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Complex Dynamics in Quantum Systems
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HGSFP
Heidelberg Graduate School of Fundamental Physics

Tutorial on Quantum Chaos

nHS, Phil.weg 12, Wednesday 14:15 - 16:00 (SS 2012)

Problem Sheet 7

Problem 15 – WKB approximation

Given the 1D potentials

$$V_1(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2 \quad , \quad x > 0 \\ \infty \quad , \quad x < 0 \end{cases} \tag{1}$$

$$V_2(x) = \begin{cases} \alpha x \quad , \quad x > 0 \quad (\alpha > 0) \\ \infty \quad , \quad x < 0 \end{cases} \tag{2}$$

a) show that for the outer turning point a

$$\int_0^a p(x)\, dx = \pi\hbar \left(n + \frac{3}{4}\right) \qquad (n \in \mathbb{N}_0)$$

implicitly defines the WKB-quantized energy levels.

b) compute the two series of WKB energy levels $E(n)$ and extract the scaling of the density of states (the inverse level distance) with the energy E.

c) Imagine a point mass with $m = 0.1\,\mathrm{kg}$; in which quantum state n would it be if you let it fall from a height of $h = 1\,\mathrm{m}$ above the floor? How large would h be in the quantum mechanical ground state with $n = 0$?

Problem 16 – EKB quantization

For $V(r) = -\frac{e^2}{r}$ we get in spherical coordinates (r, θ, ϕ), with $r \in [0, \infty), \theta \in [0, \pi], \phi \in [0, 2\pi)]$, the following Hamiltonian

$$H = \frac{\vec{p}^{\,2}}{2m} - \frac{e^2}{r} = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) - \frac{e^2}{r}\,.$$

a) Show that

$$p_r = m\dot{r} \quad , \quad p_\theta = mr^2\dot{\theta} \quad , \quad p_\phi = mr^2\sin(\theta)\dot{\phi} \;\; (= L_z)\,.$$

b) Show that the system is integrable by finding three constants of the motion.

c) Compute for $E < 0$:

$$I_r = \frac{1}{2\pi}\oint p_r\,dr \quad , \quad I_\theta = \frac{1}{2\pi}\oint p_\theta\,d\theta \quad , \quad I_\phi = \frac{1}{2\pi}\oint p_\phi\,d\phi\,.$$

Hints:
Determine first the turning points for the motion corresponding to the tori actions $I_{r,\theta,\phi}$, i.e. for $V_{\mathrm{eff}}(r) = -\frac{e^2}{r} + \frac{L^2}{2mr^2}$ with $(\vec{L}^2 = L^2)$, from $\sin^2\theta_{1,2} = (L_z/L)^2 \leq 1$, and for the cyclic (!) coordinate ϕ.

d) Use the identities

$$\vec{L}^2 = L^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} = p_\theta^2 + \frac{L_z^2}{\sin^2\theta} \qquad (L \leq 0, \;\; -L \leq L_z \leq L)$$

to obtain the quantized energies in the form

$$E = -\frac{me^4}{2} \frac{1}{(I_r + I_\theta + I_\phi)^2}\,.$$

Use now the quantization conditions of EKB to arrive at the usual well-known quantum mechanical eigenenergies of the Coulomb problem.