Sharp heat kernel estimates for higher-order operators with singular coefficients

G. Barbatis

Abstract

We obtain heat kernel estimates for higher order operators with singular/degenerate operators with measurable coefficients. Precise contants are given, which are sharp for small times.
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1 Introduction

Let

$$Hf(x) = (-1)^m \sum_{\substack{|\alpha|=m \\ |\beta|=m}} D^\alpha \{a_{\alpha\beta}(x) D^\beta f(x)\}, \quad x \in \Omega \subset \mathbf{R}^N$$

be a self-adjoint uniformly elliptic operator of order $2m$ with measurable coefficients and subject to Dirichlet boundary conditions on $\partial\Omega$. In [D2] it was shown that if $2m > N$ then the associated heat semigroup e^{-Ht} has a kernel $K(t,x,y)$ which satisfies the estimate

$$|K(t,x,y)| < c_1 t^{-N/2m} \exp\left\{ -c_2 \frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}} + c_3 t \right\}$$

for some positive constants c_i. Under suitable conditions this was recently [B2] sharpened to

$$|K(t,x,y)| < c_\epsilon t^{-N/2m} \exp\left\{ -(\sigma_m - cD - \epsilon) \frac{d_M(x,y)^{2m/(2m-1)}}{t^{1/(2m-1)}} + c_{\epsilon,M} t \right\} \qquad (1)$$

where $\sigma_m = (2m-1)(2m)^{-2m/(2m-1)} \sin(\pi/(4m-2))$, $D \geq 0$ depends on the regularity of the coefficients and $d_M(x,y)$ is a Finsler-type metric induced by the principal symbol of H and depending on the arbitrarily large parameter M; as $M \to \infty$, $d_M(x,y)$ increases to a Finsler distance $d(x,y)$, but (1) is valid only for $M < \infty$. This estimate is sharp as is seen by comparison against the small-time asymptotics for operators with smooth coefficients obtained in [T] – see (13) below. In the same direction Dungey [Du] used resolvent estimates to obtain a better esimate than (1) for powers of second order operators. He showed in a general framework that if the self-adjoint operator H satisfies a standard Gaussian esimate with exponential constant $\frac{1}{4}-\epsilon$ then the heat kernel of H^m satisfies (1) with $D=0$ and $M=+\infty$. For an alternative approach valid also for higher order systems see [AQ].

In the main theorem of this article we extend (1) in two directions. Primarily, we consider operators whose coefficients can be singular and/or degenerate on $\partial\Omega$; moreover, we do not assume H to be self-adjoint. Concerning the singularity or degeneracy of H, we assume that there is a positive function $a(x)$ that controls in a suitable sense the behaviour of the coefficient matrix $\{a_{\alpha\beta}\}$ and we then impose two conditions (H1) and (H2) on $a(x)$. The first is a weighted Sobolev embedding and the second is a weighted interpolation inequality. These conditions were introduced in [B1] and led to (non-sharp) off diagonal estimates on the heat kernel of non-uniformly elliptic self-adjoint operators. Besides conditions (H1) and (H2) we shall assume that the symbol $A(x,\xi)$ is close – in a suitable sense – to a certain class of 'good' symbols denoted by $\mathcal{G}_a$. These symbols, besides satisfying (H1) and (H2) correspond to operators that are self-adjoint, their coefficients have some local regularity, and are strongly convex in the sense of [EP]. We make use of a certain stability property inherent in our approach and obtain bounds that are asymptotically sharp: they involve the exponential constant $\sigma_m - cD$ where c is an absolute constant and D is the distance of the symbol $A(x,\xi)$ from the class $\mathcal{G}_a$ in a certain weighted norm. In particular the constant σ_m is obtained for symbols in $\mathcal{G}_a$. To our knowledge such estimates are new even if the coefficients are assumed to be smooth and the symbol lies in $\mathcal{G}_a$.

2 Formulation of results

We first fix some notation. Given a multi-index $\alpha=(\alpha_1,\ldots,\alpha_N)$ we write $\alpha! = \alpha_1!\ldots\alpha_n!$ and $|\alpha|=\alpha_1+\cdots+\alpha_n$. We write $\gamma\leq\alpha$ to indicate that $\gamma_i\leq\alpha_i$ for all i, in which case we also set $c^\alpha_\gamma=\alpha!/\gamma!(\alpha-\gamma)!$. We use the standard notation D^α for the differential expression $(\partial/\partial x_1)^{\alpha_1}\ldots(\partial/\partial x_N)^{\alpha_N}$ and for $k\geq 0$ we denote by $\nabla^k f$ the vector $(D^\alpha f)_{|\alpha|=k}$. We denote by $\hat{f}$ the Fourier transform of a function f, $\hat{f}(\xi)=(2\pi)^{-N/2}\int e^{i\xi\cdot x}f(x)dx$. We shall denote by $\|A\|_{p\to q}$ the norm of an operator A from $L^p(\Omega)$ to $L^q(\Omega)$. The letter c will stand for a positive constant whose value may change from line to line.

Let Ω be a domain in $\mathbf{R}^N$. We fix an integer $m\geq 1$ and consider the operator

$$Hf(x)=(-1)^m\sum_{\substack{|\alpha|=m\\|\beta|=m}}D^\alpha\{a_{\alpha\beta}(x)D^\beta f(x)\} \qquad (2)$$

subject to Dirichlet boundary conditions on $\partial\Omega$; the precise definition shall be given below. The matrix-valued function $\{a_{\alpha\beta}\}$ is assumed to be measurable and to take its values in the set of all complex, $\nu\times\nu$-matrices, ν being the number of multi-indices α of length $|\alpha|=m$. We assume that each $a_{\alpha\beta}$ lies in $L^\infty_{loc}(\Omega)$; we do not assume $\{a_{\alpha\beta}\}$ to be self-adjoint.

We define a quadratic form $Q(\cdot)$ on $C_c^\infty(\Omega)$ by

$$Q(f) = \int_\Omega \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x) D^\alpha f(x) D^\beta \bar{f}(x)\, dx, \quad f \in C_c^\infty(\Omega).$$

We assume that there exists a positive weight $a(x)$ with $a^{\pm 1} \in L^\infty_{loc}(\Omega)$ that controls the size of the matrix $\{a_{\alpha\beta}\}$ in the following sense: first,

$$|a_{\alpha\beta}(x)| \leq c a(x), \qquad x \in \Omega, \tag{3}$$

for all multi-indices α, β; and second, the weighted Gårding's inequality

$$\text{Re } Q(f) \geq c \int_\Omega a(x) |\nabla^m f|^2 dx, \quad f \in C_c^\infty(\Omega) \tag{4}$$

is valid for some $c > 0$. We also assume the symbol-version of (4), namely

$$\text{Re } A(x, \xi) \geq c\, a(x) |\xi|^{2m}, \quad x \in \Omega,\ \xi \in \mathbf{R}^N, \tag{5}$$

where $A(x,\xi) := \sum a_{\alpha\beta}(x) \xi^{\alpha+\beta}$. Relations (3) and (4) imply in particular that there exists $\beta > 0$ such that

$$|Q(f)| \leq \beta \, \text{Re } Q(f), \qquad f \in C_c^\infty(\Omega). \tag{6}$$

It is easily seen that Q is closable [B1]. The domain of its closure is a weighted Sobolev space which we denote by $W^{m,2}_{a,0}(\Omega)$. We retain the same symbol, Q, for the closure of the above form and denote by H the associated accretive operator on $L^2(\Omega)$, so that $\langle Hf, f \rangle = Q(f)$, $f \in \text{Dom}(H)$, and (2) is valid in a weak sense.

We make two hypotheses on the weight a: the first is a weighted Sobolev inequality and the second is a weighted interpolation inequality.

(H1) There exists $s \in [N/2m, 1]$ and $c > 0$ such that

$$\|f\|_\infty \leq c[\text{Re } Q(f)]^{s/2} \|f\|_2^{1-s}, \quad f \in C_c^\infty(\Omega). \tag{7}$$

(H2) There exists a constant c such that

$$\int_\Omega a^{k/m} |\nabla^k f|^2 dx < \epsilon \int_\Omega a |\nabla^m f|^2 dx + c\epsilon^{-k/(m-k)} \int_\Omega |f|^2 dx, \tag{8}$$

for all $0 < \epsilon < 1$, $0 \leq k < m$ and all $f \in C_c^\infty(\Omega)$.

Both (H1) and (H2) are satisfied when H is uniformly elliptic, in which case the best value for the constant s is $s = N/2m$, showing that in the general case we cannot expect any value that is better (smaller) than $N/2m$; in particular (H1) is valid trivially with $s = N/2m$ if $a(x)$ is bounded away from zero. We refer to [B1] for non-trivial examples for which (H1) and (H2) are satisfied; they involve suitable powers of either $1 + |x|$ or $\text{dist}(x, K)$ where K is a smooth surface of lower dimension.

We note that condition (H2) implies that for any k, l with $0 \leq k, l \leq m$, $k + l < 2m$, there exists a constant c so that

$$(1 + \lambda^{2m-k-l}) \int_\Omega a^{(k+l)/2m} |\nabla^k f| |\nabla^l f| dx < \epsilon \text{Re } Q(f) + c\epsilon^{-\frac{k+l}{2m-k-l}} (1 + \lambda^{2m}) \|f\|_2^2, \tag{9}$$

for all $\epsilon \in (0,1)$, $\lambda > 0$ and all $f \in C_c^\infty(\Omega)$. Indeed, for $\lambda = 1$ (9) is a consequence of (H2) and the Cauchy-Schwarz inequality; the case $\lambda < 1$ follows trivially from the case $\lambda = 1$; finally writing (9) for $\lambda = 1$ and replacing ϵ by $\epsilon\lambda^{k+l-2m}$ we obtain the result for $\lambda > 1$.

We next introduce the distance that shall be used in the heat kernel estimates. Consider the set

$$\mathcal{E}_a = \{\phi \in C^\infty(\Omega) \cap L^\infty(\Omega) \ : \ a^{k/2m}\nabla^k\phi \in L^\infty(\Omega), \ 1 \le k \le m\}$$

and its subset (recall (5))

$$\begin{aligned}\mathcal{E}_{A,M} = \{\phi \in C^\infty(\Omega) \cap L^\infty(\Omega) : \ & \mathrm{Re}\, A(x, \nabla\phi(x)) \le 1, \\ & |\nabla^k\phi(x)| \le Ma(x)^{-k/2m}, \ 2 \le k \le m, \ \text{a.e. } x \in \Omega\}. \end{aligned} \tag{10}$$

Our estimates will be expressed in terms of the distance

$$d_M(x,y) = \sup\{\phi(y) - \phi(x) \ : \ \phi \in \mathcal{E}_{A,M}\} \tag{11}$$

for arbitrarily large but finite M. For $M = +\infty$ this reduces to the distance

$$d_\infty(x,y) = \sup\{\phi(y) - \phi(x) \ : \ \mathrm{Re}\, A(x, \nabla\phi(x)) \le 1, \ x \in \Omega\}.$$

This is a Finsler distance, induced by the (singular/degenerate) Finsler metric with length element

$$ds = ds(x, dx) = \sup_{\substack{\eta \in \mathbf{R}^N \\ \eta \ne 0}} \frac{\langle dx, \eta \rangle}{(\mathrm{Re}\, A(x,\eta))^{1/2m}}. \tag{12}$$

We refer the reader to the recent book [BCS] for a comprehensive introduction to Finsler geometry. The distance $d_\infty(x,y)$ relates to the short-time off-diagonal behaviour of the heat kernel: it was shown in [T] that if $\Omega = \mathbf{R}^N$ and H is self adjoint uniformly elliptic with strongly convex symbol (see 14)), then $d_\infty(\cdot,\cdot)$ controls the small-time behaviour of $K(t,x,y)$ in the sense that

$$\log t^{N/2m} K(t,x,y) = -\sigma_m \frac{d_\infty(x,y)^{2m/(2m-1)}}{t^{1/(2m-1)}}(1 + o(1)), \qquad \text{as } t \to 0 \tag{13}$$

for x, y fixed and close enough; here and below we have

$$\sigma_m = (2m-1)(2m)^{-2m/(2m-1)} \sin(\pi/(4m-2)).$$

Let us now proceed with the definition of the class $\mathcal{G}_a$. Let the functions $a_\gamma(\cdot)$, $|\gamma| = 2m$, be defined by requiring that

$$\sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x)\xi^{\alpha+\beta} = \sum_{|\gamma|=2m} c_\gamma^{2m} a_\gamma(x)\xi^\gamma, \quad x \in \Omega, \quad \xi \in \mathbf{R}^N;$$

(recall that $c_\gamma^{2m} = (2m)!/\gamma!$). Following [EP] we say that the principal symbol $A(x,\xi)$ of H is strongly convex if the quadratic form

$$\Gamma(x,p) = \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha+\beta}(x) p_\alpha \overline{p_\beta}, \quad p = (p_\alpha) \in \mathbf{C}^\nu, \tag{14}$$

is positive semidefinite for a.e. $x \in \Omega$.

Induced by the weight $a(x)$ is the weighted Sobolev space

$$W_a^{m-1,\infty}(\Omega) = \{f \in W_{loc}^{m-1,\infty}(\Omega) \,:\, |\nabla^i f(x)| \le c a(x)^{\frac{2m-i}{2m}},\ \text{a.e. } x \in \Omega, i \le m-1\}. \tag{15}$$

Definition. We say that the symbol $A(x,\xi)$ lies in $\mathcal{G}_a$ if

(i) $A(x,\xi)$ is strongly convex

(ii) $\{a_{\alpha\beta}\}$ is real and symmetric

(iii) the coefficients $a_{\alpha\beta}$ lie in $W_a^{m-1,\infty}(\Omega)$.

We denote by D the distance of the coefficient matrix $\{a_{\alpha\beta}\}$ from $\mathcal{G}_a$ in the weighted uniform norm

$$\|f\|_{a,\infty} := \sup_{x\in\Omega} |f(x)/a(x)|,$$

that is

$$D = \inf_{\{\tilde{a}_{\alpha\beta}\}\in W_{a,\mathrm{Re}}^{m-1,\infty}} \|\{a_{\alpha\beta}\} - \{\tilde{a}_{\alpha\beta}\}\|_{a,\infty}, \tag{16}$$

where, as usual, $\|\{a_{\alpha\beta}\}\|_{a,\infty} := \|\, |\{a_{\alpha\beta}\}|_{M(\nu\times\nu)}\, \|_{a,\infty}$.

Our main result is the following:

Theorem 1 *Assume that (H1) and (H2) are satisfied. Then for all $\delta \in (0,1)$ and all M large there exist positive constants $c_\delta, c_{\delta,M}$ such that*

$$|\, K(t,x,y)\,| < c_\delta t^{-s} \exp\Big\{-(\sigma_m - cD - \delta) d_M(x,y)^{\frac{2m}{2m-1}} t^{-\frac{1}{2m-1}} + c_{\delta,M} t\Big\} \tag{17}$$

for all $x,y \in \Omega$ and $t > 0$; the constant c is independent of x, y, t, δ, D and M.

In the special case where H is uniformly elliptic and self-adjoint this estimate has already been obtained in [B2].

3 Proof of Theorem 1

Given $\phi \in \mathcal{E}_a$ the mapping $f \mapsto e^{\phi} f$ maps $W_{a,0}^{m,2}(\Omega)$ into itself [B1, Lemma 7]. Hence one can define a sesquilinear form $Q_\phi(\cdot,\cdot)$ with domain $W_{a,0}^{m,2}(\Omega)$ by

$$\begin{aligned} Q_\phi(f) &= Q(e^{\phi} f, e^{-\phi} f) && (18)\\ &= \int_\Omega \sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta} D^\alpha(e^{\phi} f) D^\beta(e^{-\phi}\overline{f}) b\, dx, \qquad f \in W_{a,0}^{m,2}(\Omega). && (19)\end{aligned}$$

The associated operator is $H_\phi = e^{-\phi} H e^{\phi}$ and has domain $\mathrm{Dom}(H_\phi) = e^{-\phi}\mathrm{Dom}(H)$. The form Q_ϕ is a lower order perturbation of Q (cf. (28)) and it is a consequence of (H2) [B1, Lemma 8] that for all $\epsilon > 0$ and $f \in W_{a,0}^{m,2}(\Omega)$ there holds

$$|Q(f) - Q_\phi(f)| < \epsilon \mathrm{Re}\, Q(f) + c\epsilon^{-2m+1}(1 + p(\phi))^{2m} \|f\|_2^2, \tag{20}$$

where we have used the seminorm

$$p(\phi) := \sup_{1\le k\le m} \operatorname{ess\,sup}_{x\in\Omega} a(x)^{k/2m}|\nabla^k\phi(x)|. \tag{21}$$

Defining $s(\phi) = (1 + p(\phi))^{2m}$ it follows in particular that

$$\text{Re } Q_\phi(f) \ge -c\, s(\phi)\|f\|_2^2, \qquad f \in C_c^\infty(\Omega), \tag{22}$$

where c is independent of ϕ, and this justifies the definition

$$-k_\phi = \inf\{\text{Re } Q_\phi(f) \ : \ f \in C_c^\infty(\Omega), \ \ \|f\|_2 = 1\}. \tag{23}$$

The next lemma follows closely an argument used in [BD].

Lemma 2 *Assume that (H2) is satisfied. Then for any $\phi \in \mathcal{E}_a$ there holds*

$$\text{(i)} \qquad \|e^{-H_\phi t}\|_{2\to 2} \le e^{k_\phi t}; \tag{24}$$

$$\text{(ii)} \qquad \|H_\phi e^{-H_\phi t}\|_{2\to 2} \le \frac{c_\delta}{t} e^{k_\phi t} e^{\delta s(\phi)t}, \ \textit{for all } \delta > 0, \tag{25}$$

where the constant c_δ is independent of $\phi \in \mathcal{E}_a$ and $t > 0$.

Proof. Part (i) is the standard energy estimate that follows by integrating

$$\frac{d}{dt}\|e^{-H_\phi t}f\|_2^2 = -2\text{Re } \langle H_\phi e^{-H_\phi t}f, e^{-H_\phi t}f\rangle \le 2k_\phi\|e^{-H_\phi t}f\|_2^2.$$

Now by (20) there holds

$$|Q_\phi(f) - Q(f)| \le \frac{1}{2}\text{Re } Q(f) + s(\phi)\|f\|_2^2, \quad f \in C_c^\infty(\Omega), \tag{26}$$

where, we recall, $s(\phi) = c(1+p(\phi)^{2m})$ for some fixed $c > 0$. Hence for any $\epsilon \in (0, 1)$

$$\begin{aligned} \text{Re } Q_\phi(f) &= \epsilon\text{Re } Q_\phi(f) + (1-\epsilon)\text{Re } Q_\phi(f) \\ &\ge \frac{\epsilon}{2}\text{Re } Q(f) - [\epsilon s(\phi) + (1-\epsilon)k_\phi]\,\|f\|_2^2 \end{aligned}$$

and hence

$$\text{Re } [Q(f) - Q_\phi(f)] \le (1 - \frac{\epsilon}{2})\text{Re } Q(f) + [\epsilon s(\phi) + (1-\epsilon)k_\phi]\|f\|_2^2.$$

Fix $f \in L^2(\Omega)$ and $\theta \in (-\pi/2, \pi/2)$ and for $\rho > 0$ set $f_\rho = \exp(-H_\phi\rho e^{i\theta})f$. We then have

$$\begin{aligned} \frac{d}{d\rho}\|f_\rho\|_2^2 &= -2\text{Re } [e^{i\theta}Q_\phi(f_\rho)] \\ &= -2\cos\theta\text{Re } Q(f_\rho) + 2\sin\theta\text{Im } Q_\phi(f_\rho) + \\ &\quad +2\cos\theta[\text{Re } Q(f_\rho) - \text{Re } Q_\phi(f_\rho)] + \\ &\le -2\cos\theta\text{Re } Q(f_\rho) + 2\sin|\theta| \left[(\frac{1}{2} + \beta)\text{Re } Q(f_\rho) + s(\phi)\|f_\rho\|_2^2\right] + \\ &\quad +2\cos\theta\left[(1 - \frac{\epsilon}{2})\text{Re } Q(f_\rho) + [\epsilon s(\phi) + (1-\epsilon)k_\phi]\|f_\rho\|_2^2\right] \\ &= [-\epsilon\cos\theta + (2\beta+1)\sin|\theta|]\,\text{Re } Q(f_\rho) + \\ &\quad + [2\cos\theta\{\epsilon s(\phi) + (1-\epsilon)k_\phi\} + 2\sin|\theta| s(\phi)]\,\|f_\rho\|_2^2. \end{aligned}$$

Let $\alpha \in (0,\pi/2)$ be such that $\tan\alpha = \epsilon/(2\beta+1)$. For $|\theta| \leq \alpha$ we then have $-\epsilon\cos\theta + (2\beta+1)\sin|\theta| \leq 0$ and hence

$$\begin{aligned}
\frac{d}{d\rho}\|f_\rho\|_2^2 &\leq 2\cos\theta[\epsilon s(\phi) + (1-\epsilon)k_\phi + s(\phi)\frac{\epsilon}{2\beta+1}]\|f_\rho\|_2^2 \\
&\leq 2\left(k_\phi + 2\epsilon s(\phi)\right)\|f_\rho\|_2^2 \\
&=: 2A_\epsilon\|f_\rho\|_2^2.
\end{aligned}$$

It follows that $\|e^{-H_\phi z}\|_{2\to 2} \leq e^{A_\epsilon |z|}$ in the sector $|\arg z| \leq \alpha$. We conclude that letting $\tau_\epsilon = \frac{A_\epsilon}{\cos\alpha}$ we have

$$\|\exp\{-(H_\phi + \tau_\epsilon)z\}\|_{2\to 2} \leq 1,$$

and hence [D1, Lemma 2.38]

$$\|(H_\phi + \tau_\epsilon)e^{-(H_\phi+\tau_\epsilon)t}\| \leq \frac{c}{\alpha t},$$

for all $t > 0$. Multiplying both sides by $e^{\tau_\epsilon t}$ and using the triangle inequality we obtain

$$\|H_\phi e^{-H_\phi t}\|_{2\to 2} \leq \frac{c}{\alpha t}\exp\{\frac{k_\phi + 2\epsilon s(\phi)}{\cos\alpha}t\} + \tau_\epsilon e^{k_\phi t}.$$

This last expression can be made smaller than the right hand side of (25) provided ϵ is chosen small enough; this completes the proof. //

Proposition 3 *Assume that (H1) and (H2) are satisfied. Then for any $\delta > 0$ there exists $c_\delta > 0$ independent of $\phi \in \mathcal{E}_a$ such that*

$$\|e^{-H_\phi t}\|_{1\to\infty} \leq c_\delta t^{-s} e^{k_\phi t} e^{\delta s(\phi) t}. \tag{27}$$

Proof. Let $f \in L^2(\Omega)$ and set $f_t = e^{-H_\phi t} f$, $t > 0$. Using (H1) we have

$$\begin{aligned}
\|f_t\|_\infty &\leq c[\mathrm{Re}\ Q(f_t)]^{s/2}\|f_t\|_2^{1-s} \\
\text{(by (26))} \quad &\leq c\left[\mathrm{Re}\ Q_\phi(f_t) + s(\phi)\|f_t\|_2^2\right]^{s/2}\|f_t\|_2^{1-s} \\
&\leq c\left[\|H_\phi f_t\|_2\|f_t\|_2 + s(\phi)\|f_t\|_2^2\right]^{s/2}\|f_t\|_2^{1-s} \\
\text{(by (25), (24))} \quad &\leq c\left[\frac{c_\epsilon}{t}e^{\epsilon s(\phi)t} + s(\phi)\right]^{s/2} e^{k_\phi t}\|f\|_2 \\
&= ct^{-s/2}\left[c_\epsilon e^{\epsilon s(\phi)t} + s(\phi)t\right]^{s/2} e^{k_\phi t}\|f\|_2.
\end{aligned}$$

Taking ϵ to be small enough we conclude that given $\delta > 0$ there exists c_δ such that

$$\|e^{-H_\phi t}\|_{2\to\infty} \leq c_\delta t^{-s/2} e^{k_\phi t} e^{\delta s(\phi)t}.$$

The same arguments are valid for $H_\phi^* = H_{-\phi}$, the constant k_ϕ clearly staying the same. Hence by duality and the semigroup property (27) follows. //

In order for Proposition 3 to be useful we need a precise upper estimate on k_ϕ, which amounts to a precise lower estimate on Re $Q_\phi(\cdot)$, cf. (23). This will be established in Lemma 10 following a series of intermediate lemmas. Recalling that $c^\alpha_\gamma = \alpha!/\gamma!(\alpha-\gamma)!$ it follows immediately from (19) that for $\lambda >$, $\phi \in \mathcal{E}_a$ we have

$$Q_{\lambda\phi}(f) = \int_\Omega \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} \sum_{\substack{\gamma\le\alpha \\ \delta\le\beta}} c^\alpha_\gamma c^\beta_\delta P_{\gamma,\lambda\phi} P_{\delta,-\lambda\phi} D^{\alpha-\gamma} f D^{\beta-\delta} \bar{f}\, dx, \tag{28}$$

where

$$P_{\gamma,\lambda\phi}(x) := e^{-\lambda\phi(x)} D^\gamma [e^{\lambda\phi(x)}]$$

is a polynomial in various derivatives of $\lambda\phi$. Now, the induction relation $P_{\gamma+e_j,\lambda\phi} = (\lambda\partial_j\phi + \partial_j)P_{\gamma,\lambda\phi}$ implies that $P_{\gamma,\lambda\phi}$ has the form

$$P_{\gamma,\lambda\phi}(x) = \sum_{k=1}^{|\gamma|} \lambda^k \sum c_{\gamma;\gamma_1,\ldots,\gamma_k} (D^{\gamma_1}\phi)\ldots(D^{\gamma_k}\phi), \tag{29}$$

where the second sum is taken over all non-zero multiindices $\gamma_1,\ldots,\gamma_k$ such that $\gamma_1+\cdots+\gamma_k = \gamma$ and $c_{\gamma;\gamma_1,\ldots,\gamma_k}$ are constants. Hence, recalling that $|\nabla^k\phi| \le ca^{-k/2m}$, we can write $P_{\gamma,\lambda\phi}(x) = \sum_{k=1}^{|\gamma|} \lambda^k \tilde{P}_{k,\phi}(x)$ where $|\tilde{P}_{k,\phi}(x)| \le ca^{-|\gamma|/2m}$. It follows from (28) that

$$Q_{\lambda\phi}(f) = \int_\Omega \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\substack{\gamma\le\alpha \\ \delta\le\beta}} \sum_{\substack{k\le|\gamma| \\ j\le|\delta|}} \lambda^{k+j} w_{\alpha\beta\gamma\delta kj}(x) D^{\alpha-\gamma} f D^{\beta-\delta} \bar{f}\, b\, dx, \tag{30}$$

where $w_{\alpha\beta\gamma\delta kj} := a_{\alpha\beta} c^\alpha_\gamma c^\beta_\delta \tilde{P}_{k,\phi}\tilde{P}_{j,-\phi}$ satisfies $|w_{\alpha\beta\gamma\delta kj}| \le ca^{(2m-|\gamma+\delta|)/2m}$. Replacing γ and δ by $\alpha-\gamma$ and $\beta-\delta$ correspondingly we conclude from (30) the following

Lemma 4 *$Q_{\lambda\phi}(f)$ is a linear combination of terms of the form*

$$T(f) = \lambda^s \int_\Omega w(x) D^\gamma f D^\delta \overline{f}\, b\, dx, \tag{31}$$

where $|w| \le ca^{\frac{|\gamma+\delta|}{2m}}$ on Ω and

(i) *s is an integer between 0 and $2m$;*
(ii) *γ and δ are multiindices with $|\gamma|,|\delta| \le m$;*
(iii) *$s + |\gamma+\delta| \le 2m$.*

Definition. We call the number $s + |\gamma+\delta|$ the *essential order* of T.
Hence the essential order is an integer between 0 and $2m$. We denote by $\mathcal{L}_{a,m}$ the linear space consisting of (finite) linear combinations of forms whose essential order is smaller than $2m$. In Lemma 9 we will see that terms in $\mathcal{L}_{a,m}$ are in a sense negligible. We also point out for later use that (9) implies the interpolation inequality

$$|T(f)| < c\{\mathrm{Re}\, Q(f) + \lambda^{2m}\|f\|_2^2\}, \quad f \in W^{m,2}_{a,0}(\Omega), \tag{32}$$

valid for all terms $T(\cdot)$ of essential order $2m$.

We have the following

Lemma 5 *Given* $\phi \in \mathcal{E}_a$ *and* $\lambda > 0$ *define*

$$Q_{1,\lambda\phi}(f) = \int_\Omega \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\substack{\gamma\leq\alpha \\ \delta\leq\beta}} a_{\alpha\beta} c^\alpha_\gamma c^\beta_\delta (\lambda\nabla\phi)^\gamma (-\lambda\nabla\phi)^\delta D^{\alpha-\gamma} f D^{\beta-\delta} \bar{f}\, dx.$$

Then the difference $Q_{\lambda\phi}(f) - Q_{1,\lambda\phi}(f)$ *lies in* $\mathcal{L}_{a,m}$.

Proof. One simply has to recall (28) and observe from (29) that $P_{\gamma,\lambda\phi}$, considered as a polynomial in λ, has $\lambda^{|\gamma|}(\nabla\phi)^\gamma$ as its highest-degree term. //

3.1 Symbols in $\mathcal{G}_a$

At this point and for the whole of this subsection we restrict our attention to operators H whose symbol belongs to $\mathcal{G}_a$. For $x \in \Omega$, $\xi, \eta \in \mathbf{C}^N$ and $\zeta \in \mathbf{R}^N$ let us define

$$\begin{aligned} k_m &= [\sin(\pi/(4m-2))]^{-2m+1} \\ A(x,\xi,\eta) &= \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) \xi^\alpha \bar{\eta}^\beta, \\ S(x,\zeta;\xi,\eta) &= \mathrm{Re}\, A(x, \xi - i\zeta, \eta + i\zeta) + k_m \mathrm{Re}\, A(x,\zeta). \end{aligned}$$

Lemma 6 *Assume that the symbol* $A(x,\xi)$ *lies in* $\mathcal{G}_a$. *Then*

$$\begin{aligned} &\mathrm{Re}\, Q_{1,\lambda\phi}(f) + k_m \lambda^{2m} \int_\Omega \mathrm{Re}\, A(x, \nabla\phi(x)) |f|^2 dx = \\ = \;& (2\pi)^{-N} \iiint_{\Omega\times\mathbf{R}^N\times\mathbf{R}^N} S(x, \lambda\nabla\phi; \xi, \eta) e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)}\, dx\, d\xi\, d\eta \qquad (33) \end{aligned}$$

for all $\phi \in \mathcal{E}_a$, $\lambda > 0$ *and* $f \in C_c^\infty(\Omega)$.

Proof. Writing $D^\gamma f(x) = (2\pi)^{-N/2} \int_{\mathbf{R}^N} (i\xi)^\gamma e^{i\xi\cdot x} \hat{f}(\xi) d\xi$ we have

$$\begin{aligned} Q_{1,\lambda\phi}(f) &= (2\pi)^{-N} \iiint_{\Omega\times\mathbf{R}^N\times\mathbf{R}^N} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} \sum_{\substack{\gamma\leq\alpha \\ \delta\leq\beta}} c^\alpha_\gamma c^\beta_\delta (-i\lambda\nabla\phi)^\gamma (-i\lambda\nabla\phi)^\delta \times \\ &\qquad \times \xi^{\alpha-\gamma} \eta^{\beta-\delta} e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)}\, d\xi\, d\eta\, dx \\ &= (2\pi)^{-N} \iiint_{\Omega\times\mathbf{R}^N\times\mathbf{R}^N} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} (\xi - i\lambda\nabla\phi)^\alpha (\eta - i\lambda\nabla\phi)^\beta \times \\ &\qquad \times e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)}\, d\xi\, d\eta\, dx \\ &= (2\pi)^{-N} \iiint_{\Omega\times\mathbf{R}^N\times\mathbf{R}^N} A(x, \xi - i\lambda\nabla\phi(x), \eta + i\lambda\nabla\phi(x)) \times \\ &\qquad \times e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)}\, d\xi\, d\eta\, dx. \end{aligned}$$

This last integral has the form $\int_\Omega q[g] dx$ where for fixed $x \in \Omega$

$$\left\{ \begin{array}{l} g(\xi) = e^{i\xi\cdot x} \hat{f}(\xi) d\xi \\ q[g] = \int_{\mathbf{R}^N\times\mathbf{R}^N} p(\xi,\eta) g(\xi) \overline{g(\eta)} d\xi\, d\eta \\ p(\xi,\eta) = A(x, \xi - i\lambda\nabla\phi(x), \eta + i\lambda\nabla\phi(x)) \end{array} \right.$$

Since the matrix $\{a_{\alpha\beta}\}$ is symmetric we have $p(\xi,\eta)=p(\eta,\xi)$ and therefore $\overline{q(g)}=\int_{\mathbf{R}^N\times\mathbf{R}^N}\overline{p(\xi,\eta)g(\xi)}g(\eta)d\xi\,d\eta$. Hence $\mathrm{Re}\; q(g)=\int_{\mathbf{R}^N\times\mathbf{R}^N}\mathrm{Re}\; p(\xi,\eta)d\xi\,d\eta$ and integration over $x\in\Omega$ yields

$$
\begin{aligned}
&\mathrm{Re}\; Q_{1,\lambda\phi}(f)+k_m\int_\Omega \mathrm{Re}\; A(x,\lambda\nabla\phi(x))|f|^2\,dx \\
=\;& (2\pi)^{-N}\iiint_{\Omega\times\mathbf{R}^N\times\mathbf{R}^N}\mathrm{Re}\,[A(x,\xi-i\lambda\nabla\phi(x),\eta+i\lambda\nabla\phi(x))+k_mA(x,\lambda\nabla\phi)]\times \\
&\qquad\qquad\times e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}\,d\xi\,d\eta\,dx \\
=\;& (2\pi)^{-N}\iiint_{\Omega\times\mathbf{R}^N\times\mathbf{R}^N}S(x,\lambda\nabla\phi;\xi,\eta)e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}\,d\xi\,d\eta\,dx. \qquad //
\end{aligned}
$$

We now proceed to estimate the triple integral in the right hand side of (33). It is shown in [EP, Theorem 2.1] that there exist positive numbers $w_0,\ldots,w_{m-2}$ such that

$$
S(x,\zeta;\xi,\xi)=\sum_{s=0}^{m-2}w_s\Gamma(x,p^{(s)}_{\xi,\zeta}),\quad x\in\Omega\;\;\zeta,\xi\in\mathbf{R}^N, \tag{34}
$$

where $\Gamma(x,\cdot)$ is the quadratic form associated to the principal symbol of H (cf. (14)) and $p^{(s)}_{\xi,\zeta}$ is the vector in $\mathbf{R}^\nu$ defined for fixed $\xi,\zeta\in\mathbf{R}^N$ by requiring that

$$
\sum_{|\alpha|=m}p^{(s)}_{\xi,\zeta,\alpha}a^\alpha=(\sin\theta_m)^{-s-2}(\xi\cdot a)^{m-s-2}(\zeta\cdot a)^s\left\{(\sin\theta_m)^2(\xi\cdot a)^2-(\cos\theta_m)^2(\zeta\cdot a)^2\right\} \tag{35}
$$

for all $a\in\mathbf{R}^N$; here $\theta_m=\pi/(4m-2)$. To simplify the notation let us define the sesquilinear forms $\mathbf{\Gamma}(x,\cdot,\cdot)$ on $\mathbf{C}^{m-1}\otimes\mathbf{C}^\nu\simeq\mathbf{C}^{\nu(m-1)}$ by

$$
\mathbf{\Gamma}(x,u,v)=\sum_{s=0}^{m-2}w_s\Gamma(x,u^{(s)},v^{(s)})=\sum_{s=0}^{m-2}\sum_{\substack{|\alpha|=m\\|\beta|=m}}w_sa_{\alpha+\beta}(x)u^{(s)}_\alpha\overline{v^{(s)}_\beta}
$$

for all $u=(u^{(s)}_\alpha)$, $v=(v^{(s)}_\beta)\in\mathbf{C}^{\nu(m-1)}$. Then $\mathbf{\Gamma}$ is positive semi-definite by the strong convexity of $A(x,\xi)$. To handle the above expressions we introduce two auxiliary elliptic differential forms $S_{\lambda\phi}$ and $\Gamma_{\lambda\phi}$ on $L^2(\Omega)$. They have common domain $W^{m,2}_{a,0}(\Omega)$ and are given by

$$
S_{\lambda\phi}(f)=(2\pi)^{-N}\iiint_{\Omega\times\mathbf{R}^N\times\mathbf{R}^N}S(x,\lambda\nabla\phi;\xi,\eta)e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}\,d\xi\,d\eta\,dx, \tag{36}
$$

$$
\Gamma_{\lambda\phi}(f)=(2\pi)^{-N}\iiint_{\Omega\times\mathbf{R}^N\times\mathbf{R}^N}\mathbf{\Gamma}(x,p_{\xi,\lambda\nabla\phi},p_{\eta,\lambda\nabla\phi})e^{i(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}\,d\xi\,d\eta\,dx \tag{37}
$$

where $p_{\xi,\lambda\nabla\phi}=(p^{(s)}_{\xi,\lambda\nabla\phi,\alpha})^{|\alpha|=m}_{0\le s\le m-2}\in\mathbf{C}^{\nu(m-1)}$ is defined by (35).

Lemma 7 *Assume that the symbol $A(x,\xi)$ lies in $\mathcal{G}_a$. Then the form $S_{\lambda\phi}(\cdot)-\Gamma_{\lambda\phi}(\cdot)$ lies in $\mathcal{L}_{a,m}$.*

Proof. It follows from (34) that $S_{\lambda\phi}$ and $\Gamma_{\lambda\phi}$ have integral kernels which are polynomials of ξ and η and whose values coincide for $\xi = \eta$. Using the inverse Fourier transform this implies that the difference $S_{\lambda\phi}(f) - \Gamma_{\lambda\phi}(f)$ is a linear combination of terms of the form

$$T(f) = \lambda^s \int_\Omega w(x)[D^{\gamma+\kappa} f D^\delta \overline{f} - (-1)^\kappa D^\gamma f D^{\delta+\kappa}\overline{f}]dx, \tag{38}$$

where w is some function and κ is a multi-index of length $|\kappa| \leq m-1$. In fact, recalling (33) and the definition of $Q_{1,\lambda\phi}$ we see that $w = a_{\alpha\beta}(\nabla\phi)^\mu$ where $|\mu| = s$ and $\gamma+\delta+\kappa+\mu = \alpha+\beta$. Since $a_{\alpha\beta} \in W_a^{m-1,\infty}(\Omega) \subset W_{\rm loc}^{m-1,\infty}(\Omega)$ we can integrate by parts $|\kappa|$ times and use Leibnitz' rule to obtain

$$T(f) = (-1)^{|\kappa|}\lambda^s \sum_{0<\kappa_1\leq\kappa} c^\kappa_{\kappa_1} \int_\Omega D^{\kappa_1} w\, D^\gamma f\, D^{\delta+\kappa-\kappa_1}\overline{f} dx. \tag{39}$$

We estimate $D^{\kappa_1}w$: clearly

$$|D^{\kappa_1}(a_{\alpha\beta}(\nabla\phi)^\mu)| \leq c\sum_{i=0}^{|\kappa_1|} |\nabla^{|\kappa_1|-i} a_{\alpha\beta}|\, |\nabla^i(\nabla\phi)^\mu| \qquad \text{in } \Omega.$$

Recalling the definition of $\mathcal{E}_{A,M}$ it is easily seen that $|\nabla^i(\nabla\phi)^\mu| \leq c a^{-(|\mu|+i)/2m}$; recalling also from (15) the definition of the space $W_a^{m-1,\infty}(\Omega)$ where the $a_{\alpha\beta}$ lie we conclude that

$$|D^{\kappa_1}(a_{\alpha\beta}(\nabla\phi)^\mu)| \leq c_M a(x)^{\frac{2m-|\kappa_1+\mu|}{2m}} = c_M a^{\frac{|\gamma+\delta+\kappa-\kappa_1|}{2m}}.$$

Hence (39) implies that T has essential order $s + |\gamma+\delta+\kappa-\kappa_1| < 2m$, as required. //

Proposition 8 *Let $A(x,\xi) \in \mathcal{G}_a$. Then for any $\phi \in \mathcal{E}_a$, $\lambda > 0$ and all $f \in C_c^\infty(\Omega)$, there holds*

$$\text{Re } Q_{\lambda\phi}(f) \geq -k_m\lambda^{2m} \int_\Omega A(x,\nabla\phi(x))|f|^2 dx + T(f) \tag{40}$$

where $T(\cdot) \in \mathcal{L}_{a,m}$.

Proof. Combining Lemmas 5, 6 and 7 we have

$$\text{Re } Q_{\lambda\phi}(f) + k_m \int_\Omega \text{Re } A(x,\lambda\nabla\phi(x))|f|^2 dx = \Gamma_{\lambda\phi}(f) + T(f), \tag{41}$$

for a form $T(\cdot) \in \mathcal{L}_{a,m}$. Now let $u(x) = \int_{\mathbf{R}^N} p_{\xi,\lambda\nabla\phi} e^{i\xi\cdot x}\hat{f}(\xi)d\xi$ (a $\mathbf{C}^{\nu(m-1)}$-valued integral defined component-wise); it follows immediately from definition (37) that

$$\Gamma_{\lambda\phi}(f) = \int_\Omega \mathbf{\Gamma}(x, u(x), u(x))dx \tag{42}$$

and hence $\Gamma_{\lambda\phi}(\cdot)$ is non-negative by the strong convexity of $A(x,\xi)$. //

3.2 The general case

We now remove the assumption $A \in \mathcal{G}_a$ and return to the general setting described in Section 2. We recall that the quantity D measures the distance of A from $\mathcal{G}_a$ and has been defined in (16).

Lemma 9 *Let $T \in \mathcal{L}_{a,m}$. Then for any $\epsilon \in (0,1)$ there holds*

$$|T(f)| < \epsilon\{\mathrm{Re}\, Q(f) + \lambda^{2m}\|f\|_2^2\} + c_\epsilon \|f\|_2^2 \tag{43}$$

for all $\lambda > 0$ and $f \in C_c^\infty(\Omega)$.

Proof. By definition, $T(f)$ is a finite linear combination of expressions of the form

$$I(f) = \lambda^s \int_\Omega w(x) D^\gamma f(x) D^\delta \bar{f}(x) dx,$$

where $|w(x)| \leq c a(x)^{|\gamma+\delta|/2m}$ and $s + |\gamma+\delta| \leq 2m-1$. Setting $\mu^{2m-|\gamma+\delta|} = \lambda^s$ and recalling (9) we have

$$\begin{aligned}
|I(f)| &\leq c\mu^{2m-|\gamma+\delta|} \int_\Omega a(x)^{|\gamma+\delta|/2m} |D^\gamma f||D^\delta f| dx \\
&\leq \epsilon \mathrm{Re}\, Q(f) + c\epsilon^{-2m+1}(1+\mu^{2m})\|f\|_2^2 \\
&\leq \epsilon \mathrm{Re}\, Q(f) + c\epsilon^{-2m+1}(1+\lambda^{2m-1})\|f\|_2^2 \\
&\leq \epsilon\{\mathrm{Re}\, Q(f) + \lambda^{2m}\|f\|_2^2\} + c\epsilon^{-4m^2+1}\|f\|_2^2. \qquad //
\end{aligned}$$

Remark. It is seen from the proof that the size of the constant c_ϵ in (43) depends only on $\epsilon > 0$ and the (finite) quantity $\max_I \sup\{|w(x)|a(x)^{-|\gamma+\delta|/2m}\}$ where the max is taken over all forms $I(\cdot)$ that make up $T(\cdot)$. In particular, when we restrict our attention to functions $\phi \in \mathcal{E}_{A,M}$ we obtain a constant $c_\epsilon = c_{\epsilon,M}$ which is otherwise independent of ϕ.

Lemma 10 *For any $\phi \in \mathcal{E}_{A,M}$, $\lambda > 0$ and $\epsilon > 0$ and all , there holds*

$$\mathrm{Re}\, Q_{\lambda\phi}(f) \geq -\Big\{(k_m + cD + \epsilon)\lambda^{2m} + c_{\epsilon,M}\Big\}\|f\|_2^2, \quad f \in C_c^\infty(\Omega). \tag{44}$$

where the constant c is independent of D, M, ϵ, λ and ϕ and the constant $c_{M,D,\epsilon}$ is independent of λ and ϕ.

Proof. Let $\tilde{A} \in \mathcal{G}_a$ be such that $\|A - \tilde{A}\|_{a,\infty} \leq 2D$. It follows from (32) that

$$\begin{cases} |\mathrm{Re}\, \tilde{Q}_{\lambda\phi}(f) - \mathrm{Re}\, Q_{\lambda\phi}(f)| < cD\{\mathrm{Re}\, Q(f) + \lambda^{2m}\|f\|_2^2\} \\ \left|\lambda^{2m} \int_\Omega [A(x,\nabla\phi(x)) - \tilde{A}(x,\nabla\phi(x))]dx\right| < cD\{\mathrm{Re}\, Q(f) + \lambda^{2m}\|f\|_2^2\}. \end{cases}$$

Combining these relations with (40) – as applied to the operator $\tilde{H}$ – we obtain

$$\mathrm{Re}\, Q_{\lambda\phi}(f) \geq -k_m\lambda^{2m} \int_\Omega \mathrm{Re}\, A(x,\nabla\phi(x))|f|^2 dx - cD\{\mathrm{Re}\, Q(f) + \lambda^{2m}\|f\|_2^2\} + T(f).$$

We have Re $A(x, \nabla\phi(x)) \le 1$ and therefore (allowing c to change from line to line and ϵ to rescale)

$$\begin{array}{rcl} \text{Re } Q_{\lambda\phi}(f) & \ge & -k_m\lambda^{2m}\|f\|_2^2 - cD\{\text{Re } Q(f) + \lambda^{2m}\|f\|_2^2\} + T(f) \\ \text{(by (43))} & \ge & -k_m\lambda^{2m}\|f\|_2^2 - (cD+\epsilon)\{\text{Re } Q(f) + \lambda^{2m}\|f\|_2^2\} - c_{\epsilon,M}\|f\|_2^2 \\ \text{(by (26))} & \ge & -k_m\lambda^{2m}\|f\|_2^2 - (cD+\epsilon)\{\text{Re } Q_{\lambda\phi}(f) + \lambda^{2m}\|f\|_2^2\} - c_{\epsilon,M}\|f\|_2^2. \end{array}$$

Now, either Re $Q_{\lambda\phi}(f)$ is positive, in which case (44) is true, or it is non-positive, in which case it can be discarded from the right hand side of the last inequality. This completes the proof. //

Proof of Theorem 1. The rest of the proof is standard. Combining Proposition 3 with (44) and using the relation $K_{\lambda\phi}(t,x,y) = e^{-\lambda\phi(x)}K(t,x,y)e^{-\lambda\phi(y)}$ we obtain

$$|K(t,x,y)| < c_\delta t^{-s} \exp\left\{\lambda[\phi(y) - \phi(x)] + \left[(k_m + cD + \delta)\lambda^{2m} + c_{\delta,M}\right] t\right\}$$

Optimizing over $\phi \in \mathcal{E}_{A,M}$ introduces $d_M(x,y)$ and choosing $\lambda = (\frac{d_M(x,y)}{2mk_m t})^{1/(2m-1)}$ we obtain

$$-\lambda d_M(x,y) + k_m\lambda^{2m}t = -\sigma_m \frac{d_M(x,y)^{2m/(2m-1)}}{t^{1/(2m-1)}},$$

which completes the proof. //

Remark. It is shown in [B2] that the term cD cannot be eliminated from (44). Thus for it to be removed from Theorem 1 a radically different approach is needed – if indeed the term is removable at all.

Remark. We point out that the above method can also work for operators of the form $H + W$, where W is a lower-order perturbation of H. It is clear that the estimate of Theorem 1 is valid for $H + W$ provided $W_{\lambda\phi}$ can be estimated as a form by

$$|W_{\lambda\phi}(f)| < \epsilon\{\text{Re } Q(f) + \lambda^{2m}\|f\|_2^2\} + c_\epsilon\|f\|_2^2$$

for all $\phi \in \mathcal{E}_a$ and $\lambda > 0$ and any $\epsilon > 0$. Such estimates can be obtained by means of weighted Hardy- and Sobolev-type inequalities. We do not elaborate on this and prove a theorem for zero-order perturbations.

Proposition 11 *Let $V = V_+ - V_-$ where $V_+ \in L^1_{loc}(\Omega)$ and $V_- \in L^1(\Omega)$. Then the heat kernel of $H + V$ satisfies the estimate of Theorem 1.*

Proof. We have

$$\begin{array}{rcl} \int_\Omega V_-|f|^2 & \le & \|V_-\|_1\|f\|_\infty^2 \\ \text{(by (H1))} & \le & c\|V_-\|_1[\text{Re } Q(f)]^s\|f\|_2^{2-2s} \\ & \le & \epsilon \text{Re } Q(f) + c_{\epsilon,V}\|f\|_2^2 \end{array}$$

(hence $H + V$ is defined with form domain the same as for $H + V_+$). Moreover $(H+V)_{\lambda\phi} = H_{\lambda\phi} + V \ge H_{\lambda\phi} - V_-$. Hence the estimate of Lemma 10 is also valid for $H + V$ and the rest of the argument goes through. //

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Gerassimos Barbatis
Department of Mathematics
University of Ioannina
45110 Ioannina
Greece
gbarbati@cc.uoi.gr