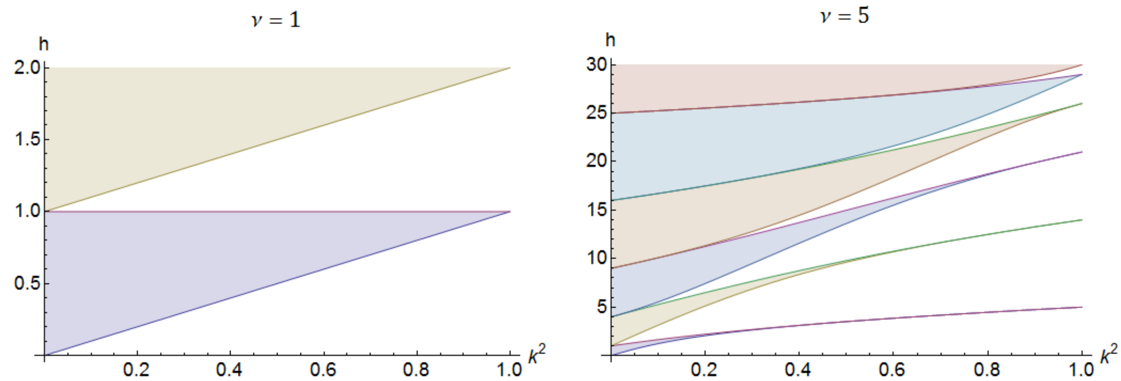


Figure 25: Stability bands (in color) of Lamé's equation (17.34) for $\nu = 1$ (left) and $\nu = 5$ (right).

where ρ_0 can be found from $4e^2 = \sinh^2 2\rho_0$ and e was defined in equation (6.86). The Lamé potential (17.39) has been plotted for various values of ρ_0 in figure 24. The Lagrange multipliers Λ_0 and $\widetilde{\Lambda}_0$ are found to be:

$$\Lambda_0 = -2\sinh^2\rho \quad \& \quad \widetilde{\Lambda}_0 = 4\sinh^2\rho. \tag{17.40}$$

The fluctuations along the transverse directions $Y^\mu = X^i = 0$ (17.36)–(17.37), again obey Lamé's equation (17.34). To bring the latter to the Jacobi form, we write the potential (17.39) as follows:

$$\sinh^2\rho\,(\tau) = \sinh^2\rho_0 \cdot \left(1 - sn^2\left[\tau\cdot\sqrt{\cosh 2\rho_0} + \mathbb{K}\left(\frac{\sinh^2\rho_0}{\cosh 2\rho_0}\right)\,\Big|\,\frac{\sinh^2\rho_0}{\cosh 2\rho_0}\right]\right) \tag{17.41}$$

and substitute $u = \tau\cdot\sqrt{\cosh 2\rho_0} + \mathbb{K}\left(k^2\right)$ and

$$z = \widetilde{y}^\mu(\tau)\,,\; h = \frac{n^2}{\cosh 2\rho_0} + 2k^2\left(2\,m^2+1\right)\,,\; \nu\,(\nu+1) = 4m^2+2\,,\; k = \frac{\sinh\rho_0}{\sqrt{\cosh 2\rho_0}}$$

$$z = \widetilde{x}^i(\tau)\,,\; h = \frac{n^2}{\cosh 2\rho_0} + 4k^2\left(m^2-1\right)\,,\; \nu\,(\nu+1) = 4m^2-4\,,\; k = \frac{\sinh\rho_0}{\sqrt{\cosh 2\rho_0}},$$

in equation (17.34).

We have found that the transverse fluctuations ($Y_0^\mu = X_0^i = 0$) of stringy membranes in $\text{AdS}_7\times\text{S}^4$ (16.1)–(16.10) fall under Lamé's equation:

$$\frac{d^2z}{du^2} + \left[h - \nu\,(\nu+1)\,k^2 sn^2\left(u|k^2\right)\right] z = 0. \tag{17.42}$$

As it is explained in appendix K, when $\nu\,(\nu+1)\in\mathbb{R}$ and $0<k<1$, Lamé's equation (17.42) always has an infinite set of real eigenvalues $a_\nu^s\left(k^2\right)$ and $b_\nu^s\left(k^2\right)$ that correspond to the equation's periodic eigenfunctions.[78] The Lamé eigenvalues can be classified into four groups, according to the parity (even or odd) and the period (equal to $2\mathbb{K}$ or $4\mathbb{K}$) of the corresponding eigenfunctions. For a generic Lamé eigenvalue h (not necessarily of a periodic eigenfunction), Lamé's equation (17.42) is stable if

[78]Note also that Lamé's equation (17.42) is symmetric under the exchange $\nu \leftrightarrow -\nu-1$, so that we only need to consider $\nu > -1/2$ and $\nu(\nu+1) > -1/4$.

Ansatz	u	k	h	z	$\nu(\nu+1)$
(16.1) $\mathrm{AdS}_7 \times \mathrm{S}^4$	$\kappa\omega\left(\sigma+\frac{\pi}{2}\right)$	$\frac{1}{\omega}$	$\left(\frac{r}{\kappa\omega}\right)^2$	$\widetilde{y}$ $\widetilde{x}$	$4\,m^2+2$ $4\left(m^2-1\right)$
(16.1) $\mathrm{AdS}_4 \times \mathrm{S}^7$	$\kappa\omega\left(\sigma+\frac{\pi}{2}\right)$	$\frac{1}{\omega}$	$\left(\frac{r}{\kappa\omega}\right)^2$	$\widetilde{y}$ $\widetilde{x}$	$m^2/4+2$ $\frac{1}{4}\left(m^2-1\right)$
(16.10) $\mathrm{AdS}_7 \times \mathrm{S}^4$	$\tau\cdot\sqrt{\cosh 2\rho_0}+\mathbb{K}\left(k^2\right)$	$\frac{\sinh\rho_0}{\sqrt{\cosh 2\rho_0}}$	$\frac{n^2}{\cosh 2\rho_0}+k^2\left(4\,m^2+2\right)$ $\frac{n^2}{\cosh 2\rho_0}+4k^2\left(m^2-1\right)$	$\widetilde{y}$ $\widetilde{x}$	$4\,m^2+2$ $4\left(m^2-1\right)$
(16.10) $\mathrm{AdS}_4 \times \mathrm{S}^7$	$\tau\cdot\sqrt{\cosh 2\rho_0}+\mathbb{K}\left(k^2\right)$	$\frac{\sinh\rho_0}{\sqrt{\cosh 2\rho_0}}$	$\frac{n^2}{\cosh 2\rho_0}+k^2\left(m^2/4+2\right)$ $\frac{n^2}{\cosh 2\rho_0}+\frac{k^2}{4}\left(m^2-1\right)$	$\widetilde{y}$ $\widetilde{x}$	$m^2/4+2$ $\frac{1}{4}\left(m^2-1\right)$

Table 2: Lamé fluctuation parameters (17.42) for stringy membranes (16.1)–(16.10) in $\mathrm{AdS}_{7/4} \times \mathrm{S}^{4/7}$.

and only if (iff) all corresponding eigenfunctions $z\,(u,h)$ are bounded. Otherwise the equation is unstable. The intervals of stability are determined by the eigenvalues of the equation's periodic solutions which are ordered as follows:

$$(a_\nu^0\,,\,a_\nu^1)\,\cup\,(b_\nu^1\,,\,b_\nu^2)\,\cup\,(a_\nu^2\,,\,a_\nu^3)\,\cup\,(b_\nu^3\,,\,b_\nu^4)\,\cup\,\ldots \tag{17.43}$$

The solutions of Lamé's equation are stable inside the above intervals and unstable outside them. The contraction symbols under adjacent eigenvalues mean that the relative order of the two consecutive eigenvalues is not generally known and may well be the opposite one for different values of $\nu \in \mathbb{R}$, $s = 0, 1, 2, \ldots$ and $k \in (0,1)$. The Lamé eigenvalues have another interesting property that is known as "coexistence". The property of coexistence implies that $\nu \in \mathbb{N}$ iff Lamé's equation has exactly $\nu+1$ intervals of stability (bands) that follow exactly $\nu+1$ intervals of instability (gaps). The plot of the Lamé bands (colored) and gaps (white) for $\nu = 1$ and $\nu = 5$ can be found in figure 25.

Summing up, the stability of Lamé solutions is organized in (stable) bands and (unstable) gaps. The parameters of Lamé's equation (17.42), for each of the stringy membrane ansätze (17.31)–(17.38), are given in table 2 (for the definitions of m, r and n, see (17.28)–(17.35)). It is rather straightforward to extend our results from $\mathrm{AdS}_7 \times \mathrm{S}^4$ to $\mathrm{AdS}_4 \times \mathrm{S}^7$ (where $\mathfrak{k} = \ell/R = 1/2$ and $\Lambda_0 = -8\widetilde{\Lambda}_0$). Table 2 includes both cases. The data for the $\mathrm{AdS}_{7/4}$ fluctuations $\widetilde{y} \equiv \{\widetilde{y}^\mu_{r,m}(\sigma)\,,\,\widetilde{y}^\mu_{m,n}(\tau)\}$ occupy the first row of each entry, while the second row contains the data for the fluctuations on $\mathrm{S}^{4/7}$, $\widetilde{x} \equiv \left\{\widetilde{x}^i_{r,m}(\sigma)\,,\,\widetilde{x}^i_{m,n}(\tau)\right\}$. Given ω, ρ_0, and $m \in \mathbb{Z}$ ($\kappa = \kappa(\omega) = 2/\pi\omega\cdot\mathbb{K}\left(1/\omega^2\right)$), the allowed values of r, $n \in \mathbb{R}$ are determined by the overlap of the $\widetilde{y}$ and $\widetilde{x}$ bands, the lowest endpoint of which satisfies:

$$h_{\min} \geq 0, \text{ in the ansatz (16.1) \& } h_{\min} \geq \left(4\,m^2+2\right)\frac{\sinh^2\rho_0}{\cosh 2\rho_0}, \text{ in the ansatz (16.10) } (\mathrm{AdS}_7\times\mathrm{S}^4)$$

$$h_{\min} \geq \left(m^2/4+2\right)\frac{\sinh^2\rho_0}{\cosh 2\rho_0}, \text{ in the ansatz (16.10) } (\mathrm{AdS}_4\times\mathrm{S}^7). \tag{17.44}$$

18 Part IV Summary and Discussion

The final part IV of this thesis (§14–§17) was devoted to the study of membranes from the perspective of the AdS/CFT correspondence. After some rudiments of p-branes and M-theory, we introduced the concept of the "stringy membrane" and studied some simple stringy membrane configurations in $AdS_7 \times S^4$ and $AdS_4 \times S^7/\mathbb{Z}_k$. In order to learn about the stringy properties of uncharged classical bosonic membranes in $AdS_m \times S^n$ spacetimes, we have asked ourselves the question what are the basic premises that allow us to embed the string sigma model in $AdS_5 \times S^5$ into the membrane sigma model in $AdS_{4/7} \times S^{7/4}$. We found that all string configurations inside AdS_5 (which is the non-compact part of $AdS_5 \times S^5$) may be reproduced by membranes that live inside $AdS_7 \times S^4$. Moreover, the behavior of any string configuration living inside $AdS_4 \subset AdS_5$ may be reproduced by a membrane of $AdS_4 \times S^7$.

The construction of stringy membranes in $AdS_m \times S^n$ is extremely simple. Two basic ingredients are needed in order to define stringy membranes: a compact and a non-compact counterpart in the background. The two world-volume coordinates of the stringy membrane are shared between the two component manifolds, so that the configuration is essentially one-dimensional in each of the two product spaces. Although the gauge-fixed Polyakov action of bosonic membranes (15.3) has a completely different structure than the corresponding string action (5.3), we prove that the former may reduce to the latter when the membrane coordinates are shared between two product spaces and the coordinate of the compact space is static. This treatment is in many ways very reminiscent of that of Duff, Howe, Inami and Stelle in [223], although our motivation is somewhat closer to the papers [149, 245, 246]. Apart from studying only bosonic membranes in $AdS_m \times S^n$, at no point did we perform a double dimensional reduction (DDR) à la [223]. Our aim was to reproduce the behavior of the GKP string from the membrane point of view. Other papers with similar considerations are [247].

The stability of stringy membranes in the linearized approximation was examined in §17. One important outcome stemming from the stability analysis, is that the similarities between stringy membranes and strings cannot be extended beyond the leading order. This is due to the fact that the perturbation equations of stringy membranes depend on the second world-volume coordinate δ which cannot be eliminated. In the same context, it was also found that the stability of stringy membranes along their transverse directions is governed by Lamé's equation. As a consequence, stringy membranes exhibit the standard stability/instability pattern of bands and gaps that is a commonplace property of the Lamé spectrum. The typical single-band/single-gap structure of classical bosonic strings in AdS_3 [81, 105] is recovered from stringy membranes as a special limiting case (entry $m = 0$ in table 2). The Lamé structure gives rise to various interesting issues of interpretation for both strings and membranes. One is whether the Lamé band/gap structure of AdS strings and membranes affords an interpretation as explosive particle production, in close analogy with the parametric resonance phenomenon that is encountered in post-inflationary universe. Secondly, we can ask what is the holographic dual, as well as what is the meaning of the Lamé band/gap structure from the point of view of the dual SCFT.

Part IV will conclude with a discussion of our results on stringy membranes along with some interesting further prospects on diverse emerging problems.

• *Scaling dimensions and stringy membranes.*

The stringy membrane (16.1) is essentially the same with the "type-I" solution of Hartnoll and Nuñez in $AdS_4 \times S^7$ [245], expressed in terms of the conformal gauge Polyakov action in $AdS_7 \times S^4$ (see §16.2–§16.3 for $AdS_4 \times S^7/\mathbb{Z}_k$). In §6 we saw that the folded closed GKP string (I) of AdS_3 is dual to the twist-2 operators $\mathrm{Tr}\big[\mathcal{Z}\,\mathcal{D}_+^S\,\mathcal{Z}\big]$ of the $\mathfrak{sl}(2)$ sector of $\mathcal{N} = 4$ SYM. Therefore, in complete agreement with GKP [11] and Hartnoll–Nuñez [245], the stringy membrane (16.1) is expected to be dual to the above twist-2 operators, given also in (6.2). The classical part of the corresponding anomalous scaling dimensions will then be given by (6.30) for small values of the spin S and by (7.5)–(7.95) for large spins S:

$$E^2 = 2\sqrt{\lambda'}\, S + \ldots \quad \left(\text{Short Stringy Membranes, } S \ll \sqrt{\lambda'}\right) \tag{18.1}$$

$$E - S = f\left(\lambda'\right) \ln \frac{S}{\sqrt{\lambda'}} + \ldots \quad \left(\text{Long Stringy Membranes, } S \gg \sqrt{\lambda'}\right). \tag{18.2}$$

where S equals the stringy membrane charge $S_1 = S^{12}$ of (15.9) and the effective coupling constant λ' is defined as $\sqrt{\lambda'} \equiv R\,\ell^2/g_s\ell_s^3$.

The full classical "short" series (18.1) has been obtained in §6.1.1, see equations (6.29)–(6.30). The classical part of the "long" series (18.2) was the subject of the whole section §7.2, where it was explicitly shown how to calculate the series terms with the Lambert W-function. In §6.1.3 we proved a formula (originally proposed in [12]) that links the classical "short" and "long" scaling dimensions (18.1)–(18.2). In §7.3 we showed that the terms of the long series (7.5)–(7.95) satisfy the Moch-Vermaseren-Vogt (MVV) relations that follow from the property of "reciprocity" or parity-preservation. Reciprocity was originally proposed by Gribov and Lipatov [110] in the context of deep inelastic scattering (DIS) and it has been verified for twist-2 operators up to three loops in perturbative QCD [109] and up to four loops in weakly coupled $\mathcal{N} = 4$ SYM [108, 248]. Naturally, all of the above statements are expected to carry over to stringy membranes.

Conversely, it turns out that the above statements cannot be extended to the quantum level. The "cusp anomalous dimension" $f(\lambda)$ receives quantum corrections that we may compute in superstring theory by evaluating the Lamé fluctuation determinants, as in [105]. However, the quadratic supermembrane sigma model on $\text{AdS}_{7/4} \times \text{S}^{4/7}$ is completely different from the corresponding superstring model. This picture was confirmed in §17 of the present thesis, where the transverse fluctuations of the AdS_3 stringy membranes were studied and were found to have a much richer Lamé band/gap structure than the corresponding GKP strings. Therefore, we generally expect that the quantum corrections to the anomalous dimensions of twist-2 operators that are dual to $\text{AdS}_{7/4} \times \text{S}^{4/7}$ stringy membranes, will be different from the quantum corrections of the corresponding GKP strings.

• *Integrability.*

The main result of §16 was that all the classical strings of AdS_5 can be reproduced by a stringy membrane of $\text{AdS}_7 \times \text{S}^4$ and all the classical strings of AdS_4 can be reproduced by a stringy membrane of $\text{AdS}_4 \times \text{S}^7/\mathbb{Z}_k$. As we have seen in §5.2 the classical string sigma model on $\text{AdS}_{2/3/4}$ is equivalent to the Liouville, sinh-Gordon and B_2-Toda equation respectively. Therefore, the stringy membranes that reproduce the classical strings of $\text{AdS}_{2/3/4}$ are expected to be classically equivalent to the Liouville, sinh-Gordon and B_2-Toda model respectively.

Our analysis also has important consequences for the dual gauge theories. The fact that two or more completely different bulk theories have excitations with similar spectra, implies that the dual SCFTs (however different they may be, e.g. they may have different dimensionalities) ought to include sectors with a similar underlying structure. Take for example the GKP string (I) that rotates in AdS_3. We saw in §7 that the GKP string (I) is dual to twist-2 operators and its energy that is equal to the operator scaling dimensions scales like the logarithm of the string's spin. The fact that we can find a stringy membrane of a different bulk theory that is dual to a different SCFT than the original GKP string but both nevertheless have the same dispersion relations (7.5)–(18.2), implies that the two SCFTs contain the same twist-2 operators, with the same spectra. In sum, we have shown that the following gauge/gravity dualities contain states/operators with anomalous dimensions that scale like $\Delta - S \sim \ln S$:

Gauge Theory	dual Gravity Theory
$\mathcal{N}=4$ $\mathfrak{su}(N)$ Super Y-M Theory	IIB String Theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$
$\mathcal{N}=8$ SCFT $\big/$ $A_{N-1}(2,0)$ SCFT	M-Theory on $\mathrm{AdS}_{4/7} \times \mathrm{S}^{7/4}$
$\mathcal{N}=6$ $U(N)_k \times U(N)_{-k}$ Super C-S Theory	
$N \to \infty$	M-Theory on $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$
$k^5 \gg N \to \infty$, $\lambda \equiv 2\pi^2 N/k = \text{const.}$	IIA String Theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$

The study of stringy membranes seems to strengthen the following conjecture that was put forward by Bozhilov in [201]. The SCFTs:

(a) $\mathcal{N}=4$ $\mathfrak{su}(N)$ SYM theory (dual to IIB String Theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$)

(b) $A_{N-1}(2,0)$ SCFT (dual to M-theory on $\mathrm{AdS}_7 \times \mathrm{S}^4$)

(c) $\mathcal{N}=8$ SCFT (dual to M-theory on $\mathrm{AdS}_4 \times \mathrm{S}^7$),

might all possess common integrable sectors. Stringy membranes further imply that the above "family" could contain more members (e.g. QCD, $\mathcal{N}=6$ quiver Super Chern-Simons, $\mathcal{N}=1$ SYM [245, 249], etc.). A similar result is that $\mathcal{N}=0,1,2,4$ SYM theories possess a common universal one-loop dilatation operator [250]. Analogous considerations are presently being put forward by the QSC (quantum spectral curve) community, where a "mysterious relation" between the integrable structures of ABJM and $\mathcal{N}=4$ SYM theories has been reported [251]. Elli Pomoni is also currently putting forward very interesting observations in the same direction [252].

• *Possible generalizations.*

We finish this section with some further remarks. We have tried to think of a general argument that demonstrates that all the (super-) string theories that can be formulated on AdS_5 as well as the corresponding sector of its dual $\mathcal{N}=4$ SYM, may respectively be embedded in (super-) membrane theory in $\mathrm{AdS}_7 \times \mathrm{S}^4$ and its dual SCFT. However it is known that double dimensional reduction (DDR) [223] is generally impossible in the case

$$\Big\{\text{membranes}/\mathrm{AdS}_{4/7} \times \mathrm{S}^{7/4}\Big\} \longrightarrow \big\{\text{strings}/\mathrm{AdS}_5 \times \mathrm{S}^5\big\},$$

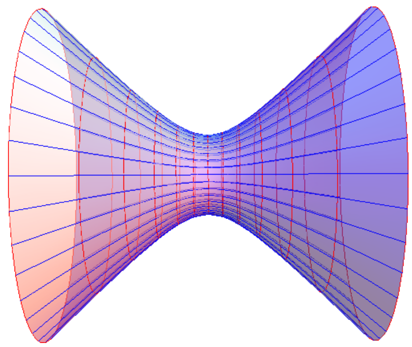
therefore at no circumstances should we expect that string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is embeddable in M-theory on $\mathrm{AdS}_{4/7} \times \mathrm{S}^{7/4}$. This doesn't mean that the results of Duff, Howe, Inami and Stelle [223] cannot be applied to $\mathrm{AdS}_{4/7} \times \mathrm{S}^{7/4}$. There could exist certain embeddings of the full Green-Schwarz action on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [38, 253] into the full supermembrane action on $\mathrm{AdS}_{4/7} \times \mathrm{S}^{7/4}$ [213] that are allowed. It would be interesting to investigate the degree up to which this is true.

Finally, we could attempt to study more rigorously the functional difference of the membrane and the string Polyakov actions $S_2 - S_1$, in more complicated situations. Mathematically, it should be possible to prove that any membrane configuration can be obtained by considering a higher-dimensional extended object (e.g. a 3-brane or a 5-brane) that lives in a higher-dimensional spacetime. More generally, any p-brane solution that lives in AdS_m should be obtainable from a $(p+1)$-brane that lives in $\mathrm{AdS}_{m'} \times \mathrm{S}^{m+n+1-m'}$ or from a $(p+q)$-brane that lives in a higher-dimensional spacetime.

Part V

Appendixes

A Anti-de Sitter Space



In this appendix we review the basic properties of AdS spacetimes.[79] Anti-de Sitter space in $p+2$ dimensions (denoted as AdS_{p+2}) consists of the hyperboloid

$$-\eta_{\mu\nu}Y^\mu Y^\nu = Y_0^2 - \sum_{i=1}^{p+1} Y_i^2 + Y_{p+2}^2 = \ell^2, \qquad \eta_{\mu\nu} = (-,+,\ldots,+,-)\,, \tag{A.1}$$

isometrically embedded in flat $p+3$ dimensional spacetime:

$$ds^2 = \eta_{\mu\nu}dY^\mu dY^\nu = -dY_0^2 + \sum_{i=1}^{p+1} dY_i^2 - dY_{p+2}^2. \tag{A.2}$$

Anti-de Sitter space is a maximally symmetric solution of Einstein's equations in vacuum, with a (negative) cosmological constant Λ:

$$S = \frac{1}{2\kappa_{d+1}^2}\int d^{d+1}x\sqrt{-g}\,(R-2\Lambda) \quad \longrightarrow \quad R_{mn} - \frac{R}{2}\,g_{mn} + \Lambda\,g_{mn} = 0. \tag{A.3}$$

Maximally symmetric spaces (or spaces of constant curvature) enjoy a number of very appealing features:[80]

1. Their metric admits the maximum number of bosonic/Killing symmetries.

2. They are homogeneous and isotropic about their every point.

3. They are uniquely characterized by their (constant) curvatures R.

All maximally symmetric spaces in $D = d+1 = p+2$ dimensions are conformally flat (i.e. they have a vanishing Weyl tensor), Einstein spaces (i.e. their Ricci tensor is proportional to the metric tensor). Their basic metric properties are:

$$R_{mnrs} = \frac{R}{d\,(d+1)}\,(g_{mr}g_{ns} - g_{nr}g_{ms}) \quad \Rightarrow \quad W_{mnrs} = 0 \tag{A.4}$$

[79] Two basic references that we follow are [254, 255].

[80] Weinberg's book [256] contains a complete discussion of maximally symmetric spaces.

$$R_{mn} = \frac{R}{d+1}\, g_{mn} \tag{A.5}$$

$$R = -d\,(d+1)\,K = \text{constant}, \quad \Lambda = \frac{d-1}{2\,(d+1)}\,R, \tag{A.6}$$

where $K \equiv R_{1212}/g$ is the Gaussian curvature and W_{mnrs} is the Weyl tensor. The curvature of AdS_{p+2} is constant and negative ($d = p+1$):

$$R = -\frac{(p+1)\,(p+2)}{\ell^2} \quad \Leftrightarrow \quad \Lambda = -\frac{p\,(p+1)}{2\ell^2}. \tag{A.7}$$

Therefore anti-de Sitter space is a maximally symmetric space with the maximum allowed number of bosonic symmetries, $(p+2)(p+3)/2$ (determined by the corresponding symmetry group $\mathfrak{so}\,(p+1,2)$). For example in AdS_5 it's $p = 3$, so that $R = -20/\ell^2$ and $\Lambda = -6/\ell^2$.

The isometry group of AdS_{p+2} is the orthogonal group $\mathfrak{so}(p+1,2)$, which is isomorphic to the conformal group in $d = p+1$ dimensions. Some of the distinguishing properties of AdS are:

- It is not Globally Hyperbolic.
- It possesses Closed Timelike Curves (CTCs).
- It is not Geodesically Convex.
- It is "Holographic".

The topology $\mathrm{S}^1 \times \mathbb{R}^d$ of AdS_{d+1} is responsible for the existence of closed timelike curves (CTCs) and closed timelike geodesics (CTGs) in anti-de Sitter space. CTCs and CTGs can be avoided by passing to the universal covering space of AdS (CAdS), by simply ignoring time periodicity. Equivalently we say that anti-de Sitter space is not globally hyperbolic or that it does not have a Cauchy surface. This means that the future and the past cannot be defined in AdS in a deterministic way. Moreover, temporal evolution in AdS can always be controlled by the information that flows into it from spatial infinity. This state of affairs can be avoided by imposing appropriate boundary conditions on the boundary of AdS.

Another special feature of AdS is that it is not geodesically convex, meaning that not all of its points can be connected with a geodesic. AdS is also inherently "holographic". The exact formulation of the holographic property of AdS will be given with the help of the "sausage" coordinate system in §A.2. As a preview, it can be proven that the total volume of AdS_{p+2} scales as its total area and consequently, the degrees of freedom of any theory that is defined in AdS can be mapped to its boundary. Therefore the boundary of AdS assumes the role of the holographic screen that we saw in §2.2.

Note however that the picture of a CFT that lives on the boundary of anti-de Sitter space is not entirely correct in AdS/CFT correspondence.[81] As we will see below, the boundary geometries of anti-de Sitter space and its universal covering space ∂AdS and ∂CAdS, depend on the coordinate system with which we describe the AdS bulk. Specifically, the boundary geometry is flat Minkowski space $\mathbb{R}^{1,p}$ in the system of global AdS coordinates and the Einstein static universe (ESU) $\mathbb{R} \times \mathrm{S}^p$ in the Poincaré coordinate system. Therefore the bulk geometry will either have a Poincaré horizon (if it is described in Poincaré coordinates) or not (if it is described in a global coordinate system) and as a consequence, the dual CFT may develop a mass gap or it may not. The following table contains a

[81] See e.g. [9], §3.1.3.

brief summary of the various topologies and geometries of anti-de Sitter space.

	Topology	Boundary
Spacetime AdS_{p+2}:	$\mathrm{S}^1 \times \mathbb{R}^{p+1}$	$\partial\mathrm{AdS}$: $\mathrm{S}^1 \times \mathrm{S}^p$ or $\mathrm{S}^1 \times \mathbb{R}^p$
Universal Covering Space CAdS_{p+2}:	$\mathbb{R}^{p+2}$	$\partial\mathrm{CAdS}$: $\mathbb{R} \times \mathrm{S}^p$ (ESU) or $\mathbb{R}^{1,p}$ (Minkowski)

Let us consider the bosonic coset space representation of AdS_{d+1} [257]:

$$\mathrm{AdS}_{d+1} = \frac{\mathfrak{so}\,(d,2)}{\mathfrak{so}\,(d,1)} \tag{A.8}$$

In this representation, AdS_{d+1} is generated by acting the group $\mathfrak{so}\,(d,2)$ on either of its two temporal directions, $\hat{Y}_0 = \{1,0,\ldots,0\}$ or $\hat{Y}_{d+1} = \{0,0,\ldots,1\}$, while $\mathfrak{so}\,(d,1)$ is its stability group w.r.t. the chosen temporal direction. We may use the classical group isomorphisms to express the first few dimensionalities, as follows:

AdS_{p+2}	$d = p+1$	p	Coset Space
AdS_1	0	–	$\frac{\mathfrak{so}\,(2)}{\mathfrak{so}\,(1)}$
AdS_2	1	0	$\frac{\mathfrak{su}\,(1,1)}{\mathfrak{so}\,(1,1)}$
AdS_3	2	1	$\frac{\mathfrak{sl}\,(2) \times \mathfrak{sl}\,(2)}{\mathfrak{so}\,(2,1)}$
AdS_4	3	2	$\frac{\mathfrak{sp}\,(4)}{\mathfrak{sl}\,(2)}$
AdS_5	4	3	$\frac{\mathfrak{su}\,(2,2)}{\mathfrak{usp}\,(2,2)}$
AdS_6	5	4	$\frac{\mathfrak{so}\,(5,2)}{\mathfrak{su}\,(4)}$

We will now present the most commonly used coordinate systems of AdS space. A nice collection of coordinate systems in AdS_3, enriched with some extra possibilities that are not presented here, can be found in appendix A of the paper [258].

A.1 Global Coordinates

To pass from embedding coordinates to the global coordinate system of AdS, we perform the following change of variables:

$$\begin{aligned} &Y_0 = \ell \cosh\rho\, \cos\tau\,, \qquad \rho \geq 0, \quad 0 \leq \tau \leq 2\pi \\ &Y_i = \ell \sinh\rho\, \Omega_i\,, \qquad i = 1,2,\ldots,p+1 \\ &Y_{p+2} = \ell \cosh\rho\, \sin\tau \end{aligned} \tag{A.9}$$

$$\boxed{ds^2 = \ell^2 \left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_p^2\right)}. \tag{A.10}$$

Another often employed version of global AdS is the following:

$$\begin{aligned}
&Y_0 = \ell \sec\varphi\, \cos\tau\,, \quad 0 \le \varphi \le \tfrac{\pi}{2}, \quad 0 \le \tau \le 2\pi \\
&Y_i = \ell \tan\varphi\, \Omega_i\,, \qquad i = 1, 2, \ldots, p+1 \\
&Y_{p+2} = \ell \sec\varphi\, \sin\tau
\end{aligned} \tag{A.11}$$

$$\boxed{ds^2 = \frac{\ell^2}{\cos^2\varphi} \left(-d\tau^2 + d\varphi^2 + \sin^2\varphi\, d\Omega_p^2\right)}. \tag{A.12}$$

To change between the two global descriptions we must set:

$$\tanh\rho = \sin\varphi \Leftrightarrow \sinh\rho = \tan\varphi. \tag{A.13}$$

In the global coordinate system, the topology $S^1 \times \mathbb{R}^{p+1}$ of AdS, as well as the existence of CTCs (due to time periodicity) are made manifest. The AdS boundary (∂AdS) is approached for $\rho \to \infty$ and $\varphi \to \pi/2$. According to (A.12), $\partial\text{AdS} = S^1 \times S^p$ in global coordinates.

By unwrapping the periodic time (i.e. by sending $S^1 \to \mathbb{R}$), we may reduce AdS to its universal covering space CAdS that has the topology $\mathbb{R} \times \mathbb{R}^{p+1}$. The boundary of CAdS is the Einstein static universe ESU_{p+1}: $\partial\text{CAdS} = \mathbb{R} \times S^p$.

The system (A.11)–(A.12) is also suited for the study of the causal structure of AdS, which is preserved under conformal rescalings. According to (A.12), the AdS spacetime is conformally equivalent to one-half the Einstein static universe $\text{ESU}_{p+2} = \mathbb{R} \times S^{p+1}$:[82]

$$ds^2 = -d\tau^2 + d\varphi^2 + \sin^2\varphi\, d\Omega_p^2 = -d\tau^2 + d\Omega_{p+1}^2. \tag{A.14}$$

A.2 "Sausage" Coordinates

"Sausage" coordinates, are defined as follows ($i = 1, 2, \ldots, p+1$):

$$\begin{aligned}
&Y_0 = \ell \cos\tau \left(\frac{1+v^2}{1-v^2}\right) \\
&Y_i = \ell\, \Omega_i \left(\frac{2v}{1-v^2}\right) \longrightarrow \boxed{ds^2 = \ell^2 \left[-\left(\frac{1+v^2}{1-v^2}\right)^2 d\tau^2 + \frac{4}{(1-v^2)^2}\left(dv^2 + v^2 d\Omega_p^2\right)\right]}. \\
&Y_{p+2} = \ell \sin\tau \left(\frac{1+v^2}{1-v^2}\right)
\end{aligned} \tag{A.15}$$

We may switch between "sausage" and global coordinates with the following change of variables:

$$\sinh\rho = \frac{2v}{1-v^2} \tag{A.16}$$

[82] Since here it's $0 \le \varphi \le \pi/2$, instead of the usual range $0 \le \varphi \le \pi$ of spherical angles (cf. appendix B).

$$\cosh\rho = \frac{1+v^2}{1-v^2}. \tag{A.17}$$

Equivalently we may use the variables ζ_k:

$$\zeta_k = 2v\cdot\Omega_k \quad \& \quad \zeta^2 = \zeta_k\cdot\zeta_k = 4v^2, \tag{A.18}$$

with which (A.15) becomes:

$$\boxed{ds^2 = \ell^2\left[-\left(\frac{1+\zeta^2/4}{1-\zeta^2/4}\right)^2 d\tau^2 + \frac{d\zeta_k\cdot d\zeta^k}{(1-\zeta^2/4)^2}\right]}. \tag{A.19}$$

In the sausage coordinate system the boundary of anti-de Sitter space (∂AdS) is located at $v\to 1$. Sausage coordinates are suitable for proving the following two propositions about anti-de Sitter space.

Proposition I [259]

The geodesic distance between two points x_1, x_2 near the AdS boundary scales as $\log|x_{12}|/\epsilon$, where $x_{12}\equiv x_1 - x_2$ and $\epsilon \ll 1$.

Proposition II

The ratio of the area over the volume of AdS_{p+2} approaches p/ℓ:

$$\lim_{v\to 1}\left[\frac{\mathrm{Area}\,(\mathrm{AdS}_{p+2})}{\mathrm{Vol}\,(\mathrm{AdS}_{p+2})}\right] = \frac{p}{\ell}\,, \quad p = 0,\ 1,\ 2,\ \ldots, \tag{A.20}$$

where ℓ is the radius of anti-de Sitter space.

A.3 Horospheric/Poincaré Coordinates

If we go over to light-cone coordinates,

$$\left\{\frac{1}{2}(Y_0 - Y_{p+1}) = \frac{\ell^2}{2y}\,,\ \frac{1}{2}(Y_0 + Y_{p+1}) = \frac{s^2}{2y}\,,\ Y_i = \frac{\ell x_i}{y}\,,\ Y_{p+2} = \frac{\ell t}{y}\right\},\ s^2 \equiv -t^2 + \mathbf{x}_p^2 + y^2, \tag{A.21}$$

we can set up the horospheric or Poincaré coordinate system as follows [260]:

$$\begin{aligned}
Y_0 &= \frac{1}{2y}\left[-t^2+\mathbf{x}_p^2+y^2+\ell^2\right],\ y\in[0,\infty) & & Y_0 = \frac{1}{2u}\left[1+u^2\left(-t^2+\mathbf{x}_p^2+\ell^2\right)\right],\ u\in[0,\infty)\\
Y_i &= \frac{\ell x_i}{y} \quad (i=1,\ 2,\ldots,\ p) & \xleftrightarrow{u=1/y}\ & Y_i = \ell u x_i \quad (i=1,\ 2,\ldots,\ p)\\
Y_{p+1} &= \frac{1}{2y}\left[-t^2+\mathbf{x}_p^2+y^2-\ell^2\right] & & Y_{p+1} = \frac{1}{2u}\left[1+u^2\left(-t^2+\mathbf{x}_p^2-\ell^2\right)\right]\\
Y_{p+2} &= \frac{\ell t}{y} & & Y_{p+2} = \ell u t.
\end{aligned} \tag{A.22}$$

There's a small web of equivalent representations of Poincaré/horospheric coordinates:

$$\boxed{ds^2 = \ell^2 \left(dr^2 + e^{2r}\left(-dt^2 + d\mathbf{x}_p^2\right)\right)}$$

(Klebanov-Maldacena [261])

$$\uparrow\; u = e^r$$

$$\boxed{ds^2 = \ell^2 \left(\frac{du^2}{u^2} + u^2\left(-dt^2 + d\mathbf{x}_p^2\right)\right)} \quad \xrightarrow{u=1/y} \quad \boxed{ds^2 = \frac{\ell^2}{y^2}\left(-dt^2 + d\mathbf{x}_p^2 + dy^2\right)} \tag{A.23}$$

(Poincaré Frame) (Conformal Frame)

$$\downarrow\; u = \frac{z}{\ell^2} \qquad\qquad \downarrow\; y = \ell e^{-\widetilde{u}/\ell}$$

$$\boxed{ds^2 = \frac{z^2}{\ell^2}\left(-dt^2 + d\mathbf{x}_p^2\right) + \frac{\ell^2}{z^2}dz^2} \qquad \boxed{ds^2 = e^{2\widetilde{u}/\ell}\left(-dt^2 + d\mathbf{x}_p^2\right) + d\widetilde{u}^2}$$

(Maldacena [6]) (Domain-Wall Frame)

In the conformal frame, the metric is manifestly invariant under the following transformations:

$$\text{Poincaré:}\;\; x'_\mu = M^\nu_\mu x_\mu + a_\mu \quad (\mathfrak{iso}(p,1)) \tag{A.24}$$

$$\text{Dilations:}\;\; x'_m = \alpha \cdot x_m, \tag{A.25}$$

where $x_\mu = (t,\, x_i)$ and $x_m = (t,\, x_i,\, y)$, $i = 1, 2, \ldots, p$. Together with inversions

$$\frac{x'_m}{x'^2} = \frac{x_m}{x^2}, \tag{A.26}$$

this gives a total of $(d+2)(d+1)/2$ conservation laws.

The boundary of anti-de Sitter space in horospheric/Poincaré coordinates is the flat $p+1$ dimensional Minkowski spacetime $\mathbb{R}^{1,p}$, that is obtained for $y \to 0$ and $u,\, z,\, r,\, \widetilde{u} \to \infty$.[83] A Poincaré horizon is approached for $u,\, z \to 0$.

The Poincaré coordinate system is just a patch of the full anti-de Sitter space, since it covers only one-half of it. To see this, notice that in Poincaré coordinates $z \in [0, +\infty)$, while in the case of the full AdS, $z \in (-\infty, 0] \bigcup [0, +\infty)$. To illustrate this point better, we express the Poincaré coordinate z in the global coordinate system (A.11):

$$z = \frac{\ell^2}{y} = Y_0 - Y_{p+1} = \ell \sec\varphi \cos\tau - \ell \tan\varphi \cos\theta \geq 0 \Rightarrow \cos\tau \geq \sin\varphi\cos\theta, \tag{A.27}$$

so that it describes one-half of the AdS hyperboloid. Summarizing,

$$\text{AdS}_{p+2} \mapsto \frac{1}{2} \cdot \text{ESU}_{p+2} \quad \& \quad \text{Poincaré patch} = \frac{1}{2} \cdot \text{AdS}, \tag{A.28}$$

since anti-de Sitter space can be conformally mapped to one-half the Einstein static universe (ESU), as we saw in §A.1 by using global coordinates in the form (A.12).

[83] The transformation between the global and the Poincaré boundaries of AdS has been studied in [262].

A.4 Stereographic Coordinates

The stereographic coordinate system of anti-de Sitter space is defined as follows [254]:

$$\begin{aligned} Y_0 &= \frac{2\ell t}{1-s^2}, \qquad s^2 \equiv -t^2 + \mathbf{x}_{p+1}^2 \\ Y_i &= \frac{2\ell x_i}{1-s^2}, \qquad i = 1,2,\ldots,p+1 \quad \longrightarrow \quad \boxed{ds^2 = \frac{4\ell^2}{(1-s^2)^2}\left(-dt^2 + d\mathbf{x}_{p+1}^2\right)}. \\ Y_{p+2} &= \ell \cdot \frac{1+s^2}{1-s^2} \end{aligned} \tag{A.29}$$

All the frames that are conformally equivalent to Minkowski spacetimes (like the stereographic coordinates and (A.22)) have the nice property that they preserve the light cone structure.

A.5 "Static" Coordinates

In [263], Hawking and Page used the following metric for the universal covering space of anti-de Sitter space (CAdS):

$$\boxed{ds^2 = -\left[\frac{r^2}{\ell^2}+1\right] d\widetilde{\tau}^2 + \frac{dr^2}{\left[\frac{r^2}{\ell^2}+1\right]} + r^2 d\Omega_p^2}. \tag{A.30}$$

This system is known as "static" coordinates. It is related to the system of global coordinates (A.10) as follows:

$$r = \ell \sinh\rho, \qquad \widetilde{\tau} = \frac{\tau}{\ell} \tag{A.31}$$

To transform this metric to the conformal frame in (A.23), we perform the following change of variables (for the case of AdS_4, see [264]):

$$\begin{aligned} t &= \frac{\sqrt{r^2+\ell^2}\,\sin(\widetilde{\tau}/\ell)}{\sqrt{r^2+\ell^2}\,\cos(\widetilde{\tau}/\ell) + r\,\Omega_1} \\ x_i &= \frac{r\,\Omega_{i+1}}{\sqrt{r^2+\ell^2}\,\cos(\widetilde{\tau}/\ell) + r\,\Omega_1}, \qquad i = 1,2,\ldots,p \\ y &= \frac{\ell}{\sqrt{r^2+\ell^2}\,\cos(\widetilde{\tau}/\ell) + r\,\Omega_1}. \end{aligned} \tag{A.32}$$

The boundary ∂CAdS is reached for $r \to \infty$. It is the Einstein static universe (ESU), $\mathbb{R} \times S^p$.

A.6 AdS as a Ruled Surface[84]

In [265], we find the following coordinate system of AdS_{p+2} :

$$Y_0 = \ell\,(\cos\phi - M\sin\phi) \tag{A.33}$$

[84] This subsection is based on unpublished research material by the author's PhD supervisor, E. Floratos.

$$Y_i = \ell\, M\, \Omega_i, \quad i = 1, 2, \ldots, p+1 \tag{A.34}$$

$$Y_{p+2} = \ell \left(\sin\phi + M \cos\phi \right), \tag{A.35}$$

the line element of which is given by

$$\boxed{ds^2 = \ell^2 \left(\left(1 + M^2\right) d\phi^2 + 2\, d\phi\, dM - M^2 d\Omega_p^2 \right)}. \tag{A.36}$$

If we complete the square we obtain the line element,

$$ds^2 = \ell^2 \left\{ \left(1 + M^2\right) \left(d\phi + \frac{dM}{1+M^2} \right)^2 - \frac{dM^2}{1+M^2} - M^2 d\Omega_p^2 \right\} \tag{A.37}$$

and by setting $\chi \equiv \arctan M$, we may bring the above metrics in the following forms:

$$ds^2 = \frac{\ell^2}{\cos^2\chi} \left\{ d(\phi + \chi)^2 - d\chi^2 - \sin^2\chi\, d\Omega_p^2 \right\}. \tag{A.38}$$

A.6.1 Light-Cone Frame

We may now pass to light-cone coordinates:

$$Y_{\pm} \equiv \frac{1}{\sqrt{2}} \left(Y_0 \pm Y_{p+1} \right), \tag{A.39}$$

which we may invert and obtain

$$Y_0 = \frac{1}{\sqrt{2}} \left(Y_+ + Y_- \right) \tag{A.40}$$

$$Y_{p+1} = \frac{1}{\sqrt{2}} \left(Y_+ - Y_- \right). \tag{A.41}$$

Since Y_0 and Y_{p+1} lie on the AdS hyperboloid (A.1), the remaining AdS coordinates are constrained:

$$\ell^2 = Y_0^2 - \sum_{i=1}^{p+1} Y_i^2 + Y_{p+2}^2 = 2Y_+Y_- - \sum_{i=1}^{p} Y_i^2 + Y_{p+2}^2. \tag{A.42}$$

In the case of AdS_2 we get:

$$\ell^2 = 2Y_+Y_- + Y_2^2 \Rightarrow Y_2 = \pm\sqrt{\ell^2 - 2Y_+Y_-} \tag{A.43}$$

and the corresponding line element is given by:

$$\boxed{ds^2 = \frac{1}{\ell^2 - 2Y_+Y_-} \left[Y_-^2 dY_+^2 + Y_+^2 dY_-^2 + 2\left(\ell^2 - Y_+Y_-\right) dY_+ dY_- \right]}. \tag{A.44}$$

A.7 AdS Coordinate Systems Summary

Here's a summary of all the coordinate systems of AdS_{p+2} that we saw above:

	Metric	∂AdS	∂CAdS
1. Embedding Coordinates:	$ds^2 = -dY_0^2 + \sum_{i=1}^{p+1} dY_i^2 - dY_{p+2}^2$	$Y \longrightarrow \infty$	ESU
	$Y_0^2 - \sum_{i=1}^{p+1} Y_i^2 + Y_{p+2}^2 = \ell^2$		
2. Global Coordinates:	$ds^2 = \ell^2 \left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_p^2\right)$	$\rho \to \infty$	ESU
	$ds^2 = \frac{\ell^2}{\cos^2\varphi}\left(-d\tau^2 + d\varphi^2 + \sin^2\varphi\, d\Omega_p^2\right)$	$\varphi \to \frac{\pi}{2}$	ESU
3. Sausage Coordinates:	$ds^2 = \ell^2 \left[-\left(\frac{1+v^2}{1-v^2}\right)^2 d\tau^2 + \frac{4}{\left(1-v^2\right)^2}\left(dv^2 + v^2 d\Omega_p^2\right)\right]$	$v \to 1$	ESU
4. Poincaré Frame:	$ds^2 = \ell^2 \left(\frac{du^2}{u^2} + u^2\left(-dt^2 + d\mathbf{x}_p^2\right)\right)$	$u \to \infty$	Minkowski
	$ds^2 = \frac{z^2}{\ell^2}\left(-dt^2 + d\mathbf{x}_p^2\right) + \frac{\ell^2}{z^2} dz^2$	$z \to \infty$	Minkowski
5. Conformal Frame:	$ds^2 = \frac{\ell^2}{y^2}\left(-dt^2 + d\mathbf{x}_p^2 + dy^2\right)$	$y \to 0$	Minkowski
6. Domain-Wall Frame:	$ds^2 = e^{2\widetilde{u}/\ell}\left(-dt^2 + d\mathbf{x}_p^2\right) + d\widetilde{u}^2$	$\widetilde{u} \to \infty$	Minkowski
	$ds^2 = \ell^2\left(dr^2 + e^{2r}\left(-dt^2 + d\mathbf{x}_p^2\right)\right)$	$r \to \infty$	Minkowski
7. Stereographic Projection:	$ds^2 = \frac{4\ell^2}{\left(1-s^2\right)^2}\left(-dt^2 + d\mathbf{x}_{p+1}^2\right)$	$s \to 1$	Minkowski
8. "Static" Coordinates:	$ds^2 = -\left[\frac{r^2}{\ell^2} + 1\right] d\widetilde{\tau}^2 + \frac{dr^2}{\left[\frac{r^2}{\ell^2} + 1\right]} + r^2 d\Omega_p^2$	$r \to \infty$	ESU

B Parametrizations of S^n

The purpose of the present appendix is to briefly review the various parametrizations of the n-dimensional unit sphere S^n that are used in this thesis.

B.1 Standard Parametrizations

The standard parametrization of the unit n-sphere comes in two main flavors, one consisting mostly of sines and one having basically cosines. To obtain either one of them we set:

B.1.1 Sine Parametrization

$$\begin{aligned}
&\Omega_1 = \cos x_1 \\
&\Omega_2 = \sin x_1 \, \cos x_2 \\
&\Omega_3 = \sin x_1 \, \sin x_2 \, \cos x_3 \\
&\vdots \qquad\qquad , \; x_1, x_2, \ldots, x_{n-1} \in [0,\pi] \, , \; x_n \in [0, 2\pi) \, , \quad \sum_{j=1}^{n+1} \Omega_j^2 = 1 \\
&\Omega_n = \sin x_1 \, \sin x_2 \, \sin x_3 \ldots \sin x_{n-1} \cos x_n \\
&\Omega_{n+1} = \sin x_1 \, \sin x_2 \, \sin x_3 \ldots \sin x_{n-1} \sin x_n
\end{aligned}$$

$$\text{Induced Metric: } \boxed{d\Omega_n^2 = dx_1^2 + s_1^2 \, dx_2^2 + s_1^2 \, s_2^2 \, dx_3^2 + \ldots + s_1^2 \, s_2^2 \, s_3^2 \ldots s_{n-1}^2 \, dx_n^2} \, . \tag{B.1}$$

B.1.2 Cosine Parametrization

$$\begin{aligned}
&\Omega_1 = \sin x_1 \\
&\Omega_2 = \cos x_1 \, \sin x_2 \\
&\Omega_3 = \cos x_1 \, \cos x_2 \, \sin x_3 \\
&\vdots \qquad\qquad , \; x_1, x_2, \ldots, x_{n-1} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] , \; x_n \in [0, 2\pi) \, , \quad \sum_{j=1}^{n+1} \Omega_j^2 = 1 \\
&\Omega_n = \cos x_1 \, \cos x_2 \, \cos x_3 \ldots \cos x_{n-1} \sin x_n \\
&\Omega_{n+1} = \cos x_1 \, \cos x_2 \, \cos x_3 \ldots \cos x_{n-1} \cos x_n
\end{aligned}$$

$$\text{Induced Metric: } \boxed{d\Omega_n^2 = dx_1^2 + c_1^2 \, dx_2^2 + c_1^2 \, c_2^2 \, dx_3^2 + \ldots + c_1^2 \, c_2^2 \, c_3^2 \ldots c_{n-1}^2 \, dx_n^2} \, . \tag{B.2}$$

B.2 Complex Parametrizations

The complex parametrization of the unit n-sphere depends crucially on whether the sphere is odd or even-dimensional. To obtain the complex parametrization, we must divide all of its points $\Omega_1, \ldots, \Omega_{n+1}$ into two main sets of coordinates that are labelled X_j and Y_j, $j = 1, 2, \ldots, \lfloor (n+1)/2 \rfloor$. The two sets are then arranged into pairs (X_j, Y_j) which serve as components of the complex coordinates of the unit n-sphere $Z_j \equiv X_j + iY_j$.

More concretely, for each of the two cases (odd and even-dimensional) we set:

$\underline{n=2k+1}$ (odd): $Z_j = X_j + i\,Y_j = \Omega_j \cdot e^{iy_j}$

$\underline{n=2k}$ (even): $Z_j = X_j + i\,Y_j = \Omega_j \cdot e^{iy_j}, \quad y_j \in [0,2\pi)\,,\; j=1,2,\ldots,\left\lfloor \frac{n+1}{2} \right\rfloor$

$$Z_{k+1} = X_{k+1} = \Omega_{k+1}, \qquad \sum_{j=1}^{k+1} |Z_j|^2 = \sum_{j=1}^{k+1} X_j^2 + Y_j^2 = 1$$

Induced Metric:
$$\boxed{ds^2 = \sum_{j=1}^{k+1} |dZ_j|^2 = \sum_{j=1}^{k+1} dX_j^2 + dY_j^2 = d\Omega_k^2 + \sum_{j=1}^{\lfloor n+1/2 \rfloor} (\Omega_j\, dy_j)^2}, \tag{B.3}$$

Let us give some examples of the complex parametrization in both its odd and even-dimensional instances. The majority of them are employed in this thesis time and again.

$\mathbf{S}^3$:
$$Z_1 = \cos\theta\, e^{\phi_1} \longrightarrow ds^2 = d\theta^2 + \cos^2\theta\, d\phi_1^2 + \sin^2\theta\, d\phi_2^2$$
$$Z_2 = \sin\theta\, e^{\phi_2}$$

$\mathbf{S}^4$:
$$Z_1 = \cos\theta_1\, e^{\phi_1}$$
$$Z_2 = \sin\theta_1 \cos\theta_2\, e^{\phi_2} \longrightarrow ds^2 = d\theta_1^2 + \cos^2\theta_1 d\phi_1^2 + \sin^2\theta_1 \left(d\theta_2^2 + \cos^2\theta_2\, d\phi_2^2\right)$$
$$Z_3 = \sin\theta_1 \sin\theta_2$$

$$Z_1 = \sin\theta_1\, e^{\phi_1}$$
$$Z_2 = \cos\theta_1 \sin\theta_2\, e^{\phi_2} \longrightarrow ds^2 = d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + \cos^2\theta_1 \left(d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2\right)$$
$$Z_3 = \cos\theta_1 \cos\theta_2$$

$\mathbf{S}^5$:
$$Z_1 = \cos\theta_1\, e^{\phi_1}$$
$$Z_2 = \sin\theta_1 \cos\theta_2\, e^{\phi_2} \longrightarrow ds^2 = d\theta_1^2 + \cos^2\theta_1\, d\phi_1^2 + \sin^2\theta_1 \left(d\theta_2^2 + \cos^2\theta_2\, d\phi_2^2 + \sin^2\theta_2\, d\phi_3^2\right)$$
$$Z_3 = \sin\theta_1 \sin\theta_2\, e^{\phi_3}$$

$$Z_1 = \sin\theta_1\, e^{\phi_1}$$
$$Z_2 = \cos\theta_1 \sin\theta_2\, e^{\phi_2} \longrightarrow ds^2 = d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2 + \cos^2\theta_1 \left(d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2 + \cos^2\theta_2\, d\phi_3^2\right)$$
$$Z_3 = \cos\theta_1 \cos\theta_2\, e^{\phi_3}$$

$\mathbf{S}^7$:
$$Z_1 = \cos\theta_1\, e^{\phi_1}$$
$$Z_2 = \sin\theta_1 \cos\theta_2\, e^{\phi_2}$$
$$Z_3 = \sin\theta_1 \sin\theta_2 \cos\theta_3\, e^{\phi_3}$$

$$Z_4 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \, e^{\phi_4}$$

$$ds^2 = d\theta_1^2 + \cos^2\theta_1 \, d\phi_1^2 + \sin^2\theta_1 \left(d\theta_2^2 + \cos^2\theta_2 \, d\phi_2^2 + \sin^2\theta_2 \left(d\theta_3^2 + \cos^2\theta_3 \, d\phi_3^2 + \sin^2\theta_4 \, d\phi_4^2\right)\right)$$

$$Z_1 = \sin\theta_1 \, e^{\phi_1}$$

$$Z_2 = \cos\theta_1 \sin\theta_2 \, e^{\phi_2}$$

$$Z_3 = \cos\theta_1 \cos\theta_2 \sin\theta_3 \, e^{\phi_3}$$

$$Z_4 = \cos\theta_1 \cos\theta_2 \cos\theta_3 \, e^{\phi_4}$$

$$ds^2 = d\theta_1^2 + \sin^2\theta_1 \, d\phi_1^2 + \cos^2\theta_1 \left(d\theta_2^2 + \sin^2\theta_2 \, d\phi_2^2 + \cos^2\theta_2 \left(d\theta_3^2 + \sin^2\theta_3 \, d\phi_3^2 + \cos^2\theta_4 \, d\phi_4^2\right)\right).$$

B.3 "Sausage" Coordinates

We may also write down the analog of the "sausage" metric (A.15) for the unit n-sphere:

$$\Omega_1 = \cos\phi \left(\frac{1-v^2}{1+v^2}\right)$$

$$\Omega_i = \widetilde{\Omega}_i \left(\frac{2v}{1+v^2}\right) \quad \longrightarrow \quad \boxed{ds^2 = \left(\frac{1-v^2}{1+v^2}\right)^2 d\phi^2 + \frac{4}{(1+v^2)^2}\left(dv^2 + v^2 d\widetilde{\Omega}_{n-2}^2\right)}, \tag{B.4}$$

$$\Omega_{n+1} = \sin\phi \left(\frac{1-v^2}{1+v^2}\right)$$

where

$$\sum_{j=1}^{n+1} \Omega_j^2 = \sum_{i=2}^{n} \widetilde{\Omega}_i^2 = 1 \quad \& \quad i = 2, 3, \ldots, n. \tag{B.5}$$

B.4 Stereographic Coordinates

The stereographic coordinates of the unit n-sphere S^n are defined as follows:

$$\Omega_i = \frac{2x_i}{s^2+1}, \qquad i = 1, 2, \ldots, n \tag{B.6}$$

$$\Omega_{n+1} = \frac{s^2-1}{s^2+1}, \quad s^2 \equiv \sum_{i=1}^{n} x_i^2 = \mathbf{x}_n^2 \quad \longrightarrow \quad \boxed{ds^2 = \frac{4d\mathbf{x}_n^2}{(1+s^2)^2}}. \tag{B.7}$$

For completeness in our presentation let us also write down the coset representation of the n-sphere [257]:

$$\mathrm{S}^n = \frac{\mathfrak{so}(n+1)}{\mathfrak{so}(n)}, \tag{B.8}$$

which is the analog of (A.8) for S^n.

C Plane-Wave Backgrounds & Penrose Limits

C.1 Plane-Wave Backgrounds

Plane-waves in $d+1$ dimensions are special cases of pp-wave spacetimes:

$$ds^2 = -2dudv - F(u,x^i)du^2 + 2A_j(u,x^i)dudx^j + g_{jk}(u,x^i)dx^j dx^k, \tag{C.1}$$

where u and v are the light-cone coordinates:

$$u = \frac{1}{\sqrt{2}}\left(x^0 + x^d\right), \quad v = \frac{1}{\sqrt{2}}\left(x^0 - x^d\right), \quad i = 1, 2, \ldots, d-1. \tag{C.2}$$

The functions $F(u,x^i)$, $A_j(u,x^i)$, $g_{jk}(u,x^i)$ (= metric of transverse spacetime) are determined from the supergravity equations of motion. Pp-waves admit a covariantly constant and null Killing vector field, while for $A_j = 0$, $g_{jk} = \delta_{jk}$, they are α'-exact solutions of supergravity [42]:

$$ds^2 = -2dudv - F(u,x^i)du^2 + dx^i dx^i. \tag{C.3}$$

Plane-waves are pp-waves for which $F(u,x^i) = f_{ij}(u)x^i x^j$, $A_j = 0$ and $g_{jk} = \delta_{jk}$:

$$ds^2 = -2dudv - f_{ij}(u)x^i x^j du^2 + dx^i dx^i. \tag{C.4}$$

Homogeneous plane-waves have constant $f_{ij}(u) = \mu_{ij}^2$:

$$ds^2 = -2dudv - \mu_{ij}^2 x^i x^j du^2 + dx^i dx^i. \tag{C.5}$$

Homogeneous and isotropic plane-waves are given by:

$$ds^2 = -2dudv - \mu^2 x^i x^i du^2 + dx^i dx^i. \tag{C.6}$$

One of the most important properties of plane-wave spacetimes is that they are the Penrose limits of any given spacetime. It can be proven that the plane-wave limits of the maximally supersymmetric backgrounds $\mathrm{AdS}_{4/5/7} \times \mathrm{S}^{7/5/4}$ of type IIB supergravity, are also backgrounds of maximally supersymmetric solutions that preserve the maximum number of 32 supersymmetries.[85] As we saw in §3.6, type IIB string theory can be exactly solved on the homogeneous and isotropic plane-wave background (C.6) that is the Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$. The plane wave/SYM duality states that IIB string theory on the plane-wave limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$ is the AdS/CFT dual of the BMN sector of $\mathcal{N} = 4$ SYM theory [43].

[85] In type IIB supergravity, the only maximally supersymmetric backgrounds are the flat spacetime, $\mathrm{AdS}_{4/5/7} \times \mathrm{S}^{7/5/4}$ and their Penrose limits. In type IIA supergravity it is only flat space. See [266]. A summary can be found in §C.3.

C.2 Penrose Limits

According a theorem of Penrose [267], every spacetime has a plane wave as a limit. Starting from any given metric we may obtain its plane-wave limit in two steps:

(a). Consider only a small neighbourhood of the spacetime near a null geodesic.

(b). Blow up spacetime near the geodesic to fill all the spacetime.

The resulting metric is a plane wave that is known as the Penrose limit of the original spacetime. Güven [268] generalized Penrose's theorem and limiting procedure to supergravity. Below we shall obtain the Penrose limits of $\mathrm{AdS}_{p+2} \times \mathrm{S}^{q+2}$ and $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$.

C.2.1 Penrose Limits of $\mathrm{AdS}_{p+2} \times \mathrm{S}^{q+2}$

Let us first consider the Penrose limit of $\mathrm{AdS}_{p+2} \times \mathrm{S}^{q+2}$ (more can be found in [44]). Begin from the line element of $\mathrm{AdS}_{p+2} \times \mathrm{S}^{q+2}$ in global coordinates:

$$ds^2 = \ell^2 \left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_p^2\right) + R^2 \left(d\theta^2 + \cos^2\theta\, d\phi^2 + \sin^2\theta\, d\widetilde{\Omega}_q^2\right). \tag{C.7}$$

If we perform the change of coordinates

$$u = \frac{1}{2}\left(t + \frac{R}{\ell}\phi\right), \quad v = \ell^2\left(t - \frac{R}{\ell}\phi\right) \quad \& \quad x^2 = x^i x^i = \ell^2 \sinh^2\rho\,, \quad y^2 = y^j y^j = R^2 \sin^2\theta, \tag{C.8}$$

for $i = 1, 2, \ldots, p+1$ and $j = 1, 2, \ldots, q+1$, the $\mathrm{AdS}_{p+2} \times \mathrm{S}^{q+2}$ metric (C.7) becomes:

$$\begin{aligned} ds^2 = &-\left(\ell^2 + x^2\right)\left(du^2 + \frac{dv^2}{4\ell^4} + \frac{dudv}{\ell^2}\right) + \frac{\ell^2 dx^2}{\ell^2 + x^2} + x^2 d\Omega_p^2 + \\ &+ \frac{R^2 dy^2}{R^2 - y^2} + \frac{\ell^2}{R^2}\left(R^2 - y^2\right)\left(du^2 + \frac{dv^2}{4\ell^4} - \frac{dudv}{\ell^2}\right) + y^2 d\widetilde{\Omega}_q^2. \end{aligned} \tag{C.9}$$

To take the Penrose limit we must let $\ell,\, R \to \infty$ with $\ell^2/R^2 = \mathfrak{k}^2 = \text{const.}$ (C.9) becomes:

$$ds^2 = -\left(\ell^2 + x^2\right) du^2 - dudv + dx^2 + x^2 d\Omega_p^2 + dy^2 + \left(\ell^2 - \mathfrak{k}^2 y^2\right) du^2 - dudv + y^2 d\widetilde{\Omega}_q^2, \tag{C.10}$$

or equivalently,

$$ds^2 = -2dudv - \left(x^2 + \mathfrak{k}^2 y^2\right) du^2 + dx^i dx^i + dy^j dy^j. \tag{C.11}$$

This is just the metric of a homogeneous anisotropic plane wave (C.5). For $x^i = 0$ or $y^j = 0$, the metric of homogeneous and isotropic plane waves (C.6) is obtained.

C.2.2 Penrose Limit of $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$

We will now examine the Penrose limits of the orbifolded spacetime $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$. More on the Penrose limits of $\mathrm{AdS}_{p+2} \times \mathrm{S}^{q+2}$ orbifolds and orientifolds can be found in [269] and references therein.

The metric of $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ in global coordinates is:

$$ds^2 = \ell^2 \left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_2^2\right) + R^2 \left(d\alpha^2 + \cos^2\alpha\, d\overline{\Omega}_3^2 + \sin^2\alpha\, d\widetilde{\Omega}_3^2\right), \quad (C.12)$$

where $R = 2\ell$ and

$$d\overline{\Omega}_3^2 = d\beta^2 + \cos^2\beta d\phi_1^2 + \sin^2\beta d\phi_2^2 \quad \& \quad d\widetilde{\Omega}_3^2 = \frac{1}{4}d\overline{\Omega}_2^2 + \left[\frac{d\chi}{k} + \frac{1}{2}(\cos\gamma - 1)\, d\delta\right]^2 \quad (C.13)$$

$$d\overline{\Omega}_2^2 = d\gamma^2 + \sin^2\gamma d\delta^2. \quad (C.14)$$

There exist two distinct ways to take the Penrose limit of $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$. Either we can boost along the (t, ϕ_1) direction or along the direction (t, χ). The former is very similar to the case of $\mathrm{AdS}_{p+2} \times \mathrm{S}^{q+2}$ that we studied above.

- Boost along (t, ϕ_1). Let us make the following change of coordinates,

$$u = \frac{1}{2}\left(t + \frac{R}{\ell}\phi_1\right), \quad v = \ell^2\left(t - \frac{R}{\ell}\phi_1\right) \quad \& \quad x^2 = x^i x^i = \ell^2 \sinh^2\rho, \quad y^2 = y^j y^j = R^2 \sin^2\alpha,$$
$$z^2 = z^k z^k = R^2 \sin^2\beta.$$

The metric (C.12) then becomes:

$$\begin{aligned} ds^2 =& -\left(\ell^2 + x^2\right)\left(du^2 + \frac{dv^2}{4\ell^4} + \frac{dudv}{\ell^2}\right) + \frac{\ell^2 dx^2}{\ell^2 + x^2} + x^2 d\Omega_2^2 + \frac{R^2 dy^2}{R^2 - y^2} + \\ &+ \left(R^2 - y^2\right)\left\{\frac{dz^2}{R^2 - z^2} + \mathfrak{k}^2\left(1 - \frac{z^2}{R^2}\right)\left[du^2 + \frac{dv^2}{4\ell^4} - \frac{dudv}{\ell^2}\right] + \frac{z^2 d\phi_2^2}{R^2}\right\} + y^2 d\widetilde{\Omega}_3^2. \end{aligned} \quad (C.15)$$

If we take the Penrose limit ℓ, $R \to \infty$ with $\ell^2/R^2 = \mathfrak{k}^2 = 1/4$ we obtain

$$ds^2 = -2dudv - \left[x^2 + \mathfrak{k}^2\left(z^2 + y^2\right)\right]du^2 + dx^2 + x^2 d\Omega_2^2 + dz^2 + z^2 d\phi_2^2 + dy^2 + y^2 d\widetilde{\Omega}_3^2. \quad (C.16)$$

For $i = 1, 2, 3$, $j = 1, 2, 3, 4$ and $k = 1, 2$ the Penrose limit along (t, ϕ_1) takes the following form:

$$ds^2 = -2dudv - \left[x^2 + \frac{1}{4}\left(y^2 + z^2\right)\right]du^2 + dx^i dx^i + dy^j dy^j + dz^k dz^k. \quad (C.17)$$

- Boost along (t, χ). We make the following change of coordinates:

$$u = \frac{1}{2}\left(t + \frac{R}{\ell}\cdot\frac{\chi}{k}\right), \quad v = \ell^2\left(t - \frac{R}{\ell}\cdot\frac{\chi}{k}\right) \quad \& \quad x^2 = x^i x^i = \ell^2 \sinh^2\rho, \quad y^2 = y^j y^j = R^2 \cos^2\alpha,$$
$$z^2 = z^k z^k = R^2 \sin^2\frac{\gamma}{2}.$$

The metric (C.12) takes the form:

$$ds^2 = -\left(\ell^2 + x^2\right)\left(du^2 + \frac{dv^2}{4\ell^4} + \frac{dudv}{\ell^2}\right) + \frac{\ell^2 dx^2}{\ell^2 + x^2} + x^2 d\Omega_2^2 + \frac{R^2 dy^2}{R^2 - y^2} + y^2 d\overline{\Omega}_3^2$$
$$+ \left(R^2 - y^2\right)\left\{\frac{dz^2}{R^2 - z^2} + \frac{z^2}{R^2}\left(1 - \frac{z^2}{R^2}\right)d\delta^2 + \left[\mathfrak{k}\left(du - \frac{dv}{2\ell^2}\right) - \frac{x^2 d\delta}{R^2}\right]^2\right\}. \quad \text{(C.18)}$$

Taking the Penrose limit ℓ, $R \to \infty$ with $\ell^2/R^2 = \mathfrak{k}^2 = 1/4$ we obtain

$$ds^2 = -2dudv - \left[x^2 + \mathfrak{k}^2\left(y^2 + z^2\right)\right]du^2 + dx^2 + x^2 d\Omega_2^2 + dy^2 + y^2 d\overline{\Omega}_3^2 + dz^2 + z^2\left(d\delta - \mathfrak{k}du\right)^2 \quad \text{(C.19)}$$

For $\widetilde{\delta} \equiv \delta - \mathfrak{k} \cdot u$, the Penrose limit along (t, χ) takes the form (C.17):

$$ds^2 = -2dudv - \left[x^2 + \frac{1}{4}\left(y^2 + z^2\right)\right]du^2 + dx^i dx^i + dy^j dy^j + dz^k dz^k. \quad \text{(C.20)}$$

where again $i = 1, 2, 3$, $j = 1, 2, 3, 4$ and $k = 1, 2$. The metric (C.20) describes a homogeneous and anisotropic plane wave (C.5). For $x^i = 0$ or $y^i = z^i = 0$ it reduces to the isotropic case (C.6).

C.3 Freund-Rubin Ansatz

The Freund-Rubin ansatz [270] is a very strong theorem that allows to obtain solutions of supergravity by taking advantage of the symmetries of spacetime. The theorem generally states that there exists a natural way to compactify a $(d+1)$-dimensional supergravity theory with an s-form antisymmetric field F_s, either as $\mathrm{X}_{(d+1-s)} \times \mathrm{M}_s$, where M_s is a compact s-dimensional manifold and $\mathrm{X}_{(d+1-s)}$ is a $(d+1-s)$ dimensional manifold of negative curvature, or as $\mathrm{X}_s \times \mathrm{M}_{(d+1-s)}$. Applying the Freund-Rubin ansatz to 1_{11} supergravity (which has a 4-form field—see §13.3.1) and IIB supergravity (with 3-form and 5-form fields), the following exact supergravity solutions are obtained [271]:

$$\mathrm{AdS}_{4/7} \times \mathrm{S}^{7/4} \quad \& \quad \mathrm{AdS}_5 \times \mathrm{S}^5, \quad \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{M}^4.$$

These solutions have also been encountered in the context of the Maldacena dualities in §3.7. Generally, any compact manifold can take the place of the p-sphere in the above compactifications. As proven by Figueroa-O'Farrill and Papadopoulos [266], only the p-sphere guarantees maximal supersymmetry. $\mathrm{AdS}_{4/5/7} \times \mathrm{S}^{7/5/4}$ spacetimes (along with flat space and a special type of plane-wave background in 10 and 11 dimensions) are maximally supersymmetric backgrounds of 1_{11} and IIB supergravity, preserving 32 supersymmetries. Conversely, IIA supergravity only admits flat space as maximally supersymmetric background.

D Strings in Flat Spacetime

When strings are infinitesimally small, the curvature of spacetime is expected to have a negligible effect in their motion, which will essentially take place in a flat background. In §6, the GKP strings were studied in great detail. According to what we have just said, the negative curvature of anti-de Sitter space and the positive curvature of the sphere will only have a subleading contribution to the short-string limits of the GKP strings, which will essentially "see" an almost flat spacetime. In this appendix we are going to study the analogues of the GKP configurations in flat space:

$$ds^2 = \ell^2 \Big[- dt^2 + d\rho^2 + \rho^2 \left(d\theta^2 + \cos^2\theta \, d\phi_1^2 + \sin^2\theta \, d\phi_2^2 \right) \Big].^{86} \tag{D.1}$$

In particular we are going to derive the dispersion relations of rotating and pulsating strings.

D.1 Rotating String

Consider the rotating configuration (6.8):

$$\Big\{ t = \kappa\tau, \, \rho = \rho(\sigma), \, \theta = \kappa\omega\tau, \, \phi_1 = \phi_2 = 0 \Big\}, \tag{D.2}$$

inside the 5-dimensional flat background (D.1). The ansatz (6.40) can be obtained from (D.2) for $\rho \to \overline{\theta}_1$. The conformal gauge ($\gamma_{ab} = \eta_{ab}$) Polyakov action is given by:

$$S_P = \frac{T\ell^2}{2} \int \Big(- \dot{t}^2 + \rho^2 \, \dot{\theta}^2 - \rho'^2 \Big) d\tau d\sigma = \frac{T\ell^2}{2} \int \Big(- \kappa^2 + \kappa^2 \, \omega^2 \, \rho^2 - \rho'^2 \Big) d\tau d\sigma. \tag{D.3}$$

This is essentially the same as taking $\rho, \overline{\theta}_1 \to 0$ in the actions (6.11)–(6.42). κ is again a factor that guarantees $\sigma\left(\rho_0\right) = \pi/2$:

$$\sigma\left(\rho_0\right) = \frac{\pi}{2} = \int_0^{\rho_0} \frac{d\rho}{\kappa\sqrt{1 - \omega^2 \, \rho^2}} = \frac{\pi}{2\kappa\,\omega} \Rightarrow \kappa = \frac{1}{\omega} = \rho_0. \tag{D.4}$$

The conserved charges can be calculated either from the Polyakov action (D.3) or as the $\rho, \overline{\theta}_1 \to 0$ limits of the charges (6.15)–(6.16) and (6.47)–(6.48):

$$E = \frac{\ell^2}{2\pi\alpha'} \int_0^{2\pi} \kappa \, \cosh^2\rho \, d\sigma \xrightarrow{\rho \to 0} \frac{\ell^2}{2\pi\alpha'} \int_0^{2\pi} \frac{1}{\omega} \, d\sigma = \frac{\ell^2}{\omega\,\alpha'} = \frac{\sqrt{\lambda}}{\omega} \tag{D.5}$$

$$S = \frac{\ell^2}{2\pi\alpha'} \int_0^{2\pi} \kappa\,\omega \, \sinh^2\rho \, d\sigma \xrightarrow{\rho \to 0} \frac{\ell^2}{2\pi\alpha'} \int_0^{2\pi} \rho^2 \, d\sigma = \frac{\ell^2}{2\pi\alpha'} \int_0^{\rho_0} \frac{4\omega\rho^2 \, d\rho}{\sqrt{1 - \omega^2 \, \rho^2}} = \frac{\ell^2}{2\alpha'\omega^2} = \frac{\sqrt{\lambda}}{2\omega^2}. \tag{D.6}$$

We thus obtain the energy of the string as a function of its spin:

$$E = \left(2\sqrt{\lambda}\, S \right)^{1/2}. \tag{D.7}$$

[86] The $\ell^2 = \alpha'\sqrt{\lambda}$ factor in front of the flat metric has been included in order to enable the comparison between the flat spacetime results and those from $AdS_5 \times S^5$.

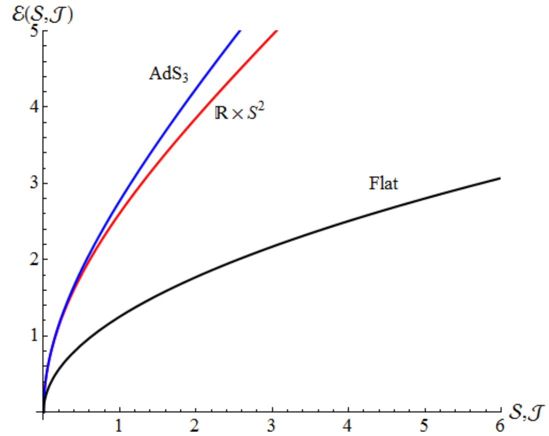


Figure 26: $\mathcal{E} = \mathcal{E}(\mathcal{S}, \mathcal{J})$ for rotating strings in AdS$_3$, $\mathbb{R}\times S^2$ and flat spacetimes.

Notice that (D.7) coincides with the leading term of the short series (6.30)–(6.62) of the GKP strings in AdS$_3$ and $\mathbb{R}\times S^2$. The subleading terms of these series are due to the curvature of spacetime and quantify the deviation of the string background from the flat metric (D.1). In figure 26 we have plotted in a common diagram the energy as a function of the spin for folded closed strings that rotate in either AdS$_3$ (6.22)–(6.23), $\mathbb{R}\times S^2$ (6.55)–(6.56), or the flat spacetime (D.7).

The universal scaling (D.7) for the leading contribution to the energy of small excitations inside anti-de Sitter space may also be obtained from the scaling dimensions (3.42) of scalar operators that are coupled to massive string states [7]. At strong coupling $\lambda \to \infty$, the scaling dimension of a generic scalar field of mass m in AdS$_{p+2}$ is given by (3.42):

$$\Delta_\pm = \frac{1}{2}\left(p+1\pm\sqrt{(p+1)^2+(2\,m\,\ell)^2}\right) = \frac{1}{2}\left(p+1\pm\sqrt{(p+1)^2+16\sqrt{\lambda}\,n}\right) \xrightarrow{\lambda\to\infty} 2\left(\sqrt{\lambda}n\right)^{1/2},$$

where $m^2 = 4\,n/\alpha'$ is the excitation level of the string and $S = 2n$. The scaling $E = 2\left(\sqrt{\lambda}n\right)^{1/2}$ of the string energy is valid for small n's.

D.2 Pulsating String

The pulsating GKP string configuration (6.78)

$$\Big\{t = t\left(\tau\right),\ \rho = \rho(\tau),\ \theta = 0,\ \phi_1 = w\sigma,\ \phi_2 = 0\Big\} \tag{D.8}$$

inside the flat background (D.1), is expected to reproduce the leading contribution to the energy of the pulsating GKP string (6.102) in the limit of small excitation levels n. The corresponding Polyakov action (in the conformal gauge, $\gamma_{ab} = \eta_{ab}$) is:

$$S_P = \frac{\ell^2}{4\pi\alpha'}\int\Big(-\dot{t}^2+\dot{\rho}^2-\rho^2\,\phi_1'^2\Big)d\tau d\sigma = \frac{\sqrt{\lambda}}{2}\int\Big(-\dot{t}^2+\dot{\rho}^2-w^2\,\rho^2\Big)d\tau. \tag{D.9}$$

The equations of motion and the Virasoro constraints correspond to harmonic motion:

$$\ddot{t} = 0 \Rightarrow t = \kappa\,\tau\,, \quad \ddot{\rho}+w^2\rho = 0\,, \quad \dot{\rho}^2-\kappa^2+w^2\rho^2 = 0. \tag{D.10}$$

Denoting by ρ_0 the classical turning point, we obtain the string length and the conserved energy:

$$E = \left|\frac{\partial L}{\partial \dot{t}}\right| = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} \dot{t}\, d\sigma = \kappa\sqrt{\lambda} \tag{D.11}$$

$$\tau\left(\rho\right) = \int_0^{\rho} \frac{d\rho}{\sqrt{\kappa^2 - w^2\,\rho^2}} = \frac{1}{w}\arcsin\frac{w\rho}{\kappa} \Leftrightarrow \rho\left(\tau\right) = \frac{\kappa}{w}\,\sin w\tau\,, \quad \rho_0 = \frac{\kappa}{w} = \frac{E}{w\sqrt{\lambda}} = e. \tag{D.12}$$

The system may now be first-quantized, as it was done in §6.3.1. The corresponding wave equation is:

$$-\hbar^2\,\psi''\left(\rho\right) = \left(E^2 - w^2\lambda\,\rho^2\right)\cdot\psi\left(\rho\right)\,, \quad \Psi\left(t,\rho\right) = e^{-i\,E\,t/\hbar}\cdot\psi\left(\rho\right). \tag{D.13}$$

This is a "half" harmonic oscillator. Imposing the boundary condition, $\psi\left(0\right) = \pm 1$, its eigenenergies are:

$$E = 2\left(\hbar\sqrt{\lambda}\,w\right)^{1/2}\cdot\left(n + \frac{1}{4}\right)^{1/2}\,, \quad n = 0\,,1\,,2\,,\ldots \tag{D.14}$$

which is (6.102) to lowest order. Another way to obtain this result has been given in [80].

E More Short-Long Dualities

In this appendix we will formulate some additional short-long dualities for the two rotating GKP configurations (I–II) and provide some classical expressions that link the conserved charges of strings that spin in AdS_3 to the charges of strings that rotate in $\mathbb{R}\times\text{S}^2$. Let us start with a few definitions:

Folded Strings in AdS_3

$$\mathcal{E}_1 \equiv \frac{\pi E_1}{\sqrt{\lambda}} = \frac{2\,\omega}{\omega^2-1}\,\mathbb{E} = \frac{2\sqrt{1-x}}{x}\,\mathbb{E} = \frac{2}{3}\sqrt{1-x}\Big(\mathbb{R}_D\left(0,x,1\right)+\mathbb{R}_D\left(0,1,x\right)\Big)$$

$$\mathcal{S}_1 \equiv \frac{\pi S_1}{\sqrt{\lambda}} = 2\left[\frac{\omega^2}{\omega^2-1}\,\mathbb{E}-\mathbb{K}\right] = 2\left[\frac{1}{x}\,\mathbb{E}-\mathbb{K}\right] = \frac{2}{3}\left(1-x\right)\mathbb{R}_D\left(0,1,x\right)$$

$$\gamma_1 = 2\left[\frac{\sqrt{1-x}-1}{x}\,\mathbb{E}+\mathbb{K}\right] = 2\left[\frac{\sqrt{1-x}-1}{3}\Big(\mathbb{R}_D\left(0,x,1\right)+\mathbb{R}_D\left(0,x,1\right)\Big)+\mathbb{R}_F\left(0,x,1\right)\right]$$

Folded Strings in $\mathbb{R}\times\text{S}^2$

$$\mathcal{E}_2 \equiv \frac{\pi E_2}{\sqrt{\lambda}} = \frac{2}{\omega}\,\mathbb{K} = 2\sqrt{1-x}\,\mathbb{K} = 2\sqrt{1-x}\,\mathbb{R}_F\left(0,x,1\right)$$

$$\mathcal{J}_2 \equiv \frac{\pi J_2}{\sqrt{\lambda}} = 2\left(\mathbb{K}-\mathbb{E}\right) = \frac{2}{3}\left(1-x\right)\mathbb{R}_D\left(0,x,1\right)$$

$$\gamma_2 = 2\left[\left(\sqrt{1-x}-1\right)\,\mathbb{K}+\mathbb{E}\right] = 2\left(\sqrt{1-x}-1\right)\mathbb{R}_F\left(0,x,1\right)+\frac{2x}{3}\Big(\mathbb{R}_D\left(0,x,1\right)+\mathbb{R}_D\left(0,x,1\right)\Big),$$

where the arguments of all the elliptic functions are $1/\omega^2 \equiv 1-x$. We find:

$$\mathcal{E}_1 = -\omega\,\frac{d\mathcal{E}_2}{d\omega} \qquad \& \qquad \mathcal{S}_1 = -\omega\,\frac{d\left(\omega\,\mathcal{E}_2\right)}{d\omega} = -\frac{d\left(\omega\,\mathcal{J}_2\right)}{d\omega} \tag{E.1}$$

$$\omega\,\mathcal{E}_2 = \omega\,\mathcal{E}_1-\mathcal{S}_1 = \mathcal{J}_2+\left(\omega-\frac{1}{\omega}\right)\mathcal{E}_1 = \omega^2\,\mathcal{J}_2+\left(\omega^2-1\right)\mathcal{S}_1 = 2\,\mathbb{K}\left(\frac{1}{\omega^2}\right) \tag{E.2}$$

$$\left(\omega-\frac{1}{\omega}\right)\mathcal{E}_1 = \left(1-\frac{1}{\omega^2}\right)\mathcal{S}_1+\left(\omega-\frac{1}{\omega}\right)\mathcal{E}_2 = \omega\,\mathcal{E}_2-\mathcal{J}_2 = \left(\omega^2-1\right)\left(\mathcal{S}_1+\mathcal{J}_2\right) = 2\,\mathbb{E}\left(\frac{1}{\omega^2}\right). \tag{E.3}$$

Plugging some of these relations into Legendre's relation (6.37), we find the following additional short-long formulas:

$$\frac{\omega'}{\omega}\,\mathcal{E}_1'\,\mathcal{E}_2+\frac{\omega}{\omega'}\,\mathcal{E}_1\,\mathcal{E}_2'-\omega\,\omega'\,\mathcal{E}_2\,\mathcal{E}_2' = 2\pi \tag{E.4}$$

$$\frac{1}{\omega'}\,\mathcal{S}_1\,\mathcal{E}_2'+\frac{1}{\omega}\,\mathcal{S}_1'\,\mathcal{E}_2 = 2\pi \quad \& \quad \frac{1}{\omega\,\omega'}\,\mathcal{E}_1\,\mathcal{E}_1'-\mathcal{J}_2\,\mathcal{J}_2' = 2\pi \tag{E.5}$$

$$\mathcal{S}_1\,\mathcal{J}_2'+\mathcal{S}_1'\,\mathcal{J}_2+\mathcal{S}_1\,\mathcal{S}_1' = 2\pi. \tag{E.6}$$

F Mathematica Code

This appendix contains Mathematica codes that generate the inverse spin functions x and the anomalous dimensions γ of the GKP strings (I–II), giant magnons and single spikes (elementary or doubled) in terms of their conserved spin/angular momentum $\mathcal{J}$, $\mathcal{S}$ and linear momentum p. The code can be directly copy-pasted and run with Mathematica. Some of the results that have been obtained with these algorithms have been put in the following appendix G.

F.1 GKP Strings in $\mathbb{R} \times \mathrm{S}^2$

F.1.1 Long Folded Strings ($\omega \to 1^+$)

Let us start from the long folded ($\omega \to 1^+$) string in $\mathbb{R} \times \mathrm{S}^2$. The inverse spin function $x = x(\mathcal{J})$ is given by x[m, J, v], where m is the number of terms in the series, the variable J corresponds to the (rescaled) angular momentum $\mathcal{J} = \pi J/\sqrt{\lambda}$ and v stands for $e^{-\mathcal{J}-2}$. The anomalous dimensions $\gamma = \mathcal{E} - \mathcal{J} = \gamma(\mathcal{J})$ are given by gamma[m, J, z], with m and J as before and z being the computed value of x, x[m, J, v].

The last three lines of the code are actually the ones that produce the output. The number of nn = 10 terms in x[nn, J, v] and gamma[nn, J, z] is adjustable. The reader may well-change this value, depending on the desired output length and the available computer power. As an indication, nn = 13 terms take about 42s in our system. The equations (G.2)–(G.3) of appendix G, contain the first few terms of the result.

```
d[n_]:=-(1/2)((2n-1)!!/(2n)!!)^2;
h[n_]:=(-d[n])*(4*Log[2]+2*Sum[1/k-2/(2*k-1),{k,1,n}]);
c[n_]:=-(d[n]/(2*n-1));
b[n_]:=(-c[n])*(4*Log[2]+2*Sum[1/k-2/(2*k-1),{k,1,n}]+2/(2*n-1));
f[n_]:=-c[n]-Sum[((2*k-3)!!/(2*k)!!)*d[n-k],{k,0,n}];
g[n_]:=-b[n]-Sum[((2*k-3)!!/(2*k)!!)*h[n-k],{k,0,n}];
A[n_,J_]:=g[n]+f[n]*(4*Log[2]-J-2);
y[m_,J_,x_]:=Series[x*Exp[Sum[(b[n]/c[0])*x^n,{n,1,m}]-(((J/2)-b[0])/c[0]
            -Sum[(b[n]/c[0])*x^n,{n,1,m}])*Sum[(-1)^k*Sum[(c[l]/c[0])*x^l,{l,1,m}]
            ^k,{k,1,m}]],{x,0,m}];
x[m_,J_,v_]:=InverseSeries[(1/16)*y[m,J,x],v];
gamma[m_,J_,z_]:=2*Sum[z^p*(A[p,J]+f[p]*Log[z/(16*v)]),{p,0,m}];
nn = 10;
x[nn,J,v];
Normal[%]/.v->E^(-J-2)
Simplify[Collect[FullSimplify[gamma[nn,J,z]/.z->%%],{v,J}]]/.v->E^(-J-2)
```

One may recognize d[n], h[n], c[n], b[n] as the series coefficients d_n, h_n, c_n, b_n given in (6.67). f[n], g[n] and A[n, J] are respectively the coefficients f_n, g_n, A_n of (7.30)–(7.32). The series y[m, J, x] is derived by exponentiating and rearranging equation (7.18).

F.1.2 Fast Circular Strings ($\omega \to 1^-$)

With appropriate modifications, the previous algorithm can also be applied to the case of fast circular ($\omega \to 1^-$) strings in $\mathbb{R} \times \mathrm{S}^2$. Here again, `d[n]`, `h[n]`, `c[n]`, `b[n]` are the series coefficients d_n, h_n, c_n, b_n of (6.67), however the coefficients that appear in equation (6.75) are slightly different and are actually given by `cc[n]` and `bb[n]`. The coefficients `ff[n]`, `gg[n]` and `AA[n, J]` stand for f_n, g_n and A_n respectively, written down in equations (7.67)–(7.68). This algorithm (with `nn = 13` terms) took about 40s to run in our system. The first few terms of the output appear in equations (G.4)–(G.5) of appendix G.

```
d[n_]:=(-(1/2))*((2*n-1)!!/(2*n)!!)^2;

h[n_]:=(-d[n])*(4*Log[2]+2*Sum[1/k-2/(2*k-1),{k,1,n}]);

c[n_]:=-(d[n]/(2*n-1));

b[n_]:=(-c[n])*(4*Log[2]+2*Sum[1/k-2/(2*k-1),{k,1,n}]+2/(2*n-1));

cc[n_]:=Sum[((2*k-1)!!/(2*k)!!)*c[n-k],{k,0,n}];

bb[n_]:=Sum[((2*k-1)!!/(2*k)!!)*b[n-k],{k,0,n}];

ff[n_]:=d[n]-cc[n];

gg[n_]:=h[n]-bb[n];

AA[n_,J_]:=gg[n]+ff[n]*(4*Log[2]-J-2);

y[m_,J_,x_]:=Series[(x/16)*Exp[Sum[(bb[n]/cc[0])*x^n,{n,1,m}]-((J-2*bb[0])/
              (2*cc[0])-Sum[(bb[n]/cc[0])*x^n,{n,1,m}])*Sum[(-1)^k*
              Sum[(cc[l]/cc[0])*x^l,{l,1,m}]^k,{k,1,m}]],{x,0,m}];

x[m_,J_,v_]:=InverseSeries[y[m,J,x],v];

gamma[m_,J_,z_]:=2*Sum[z^p*(AA[p,J]+ff[p]*Log[z/(16*v)]),{p,0,m}];

nn = 13;

x[nn,J,v];

Normal[%]/.v->E^(-J-2)

Simplify[Collect[FullSimplify[gamma[nn,J,z]/.z->%%],{v,J}]]/.v->E^(-J-2)
```

F.2 GKP Strings in AdS_3

Mathematica may also be put to invert equation (7.71) for long, closed and folded single-spin strings that spin inside AdS_3, GKP case (I). This way, exact expressions for the inverse spin function $x = x(\mathcal{S})$ and the anomalous dimensions $\gamma = \mathcal{E} - \mathcal{S} = \gamma(\mathcal{S})$ can be obtained, see equations (G.6)–(G.7) in appendix G. However, due to the presence of logarithms in the corresponding expansions, a rather different approach than that for strings in $\mathbb{R} \times \mathrm{S}^5$ must be followed. We make the following change of variables in equation (7.71):

$$x = \frac{2e^u}{\mathcal{S}}, \tag{F.1}$$

so that (7.71) becomes:

$$\ln \mathcal{S} = u + \ln 2 + \left[\frac{b_0}{c_0} + \sum_{n=1}^{\infty}\left[\frac{\mathcal{S}}{2c_0}\frac{(-u)^n}{n!} + \frac{2^n b_n}{c_0}\frac{e^{nu}}{\mathcal{S}^n}\right]\right] \cdot \sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{2^k c_k}{c_0}\frac{e^{ku}}{\mathcal{S}^k}\right)^n. \tag{F.2}$$

If we invert this equation for u, the variable $x = x(\mathcal{S})$ can be obtained from equation (F.1). Then, $x = x(\mathcal{S})$ may be inserted into the equation (7.85) and give the anomalous dimensions $\gamma = \gamma(\mathcal{S})$. Here's the Mathematica code:

```
d[n_]:=(-(1/4))*((2*n-1)!!/(2*n)!!)^2*((2*n+1)/(n+1));

h[n_]:=(-d[n])*(4*Log[2]+2*Sum[1/k-2/(2*k-1),{k,1,n}]+1/(n+1)-2/(2*n+1));

c[n_]:=-(d[n]/(2*n+1));

b[n_]:=(-c[n])*(4*Log[2]+2*Sum[1/k-2/(2*k-1),{k,1,n}]+1/(n+1));

f[n_]:=-c[n]-Sum[((2*k-3)!!/(2*k)!!)*d[n-k],{k,0,n}];

g[n_]:=-b[n]-(2*n-1)!!/(2*n+2)!!-Sum[((2*k-3)!!/(2*k)!!)*h[n-k],{k,0,n}];

A[n_,S_]:=g[n]+f[n]*(((S/2)-b[0])/c[0]+c[1]/c[0]^2);

y[m_,S_,u_]:=Series[u+Log[2]+(b[0]/c[0]+(1/c[0])*Sum[(S*(-u)^n)/(2*n!)+(2^n*b[n]*
             Exp[n*u])/S^n,{n,1,m}])/((1/c[0])*Sum[(2^k*c[k]*Exp[k*u])/S^k,
             {k,0,m}]),{u,0,m}];

x[m_,S_,v_]:=Series[(2/S)*Exp[InverseSeries[y[m,S,u],v]],{S,Infinity,m}];

SpinSeries[x_,S_,m_]:=Series[(-(1/x)+(S/2)-Sum[b[n]*x^n,{n,0,m}])/Sum[c[n]*x^n,
                      {n,0,m}],{x,0,m}];

a[n_,S_,m_]:=Coefficient[SpinSeries[x,S,m],x^n];

gamma[m_,S_,z_]:=2*Series[-((4*f[0])/z)+A[0,S]+Sum[z^n*(A[n,S]-4*f[n+1]+Sum[f[n-k-1]
                 *a[k+1,S,m],{k,0,n-1}]),{n,1,m}],{z,0,m}];

nn = 7;

x[nn,S,v];

Collect[Refine[Collect[%/.{v->Log[S]},{S,Log[2]}],S>0],{S,Log[S]}]

Collect[Refine[Collect[Normal[gamma[nn,S,z]]/.{z->%%}/.{v->Log[S]},
                       {S,Log[S],Log[2]}],S>0],{S,Log[S]}]
```

In the above algorithm, d[n], h[n], c[n], b[n] are the series coefficients d_n, h_n, c_n, b_n given in (6.35) and f[n], g[n], A[n, S] are respectively the coefficients f_n, g_n, A_n of (7.87)–(7.89). The series y[m, S, u] and x[m, S, v] parametrize the equations (F.1)–(F.2) that we saw above. The anomalous dimensions are computed from equation (7.91) with the variable gamma[m, S, z]. To compute the latter we need the coefficients a_n from equation (7.72), which we write as a[n, S, m] and we find from equation (7.71), SpinSeries[x, S, m] in Mathematica. The output is again generated from the last three lines. For nn = 7 terms, the program took about 30s to run in our system.

F.3 Giant Magnons

F.3.1 Giant Magnon, Elementary Region: $0 \le |v| < 1/\omega \le 1$

Here's the Mathematica code for giant magnons in the elementary region ($0 \le |v| < 1/\omega \le 1$):

```
d[n_]:=(-(1/2))*((2*n-1)!!/(2*n)!!)^2

h[n_]:=-4*d[n]*(Log[2]+HarmonicNumber[n]-HarmonicNumber[2*n])

c[n_]:=-(d[n]/(2*n-1))

b[n_]:=-4*c[n]*(Log[2]+HarmonicNumber[n]-HarmonicNumber[2*n]+1/(2*(2*n-1)))

momentum[m_,a_,x_]:=Series[(Pi*EllipticF[a,x])/EllipticK[x]+((2*(1-x)*Tan[a])/
                    (EllipticK[x]*Sqrt[1-x*Sin[a]^2]))*(EllipticK[x]-
                    EllipticPi[(x*Cos[a]^2)/(1-x*Sin[a]^2),y])*(Sum[x^n*h[n],{n,0,m}]
                    +(Sum[x^n*d[n],{n,0,m}]/Sum[x^n*c[n],{n,0,m}])*(J/Sin[a]-
                    Sum[x^n*b[n],{n,0,m}])),{x,0,m}]/.y->x

velocity[m_,p_,x_]:=Series[Sin[Normal[InverseSeries[Series[Normal[
                    FullSimplify[momentum[m,a,x]]],{a,p/2,m}]-p]]/.a->0],{x,0,m}]

prefactor[m_,p_,x_]:=Series[((1-x)*z)/Sqrt[1-x*z^2],{x,0,m}]/.z->velocity[m,p,x]

energy[m_,p_,x_]:=prefactor[m,p,x]*Series[Sum[x^n*(d[n]*Log[x]+h[n]),{n,0,m}],{x,0,m}]

spin[m_,p_,x_]:=velocity[m,p,x]*Series[Sum[x^n*(c[n]*Log[x]+b[n]),{n,0,m}],{x,0,m}]

adimension[m_,p_,x_]:=energy[m,p,x]-spin[m,p,x]

A[n_,J_,p_]:=gg[n]+2*ff[n]*(2*Log[2]-J/Sin[p/2]-1)

x1[m_,J_,x_]:=Series[(x/16)*Exp[Sum[(bb[n]/cc[0])*x^n,{n,1,m}]-((J-bb[0])/cc[0]-
              Sum[(bb[n]/cc[0])*x^n,{n,1,m}])*Sum[(-1)^k*Sum[(cc[l]/cc[0])*x^l,
              {l,1,m}]^k,{k,1,m}]],{x,0,m}]

x2[m_,J_,v_]:=InverseSeries[x1[m,J,x],v]

\[Gamma][m_,p_,J_,v_]:=Sum[z^n*(A[n,J,p]+ff[n]*Log[z/(16*v)]),{n,0,m}]/.z->x2[m,J,v]

nn = 3; Collect[spin[nn,p,x]/.{Log[x]->y},{x,y,J}];

Do[cc[n]=Collect[If[n==0,Coefficient[%/.x->0,y],Coefficient[%,x^n*y]],J],{n,0,nn}];

Do[bb[n]=Collect[If[n==0,%%/.{x->0,y->0},Coefficient[%%,x^n]/.y->0],J],{n,0,nn}];

Collect[adimension[nn,p,x]/.{Log[x]->y},{x,y,J}];

Do[ff[n]=Collect[If[n==0,Coefficient[%/.x->0,y],Coefficient[%,x^n*y]],J],{n,0,nn}];

Do[gg[n]=Collect[If[n==0,%%/.{x->0,y->0},Coefficient[%%,x^n]/.y->0],J],{n,0,nn}];

Collect[velocity[nn,p,x],{x,J}]

Collect[x2[nn,J,v],{v,J},FullSimplify]/.v->E^(-L)

Collect[FullSimplify[\[Gamma][nn,p,J,v]],{v,J},FullSimplify]/.v->E^(-L)
```

Let us describe what the above code does. The goal is to obtain the dispersion relation of elementary region giant magnons in terms of the string's conserved charges p and $\mathcal{J}$. We originally know it (10.25) as a function of the GM linear and angular velocities v and ω, which appear in our system through the associated variables $v = \cos a$ and x, defined in equation (10.22). We need to eliminate these in favor of the charges p and $\mathcal{J}$, given in equations (10.24)–(10.26). This entails the following steps. First we eliminate the logarithms from the equations (10.24)–(10.26). The resulting equation (10.30), given by function `momentum[m, a, x]`, is expanded in a double series in `a` and `x` around $a = p/2$ and $x = 0$. The variable `m` denotes the number of terms that we keep in our expansions. The series `momentum[m, a, x]` is subsequently inverted for the variable a. The result for $\sin a$ is encoded in the function `velocity[m, p, x]`.

$\sin a$ that we found is then inserted into the expression for the angular momentum $\mathcal{J}$ that is given in equation (10.31). We get the function `spin[m, p, x]`, which we expand for `x` in order to compute the coefficients `cc[n]` and `bb[n]`. Now we know $\mathcal{J}$ in a form like (7.18), which we may invert for the inverse spin function `x2[m, J, v]` à la GKP.[87] As an intermediate step we must compute the function `x1[m, J, v]` that encodes the second line of (7.18).

The final step is to insert the inverse spin function that we found into the relation for the energy minus the spin (10.45), which we write as `\[Gamma][m, p, J, v]`. In order to be able to do this, we need the coefficients f_n, g_n and A_n, (10.44)–(10.46) of the series (10.25). The corresponding function is `adimension[m, p, x]`, defined in terms of the functions `prefactor[m, p, x]`, `energy[m, p, x]` and `spin[m, p, x]`. In the above code, the coefficients f_n, g_n and A_n are denoted by `ff[n]`, `gg[n]` and `A[n]` respectively. For completeness, let us also mention that the coefficients d_n, h_n, c_n, b_n in (10.29), are given by the Mathematica variables `d[n]`, `h[n]`, `c[n]` and `b[n]`.

Some of the results that can be obtained with our code have been placed the appendix G.2. The output contains the inverse momentum $\sin a = \sqrt{1 - v^2}$ as a function of the magnon's momentum p, spin $\mathcal{J}$ and the inverse spin function x, the inverse spin function x as a function of the conserved charges $\mathcal{J}$ and p and the dispersion relation $\gamma = \gamma\,(p, \mathcal{J})$. See (G.10)–(G.12). The number of terms in the output `nn` is adjustable. For example `nn = 3` took about 30s to run in our system.

F.3.2 Giant Magnon, Doubled Region: $0 \leq |v| \leq 1 \leq 1/\omega$

The skeleton of the code for finite-size giant magnons in the doubled region is the same as in the elementary region. We only have to change the values of the series coefficients of the conserved charges `d[n]`, `h[n]`, `c[n]`, `b[n]`, the expressions of the conserved charges, as well as the equations that we have to invert in order to eliminate the variables v and ω in favor of p and $\mathcal{J}$.

As before, the code calculates the inverse momentum $\sin a = \sqrt{1 - v^2}$, the inverse spin function x and the dispersion relation $\gamma = \gamma\,(p, \mathcal{J})$. For the latter, see equation (G.13). Again, the number of terms `nn` in the output is adjustable. E.g. for the proposed value of `nn = 3`, our system took about 30s to run. Here's the Mathematica code:

```
d[n_]:=(-(1/2))*((2*n-1)!!/(2*n)!!)^2

h[n_]:=-4*d[n]*(Log[2]+HarmonicNumber[n]-HarmonicNumber[2*n])

c[n_]:=-(d[n]/(2*n-1))

b[n_]:=-4*c[n]*(Log[2]+HarmonicNumber[n]-HarmonicNumber[2*n]+1/(2*(2*n-1)))
```

[87] The variable `v` does not stand for the GM's velocity but for $\exp(-\mathcal{L})$, where $\mathcal{L}$ is given by:

$$\mathcal{L} \equiv 2\mathcal{J} \csc\frac{p}{2} + 2.$$

.

```
momentum[m_,a_,x_]:=FullSimplify[Series[(Pi*EllipticF[ArcSin[Sin[a]/
                    Sqrt[1-x*Cos[a]^2]],x])/EllipticK[x]+((2*Tan[a])/
                    Sqrt[1-x*Cos[a]^2])*(1-(1-x*Cos[a]^2)*(EllipticPi[x*Cos[a]^2,y]/
                    EllipticK[x]))*(Sum[x^n*h[n],{n,0,m}]+(Sum[x^n*d[n],{n,0,m}]/
                    Sum[x^n*c[n],{n,0,m}])*(Sqrt[1-x]*(J/Sin[a])-Sum[x^n*b[n],
                    {n,0,m}])),{x,0,m}]/.y->x]
velocity[m_,p_,x_]:=Series[Sin[Normal[InverseSeries[FullSimplify[Series[Normal[
                    momentum[m,a,x]],{a,p/2,m}],{p>0,p<Pi}]-p]]/.a->0],{x,0,m}]
prefactor[m_,p_,x_]:=Series[z/Sqrt[1-x*(1-z^2)],{x,0,m}]/.z->velocity[m,p,x]
energy[m_,p_,x_]:=prefactor[m,p,x]*Series[Sum[x^n*(d[n]*Log[x]+h[n]),{n,0,m}],{x,0,m}]
spin[m_,p_,x_]:=(velocity[m,p,x]/Sqrt[1-x])*Series[Sum[x^n*(c[n]*Log[x]+b[n]),{n,0,m}]
                ,{x,0,m}]
adimension[m_,p_,x_]:=energy[m,p,x]-spin[m,p,x]
A[n_,J_,p_]:=gg[n]+2*ff[n]*(2*Log[2]-J/Sin[p/2]-1)
x1[m_,J_,x_]:=Series[(x/16)*Exp[Sum[(bb[n]/cc[0])*x^n,{n,1,m}]-((J-bb[0])/cc[0]-
              Sum[(bb[n]/cc[0])*x^n,{n,1,m}])*Sum[(-1)^k*Sum[(cc[l]/cc[0])*x^l,
              {l,1,m}]^k,{k,1,m}]],{x,0,m}]
x2[m_,J_,v_]:=InverseSeries[x1[m,J,x],v]
\[Gamma][m_,p_,J_,v_]:=Sum[z^n*(A[n,J,p]+ff[n]*Log[z/(16*v)]),{n,0,m}]/.z->x2[m,J,v]
nn = 3; Collect[spin[nn,p,x]/.{Log[x]->y},{x,y,J}];
Do[cc[n]=Collect[If[n==0,Coefficient[%/.x->0,y],Coefficient[%,x^n*y]],J],{n,0,nn}];
Do[bb[n]=Collect[If[n==0,%%/.{x->0,y->0},Coefficient[%%,x^n]/.y->0],J],{n,0,nn}];
Collect[adimension[nn,p,x]/.{Log[x]->y},{x,y,J}];
Do[ff[n]=Collect[If[n==0,Coefficient[%/.x->0,y],Coefficient[%,x^n*y]],J],{n,0,nn}];
Do[gg[n]=Collect[If[n==0,%%/.{x->0,y->0},Coefficient[%%,x^n]/.y->0],J],{n,0,nn}];
Collect[velocity[nn,p,x],{x,J}]
Collect[x2[nn,J,v],{v,J},FullSimplify]/.v->E^(-L)
Collect[FullSimplify[\[Gamma][nn,p,J,v]],{v,J},FullSimplify]/.v->E^(-L)
```

F.4 Single Spikes

F.4.1 Single Spike, Elementary Region: $0 \leq 1/\omega < |v| \leq 1$

Just as the algorithm that we followed in order to obtain the analytic dispersion relations of single spikes (10.53)–(10.54) was rather different from the ones for giant magnons (10.49)–(10.51), the numeric procedure with Mathematica is also expected to be a little different.

As before, the logarithms must be eliminated from the equations (9.30)–(9.27). The resulting equation is `spin[m, a, x, p]`, which is expanded in a double series in `a` and `x` around $a = q/2$ and $x = 0$.[88] `spin[m, a, x, p]` is subsequently inverted for the variable a. We find $\sin a$ as `\[Omega][m, p, x]`.

$\sin a$ is then inserted into the expression for the linear momentum p. We obtain `momentum[m, p, x]` which we expand for `x` in order to compute the coefficients `cc[n]` and `bb[n]`. The result is p in a form like (7.18), which we invert for `x2[m, J, v]` à la GKP.[89] The function `x1[m, J, v]` encodes again the second line of (7.18).

The inverse spin function `x2[m, J, v]` that we have found is then inserted into the relation that gives the energy minus half the single spikes's linear momentum $\mathcal{E} - p/2$. In Mathematica this is written as `\[Gamma][m, p, \[Theta], v]`. But before carrying out this step, the coefficients f_n, g_n and A_n (`ff[n]`, `gg[n]` and `A[n]` in Mathematica) must be calculated. These are the single spike analogues of the giant magnon coefficients (10.44)–(10.46). They are calculated from the function `adimension[m, p, x]` which is defined in terms of `prefactor[m, p, x]`, `energy[m, p, x]` and `momentum[m, p, x]`.

The following code calculates the inverse momentum $\sin a = \sqrt{1 - 1/\omega^2}$, the inverse spin function x and the dispersion relation $\gamma = \gamma\,(p, \mathcal{J})$. See equation (G.14), in appendix G.2. The number of terms in the output is `nn`. For example `nn = 3` took about 60s to run in our system.

```
d[n_]:=(-(1/2))*((2*n-1)!!/(2*n)!!)^2

h[n_]:=-4*d[n]*(Log[2]+HarmonicNumber[n]-HarmonicNumber[2*n])

c[n_]:=If[n==0,0,-(d[n-1]/(2*n))]

b[n_]:=If[n==0,1,((2*d[n-1])/n)*(Log[2]+HarmonicNumber[n-1]-HarmonicNumber[2*n-2]+
        1/(4*n))]

spin[m_,a_,x_,p_]:=FullSimplify[Series[Sin[a]*(Sum[x^n*b[n],{n,0,m}]+(Sum[x^n*c[n],
                    {n,0,m}]/Sum[x^n*d[n],{n,0,m}])*(((p+Pi*(EllipticF[a,x]/
                    EllipticK[x]))/EllipticPi[(x*Cos[a]^2)/(1-x*Sin[a]^2),y])*
                    ((EllipticK[x]*Sqrt[1-x*Sin[a]^2])/(2*(1-x)*Tan[a]))-Sum[x^n*
                    h[n],{n,0,m}])),{x,0,m}]/.y->x]

\[Omega][m_,p_,x_]:=Series[Normal[Sin[InverseSeries[Series[Normal[FullSimplify[
                     spin[m,a,x,p]]],{a,\[Theta],m}]-Sin[\[Theta]]]]]/.{a->0},
                     {x,0,m}]

momentum[m_,p_,x_]:=Series[(-2/(z*Sqrt[1-x*z^2]*EllipticK[x]))*((Pi/2)*z*Sqrt[1-
                     x*z^2]*EllipticF[ArcSin[z],x]-(((1-x)*z^2)/Sqrt[1-z^2])*Normal
                     [Series[EllipticPi[(x*(1-z^2))/(1-x*z^2),y],{x,0,m}]/.y->x]*Sum
                     [x^n*(d[n]*Log[x]+h[n]),{n,0,m}]),{x,0,m}]/.z->\[Omega][m,p,x]
```

[88] q is defined as $\mathcal{J} \equiv \sin q/2$ and it is encoded in the Mathematica variable `\[Theta]` which stands for $q/2$. We have also set $1/\omega \equiv \cos a$.

[89] Here the variable `v` stands for $\exp(-\mathcal{R})$, where $\mathcal{R}$ is given by:

$$\mathcal{R} \equiv \sqrt{\frac{1}{\mathcal{J}^2} - 1} \cdot (p + 2 \arcsin \mathcal{J}) = (p + q) \cdot \cot \frac{q}{2}.$$

.

```
prefactor[m_,p_,x_]:=Series[z/Sqrt[1-z^2],{x,0,m}]/.z->\[Omega][m,p,x]
energy[m_,p_,x_]:=prefactor[m,p,x]*(1-x)*Series[Sum[x^n*(d[n]*Log[x]+h[n]),{n,0,m}],
              {x,0,m}]
adimension[m_,p_,x_]:=energy[m,p,x]-(1/2)*momentum[m,p,x]
A[n_,\[Theta]_,p_]:=gg[n]+ff[n]*((-p)*Cot[\[Theta]]-2*\[Theta]*Cot[\[Theta]]+Log[16])
x1[m_,p_,x_]:=Series[(x/16)*Exp[Sum[(bb[n]/cc[0])*x^n,{n,1,m}]-((p-bb[0])/cc[0]-
           Sum[(bb[n]/cc[0])*x^n,{n,1,m}])*Sum[(-1)^k*Sum[(cc[l]/cc[0])*x^l,
           {l,1,m}]^k,{k,1,m}]],{x,0,m}]
x2[m_,p_,v_]:=InverseSeries[x1[m,p,x],v]
\[Gamma][m_,p_,\[Theta]_,v_]:=Sum[z^n*(A[n,\[Theta],p]+ff[n]*Log[z/(16*v)]),
                        {n,0,m}]/.z->x2[m,p,v]
nn = 3;Refine[Collect[momentum[nn,p,x]/.{Log[x]->y},{x,y},Simplify],
          {\[Theta]>0,\[Theta]<Pi/2}];
Do[cc[n]=Collect[If[n==0,Coefficient[%/.x->0,y],Coefficient[%,x^n*y]],J],{n,0,nn}];
Do[bb[n]=Collect[If[n==0,%%/.{x->0,y->0},Coefficient[%%,x^n]/.y->0],J],{n,0,nn}];
Refine[Collect[adimension[nn,p,x]/.{Log[x]->y},{x,y},Simplify],
     {\[Theta]>0,\[Theta]<Pi/2}];
Do[ff[n]=Collect[If[n==0,Coefficient[%/.x->0,y],Coefficient[%,x^n*y]],J],{n,0,nn}];
Do[gg[n]=Collect[If[n==0,%%/.{x->0,y->0},Coefficient[%%,x^n]/.y->0],J],{n,0,nn}];
Collect[\[Omega][nn,p,x],{x,p}]/.\[Theta]->q/2
Collect[x2[nn,p,v],{v,p},FullSimplify]/.{v->E^(-R),\[Theta]->q/2}
Collect[\[Gamma][nn,p,\[Theta],v],{v,p,\[Theta]},FullSimplify]/.{v->E^(-R),
      \[Theta]->q/2}
```

F.4.2 Single Spike, Doubled Region: $0 \leq 1/\omega \leq 1 \leq |v|$

For single spikes in the doubled region, the above algorithm takes the following form:

```
d[n_]:=(-(1/2))*((2*n-1)!!/(2*n)!!)^2
h[n_]:=-4*d[n]*(Log[2]+HarmonicNumber[n]-HarmonicNumber[2*n])
c[n_]:=If[n==0,0,((2*n)/(2*n-1))*d[n]]
b[n_]:=If[n==0,1,(-((4*n)/(2*n-1)))*d[n]*(2*Log[2]+2*HarmonicNumber[n-1]-2*
      HarmonicNumber[2*n-2]-1/(2*n*(2*n-1)))]
\[Omega][m_,p_,x_]:=Series[Normal[Sin[InverseSeries[Series[Normal[FullSimplify[
                   spin[m,a,x,p]]],{a,\[Theta],m}]-Sin[\[Theta]]]]]/.{a->0},
                   {x,0,m}]
```

```
spin[m_,a_,x_,p_]:=FullSimplify[Series[(Sin[a]/Sqrt[1-x])*(Sum[x^n*b[n],
                   {n,0,m}]+(Sum[x^n*c[n],{n,0,m}]/Sum[x^n*d[n],{n,0,m}])*
                   (((p+Pi*(EllipticF[a,x]/EllipticK[x]))/EllipticPi[x*Cos[a]^2,y])
                   *(EllipticK[x]/(2*Sqrt[1-x*Cos[a]^2]*Tan[a]))-Sum[x^n*h[n]
                   ,{n,0,m}])),{x,0,m}]/.y-> x]
momentum[m_,p_,x_]:=Series[(2/EllipticK[x])*(\[Omega][m,p,x]*(Sqrt[1-x*(1-
                    \[Omega][m,p,x]^2)]/Sqrt[1-\[Omega][m,p,x]^2])*Normal[
                    Series[EllipticPi[x*(1-\[Omega][m,p,x]^2),y],{x,0,m}]/.y->x]
                    *Sum[x^n*(d[n]*Log[x]+h[n]),{n,0,m}]-(Pi/2)*EllipticF[ArcSin[
                    \[Omega][m,p,x]/Sqrt[1-x*(1-\[Omega][m,p,x]^2)]],x]),{x,0,m}]
prefactor[m_,p_,x_]:=Series[\[Omega][m,p,x]/Sqrt[1-\[Omega][m,p,x]^2],{x,0,m}]
energy[m_,p_,x_]:=(prefactor[m,p,x]/Sqrt[1-x])*Series[Sum[x^n*(d[n]*Log[x]+h[n]),
                  {n,0,m}],{x,0,m}]
adimension[m_,p_,x_]:=energy[m,p,x]-(1/2)*momentum[m,p,x]
A[n_,\[Theta]_,p_]:=gg[n]+ff[n]*((-p)*Cot[\[Theta]]-2*\[Theta]*Cot[\[Theta]]+Log[16])
x1[m_,p_,x_]:=Series[(x/16)*Exp[Sum[(bb[n]/cc[0])*x^n,{n,1,m}]-((p-bb[0])/cc[0]-
              Sum[(bb[n]/cc[0])*x^n,{n,1,m}])*Sum[(-1)^k*Sum[(cc[l]/cc[0])*x^l,
              {l,1,m}]^k,{k,1,m}]],{x,0,m}];
x2[m_,p_,v_]:=InverseSeries[x1[m,p,x],v]
\[Gamma][m_,p_,\[Theta]_,v_]:=Sum[z^n*(A[n,\[Theta],p]+ff[n]*Log[z/(16*v)]),
                              {n,0,m}]/.z->x2[m,p,v]
nn = 3;Refine[Collect[momentum[nn,p,x]/.{Log[x]->y},{x,y},Simplify],
             {\[Theta]>0,\[Theta]< Pi/2}];
Do[cc[n]=Collect[If[n==0,Coefficient[%/.x->0,y],Coefficient[%,x^n*y]],J],{n,0,nn}];
Do[bb[n]=Collect[If[n==0,%%/.{x->0,y->0},Coefficient[%%,x^n]/.y->0],J],{n,0,nn}]
Refine[Collect[adimension[nn,p,x]/.{Log[x]->y},{x,y},Simplify],{\[Theta]>0,
      \[Theta]<Pi/2}];
Do[ff[n]=Collect[If[n==0,Coefficient[%/.x->0,y],Coefficient[%,x^n*y]],J],{n,0,nn}];
Do[gg[n]=Collect[If[n==0,%%/.{x->0,y->0},Coefficient[%%,x^n]/.y->0],J],{n,0,nn}];
Collect[\[Omega][nn,p,x],{x,p}]/.\[Theta]->q/2
Collect[x2[nn,p,v],{v,p},FullSimplify]/.{v->E^(-R),\[Theta]->q/2}
Collect[\[Gamma][nn,p,\[Theta],v],{v,p,\[Theta]},FullSimplify]/.{v->E^(-R),
        \[Theta]->q/2}
```

G Symbolic Computations

This appendix contains some results of the symbolic computations that were performed with the Mathematica codes of the previous appendix F. They can be used to verify the Lambert W-function expressions of the string dispersion relations that were derived in §7 and §10.

G.1 Long and Fast GKP Strings

Let us begin with long and fast GKP strings in $\mathbb{R}\times\mathrm{S}^2$ and AdS_3. Here are the Mathematica results:

- Folded string in $\mathbb{R}\times\mathrm{S}^2$ $(\omega>1)$.[90]

$$x = 16\,e^{-\mathcal{J}-2} - 64\,(\mathcal{J}+2)\;e^{-2\mathcal{J}-4} + 64\left(6\mathcal{J}^2+17\mathcal{J}+15\right)\,e^{-3\mathcal{J}-6} - \frac{256}{3}(32\mathcal{J}^3+108\mathcal{J}^2+153\mathcal{J}+84)\,e^{-4\mathcal{J}-8}+ + \frac{32}{3}(2000\mathcal{J}^4+7600\mathcal{J}^3+13.740\mathcal{J}^2+12.726\mathcal{J}+4989)\,e^{-5\mathcal{J}-10} - \frac{512}{5}(1728\mathcal{J}^5+7200\mathcal{J}^4+15300\mathcal{J}^3+ +18.615\mathcal{J}^2+12.740\mathcal{J}+3855)\,e^{-6\mathcal{J}-12}+\ldots \tag{G.2}$$

$$\mathcal{E}-\mathcal{J} = 2-8e^{-\mathcal{J}-2} + 8\,(2\mathcal{J}-1)\;e^{-2\mathcal{J}-4} - 32\left(2\mathcal{J}^2-\mathcal{J}+2\right)\,e^{-3\mathcal{J}-6} + \frac{8}{3}\left(128\mathcal{J}^3-48\mathcal{J}^2+228\mathcal{J}-63\right)\,e^{-4\mathcal{J}-8}- -\frac{16}{3}(400\mathcal{J}^4-80\mathcal{J}^3+972\mathcal{J}^2-330\mathcal{J}+279)\,e^{-5\mathcal{J}-10} + \frac{64}{5}(1152\mathcal{J}^5+3480\mathcal{J}^3-1010\mathcal{J}^2+2080\mathcal{J}- -405)\,e^{-6\mathcal{J}-12}-\ldots \tag{G.3}$$

- Circular string in $\mathbb{R}\times\mathrm{S}^2$ $(\omega<1)$.

$$\widetilde{x} = 16\,e^{-\mathcal{J}-2} + 64\,(\mathcal{J}-2)\;e^{-2\mathcal{J}-4} + 192\left(2\mathcal{J}^2-5\mathcal{J}+5\right)\,e^{-3\mathcal{J}-6} + \frac{256}{3}(32\mathcal{J}^3-84\mathcal{J}^2+129\mathcal{J}-84)\,e^{-4\mathcal{J}-8}+ +\frac{32}{3}(2000\mathcal{J}^4-5200\mathcal{J}^3+9900\mathcal{J}^2-10.316\mathcal{J}+4989)\,e^{-5\mathcal{J}-10} + \frac{1536}{5}(576\mathcal{J}^5-1440\mathcal{J}^4+3180\mathcal{J}^3- -4115\mathcal{J}^2+3360\mathcal{J}-1285)\,e^{-6\mathcal{J}-12}+\ldots \tag{G.4}$$

$$\mathcal{E}-\mathcal{J} = 2+8e^{-\mathcal{J}-2} + 8\,(2\mathcal{J}-1)\;e^{-2\mathcal{J}-4} + 32\left(2\mathcal{J}^2-\mathcal{J}+2\right)\,e^{-3\mathcal{J}-6} + \frac{8}{3}\left(128\mathcal{J}^3-48\mathcal{J}^2+228\mathcal{J}-63\right)\,e^{-4\mathcal{J}-8} +\frac{16}{3}(400\mathcal{J}^4-80\mathcal{J}^3+972\mathcal{J}^2-330\mathcal{J}+279)\,e^{-5\mathcal{J}-10} + \frac{64}{5}(1152\mathcal{J}^5+3480\mathcal{J}^3-1010\mathcal{J}^2+2080\mathcal{J}- -405)\,e^{-6\mathcal{J}-12}+\ldots \tag{G.5}$$

[90] As we have already noted in §7.3, the transformation

$$\mathcal{S}\equiv\frac{1}{16}\,e^{\mathcal{J}+2}\Leftrightarrow\mathcal{J}=\ln\mathcal{S}+4\ln2-2 \tag{G.1}$$

makes the inverse spin functions and the anomalous dimensions of long folded strings in $\mathbb{R}\times\mathrm{S}^2$ (GKP II) and AdS_3 (GKP I) look alike and allows to compare them. See equation (7.127).

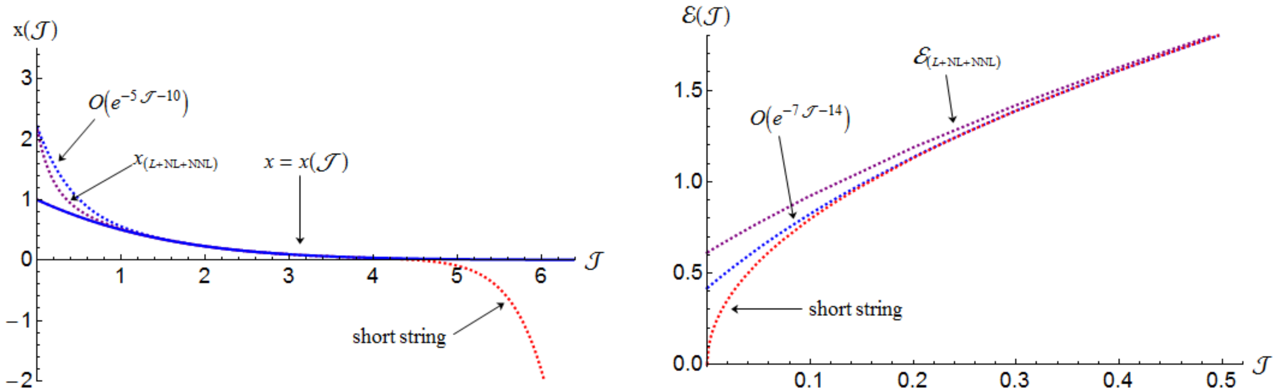


Figure 27: Short & long approximations to the folded GKP string in $\mathbb{R}\times \mathrm{S}^2$. The plot on the left contains the parametric plot of the inverse spin function $x = x\left(\mathcal{J}\right)$ according to equation (6.56) (thick blue line), its "short" string approximation (6.60) (red dashed line), the NNL formula (7.64) (purple dashed line) and the first 5 terms of the "long" string approximation (G.2) (blue dashed line). The plot on the right contains the "short" string approximation to the anomalous dimensions (6.61) (red dashed line), the NNL formula (7.65) (purple dashed line) and the first eight terms of the "long" approximation (G.3). Compare with the plot of $\mathcal{E} = \mathcal{E}\left(\mathcal{J}\right)$ in figure 8.

- Folded string in AdS_3 ($\omega > 1$).

$$x = \frac{2}{\mathcal{S}} - \left[\ln\mathcal{S} + \left(3\ln 2 + 1\right)\right]\frac{1}{\mathcal{S}^2} + \left[\frac{\ln^2\mathcal{S}}{2} + \left(3\ln 2 + \frac{1}{4}\right)\ln\mathcal{S} + \left(\frac{9\ln^2 2}{2} + \frac{3\ln 2}{4} + \frac{3}{8}\right)\right]\frac{1}{\mathcal{S}^3} - \left[\frac{\ln^3\mathcal{S}}{4} + \left(\frac{9\ln 2}{4} - \frac{1}{4}\right)\ln^2\mathcal{S} + \left(\frac{27\ln^2 2}{4} - \frac{3\ln 2}{2} + \frac{3}{8}\right)\ln\mathcal{S} + \left(\frac{27\ln^3 2}{4} - \frac{9\ln^2 2}{4} + \frac{9\ln 2}{8}\right)\right]\frac{1}{\mathcal{S}^4} + \ldots \tag{G.6}$$

$$\gamma = \ln\mathcal{S} + \left[3\ln 2 - 1\right] + \left[\frac{\ln\mathcal{S}}{2} + \left(\frac{3\ln 2}{2} - \frac{1}{2}\right)\right]\frac{1}{\mathcal{S}} - \left[\frac{\ln^2\mathcal{S}}{8} + \left(\frac{3\ln 2}{4} - \frac{9}{16}\right)\ln\mathcal{S} + \left(\frac{9\ln^2 2}{8} - \frac{27\ln 2}{16} + \frac{5}{16}\right)\right]\frac{1}{\mathcal{S}^2} + \left[\frac{\ln^3\mathcal{S}}{24} + \left(\frac{3\ln 2}{8} - \frac{3}{8}\right)\ln^2\mathcal{S} + \left(\frac{9\ln^2 2}{8} - \frac{9\ln 2}{4} + \frac{11}{16}\right)\ln\mathcal{S} + \left(\frac{9\ln^3 2}{8} - \frac{27\ln^2 2}{8} + \frac{33\ln 2}{16} - \frac{7}{24}\right)\right]\frac{1}{\mathcal{S}^3} - \left[\frac{\ln^4\mathcal{S}}{64} + \left(\frac{3\ln 2}{16} - \frac{43}{192}\right)\ln^3\mathcal{S} + \left(\frac{27\ln^2 2}{32} - \frac{129\ln 2}{64} + \frac{51}{64}\right)\ln^2\mathcal{S} + \left(\frac{27\ln^3 2}{16} - \frac{387\ln^2 2}{64} + \frac{153\ln 2}{32} - \frac{937}{1024}\right)\cdot \ln\mathcal{S} + \left(\frac{81\ln^4 2}{64} - \frac{387\ln^3 2}{64} + \frac{459\ln^2 2}{64} - \frac{2811\ln 2}{1024} + \frac{1919}{6144}\right)\right]\frac{1}{\mathcal{S}^4} + \ldots \tag{G.7}$$

All of these results agree with the W-function formulas and the coefficients that we have derived. (G.2)–(G.3) agree with (7.64)–(7.65) and (G.4)–(G.5) agree with (7.69)–(7.70). The formulas (G.6)–(G.7) agree with all the coefficients (7.6)–(7.9) but also the W-function expressions (7.112)–(7.113). In figures 27–28 we have plotted all the Mathematica results of this appendix (G.2)–(G.7), the corresponding parametric plots, the Lambert W-function expressions of §7 and the respective short string approximations of §6.

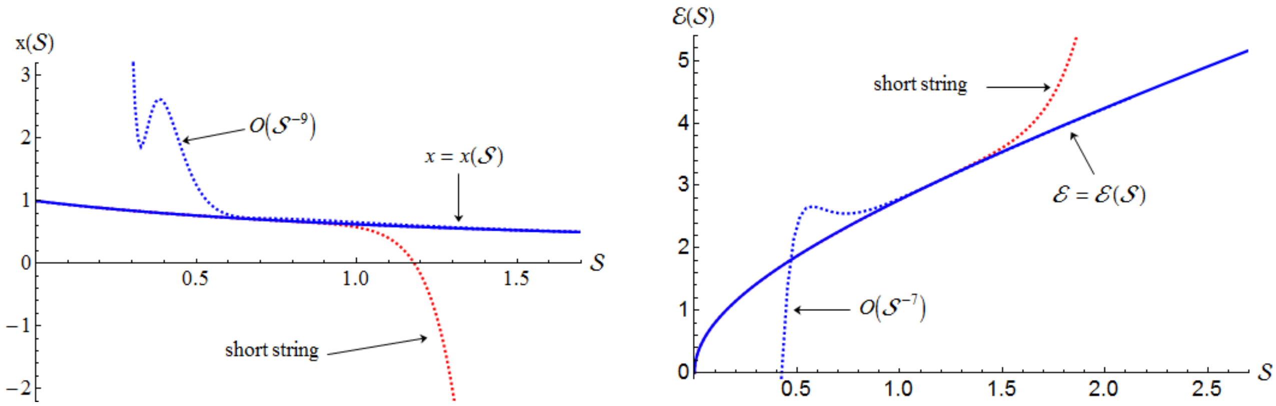


Figure 28: Short & long approximations to the folded GKP string in AdS_3. On the left we have plotted the inverse spin function $x(\mathcal{S})$ parametrically according to (6.23) (thick blue line), as well as its short (6.28) (red dashed line) and long approximations (G.6) (blue dashed line). In the latter, much more terms (up to order $\mathcal{S}^{-9}$) than those contained in equation (G.6) have been used. The plot on the right is a parametric plot of the anomalous dimensions $\mathcal{E}(\mathcal{S})$ from equation (6.22) (thick blue line), along with the "short" approximation (6.29) (red dashed line) and the "long" approximation (G.7) (up to terms $\mathcal{S}^{-7}$). Compare with the plot in figure 4.

G.2 Giant Magnons & Single Spikes

For giant magnons we set,

$$\mathcal{L} \equiv 2\mathcal{J}\csc\frac{p}{2} + 2 \tag{G.8}$$

and for single spikes we set,

$$\mathcal{R} \equiv \sqrt{\frac{1}{\mathcal{J}^2} - 1}\cdot\left(p + 2\arcsin\mathcal{J}\right) = (p+q)\cdot\cot\frac{q}{2}\,, \quad \mathcal{J} \equiv \sin\frac{q}{2}. \tag{G.9}$$

We obtain the following results with Mathematica:

• Finite-Size Giant Magnons: Elementary Region, $0 \leq |v| < 1/\omega \leq 1$.

$$\sqrt{1-v^2} = \sin a = \sin\frac{p}{2} + \frac{1}{4}\cos^2\frac{p}{2}\left[2\mathcal{J} + 3\sin\frac{p}{2}\right]x - \frac{3}{64}\cos^2\frac{p}{2}\left[8\mathcal{J}^2\sin\frac{p}{2} - 12\mathcal{J}\cos p - 5\sin\frac{3p}{2}\right]x^2 - \frac{1}{3072}\cos^2\frac{p}{2}\cdot$$
$$\cdot\left[\mathcal{J}^3(512\cos p - 256) + 216\mathcal{J}^2\left(5\sin\frac{3p}{2} + \sin\frac{p}{2}\right) - 12\mathcal{J}\left(73\cos 2p + 66\cos p + 11\right) - 259\sin\frac{5p}{2} - \right.$$
$$\left. -272\sin\frac{3p}{2} + 11\sin\frac{p}{2}\right]x^3 + \ldots \tag{G.10}$$

$$x = 16\,e^{-\mathcal{L}} + \left[256\mathcal{J}^2\cot^2\frac{p}{2} + 64\mathcal{J}\left(3\cos p + 1\right)\csc\frac{p}{2} - 128\right]e^{-2\mathcal{L}} + \left[6144\mathcal{J}^4\cot^4\frac{p}{2} + 512\mathcal{J}^3\left(19\cos p + 1\right)\cot^2\frac{p}{2}\cdot\right.$$
$$\left.\cdot\csc\frac{p}{2} - 256\mathcal{J}^2\left(2\csc^2\frac{p}{2} + 33\cos p + 25\right) + 64\mathcal{J}\left(6\cos 2p - 51\cos p - 23\right)\csc\frac{p}{2} + 960\right]e^{-3\mathcal{L}} + \left[\frac{524\,288}{3}\mathcal{J}^6\cot^6\frac{p}{2} + \right.$$
$$+32\,768\mathcal{J}^5\left(13\cos p - 1\right)\cot^4\frac{p}{2}\csc\frac{p}{2} + \frac{8192}{3}\mathcal{J}^4\left(68\cos 2p - 27\cos p + 1\right)\cot^2\frac{p}{2}\csc^2\frac{p}{2} + \frac{128}{3}\mathcal{J}^3(819\cos 3p -$$

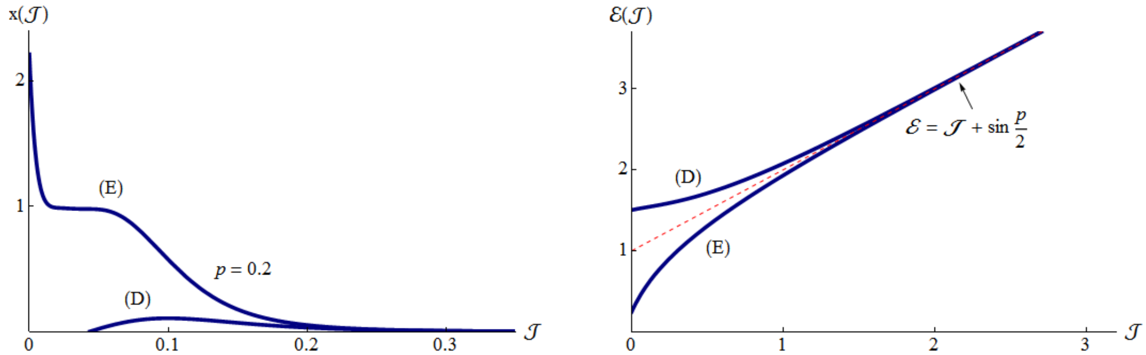


Figure 29: Inverse spin function and energy of finite-size giant magnons. On the left we have plotted $x\,(p = 0.2, \mathcal{J})$ and $\widetilde{x}\,(p = 0.2, \mathcal{J})$ in the elementary (G.11) and doubled region of giant magnons. On the right we have plotted $\mathcal{E}\,(p = 3.0, \mathcal{J})$ in the GM elementary (G.12) and doubled regions (G.13). The curves in the elementary region have been labelled with an (E) and the curves in the doubled region with a (D). The approximations become more trustworthy as the angular momentum $\mathcal{J}$ gets larger and the infinite-size result of Hofman-Maldacena (8.6) is approached.

$$-786\cos 2p - 3027\cos p - 1934)\csc^3\frac{p}{2} + 1024\mathcal{J}^2\left(11\cos 3p - 44\cos 2p - 18\cos p + 1\right)\csc^2\frac{p}{2} + \frac{64}{3}\mathcal{J}(70\cos 3p-$$

$$-319\cos 2p + 1742\cos p + 907)\csc\frac{p}{2} - 7168\Big]\,e^{-4\mathcal{L}} + \ldots \tag{G.11}$$

$$\mathcal{E} - \mathcal{J} = \sin\frac{p}{2} - 4\sin^3\frac{p}{2}e^{-\mathcal{L}} - \left[8\mathcal{J}^2\csc\frac{p}{2}\sin^2 p - \mathcal{J}\left(12\cos 2p - 8\cos p - 4\right) + 4\left(6\cos p + 7\right)\sin^3\frac{p}{2}\right]e^{-2\mathcal{L}} -$$

$$-\Bigg[32\mathcal{J}^4\csc^5\frac{p}{2}\sin^4 p + \frac{32}{3}\mathcal{J}^3\left(31\cos 2p + 88\cos p + 57\right) + 32\mathcal{J}^2\left(9\sin\frac{5p}{2} + 11\sin\frac{3p}{2} + 6\sin\frac{p}{2}\right) -$$

$$-\mathcal{J}\left(96\cos 3p + 44\cos 2p - 112\cos p - 28\right) + \frac{8}{3}\left(37\cos 2p + 97\cos p + 72\right)\sin^3\frac{p}{2}\Bigg]e^{-3\mathcal{L}} -$$

$$-\Bigg[\frac{512}{3}\mathcal{J}^6\csc^9\frac{p}{2}\sin^6 p + 2048\mathcal{J}^5\left(19\cos p + 5\right)\cos^2\frac{p}{2}\cot^2\frac{p}{2} + \frac{64}{3}\mathcal{J}^4\left(1273\cos 2p + 1824\cos p + 1319\right)\cdot$$

$$\cdot\cos\frac{p}{2}\cot\frac{p}{2} + \frac{64}{3}\mathcal{J}^3\left(441\cos 3p + 1242\cos 2p + 1983\cos p + 1118\right) + 8\mathcal{J}^2\Big(431\sin\frac{7p}{2} + 734\sin\frac{5p}{2} +$$

$$+544\sin\frac{3p}{2} + 273\sin\frac{p}{2}\Big) - \frac{4}{3}\mathcal{J}\left(511\cos 4p + 360\cos 3p - 88\cos 2p - 588\cos p - 195\right) + 4(118\cos 3p +$$

$$+322\cos 2p + 532\cos p + 349)\sin^3\frac{p}{2}\Bigg]e^{-4\mathcal{L}} - \ldots \tag{G.12}$$

- Finite-Size Giant Magnons: Doubled Region, $0 \leq |v| \leq 1 \leq 1/\omega$.

$$\mathcal{E} - \mathcal{J} = \sin\frac{p}{2} + 4\sin^3\frac{p}{2}e^{-\mathcal{L}} - \left[8\mathcal{J}^2\csc\frac{p}{2}\sin^2 p - \mathcal{J}\left(12\cos 2p - 8\cos p - 4\right) + 4\left(6\cos p + 7\right)\sin^3\frac{p}{2}\right]e^{-2\mathcal{L}} +$$

$$+\Bigg[32\mathcal{J}^4\csc^5\frac{p}{2}\sin^4 p + \frac{32}{3}\mathcal{J}^3\left(31\cos 2p + 88\cos p + 57\right) + 32\mathcal{J}^2\left(9\sin\frac{5p}{2} + 11\sin\frac{3p}{2} + 6\sin\frac{p}{2}\right) -$$

$$-\mathcal{J}\left(96\cos 3p+44\cos 2p-112\cos p-28\right)+\frac{8}{3}\left(37\cos 2p+97\cos p+72\right)\sin^3\frac{p}{2}\Big]e^{-3\mathcal{L}}-$$

$$-\Big[\frac{512}{3}\mathcal{J}^6\csc^9\frac{p}{2}\sin^6 p+2048\mathcal{J}^5\left(19\cos p+5\right)\cos^2\frac{p}{2}\cot^2\frac{p}{2}+\frac{64}{3}\mathcal{J}^4\left(1273\cos 2p+1824\cos p+1319\right)\cdot$$

$$\cdot\cos\frac{p}{2}\cot\frac{p}{2}+\frac{64}{3}\mathcal{J}^3\left(441\cos 3p+1242\cos 2p+1983\cos p+1118\right)+8\mathcal{J}^2\Big(431\sin\frac{7p}{2}+734\sin\frac{5p}{2}+$$

$$+544\sin\frac{3p}{2}+273\sin\frac{p}{2}\Big)-\frac{4}{3}\mathcal{J}\left(511\cos 4p+360\cos 3p-88\cos 2p-588\cos p-195\right)+4(118\cos 3p+$$

$$+322\cos 2p+532\cos p+349)\sin^3\frac{p}{2}\Big]e^{-4\mathcal{L}}+\ldots \tag{G.13}$$

- Finite-Size Single Spikes: Elementary Region, $0\le 1/\omega<|v|\le 1$.

$$\mathcal{E}-\frac{p}{2}=\frac{q}{2}+4\sin^2\frac{q}{2}\tan\frac{q}{2}\cdot e^{-\mathcal{R}}+\Big\{8p^2\cos^2\frac{q}{2}+2p\cos\frac{q}{2}\Big(8q\cos\frac{q}{2}-\sin\frac{3q}{2}+7\sin\frac{q}{2}\Big)+8q^2\cos^2\frac{q}{2}-2q\sin q\big(\cos q-$$

$$-3\big)+\sin^2\frac{q}{2}\left(\cos 2q-2\cos q+5\right)\Big\}\sec^2\frac{q}{2}\tan\frac{q}{2}\cdot e^{-2\mathcal{R}}+\Big\{32p^4\cos^4\frac{q}{2}+\frac{8p^3}{3}\cos^3\frac{q}{2}\Big(48q\cos\frac{q}{2}-11\sin\frac{3q}{2}+$$

$$+25\sin\frac{q}{2}\Big)+p^2\cos^2\frac{q}{2}\Big[192q^2\cos^2\frac{q}{2}-8q\sin q\left(11\cos q-7\right)-5\cos 3q+22\cos 2q-59\cos q+42\Big]+$$

$$+\frac{1}{4}p\cos\frac{q}{2}\Big[512q^3\cos^3\frac{q}{2}-32q^2\sin q\,\cos\frac{q}{2}\left(11\cos q-7\right)+16q\sin q\,\sin\frac{q}{2}\left(5\cos 2q-12\cos q+15\right)-8\sin^3\frac{q}{2}\cdot$$

$$\cdot\left(\cos 3q-5\cos 2q+15\cos q-27\right)\Big]+32q^4\cos^4\frac{q}{2}-\frac{8}{3}q^3\cos^2\frac{q}{2}\sin q\left(11\cos q-7\right)+q^2\sin^2 q\big(5\cos 2q-$$

$$-12\cos q+15\big)-q\sin q\sin^2\frac{q}{2}\left(\cos 3q-5\cos 2q+15\cos q-27\right)+\frac{1}{6}\sin^4\frac{q}{2}\big(\cos 4q+2\cos 3q+16\cos 2q-$$

$$-50\cos q+127\big)\Big\}\csc\frac{q}{2}\sec^5\frac{q}{2}\cdot e^{-3\mathcal{R}}+\ldots \tag{G.14}$$

- Finite-Size Single Spikes: Doubled Region, $0\le 1/\omega\le 1\le|v|$.

$$\mathcal{E}-\frac{p}{2}=\frac{q}{2}-4\sin^2\frac{q}{2}\tan\frac{q}{2}\cdot e^{-\mathcal{R}}+\Big\{8p^2\cos^2\frac{q}{2}+2p\cos\frac{q}{2}\Big(8q\cos\frac{q}{2}-\sin\frac{3q}{2}+7\sin\frac{q}{2}\Big)+8q^2\cos^2\frac{q}{2}-2q\sin q\big(\cos q-$$

$$-3\big)+\sin^2\frac{q}{2}\Big(\cos 2q-34\cos q-91+64\csc^2\frac{q}{2}\Big)\Big\}\sec^2\frac{q}{2}\tan\frac{q}{2}\cdot e^{-2\mathcal{R}}-\Big\{32p^4\cos^4\frac{q}{2}+\frac{8p^3}{3}\cos^3\frac{q}{2}\Big(48q\cos\frac{q}{2}$$

$$-11\sin\frac{3q}{2}+25\sin\frac{q}{2}\Big)+p^2\cos^2\frac{q}{2}\Big[192q^2\cos^2\frac{q}{2}-8q\sin q\left(11\cos q-7\right)-5\cos 3q+86\cos 2q+197\cos q+$$

$$+234\Big]+\frac{1}{4}p\cos\frac{q}{2}\Big[512q^3\cos^3\frac{q}{2}-32q^2\sin q\,\cos\frac{q}{2}\left(11\cos q-7\right)+16q\sin q\,\sin\frac{q}{2}\Big(5\cos 2q-76\cos q-177+$$

$$+128\csc^2\frac{q}{2}\Big)-8\sin^3\frac{q}{2}\cdot\Big(\cos 3q-69\cos 2q-433\cos q-795+384\csc^2\frac{q}{2}\Big)\Big]+32q^4\cos^4\frac{q}{2}-\frac{8}{3}q^3\cos^2\frac{q}{2}\sin q$$

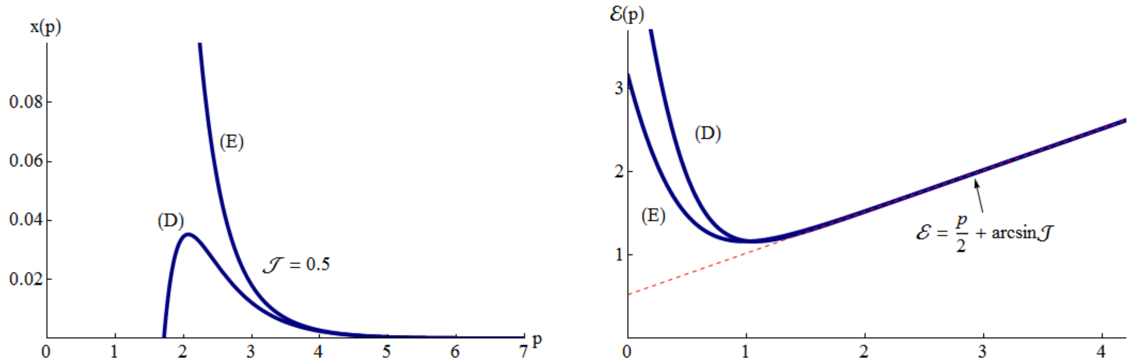


Figure 30: Inverse spin function and energy of finite-size single spikes. On the left we have plotted $x\,(p, \mathcal{J} = 0.5)$ and $\widetilde{x}\,(p, \mathcal{J} = 0.5)$ in the elementary and doubled region of single spikes. On the right we have plotted $\mathcal{E}\,(p, \mathcal{J} = 0.5)$ in the SS elementary (G.14) and doubled regions (G.15). The curves in the elementary region have been labelled with an (E) and the curves in the doubled region with a (D). The approximations become more trustworthy as the linear momentum p gets larger and (8.8) is approached.

$$\cdot (11\cos q - 7) + q^2 \sin^2 q \left(5\cos 2q - 76\cos q - 177 + 128\csc^2\frac{q}{2}\right) - q\sin q \sin^2\frac{q}{2}\Big(\cos 3q - 69\cos 2q - 433\cos q$$
$$-795 + 384\csc^2\frac{q}{2}\Big) + \frac{1}{6}\sin^4\frac{q}{2}\left(\cos 4q - 190\cos 3q - 1424\cos 2q - 4466\cos q - 3809 + 768\csc^2\frac{q}{2}\right)\Bigg\}\csc\frac{q}{2}\cdot$$
$$\cdot \sec^5\frac{q}{2}\cdot e^{-3\mathcal{R}} + \ldots \tag{G.15}$$

All of our results agree with the Lambert W-function formulas that were derived in §10. For giant magnons, (G.11) and (G.12) agree with (10.39) and (10.49). (G.13) agrees with (10.51). Notice that the only difference between the dispersion relations of giant magnons in the elementary and doubled regions (G.12)–(G.13) is the sign of all odd-powered exponential corrections.

For single spikes the dispersion relations in the elementary and doubled regions are quite different. We have marked the terms of (G.15) that are absent from the corresponding dispersion relation in the elementary region (G.14) with red color. Again, the Mathematica results (G.15)–(G.14) are in complete agreement with the Lambert W-function formulae (10.53)–(10.54). In figures 29–30 we have plotted all the Mathematica results of this appendix (G.11)–(G.15) for giant magnons and single spikes.

H Elliptic Integrals and Jacobian Elliptic Functions

This appendix contains the definitions and some basic properties of elliptic integrals and Jacobian elliptic functions that we employ in the text. Our conventions mainly follow Abramowitz-Stegun [85].

Jacobian Elliptic Functions

$$u \equiv \int_0^{\varphi} \frac{d\theta}{\left(1 - m\sin^2\theta\right)^{1/2}}\,, \quad \varphi \equiv am(u|m)\,, \quad \Delta(\varphi) \equiv (1-\sin^2\theta)^{1/2} \equiv dn(u|m) \tag{H.1}$$

$$x = \sin\varphi \equiv sn(u|m)\,, \quad \cos\varphi \equiv cn(u|m).$$

Elliptic Integral of the First Kind

$$\mathbb{F}\left(\varphi\middle|m\right) \equiv \int_0^{\varphi} \left(1 - m\,\sin^2\theta\right)^{-1/2} d\theta = \int_0^x \left[\left(1-t^2\right)\left(1-m\,t^2\right)\right]^{-1/2} dt = u \tag{H.2}$$

$$\mathbb{K}\left(m\right) \equiv \mathbb{F}\left(\frac{\pi}{2}\middle|m\right) = \frac{\pi}{2}\cdot {}_2\mathcal{F}_1\left[\frac{1}{2},\frac{1}{2};1;m\right] \quad \text{(complete)} \tag{H.3}$$

$$\mathbb{K}\left(m\right) = \frac{\pi}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 m^n =$$

$$= \frac{\pi}{2}\cdot\left[1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 m^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 m^3 + \ldots\right]\,, \quad |m|<1 \tag{H.4}$$

$$\mathbb{K}\left(m\right) = \frac{1}{2\pi}\cdot\sum_{n=0}^{\infty}\left(\frac{\Gamma\left(n+1/2\right)}{n!}\right)^2 \left[2\psi\left(n+1\right) - 2\psi\left(n+1/2\right) - \ln\left(1-m\right)\right]\,\left(1-m\right)^n =$$

$$= \sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \left[\psi\left(n+1\right) - \psi\left(n+1/2\right) - \frac{1}{2}\ln\left(1-m\right)\right]\,\left(1-m\right)^n\,, \quad |1-m|<1, \tag{H.5}$$

where $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ is the psi/digamma function.

Elliptic Integral of the Second Kind

$$\mathbb{E}\left(\varphi\middle|m\right) \equiv \int_0^{\varphi} \left(1 - m\,\sin^2\theta\right)^{1/2} d\theta = \int_0^x \left(1-t^2\right)^{-1/2}\left(1-m\,t^2\right)^{1/2} dt \tag{H.6}$$

$$\mathbb{E}\left(m\right) \equiv \mathbb{E}\left(\frac{\pi}{2}\middle|m\right) = \frac{\pi}{2}\cdot {}_2\mathcal{F}_1\left[-\frac{1}{2},\frac{1}{2};1;m\right] \quad \text{(complete)} \tag{H.7}$$

$$\mathbb{E}\left(m\right) = -\frac{\pi}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{m^n}{2n-1} =$$

$$= \frac{\pi}{2} \cdot \left[1 - \left(\frac{1}{2}\right)^2 \frac{m}{1} - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{m^2}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{m^3}{5} + \ldots \right], \quad |m| < 1 \quad \text{(H.8)}$$

$$\mathbb{E}\,(m) = 1 - \frac{1}{2\pi} \cdot \sum_{n=0}^{\infty} \frac{\Gamma\,(n+1/2)\,\Gamma\,(n+3/2)}{n!\,(n+1)!} \left[\ln(1-m) + \psi\,(n+1/2) + \psi\,(n+3/2) - \psi\,(n+1) - \right.$$

$$\left. -\psi\,(n+2) \right] (1-m)^{n+1} =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(2n-1)\,[(2n-3)!!]^2}{(2n-2)!!\,(2n)!!} \left[\psi\,(n) - \psi\,(n-1/2) - \frac{1}{2n\,(2n-1)} - \frac{1}{2} \ln(1-m) \right] (1-m)^n \,,$$

$|1-m| < 1.$[91] (H.9)

One could also define an elliptic-D function as follows [272]:

$$\mathbb{D}\,(\varphi|m) \equiv \int_0^{\varphi} \frac{\sin^2\theta\, d\theta}{\sqrt{1 - m\,\sin^2\theta}} = \int_0^x \frac{t^2\, dt}{\sqrt{(1-t^2)\,(1-m\,t^2)}} = \frac{1}{m} \Big[\mathbb{F}\,(\varphi|m) - \mathbb{E}\,(\varphi|m) \Big] \quad \text{(H.10)}$$

$$\mathbb{D}\,(m) \equiv \mathbb{D}\left(\frac{\pi}{2}\Big|m\right) = \frac{1}{m} \Big[\mathbb{K}\,(m) - \mathbb{E}\,(m) \Big] \quad \text{(complete).} \quad \text{(H.11)}$$

Elliptic Integral of the Third Kind

$$\mathbf{\Pi}(n, \varphi|m) \equiv \int_0^{\varphi} \left(1 - n\sin^2\theta\right)^{-1} \left(1 - m\sin^2\theta\right)^{-1/2} =$$

$$= \int_0^x \left(1 - nt^2\right)^{-1} \left[\left(1-t^2\right)\left(1 - m\,t^2\right)\right]^{-1/2} dt \quad \text{(H.12)}$$

$$\mathbf{\Pi}(n; m) \equiv \mathbf{\Pi}(n, \frac{\pi}{2}\Big|m) \quad \text{(complete).} \quad \text{(H.13)}$$

A very useful addition formula for the complete elliptic integrals of the third kind, allows to isolate their logarithmic singularities [273]:

$$\mathbf{\Pi}(n; m) = \frac{1}{(1-n)\,\mathbb{K}\,(m_1)} \cdot \left\{ \frac{\pi}{2} \sqrt{\frac{n\,(n-1)}{m-n}} \cdot \mathbb{F}\left(\arcsin\sqrt{\frac{n}{n-m}}, m_1\right) - \mathbb{K}\,(m) \cdot \left[(n-1)\,\mathbb{K}\,(m_1) - \right.\right.$$

$$\left.\left. -n \cdot \mathbf{\Pi}\left(\frac{1-m}{1-n}; m_1\right) \right] \right\}, \quad m + m_1 = 1\,, \quad 0 < -n < \infty. \quad \text{(H.14)}$$

[91] We repeat here some useful values of the double factorial: $0!! = 1$, $(-1)!! = 1$, $(-3)!! = -1$.

Carlson Elliptic Integrals

There is a sophisticated way to express Legendre's forms (H.2)–(H.10) with the aid of what is known as Carlson's symmetric forms. Briefly, Carlson's complete set of integrals is defined as follows [272, 274]:

$$\mathbb{R}_F(x,y,z) = \frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}} \tag{H.15}$$

$$\mathbb{R}_J(x,y,z,p) = \frac{3}{2}\int_0^\infty \frac{dt}{(t+p)\sqrt{(t+x)(t+y)(t+z)}} \tag{H.16}$$

$$\mathbb{R}_C(x,y) = \mathbb{R}_F(x,y,y) = \frac{1}{2}\int_0^\infty \frac{dt}{(t+y)\sqrt{(t+x)}} \tag{H.17}$$

$$\mathbb{R}_D(x,y,z) = \mathbb{R}_J(x,y,z,z) = \frac{3}{2}\int_0^\infty \frac{dt}{(t+z)\sqrt{(t+x)(t+y)(t+z)}}. \tag{H.18}$$

Carlson's symmetric forms owe much of their usefulness and elegance to the fact that, in contrast to the Legendre's forms, they are completely symmetric in all or some of their arguments. As it turns out, all of the incomplete elliptic integrals may be expressed in terms of Carlson's forms. The deeper reason for this can be traced back to the fact that all elliptic integrals are descendants of a multivariate hypergeometric function called Lauricella's $\mathcal{F}_D$. The complete elliptic integrals in particular, are given in terms of Carlson's forms as follows:

$$\mathbb{K}(m) = \mathbb{R}_F(0, 1-m, 1) \tag{H.19}$$

$$\mathbb{E}(m) = \frac{1}{3}(1-m)\cdot\Big[\mathbb{R}_D(0,1-m,1) + \mathbb{R}_D(0,1,1-m)\Big] \tag{H.20}$$

$$\mathbb{K}(m) - \mathbb{E}(m) = m\,\mathbb{D}(m) = \frac{1}{3}m\cdot\mathbb{R}_D(0,1-m,1) \tag{H.21}$$

$$\mathbb{E}(m) - (1-m)\,\mathbb{K}(m) = \frac{1}{3}m\,(1-m)\cdot\mathbb{R}_D(0,1,1-m)\,. \tag{H.22}$$

I Lambert's W-Function

One of the main outcomes of our work in part II has been the parametrization of the dispersion relations of certain $AdS_5 \times S^5$ strings in terms of Lambert's W-function. Lambert's W-function is defined by the following formula:

$$W(z)\, e^{W(z)} = z \Leftrightarrow W(z\, e^z) = z. \tag{I.1}$$

The function is named after Johann Heinrich Lambert, but it was Euler who first wrote down the series expansion of $-W(-z)$ that is known today as the tree function $T(z)$. Euler essentially generalized an algebraic equation that had been studied earlier by Lambert [275] and then solved it in some special cases, one of which was a variant of equation (I.1) [276].[92] According to [277], the symbol W originates from the function's early Maple usage (most computer algebra systems, including Maple, call it LambertW, whereas in Mathematica it is denoted as ProductLog).[93]

Lambert's W-function appears in numerous contexts in both mathematics and physics. Its applications can be found in fields such as combinatorics, algorithms and graphs, algebraic and differential equations, analysis and fractals (see e.g. [277, 280]) but also statistical physics, fluid dynamics, optics, astrophysics, general relativity, inflationary cosmology, etc. (see [281]). As concrete examples, one could single out the exact expression of Wien's displacement law in terms of W-function [282], the solution of the double Dirac delta potential well [283] and that of the two-body problem in $(1+1)$-dimensional dilaton gravity [284], or the inversion of Schwarzschild coordinates in terms of the Kruskal-Szekeres ones [285, 255].

More pertinent to our point of view is the exact solution of the renormalization group equations in terms of Lambert's W-function [286]. In the case of QCD it is known that the exact 3-loop running coupling, can be expressed in terms of the W-function [287, 288]:[94]

$$\alpha_s\left(Q^2\right) = \frac{-\pi/c}{1 - c_2/c^2 + W_{-1}(z)}, \tag{I.2}$$

where c_2 is a renormalization scheme-dependent constant and

$$\beta_0 \equiv \frac{1}{4}\left(11 - \frac{2}{3}N_f\right), \quad c \equiv \frac{1}{4\beta_0}\left[102 - \frac{38}{3}N_f\right], \quad z \equiv -\frac{1}{c}\exp\left[-1 + \frac{c_2}{c^2} - \frac{\beta_0\, t}{c}\right], \quad t \equiv \ln\left(\frac{Q^2}{\Lambda^2}\right).$$

$W(x)$ has two real branches, $W_0(x)$ in $\left[-e^{-1}, \infty\right)$ and $W_{-1}(x)$ in $\left[-e^{-1}, 0\right]$, drawn in figure 31.[95] The branch point is $\left(-e^{-1}, -1\right)$. The Taylor series at $x = 0$ in each of the two branches is [277]:

$$W_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)^n}{(n+1)!} \cdot x^{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{n-1}}{n!} \cdot x^n\,, \quad |x| \leq e^{-1} \tag{I.3}$$

$$W_{-1}(x) = \ln|x| - \ln|\ln|x|| + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^n}{m!} \begin{bmatrix} n+m \\ n+1 \end{bmatrix} (\ln|x|)^{-n-m} (\ln|\ln|x||)^m\,, \tag{I.4}$$

[92] In his 1783 paper, Euler refers to Lambert as "the ingenious engineer Lambert". For more on the very interesting history of Lambert's function see the article [277].

[93] More on Lambert onomastics can be found in the article [278]. Closely related definitions are those of *glog* and Wright's ω-function [279].

[94] For a reviews see [289].

[95] The branch of the W-function in the formula for the running coupling of QCD (I.2) depends on the number of flavors N_f. For $c > 0 \Leftrightarrow z < 0$ the relevant branch is W_{-1}, while for $c < 0 \Leftrightarrow z > 0$ the branch is W_0 [288].

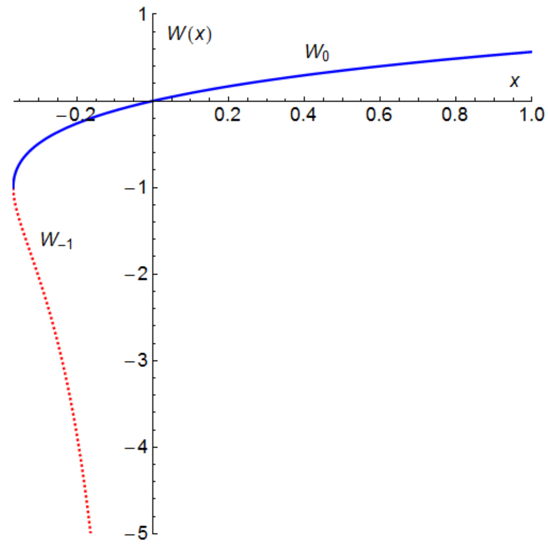


Figure 31: The two real branches of Lambert's W-function (I.1).

with the unsigned Stirling numbers of the first kind $\left[\begin{matrix} n+m \\ n+1 \end{matrix} \right]$, defined recursively as [290]:

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right] + (n-1) \left[\begin{matrix} n-1 \\ k \end{matrix} \right] \quad \& \quad \left[\begin{matrix} n \\ 0 \end{matrix} \right] = \left[\begin{matrix} 0 \\ k \end{matrix} \right] = 0\,, \ \left[\begin{matrix} 0 \\ 0 \end{matrix} \right] = 1\,, \ n,k \geq 1. \tag{I.5}$$

The following identities of unsigned Stirling numbers are often used:

$$\left[\begin{matrix} n \\ 1 \end{matrix} \right] = (n-1)!\,, \quad \left[\begin{matrix} n \\ 2 \end{matrix} \right] = (n-1)!\, H_{n-1}\,, \quad \left[\begin{matrix} n \\ 3 \end{matrix} \right] = \frac{1}{2} (n-1)! \left[H_{n-1}^2 - H_{n-1}^{(2)} \right]. \tag{I.6}$$

The W-function provides a nice series parametrization of the tetration $x^{x^{x^{\cdots}}}$:

$$x^{x^{x^{\cdots}}} = {}^{\infty}(x^z) = \frac{W(-\ln x)}{-\ln x}. \tag{I.7}$$

By using the defining property (I.1) of Lambert's W-function, its derivatives and antiderivatives can be significantly simplified. Here's a list of some useful identities of the W_0 function:

$$W'(x) = \frac{W(x)}{x\,(1 + W(x))} \tag{I.8}$$

$$x\,W'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^n}{n!} \cdot x^n = \frac{W(x)}{1 + W(x)} \tag{I.9}$$

$$x\,\left(x\,W'(x) \right)' = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{n+1}}{n!} \cdot x^n = \frac{W(x)}{(1 + W(x))^3} \tag{I.10}$$

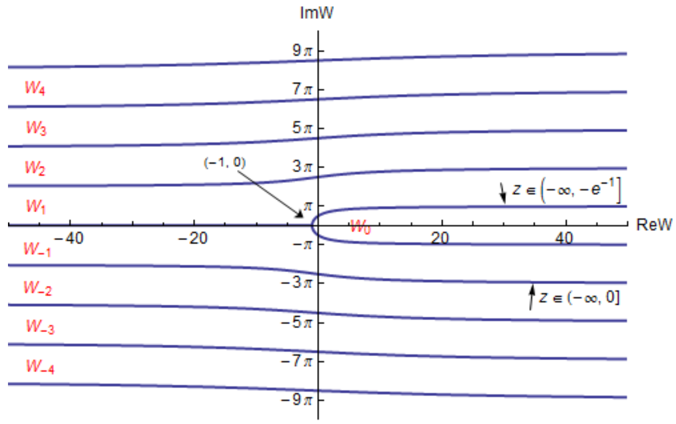


Figure 32: Branch diagram of Lambert's W-function.

$$\int W(x)\ dx = x\left(W(x) - 1 + \frac{1}{W(x)}\right) \tag{I.11}$$

$$\int \frac{W(x)}{x}\ dx = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^{n-2}}{n!}\cdot x^n = W(x) + \frac{W^2(x)}{2} \tag{I.12}$$

$$\int \frac{1}{x}\int \frac{W(x)}{x}\ dx^2 = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^{n-3}}{n!}\cdot x^n = W(x) + \frac{3W^2(x)}{4} + \frac{W^3(x)}{6}. \tag{I.13}$$

The branch structure of Lambert's W-function is very reminiscent of that of the logarithm. Besides, the W-function is a generalization of the logarithmic function. However, instead of the familiar straight lines that separate the adjacent branches of the logarithm, the neighboring branches of Lambert's function are separated by a family of curves that is known as "Quadratrix of Hippias":

$$\left\{-\eta\cot\eta + i\eta, \quad -\pi < \eta < \pi \quad \text{or} \quad 2k\pi < \pm\eta < (2k+1)\pi, \quad k = 1, 2, 3, \dots\right\} \tag{I.14}$$

(I.4) gives the asymptotics of all the branches around $z = \infty$, and all non-principal branches around $z = 0$. Other notable features of the branch diagram of the W-function, apart from the two real branches that we have already talked about, is the triple branch point at $W_{\{0,\pm1\}}\left(-e^{-1}\right) = -1$ and the branch cuts $\left(-\infty, -e^{-1}\right]$ of $W_{0,\pm1}$ and $(-\infty, 0]$ of $W_{k\neq0}$. The branch diagram of the W-function has been drawn in figure 32.

J Partition Polynomials

J.1 Bell Polynomials

The exponential complete Bell polynomials $\mathbf{B}_n(x_1, x_2, \ldots, x_n)$ are defined by the formula [290]:

$$\exp\left[\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right] = \sum_{n=0}^{\infty} \mathbf{B}_n(x_1, x_2, \ldots, x_n) \frac{t^n}{n!}. \tag{J.1}$$

The exponential partial Bell polynomials $\mathbf{B}_{n,k} = \mathbf{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ are defined as follows:

$$\mathbf{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_j \frac{n!}{j_1! j_2! \ldots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \ldots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \tag{J.2}$$

where $j_1 + j_2 + \ldots = k$ and $j_1 + 2\, j_2 + \ldots = n$. One finds,

$$\mathbf{B}_0(x_1, x_2, \ldots, x_n) = 1\,, \quad \mathbf{B}_n(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \mathbf{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1})\,. \tag{J.3}$$

Note that the unsigned Stirling numbers of the first kind are given in terms of the partial Bell polynomials as $\begin{bmatrix} n \\ k \end{bmatrix} = \mathbf{B}_{n,k}(0!, 1!, \ldots, (n-k)!)$. The ordinary partial Bell polynomials are defined as:

$$\widehat{\mathbf{B}}_{n,k}(x_1, x_2, \ldots, x_n) = \sum_j \frac{n!}{j_1! j_2! \ldots j_n!} {x_1}^{j_1} {x_2}^{j_2} \ldots x_n^{j_{n-k+1}}. \tag{J.4}$$

J.2 Potential Polynomials

The potential polynomials $\mathbf{P}_n^{(r)}$ are defined by means of the formula:

$$\left[\sum_{m=0}^{\infty} x_m \frac{t^m}{m!}\right]^r = \sum_{n=0}^{\infty} \mathbf{P}_n^{(r)}(x_1, x_2, \ldots, x_n) \frac{t^n}{n!}. \tag{J.5}$$

J.3 Logarithmic Polynomials

The definition of the logarithmic polynomials $\mathbf{L}_n^{(r)}$ is similar:

$$\ln\left[\sum_{m=0}^{\infty} x_m \frac{t^m}{m!}\right] = \sum_{n=0}^{\infty} \mathbf{L}_n^{(r)}(x_1, x_2, \ldots, x_n) \frac{t^n}{n!}. \tag{J.6}$$

K Lamé's Equation

We saw in §17 that the equations for the transverse fluctuations of the stringy membranes (16.1)–(16.10) can be reduced to the Jacobi form of Lamé's equation (17.34):

$$\frac{d^2z}{du^2} + \left[h - \nu\left(\nu+1\right)k^2 sn^2\left(u|k^2\right)\right]z = 0, \tag{K.1}$$

where $\nu\left(\nu+1\right) \in \mathbb{R}$ and $0 < k < 1$. The potential of Lamé's equation (K.1), $sn^2\left(u|k^2\right)$ is a doubly periodic function with (primitive) real and imaginary periods equal to $2\mathbb{K}\left(k^2\right)$ and $2i\mathbb{K}'\left(k^2\right)$ respectively. Its real and imaginary parts have been drawn in a contour plot in figure 33. The eigenfunctions of Lamé's equation (aka Lamé functions) that have real periods are the following:

eigenfunction $z\left(u\right)$	eigenvalue h	parity of $z\left(u\right)$	parity of $z\left(u-\mathbb{K}\right)$	period of $z\left(u\right)$
$Ec_\nu^{2n}\left(u,k^2\right)$	$a_\nu^{2n}\left(k^2\right)$	even	even	$2\mathbb{K}$
$Ec_\nu^{2n+1}\left(u,k^2\right)$	$a_\nu^{2n+1}\left(k^2\right)$	odd	even	$4\mathbb{K}$
$Es_\nu^{2n+1}\left(u,k^2\right)$	$b_\nu^{2n+1}\left(k^2\right)$	even	odd	$4\mathbb{K}$
$Es_\nu^{2n+2}\left(u,k^2\right)$	$b_\nu^{2n+2}\left(k^2\right)$	odd	odd	$2\mathbb{K}$

where $n = 0, 1, 2, \ldots$ The Lamé eigenvalues a_ν^n and b_ν^n have the following ordering properties [272, 291]:

$$\begin{aligned}
&a_\nu^0 < a_\nu^1 < a_\nu^2 < a_\nu^3 \ldots, \quad a_\nu^n \to \infty \text{ as } n \to \infty \\
&b_\nu^1 < b_\nu^2 < b_\nu^3 < b_\nu^4 \ldots, \quad b_\nu^n \to \infty \text{ as } n \to \infty \\
&a_\nu^0 < b_\nu^1 < a_\nu^2 < b_\nu^3 \ldots \\
&a_\nu^1 < b_\nu^2 < a_\nu^3 < b_\nu^4 \ldots
\end{aligned}$$

The intervals of stability of Lamé's equation (K.1) follow from the *oscillation theorem* [292]. They are:

$$\left(a_\nu^0\,,\ \underbrace{a_\nu^1\right)\ \cup\ \left(b_\nu^1}\,,\ \underbrace{b_\nu^2\right)\ \cup\ \left(a_\nu^2}\,,\ \underbrace{a_\nu^3\right)\ \cup\ \left(b_\nu^3}\,,\ \underbrace{b_\nu^4\right)\ \cup\ \ldots}, \tag{K.2}$$

where the contractions between consecutive eigenvalues mean that the relative order of the two contracted eigenvalues is not generally known and may therefore be reversed, for given values of ν and k^2.

For $\nu \in \mathbb{R}$, the expression $\nu\left(\nu+1\right) \in \mathbb{R}$ is symmetric under the map $\nu \leftrightarrow -\nu-1$ so that without loss of generality, we may consider $\nu \geq -1/2$ and $\nu\left(\nu+1\right) \geq -1/4$. If further $\nu \in \mathbb{N}$, then the first $2\nu+1$ of the Lamé functions are polynomials (aka Lamé polynomials), while the remaining transcendental solutions of Lamé's equation *coexist*, that is:

$$a_\nu^n = b_\nu^n, \text{ for n, } \nu \in \mathbb{N} \text{ and } n \geq \nu+1. \tag{K.3}$$

The above picture is nicely summarized by the following theorem [292]:

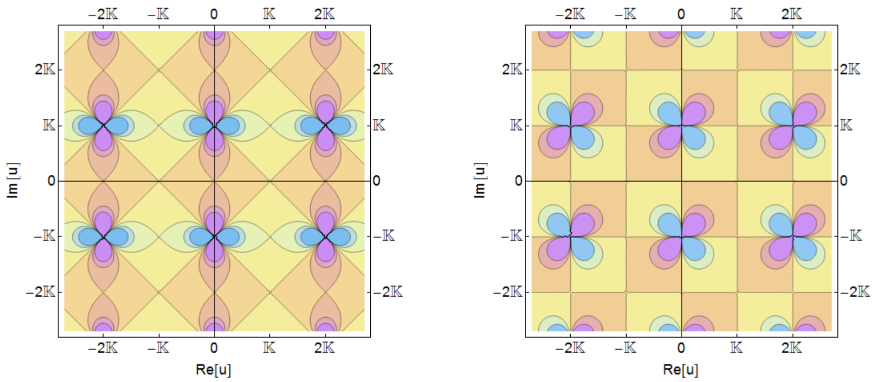


Figure 33: Real part (left) and imaginary part (right) of the Lamé potential, $\mathrm{sn}^2\,(u|1/2)$.

■ **Theorem 1.** *Lamé's equation* (K.1)*, displays coexistence iff* $\nu \in \mathbb{Z}$*. It has exactly* $\nu+1$ *instabilities if* $\nu \in \mathbb{N}$ *and exactly* $|\nu|$ *instabilities if* $\nu \in \mathbb{Z}^-$*.*

The stability intervals when $\nu \in \mathbb{N}$ are given by [272]:

$$(a_\nu^0\,,\,b_\nu^1)\,\cup\,(a_\nu^1\,,\,b_\nu^2)\,\cup\,(a_\nu^2\,,\,b_\nu^3)\,\cup\,\ldots\,\cup\,(a_\nu^{\nu-1}\,,\,b_\nu^\nu)\,\cup\,(a_\nu^\nu\,,\,+\infty)\quad,\quad\nu\in\mathbb{N}. \tag{K.4}$$

Finally, let us mention a few things about Lamé functions that have imaginary periods. First observe that Lamé's equation (K.1) has the following symmetry [244, 272, 291]:

$$\begin{aligned} u' &= i\left(u - \mathbb{K}\left(k^2\right) - i\,\mathbb{K}'\left(k^2\right)\right)\\ h' &= \nu\left(\nu+1\right) - h \quad,\quad k'^{\,2} = 1-k^2, \end{aligned} \tag{K.5}$$

so that, when the solution of Lamé's equation $z\,(u)$ has a real period equal to $2\,p\,\mathbb{K}$ (with $p=1,2$), then the function $z'\,(u') \equiv z\,(u)$ will have an imaginary period equal to $2\,i\,p\,\mathbb{K}$ and will satisfy the following transformed equation:

$$\frac{d^2z}{du'^2} + \left[h' - \nu\left(\nu+1\right)k'^{\,2}sn^2\left(u'|k'^{\,2}\right)\right]z = 0. \tag{K.6}$$

It can be proven that the duality (K.5) interchanges the bands of stability with the gaps of instability in (K.2) [244].

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