



NATIONAL & KAPODISTRIAN UNIVERSITY
OF ATHENS

DEPARTMENT OF PHYSICS

SECTION OF NUCLEAR & ELEMENTARY PARTICLE
PHYSICS

**The Closed Complex Time
Evolution in Open Quantum
Systems**

Supervisors:

Alexandros Karanikas
Fotios Diakonos
Emmanuel Floratos

A Master Thesis by:
Georgios Kordas

Preface

In recent years there has been increasing interest in the consistent description of the dynamics of open quantum systems [4, 5, 9, 12]. Quantum decoherence and dissipation are very important phenomena in many different areas of physics. A non-exhaustive list includes problems from quantum optics to many body and field-theoretical systems. Dissipative processes play a basic role in the quantum theory of lasers and photon detection, and they are equally important in nuclear fission and the deep inelastic collisions of heavy ions. More recently, the influence of the environment on a quantum system emerged as an issue of crucial importance, not only due to its fundamental implications, but also due to its practical applications in quantum information theory [2, 3].

Theoretical studies of decoherence and dissipation in quantum mechanics are centered on the time evolution of the reduced density matrix of a system embedded in a specific environment. The basic tools for studying the reduced dynamics are either effective equations of motion, in which the dynamics of the environment is eliminated, such as the Lindblad master equation [6, 7], or the influence functional technique introduced by Feynman and Vernon [13]. The latter is based on the path integral approach, and was used by A. Caldeira and A. Leggett [14] in the study of the quantum Brownian motion more than twenty years ago. In most cases however, neither the Lindblad equation nor the influence functional can be exactly evaluated, since the interaction between the system and the environment is too complicated. Therefore, if one is interested in exact expressions, one must rely on some specific system-environment simple models: a harmonic oscillator or a two-level quantum mechanical system embedded in a (thermal) bath of other harmonic oscillators or other spin systems. The main issue of this thesis is to investigate the possibility to extend the calculational capability of the Feynman-Vernon path integral approach by adopting and combining definite functional methodological tools already known from different research fields. The first such tool is a combination of the well-known “closed (real) time formalism” [18] with the (equally well-known) imaginary time formulation [12] in the context of path integration. The compound result, called “closed complex time formalism” (or CCT), enables us to isolate, in a simple and compact expression, the influence of the environment on the evolution of the system. Our second suggestion is the application of the so-called “cluster expansion” in the CCT context. In this way we are led to write down an expression for the “effective action” that governs the dynamics of the system after the elimination of the environmental degrees of freedom, which sets the

scene for exact or approximate calculations. Our ultimate aim is the transfer of the existent richness of perturbative and non-perturbative path integral techniques into the realm of open quantum systems. It is worth noting that our proposal can be extended to systems with an infinite number of degrees of freedom, such as the electromagnetic field interacting with matter or other field-theoretical systems.


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Chapter 1

Fundamental Concepts

n this chapter we discuss some of the basic concepts of open quantum systems, such as mixed states, density matrix, entanglement, Von Neuman entropy and decoherence. More details one can find in the following [1–3, 8, 11].

1.1 Pure and Mixed States

A pure ensemble is a collection of physical systems each of which is characterized by the same state $|\psi\rangle$. As a trivial example consider a beam of non-interacting spin 1/2 particles in the state $|\psi\rangle = c_+|+\rangle + c_-|-\rangle$. This state characterizes a particle whose spin is pointing in some definite direction: if β and α , is the polar and azimuthal angles, respectively, then we can obtain c_{\pm} by solving the equations

$$\frac{c_+}{c_-} = \frac{\cos(\beta/2)}{e^{i\alpha} \sin(\beta/2)}, \quad |c_+|^2 + |c_-|^2 = 1. \quad (1.1)$$

We say that the beam is polarized along the specific direction. The above beam of particles is an example of a system in a pure state.

Now suppose that we have a beam of particles with completely random spin orientation. For example half of the particles can be polarized along the positive x direction and the other half along the negative z direction. It is obvious that it is impossible to find a state vector describing this system. The only thing we can say is that we have $w_+ = 0.5$ probability to find a particle of the beam in the state $|x_+\rangle$ and $w_- = 0.5$ to find it in the state $|-\rangle \equiv |z_-\rangle$. In this example, our system is in a **mixed state**. Here we must notice that the numbers $w_+, w_- \in \mathbb{R}$ refer to classical probabilities (we have

the normalization condition: $w_+ + w_- = 1$) and we must not confuse them with the complex numbers $c_+, c_- \in \mathbb{C}$ of the previous paragraph, because $|c_+|^2, |c_-|^2$ refer to “quantum” probabilities. Mixed state we can, also when we don’t know exactly the state that characterizes a particle. Let’s see an example for this case: Suppose we have a particle in the state $|\psi\rangle = c_+|+\rangle + c_-|-\rangle$, and we know that someone did a measurement in the third axis, but we don’t know the result of the measurement. The only thing we can say for sure is that if we do a new measurement in the third axis we have $w_+ = |c_+|^2$ probability to find it with spin-up and $w_- = |c_-|^2$ probability to find it with spin-down (attention, these are classical probabilities, because after the first measurement - the result of which is unknown from us - the initial wavefunction has collapsed). Lets clear the situation: before the first measurement the spin of the particle pointed in the direction defined by equation (1.1), after the first measurement the particle has spin-up or spin-down in the third axis with probability w_+ or w_- respectively.

The appropriate formalism to deal with mixed states, is that of **density matrix** which is the right tool for describing a system the exact state of which is not known.

1.2 The Density Matrix

Suppose we have an ensemble of pure states $|\psi_i\rangle$, each of which characterizes our system with probability w_i . We define the density matrix as:

$$\hat{\rho} = \sum_i w_i |\psi_i\rangle\langle\psi_i|, \quad (1.2)$$

with the constrain $\sum w_i = 1$. Here we have to mention that the states $|\psi_i\rangle$ need not to be orthogonal. In the case where our system is in a pure state $|\psi_j\rangle$, then it has the density matrix: $\hat{\rho} = |\psi_j\rangle\langle\psi_j|$. In a complete basis $\{|x\rangle\}$, the density matrix takes the form

$$\rho(x, x') = \langle x|\hat{\rho}|x'\rangle = \sum_i w_i \langle x|\psi_i\rangle\langle\psi_i|x'\rangle. \quad (1.3)$$

It is easy to see that for the density matrix we have the normalization condition $Tr(\rho) = 1$, indeed

$$Tr(\rho) \equiv \sum_x \rho(x, x) = \sum_i \sum_x w_i \langle x|\psi_i\rangle\langle\psi_i|x\rangle = \sum_i w_i \langle\psi_i|\psi_i\rangle = \sum_i w_i = 1.$$

We can prove that the density matrix is a positive operator, since if we consider the arbitrary state $|\varphi\rangle$ we have

$$\langle\varphi|\hat{\rho}|\varphi\rangle = \sum_i w_i \langle\varphi|\psi_i\rangle\langle\psi_i|\varphi\rangle = \sum_i w_i |\langle\varphi|\psi_i\rangle|^2 \geq 0.$$

Now we will see the following “purity” criterion: If our system is in the state $|\psi_j\rangle$ then we have

$$Tr(\rho^2) = Tr(|\psi_j\rangle\langle\psi_j|\psi_j\rangle\langle\psi_j|) = Tr(|\psi_j\rangle\langle\psi_j|) = Tr(\rho) = 1 \Rightarrow$$

$$Tr(\rho^2) = 1 \tag{1.4}$$

that is, when we are in a pure state we have the relation (1.4). Suppose now that our system is in a mixed state with density matrix (1.2), then

$$Tr(\rho^2) = \sum_x \sum_y \rho(x, y)\rho(y, x) = \sum_x \sum_y \sum_i \sum_j w_i w_j \langle x|\psi_i\rangle\langle\psi_i|y\rangle\langle y|\psi_j\rangle\langle\psi_j|x\rangle =$$

$$= \sum_i \sum_j w_i w_j \langle\psi_i|\psi_j\rangle\langle\psi_i|\psi_j\rangle = \sum_i \sum_j w_i w_j \delta_{ij} = \sum_i w_i^2 < \left(\sum_i w_i\right)^2 = 1 \Rightarrow$$

$$Tr(\rho^2) < 1 \tag{1.5}$$

that is, when we are in a mixed state we have the relation (1.5). Let’s see two examples:

Example 1.2.1. *Suppose we have a completely polarized beam in the positive direction of z , then*

$$\rho = |+\rangle\langle+| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow Tr(\rho^2) = 1. \diamond$$

Example 1.2.2. *Now suppose we have an unpolarized beam. This can be regarded as an incoherent mixture of a spin-up ensemble and a spin down ensemble with equal weights :*

$$\rho = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \Rightarrow$$

$$Tr(\rho^2) = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} \Rightarrow Tr(\rho^2) = \frac{1}{2} < 1. \quad \diamond$$

Here we must say that the same density matrix may correspond to more than one ensembles of quantum states. The eigenstates and eigenvalues of the density matrix indicate one of the many possible ensembles giving rise to the specific density matrix, and there is no reason to suppose there exists an especially privileged ensemble. Let's see an example :

Example 1.2.3. *Suppose we have a quantum system with density matrix*

$$\rho = \frac{1}{4}|+\rangle\langle+| + \frac{3}{4}|-\rangle\langle-|.$$

One may consider that our system is in the state $|+\rangle$ with probability 0.25 and in the state $|-\rangle$ with probability 0.75. However, we would have the same density matrix if our system was in the state $|\alpha\rangle$ with probability 0.5 and in the state $|\beta\rangle$ with probability 0.5, where

$$|\alpha\rangle \equiv \sqrt{\frac{1}{4}}|+\rangle + \sqrt{\frac{3}{4}}|-\rangle$$

$$|\beta\rangle \equiv \sqrt{\frac{1}{4}}|+\rangle - \sqrt{\frac{3}{4}}|-\rangle,$$

because

$$\rho = \frac{1}{2}|\alpha\rangle\langle\alpha| + \frac{1}{2}|\beta\rangle\langle\beta| = \frac{1}{4}|+\rangle\langle+| + \frac{3}{4}|-\rangle\langle-|. \quad \diamond$$

As we are going to see, we can rewrite all the principles of quantum mechanics, in the language of the density matrix. Suppose that the time evolution of a closed quantum system is described by the unitary operator U . If the system was initially in the state $|\psi_i\rangle$ with probability p_i , then after the evolution has occurred the system will be in the state $U|\psi_i\rangle$ with probability p_i . Thus, the evolution of the density operator is described by the equation

$$\rho(t_0) = \sum_i p_i |\psi_i\rangle\langle\psi_i| \xrightarrow{U} \rho(t) = \sum_i p_i U(t, t_0) |\psi_i\rangle\langle\psi_i| U^\dagger(t, t_0) \Rightarrow$$

$$\rho(t) = U(t, t_0) \rho(t_0) U^\dagger(t, t_0). \quad (1.6)$$

From the relation (1.6) with differentiation we have the equation of motion for the density matrix

$$\frac{d}{dt} \rho(t) = -\frac{i}{\hbar} [H(t), \rho(t)]. \quad (1.7)$$

If we write the Hamiltonian of the system in the form $H(t) = H_0 + H_I(t)$, then we can write the equation of motion in the interaction picture

$$\frac{d}{dt}\tilde{\rho}(t) = -\frac{i}{\hbar}[H_I(t), \tilde{\rho}(t)], \quad (1.8)$$

where

$$\tilde{\rho}(t) \equiv e^{-\frac{i}{\hbar}H_I(t-t_0)}\rho(t_0)e^{\frac{i}{\hbar}H_I(t-t_0)} \quad (1.9)$$

the density matrix in the interaction picture.

Measurements are also described in the density operator language. Suppose we perform a measurement described by measurement operators M_m , for which we have the completeness relation

$$\sum_m M_m^\dagger M_m = I.$$

If the initial state was $|\psi_i\rangle$, then the probability of getting result m is

$$p(m|i) = \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle = \text{Tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|). \quad (1.10)$$

The probability of obtaining result m is

$$p(m) = \sum_i p(m|i)p_i = \sum_i p_i \text{Tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|) = \text{Tr}(M_m^\dagger M_m\rho). \quad (1.11)$$

The density matrix after the measurement changes: Suppose the initial state was $|\psi_i\rangle$. After a measurement with result m , the state is

$$|\psi_i^m\rangle = \frac{M_m|\psi_i\rangle}{\sqrt{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}}.$$

Thus, after the measurement we have an ensemble of states $|\psi_i^m\rangle$ with corresponding probabilities $p(m|i)$. Therefore, the density matrix is

$$\rho_m = \sum_i p(m|i)|\psi_i^m\rangle\langle\psi_i^m| = \sum_i p(m|i)\frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}.$$

Taking into account that $p(i|m)/p(m|i) = p_i/p(m)$, and using (1.10) and (1.11) we get

$$\rho_m = \sum_i p(m|i)\frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\text{Tr}(M_m^\dagger M_m\rho)} = \frac{M_m\rho M_m^\dagger}{\text{Tr}(M_m^\dagger M_m\rho)}. \quad (1.12)$$

1.2.1 The Reduced Density Matrix

One of the most important advantages of the density matrix formalism, is that it can deal with subsystems of a composite quantum system. The tool we use in this case is the **reduced density matrix**. Suppose we have two quantum systems A and B. The composite system is described by a state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, and consequently by a density matrix $\rho^{AB} = |\psi\rangle\langle\psi|$. The reduced density matrix for the subsystem A is defined as follows

$$\rho^A = Tr_B(\rho^{AB}), \quad (1.13)$$

where Tr_B is the *partial trace* over the system B. We defined the partial trace as

$$Tr_B(|\alpha_1\rangle\langle\alpha_2| \otimes |\beta_1\rangle\langle\beta_2|) \equiv |\alpha_1\rangle\langle\alpha_2| Tr(|\beta_1\rangle\langle\beta_2|),$$

where $|\alpha_1\rangle, |\alpha_2\rangle \in \mathcal{H}_A$ and $|\beta_1\rangle, |\beta_2\rangle \in \mathcal{H}_B$.

Here we must notice that it is not obvious that the reduced density matrix can describe the subsystem A. As we are going to prove the partial trace is the *only* operation which gives rise to the consistent description of observable quantities for subsystems of a composite system. Let's see first an example:

Example 1.2.4. *Suppose we have a two electron system in the state $|\psi\rangle = (|++\rangle + |--\rangle)/\sqrt{2}$. In this case the density matrix is*

$$\begin{aligned} \rho &= \left(\frac{|++\rangle + |--\rangle}{\sqrt{2}} \right) \left(\frac{\langle++| + \langle--|}{\sqrt{2}} \right) = \\ &= \frac{|++\rangle\langle++| + |++\rangle\langle--| + |--\rangle\langle++| + |--\rangle\langle--|}{2}. \end{aligned}$$

If we take the partial trace over the second electron, we have the reduced density matrix for the first

$$\begin{aligned} \rho^1 &= Tr_2(\rho) = \frac{Tr_2(|++\rangle\langle++|) + Tr_2(|++\rangle\langle--|)}{2} + \\ &+ \frac{Tr_2(|--\rangle\langle++|) + Tr_2(|--\rangle\langle--|)}{2} = \\ &= \frac{|+\rangle\langle+| + |+\rangle\langle+| + |+\rangle\langle-|\langle-+| + |-\rangle\langle+|\langle+ -| + |-\rangle\langle-|\langle- -|}{2} = \\ &= \frac{|+\rangle\langle+| + |-\rangle\langle-|}{2} = \frac{I}{2}. \quad \diamond \end{aligned} \quad (1.14)$$

Here we can observe that the reduced Density matrix (1.14) correspond to a mixed state, since $Tr(\rho_1^2) = 1/2 < 1$. While the total system is in a pure state (we know exactly its state), the subsystem is in a mixed state (we don't have exact knowledge of its state). We will discuss this in detail when we talk about **Entanglement**.

Now, let's see why the reduced density matrix consistently describes a subsystem. Suppose that M is an observable on the system A, and we have a measuring device which can measure it. Let \tilde{M} denote the corresponding observable for the same measurement, performed on the composite system. First we must argue that $\tilde{M} = M \otimes I_B$. If the system AB is prepared in the state $|m\rangle|\psi\rangle$, where $|m\rangle$ is an eigenstate of M with eigenvalue m and $|\psi\rangle$ is any state of B, then the measuring device yield the result m for the measurement, with probability one. Thus, if $P_m = |m\rangle\langle m|$ is the projector onto the m eigenspace of the observable M , then the corresponding projector for \tilde{M} is $P_m \otimes I_B$. We therefore have

$$\tilde{M} = \sum_m m P_m \otimes I_B = M \otimes I_B.$$

Suppose now that we measure on A the observable M . The averages computed by using ρ_A or ρ_{AB} must be the same

$$Tr_A(M\rho^A) = Tr(\tilde{M}\rho^{AB}) = Tr((M \otimes I_B)\rho^{AB}). \quad (1.15)$$

This equation is satisfied if we choose $\rho^A = Tr_B(\rho^{AB})$. In fact, the partial trace is the unique operation that has this property. To see this, let f be any map of density operators on AB to density operators on A, such that

$$Tr(Mf(\rho^{AB})) = Tr((M \otimes I_B)\rho^{AB}),$$

for all observables M . Let M_i be an orthonormal basis of operators for the space of Hermitian operators, then expanding $f(\rho_{AB})$ in this basis gives

$$f(\rho^{AB}) = \sum_i M_i Tr(M_i f(\rho^{AB})) = \sum_i M_i Tr((M_i \otimes I_B f(\rho^{AB}))).$$

It follows that f is uniquely determined by equation (1.15). Moreover, the partial trace satisfies (1.15), so it is the unique function having this property.

The time evolution of the reduced density matrix is very interesting since we can determine how a subsystem is affected from the rest of the system. We will study this issue in detail below.

1.3 The Entanglement

Entanglement is one of the most important properties of Nature. It is not describable in the context of classical physics but it appears naturally within the mathematical structure of quantum mechanics. A pure state $|\psi\rangle$ of a composite quantum system, lives in a linear Hilbert space, constructed by a tensor product of Hilbert spaces referring to its subsystems. Such composite spaces contain state vectors that cannot be written as a tensor product of vectors that belong to Hilbert subspaces of the subsystems. These states are called *entangled*. Speaking more sharply, a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \dots \otimes \mathcal{H}_N$ is called entangled if and only if

$$\forall |\varphi_A\rangle \in \mathcal{H}_A, \forall |\varphi_B\rangle \in \mathcal{H}_B, \dots, \forall |\varphi_N\rangle \in \mathcal{H}_N$$

we have :

$$|\psi\rangle \neq |\varphi_A\rangle \otimes |\varphi_B\rangle \otimes \dots \otimes |\varphi_N\rangle.$$

For the mixed states we say that the density matrix ρ is entangled, if the ensemble of states, describing the system, contains at least one pure entangled state.

The following theorem will help us to understand the mathematical basis of the entanglement:

Theorem 1.3.1. (*Schmidt Decomposition*) *Let a composite system AB that it is described by a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Moreover, let $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ orthonormal basis set of \mathcal{H}_A and \mathcal{H}_B , respectively. Then the vectors $\{|i_A\rangle|i_B\rangle\}$ forms a basis in $\mathcal{H}_A \otimes \mathcal{H}_B$, so we can write*

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} |i_A\rangle |i_B\rangle, \quad (1.16)$$

where λ_i is non negative real numbers that satisfy the relation $\sum_i \lambda_i = 1$ and is called *Schmidt coefficients*.

The basis $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ are called Schmidt basis for A and B respectively, and the number of non-zero λ_i is called *Schmidt number* for the state $|\psi\rangle$. A state is entangled if the Schmidt number is greater than one.

The above mathematical structure describing entanglement has an unexpected consequence : the non locality of quantum mechanics. Suppose we have of two particles described by the entangled state $|\psi\rangle$. Next we separate the particles minimizing any interaction between them and we bring the first

in the lab A and the second in the lab B. We write the pure state of the total system with the help of Schmidt decomposition

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} |i_A\rangle |i_B\rangle, \quad \sum_i \lambda_i = 1.$$

Now we perform a measurement in the Schmidt basis in each of the labs. The results of the measurements are random variables, and will be denoted X_A and X_B for the outcomes in lab A and B, respectively. In the limit of many repeated measurements, the relative weight of the outcomes will converge to the following probability distributions

$$p(X_A = k) = \langle k_A | (Tr_B |\psi\rangle\langle\psi|) |k_A\rangle = \lambda_k$$

$$p(X_B = k) = \langle k_B | (Tr_A |\psi\rangle\langle\psi|) |k_B\rangle = \lambda_k$$

Now, the two labs compare their results in order to evaluate the joint probability :

$$\begin{aligned} p(X_A, X_B = i, j) &= \langle i_A | \langle j_B | (|\psi\rangle\langle\psi|) |j_B\rangle |i_A\rangle = \delta_{ij} \lambda_i \\ &\neq \lambda_i \lambda_j = p(X_A = i) p(X_B = j) \end{aligned}$$

where the inequality, in the last line, holds if at least two of the Schmidt coefficients are larger than zero. The above result means that the results of the measurements X_A and X_B are not independent, although there is no interaction between the two particles-subsystems! This phenomenon is called non locality. Let's see an example

Example 1.3.1. *Suppose we let two spin 1/2 particles interact so the total system is described by the pure state*

$$|\psi\rangle = \frac{\sqrt{3}}{2} |++\rangle + \frac{1}{2} |--\rangle \equiv \frac{\sqrt{3}}{2} |+\rangle_A |+\rangle_B + \frac{1}{2} |-\rangle_A |-\rangle_B,$$

where $|\pm\rangle$ are the eigenstates of S_z . The Schmidt coefficients for this state are $\lambda_+ = \sqrt{3}/2$ and $\lambda_- = 1/2$. Now we separate the two particles, so there is no interaction between them, and bring them to the labs A and B. Now, the experimentalists in the lab A perform a measurement in the S_z axis and they find that their particle is in the $|+\rangle_A$ state. After the measurement the total wavefunction has collapsed into the state

$$|\psi'\rangle = |++\rangle \equiv |+\rangle_A |+\rangle_B.$$

This lead them to the definite conclusion that if the experimentalists in lab B perform the same measurement they will find, with probability one, that their

particle is in the state $|+\rangle_B$. One could say that we would have propagation of information with speed greater than the light speed if, the experimentalists in lab B had a way to know what had happened in the lab A. \diamond

What we discuss above concern the case where the total system is in a pure state. But what happens if the total system is in a mixed state? In such a case, as we will see in the next example, there are also classical correlations that do not have their origin in entanglement.

Example 1.3.2. Suppose we have a system of two spin 1/2 particles, that they are prepared with probability 1/2 in the state $|++\rangle$ and with probability 1/2 in the state $|--\rangle$ (attention these are classical probabilities). The density matrix of the system is written

$$\rho = \frac{1}{2}|++\rangle\langle ++| + \frac{1}{2}|--\rangle\langle --| = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let X_A and X_B be random variables that represent the measurement outcomes of spin A and B, respectively. Then we have

$$p(x_A) = p(x_B) = \langle x_A | \text{Tr}_B \rho | x_A \rangle = \langle x_A | \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} | x_A \rangle = \frac{1}{2}, \quad \forall x_A, x_B = 0, 1.$$

Also, we have

$$p(x_A, x_B = i, j) = \langle ij | \rho | ij \rangle = \frac{1}{2} \delta_{ij} \neq p(x_A = i) p(x_B = j) = \frac{1}{4}.$$

We observe that, in this particular basis, we have the same correlation as we would have starting from the Bell state

$$|\psi\rangle = \frac{|++\rangle + |--\rangle}{\sqrt{2}},$$

but these correlations have not their origin in entanglement, they are classical correlations. \diamond

1.4 The Quantum Entropies

The aim of this section is to study the quantum entropies, that play important role in quantum statistical mechanics and quantum information theory. We will define the Von Neumann entropy, the relative entropy and the linear entropy, and we will present their important properties. More details about the subject may be found in [2, 11].

1.4.1 The Von Neumann Entropy

The Von Neumann or entanglement entropy for a density matrix is defined as follows

$$S(\hat{\rho}) \equiv -\text{Tr}\{\hat{\rho} \ln \hat{\rho}\}, \quad (1.17)$$

using the spectral decomposition of the density matrix (1.2) we have

$$S(\hat{\rho}) = -\sum_i w_i \ln w_i. \quad (1.18)$$

A statistical mixture which is described by the density matrix (1.2) can be obtained by mixing pure ensembles described by states $|\psi_i\rangle$ with corresponding weights w_i . Then the entropy $S(\rho)$ expresses our uncertainty, or lack of knowledge about the realization of a particular state $|\psi_i\rangle$ in the mixture. Let's see, now, some important properties of the entropy

1. For all density matrices one has

$$S(\rho) \geq 0, \quad (1.19)$$

where the equality sign holds if and only if ρ is a pure state, otherwise the system is in a mixed state.

2. If the dimension of the Hilbert space is finite, $\dim\mathcal{H} = D < \infty$, then the entropy is bounded from above $S(\rho) \leq \ln D$, where the equality holds if and only if ρ is completely mixed $\rho = I/D$.
3. The Von Neumann entropy is invariant with respect to unitary transformations U of the Hilbert space, that is $S(U\rho U^\dagger) = S(\rho)$.
4. The Von Neumann entropy is a concave functional $\rho \mapsto S(\rho)$ on the space of density matrices. This means that for any collection of densities ρ_i and numbers $\lambda_i \geq 0$, satisfying $\sum_i \lambda_i = 1$, one has the inequality

$$S\left(\sum_i \lambda_i \rho_i\right) \geq \sum_i \lambda_i S(\rho_i).$$

The equality holds if and only if all ρ_i with vanishing λ_i are equal to each other. This property means that our uncertainty about the state $\rho = \sum_i \lambda_i \rho_i$ is greater than or equal to the average uncertainty of the states ρ_i that constitute the total mixture.

5. Consider a composite system with Hilbert space, $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$, described by the density matrix ρ^{AB} . The two subsystems are described by the density matrices $\rho^A = Tr_B \rho^{AB}$ and $\rho^B = Tr_A \rho^{AB}$. Then we have for the entropy

$$S(\rho^{AB}) \leq S(\rho^A) + S(\rho^B),$$

where the equality holds if and only if the density matrix of the total system is of the form $\rho^{AB} = \rho^A \otimes \rho^B$. Thus, our uncertainty about the product state $\rho^A \otimes \rho^B$ is, in general, greater than the uncertainty about the state ρ^{AB} of the total system. In other words, the partial trace has as a consequence to lose information about correlations between the subsystems and thus the entropy is increased. If the total system is in a pure state then the two density matrices ρ^A and ρ^B have the same eigenvalues, so they have equal entropies

$$S(\rho^A) = S(\rho^B) \geq 0,$$

where the sign of greater than holds strictly if and only if the state of the total system is an entangled state and the equality holds if the state is disentangled.

Let's see a simple example

Example 1.4.1. *Suppose we have the pure state*

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|+\rangle + e^{i\varphi} |-\rangle)$$

thus the density matrix is

$$\rho = |\chi\rangle\langle\chi| = \frac{1}{2} \begin{pmatrix} 1 & e^{-i\varphi} \\ e^{i\varphi} & 1 \end{pmatrix}.$$

In order to calculate the entropy we must write the density matrix in a diagonal form

$$\rho_{diag} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

so, the entropy is

$$S(\rho) = - \sum_{i=1}^2 w_i \ln w_i = -0 \cdot \ln 0 - 1 \cdot \ln 1 = 0.$$

Suppose now we a mixed state, in the following form

$$\sigma = \sum_i w_i |\varphi_i\rangle\langle\varphi_i|,$$

where $w_i \geq 0$ and $\sum_i w_i = 1$. Thus, the entropy is

$$\begin{aligned} S(\sigma) &= -Tr\{\sigma \ln \sigma\} = -Tr \left\{ \sum_i w_i |\varphi_i\rangle\langle\varphi_i| \ln \sum_i w_i |\varphi_i\rangle\langle\varphi_i| \right\} = \\ &= -Tr \left\{ \sum_i w_i \sum_j \sum_k c_k(p_j)^k |\varphi_i\rangle\langle\varphi_i|\varphi_j\rangle\langle\varphi_j|\dots \right\} = - \sum_i w_i \ln w_i. \quad \diamond \end{aligned}$$

1.4.2 The Relative Entropy

For a given pair of density matrices ρ and σ the relative entropy is defined by

$$S(\rho||\sigma) \equiv Tr\{\rho \ln \rho\} - Tr\{\rho \ln \sigma\}. \quad (1.20)$$

A physical interpretation for the relative entropy can be given if we consider a composite system that is described by the state ρ . This subsystem consists of two subsystems A and B that are described by the density matrices $\rho^A = Tr_B \rho^{AB}$ and $\rho^B = Tr_A \rho^{AB}$, respectively. Thus, we have

$$S(\rho^{AB}||\rho^A \otimes \rho^B) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}).$$

So, this entropy is a measure of the information encoded into the subsystems correlation.

Let's see some important properties of this entropy :

1. The relative entropy fulfils the inequality

$$S(\rho||\sigma) \geq 0$$

for all density matrices ρ and σ . The equality holds if and only if $\rho = \sigma$.

2. The relative entropy is invariant with respect to unitary transformations U ,

$$S(U\rho U^\dagger||U\sigma U^\dagger) = S(\rho||\sigma) = S(\rho||\sigma).$$

3. The relative entropy is jointly convex in its arguments. This means that for $0 \leq \lambda \leq 1$ we have the inequality

$$S(\rho||\sigma) \leq \lambda S(\rho_1||\sigma_1) + (1 - \lambda)S(\rho_2||\sigma_2),$$

where $\rho = \lambda\rho_1 + (1 - \lambda)\rho_2$ and $\sigma = \lambda\sigma_1 + (1 - \lambda)\sigma_2$.

4. If $\rho^A = \text{Tr}_B \rho^{AB}$ and $\sigma^A = \text{Tr}_B \sigma^{AB}$ then we have

$$S(\rho^A || \sigma^A) \leq S(\rho || \sigma)$$

that is, the partial trace reduces the relative entropy. If the state ρ^{AB} is a tensor product, we have

$$S(\rho^A || \sigma^A) = S(\rho^A \otimes \rho^B || \sigma^A \otimes \sigma^B).$$

1.4.3 The Linear Entropy

The linear entropy for a density matrix ρ , is defined by

$$S_l(\rho) = \text{Tr}\{\rho - \rho^2\} = 1 - \text{Tr}\rho^2. \quad (1.21)$$

We can immediately give an upper and a lower bound for this functional

$$0 \leq S_l(\rho) \leq 1,$$

where the equality with zero holds if and only if the state is pure. The second inequality follows from the fact that $\text{Tr}\rho^2$ is a positive operator.

For a D dimensional Hilbert space we have the upper bound

$$S_l(\rho) \leq 1 - \frac{1}{D}.$$

1.5 Time Evolution of Open Quantum Systems : An Introduction

In this section we will present the two main formalisms that refer to the time evolution of the density matrix : the *master equation* formalism [6, 7, 10] and the *influence functional* formalism [12–14]. The present thesis is based on the influence functional formalism so we won't present in detail the master equation formalism except from what is needed for the comparison of the two formalisms. In the last subsection, we will present a sort introduction to **Decoherence**.

1.5.1 The Master Equation

Here we are going to present an equation giving the time evolution of the reduced density matrix of a quantum system coupled to an environment. A schematic picture of the typical situation under study is shown in figure 1.1.

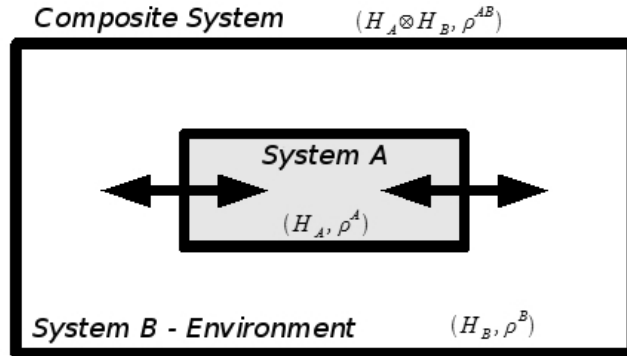


Figure 1.1: Schematic picture of an open quantum system.

An equation like this, can be obtained from the relation (1.7), if we take the partial trace, over the degrees of freedom of the environment, on both sides

$$\frac{d}{dt}\hat{\rho}_s(t) = -\frac{i}{\hbar}Tr_e[\hat{H}(t), \hat{\rho}(t)], \quad (1.22)$$

where $\rho_s(t)$, $\rho(t)$ are the reduced density matrix of the system and the density matrix of the total system respectively and

$$\hat{H}(t) = \hat{H}_s \otimes \hat{I}_e + \hat{I}_s \otimes \hat{H}_e + \hat{H}_I(t) \quad (1.23)$$

is the Hamiltonian of the total system. The equation (1.22) is exact, however in its right side there is the density matrix of the total system (system + environment), a fact that complicates things : the determination of ρ_s in some instant demands the knowledge of $\rho_e(t_0)$, in which we have no access. In other words, the $\rho_s(t_0 + dt)$ does not depend only from the $\rho_s(t_0)$, but also from the ρ_s in previous times, since the environment “remembers” that information and can transfer it back to the system. Thus, we need an equation that does not incorporate the environment.

In order to construct such an equation, from the underlying Hamiltonian dynamics of the total system, we will need the following assumptions [4, 10]:

- We assume that the initial state of the total system can be written as a tensor product and it is pure, $\hat{\rho}(t_0) = \hat{\rho}_s(t_0) \otimes \hat{\rho}_e(t_0)$. That is, initially the environment and the system are uncorrelated.
- Markov Approximation. This approximation allows to make the master equation local in time. We assume that there is a characteristic time τ_e for the environment, which is the typical time during which the internal correlations in the environment exist. This is such a time, that when it elapses, the state of the environment is practically independent of its initial state. In other words, τ_e is the time in which the environment "forgets" the information that the system gives to it. We are interesting for times, such that

$$\tau_e \ll \Delta t. \quad (1.24)$$

- Born Approximation. The weak-coupling assumption allows us to expand the exact equation of motion (1.22) for the density matrix to second order. Also, allows us to write the density matrix of the total system in the form $\hat{\rho}(t) \approx \hat{\rho}_s(t) \otimes \hat{\rho}_e(t)$, for every $t > 0$.
- Secular approximation. This approximation is consequence of the fact that we interesting for times, such that

$$\Delta t \ll \tau_i, \quad (1.25)$$

where τ_i is a characteristic relaxation time in system s due to interaction with environment. This approximation ensures that the master equation is in Lindblad form (see below).

With these assumptions at hand, we can derive from (1.22) the following equation of motion for the reduced density matrix

$$\frac{d}{dt}\hat{\rho}_s(t) = -\frac{i}{\hbar}[\hat{H}_{LS}, \hat{\rho}_s(t)] + \mathfrak{D}(\hat{\rho}_s(t)). \quad (1.26)$$

which is known as Lindblad equation. The first term, in the right-hand side of this equation, provides a Hamiltonian contribution to the dynamics. This term is often called the Lamb shift Hamiltonian since it leads to a Lamb-type renormalization of the unperturbed energy levels induced by the system-environment coupling. The second term is called dissipator and it is responsible for dissipation and decoherence (see below).

1.5.2 The Feynman-Vernon Theory

The time evolution of an open quantum system can also be examined in the framework of the well-known Feynman-Vernon path integral formalism. The total Hamiltonian is of the form (1.23), while the initial state is assumed to be pure and disentangled. The total system being closed the evolution of the total density matrix is

$$\hat{\rho}(t) = \hat{U}(t)[\hat{\rho}^s(0) \otimes \hat{\rho}^e(0)]\hat{U}^\dagger(t) \quad (1.27)$$

which in the coordinate representation is written

$$\begin{aligned} \rho_{X',X}(t) &= \langle X' | \hat{U}(t) \hat{\rho}^s(0) \otimes \hat{\rho}^e(0) \hat{U}^\dagger(t) | X \rangle \equiv \\ &\equiv \langle q', x' | \hat{U}(t) \hat{\rho}^s(0) \otimes \hat{\rho}^e(0) \hat{U}^\dagger(t) | q, x \rangle \Rightarrow \\ \rho_{X',X}(t) &= \int dX'' \int dX''' \langle X' | \hat{U}(t) | X'' \rangle \langle X'' | \hat{\rho}^s(0) \otimes \hat{\rho}^e(0) | X''' \rangle \langle X''' | \hat{U}^\dagger(t) | X \rangle \end{aligned}$$

or

$$\rho_{X',X}(t) = \int dX'' \int dX''' \rho_{x''x'''}^s(0) \rho_{q''q'''}^e(0) \langle X' | \hat{U}(t) | X'' \rangle \langle X''' | \hat{U}^\dagger(t) | X \rangle, \quad (1.28)$$

where $X \equiv (q, x)$, with q and x be the coordinates of the environment and system, respectively. We are interesting in the reduced density matrix of the open system, so we integrate the degrees of freedom of the environment

$$\begin{aligned} \hat{\rho}^R(t) &= Tr_e[\hat{U}(t)[\hat{\rho}^s(0) \otimes \hat{\rho}^e(0)]\hat{U}^\dagger(t)] \Rightarrow \quad (1.29) \\ \rho_{x'x}^R(t) &= \int dq \int dx'' \int dq'' \int dx''' \int dq''' \rho_{x''x'''}^s(0) \rho_{q''q'''}^e(0) \times \\ &\times \langle x', q | \hat{U}(t) | x'', q'' \rangle \langle x''', q''' | \hat{U}^\dagger(t) | x, q \rangle. \quad (1.30) \end{aligned}$$

Let us write in detail, each of the factors of the above relation: For the two propagators we have

$$\begin{aligned} \langle x', q | \hat{U}(t) | x'', q'' \rangle &= \int_{x^{(4)}(0)=x''}^{x^{(4)}(t)=x'} \mathcal{D}x^{(4)}(t) \int \mathcal{D}q^{(4)}(t) \delta[q^{(4)}(t) - q] \delta[q^{(4)}(0) - q''] \times \\ &\times \exp \left\{ \frac{i}{\hbar} \int_0^t dt' \mathcal{L}[q^{(4)}, x^{(4)}(t')] \right\}, \quad (1.31) \end{aligned}$$

$$\begin{aligned}
\langle x''', q''' | \hat{U}^\dagger(t) | x, q \rangle &= \int_{x^{(4)}(0)=x'''}^{x^{(4)}(t)=x} \mathcal{D}x^{(1)}(t) \int \mathcal{D}q^{(1)}(t) \delta[q^{(1)}(t) - q] \delta[q^{(1)}(0) - q'''] \times \\
&\times \exp \left\{ \frac{i}{\hbar} \int_t^0 dt' \mathcal{L}[q^{(1)}(t'), x^{(1)}(t')] \right\}, \tag{1.32}
\end{aligned}$$

where \mathcal{L} is the Lagrangian of the total system. The numbering we used is for our later convenience. Now we write the total Lagrangian as $\mathcal{L} = \mathcal{L}_s + \mathcal{L}_e + \mathcal{L}_I$, where the system-environment interaction is incorporated in \mathcal{L}_I . Combining the equations (1.31), (1.32) and (1.30) and reorder the terms, so we can write

$$\rho_{x'x}^R(t) = \int dx'' \int dx''' J(x, x', x'', x'''; t) \rho_{x''x'''}^s(0), \tag{1.33}$$

where

$$\begin{aligned}
J(x, x', x'', x'''; t) &\equiv \int_{x^{(4)}(0)=x'''}^{x^{(4)}(t)=x'} \mathcal{D}x^{(4)}(t) \int_{x^{(1)}(0)=x'''}^{x^{(1)}(t)=x} \mathcal{D}x^{(1)}(t) \exp \left\{ \frac{i}{\hbar} \int_0^t dt' \mathcal{L}_s[x^{(4)}(t')] \right\} \times \\
&\times \exp \left\{ - \frac{i}{\hbar} \int_0^t dt' \mathcal{L}_s[x^{(1)}(t')] \right\} \mathcal{F}[x^{(4)}(t), x^{(1)}(t); t]. \tag{1.34}
\end{aligned}$$

The last factor is the well-known *influence functional* which assumes the form

$$\begin{aligned}
\mathcal{F}[x^{(4)}(t), x^{(1)}(t); t] &\equiv \int dq \int dq'' \int dq''' \rho_{q''q'''}^e(0) \times \\
&\times \int_{q^{(4)}(0)=q''}^{q^{(4)}(t)=q} \mathcal{D}q^{(4)}(t) \int_{q^{(1)}(0)=q'''}^{q^{(1)}(t)=q} \mathcal{D}q^{(1)}(t) \exp \left\{ \frac{i}{\hbar} \int_0^t dt' \mathcal{L}_{e+I}[q^{(4)}(t'), x^{(4)}(t')] - \right. \\
&\quad \left. - \frac{i}{\hbar} \int_0^t dt' \mathcal{L}_{e+I}[q^{(1)}(t'), x^{(1)}(t')] \right\}. \tag{1.35}
\end{aligned}$$

If there were no interaction between system and environment, the influence functional would be unit. Then the relation (1.34) would be written as the product of two propagators, one forward in time and one backward in time. In case of interaction the two propagations are coupled by the influence functional.

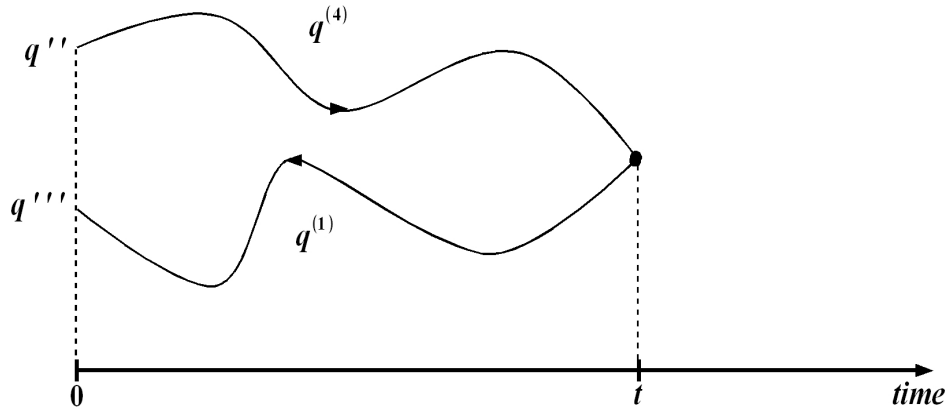


Figure 1.2: The paths $q^{(1)}$ and $q^{(4)}$.

Here we must notice that the only assumption being made, in order to derive the relation (1.33), is that the initial state can be written as a tensor product. This is a limiting assumption, but we can easily extend the path integral formalism (not the influence functional) to deal with entangled initial conditions [15–17]. Because of the fact that we don't need many assumptions, this formalism have used in order to derive exact master equations for the reduced density matrix (see for example [20–22]).

1.5.3 Decoherence

The interaction of an open quantum system with its environment creates correlations between the states of the system and those of the environment. The environment exchanges information with the open system in the form of these correlations. This process has as a consequence that a certain set of states of the open system Hilbert space exhibits strong stability properties, while superpositions of these states are destroyed in the course of time. This dynamical destruction of quantum coherence is called decoherence.

First, let's see a simple example showing the results of the decoherence in a quantum system. Consider the interference effect¹ in the two slit experiment. In the case that there is no environment (see figure 1.3 (a)), the electrons are emitted from the source S and they create the well-known interference pattern on the screen. The probability to find an electron in the position x

¹Interference experiments has been used in order to study the phenomenon of decoherence, see for example [27, 28].

is

$$|\psi_s(x)|^2 = |\psi_1(x) + \psi_2(x)|^2 = |\psi_1(x)|^2 + |\psi_2(x)|^2 + \psi_1(x)\psi_2^*(x) + \psi_1^*(x)\psi_2(x),$$

where ψ_1 is the wavefunction that characterizes the electrons which come from the slit S_1 respectively for ψ_2 . In the language of the density matrix the electrons, between the wall with the two slits and the screen, will be described by the density matrix

$$\hat{\rho}_s = \frac{1}{N} \sum_{n,m=1}^2 |\psi_n\rangle\langle\psi_m|. \quad (1.36)$$

Suppose, now, that the electrons, in the area between the wall and the screen, interact with an environment (a photon bath for example) in which we have no access. In this case the density matrix of the electrons in that area, will be

$$\hat{\rho}_s = \text{tr}_e \left\{ |\psi_s\rangle|\psi_e\rangle\langle\psi_e|\langle\psi_s| \right\}, \quad (1.37)$$

where $|\psi_s\rangle$ characterizes the electrons and $|\psi_e\rangle$ the environment. The time evolution of the reduced density matrix (1.37) will be described from the equation (1.26) or (1.33). The decoherence theory says that very fast the density matrix of the electrons will be

$$\frac{1}{N} \sum_{n=1}^2 |\psi_n\rangle\langle\psi_n| = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|. \quad (1.38)$$

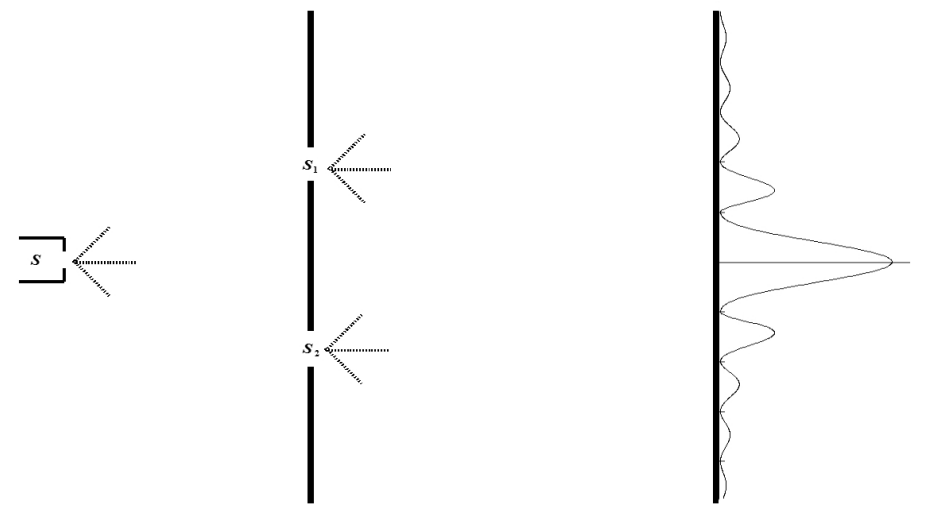
That is, the interference terms will be destroyed very fast, so we will have on the screen the pattern showed in figure 1.3 (b). We would expect such an image, if the electrons was classical particles.

Let's examine the situation in detail [4]. Suppose that the total system is described by a Hamiltonian of the form (1.23). We write the interaction term in the form

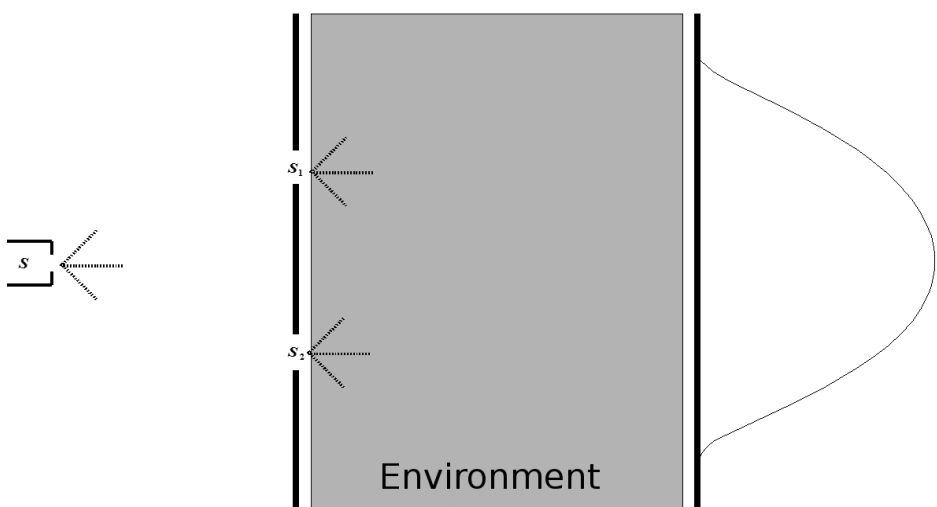
$$\hat{H}_I = \sum_n |n\rangle\langle n| \otimes \hat{B}_n \equiv \sum_n \hat{A}_n \otimes \hat{B}_n, \quad (1.39)$$

where $|n\rangle$ is an orthogonal basis of our system and $B_n = B_n^\dagger$ are arbitrary operators of the environment. We assume further that the system operators A_n are conserved quantities

$$[\hat{H}_s + \hat{H}_e + \hat{H}_I, \hat{A}_n] = [\hat{H}_s + \hat{H}_e, \hat{A}_n] = 0. \quad (1.40)$$



(a)



(b)

Figure 1.3: *The two slit experiment in the case of (a) the absence of environment (b) presence of environment.*

Thus, in the interaction picture, the evolution operator is written

$$\hat{U}(t) = \hat{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t dt' \sum_n |n\rangle \langle n| \otimes \hat{B}_n(t') \right\}. \quad (1.41)$$

This expression says that the basis $|n\rangle$ is not affected by the coupled dynamics and that the initial state

$$|\psi(0)\rangle = \sum_n c_n |n\rangle \otimes |\phi\rangle, \quad (1.42)$$

where $|\phi\rangle$ is an arbitrary environment state, evolves into

$$|\psi(t)\rangle = \sum_n c_n |n\rangle \otimes |\phi_n(t)\rangle, \quad (1.43)$$

where

$$|\phi_n(t)\rangle = \hat{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t dt' \hat{B}_n(t') \right\} |\phi\rangle. \quad (1.44)$$

The state (1.43) is an entangled system-environment state given by a superposition of the states $|n\rangle \otimes |\phi_n(t)\rangle$. That is, the environment carries information onto the system state. Thus, the reduced density matrix at a time t is given by

$$\hat{\rho}_s(t) = Tr_e \{ |\psi(t)\rangle \langle \psi(t)| \} = \sum_{n,m} c_n c_m^* |n\rangle \langle m| \langle \phi_m(t) | \phi_n(t) \rangle.$$

It follows from (1.44) that $\langle \phi_n(t) | \phi_n(t) \rangle = 1$, and thus, the diagonal elements of $\hat{\rho}_s(t)$ are constant in time. However, the off-diagonal elements $\hat{\rho}_s(t)$ do change with time, in general. The time dependence of the matrix element $\langle n | \hat{\rho}_s(t) | m \rangle$ is given by the overlap of the corresponding environment states $|\phi_n(t)\rangle$ and $|\phi_m(t)\rangle$ which will be written as

$$|\langle \phi_n(t) | \phi_m(t) \rangle| = \exp\{\Gamma_{nm}(t)\}, \quad \Gamma_{nm}(t) \leq 0. \quad (1.45)$$

For $n \neq m$ the quantity $\Gamma_{nm}(t)$ describes the behavior of the off-diagonal elements of the reduced density matrix.

The time dependence of the function $\Gamma_{nm}(t)$ strongly depends, in general, on the specific form of the system-environment coupling, on the various parameters of the underlying microscopic model, and also on the properties of the initial state. For many physical systems it turns out that the irreversible dynamics induced by the system-environment interaction leads to a rapid decrease of the overlap $\langle \phi_m(t) | \phi_n(t) \rangle$ when $n \neq m$. Consider the extreme case

in which the overlap of the states $|\phi_n(t)\rangle$ and $|\phi_m(t)\rangle$ decreases, for $n \neq m$, very fast to zero after a time interval large compared to a typical scale τ_D , the so-called decoherence time,

$$\langle \phi_n(t) | \phi_m(t) \rangle \longrightarrow \delta_{nm}, \quad \gamma \iota \alpha \quad t \gg \tau_D. \quad (1.46)$$

This leads to a reduced density matrix of the form:

$$\hat{\rho}_s(t) \rightarrow \sum_n |c_n|^2 |n\rangle \langle n|. \quad (1.47)$$

The coherences of the density matrix in the basis $|n\rangle$ have disappeared as a result of the interaction with the environment : After a time $t \gg \tau_D$ the state $\hat{\rho}_s(t)$ of the reduced system behaves as an incoherent mixture of the state $|n\rangle$, in the sense that interference terms of the form $\langle m | \hat{A} | n \rangle$, $n \neq m$, no longer appear in the expectation value of any observable \hat{A} . Superpositions of the states $|n\rangle$ are therefore effectively destroyed locally which means that they are unobservable for all measurements performed solely on the system S .

Summarizing, we can say that the interaction between a quantum system with its environment makes the system to behave more “classical”.

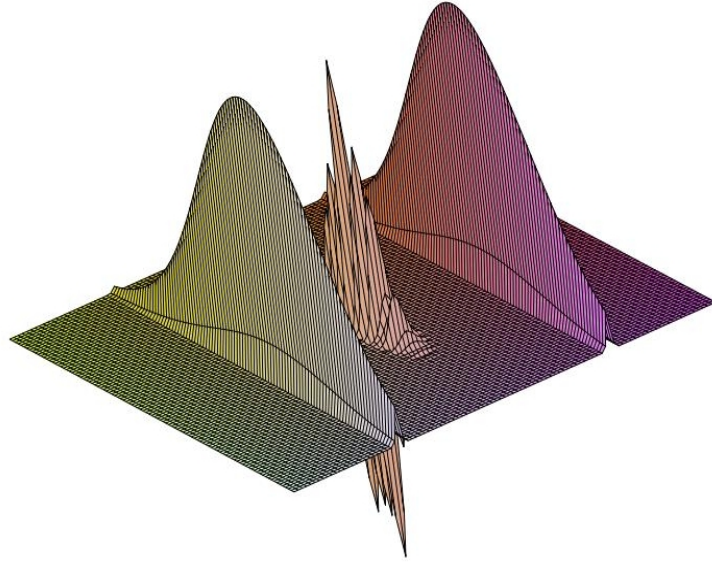
Before we close this paragraph we will introduce a useful tool which allows us to “see” the decoherence : the Wigner function [9,30]. Let’s examine this function, first, in a classical level. Consider a classical harmonic oscillator. Its motion can be completely described by a point in the phase space. For a large number of identical classical oscillators, one can define the phase-space probability distribution : a function $W(x, p)$ which indicates the probability of finding a particle at a certain point in the phase space. This function must, of course, be **non-negative** and normalized: its integral over the entire phase space must be equal to one.

In a quantum mechanical level, however, the notion of a certain point in the phase space does not make sense because the position and the momentum cannot be measured simultaneously (Heisenberg’s uncertainty principle). Nevertheless a quantum mechanical Wigner function can be defined. It is real and normalized but not positive defined. We can calculate this function from the density matrix, using the relation

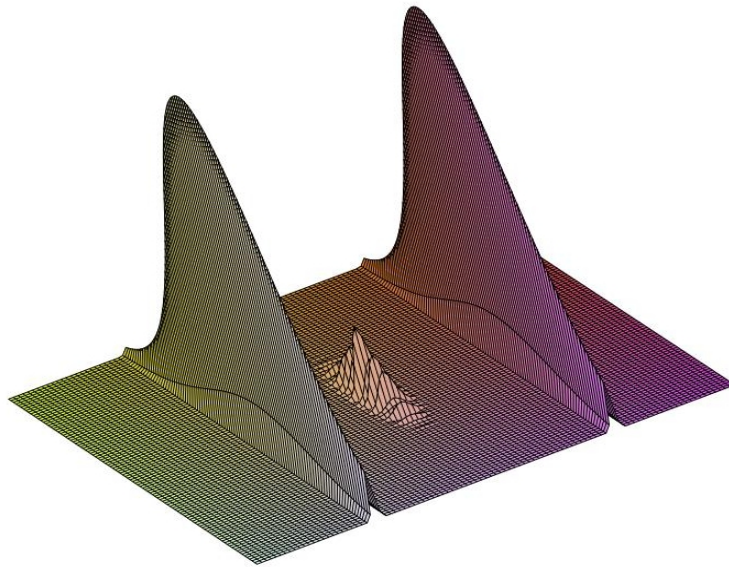
$$W(x, p) = \frac{1}{\pi} \int_{-\infty}^{\infty} dy \exp \left\{ \frac{2i}{\hbar} py \right\} \rho(x - y, x + y). \quad (1.48)$$

Let’s see, now, how we can use the Wigner function in order to “visualize” the quantum behavior and decoherence. Suppose we have a superposition of two Gaussian wave-packets [9]. In this case, the Wigner function is shown

in figure 1.4 (a). As we can see, in this graph, there are oscillations which are characteristic for the quantum behavior : the function takes negative values. In graph 1.4 (b) we have a snapshot of the time evolution of the Wigner function, when there is an environment : the oscillations are damped in the passage of time and the function reminds us the classical phase space. This behavior is characteristic for the decoherence.



(a)



(b)

Figure 1.4: *The Wigner function of the superposition of two Gaussian wavepackets (a) in the absence of environment (b) the time evolution in the presence of environment.*

Chapter 2

The Von Neumann Entropy and The Replica Method

In this chapter we will present a technique that allows us to calculate the Von Neuman entropy of a density matrix in the position representation. This calculation technique is known as **replica method** [32–34]. We will, also, give some analytic examples.

2.1 The Replica Method

The entanglement entropy (1.18), as we have seen, is produced from the eigenvalues of the density matrix:

$$S = - \sum_i w_i \ln w_i = -Tr(\hat{\rho} \ln \hat{\rho}). \quad (2.1)$$

The calculation of entropy is not an easy task even for the simple case of the two harmonic oscillators (as we will see in the example that follow). The problem appears to be very hard when you deal with many (or even infinite [31–34]) degrees of freedom. In this cases the replica method can be proved very helpful. In order to understand how the replica method works, we will use a quantum system with one degree of freedom. To calculate the entropy, we will based on the following observation:

Suppose that we can calculate the trace $Tr \rho^n$:

$$Tr \rho^n = \int dx_2^{(1)} \dots \int dx_2^{(n)} \rho(x_2^{(1)}, x_2^{(2)}) \rho(x_2^{(2)}, x_2^{(3)}) \dots \rho(x_2^{(n)}, x_2^{(1)}). \quad (2.2)$$

After the calculation of the function $f(n) = Tr \hat{\rho}^n$ for integer n , one considers the function $f(\nu) = Tr \hat{\rho}^\nu$ where $\nu > 0$. Using analytic continuation one then

can find the Von Neumann entropy from the relation:

$$-\left. \frac{\partial}{\partial \nu} f(\nu) \right|_{\nu=1} = -\lim_{\nu \rightarrow 1} \frac{Tr \hat{\rho}^\nu - 1}{\nu - 1} = -Tr(\hat{\rho} \ln \hat{\rho}) \equiv S \quad (2.3)$$

since

$$\begin{aligned} Tr \hat{\rho}^\nu &= Tre^{\nu \ln \hat{\rho}} = Tre^{(\nu-1) \ln \hat{\rho}} \hat{\rho} = Tr[\hat{\rho} + (\nu-1) \ln \hat{\rho} + O((\nu-1)^2)] = \\ &= 1 + (\nu-1)Tr(\hat{\rho} \ln \hat{\rho}) + O((\nu-1)^2) \Rightarrow \lim_{\nu \rightarrow 1} \frac{Tr \hat{\rho}^\nu - 1}{\nu - 1} = Tr(\hat{\rho} \ln \hat{\rho}) \Rightarrow \\ &S = -\left. \frac{\partial}{\partial \nu} Tr \hat{\rho}^\nu \right|_{\nu=1}. \end{aligned}$$

2.2 A Simple Example: The Two Harmonic Oscillators

In this section we will see how this method applied in the simple case of the two coupled harmonic oscillators. In all three examples we are going to present, we consider that the one harmonic oscillator is the system and the other is the environment and the total system (system and environment) is in its ground state. In the first example we consider the coupling to be time independent, in the second we suppose that the coupling changes suddenly at $t = 0$ and in the third the coupling changes adiabatically. In all the examples we have calculated the reduced density matrix and the Von Neumann entropy.

2.2.1 Time Independent Coupling Constant

We consider a system of two coupled harmonic oscillators [31]:

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}k_0(x_1^2 + x_2^2) + \frac{1}{2}k_1(x_1 - x_2)^2. \quad (2.4)$$

The normalized ground state wave function is

$$\psi_0(x_1, x_2) = \sqrt[4]{\frac{\omega_+ \omega_-}{\pi^2}} \exp \left\{ -\frac{1}{2}(\omega_+ x_+^2 + \omega_- x_-^2) \right\}, \quad (2.5)$$

where $x_\pm = (x_1 \pm x_2)/\sqrt{2}$, $\omega_+ = \sqrt{k_0}$ and $\omega_- = \sqrt{k_0 + 2k_1}$. We now form the ground state density matrix, and trace over the first oscillator, resulting in a reduced density matrix for the second oscillator

$$\rho_R(x'_2, x_2) = \sqrt{\frac{\gamma - \beta}{\pi}} \exp \left\{ -\frac{\gamma}{2}(x_2^2 + x_2'^2) + \beta x_2 x_2' \right\} \quad (2.6)$$

where

$$\beta = \frac{(\omega_+ - \omega_-)^2}{4(\omega_+ + \omega_-)}, \gamma - \beta = \frac{2\omega_+\omega_-}{\omega_+ + \omega_-}.$$

Let's find, now, the Von Neumann entropy for the reduced density matrix (2.6). In order to use the relation (2.3) we must first calculate the trace (2.17):

$$\begin{aligned} Tr \rho_R^n &= \left(\frac{\gamma - \beta}{\pi} \right)^{n/2} \int dx_2^{(1)} \dots \int dx_2^{(n)} \exp \left\{ -\frac{\gamma}{2}(x_2^{(1)2} + x_2^{(2)2}) + \beta x_2^{(1)} x_2^{(2)} \right\} \times \\ &\quad \times \dots \times \exp \left\{ -\frac{\gamma}{2}(x_2^{(n)2} + x_2^{(1)2}) + \beta x_2^{(n)} x_2^{(1)} \right\} = \\ &= \left(\frac{\gamma - \beta}{\pi} \right)^{n/2} \int dx_2^{(1)} \dots \int dx_2^{(n)} \exp \left\{ -\gamma(x_2^{(1)2} + x_2^{(2)2} + \dots + x_2^{(n)2}) + \right. \\ &\quad \left. + \beta(x_2^{(1)} x_2^{(2)} + x_2^{(2)} x_2^{(3)} + \dots + x_2^{(n)} x_2^{(1)}) \right\} = \\ &= \left(\frac{\gamma - \beta}{\pi} \right)^{n/2} \int dx_2^{(1)} \dots \int dx_2^{(n)} \exp \left\{ -\mathbf{x}^T \mathbf{M}_n \mathbf{x} \right\}, \end{aligned}$$

where

$$\mathbf{M}_n = \begin{pmatrix} \gamma & -\beta/2 & 0 & \dots & 0 & -\beta/2 \\ -\beta/2 & \gamma & -\beta/2 & \dots & 0 & 0 \\ 0 & -\beta/2 & \gamma & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \gamma & -\beta/2 \\ -\beta/2 & 0 & 0 & \dots & -\beta/2 & \gamma \end{pmatrix} \quad (2.7)$$

So the trace takes the form

$$Tr \rho_R^n = \left(\frac{\gamma - \beta}{\pi} \right)^{n/2} \frac{(\pi)^{n/2}}{\sqrt{\det \mathbf{M}_n}} = \frac{(\gamma - \beta)^{n/2}}{\sqrt{\det \mathbf{M}_n}}. \quad (2.8)$$

In order to calculate the above determinant, we define the matrix [32]

$$\tilde{\mathbf{M}}_n = \begin{pmatrix} 2 & -C & 0 & \dots & 0 & -C \\ -C & 2 & -C & \dots & 0 & 0 \\ 0 & -C & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -C \\ -C & 0 & 0 & \dots & -C & 2 \end{pmatrix} \quad (2.9)$$

for which we have

$$\det \tilde{\mathbf{M}}_{\mathbf{n}} = \prod_{r=1}^n \left[2 - 2C \cos \left(\frac{2\pi r}{n} \right) \right] = 2^n \frac{(1 - \xi^n)^2}{(1 + \xi^2)^n}, \quad (2.10)$$

where $C = \beta/\gamma = \frac{2\xi}{\xi^2+1}$ with $\xi = \frac{\beta}{\gamma+\alpha}$ and $\alpha = \sqrt{\gamma^2 - \beta^2}$. Thus the determinant is

$$\det \mathbf{M}_{\mathbf{n}} = \left(\frac{\gamma}{2} \right)^n \det \tilde{\mathbf{M}}_{\mathbf{n}} = \gamma^n \frac{(1 - \xi)^2}{(1 + \xi^2)^n}.$$

So from the relation (2.8) we have

$$\text{Tr} \rho_R^n = \left(\frac{\gamma - \beta}{\gamma} \right)^{n/2} \frac{(1 + \xi^2)^{n/2}}{(1 - \xi^n)} = \frac{(1 - \xi)^n}{1 - \xi^n}. \quad (2.11)$$

The asked entropy arise from the relation (2.11) if we put it in (2.3), thus

$$S = -\ln(1 - \xi) - \frac{\xi}{1 - \xi} \ln \xi. \quad (2.12)$$

The above entropy is of course time independent since the total system is in its ground state, and as we expect it has non zero value.

2.2.2 Sudden Change of the Coupling Constant

Here we will study the time evolution of the Von Neumann entropy, in the case where the Hamiltonian (2.4), at t_0 , suddenly changes to :

$$\tilde{H} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\tilde{k}_0(x_1^2 + x_2^2) + \frac{1}{2}\tilde{k}_1(x_1 - x_2)^2, \quad (2.13)$$

that is, we have a suddenly change of the constants $k_0 \rightarrow \tilde{k}_0$ and $k_1 \rightarrow \tilde{k}_1$. We suppose again that our system is in its ground state. The density matrix in a time $t > t_0$ will be

$$\begin{aligned} \rho_t(x'_1, x'_2; x_1, x_2) &= \langle x'_1, x'_2 | \hat{U}(t) | \psi_0 \rangle \langle \psi_0 | \hat{U}^\dagger(t) | x_1, x_2 \rangle = \\ &= \int dz'_1 dz'_2 \int dz_1 dz_2 \langle x'_1, x'_2 | \hat{U}(t) | z'_1, z'_2 \rangle \rho_0(z'_1, z'_2; z_1, z_2) \langle z_1, z_2 | \hat{U}^\dagger(t) | x_1, x_2 \rangle, \end{aligned} \quad (2.14)$$

where ρ_0 is the density matrix calculated in the previous subsection. Changing variables and writing the Hamiltonian as a sum of two independent oscillators $\hat{H} = \hat{H}_+ + \hat{H}_-$ we have

$$\langle x'_1, x'_2 | \hat{U}(t) | z'_1, z'_2 \rangle = \langle y'_1, y'_2 | e^{-\frac{i}{\hbar} \hat{H} t} | z'_+, z'_- \rangle = \langle y'_1 | e^{-\frac{i}{\hbar} \hat{H}_+ t} | z'_+ \rangle \langle y'_2 | e^{-\frac{i}{\hbar} \hat{H}_- t} | z'_- \rangle =$$

$$= \left(\frac{-\tilde{\omega}_+\tilde{\omega}_-}{4\pi^2\hbar^2 \sin(\tilde{\omega}_+t) \sin(\tilde{\omega}_-t)} \right)^{1/2} \exp \left\{ \frac{i\tilde{\omega}_+}{2\hbar \sin(\tilde{\omega}_+t)} [(y_1'^2 + z_+'^2) \cos(\tilde{\omega}_+t) - 2y_1'z_+'] \right\} \times \\ \times \exp \left\{ \frac{i\tilde{\omega}_-}{2\hbar \sin(\tilde{\omega}_-t)} [(y_2'^2 + z_-'^2) \cos(\tilde{\omega}_-t) - 2y_2'z_-'] \right\},$$

and

$$\langle z_1, z_2 | \hat{U}^\dagger(t) | x_1, x_2 \rangle = \langle y_1, y_2 | \hat{U}(t) | z_+, z_- \rangle^* = \\ = \left(\frac{-\tilde{\omega}_+\tilde{\omega}_-}{4\pi^2\hbar^2 \sin(\tilde{\omega}_+t) \sin(\tilde{\omega}_-t)} \right)^{1/2} \exp \left\{ \frac{i\tilde{\omega}_+}{2\hbar \sin(\tilde{\omega}_+t)} [(y_1^2 + z_+^2) \cos(\tilde{\omega}_+t) - 2y_1z_+] \right\} \times \\ \times \exp \left\{ \frac{i\tilde{\omega}_-}{2\hbar \sin(\tilde{\omega}_-t)} [(y_2^2 + z_-^2) \cos(\tilde{\omega}_-t) - 2y_2z_-] \right\},$$

finally the density matrix ρ_0 is

$$\rho_0(z_+', z_-'; z_+, z_-) = \left(\frac{\omega_+\omega_- \sinh^2(\omega_+T_E) \sinh^2(\omega_-T_E)}{4\pi^2\hbar^2 \sinh(2\omega_+T_E) \sinh(2\omega_-T_E)} \right)^{1/2} \times \\ \times \exp \left\{ \frac{-\omega_+}{2\hbar \sinh(2\omega_+T_E)} [(z_+^2 + z_+'^2) \cosh(2\omega_+T_E) - 2z_+z_+'] \right\} \times \\ \times \exp \left\{ \frac{-\omega_-}{2\hbar \sinh(2\omega_-T_E)} [(z_-^2 + z_-'^2) \cosh(2\omega_-T_E) - 2z_-z_-'] \right\}.$$

Substituting the above result in (2.14), integrating z_\pm, z'_\pm and taking the limit $T_E \rightarrow \infty$ we find for the density matrix:

$$\rho_t(y_1', y_2'; y_1, y_2) = C(t) \exp \left\{ -(\Omega_+ y_1'^2 + \Omega_+^* y_1^2 + \Omega_- y_2'^2 + \Omega_-^* y_2^2) \right\},$$

where we denoted

$$\Omega_\pm = \frac{\tilde{\omega}_\pm^2 \tan(\tilde{\omega}_\pm t)}{\sin^2(\tilde{\omega}_\pm t) [\omega_\pm \tan(\tilde{\omega}_\pm t) - i\tilde{\omega}_\pm]} - i \frac{\tilde{\omega}_\pm}{\tan(\tilde{\omega}_\pm t)}$$

$C(t)$ is a function of time, which is not necessary for the calculation of entropy. In order, now, to calculate the reduced density matrix for oscillator 2, we go back to the old coordinates and we integrate the coordinates of the oscillator 1

$$\langle x_2' | \hat{\rho}_R(t) | x_2 \rangle = \int dx_1 \rho_t(x_1, x_2'; x_1, x_2) \Rightarrow$$

$$\langle x'_2 | \hat{\rho}_R(t) | x_2 \rangle = \tilde{C}(t) \exp \left\{ -\frac{1}{2}(\gamma(t)x_2'^2 + \gamma^*(t)x_2^2) + \beta(t)x_2'x_2 \right\},$$

where

$$\gamma(t) = \frac{1}{2} \left[(\Omega_+ + \Omega_-) - \frac{(\Omega_+ - \Omega_-)^2}{(\Omega_+ + \Omega_-) + (\Omega_+ + \Omega_-)^*} \right]$$

and

$$\beta(t) = \frac{1}{2} \frac{(\Omega_+ - \Omega_-)(\Omega_+ - \Omega_-)^*}{(\Omega_+ + \Omega_-) + (\Omega_+ + \Omega_-)^*}.$$

Now to find the entropy we will need the trace (2.17), so we have

$$\begin{aligned} Tr \rho_R^n(t) &= [\tilde{C}(t)]^n \int dx_2^{(1)} \dots \int dx_2^{(n)} \exp \left\{ -Re(\gamma(t))(x_2^{(1)2} + \dots + x_2^{(n)2}) + \right. \\ &\quad \left. + \beta(t)(x_2^{(1)}x_2^{(2)} + \dots + x_2^{(n)}x_2^{(1)}) \right\} = [\tilde{C}(t)]^n \int dx_2^{(1)} \dots \int dx_2^{(n)} e^{-\mathbf{x}^T \mathbf{M}_n(t) \mathbf{x}}. \end{aligned}$$

Thus, we will find the entropy as we did in the previous subsection, that is

$$S(t) = -\ln(1 - \xi(t)) - \frac{\xi(t)}{1 - \xi(t)} \ln \xi(t),$$

where

$$\begin{aligned} \xi(t) &= \frac{\beta(t)}{Re(\gamma(t)) + \sqrt{Re(\gamma(t))^2 - \beta^2(t)}} = \\ &= \frac{(x_+ - x_-)^2 + (y_+ - y_-)^2}{(x_+ + x_-)^2 + (y_+ - y_-)^2 + 4x_+x_- + 4\sqrt{x_+x_-}[(x_+ + x_-)^2 + (y_+ - y_-)^2]}, \end{aligned}$$

with

$$x_{\pm} = \frac{\tilde{\omega}_{\pm}^2 \omega_{\pm}}{\omega_{\pm}^2 \sin^2(\tilde{\omega}_{\pm}t) + \tilde{\omega}_{\pm}^2 \cos^2(\tilde{\omega}_{\pm}t)}$$

and

$$y_{\pm} = \frac{\tilde{\omega}_{\pm}^3 - \tilde{\omega}_{\pm}[\omega_{\pm}^2 \sin^2(\tilde{\omega}_{\pm}t) + \tilde{\omega}_{\pm}^2 \cos^2(\tilde{\omega}_{\pm}t)]}{\tan(\tilde{\omega}_{\pm}t)[\omega_{\pm}^2 \sin^2(\tilde{\omega}_{\pm}t) + \tilde{\omega}_{\pm}^2 \cos^2(\tilde{\omega}_{\pm}t)]}.$$

In figure 2.1 we see the graph of entropy as a function of time, for several values of the Hamiltonian's parameters. In graphs (a) and (b) we have choose $k_1 = 0$. In such a case initially there is no coupling between the two oscillators, and on time $t_0 = 0$ we "switch on" the coupling. In graph (a) we kept the constant k_0 unchanged, so the entropy has one period (as we can see in graph 2.2 (a)). In graph (b) we changed also the constant k_0 , so we have two periods (see graph 2.2 (b)), as we expect since our problem has now two frequencies. The red line indicates the mean value of the entropy. As we can see the mean entropy for $t > 0$ takes higher value than the entropy

	(a)	(b)	(c)	(d)	(e)	(f)
k_0	1	1	1	1	4	1
k_1	0	0	4	4	2.5	4
\tilde{k}_0	1	4	1	4	1	1.0201
\tilde{k}_1	7.5	6	12	6	17.5	4.02
\tilde{k}_0/k_0	1	4	1	4	0.25	1.0201
\tilde{k}_1/k_1			3	1.5	7	1.005

Table 2.1: The parameters of the Hamiltonian.

for $t < 0$ (in that case $S = 0$). In graph 2.1 (c) and (d) we have non zero coupling constant for $t < 0$. In this case the entropy initially has a constant non zero value. In graph (d) we see that the mean entropy has lower value than the entropy for $t < 0$. This is because k_0 increases more than k_1 (we have $\tilde{k}_0/k_0 = 4$ and $\tilde{k}_1/k_1 = 1.5$). In graph (e) we have choose the constant k_0 to decrease, while k_1 increases. Now we have several periods (as we can see in graph 2.2 (e)) and, of course, the mean entropy increases. Finally, in graph (d) we choose the constants to increase a little, with k_0 to increase more than k_1 (we have $\tilde{k}_0/k_0 = 1.0201$ and $\tilde{k}_1/k_1 = 1.005$).

Let's see now, the origin of the oscillations appearing in these graphs. We assumed that initially our total system was in the ground state of the Hamiltonian (2.4), that is $\hat{H}|\psi_0\rangle = E_0|\psi_0\rangle$, so the density matrix of the total system for $t = 0$ is $\hat{\rho}(0) = |\psi_0\rangle\langle\psi_0|$. At the time $t = 0$ the Hamiltonian changes suddenly to (2.13), so our state starts to change with time :

$$e^{-i\hat{H}t}|\psi_0\rangle = e^{-i\hat{H}t} \sum_n a_n |\tilde{\psi}_n\rangle = \sum_n a_n e^{-i\tilde{E}_n t} |\tilde{\psi}_n\rangle,$$

where $\tilde{E}_n, |\tilde{\psi}_n\rangle$ the eigenvalues and eigenstates of the Hamiltonian (2.13). Thus for times $t > 0$ the density matrix of the total system evolves according to the following equation:

$$\hat{\rho}(t) = e^{-i\hat{H}t}|\psi_0\rangle\langle\psi_0|e^{i\hat{H}t} = \sum_n \sum_m a_n a_m^* e^{-i(\tilde{E}_n - \tilde{E}_m)t} |\tilde{\psi}_n\rangle\langle\tilde{\psi}_m|,$$

so if we take the partial trace of the above density matrix, we have the reduced density matrix

$$\hat{\rho}_R(t) = Tr_e \hat{\rho}(t) = \sum_n \sum_m a_n a_m^* e^{-i(\tilde{E}_n - \tilde{E}_m)t} Tr_e (|\tilde{\psi}_n\rangle\langle\tilde{\psi}_m|).$$

From the above relation it is obvious that the time evolution, of the reduced density matrix (and consequently the entropy), has its cause in the complex exponential, which oscillates with time.

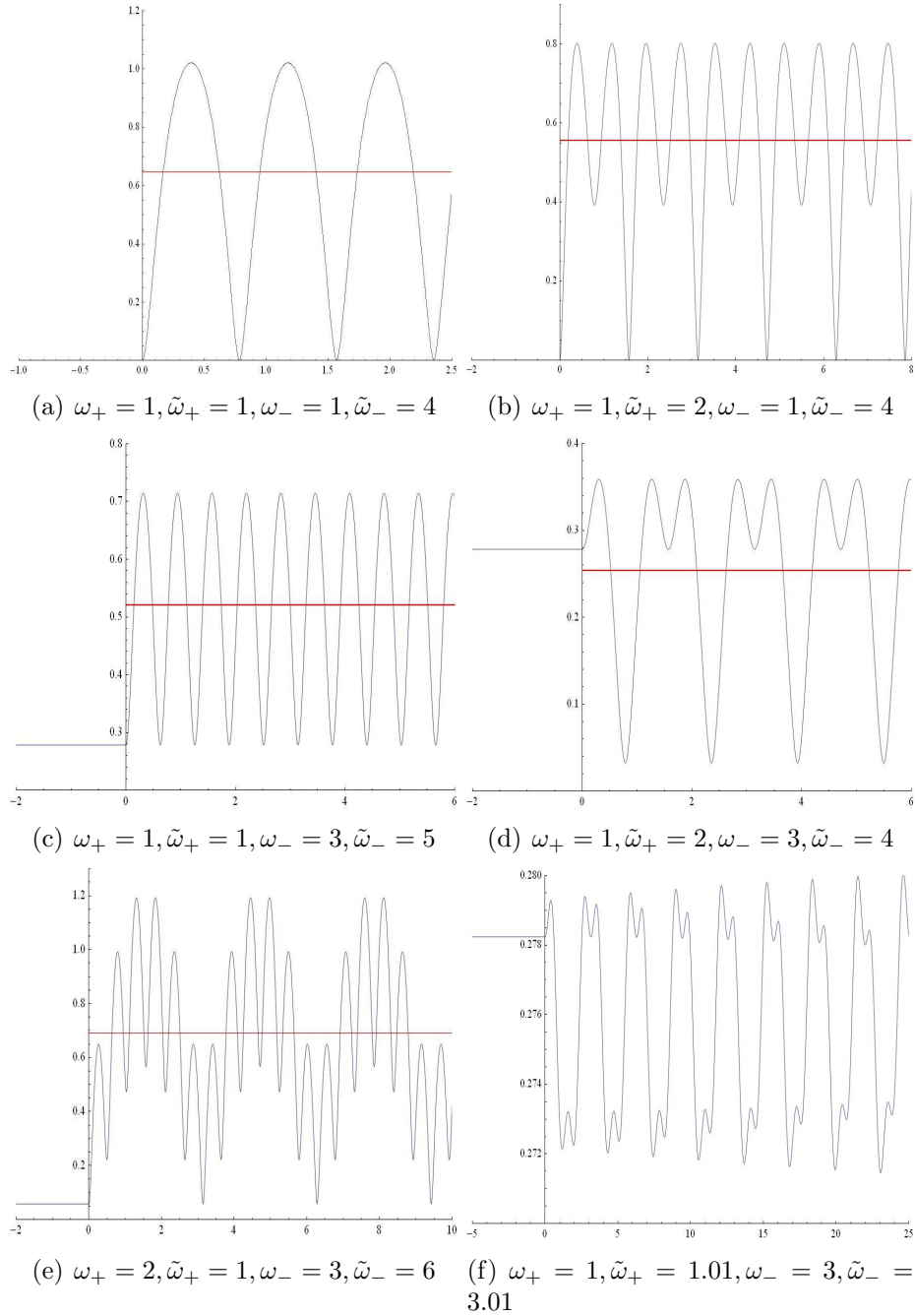


Figure 2.1: The diagrams of entropy as a function of time, for different values of the parameters. The red line is the mean entropy.

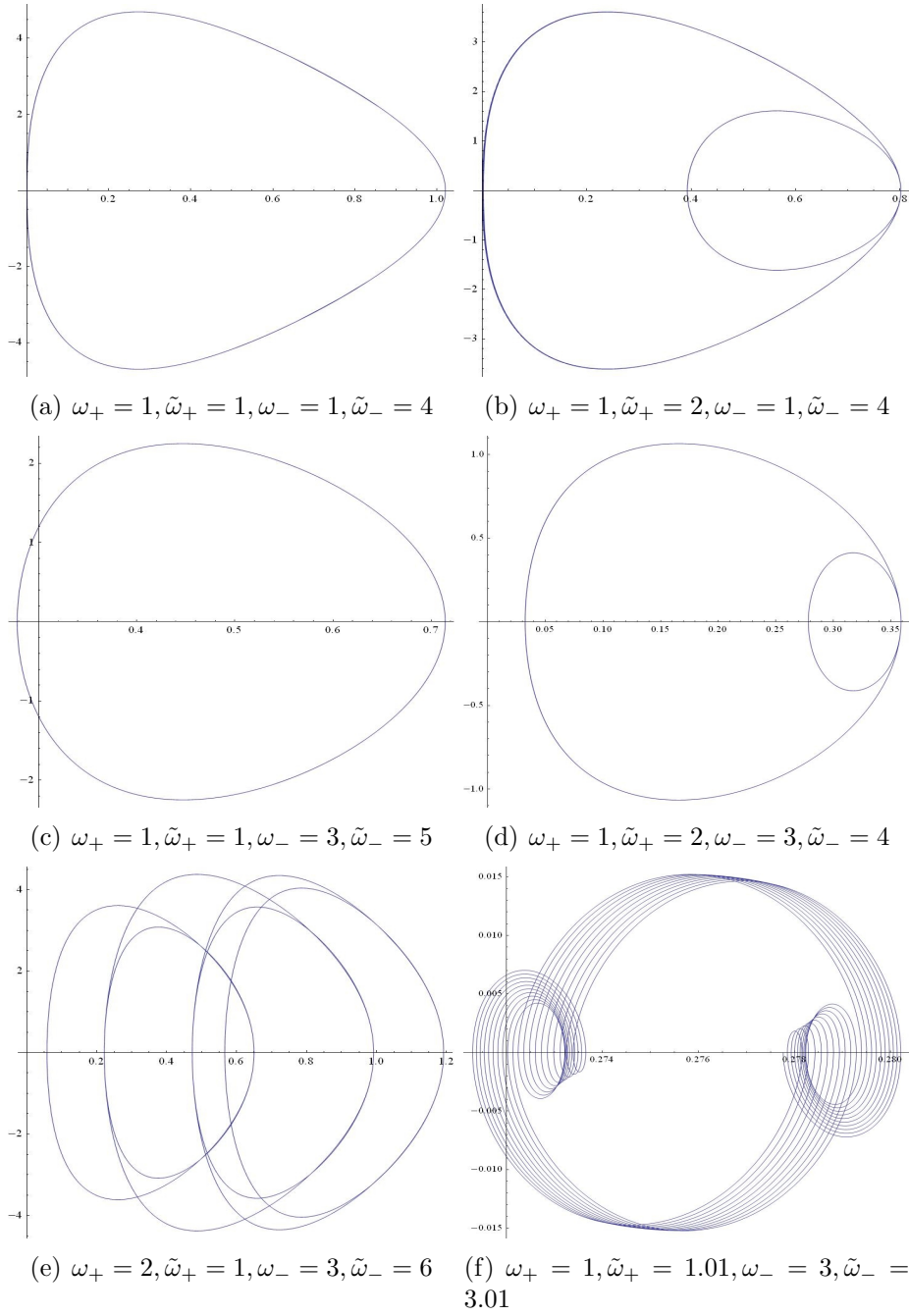


Figure 2.2: *The graphs $(S(t), \dot{S}(t))$ for several values of the parameters.*

2.2.3 Adiabatic Change of the Coupling Constant

Now we will see, how we can translate the *adiabatic theorem* (for a closed system) in the language of the reduced density matrix and entanglement entropy. Our system remains that of the two coupled harmonic oscillators, but now the coupling constant k_1 is a function of time, of the form $k_1(t) = k_1 \tanh\left(\frac{t}{\tau}\right)$ where k_1 and τ are constants. The coupling of the two oscillators for $t \leq 0$ is zero, while for $t \geq 0$ is a function that changes slow in time (for appropriate τ) and for $t \rightarrow \infty$ takes the constant value k_1 . For $t \leq 0$ our system is in the ground state $|\psi_0\rangle = |0\rangle_1 \otimes |0\rangle_2$ of the Hamiltonian

$$H_0 = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}k_0(x_1^2 + x_2^2).$$

As a first step we are going to calculate the time evolution, for $t \geq 0$, of the reduced density matrix and the entanglement entropy for one harmonic oscillator, considering the other as environment, for any function $k_1(t)$. Next, we shall examine the condition under which the adiabatic theorem can be considered as valid.

For $t \geq 0$ the Hamiltonian assumes the form

$$H(t) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}k_0(x_1^2 + x_2^2) + \frac{1}{2}k_1(t)(x_1 - x_2)^2. \quad (2.15)$$

and the density matrix can be read from (2.14). The evolution operator is now represented by

$$\langle x'_1, x'_2 | \hat{U}(t) | z'_1, z'_2 \rangle = \langle y'_1 | e^{-\frac{i}{\hbar}H_+t} | z'_+ \rangle \langle y'_2 | e^{-\frac{i}{\hbar} \int_0^t ds H_-(s)} | z'_- \rangle.$$

The left part of the right-hand side of the last equation has the standard form, while right part reads

$$\langle y'_2 | U_-(t) | z'_- \rangle = \int_{x(0)=y'_2}^{x(t)=z'_-} \mathcal{D}x(s) \exp \left\{ \frac{i}{\hbar} \int_0^t ds \frac{1}{2} [\dot{x}^2(s) - \omega_-^2(s)x^2(s)] \right\}$$

with $\omega_-(t) = \sqrt{k_0 + 2k_1(t)}$. The result of the path integration can be written in the form [24]

$$\langle y'_2 | U_-(t) | z'_- \rangle = \frac{1}{\sqrt{2\pi\hbar i D_0(t)}} \exp \left\{ \frac{i}{\hbar} \frac{1}{2D_0(t)} [z'^2_- \dot{D}_0(t) - y'^2_2 \dot{D}(0) - 2z'_- y'_2] \right\}$$

where the functions D_0 , \dot{D}_0 and D are solutions of the differential equations

$$\left(\frac{d^2}{ds^2} + \omega_-^2(s) \right) D_0(s) = 0 \quad (2.16)$$

with initial conditions $D_0(0) = 0, \dot{D}_0(0) = 1$ and

$$\left(\frac{d^2}{ds^2} + \omega_-^2(s) \right) D(s) = 0 \quad (2.17)$$

with final conditions $D(t) = 0, \dot{D}(t) = -1$.

Finally, the density matrix is written

$$\begin{aligned} \rho_t(y'_1, y'_2; y_1, y_2) &= \frac{\omega_+^2}{8\pi^3 \hbar^3 \sin(\omega_+ t) D_0(t)} \frac{\sinh^2(\omega_+ T_E)}{\sinh(2\omega_+ T_E)} \times \\ &\times \int dz'_+ dz'_- dz_+ dz_- \exp \left\{ \frac{i\omega_+}{2\hbar \sin(\omega_+ t)} [(y'_1{}^2 + z'_+{}^2) \cos(\omega_+ t) - 2y'_1 z'_+] \right\} \times \\ &\times \exp \left\{ \frac{i}{2\hbar D_0(t)} [z'_-{}^2 \dot{D}_0(t) - y'_2{}^2 \dot{D}(0) - 2z'_- y'_2] \right\} \times \\ &\times \exp \left\{ \frac{-\omega_+}{2\hbar \sinh(2\omega_+ T_E)} [(z_+{}^2 + z'_+{}^2) \cosh(2\omega_+ T_E) - 2z_+ z'_+] \right\} \times \\ &\times \exp \left\{ \frac{-\omega_-}{2\hbar \sinh(2\omega_- T_E)} [(z_-{}^2 + z'_-{}^2) \cosh(2\omega_- T_E) - 2z_- z'_-] \right\} \times \\ &\exp \left\{ \frac{-i\omega_+}{2\hbar \sin(\omega_+ t)} [(y_1^2 + z_+^2) \cos(\omega_+ t) - 2y_1 z_+] \right\} \times \\ &\times \exp \left\{ \frac{i}{2\hbar D_0(t)} [z_-^2 \dot{D}_0(t) - y_2^2 \dot{D}(0) - 2z_- y_2] \right\} \end{aligned}$$

We calculate the integrals and we take the limit $T_E \rightarrow \infty$ we have

$$\rho_t(y'_1, y'_2; y_1, y_2) = C(t) \exp \left\{ -(\Omega_+ y'_1{}^2 + \Omega_+^* y_1^2 + \Omega_- y'_2{}^2 + \Omega_-^* y_2^2) \right\},$$

where

$$\begin{aligned} \Omega_+ &= \omega_+, \\ \Omega_- &= \frac{\omega_+}{D_0^2(t) \left(\omega_+^2 + \left(\frac{\dot{D}_0(t)}{D_0(t)} \right)^2 \right)} + i \left[\frac{\dot{D}(0)}{D_0(t)} + \frac{\frac{\dot{D}_0(t)}{D_0(t)}}{D_0^2(t) \left(\omega_+^2 + \left(\frac{\dot{D}_0(t)}{D_0(t)} \right)^2 \right)} \right] \end{aligned}$$

and $C(t)$ is a function of time, not necessary for the calculation of entropy. In order, now, to calculate the reduced density matrix for the oscillator 2,

we go back to the old coordinates and we integrate the coordinates of the oscillator 1

$$\begin{aligned} \langle x'_2 | \hat{\rho}_R(t) | x_2 \rangle &= \int dx_1 \rho_t(x_1, x'_2; x_1, x_2) \Rightarrow \\ \langle x'_2 | \hat{\rho}_R(t) | x_2 \rangle &= \tilde{C}(t) \exp \left\{ -\frac{1}{2}(\gamma(t)x'_2{}^2 + \gamma^*(t)x_2^2) + \beta(t)x'_2 x_2 \right\}, \end{aligned}$$

where

$$\gamma(t) = \frac{1}{2} \left[(\Omega_+ + \Omega_-) - \frac{(\Omega_+ - \Omega_-)^2}{(\Omega_+ + \Omega_-) + (\Omega_+ + \Omega_-)^*} \right]$$

and

$$\beta(t) = \frac{1}{2} \frac{(\Omega_+ - \Omega_-)(\Omega_+ - \Omega_-)^*}{(\Omega_+ + \Omega_-) + (\Omega_+ + \Omega_-)^*}.$$

As above, we have for the Von Neumann entropy

$$S(t) = -\ln(1 - \xi(t)) - \frac{\xi(t)}{1 - \xi(t)} \ln \xi(t),$$

where

$$\xi(t) = \frac{\beta(t)}{\text{Re}(\gamma(t)) + \sqrt{\text{Re}(\gamma(t))^2 - \beta^2(t)}}.$$

For the final result we arithmetically¹ solve the differential equations (2.16) and (2.17) using the expression:

$$k_1(t) = k_1 \tanh \left(\frac{t}{\tau} \right). \quad (2.18)$$

In graphs 2.3 (a) - (f) we show the entropy as a function of time for $\tau = 1$ and $\tau = 10$ and various values for k_1 . As we can see the entropy after a definite time interval oscillates with a constant amplitude d . As τ increases the amplitude d decreases, as it can be seen in the graph 2.4 where we show the amplitude d as a function of the parameter τ , for 150 different τ . In graph 2.4 (b) and (d) we fit the points in a curve of the form $d(\tau) = b/\tau$ and we find $b = 4.29$ for $k_1 = 1.5$ and $b = 1.034$ for $k_1 = 10$ as we can see there is a very good agreement between the curve and the points (error of order 10^{-4}). Thus, we can say that in the limit $\tau \rightarrow \infty$ the amplitude d goes to zero. As we are going to see, this behavior confirms the adiabatic theorem.

Let's discuss now the adiabatic theorem. We consider the Hamiltonian $\hat{H}(\lambda_t)$, where the parameter λ_t depends from the time. We notice as $|n(\lambda_t)\rangle$ the instantaneous eigenstates of the time dependent Hamiltonian at time t :

$$\hat{H}(\lambda_t)|n(\lambda_t)\rangle = E_n(\lambda_t)|n(\lambda_t)\rangle.$$

¹The program codes we have used are in Appendix C.

The adiabatic theorem states, that if the parameter λ_t changes slow enough with time and at the time $t_0 = 0$ our (closed) system is in an eigenstate of the Hamiltonian $|n(\lambda_0)\rangle$, then at time t our system will be in a state proportional to $|n(\lambda_t)\rangle$.

In the example of the two oscillators, discussed above, time dependent is the coupling constant $k_1(t)$. At the time $t_0 = 0$ the system of the two harmonic oscillators is in the ground state of the Hamiltonian $\hat{H}(k_1(0) = 0)$, thus if the evolution of $k_1(t)$ is slow enough we expect that at time t the two harmonic oscillators will be in a state proportional to the ground state of the Hamiltonian $\hat{H}(k_1(t))$. In the language of density matrix, initially our system is characterized by the density matrix $\hat{\rho}_0 = |0(k_1(0))\rangle\langle 0(k_1(0))| = |0\rangle_2|0\rangle_{11}\langle 0|_2\langle 0|$. If the conditions of the adiabatic theorem are fulfilled, after time t the two oscillators will be described by $\hat{\rho}_0 = |0(k_1(t))\rangle\langle 0(k_1(t))|$ (The phases that appear from the adiabatic theorem it is supposed to cancel each other). What happens with the open system? Initially the open system is described by the reduced density matrix $\hat{\rho}_{R_0} = Tr_1(|0(k_1(0))\rangle\langle 0(k_1(0))|) = |0\rangle_{22}\langle 0|$, so the entanglement entropy is zero. If we use for the parameter the relation (2.18) and consider the limit $t/\tau \gg 1$ the coupling constant will take the constant value $k_1(t \rightarrow \infty) = k_1$. Thus, after this time the open system will be characterized by $\hat{\rho}_{R_t} = Tr_1(|0(k_1)\rangle\langle 0(k_1)|)$, while the Hamiltonian will not change with time. So, in this case, we expect the entropy to be a non zero constant. The results of our calculation seems to confirm this picture, since as we saw, from graphs 2.3 and 2.4, in the limit $\tau \rightarrow \infty$ the oscillation of the entropy goes to zero and the entropy takes a constant non zero value.

The oscillating behavior we observe, is related to the off-diagonal geometrical phases that disappear only at the strict limit $\tau \rightarrow \infty$ [35].

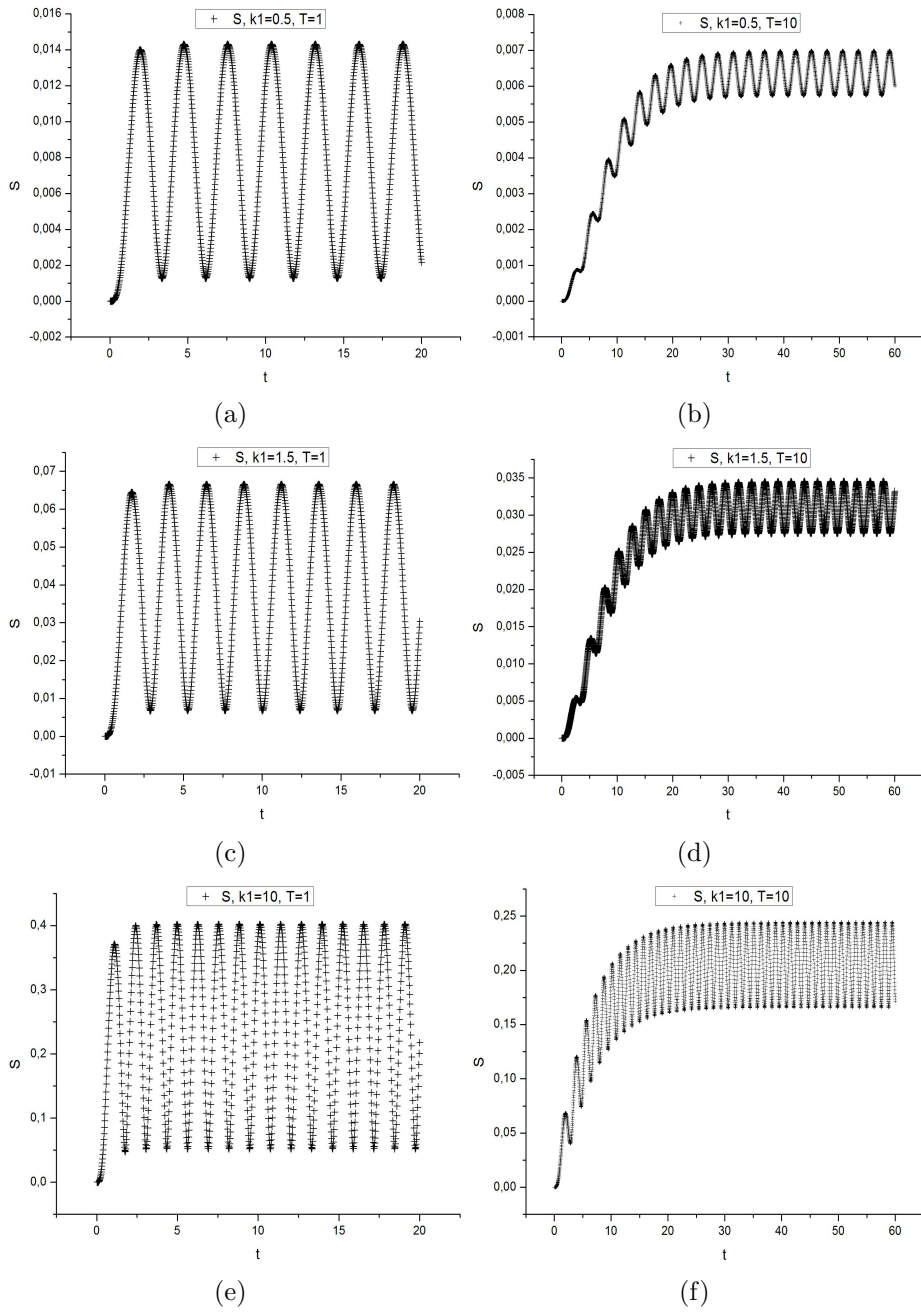
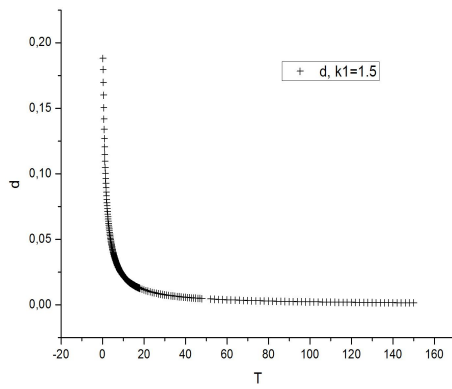
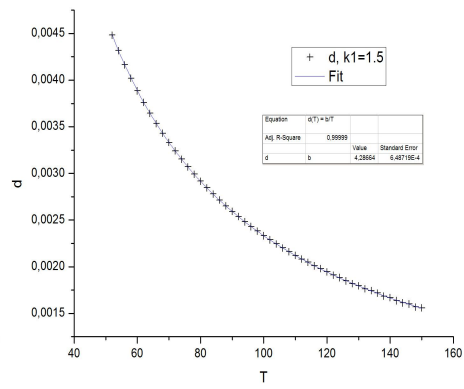


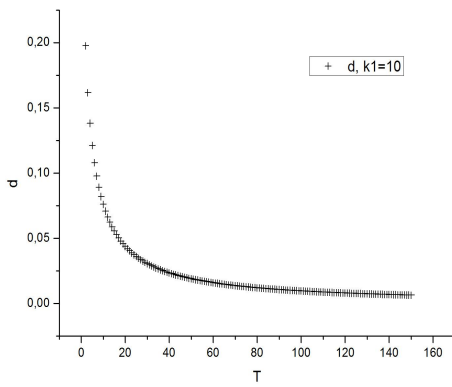
Figure 2.3: In this graphs, $S = S(t)$, we use $k_0 = 1$



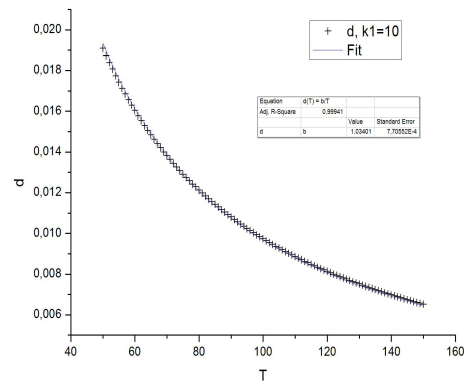
(a)



(b)



(c)




(d)

Figure 2.4: The graphs of the oscillation amplitude of the entropy d as a function of the parameter τ and the fit for $\tau \geq 50$. We use $k_0 = 1$.

Chapter 3

The Influence Functional and The Closed Complex Time Formalism

n this chapter we shall develop a systematic way for the study of the time evolution of the reduced density matrix of an open quantum system, under the influence of its environment. Our formalism is based on that of Feynman and Vernon [13], and it is an extension of the “closed time” formalism [18,19].

3.1 The Closed Complex Time Formalism

The starting point to develop our formalism is the relation (1.33). We consider an open quantum system in interaction with its environment. We suppose that the total system is closed so its evolution is unitary. We also assume that the density matrix of the total system can be written in the following form

$$\hat{\rho}(0) = \hat{\rho}_s(0) \otimes \hat{\rho}_e(0), \quad (3.1)$$

that is, our system and the environment are initially in a pure and disentangled state. The relation (1.33), gives us the reduced density matrix of the open system at any time $t > 0$.

The first assumption we will make is that the environment, initially, is in its ground state, so it is characterized (according to the relation (A.13)) by the density matrix

$$\rho_{x''x'''}^e(0) = \frac{1}{Z_e} \int_{q^{(3)}(-0)=q''} \mathcal{D}q^{(3)}(\tau) \int_{q^{(2)}(+0)=x'''} \mathcal{D}q^{(2)}(\tau) \exp \left\{ -\frac{1}{\hbar} \int_{-\infty}^{-0} d\tau \mathcal{L}_{e,E}[q^{(3)}(\tau)] \right\} \times$$

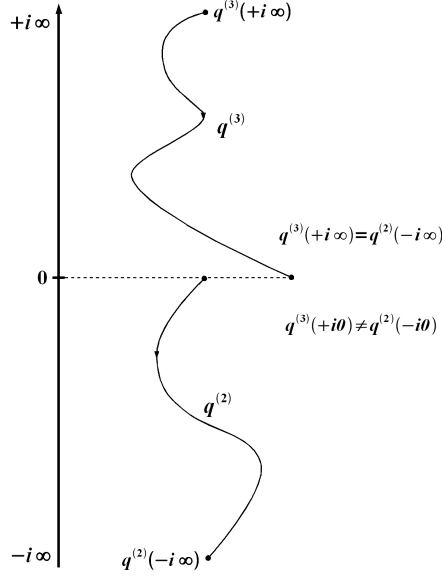


Figure 3.1: The paths $q^{(2)}$ and $q^{(3)}$. Notice the discontinuity of the two paths in zero and their equality in the complex infinite.

$$\times \exp \left\{ -\frac{1}{\hbar} \int_{+0}^{+\infty} d\tau \mathcal{L}_{e,E}[q^{(2)}(\tau)] \right\}, \quad (3.2)$$

where $\mathcal{L}_{e,E}$ is the Euclidian Lagrangian of the environment. Thus, inserting equations (1.35) into (3.2), we have for the influence functional

$$\begin{aligned} \mathcal{F}[x^{(4)}(t), x^{(1)}(t); t] &\equiv \int \mathcal{D}q^{(4)}(t) \int \mathcal{D}q^{(3)}(\tau) \int \mathcal{D}q^{(2)}(\tau) \int \mathcal{D}q^{(1)}(t) \times \\ &\times \delta[q^{(4)}(t) - q^{(1)}(t)] \delta[q^{(4)}(0) - q^{(2)}(+0)] \delta[q^{(3)}(-0) - q^{(1)}(0)] \times \\ &\times \exp \left\{ \frac{i}{\hbar} \int_0^t dt' \mathcal{L}_{e+I}[q^{(4)}(t'), x^{(4)}(t')] - \frac{1}{\hbar} \int_{-\infty}^{-0} d\tau \mathcal{L}_{e,E}[q^{(3)}(\tau)] - \right. \\ &\left. - \frac{1}{\hbar} \int_{+0}^{+\infty} d\tau \mathcal{L}_{e,E}[q^{(2)}(\tau)] + \frac{i}{\hbar} \int_t^0 dt' \mathcal{L}_{e+I}[q^{(1)}(t'), x^{(1)}(t')] \right\} \quad (3.3) \end{aligned}$$

Assuming now that the system-environment interaction is linear, so (3.3) reads:

$$\mathcal{F}[x^{(4)}(t), x^{(1)}(t); t] = \int \mathcal{D}q^{(4)}(t) \int \mathcal{D}q^{(3)}(\tau) \int \mathcal{D}q^{(2)}(\tau) \int \mathcal{D}q^{(1)}(t) \times$$

$$\begin{aligned}
& \times \delta[q^{(4)}(t) - q^{(1)}(t)] \delta[q^{(4)}(0) - q^{(2)}(+0)] \delta[q^{(3)}(-0) - q^{(1)}(0)] \times \\
& \times \exp \left\{ \frac{i}{\hbar} \int_0^t dt' \left(\mathcal{L}_e[q^{(4)}(t')] + gq^{(4)}(t')x^{(4)}(t') \right) - \frac{1}{\hbar} \int_{-\infty}^{-0} d\tau \mathcal{L}_{e,E}[q^{(3)}(\tau)] - \right. \\
& \left. - \frac{1}{\hbar} \int_{+0}^{+\infty} d\tau \mathcal{L}_{e,E}[q^{(2)}(\tau)] + \frac{i}{\hbar} \int_t^0 dt' \left\{ \mathcal{L}_e[q^{(1)}(t')] + gq^{(1)}(t')x^{(1)}(t') \right\} \right\} \quad (3.4)
\end{aligned}$$

The above expression for the influence functional can be considerably simplified if we introduce the complex variable z defined on the contour C shown in figure 3.2(a). This contour consists of four different lines :

- The line L_1 goes parallel to the real axis from the point $z = t - i0$ to point $z = 0 - i0$. For this line we have

$$\begin{aligned}
S_1 &= \frac{i}{\hbar} \int_{L_1} dz \left\{ \mathcal{L}_e[q_{L_1}(z)] + \mathcal{L}_I[q_{L_1}(z), x_{L_1}(z)] \right\} = \\
&= \frac{i}{\hbar} \int_{t-i0}^{0-i0} dz \left\{ \mathcal{L}_e[q_{L_1}(z)] + \mathcal{L}_I[q_{L_1}(z), x_{L_1}(z)] \right\} = \\
&= \frac{i}{\hbar} \int_t^0 dt' \left\{ \mathcal{L}_e[x^{(1)}(t')] + \mathcal{L}_I[x^{(1)}(t'), q^{(1)}(t')] \right\}.
\end{aligned}$$

- The Line L_2 goes parallel to the imaginary axis from the point $z = 0 - i0$ to point $z = 0 - i\infty$. For this line we have

$$\begin{aligned}
S_2 &= \frac{i}{\hbar} \int_{L_2} dz \mathcal{L}_e[q_{L_2}(z)] = \frac{i}{\hbar} \int_{0-i0}^{0-i\infty} dz \mathcal{L}_e[q_{L_2}(z)] = \\
&= \frac{i}{\hbar} \int_{z=0-i\tau}^{\infty} (-i) d\tau (-1) \mathcal{L}_{e,E}[q^{(2)}(\tau)] = -\frac{1}{\hbar} \int_0^{\infty} d\tau \mathcal{L}_{e,E}[q^{(2)}(\tau)].
\end{aligned}$$

- The line L_3 goes parallel to the imaginary axis from the point $z = 0 + i\infty$ to point $z = 0 + i0$. For this line we have

$$\begin{aligned}
S_3 &= \frac{i}{\hbar} \int_{L_3} dz \mathcal{L}_e[q_{L_3}(z)] = \frac{i}{\hbar} \int_{0+i\infty}^{0+i0} dz \mathcal{L}_e[q_{L_3}(z)] = \\
&= \frac{i}{\hbar} \int_{z=0-i\tau}^{\infty} (-i) d\tau (-1) \mathcal{L}_{e,E}[q^{(3)}(\tau)] = -\frac{1}{\hbar} \int_{-\infty}^{-0} d\tau \mathcal{L}_{e,E}[q^{(3)}(\tau)].
\end{aligned}$$

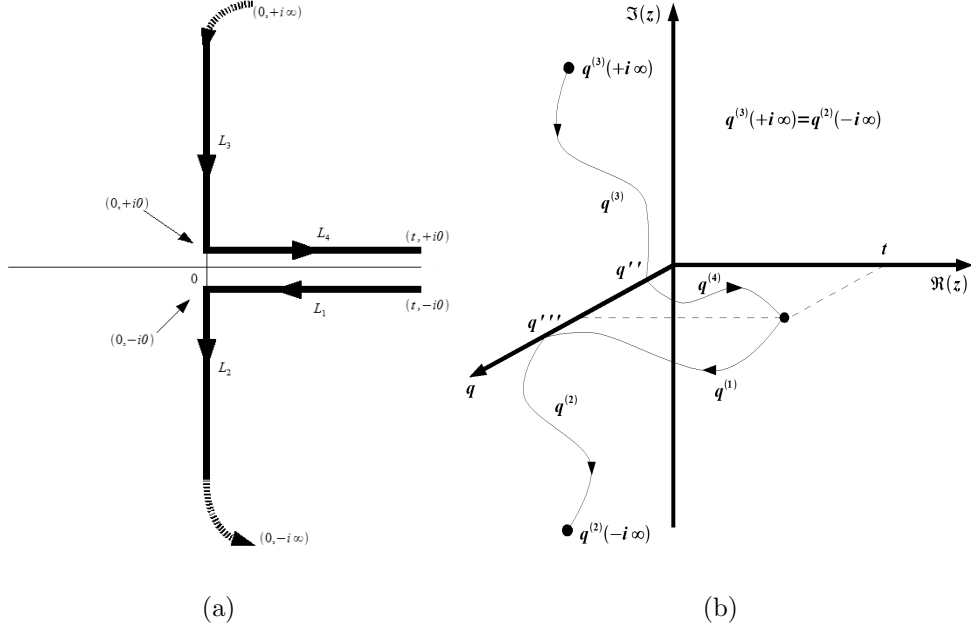


Figure 3.2: (a) The Contour C in the Complex Plane. (b) How the Paths are Stitching Together.

- The line L_4 goes parallel to the real axis from the point $z = 0 + i0$ to point $z = t + i0$. For this line we have

$$\begin{aligned}
 S_4 &= \frac{i}{\hbar} \int_{L_4} dz \left\{ \mathcal{L}[q_{L_4}(z)] + \mathcal{L}_I[q_{L_4}(z), x_{L_4}(z)] \right\} = \\
 &= \frac{i}{\hbar} \int_{0+i0}^{t+i0} dz \left\{ \mathcal{L}_e[q_{L_4}(z)] + \mathcal{L}_I[q_{L_4}(z), x_{L_4}(z)] \right\} = \\
 &= \frac{i}{\hbar} \int_{z=t'+i0}^t dt' \left\{ \mathcal{L}_e[q^{(4)}(t')] + \mathcal{L}_I[q^{(4)}(t')x^{(4)}(t')] \right\}.
 \end{aligned}$$

It is now easy to be proved that the “action” in the influence functional (3.4) can be written

$$\tilde{S} = S_1 + S_2 + S_3 + S_4 = \int_C dz \left\{ \mathcal{L}_e[q_c(z)] + \mathcal{L}_I[q_c(z), x_c(z)] \right\}. \quad (3.5)$$

Imposing the boundary condition

$$q_c(-i\infty) = q_c(+i\infty) \quad (3.6)$$

we get the following compact expression for the influence functional

$$\begin{aligned} \mathcal{F}[x^{(1)}(t), x^{(4)}(t); t] &= \frac{1}{Z_e} \int_{q_c(t-i0)=q_c(t+i0)} \mathcal{D}q_c(z) \exp \left\{ \frac{i}{\hbar} \int_C dz \{ \mathcal{L}_e[q_c(z)] + \mathcal{L}_I[q_c(z), x_c(z)] \} \right\} = \\ &\equiv \left\langle \exp \left\{ \frac{i}{\hbar} \int_C dz \mathcal{L}_I[q_c(z), x_c(z)] \right\} \right\rangle_q \equiv \exp \left\{ \frac{i}{\hbar} S_{FV}[x_c] \right\}, \end{aligned} \quad (3.7)$$

Here we can observe that, if there was no interaction, the influence functional become unit, as expected since there is no interaction between system and environment.

As it is obvious from the above expression the introduction of the complex time z defined on the contour C has enabled us to interpret the influence functional as an integral over continuous paths with periodic boundary conditions. Any step further strongly depends on the dynamical details of the environment as well as on the specific form of the interaction between the latter and the system (here we have assumed that this interaction is linear). In any case the compact formulation, indicated in equation (3.7), can be combined with all the existent calculational technologies to produce concrete results in the field of open quantum systems.

3.1.1 The Cluster Expansion

In this framework it is very convenient to use a well-known and very powerful technique: The so-called **cluster or cumulant expansion**¹. This fundamental technique is widely used in a great variety of problems, from statistical physics to quantum field theories [36–38]. The methodology has been extensively used in areas such as the resummation of perturbative series and non-perturbative estimations, among others, and has been proven to be a very useful tool.

In our case, the cluster expansion theorem can be read from the relation

$$\begin{aligned} &\left\langle \exp \left\{ \frac{i}{\hbar} \int_C dz \mathcal{L}_I[q_c(z)] \right\} \right\rangle_q = \\ &= \exp \left\{ \sum_{n=1}^{\infty} \left(\frac{i}{\hbar} \right)^n \int_C dz_n \dots \int_C dz_1 \theta_c(z_n, \dots, z_1) K^{(n)}(z_n, \dots, z_1) \right\}, \end{aligned} \quad (3.8)$$

¹The cumulant expansion have been used in the theory of open quantum systems in a completely different context [29].

where the functions $K^{(i)}$ are the cumulants, the form of which depends in the form of the interaction term. We shall make the rather general (and convenient) assumption that the coupling between the system and the environment is the minimal one

$$\mathcal{L}_I[q_c, x_c] = g_c q_c x_c,$$

where g_c is a contour dependent coupling, such that $g_{L_1} = g_{L_4} = g$ and $g_{L_2} = g_{L_3} = 0$. In this case the cumulants are given from the expressions:

$$K^{(1)}(z_1) \equiv \langle \mathcal{L}_I[q_c(z_1)] \rangle_q = g_c(z_1) x_c(z_1) \langle q_c(z_1) \rangle_q,$$

$$\begin{aligned} K^{(2)}(z_2, z_1) &\equiv \langle \mathcal{L}_I[q_c(z_2)] \mathcal{L}_I[q_c(z_1)] \rangle_q - \langle \mathcal{L}_I[q_c(z_2)] \rangle_q \langle \mathcal{L}_I[q_c(z_1)] \rangle_q = \\ &= g_c(z_2) g_c(z_1) x_c(z_2) x_c(z_1) \left[\langle q_c(z_2) q_c(z_1) \rangle_q - \langle q_c(z_2) \rangle_q \langle q_c(z_1) \rangle_q \right], \end{aligned}$$

$$\begin{aligned} K^{(3)}(z_3, z_2, z_1) &\equiv \langle \mathcal{L}_I[q_c(z_3)] \mathcal{L}_I[q_c(z_2)] \mathcal{L}_I[q_c(z_1)] \rangle_q - \\ &- \left\{ \langle \mathcal{L}_I[q_c(z_3)] \rangle_q \langle \mathcal{L}_I[q_c(z_2)] \mathcal{L}_I[q_c(z_1)] \rangle_q + perms \right\} + \\ &+ \langle \mathcal{L}_I[q_c(z_3)] \rangle_q \langle \mathcal{L}_I[q_c(z_2)] \rangle_q \langle \mathcal{L}_I[q_c(z_1)] \rangle_q, \end{aligned}$$

$$\dots\dots\dots etc \dots\dots\dots \tag{3.9}$$

In equation (3.8) we have introduced the chain of path dependent step functions

$$\theta_c(z_n, \dots, z_1) = \theta_c(z_n - z_{n-1}) \dots \theta_c(z_2 - z_1), \tag{3.10}$$

which takes care of the time ordering needed whenever the variables z_i are integrated along the same line. The path dependent step functions that appear in the above expression can be defined with the help of a proper parametrization of the contour C with a real parameter $\sigma \in [0, 1]$, such that $z(0) = t - i0$ and $z(1) = t + i0$:

$$0 \leq \sigma \leq \sigma_1 : x_{L_1}, z = \frac{\sigma_1 - \sigma}{\sigma_1} t - i0, z(0) = t - i0, z(\sigma_1) = 0 - i0$$

$$\sigma_1 \leq \sigma \leq \sigma_2 : x_{L_2}, z = 0 - iT_E \frac{\sigma_1 - \sigma}{\sigma_1 - \sigma_2}, z(\sigma_1) = 0 - i0, z(\sigma_2) = 0 - iT_E$$

$$\sigma_3 \leq \sigma \leq \sigma_4 : x_{L_3}, z = 0 + iT_E \frac{\sigma - \sigma_4}{\sigma_3 - \sigma_4}, z(\sigma_3) = 0 + iT_E, z(\sigma_4) = 0 + i0$$

$$\sigma_4 \leq \sigma \leq 1 : x_{L_4}, z = \frac{\sigma - \sigma_4}{1 - \sigma_4} t + i0, z(\sigma_4) = 0 + i0, z(1) = t + i0$$

so, the total time is

$$\tilde{T} = \oint_C ds = \int_0^{\sigma_2} d\sigma \dot{z}(\sigma) + \int_{\sigma_3}^1 d\sigma \dot{z}(\sigma) = z(\sigma_2) - z(0) + z(1) - z(\sigma_3) = -2iT_E.$$

Since the time flow, follow different directions along different lines we have introduced the following definition

$$\theta_c(z - z') = \theta_c(z(\sigma) - z(\sigma')) = \begin{cases} \theta(\sigma - \sigma'), & \text{when } C = L_{2,4} \\ \theta(\sigma' - \sigma), & \text{when } C = L_{1,3} \end{cases} \quad (3.11)$$

When the variables z are integrated along different lines the step functions become identically 1 or 0: For example, if $z \in L_1$ and $z' \in L_4$ we define $\theta_{L_1 \cup L_4}(z - z') = 1$, because the time along the line L_1 decreases, and this happens *after* its growth along the line L_4 .

The validity of equation (3.8) can be readily proven by expanding the corresponding exponentials. The proof can also be easily extended to the case of non-commutating quadratic matrices with the help of a proper time ordering.

With the above conventions it is easy to transfer every well-known result of the conventional path integration into the complex time framework as is defined by the expressions (3.7), (3.8) and (3.9).

From the preceding analysis we saw that the influence of the environment has been incorporated into the correlators

$$(i\hbar)^{n-1} \Delta_C^{(n)}(z_n, \dots, z_1) \equiv \langle q_c(z_n) \dots q_c(z_1) \rangle_q. \quad (3.12)$$

As it is evident from the definition of the path integral in equation (3.7), non trivial correlations can exist only along the same line or among the lines L_1 and L_4 . However, it is apparent that the cluster expansion is useful only if the infinite series appearing in the exponent in the right-hand side of equation (3.8) can be approximated by keeping only the first few terms. This issue will be discussed in a following subsection. For now it is important to note that the complex time formalism in the form we have used it till now can also be applied in cases where the initial state cannot be factorized as we have assumed. In such a case non trivial correlations can exist among all of the lines of the contour C .

We can enlighten the properties of the fundamental functions (3.12) by discussing, at this point, some of the properties of the two point correlator $\Delta_C^{(2)}$. A first observation is that it must have a non vanishing imaginary part due to the imaginary period over which it is defined. To be concrete let us consider the propagation along the line L_1 :

$$\Delta_{L_1}^{(2)}(t_2 - t_1) \equiv G_R(t_2 - t_1) + iG_I(t_2 - t_1). \quad (3.13)$$

Along the line L_4 the time flow is reversed and consequently:

$$\Delta_{L_4}^{(2)}(t_2 - t_1) \equiv G_R(t_1 - t_2) + iG_I(t_1 - t_2) = \Delta_{L_1}^{(2)}(t_1 - t_2). \quad (3.14)$$

At this point we can appeal to the hermiticity of the density matrix: The influence functional must remain the same if we interchange $x^{(1)}$ and $x^{(4)}$ while taking the complex conjugate. The last action reverses the time ordering along the contour C and consequently the function $\Delta_{L_1}^{(2)}$ must be anti-hermitian:

$$[\Delta_{L_1}^{(2)}(t_1 - t_2)]^* = -\Delta_{L_1}^{(2)}(t_2 - t_1). \quad (3.15)$$

Thus we immediately conclude that the real part of the propagator (3.13) is an odd function while its imaginary part is an even function of time:

$$G_R(t_2 - t_1) = -G_R(t_1 - t_2), \quad G_I(t_2 - t_1) = G_I(t_1 - t_2). \quad (3.16)$$

The exchange contributions can also be deduced with the same reasoning: Since, as we have discussed, the time along L_1 is after the time along L_4 the exchange from the line L_4 to the line L_1 is controlled by a function $\Delta_{L_4 \cup L_1}^{(2)}(t_2 - t_1)$ in which $t_2 < t_1$ while the exchange from the line L_1 to the line L_4 must be controlled by a function $\Delta_{L_1 \cup L_4}^{(2)}(t_2 - t_1)$ in which $t_2 > t_1$. Clearly the relation

$$\Delta_{L_1 \cup L_4}^{(2)}(t_2 - t_1) = -[\Delta_{L_4 \cup L_1}^{(2)}(t_1 - t_2)]^* \quad (3.17)$$

must hold. The trace of the reduced density matrix must be equal to one, and, consequently, the Feynman-Vernon action must go to zero as $x^{(4)} \rightarrow x^{(1)}$. This can happen only if the (forward) propagation $L_4 \rightarrow L_1$ exactly cancels the (forward) propagation along L_4 and the (backward) propagation $L_1 \rightarrow L_4$ exactly cancels the (backward) propagation along L_1 :

$$\Delta_{L_4 \cup L_1}^{(2)}(t_2 - t_1) = G_R(t_1 - t_2) - iG_I(t_1 - t_2) \quad (3.18)$$

and

$$\Delta_{L_1 \cup L_4}^{(2)}(t_2 - t_1) = -G_R(t_2 - t_1) - iG_I(t_2 - t_1). \quad (3.19)$$

After these arguments it is clear that, quite generally, the order g^2 contribution to the Feynman-Vernon action:

$$\begin{aligned} \frac{i}{\hbar} S_{FV}^{(2)} = & -\frac{1}{\hbar} \int_0^t dt_2 \int_0^{t_2} dt_1 [x^{(1)}(t_2) - x^{(4)}(t_2)] G_I(t_2 - t_1) [x^{(1)}(t_1) - x^{(4)}(t_1)] + \\ & + \frac{i}{\hbar} \int_0^t dt_2 \int_0^{t_2} dt_1 [x^{(1)}(t_2) - x^{(4)}(t_2)] G_R(t_2 - t_1) [x^{(1)}(t_1) + x^{(4)}(t_1)]. \quad (3.20) \end{aligned}$$

It is now readily evident that the Feynman-Vernon action considerably changes the dynamics of the central quantum system. Its fluctuating part, which is connected to the imaginary part of the line propagator, lessens coherence. Its real part, which is connected to the real part of the line propagator, it is customary and convenient to be re-expressed with the help of an even function $\gamma(t_2 - t_1) = \gamma(t_1 - t_2)$ through the relation [4, 12]:

$$G_R(t_2 - t_1) = \frac{\partial}{\partial t_2} \gamma(t_2 - t_1). \quad (3.21)$$

The function γ introduces in the Feynman-Vernon action a term which, on the classical level, can be understood as a damping or “friction” term.

In the next subsection we give a simple example in order to see how this formalism works: we consider the case of one harmonic oscillator as an environment.

3.1.2 One Harmonic Oscillator as Environment

As a specific example let us compute, in the framework of the preceding analysis, the influence functional for the case in which the environment is just a simple harmonic oscillator. In this case, the environment Lagrangian is

$$\mathcal{L}_e[\dot{q}, q] = \frac{m_e}{2} \dot{q}^2 - \frac{1}{2} m_e \omega_e^2 q^2. \quad (3.22)$$

For this Lagrangian, only one term appears in the right-hand side of (3.8), so the Feynman-Vernon action is

$$\begin{aligned} \frac{i}{\hbar} S_{FV} &= -\frac{1}{\hbar^2} \int_C dz_2 g_c(z_2) x_c(z_2) \int_C dz_1 g_c(z_1) x_c(z_1) \theta_c(z_2 - z_1) \langle q_c(z_2) q_c(z_1) \rangle_q = \\ &= -\frac{1}{\hbar^2} \int_C dz_2 g_c(z_2) x_c(z_2) \int_C dz_1 g_c(z_1) x_c(z_1) \theta_c(z_2 - z_1) \Delta_C^{(2)}(z_2 - z_1). \end{aligned} \quad (3.23)$$

The Green function that appears in the last equation obeys periodic boundary conditions and assumes the well-known form

$$\Delta_C^{(2)}(z_2 - z_1) = -\frac{1}{2m_e \omega_e} \frac{\cos[\omega_e(|z_2 - z_1|_c - \tilde{T}/2)]}{\sin(\omega_e \tilde{T}/2)}, \quad (3.24)$$

with

$$|z_2 - z_1|_c = (z_2 - z_1)[\theta_c(z_2 - z_1) - \theta_c(z_1 - z_2)]. \quad (3.25)$$

The period is obviously imaginary $\tilde{T} = -2iT_E$, and consequently

$$\Delta_C^{(2)}(z_2 - z_1) = \frac{1}{2m_e \omega_e} \frac{\cos[\omega_e(|z_2 - z_1|_c + iT_E)]}{\sin(i\omega_e T_E)} =$$

$$\begin{aligned}
&= \frac{1}{2m_e\omega_e} \frac{\cos[\omega_e|z_2 - z_1|_c] \cos(i\omega_e T_E)}{\sin(i\omega_e T_E)} - \frac{1}{2m_e\omega_e} \sin[\omega_e|z_2 - z_1|_c] \xrightarrow{T_E \rightarrow \infty} \\
\Delta_C^{(2)}(z_2 - z_1) &= -\frac{1}{2m_e\omega_e} [i \cos(\omega_e|z_2 - z_1|_c) + \sin(\omega_e|z_2 - z_1|_c)]. \quad (3.26)
\end{aligned}$$

Given that $g_{L_1} = g_{L_4} = g$ and $g_{L_2} = g_{L_3} = 0$, the integral in (3.23) can be split as follows

$$\begin{aligned}
&\int_C dz_2 g_c(z_2) x_c(z_2) \int_C dz_1 g_c(z_1) x_c(z_1) \theta_c(z_2 - z_1) \Delta_C^{(2)}(z_2 - z_1) = \\
&= g^2 \int_{L_1} dz_2 \int_{L_1} dz_1 x_{L_1}(z_2) x_{L_1}(z_1) \theta_{L_1}(z_2 - z_1) \Delta_{L_1}^{(2)}(z_2 - z_1) + \\
&\quad + g^2 \int_{L_4} dz_2 \int_{L_4} dz_1 x_{L_4}(z_2) x_{L_4}(z_1) \theta_{L_4}(z_2 - z_1) \Delta_{L_4}^{(2)}(z_2 - z_1) + \\
&\quad + g^2 \int_{L_1} dz_2 \int_{L_4} dz_1 x_{L_1}(z_2) x_{L_4}(z_1) \theta_{L_1 \cup L_4}(z_2 - z_1) \Delta_{L_1 \cup L_4}^{(2)}(z_2 - z_1) + \\
&\quad + g^2 \int_{L_4} dz_2 \int_{L_1} dz_1 x_{L_4}(z_2) x_{L_1}(z_1) \theta_{L_4 \cup L_1}(z_2 - z_1) \Delta_{L_4 \cup L_1}^{(2)}(z_2 - z_1), \quad (3.27)
\end{aligned}$$

the last integral obviously is zero, since for $z_2 \in L_4$ and $z_1 \in L_1$ we have $\theta_{L_4 \cup L_1}(z_2 - z_1) = 0$. Thus, the Feynman-Vernon action can be written

$$S_{FV} = I_{11} + I_{44} + I_{14}, \quad (3.28)$$

where

$$\begin{aligned}
I_{11} &= -g^2 \int_{L_1} dz_2 \int_{L_1} dz_1 x_{L_1}(z_2) x_{L_1}(z_1) \theta_{L_1}(z_2 - z_1) \Delta_{L_1}^{(2)}(z_2 - z_1) = \\
&= g^2 \int_t^0 dt_2 \int_t^0 dt_1 \theta(t_1 - t_2) x^{(1)}(t_2) x^{(1)}(t_1) \left[i \frac{\cos \omega_e(t_2 - t_1)}{2m_e\omega_e} + \frac{\sin \omega_e(t_2 - t_1)}{2m_e\omega_e} \right] = \\
&= \int_0^t dt_2 \int_0^t dt_1 \theta(t_2 - t_1) y(t_2) y(t_1) [iG_I(t_2 - t_1) + G_R(t_2 - t_1)], \quad (3.29)
\end{aligned}$$

in the last integral we have defined

$$G_I(t_2 - t_1) \equiv \frac{g^2}{2m_e\omega_e} \cos \omega_e(t_2 - t_1) \quad (3.30)$$

and

$$G_R(t_2 - t_1) \equiv -\frac{g^2}{2m_e\omega_e} \sin \omega_e(t_2 - t_1). \quad (3.31)$$

The second integral is written

$$\begin{aligned}
I_{44} &= -g^2 \int_{L_4} dz_2 \int_{L_4} dz_1 x_{L_4}(z_2) x_{L_4}(z_1) \theta_{L_4}(z_2 - z_1) \Delta_{L_4}^{(2)}(z_2 - z_1) = \\
&= g^2 \int_0^t dt_2 \int_0^t dt_1 \theta(t_2 - t_1) x^{(4)}(t_2) x^{(4)}(t_1) \left[i \frac{\cos \omega_e(t_2 - t_1)}{2m_e \omega_e} + \frac{\sin \omega_e(t_2 - t_1)}{2m_e \omega_e} \right] = \\
&= \int_0^t dt_2 \int_0^t dt_1 \theta(t_2 - t_1) x^{(4)}(t_2) x^{(4)}(t_1) [iG_I(t_2 - t_1) - G_R(t_2 - t_1)]. \quad (3.32)
\end{aligned}$$

Finally, the last integral in equation (3.29) is

$$\begin{aligned}
I_{14} &= -g^2 \int_{L_1} dz_2 \int_{L_4} dz_1 x_{L_1}(z_2) x_{L_4}(z_1) \theta_{L_1 \cup L_4}(z_2 - z_1) \Delta_{L_1 \cup L_4}^{(2)}(z_2 - z_1) = \\
&= g^2 \int_0^t dt_2 \int_0^t dt_1 [\theta(t_2 - t_1) - \theta(t_1 - t_2)] x^{(4)}(t_2) x^{(1)}(t_1) \left[-i \frac{\cos \omega_e(t_1 - t_2)}{2m_e \omega_e} - \frac{\sin \omega_e(t_1 - t_2)}{2m_e \omega_e} \right] = \\
&= - \int_0^t dt_2 \int_0^t dt_1 \theta(t_2 - t_1) x^{(4)}(t_2) x^{(1)}(t_1) [iG_I(t_2 - t_1) + G_R(t_2 - t_1)] - \\
&\quad - \int_0^t dt_2 \int_0^t dt_1 \theta(t_1 - t_2) x^{(4)}(t_2) x^{(1)}(t_1) [iG_I(t_1 - t_2) - G_R(t_1 - t_2)]. \quad (3.33)
\end{aligned}$$

Inserting (3.29), (3.32) and (3.33) into (3.29) we confirm the general result (3.20) with the specific expressions (3.30) and (3.31) for the real and the imaginary part of the line propagator. These forms can be readily extended to the case of collection of N harmonic oscillators

$$G_R(t_2 - t_1) = - \sum_{n=1}^N \frac{g_n^2}{2m_e \omega_{ne}} \sin[\omega_{ne}(t_2 - t_1)] \quad (3.34)$$

and

$$G_I(t_2 - t_1) = \sum_{n=1}^N \frac{g_n^2}{2m_e \omega_{ne}} \cos \omega_{ne}(t_2 - t_1). \quad (3.35)$$

The last expressions are obviously the $T \rightarrow 0$ limit of the well known result for an environment consisting of a collection of harmonic oscillators in thermal equilibrium [14].

3.2 The Stochastic Environment

As we have discussed in the previous section the cluster expansion helps us to interpret the Feynman-Vernon action and consequently the influence functional as an infinite series over all possible correlations among the environmental degrees of freedom. It is, however, evident that such an interpretation can be useful only if this infinite series can be truncated with negligible error. The case of weak coupling between the system and its environment is a first an obvious example; we shall not discuss this case in this work but it is worth to be noted that the use of cluster expansion facilitates the resummation of the perturbative series. In this section we will focus on the case of an environment the dynamics of which establish a characteristic time scale τ_e after which the internal correlations decay exponentially fast

$$G_I(t) = G_I \left(\frac{|t|}{\tau_e} \right) \xrightarrow{|t| > \tau_e} 0, \quad (3.36)$$

$$\gamma(t) = \gamma \left(\frac{|t|}{\tau_e} \right) \xrightarrow{|t| > \tau_e} 0. \quad (3.37)$$

One could say that τ_e is such a time interval that when it elapses the environment returns to its initial state. Our next assumption is that there exists a second time scale τ_i , characterizing the interaction between the two parts of the entire system and, consequently, the evolution of the reduced density matrix. To be concrete let us assign an order of magnitude $\|K^{(2)}\|$ to the second order cumulant appearing in equation (3.9). We shall consider as stochastic the limit

$$\frac{\tau_e}{\hbar} \sqrt{\|K^{(2)}\|} \rightarrow 0. \quad (3.38)$$

It is apparent from its definition that $\|K^{(2)}\|$ is a measure of the average "strength" of the interaction between the system and its environment: $\sqrt{\|K^{(2)}\|} \sim \langle V \rangle$. Defining the time scale τ_i as $\tau_i \sim \hbar / \langle V \rangle$ the limit indicated in equation (3.38) can be obviously rephrased as $\tau_e / \tau_i \rightarrow 0$.

Let us now examine how the cluster expansion is formed in the stochastic limit. Assuming that $\langle q_c \rangle_q = 0$ the first term in the expansion (3.8) disappears and consequently the first non-vanishing term in the expansion is the second order term which assumes the quite general form

$$\begin{aligned} \frac{i}{\hbar} S_{FV} &= -\frac{i}{\hbar} \int_C dz_2 g_c(z_2) x_c(z_2) \int_C dz_1 g_c(z_1) x_c(z_1) \theta_c(z_2 - z_1) \Delta_C(z_2 - z_1) = \\ &= -\frac{1}{2\hbar} \int_0^t dt_2 \int_0^{t_2} dt_1 [x^{(1)}(t_2) - x^{(4)}(t_2)] G_I(t_2 - t_1) [x^{(1)}(t_1) - x^{(4)}(t_1)] + \end{aligned}$$

$$\begin{aligned}
& + \frac{i}{2\hbar} \int_0^t dt_2 \int_0^{t_2} dt_1 [x^{(1)}(t_2) - x^{(4)}(t_2)] \gamma(t_2 - t_1) [\dot{x}^{(1)}(t_1) + \dot{x}^{(4)}(t_1)] - \\
& \quad - \frac{i}{\hbar} \gamma(0) \int_0^t dt_1 \left[(x^{(1)}(t_1))^2 - (x^{(4)}(t_1))^2 \right] + \\
& \quad + \frac{i}{\hbar} [x^{(1)}(0) + x^{(4)}(0)] \int_0^t dt_1 [x^{(1)}(t_1) - x^{(4)}(t_1)] \gamma(t_1). \quad (3.39)
\end{aligned}$$

Being interested for $t \gg \tau_e$ we take into account equations (3.36), (3.37) and (3.38), we make the expansion

$$x^{(i)}(t_2) \simeq x^{(i)}(t_1) + \mathcal{O}(t_2 - t_1) \quad (3.40)$$

and we get

$$\int_0^t dt_2 [x^{(1)}(t_2) - x^{(4)}(t_2)] G_I(t_2 - t_1) \approx \sigma [x^{(1)}(t_1) - x^{(4)}(t_1)], \quad (3.41)$$

where we have written

$$\sigma \equiv \int_{-\infty}^{\infty} dt_2 G_I(t_2). \quad (3.42)$$

In the same way the second term in the right-hand side of equation (3.39) can be approximated as follows

$$\int_0^t dt_2 [x^{(1)}(t_2) - x^{(4)}(t_2)] \gamma(t_2 - t_1) \approx \lambda [x^{(1)}(t_1) - x^{(4)}(t_1)], \quad (3.43)$$

with

$$\lambda \equiv \int_{-\infty}^{\infty} dt_2 \gamma(t_2). \quad (3.44)$$

After a time rescaling $t_i = \tau_e \tilde{t}_i$, the defining relation for the γ function, (3.21) supports the estimation that

$$\lambda/\sigma \propto \tau_e \rightarrow 0. \quad (3.45)$$

After the preceding approximations the second order contribution to the Feynman-Vernon action reads

$$\begin{aligned}
\frac{i}{\hbar} S_{FV}^{(2)} = & - \frac{\sigma}{2\hbar} \int_0^t dt_1 [x^{(1)}(t_1) - x^{(4)}(t_1)]^2 + \frac{i\lambda}{2\hbar} \int_0^t dt_1 [x^{(1)}(t_1) - x^{(4)}(t_1)] [\dot{x}^{(1)}(t_1) + \dot{x}^{(4)}(t_1)] - \\
& - \frac{i}{\hbar} \gamma(0) \int_0^t dt_1 \left[(x^{(1)}(t_1))^2 - (x^{(4)}(t_1))^2 \right] + \frac{i\lambda}{2\hbar} \left[(x^{(1)}(0))^2 - (x^{(4)}(0))^2 \right]. \quad (3.46)
\end{aligned}$$

Our claim is that at the stochastic limit the above expression is that the dominant contribution to the cluster expansion (3.8) that defines the Feynman-Vernon action. Indeed, each term

$$I^{(n)} = \int_0^t dt_n \dots \int_0^t dt_1 K^{(n)}(t_n, \dots, t_1) \quad (3.47)$$

represents a cumulant the main contribution of which comes from time intervals $|t_1 - t_i| \sim \tau_e$, $i = 2, 3, \dots$. Expanding the integrand as we have done in equations (3.41) and (3.43) we conclude that

$$\frac{I^{(n)}}{I^{(n-1)}} = \mathcal{O}\left(\frac{\tau_e}{\tau_i}\right). \quad (3.48)$$

This conclusion can be used to give concrete meaning to the environment characterized as stochastic: It is the environment the influence of which can be approximated by keeping only the second order correlator in the cluster expansion.

Put it in other words, the Feynman-Vernon action, at the stochastic limit, can be approximated as follows:

$$S_{FV} \approx S_{FV}^{(2)} + \mathcal{O}\left(\frac{\tau_e}{\tau_i}\right). \quad (3.49)$$

Finally, we can write the influence functional

$$\mathcal{F}[x^{(4)}(t), x^{(1)}(t); t] \approx \frac{1}{Z_e} \exp\left\{\frac{i}{\hbar} S_{FV}^{(2)}[x^{(1)}(t), x^{(4)}(t)]\right\}. \quad (3.50)$$

The dynamics behind such a behavior are, of course, strongly depended on the specific “environment” under consideration and, no need to be said can be extremely complicated. We shall not discuss any special case in this work in which the undertaken task is, so to speak, “phenomenological”: The approximation (3.49) being given for the influence of the environment, we try to estimate the consequences on the central system.

At this point we must underline the strong resemblance of our result (3.46) to the case of the, so-called, ohmic environment. The latter case is simulated by a collection of harmonic oscillators the frequencies of which are Gaussian distributed: $\sim \omega e^{-\omega^2 \tau_e^2}$ leading to correlators of the form $G_I(t) \prec T\delta(t)$ and $\gamma(t) \prec \delta(t)$. Despite the fact that the expressions for the Feynman-Vernon action are, in both cases, formally the same our results are supposed to valid at zero temperature and for an environment exhibiting stochastic behavior. The parameters appearing in equation (3.46) are not phenomenological, but

they are strictly related to the two-point correlation function of the environment, and, in principle, can be calculated at least numerically. In the same context, the expression (3.7) which is approximated by (3.48), does not represent the introduction of a random complex-valued Gaussian stochastic force: It is the specific environment under consideration and its dynamics that justify the stochastic approximation. Having in mind the extension of our work to infinite degrees of freedom, the non-Abelian gauge theories [39] constitute the primary example of such a stochastic behavior.

In any case the result (3.49) considerably facilitates the road to the reduced density matrix even if the final result does strongly depend on the initial state of our system. In what follows we shall consider the case in which our system begins from its ground state

$$|\psi_s\rangle = |0_s\rangle, \quad (3.51)$$

in this case we can use for $\rho^s(0)$ an expression analogous to (3.2), thus we can write

$$\begin{aligned} \rho_{x''x'''}^s(0) &= \frac{1}{Z_s} \int_{x^{(3)}(-0)=x''} \mathcal{D}x^{(3)}(\tau) \int_{x^{(2)}(+0)=x'''} \mathcal{D}x^{(2)}(\tau) \exp \left\{ -\frac{1}{\hbar} \int_{-\infty}^{-0} d\tau \mathcal{L}_{s,E}[x^{(3)}(\tau)] \right\} \times \\ &\quad \times \exp \left\{ -\frac{1}{\hbar} \int_{+0}^{+\infty} d\tau \mathcal{L}_{s,E}[x^{(2)}(\tau)] \right\}. \end{aligned} \quad (3.52)$$

Inserting the relations (1.34), (3.50) and (3.52) into (1.33) we have

$$\begin{aligned} \rho_{x'x}^R(t) &\approx \frac{1}{Z_s Z_e} \int dx'' \int dx''' \int_{x(t)=x'} \mathcal{D}x(t) \int_{x^{(1)}(t)=x} \mathcal{D}x^{(1)}(t) \int \mathcal{D}x^{(2)}(\tau) \int \mathcal{D}x^{(3)}(\tau) \times \\ &\quad \times \delta[x^{(4)}(0) - x''] \delta[x^{(1)}(0) - x'''] \delta[x^{(3)}(-0) - x''] \delta[x^{(2)}(+0) - x'''] \times \\ &\quad \times \exp \left\{ \frac{i}{\hbar} \int_0^t dt' \mathcal{L}_s[x^{(4)}(t')] + \frac{i}{\hbar} \int_t^0 dt' \mathcal{L}_s[x^{(1)}(t')] - \frac{1}{\hbar} \int_{-\infty}^{-0} d\tau \mathcal{L}_{s,E}[x^{(3)}(\tau)] - \right. \\ &\quad \left. - \frac{1}{\hbar} \int_0^{\infty} d\tau \mathcal{L}_{s,E}[x^{(2)}(\tau)] + \frac{i}{\hbar} S_{FV}^{(2)}[x^{(4)}(t), x^{(1)}(t)] \right\}, \end{aligned}$$

thus, using the closed complex time formalism, we can write the reduced density matrix in the following compact form

$$\rho_{x'x}^R(t) \approx \frac{1}{Z_s Z_e} \int_{x_c(t-i0)=x}^{x_c(t+i0)=x'} \mathcal{D}x_c(z) \exp \left\{ \frac{i}{\hbar} \int_c dz \mathcal{L}_s[x_c(z)] + \frac{i}{\hbar} S_{FV}^{(2)}[x_c(z)] \right\}. \quad (3.53)$$

3.2.1 The System as One Harmonic Oscillator

The result (3.53) for the reduced density matrix as it is expressed in the last equation is simple and compact. This is due to the complex parametrization of the paths under integration. To obtain the final result, the integration over the central degrees of freedom must be performed and, obviously, this is a task that cannot be exactly accomplished in the general case: Some kind of approximation is needed. In any case equation (3.53) sets the scene on which any available approximation technique can be performed. We can demonstrate the calculational abilities of our formalism by considering the zero order approximation i.e., the simple case in which the system is just one simple harmonic oscillator:

$$\mathcal{L}_s[x_c(z)] = \frac{1}{2}m\dot{x}_c^2 - \frac{1}{2}m\omega^2 x_c^2. \quad (3.54)$$

Since the contribution from the Feynman-Vernon action is also quadratic, it is obvious that the dependence of the reduced density matrix on the boundary values x and x' can be deduced just from the classical path :

$$m \left(\frac{d^2}{dz^2} + \omega^2 \right) x_c^{cl.}(z) = \frac{\delta S_{FV}^{(2)}[x_c^{cl.}]}{\delta x_c^{cl.}(z)}, x_c^{cl.}(t+i0) = x', x_c^{cl.}(t-i0) = x. \quad (3.55)$$

The right-hand side of the last equation must be read in terms of the stochastic limit (3.38). Thus, we readily obtain

$$\begin{aligned} \rho_{x'x}^R(t) &\sim \exp \left\{ \frac{im}{2\hbar} [x'_s \dot{x}_c^{cl.}(t+i0) - x_s \dot{x}_c^{cl.}(t-i0)] \right\} \times \\ &\times \exp \left\{ -\frac{i}{2\hbar} \int_C dz x_c^{cl.}(z) \frac{\delta S_{FV}^{(2)}[x_c^{cl.}]}{\delta x_c^{cl.}(z)} + \frac{i}{\hbar} S_{FV}^{(2)}[x_c^{cl.}] \right\}. \end{aligned} \quad (3.56)$$

The last two terms appearing in the right-hand side of (3.56), cancel each other due to the quadratic nature of the truncated Feynman-Vernon action $S_{FV}^{(2)}$. Thus we conclude

$$\begin{aligned} \rho_{x'x}^R(t) &\sim \exp \left\{ \frac{im}{2\hbar} [x' \dot{x}_c^{cl.}(t+i0) - x \dot{x}_c^{cl.}(t-i0)] \right\} = \\ &= \exp \left\{ \frac{im}{2\hbar} [x' \dot{x}_{cl.}^{(4)}(t) - x \dot{x}_{cl.}^{(1)}(t)] \right\}. \end{aligned} \quad (3.57)$$

The appearance of the classical trajectory in the last equation calls for the solution of the equation of motion (3.55). This is a lengthy but straightforward task which is presented in full detail in Appendix B. At this point it is enough to observe that the dependence of the classical solution on the boundary values x and x' is easily determined using the quite general ansatz:

$$\dot{x}_{cl.}^{(4)}(t) = \frac{1}{2}\alpha(t)x' + \frac{1}{2}\beta(t)x, \quad (3.58)$$

$$\dot{x}_{cl.}^{(1)}(t) = \frac{1}{2}\gamma(t)x' + \frac{1}{2}\delta(t)x. \quad (3.59)$$

In the Appendix B we present the integral equations that determine the coefficients in the above relations. It is also confirmed, in the same Appendix, the validity of the relations

$$\delta(t) = \alpha^*(t), \gamma(t) = \beta^*(t),$$

which are necessary for the hermiticity of the reduced density matrix. Inserting expressions (3.58) in equation (3.57) we find that

$$\rho_{x'x}^R(t) = C(t) \exp \left\{ \frac{im}{2\hbar} \left[x'^2 \alpha(t) - x^2 \alpha^*(t) + x'x(\beta(t) - \beta^*(t)) \right] \right\}, \quad (3.60)$$

The suppression of the off-diagonal terms in the representation (3.60) of the reduced density matrix is obviously related to the non-zero imaginary part of the function $\alpha(t)$ which in turn, as we confirm in the Appendix A, is related to the non-vanishing imaginary part of the environmental correlations. The normalization factor in equation (3.60) is now determined by demanding:

$$C(t) \int_{-\infty}^{\infty} dx \exp \left\{ -\frac{m}{2\hbar} x^2 \Im(\alpha(t) + \beta(t)) \right\} = 1. \quad (3.61)$$

The explicit calculations presented in Appendix B, show that

$$\Im(\alpha(t) + \beta(t)) = 0 \quad (3.62)$$

yielding the conclusion that $C = 1/L \rightarrow 0$ where L is the volume of the space in which the system lives. In this case the reduced density matrix reads:

$$\rho_{x'x}^R(t) \sim \exp \left\{ \frac{im}{4\hbar} \Re \alpha(t) (x'^2 - x^2) \right\} \exp \left\{ \frac{m}{4\hbar} \Im \alpha(t) (x' - x)^2 \right\}. \quad (3.63)$$

The explicit form of the function $\alpha(t)$ is presented in Appendix B. Here it is enough to note that $\Im \alpha$ is a positive definite increasing function of time. It

is strictly related to the imaginary part of the environmental second order correlator since $\Im\alpha \propto \sigma$. Thus, the real factor of the density matrix (3.63) is formally the density matrix of a free particle in a heat bath of temperature $k_B T = \frac{1}{2}\Im\alpha \propto \sigma$.

The exact time dependence of the function $\alpha(t)$ is tied with the value of the quantity:

$$q^2 = \frac{\lambda^2}{4m} + 2\frac{\gamma(0)}{m} - \omega^2. \quad (3.64)$$

If $q^2 > 0$, $\alpha(t)$ becomes time independent for $t|q| \gg 1$ and

$$\Im\alpha \approx \frac{\sigma}{m} \frac{1}{|q|}, \quad \Re\alpha \approx \frac{\lambda}{m} + 2|q|. \quad (3.65)$$

For $q^2 = 0$ and for $(\omega - \lambda/m)t \gg 1$, $\alpha(t)$ is again time independent:

$$\Im\alpha \approx \frac{\sigma}{m} \frac{2}{\omega - \lambda/m}, \quad \Re\alpha \approx 2\omega. \quad (3.66)$$

If $q^2 \equiv -k^2 < 0$, and for $kt \gg 1$, $\Im\alpha$ remains an increasing function of time:

$$\Im\alpha(t) \approx \frac{\sigma}{m} \frac{k^2 + (\omega - \lambda/2m)^2}{[(\omega - \lambda/2m) \sin kt + k \cos kt]^2} t. \quad (3.67)$$

The Von Neumann Entropy

The calculation of the reduced density matrix can play a key role for the determination of the system properties that we interest for. Let's calculate, for example, the Von Neumann entropy

$$S(t) = -Tr_s \{ \hat{\rho}_R(t) \ln \hat{\rho}_R(t) \}, \quad (3.68)$$

for the density matrix (3.63). The calculation can be performed with the help of the replica method, we have introduced in the second chapter:

We calculate the following function $f(n)$, for the reduced density matrix (3.63)

$$\begin{aligned} f(n) &= \int dx^{(1)} \int dx^{(2)} \dots \int dx^{(n)} \rho_{x^{(1)}x^{(2)}}^R \rho_{x^{(2)}x^{(3)}}^R \dots \rho_{x^{(n)}x^{(1)}}^R = \\ &= C^n \left(\prod_{i=1}^n \int dx^{(i)} \right) \prod_{i=1}^n \exp \left\{ -\frac{m}{4\hbar} \Im\alpha(t) (x^{(i+1)} - x^{(i)})^2 \right\}, \end{aligned} \quad (3.69)$$

where $x^{(n+1)} = x^{(n)}$. Consider now the propagation of a free particle with mass m from the point x to the point x' in the Euclidean time interval $t_E = 2/\Im\alpha(t)$:

$$\int_{x(0)=x}^{x(t_E)=x'} \mathcal{D}x \exp \left\{ -\frac{m}{2} \int_0^{t_E} d\tau \dot{x}(\tau) \right\} = \sqrt{\frac{m\Im\alpha(t)}{4\pi\hbar}} \exp \left\{ -\frac{m}{4\hbar} \Im\alpha(t)(x' - x)^2 \right\}. \quad (3.70)$$

Inserting the last expression into eq. (3.69) we find that:

$$f(n) = \left[\frac{4\pi\hbar}{m\Im\alpha(t)L^2} \right]^{n/2} \int_{x(0)=x(nt_E)} \mathcal{D}x \exp \left\{ -\frac{m}{2} \int_0^{nt_E} d\tau \dot{x}(\tau) \right\}, \quad (3.71)$$

where the last integral must be performed over periodic trajectories with period nt_E . The path integral in the right-hand side of (3.71) can be calculated easily:

$$\begin{aligned} \int_{x(0)=x(nt_E)} \mathcal{D}x \exp \left\{ -\frac{m}{2} \int_0^{nt_E} d\tau \dot{x}(\tau) \right\} &= \int d\ell \int_{x(0)=\ell}^{x(nt_E)=\ell} \mathcal{D}x \exp \left\{ -\frac{m}{2} \int_0^{nt_E} d\tau \dot{x}(\tau) \right\} = \\ &= \left[\frac{m\Im\alpha(t)}{4\pi\hbar n} \right]^{1/2} \int d\ell = \left[\frac{m\Im\alpha(t)}{4\pi\hbar n} \right]^{1/2} L. \end{aligned} \quad (3.72)$$

Thus we can immediately conclude that:

$$f(n) = \left[\frac{4\pi\hbar}{m\Im\alpha(t)L^2} \right]^{n/2} \left[\frac{m\Im\alpha(t)L^2}{4\pi\hbar n} \right]^{1/2}. \quad (3.73)$$

The Von Neumann entropy is now easily computed with the help of eq. (2.3):

$$S(t) = -\frac{1}{2} \ln \left[\frac{4\pi\hbar}{m\Im\alpha(t)L^2} \right] + \frac{1}{2}. \quad (3.74)$$

It is worth to be noted the well-known fact that the entanglement entropy $S \sim \ln L$ is not an extensive quantity: Contrary to the thermal entropy is not analogous to the volume of the space in which the subsystem lives.

3.3 Conclusions

In this chapter we have introduced two basic methodological tools for calculating the time evolution of the reduced density matrix and, consequently, the

dynamics of an open quantum system. The first is the closed complex time (CCT) formalism which combines in a single set-up two known approaches: The closed time formalism and the complex time one. This formalism enabled us to express the influence functional of Feynman and Vernon (which encloses the time dependence of the reduced density matrix), in terms of a compact path integral in which the paths are parametrized on a closed contour on the complex plane. Our second suggestion was the introduction of the cluster expansion which is a very powerful tool, being tested in a variety of problems, when the environmental details can be successfully approximated by keeping only the two-point correlators. In this combined CCT-cluster expansion framework, we examined the case of the so-called stochastic environment in which the correlations are decaying very “fast”. In order to check our tools and examine the consequences of a stochastic environment, we performed a detailed “zero-order” calculation for the simple case in which the system is a harmonic oscillator. We found the explicit form of the reduced density matrix as a function of time and we calculated the entanglement entropy. Depending on the details of the environment the entropy is either a constantly increasing function of time or an increasing function of time that saturates to a constant value.

Appendix A

The Density Matrix of the Ground State in the Language of Path Integrals

The quantity which is the basis for our analysis is the propagator, that defined from the relation

$$G(x', t'; x, t) = {}_H\langle x', t' | x, t \rangle_H = \langle x' | \exp \left[-\frac{i}{\hbar} (t' - t) \hat{H} \right] | x \rangle \quad (\text{A.1})$$

In the above equation $|x, t\rangle_H$ are the eigenstates of the position operator in the Heisenberg picture. The last equality assumes that the Hamiltonian is time independent.

We suppose that the Hamiltonian has a complete system of eigenstates and we write:

$$G(x', t'; x, t) = \sum_n \langle x' | \exp \left[-\frac{i}{\hbar} (t' - t) \hat{H} \right] | n \rangle \langle n | x \rangle = \sum_n e^{-\frac{i}{\hbar} (t' - t) E_n} \phi_n(x') \phi_n^*(x) \quad (\text{A.2})$$

The above expression allows us to study the propagator as a mathematical function of the variable t . Making the change

$$t_E = it \quad (\text{A.3})$$

we rewrite equation (A.2) as:

$$G(x', t'_E; x, t_E) = \sum_n e^{(t'_E - t_E) E_n / \hbar} \phi_n(x') \phi_n^*(x) \quad (\text{A.4})$$

Now, we write $t_E = -T_E, t'_E = 0$ and consider the limit $T_E \rightarrow \infty$. In this limit the dominant term that appear in the right-hand side of (A.4) is the

term with the lowest energy, the ground state :

$$G(x', 0; x, -T_E) = \phi_0(x')\phi_0^*(x)e^{-T_E E_0/\hbar} [1 + O(e^{-T_E(E_n - E_0)/\hbar})] \quad (\text{A.5})$$

The only assumption we have used is that the ground state is unique.

From the equation (A.5) we can find immediately that:

$$\begin{aligned} \int dx G(x', 0; x, -T_E) &= \int dx \phi_0(x')\phi_0^*(x)e^{-T_E E_0/\hbar} = \\ &= \phi_0(x') \int dx \phi_0^*(x)e^{-T_E E_0/\hbar} \equiv N\phi_0(x') \Rightarrow \\ \phi_0(x') &= \frac{1}{N} \int dx G(x', 0; x, -T_E) \end{aligned} \quad (\text{A.6})$$

where we have introduced the normalization constant:

$$\begin{aligned} N &= \left(\int dx \phi_0^*(x) \right) e^{-T_E E_0/\hbar} = \left(\int dx \phi_0^*(x) \right) \int dx G(x, 0; x, -T_E) \Rightarrow \\ N &= \left(\int dx \phi_0^*(x) \right) \text{Tr} \hat{G}(0, -T_E). \end{aligned} \quad (\text{A.7})$$

For the last result we observed that

$$\begin{aligned} \int dx G(x, 0; x, -T_E) &= \int dx \phi_0(x)\phi_0^*(x)e^{-T_E E_0/\hbar} = \\ &= e^{-T_E E_0/\hbar} \int dx |\phi_0(x)|^2 = e^{-T_E E_0/\hbar} \Rightarrow \text{Tr} \hat{G}(0, -T_E) = e^{-T_E E_0/\hbar} \end{aligned}$$

The relation (A.6) indicates that the ground state , can be found through the propagator with the help of the procedure we have introduced.

The next step, is to write the propagator as a path integral :

$$\begin{aligned} G(x', 0; x, -T_E) &= \int_{\substack{x(0)=x' \\ x(-T_E)=x}} \mathcal{D}x(\tau) \exp \left\{ -\frac{1}{\hbar} \int_{-T_E}^0 d\tau \mathcal{L}_E[x(\tau)] \right\} = \\ &= \int \mathcal{D}x(\tau) \delta(x(0) - x') \delta(x(-T_E) - x) \exp \left\{ -\frac{1}{\hbar} \int_{-T_E}^0 d\tau \mathcal{L}_E[x(\tau)] \right\}. \end{aligned} \quad (\text{A.8})$$

Let's clear, here, our notation: The weight with which every path contributes to the path integral is $\exp(\frac{i}{\hbar} S[x])$ where $S[x]$ is the classical action that corresponds to the particular path. If, for example, the Hamiltonian of a particle is $H = \frac{p^2}{2m} + V(x)$ then the corresponding action will be

$S = \int_0^T (\frac{m}{2}\dot{x}^2(t) + V(x(t)))$. In our case, we have made the change (A.3) so the weight in the path integral (A.8) is

$$\begin{aligned} & \exp \left[\frac{i}{\hbar} \int_0^{iT} (-idt_E) \left(-\frac{m}{2}\dot{x}^2(t_E) + V(x(t_E)) \right) \right] = \\ & = \exp \left[-\frac{1}{\hbar} \int_0^{T_E} dt_E \left(-\frac{m}{2}\dot{x}^2(t_E) + V(x(t_E)) \right) \right] \equiv \exp \left[-\frac{1}{\hbar} \int_0^{T_E} dt_E \mathcal{L}_E[x(t_E)] \right] \end{aligned}$$

This notation is introduced in equation (A.8) together with a compactness in writing. Inserting, now, equation (A.8) into (A.6) we have:

$$\begin{aligned} \phi_0(x') &= \frac{1}{N} \int dx G(x', 0; x, -T_E) = \\ &= \frac{1}{N} \int dx \int \mathcal{D}x(t_E) \delta(x(0) - x') \delta(x(-T_E) - x) \exp \left\{ -\frac{1}{\hbar} \int_0^{T_E} dt_E \mathcal{L}_E[x(t_E)] \right\} \Rightarrow \\ \phi_0(x') &= \frac{1}{N} \int \mathcal{D}x(t_E) \delta(x(0) - x') \left\{ \int_0^{T_E} dt_E \mathcal{L}_E[x(t_E)] \right\} \quad (\text{A.9}) \end{aligned}$$

We can go, now, back to (A.4) and play the same game. We will write $t'_E = T_E \rightarrow \infty, t_E = 0$. The relation (A.5) will become:

$$G(x', T_E; x, 0) = \phi_0(x') \phi_0^*(x) e^{-T_E E_0 / \hbar} [1 + O(e^{-T_E (E_n - E_0) / \hbar})] \quad (\text{A.10})$$

and thus :

$$\phi_0^*(x) = \frac{1}{\bar{N}} \int \mathcal{D}x(\tau) \delta(x(0) - x) \exp \left\{ -\frac{1}{\hbar} \int_0^{T_E} d\tau \mathcal{L}_E[x(\tau)] \right\}. \quad (\text{A.11})$$

The normalization constant is:

$$\bar{N} = \left(\int dx \phi_0(x) \right) Tr \hat{G}(T_E, 0) \quad (\text{A.12})$$

If we combine the relations (A.9) and (A.11) we can write the density matrix of the system (: the particle that is in its ground state):

$$\begin{aligned} \rho(x', x) &\equiv \phi_0(x') \phi_0^*(x) = \frac{1}{N\bar{N}} \int \mathcal{D}x(\tau) \int \mathcal{D}\tilde{x}(\tau) \delta(x(0) - x') \delta(\tilde{x}(0) - x) \times \\ &\times \exp \left\{ -\frac{1}{\hbar} \int_{-T_E}^0 d\tau \mathcal{L}_E[x(\tau)] - \frac{1}{\hbar} \int_0^{T_E} d\tau \mathcal{L}_E[\tilde{x}(\tau)] \right\}. \quad (\text{A.13}) \end{aligned}$$

We define, now, a new function:

$$z(\tau) = \begin{cases} x(\tau), & -T_E < \tau < 0 \\ \tilde{x}(\tau), & 0 < \tau < T_E \end{cases} \quad (\text{A.14})$$

which is obviously non continuous :

$$z(-0) = x(0) = x' \neq z(+0) = \tilde{x}(0) = x. \quad (\text{A.15})$$

With the help of this function, the density matrix takes the form:

$$\begin{aligned} \rho(x', x) &= \frac{1}{Z} \int \mathcal{D}z(\tau) \delta(z(-0) - x') \delta(z(+0) - x) \times \\ &\times \exp \left\{ -\frac{1}{\hbar} \int_{-T_E}^0 d\tau \mathcal{L}_E[z(\tau)] - \frac{1}{\hbar} \int_0^{T_E} d\tau \mathcal{L}_E[z(\tau)] \right\}. \end{aligned} \quad (\text{A.16})$$

The normalization constant can be determined from the relation:

$$\begin{aligned} \int dx \rho(x, x) &= \text{Tr} \hat{\rho} = 1 \Rightarrow \\ \frac{1}{Z} \int dx \int \mathcal{D}z(\tau) \delta(z(-0) - x) \delta(z(+0) - x) \exp \left\{ -\frac{1}{\hbar} \int_{-T_E}^0 d\tau \mathcal{L}_E[z(\tau)] - \frac{1}{\hbar} \int_0^{T_E} d\tau \mathcal{L}_E[z(\tau)] \right\} &\Rightarrow \\ \frac{1}{Z} \int dx \int \mathcal{D}z(\tau) \delta(z(+0) - z(-0)) \exp \left\{ -\frac{1}{\hbar} \int_{-T_E}^0 d\tau \mathcal{L}_E[z(\tau)] - \frac{1}{\hbar} \int_0^{T_E} d\tau \mathcal{L}_E[z(\tau)] \right\} &\Rightarrow \\ Z &= \int_{z(+0)=z(-0)} \mathcal{D}z(\tau) \exp \left\{ -\frac{1}{\hbar} \int_{-T_E}^{T_E} d\tau \mathcal{L}_E[z(\tau)] \right\}. \end{aligned} \quad (\text{A.17})$$

Here we must say that, we could conclude to the same result if we use (A.4) and put $t'_E = T_E, t_E = -T_E, T_E \rightarrow \infty$:

$$G(x', T_E; x, -T_E) = \phi_0(x') \phi_0^*(x) e^{-2T_E E_0/\hbar} [1 + O(e^{-2T_E(E_n - E_0)/\hbar})]. \quad (\text{A.18})$$

Thus

$$\begin{aligned} \rho(x', x) &= \frac{1}{A} G(x', T_E; x, -T_E) = \frac{1}{Z} \int \mathcal{D}z(\tau) \delta(z(-T_E) - x) \delta(z(T_E) - x') \times \\ &\times \exp \left\{ -\frac{1}{\hbar} \int_{-T_E}^{T_E} d\tau \mathcal{L}_E[z(\tau)] \right\}. \end{aligned} \quad (\text{A.19})$$

The identity of the two expressions (A.16) and (A.19) can be shown if we write

$$\int_{-T_E}^{T_E} d\tau \mathcal{L}_E[z(\tau)] = \int_0^{T_E} d\tau \mathcal{L}_E[z(\tau)] + \int_{-T_E}^0 d\tau \mathcal{L}_E[z(\tau)].$$

and then make the change $\tau = \tilde{\tau} - T_E$ in the first integral and the change $\tau = \tilde{\tau} + T_E$ in the second:

$$\int_{-T_E}^{T_E} d\tau \mathcal{L}_E[z(\tau)] \rightarrow \int_0^{T_E} d\tilde{\tau} \mathcal{L}_E[z(\tilde{\tau})] + \int_{-T_E}^0 d\tilde{\tau} \mathcal{L}_E[z(\tilde{\tau})].$$

Of course, we must change, also, the boundary conditions indicated in (A.19): $z(-T_E) \rightarrow z(+0)$ and $z(T_E) \rightarrow z(-0)$. With these changes it is obvious that we will conclude to (A.16). After this observation it is obvious that the two expressions (A.16) and (A.19) are equivalent.

Appendix B

The Non-Local Classical Equations of Motion

In this Appendix we shall determine the functions $\alpha(t)$ and $\beta(t)$ beginning from the classical equation of motion

$$m \left(\frac{d^2}{dz^2} + \omega^2 \right) x_c^{cl.}(z) = \frac{\delta S_{FV}^{(2)}[x_c^{cl.}]}{\delta x_c^{cl.}(z)} \quad (\text{B.1})$$

Due to its nonlocal character the above equation must be examined independently in every segment of the contour C .

Along the line L_4 the classical equation takes the form:

$$m \left(\frac{d^2}{dt'^2} - \Omega^2 \right) x_{cl.}^{(4)}(t') = i\sigma x_{cl.}^{(4)}(t') - \left(\lambda \frac{d}{dt'} + i\sigma \right) x_{cl.}^{(1)}(t'), \quad (\text{B.2})$$

where we defined

$$m\Omega^2 \equiv -m\omega^2 + 2\gamma(0). \quad (\text{B.3})$$

Along the lines L_3 and L_2 we have

$$m \left(\frac{d^2}{d\tau^2} - \omega^2 \right) x_{cl.}^{(3)}(\tau) = 0 \quad (\text{B.4})$$

and

$$m \left(\frac{d^2}{d\tau^2} - \omega^2 \right) x_{cl.}^{(2)}(\tau) = 0. \quad (\text{B.5})$$

The last part of the classical equation refers to the line L_1 :

$$m \left(\frac{d^2}{dt'^2} - \Omega^2 \right) x_{cl.}^{(1)}(t') = -i\sigma x_{cl.}^{(1)}(t') - \left(\lambda \frac{d}{dt'} - i\sigma \right) x_{cl.}^{(4)}(t'). \quad (\text{B.6})$$

Seeking for continuous and differentiable solutions of the above system of classical equations we impose the following boundary conditions:

$$\begin{aligned}
x_{cl.}^{(4)}(t) &= x', & x_{cl.}^{(4)}(0) &= x_{cl.}^{(3)}(0) \\
x_{cl.}^{(3)}(-\infty) &= 0, & x_{cl.}^{(3)}(0) &= x_{cl.}^{(4)}(0) \\
x_{cl.}^{(2)}(0) &= x_{cl.}^{(1)}(0), & x_{cl.}^{(2)}(+\infty) &= 0 \\
x_{cl.}^{(1)}(t) &= x, & x_{cl.}^{(1)}(0) &= x_{cl.}^{(2)}(0)
\end{aligned} \tag{B.7}$$

and

$$\dot{x}_{cl.}^{(4)}(0) = \dot{x}_{cl.}^{(3)}(0), \quad \dot{x}_{cl.}^{(2)}(0) = -\dot{x}_{cl.}^{(1)}(0), \quad \dot{x}_{cl.}^{(3)}(-\infty) = \dot{x}_{cl.}^{(2)}(\infty). \tag{B.8}$$

Equations (B.4) and (B.5) can be readily solved with the help of the above indicated boundary conditions:

$$x_{cl.}^{(3)}(\tau) = x_{cl.}^{(4)}(0)e^{\omega\tau}, \quad x_{cl.}^{(2)}(\tau) = x_{cl.}^{(1)}(0)e^{-\omega\tau}. \tag{B.9}$$

Using once again the boundary conditions (B.8) we find that:

$$\omega x_{cl.}^{(4)}(0) = \dot{x}_{cl.}^{(4)}(0), \quad \omega x_{cl.}^{(1)}(0) = \dot{x}_{cl.}^{(1)}(0). \tag{B.10}$$

Introducing the combinations

$$y^{(\pm)} = \frac{1}{2}(x_{cl.}^{(4)} \pm x_{cl.}^{(1)}), \tag{B.11}$$

the system of eqs. (B.2) and (B.4) can be considerably simplified:

$$\left(\frac{d^2}{dt'^2} - \frac{\lambda}{m} \frac{d}{dt'} - \Omega^2 \right) y^{(+)}(t') = 2i \frac{\sigma}{m} y^{(-)}(t'), \tag{B.12}$$

$$\left(\frac{d^2}{dt'^2} - \frac{\lambda}{m} \frac{d}{dt'} - \Omega^2 \right) y^{(-)}(t') = 0. \tag{B.13}$$

The solutions $y^{(\pm)}$ of the last equations are now trivially obtained and they lead us immediately to the result:

$$x_{cl.}^{(4)}(t') = A_1 \varphi_+^{(4)}(t') + A_2 \varphi_-^{(4)}(t') + A_3 e^{\alpha+t'} + A_4 e^{-\alpha-t'}, \tag{B.14}$$

$$x_{cl.}^{(1)}(t') = A_1 \varphi_+^{(1)}(t') + A_2 \varphi_-^{(1)}(t') + A_3 e^{\alpha+t'} + A_4 e^{-\alpha-t'}. \tag{B.15}$$

In the above expression we have written:

$$\varphi_+^{(4)}(t') = e^{\alpha+t'} - 2i \frac{\sigma}{m} \int_0^t dt'' G(t', t'') e^{\alpha+t''}, \tag{B.16}$$

$$\varphi_-^{(4)}(t') = e^{-\alpha-t'} - 2i \frac{\sigma}{m} \int_0^t dt'' G(t', t'') e^{-\alpha-t''}, \quad (\text{B.17})$$

$$\varphi_+^{(1)}(t') = e^{\alpha+t'} + 2i \frac{\sigma}{m} \int_0^t dt'' G(t', t'') e^{\alpha+t''}, \quad (\text{B.18})$$

$$\varphi_-^{(1)}(t') = e^{-\alpha-t'} + 2i \frac{\sigma}{m} \int_0^t dt'' G(t', t'') e^{-\alpha-t''}, \quad (\text{B.19})$$

$$\alpha_{\pm} = \pm \frac{\lambda}{2m} + \sqrt{\frac{\lambda^2}{4m^2} + \frac{2\gamma(0)}{m} - \omega^2}. \quad (\text{B.20})$$

In eqs. (A.16) - (A.19) we used the Green's function

$$\left(\frac{d^2}{dt'^2} - \frac{\lambda}{m} \frac{d}{dt'} - \Omega^2 \right) G(t', t'') = -\delta(t' - t''), \quad G(t, t'') = G(0, t'') = 0 \quad (\text{B.21})$$

which assumes the form:

$$\begin{aligned} G(t', t'') &= \frac{e^{(\alpha_+ + \alpha_-)t/2 - \alpha_+ t''} - e^{-(\alpha_+ + \alpha_-)t/2 + \alpha_- t''}}{2(\alpha_+ + \alpha_-) \sinh[(\alpha_+ + \alpha_-)t/2]} (e^{\alpha_+ t'} - e^{-\alpha_- t'}) \theta(t'' - t') + \\ &+ \frac{e^{(\alpha_+ + \alpha_-)t/2 - \alpha_- t'} - e^{-(\alpha_+ + \alpha_-)t/2 + \alpha_+ t'}}{2(\alpha_- + \alpha_+) \sinh[(\alpha_+ + \alpha_-)t/2]} (e^{\alpha_- t''} - e^{-\alpha_+ t''}) \theta(t' - t''). \end{aligned} \quad (\text{B.22})$$

The coefficients in eqs.(B.14) and (B.15) can straightforwardly be obtained with the help of the boundary conditions (B.7) and (B.10):

$$A_1(t) = -\frac{\lambda_-(t)}{D(t)} \frac{x' - x}{2}, \quad (\text{B.23})$$

$$A_2(t) = \frac{\lambda_+(t)}{D(t)} \frac{x' - x}{2}, \quad (\text{B.24})$$

$$A_3(t) = \frac{\alpha_- + \omega}{\tilde{D}(t)} \frac{x' + x}{2} + \frac{\mu_+(t)\lambda_-(t) - \mu_-(t)\lambda_+(t)}{\tilde{D}(t)D(t)} e^{-\alpha_- t} \frac{x' - x}{2}, \quad (\text{B.25})$$

$$A_4(t) = \frac{\alpha_+ - \omega}{\tilde{D}(t)} \frac{x' + x}{2} - \frac{\mu_+(t)\lambda_-(t) - \mu_-(t)\lambda_+(t)}{\tilde{D}(t)D(t)} e^{\alpha_+ t} \frac{x' - x}{2}, \quad (\text{B.26})$$

with

$$D(t) = \lambda_+(t)e^{-\alpha_- t} - \lambda_-(t)e^{\alpha_+ t}, \quad \tilde{D}(t) = (\alpha_+ - \omega)e^{-\alpha_- t} + (\alpha_- + \omega)e^{\alpha_+ t}, \quad (\text{B.27})$$

$$\lambda_{\pm}(t) = \dot{\varphi}_{\pm}^{(4)}(0) + \dot{\varphi}_{\pm}^{(1)}(0) - 2\omega, \quad \mu_{\pm} = \frac{1}{2}(\dot{\varphi}_{\pm}^{(4)}(0) - \dot{\varphi}_{\pm}^{(1)}(0)). \quad (\text{B.28})$$

Inserting (B.22) and (B.23) into (B.14) and (B.15) we determine:

$$\begin{aligned} \alpha(t) = & \frac{\lambda_+(t)\dot{\varphi}_-^{(4)}(t) - \lambda_-(t)\dot{\varphi}_+^{(4)}(t)}{D(t)} + (\alpha_+ - \alpha_-) \frac{\mu_+(t)\lambda_-(t) - \mu_-(t)\lambda_+(t)}{\tilde{D}(t)D(t)} e^{(\alpha_+ - \alpha_-)t} + \\ & + \frac{\alpha_+(\alpha_- + \omega)e^{\alpha_+ t} - \alpha_-(\alpha_+ - \omega)e^{-\alpha_- t}}{\tilde{D}(t)}, \end{aligned} \quad (\text{B.29})$$

$$\begin{aligned} \beta(t) = & -\frac{\lambda_+(t)\dot{\varphi}_-^{(4)}(t) - \lambda_-(t)\dot{\varphi}_+^{(4)}(t)}{D(t)} - (\alpha_+ - \alpha_-) \frac{\mu_+(t)\lambda_-(t) - \mu_-(t)\lambda_+(t)}{\tilde{D}(t)D(t)} e^{(\alpha_+ - \alpha_-)t} + \\ & + \frac{\alpha_+(\alpha_- + \omega)e^{\alpha_+ t} - \alpha_-(\alpha_+ - \omega)e^{-\alpha_- t}}{\tilde{D}(t)}, \end{aligned} \quad (\text{B.30})$$

$$\begin{aligned} \gamma(t) = & -\frac{\lambda_+(t)\dot{\varphi}_-^{(1)}(t) - \lambda_-(t)\dot{\varphi}_+^{(1)}(t)}{D(t)} + (\alpha_+ - \alpha_-) \frac{\mu_+(t)\lambda_-(t) - \mu_-(t)\lambda_+(t)}{\tilde{D}(t)D(t)} e^{(\alpha_+ - \alpha_-)t} + \\ & + \frac{\alpha_+(\alpha_- + \omega)e^{\alpha_+ t} - \alpha_-(\alpha_+ - \omega)e^{-\alpha_- t}}{\tilde{D}(t)}, \end{aligned} \quad (\text{B.31})$$

$$\begin{aligned} \delta(t) = & \frac{\lambda_+(t)\dot{\varphi}_-^{(1)}(t) - \lambda_-(t)\dot{\varphi}_+^{(1)}(t)}{D(t)} - (\alpha_+ - \alpha_-) \frac{\mu_+(t)\lambda_-(t) - \mu_-(t)\lambda_+(t)}{\tilde{D}(t)D(t)} e^{(\alpha_+ - \alpha_-)t} + \\ & + \frac{\alpha_+(\alpha_- + \omega)e^{\alpha_+ t} - \alpha_-(\alpha_+ - \omega)e^{-\alpha_- t}}{\tilde{D}(t)}. \end{aligned} \quad (\text{B.32})$$

(The argument in all the functions is the instant t .)

At this point we are ready to confirm some of the claims presented in the main text. We must distinguish two cases. The first is when:

$$\frac{\lambda^2}{4m^2} \geq \omega^2 - 2\frac{\gamma(0)}{m} \quad (\text{B.33})$$

In such a case α_{\pm} are real and consequently $\varphi_{\pm}^{(4)} = (\varphi_{\pm}^{(1)})^*$. Observing that $\lambda_{\pm} = \lambda_{\pm}^*$, $\mu_{\pm} = -\mu_{\pm}^*$ we immediately see that:

$$\gamma^* = \beta, \quad \delta^* = \alpha \quad (\text{B.34})$$

and

$$\Im\alpha(t) = -\Im\beta(t). \quad (\text{B.35})$$

When

$$\frac{\lambda^2}{4m^2} < \omega^2 - 2\frac{\gamma(0)}{m} \quad (\text{B.36})$$

we observe that $\alpha_+ = -\alpha_-^*$, $\varphi_{\pm}^{(4)} = (\varphi_{\mp}^{(1)})^*$, and since λ_{\pm}, μ_{\pm} turn out to be the same as in the case (B.33), we verify once again the relations (B.34) and (B.35).

When α_{\pm} are real we straightforwardly obtain:

$$\Im\alpha(t) = \frac{\sigma}{m} \left[\frac{e^{(\alpha_+ - \alpha_-)t/2}}{D(t)} f_1(t) + \frac{e^{(\alpha_+ - \alpha_-)t}}{D^2(t)} f_2(t) \right], \quad (\text{B.37})$$

$$\Re\alpha(t) = 2\frac{d}{dt} \ln D(t), \quad (\text{B.38})$$

with

$$f_1(t) = \frac{1}{\sinh[(\alpha_+ + \alpha_-)t/2]} \left\{ (\alpha_+ - \omega) \left[t - \frac{1 - e^{-(\alpha_+ + \alpha_-)t}}{(\alpha_+ + \alpha_-)} \right] + \right. \\ \left. + (\alpha_- + \omega) \left[\frac{e^{(\alpha_+ + \alpha_-)t} - 1}{(\alpha_+ + \alpha_-)} - t \right] \right\} \quad (\text{B.39})$$

and

$$f_2(t) = \frac{\alpha_+ + \alpha_-}{\sinh[(\alpha_+ + \alpha_-)t/2]} \left\{ (\alpha_- + \omega) \left[te^{(\alpha_+ + \alpha_-)t/2} - 2\frac{\sinh[(\alpha_+ + \alpha_-)t/2]}{\alpha_+ + \alpha_-} \right] + \right. \\ \left. + (\alpha_+ - \omega) \left[2\frac{\sinh[(\alpha_+ + \alpha_-)t/2]}{\alpha_+ + \alpha_-} - te^{-(\alpha_+ + \alpha_-)t/2} \right] \right\}. \quad (\text{B.40})$$

The last relations confirm that $\Im\alpha > 0$. For $t\alpha_{\pm} \gg 1$ it is easy to check that $\Im\alpha$ and $\Re\alpha$ become constants:

$$\Im\alpha \approx \frac{2\sigma}{m} \frac{1}{\alpha_+ + \alpha_-}, \quad \Re\alpha \approx 2\alpha_+. \quad (\text{B.41})$$

The last relation holds as long as $\alpha_+ + \alpha_- \neq 0$. If $\alpha_+ + \alpha_- = 0$ that is if

$$\frac{\lambda^2}{4m^2} + 2\frac{\gamma(0)}{m} = \omega^2 \quad (\text{B.42})$$

we immediately find that

$$\Im\alpha = \frac{\sigma}{m} \frac{2t}{1 + (\omega - \lambda/2m)t} \underset{t \rightarrow \infty}{\approx} \frac{\sigma}{m} \frac{2}{\omega - \lambda/m}, \quad (\text{B.43})$$

$$\Re\alpha = 2 \frac{\omega + t(\omega - \lambda/2m)\lambda/2m}{1 + t(\omega - \lambda/2m)} \underset{t \rightarrow \infty}{\approx} 2\omega. \quad (\text{B.44})$$

When α_{\pm} are complex we find that:

$$\Im\alpha(t) = \frac{\sigma}{m} \frac{1}{[(\omega - \lambda/2m) \sin kt + k \cos kt]^2} \left\{ [k^2 + (\omega - \lambda/2m)^2] t + \right. \\ \left. + 2(\omega - \lambda/2m) \sin^2 kt + [k^2 - (\omega - \lambda/2m)^2] \frac{\sin 2kt}{2k} \right\}. \quad (\text{B.45})$$

Using the fact $x/\sin x \geq 1$ once again we can verify that $\Im\alpha(t) > 0$. It also straightforward to see that:

$$\Re\alpha(t) = 2 \frac{d}{dt} \ln D(t) = 2 \frac{[\frac{\lambda}{2m}(\omega - \lambda/2m) - k^2] \sin kt + k \cos kt}{(\omega - \lambda/2m) \sin kt + k \cos kt}, \quad (\text{B.46})$$

where we have noted

$$k^2 = \omega^2 - \frac{2\gamma(0)}{m} - \frac{\lambda^2}{4m^2}. \quad (\text{B.47})$$

Appendix C

FORTRAN Codes

In this appendix we give the FORTRAN 77 programs, we have used in the numerical calculations.

C.1 The Entropy in the Adiabatic Approach

```
program adiabatic
implicit real*8(a-h,o-z)
real*8 ksi
dimension y(2),dy(2),ytc(2)
dimension yi(2),dyi(2),ysci(2)
dimension ysol1(10000),ysol2(10000),tsol(10000)
dimension ysoli1(10000),ysoli2(10000)
common/delay/tinv
common/flag/iflag
common/param/p1,p2,tau
external derivs

open(8,file='K1L10T1.dat')

p1=1.d0
p2=5.d0
tau=1

istmax=1000
t_0=0
t_f=20 ! Is the tau we define
htry=(t_f-t_0)/dble(istmax)
```

```

        epsi=1.d-6

do i=1,2
    ysc(i)=1.d0
enddo

    t_r=0.d0
y(1)=0.d0
    y(2)=1.d0
iflag=0

istep=0
1     istep=istep+1

call derivs(t_r,y,dy)
    call bsstep(y,dy,2,t_r,htry,epsi,ysc,hdid,hnext)
tsol(istep)=t_r
    ysol1(istep)=y(1)
    ysol2(istep)=y(2)
if(istep.lt.istmax) goto 1

iflag=1

do j=1,istmax
    t_i=0.d0
    yi(1)=0.d0
    yi(2)=1.d0
    do i=1,2
        ysci(i)=1.d0
    enddo
    tinv=tsol(j)
    jstmax=j
    jstep=0
2     jstep=jstep+1
    call derivs(t_i,yi,dyi)
    call bsstep(yi,dyi,2,t_i,htry,epsi,ysci,hdid,hnext)
    if(jstep.lt.jstmax) goto 2
    ysoli1(j)=yi(1)
    ysoli2(j)=yi(2)
    print*,j,t_i,tsol(jstmax)
enddo

```

```

do i=1,istmax

! The type of entropy is :
  t=tsol(i)
  d0=ysol1(i)
  d0d=ysol2(i)
  di=ysoli1(i)
  did=-ysoli2(i)
  s1=(1.d0+d0d*did)**2
  s2=2.d0+d0d**2+did**2
  s3=d0**2+d0d**2
  sh=4.d0*dsqrt((d0**4+s1+d0**2*s2)/(d0**2*s3**2))
  snom=d0**4+s1+d0**2*(-4.d0+s2)
  sden=s1+d0**4*(1.d0+sh)+d0**2*(6.d0+did**2+d0d**2*(1.d0+sh))
  ksi=snom/sden

  s=-dlog(1.d0-ksi)-((dlog(ksi)*ksi)/(1.d0-ksi))
  write(8,15) t,ksi,s
enddo

15 format(3(4x,g13.7))
close(8)
stop
end

SUBROUTINE BSSTEP(Y,DYDX,NV,X,HTRY,EPS,YSCAL,HDID,HNEXT)
IMPLICIT REAL*8(A-H,O-Z)
PARAMETER (NMAX=10,IMAX=11,NUSE=7,ONE=1.DO,SHRINK=.95D0,GROW=1.2D0
*)
DIMENSION Y(NV),DYDX(NV),YSCAL(NV),YERR(NMAX),
* YSAV(NMAX),DYSAV(NMAX),YSEQ(NMAX),NSEQ(IMAX)
DATA NSEQ /2,4,6,8,12,16,24,32,48,64,96/
H=HTRY
XSAV=X
DO 11 I=1,NV
YSAV(I)=Y(I)
DYSAV(I)=DYDX(I)
11 CONTINUE
1 DO 10 I=1,IMAX
CALL MMID(YSAV,DYSAV,NV,XSAV,H,NSEQ(I),YSEQ)

```



```

XEST=(H/NSEQ(I))**2
CALL RZEXTR(I,XEST,YSEQ,Y,YERR,NV,NUSE)
ERRMAX=0.DO
DO 12 J=1,NV
ERRMAX=DMAX1(ERRMAX,DABS(YERR(J)/YSCAL(J)))
12 CONTINUE
ERRMAX=ERRMAX/EPS
IF(ERRMAX.LT.ONE) THEN
X=X+H
HDID=H
IF(I.EQ.NUSE) THEN
HNEXT=H*SHRINK
ELSE IF(I.EQ.NUSE-1) THEN
HNEXT=H*GROW
ELSE
HNEXT=(H*NSEQ(NUSE-1))/NSEQ(I)
ENDIF
RETURN
ENDIF
10 CONTINUE
H=0.25D0*H/2**((IMAX-NUSE)/2)
IF(X+H.EQ.X) PAUSE 'Step size underflow.'
GOTO 1
END

SUBROUTINE MMID(Y,DYDX,NVAR,XS,HTOT,NSTEP,YOUT)
  IMPLICIT REAL*8(A-H,O-Z)
  PARAMETER (NMAX=10)
  DIMENSION Y(NVAR),DYDX(NVAR),YOUT(NVAR),YM(NMAX),YN(NMAX)
  H=HTOT/NSTEP
  DO 11 I=1,NVAR
    YM(I)=Y(I)
    YN(I)=Y(I)+H*DYDX(I)
11 CONTINUE
  X=XS+H
  CALL DERIVS(X,YN,YOUT)
  H2=2.DO*H
  DO 13 N=2,NSTEP
    DO 12 I=1,NVAR
      SWAP=YM(I)+H2*YOUT(I)
      YM(I)=YN(I)

```

```

        YN(I)=SWAP
12    CONTINUE
        X=X+H
        CALL DERIVS(X, YN, YOUT)
13    CONTINUE
        DO 14 I=1, NVAR
            YOUT(I)=0.5DO*(YM(I)+YN(I)+H*YOUT(I))
14    CONTINUE
        RETURN
        END

SUBROUTINE RZEXTR( IEST, XEST, YEST, YZ, DY, NV, NUSE)
    IMPLICIT REAL*8(A-H, O-Z)
    PARAMETER (IMAX=11, NMAX=10, NCOL=7)
    DIMENSION X(IMAX), YEST(NV), YZ(NV), DY(NV), D(NMAX, NCOL), FX(NCOL)
    X(IEST)=XEST
    IF(IEST.EQ.1) THEN
        DO 11 J=1, NV
            YZ(J)=YEST(J)
            D(J,1)=YEST(J)
            DY(J)=YEST(J)
11    CONTINUE
    ELSE
        M1=MIN(IEST, NUSE)
        DO 12 K=1, M1-1
            FX(K+1)=X(IEST-K)/XEST
12    CONTINUE
        DO 14 J=1, NV
            YY=YEST(J)
            V=D(J,1)
            C=YY
            D(J,1)=YY
            DO 13 K=2, M1
                B1=FX(K)*V
                B=B1-C
                IF(B.NE.0.DO) THEN
                    B=(C-V)/B
                    DDY=C*B
                    C=B1*B
                ELSE
                    DDY=V

```

```

        ENDIF
        V=D(J,K)
        D(J,K)=DDY
        YY=YY+DDY
13      CONTINUE
        DY(J)=DDY
        YZ(J)=YY
14      CONTINUE
ENDIF
RETURN
END

SUBROUTINE DERIVS(T,YN,DYN)
  IMPLICIT REAL*8(A-H,O-Z)
  DIMENSION YN(2),DYN(2)
  common/delay/tinv
  common/flag/iflag
  common/param/p1,p2,tau
  if(iflag.eq.0) then
    DYN(1)=YN(2)
    DYN(2)=- (p1+p2*dtanh(T/tau))*YN(1)
  else
    DYN(1)=YN(2)
    DYN(2)=- (p1+p2*dtanh((tinv-T)/tau))*YN(1)
  endif
RETURN
END

```

C.2 The Oscillation Amplitude of Entropy in the Adiabatic Approximation

```

program adiamp
implicit real*8(a-h,o-z)
real*8 ksi,smin,smax,sdif
real ::ss(30000)
real ::vima
integer ::ii
dimension y(2),dy(2),y(2)
dimension yi(2),dyi(2),ysci(2)

```

```

dimension ysol1(30000),ysol2(30000),tsol(30000)
dimension ysoli1(30000),ysoli2(30000)
common/delay/tinv
common/flag/iflag
common/param/p1,p2,tau
external derivs

open (9,file='new1150.dat')
p1=1.d0
  p2=10.d0
  tau=1
vima=1

do ii=1,50
  tau=tau+vima

istmax=10000
  t_0=0.d0
  t_f=200.d0 ! Is the tau we define
  htry=(t_f-t_0)/dble(istmax)
  epsi=1.d-6

do i=1,2
  ysc(i)=1.d0
enddo

  t_r=0.d0
y(1)=0.d0
  y(2)=1.d0
iflag=0

istep=0
  1 istep=istep+1

call derivs(t_r,y,dy)
  call bsstep(y,dy,2,t_r,htry,epsi,ysc,hdid,hnext)
tsol(istep)=t_r
  ysol1(istep)=y(1)
  ysol2(istep)=y(2)
if(istep.lt.istmax) goto 1

```

```

iflag=1

do j=1,istmax
  t_i=0.d0
  yi(1)=0.d0
  yi(2)=1.d0
  do i=1,2
    ysci(i)=1.d0
  enddo
  tinv=tsol(j)
  jstmax=j
  jstep=0
2  jstep=jstep+1
  call derivs(t_i,yi,dyi)
  call bsstep(yi,dyi,2,t_i,htry,epsi,ysci,hdid,hnext)
  if(jstep.lt.jstmax) goto 2
  ysoli1(j)=yi(1)
  ysoli2(j)=yi(2)
enddo

textcolor[rgb]0.00,0.00,1.00do i=1,istmax

! The Entropy Is :
  t=tsol(i)
  d0=ysol1(i)
  d0d=ysol2(i)
  di=ysoli1(i)
  did=-ysoli2(i)
  s1=(1.d0+d0d*did)**2
  s2=2.d0+d0d**2+did**2
  s3=d0**2+d0d**2
  sh=4.d0*dsqrt((d0**4+s1+d0**2*s2)/(d0**2*s3**2))
  snom=d0**4+s1+d0**2*(-4.d0+s2)
  sden=s1+d0**4*(1.d0+sh)+d0**2*(6.d0+did**2+d0d**2*(1.d0+sh))
  ksi=snom/sden

  ss(i)=s

enddo

15 format(3(4x,g13.7))

```

```

do i=1,istmax-2
  if (ss(i)<ss(i+1).and.ss(i+1)>ss(i+2)) then
    smax=ss(i+1)
  end if
  if (ss(i)>ss(i+1).and.ss(i+1)<ss(i+2)) then
    smin=ss(i+1)
  end if
end do
sdif=(smax-smin)/2
print*,ii,tau,sdif
write (9,*) tau,sdif

end do

stop
end

SUBROUTINE BSSTEP(Y,DYDX,NV,X,HTRY,EPS,YSCAL,HDID,HNEXT)
IMPLICIT REAL*8(A-H,O-Z)
PARAMETER (NMAX=10,IMAX=11,NUSE=7,ONE=1.DO,SHRINK=.95D0,GROW=1.2D0
*)
DIMENSION Y(NV),DYDX(NV),YSCAL(NV),YERR(NMAX),
* YSAV(NMAX),DYSAV(NMAX),YSEQ(NMAX),NSEQ(IMAX)
DATA NSEQ /2,4,6,8,12,16,24,32,48,64,96/
H=HTRY
XSAV=X
DO 11 I=1,NV
YSAV(I)=Y(I)
DYSAV(I)=DYDX(I)
11 CONTINUE
1 DO 10 I=1,IMAX
CALL MMID(YSAV,DYSAV,NV,XSAV,H,NSEQ(I),YSEQ)
XEST=(H/NSEQ(I))**2
CALL RZEXTR(I,XEST,YSEQ,Y,YERR,NV,NUSE)
ERRMAX=0.DO
DO 12 J=1,NV
ERRMAX=DMAX1(ERRMAX,DABS(YERR(J)/YSCAL(J)))
12 CONTINUE
ERRMAX=ERRMAX/EPS
IF(ERRMAX.LT.ONE) THEN
X=X+H

```

```

HDID=H
IF (I.EQ.NUSE) THEN
HNEXT=H*SHRINK
ELSE IF (I.EQ.NUSE-1) THEN
HNEXT=H*GROW
ELSE
HNEXT=(H*NSEQ(NUSE-1))/NSEQ(I)
ENDIF
RETURN
ENDIF
10 CONTINUE
H=0.25D0*H/2**((IMAX-NUSE)/2)
IF(X+H.EQ.X) PAUSE 'Step size underflow.'
GOTO 1
END

SUBROUTINE MMID(Y,DYDX,NVAR,XS,HTOT,NSTEP,YOUT)
  IMPLICIT REAL*8(A-H,O-Z)
  PARAMETER (NMAX=10)
  DIMENSION Y(NVAR),DYDX(NVAR),YOUT(NVAR),YM(NMAX),YN(NMAX)
  H=HTOT/NSTEP
  DO 11 I=1,NVAR
    YM(I)=Y(I)
    YN(I)=Y(I)+H*DYDX(I)
11 CONTINUE
  X=XS+H
  CALL DERIVS(X,YN,YOUT)
  H2=2.D0*H
  DO 13 N=2,NSTEP
    DO 12 I=1,NVAR
      SWAP=YM(I)+H2*YOUT(I)
      YM(I)=YN(I)
      YN(I)=SWAP
12 CONTINUE
  X=X+H
  CALL DERIVS(X,YN,YOUT)
13 CONTINUE
  DO 14 I=1,NVAR
    YOUT(I)=0.5D0*(YM(I)+YN(I)+H*YOUT(I))
14 CONTINUE
  RETURN

```

```

END

SUBROUTINE RZEXTR( IEST, XEST, YEST, YZ, DY, NV, NUSE)
  IMPLICIT REAL*8(A-H, O-Z)
  PARAMETER (IMAX=11, NMAX=10, NCOL=7)
  DIMENSION X(IMAX), YEST(NV), YZ(NV), DY(NV), D(NMAX, NCOL), FX(NCOL)
  X(IEST)=XEST
  IF(IEST.EQ.1) THEN
    DO 11 J=1, NV
      YZ(J)=YEST(J)
      D(J,1)=YEST(J)
      DY(J)=YEST(J)
11  CONTINUE
  ELSE
    M1=MIN(IEST, NUSE)
    DO 12 K=1, M1-1
      FX(K+1)=X(IEST-K)/XEST
12  CONTINUE
    DO 14 J=1, NV
      YY=YEST(J)
      V=D(J,1)
      C=YY
      D(J,1)=YY
      DO 13 K=2, M1
        B1=FX(K)*V
        B=B1-C
          IF(B.NE.0.D0) THEN
            B=(C-V)/B
            DDY=C*B
            C=B1*B
          ELSE
            DDY=V
          ENDIF
          V=D(J,K)
          D(J,K)=DDY
          YY=YY+DDY
13  CONTINUE
      DY(J)=DDY
      YZ(J)=YY
14  CONTINUE
  ENDIF

```



```
RETURN
END

SUBROUTINE DERIVS(T,YN,DYN)
  IMPLICIT REAL*8(A-H,O-Z)
  DIMENSION YN(2),DYN(2)
  common/delay/tinv
  common/flag/iflag
  common/param/p1,p2,tau
  if(iflag.eq.0) then
    DYN(1)=YN(2)
    DYN(2)=- (p1+p2*dtanh(T/tau))*YN(1)
  else
    DYN(1)=YN(2)
    DYN(2)=- (p1+p2*dtanh((tinv-T)/tau))*YN(1)
  endif
RETURN
END
```

Bibliography

- [1] J.J. Sakurai: *Modern Quantum Mechanics*, Addison-Wesley
- [2] M.A. Nielsen, I.L. Chuang: *Quantum Computation and Quantum Information*, Cambridge University Press
- [3] J. Preskill: *Quantum Information and Computation*, Lecture Notes
- [4] H. Breuer, F. Petruccione: *The Theory of Open Quantum Systems*, Oxford
- [5] E. B. Davies: *Quantum Theory of Open Systems*, Academic Press
- [6] G. Lindblad: *On the Generator of Quantum Dynamical Semigroups*, Comm. Math. Phys. **48** (1976) 119
- [7] D.A. Lidar, Z. Bihary, K.B. Whaley: *From completely positive maps to the quantum Markovian semigroup master equation*, Chem. Phys. **268** (2001) 35
- [8] A. Buchleitner, C. Viviescas, M. Tiersch: *Entanglement and Decoherence*, Springer
- [9] D. Giulini, E. Joos, C. Kiefer, J. Kupsch, I.-O. Stamatescu, H.D. Zeh: *Decoherence and the Appearance of a Classical World in Quantum Theory*, Springer
- [10] S. Kryszewski, J. Czechowska-Kryszk: *Master Equation - Tutorial Approach*, quant-ph/0801.1757
- [11] I. Bengtsson, K. Życzkowski: *Geometry of Quantum States*, Cambridge University Press
- [12] U. Weiss: *Quantum Dissipative Systems*, World Scientific
- [13] R.P. Feynman, F.L. Vernon: *The Theory of a General Quantum System Interacting With a Linear Dissipative System*, Ann. Phys. **24** (1963) 118

- [14] A.O. Caldeira, A.J. Leggett: *Path Integral Approach to Quantum Brownian Motion*, Physica A **121** (1983) 587
- [15] H. Grabert, P. Scramm, G. Ingold: *Quantum Brownian Motion: The Functional Integral Approach*, Phys. Rep. **168** (1988) 115
- [16] P. Scramm, H. Grabert: *Low-Temperature and Long-Time Anomalies of a Damped Quantum Particle*, Jour. Stat. Phys. **49** (1987) 767
- [17] C. Morais Smith, A.O. Caldeira: *Generalized Feynman-Vernon Approach to Dissipative Quantum Systems*, Phys. Rev. A **36** (1987) 3509
- [18] K.Chou, Z. Su, B. Hao, L. Yu: *Equilibrium and Nonequilibrium Formalisms Made Unified*, Phys. Rep. **118** (1985) 1
- [19] Z. Su, L. Chen, X. Yu, K. Chou: *Influence Functional and Closed-Time-Path Green's Function*, Phys. Rev. B **37** (1988) 9810
- [20] B.L. Hu, J.P. Paz, Y. Zhang: *Quantum Brownian Motion in General Environment: Exact Master Equation with Nonlocal Dissipation and Colored Noise*, Phys. Rev. D **45** (1992) 2843
- [21] C. Anastopoulos, B.L. Hu: *Two-Level Atom-Field Interaction: Exact Master Equations for Non-Markovian Dynamics, Decoherence and Relaxation*, Phys. Rev. A **62** (2000) 033821
- [22] C. Chou, T. Yu, B.L. Hu: *Exact Master Equation and Quantum Decoherence of Two Coupled Harmonic Oscillators in a General Environment*, Phys. Rev. E **77** (2008) 011112
- [23] R.P. Feynman, A.R. Hibbs: *Quantum Mechanics and Path Integrals*, McGraw-Hill
- [24] H. Kleinert: *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, World Scientific
- [25] L.S. Schulman: *Techniques and Applications of Path Integration*, Dover
- [26] M. Chaichian, A. Demishev: *Path Integrals in Physics*, Taylor & Francis
- [27] A. Stern, Y. Aharonov, Y. Imry: *Phase Uncertainty and Loss of Interference: A General Picture*, Phys. Rev. A **41** (1990) 3436
- [28] D. Loss, K. Mullen: *Dephasing by a Dynamic Asymmetric Environment*, Phys. Rev. B **43** (1991) 13252

- [29] H. Breuer, A. Ma, F. Petruccione: *Time-Local Master Equations: Influence Functional and Cumulant Expansion*, quant-ph/0209153
- [30] E. Wigner: *On the Quantum Correction for Thermodynamic Equilibrium*, Phys. Rev. **40** (1932) 749
- [31] M. Srednicki: *Entropy and Area*, Phys. Rev. Lett. **71** (1993) 666
- [32] C. Callan, F. Wilczek: *On Geometric Entropy*, Phys. Lett. B **333** (1994) 55
- [33] P. Calabrese, J. Cardy: *Entanglement Entropy and Quantum Field Theory*, J. Stat. Mech. **06** (2004) 06002
- [34] S. Ryu, T. Takayanagi: *Aspects of Holographic Entanglement Entropy*, hep-th/0605073
- [35] K. Fujikawa: *Adiabatic Approximation in the Second Quantized Formulation*, Phys. Rev. D **77** (2008) 045006
- [36] H.G. Dosch: *Gluon Condensate and Effective Linear Potential*, Phys. Lett. B **190** (1987) 177
- [37] H.G. Dosch and Yu.A. Simonov: *The Area Law of the Wilson Loop and Vacuum Field Correlators*, Phys. Lett. B **205** (1988) 339
- [38] Yu.A. Simonov: *Vacuum Background Fields in QCD as a Source of Confinement*, Nucl. Phys. B **307** (1988) 512
- [39] A. Di Giacomo, H.G. Dosch, V.I. Shevchenko, Yu.A. Simonov: *Field correlators in QCD. Theory and applications*, Phys. Rep. **372** (2002) 319