Sharp heat kernel estimates for higher-order operators with singular coefficients

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Abstract

We obtain heat kernel estimates for higher order operators with singular/degnerate operators with measurable coefficients. Precise constants are given, which are sharp for small times.

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1 Introduction

Let

\[ Hf(x) = (-1)^m \sum_{|\alpha|=m} D^\alpha \{ a_{\alpha\beta}(x) D^{\beta} f(x) \}, \quad x \in \Omega \subset \mathbb{R}^N \]

be a self-adjoint uniformly elliptic operator of order \(2m\) with measurable coefficients and subject to Dirichlet boundary conditions on \(\partial \Omega\). In [D2] it was shown that if \(2m > N\) then the associated heat semigroup \(e^{-Ht}\) has a kernel \(K(t, x, y)\) which satisfies the estimate

\[ |K(t, x, y)| < c_1 t^{-N/(2m)} \exp \left\{ -c_2 \frac{|x - y|^{2m/(2m-1)}}{t^{1/(2m-1)}} + c_3 t \right\} \]

for some positive constants \(c_i\). Under suitable conditions this was recently [B2] sharpened to

\[ |K(t, x, y)| < c_\epsilon t^{-N/(2m)} \exp \left\{ -c_\epsilon t^{-(\sigma_m - cD - \epsilon) d_M(x, y)^{2m/(2m-1)}} + c_\epsilon t \right\} \] (1)

where \(\sigma_m = (2m - 1)(2m)^{-2m/(2m-1)} \sin(\pi/(4m - 2))\), \(D \geq 0\) depends on the regularity of the coefficients and \(d_M(x, y)\) is a Finsler-type metric induced by the principal symbol of \(H\) and depending on the arbitrarily large parameter \(M\); as \(M \to \infty\), \(d_M(x, y)\) increases to a Finsler distance \(d(x, y)\), but (1) is valid only for \(M < \infty\). This estimate is sharp as is seen by comparison against the small-time asymptotics for operators with smooth coefficients obtained in [T] – see (13) below. In the same direction Dungey [Du] used resolvent estimates to obtain a
better estimate than (1) for powers of second order operators. He showed in a general framework that if the self-adjoint operator $H$ satisfies a standard Gaussian estimate with exponential constant $\frac{1}{4} - \epsilon$ then the heat kernel of $H^m$ satisfies (1) with $D = 0$ and $M = +\infty$. For an alternative approach valid also for higher order systems see [AQ].

In the main theorem of this article we extend (1) in two directions. Primarily, we consider operators whose coefficients can be singular and/or degenerate on $\partial \Omega$; moreover, we do not assume $H$ to be self-adjoint. Concerning the singularity or degeneracy of $H$, we assume that there is a positive function $a(x)$ that controls in a suitable sense the behaviour of the coefficient matrix $\{a_{\alpha\beta}\}$ and we then impose two conditions (H1) and (H2) on $a(x)$. The first is a weighted Sobolev embedding and the second is a weighted interpolation inequality. These conditions were introduced in [B1] and led to (non-sharp) off diagonal estimates on the heat kernel of non-uniformly elliptic self-adjoint operators. Besides conditions (H1) and (H2) we shall assume that the symbol $A(x, \xi)$ is close – in a suitable sense – to a certain class of ‘good’ symbols denoted by $G_a$. These symbols, besides satisfying (H1) and (H2) correspond to operators that are self-adjoint, their coefficients have some local regularity, and are strongly convex in the sense of [EP]. We make use of a certain stability property inherent in our approach and obtain bounds that are asymptotically sharp: they involve the exponential constant $\sigma_m - cD$ where $c$ is an absolute constant and $D$ is the distance of the symbol $A(x, \xi)$ from the class $G_a$ in a certain weighted norm. In particular the constant $\sigma_m$ is obtained for symbols in $G_a$. To our knowledge such estimates are new even if the coefficients are assumed to be smooth and the symbol lies in $G_a$.

2 Formulation of results

We first fix some notation. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$ we write $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We write $\gamma \leq \alpha$ to indicate that $\gamma_i \leq \alpha_i$ for all $i$, in which case we also set $e_\alpha^\gamma = \alpha! / \alpha!(\alpha - \gamma)!$. We use the standard notation $D^\alpha$ for the differential expression $(\partial / \partial x_1)^{\alpha_1} \cdots (\partial / \partial x_N)^{\alpha_N}$ and for $k \geq 0$ we denote by $\nabla^k f$ the vector $(D^\alpha f)_{|\alpha|=k}$. We denote by $\hat{f}$ the Fourier transform of a function $f$, $\hat{f}(\xi) = (2\pi)^{-N/2} \int e^{ix \cdot \xi} f(x) dx$. We shall denote by $\|A\|_{p,q}$ the norm of an operator $A$ from $L^p(\Omega)$ to $L^q(\Omega)$. The letter $c$ will stand for a positive constant whose value may change from line to line.

Let $\Omega$ be a domain in $\mathbb{R}^N$. We fix an integer $m \geq 1$ and consider the operator

$$Hf(x) = (-1)^m \sum_{|\alpha|=m \atop |\beta|=m} D^\alpha \{a_{\alpha\beta}(x) D^\beta f(x)\}$$

subject to Dirichlet boundary conditions on $\partial \Omega$; the precise definition shall be given below. The matrix-valued function $\{a_{\alpha\beta}\}$ is assumed to be measurable and to take its values in the set of all complex, $\nu \times \nu$-matrices, $\nu$ being the number of multi-indices $\alpha$ of length $|\alpha| = m$. We assume that each $a_{\alpha\beta}$ lies in $L^\infty_{loc}(\Omega)$; we do not assume $\{a_{\alpha\beta}\}$ to be self-adjoint.
We define a quadratic form $Q(\cdot)$ on $C_c^\infty(\Omega)$ by

$$Q(f) = \int_{\Omega} \sum_{|\alpha|=m |\beta|=m} a_{\alpha\beta}(x) D^\alpha f(x) D^\beta \bar{f}(x) \, dx, \quad f \in C_c^\infty(\Omega).$$

We assume that there exists a positive weight $a(x)$ with $a^{\pm 1} \in L^1_{\text{loc}}(\Omega)$ that controls the size of the matrix $\{a_{\alpha\beta}\}$ in the following sense: first,

$$|a_{\alpha\beta}(x)| \leq ca(x), \quad x \in \Omega,$$

for all multi-indices $\alpha, \beta$; and second, the weighted Gårding’s inequality

$$\text{Re } Q(f) \geq c \int_{\Omega} a(x)|\nabla^m f|^2 \, dx, \quad f \in C_c^\infty(\Omega)$$

is valid for some $c > 0$. We also assume the symbol-version of (4), namely

$$\text{Re } A(x, \xi) \geq c a(x)|\xi|^{2m}, \quad x \in \Omega, \; \xi \in \mathbb{R}^N,$$

where $A(x, \xi) := \sum_{\alpha, \beta} a_{\alpha\beta}(x) \xi^{\alpha+\beta}$. Relations (3) and (4) imply in particular that there exists $\beta > 0$ such that

$$|Q(f)| \leq \beta \text{Re } Q(f), \quad f \in C_c^\infty(\Omega).$$

It is easily seen that $Q$ is closable [B1]. The domain of its closure is a weighted Sobolev space which we denote by $W^{m,2}_{a,0}(\Omega)$. We retain the same symbol, $Q$, for the closure of the above form and denote by $H$ the associated accretive operator on $L^2(\Omega)$, so that $\langle Hf, f \rangle = Q(f)$, $f \in \text{Dom}(H)$, and (2) is valid in a weak sense.

We make two hypotheses on the weight $a$: the first is a weighted Sobolev inequality and the second is a weighted interpolation inequality.

\begin{enumerate}
  \item [(H1)] There exists $s \in [N/2m, 1]$ and $c > 0$ such that
  $$\|f\|_\infty \leq c|\text{Re } Q(f)|^{s/2} \|f\|_2^{1-s}, \quad f \in C_c^\infty(\Omega).$$
  \item [(H2)] There exists a constant $c$ such that
  $$\int_{\Omega} a^{k/m}|\nabla^k f|^2 \, dx < \epsilon \int_{\Omega} a^m |\nabla f|^2 \, dx + ce^{-k/(m-k)} \int_{\Omega} |f|^2 \, dx,$$
  for all $0 < \epsilon < 1, \; 0 \leq k < m$ and all $f \in C_c^\infty(\Omega)$.
\end{enumerate}

Both (H1) and (H2) are satisfied when $H$ is uniformly elliptic, in which case the best value for the constant $s$ is $s = N/2m$, showing that in the general case we cannot expect any value that is better (smaller) than $N/2m$; in particular (H1) is valid trivially with $s = N/2m$ if $a(x)$ is bounded away from zero. We refer to [B1] for non-trivial examples for which (H1) and (H2) are satisfied; they involve suitable powers of either $1 + |x|$ or $\text{dist}(x, K)$ where $K$ is a smooth surface of lower dimension.

We note that condition (H2) implies that for any $k, l$ with $0 \leq k, l \leq m, \; k+l < 2m$, there exists a constant $c$ so that

$$\int_{\Omega} a^{(k+l)/2m}|\nabla^k f||\nabla^l f| \, dx < c\epsilon |\text{Re } Q(f)| + ce^{-k/(m-k)}(1 + \lambda^{2m}) \|f\|_2^2.$$
for all $\epsilon \in (0, 1)$, $\lambda > 0$ and all $f \in C^\infty_c(\Omega)$. Indeed, for $\lambda = 1$ (9) is a consequence of (H2) and the Cauchy-Schwarz inequality; the case $\lambda < 1$ follows trivially from the case $\lambda = 1$; finally writing (9) for $\lambda = 1$ and replacing $\epsilon$ by $\epsilon \lambda^{k-1-2m}$ we obtain the result for $\lambda > 1$.

We next introduce the distance that shall be used in the heat kernel estimates. Consider the set
\[
\mathcal{E}_a = \{ \phi \in C^\infty(\Omega) \cap L^\infty(\Omega) : a^{k/2m} \nabla^k \phi \in L^\infty(\Omega), 1 \leq k \leq m \}
\]
and its subset (recall (5))
\[
\mathcal{E}_{A,M} = \{ \phi \in C^\infty(\Omega) \cap L^\infty(\Omega) : \text{Re} \, A(x, \nabla \phi(x)) \leq 1, \\
|\nabla^k \phi(x)| \leq M a(x)^{-k/2m}, 2 \leq k \leq m, \text{ a.e. } x \in \Omega \}. \tag{10}
\]
Our estimates will be expressed in terms of the distance
\[
d_M(x, y) = \sup_{\phi \in \mathcal{E}_{A,M}} (\phi(y) - \phi(x)) : \phi \in \mathcal{E}_{A,M} \tag{11}
\]
for arbitrarily large but finite $M$. For $M = +\infty$ this reduces to the distance
\[
d_\infty(x, y) = \sup_{\phi \in \mathcal{E}_{A,M}} (\phi(y) - \phi(x)) : \text{Re} \, A(x, \nabla \phi(x)) \leq 1, \ x \in \Omega.
\]
This is a Finsler distance, induced by the (singular/degenerate) Finsler metric with length element
\[
ds = ds(x, dx) = \sup_{\eta \in \mathbb{R}^N \setminus \{0\}} \frac{\langle dx, \eta \rangle}{(\text{Re} \, A(x, \eta))^{1/2m}}. \tag{12}
\]
We refer the reader to the recent book [BCS] for a comprehensive introduction to Finsler geometry. The distance $d_\infty(x, y)$ relates to the short-time off-diagonal behaviour of the heat kernel: it was shown in [T] that if $\Omega = \mathbb{R}^N$ and $H$ is self adjoint uniformly elliptic with strongly convex symbol (see (14)), then $d_\infty(\cdot, \cdot)$ controls the small-time behaviour of $K(t, x, y)$ in the sense that
\[
\log t^{N/2m} K(t, x, y) = -\sigma_m \frac{d_\infty(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}} (1 + o(1)), \quad \text{as } t \to 0 \tag{13}
\]
for $x, y$ fixed and close enough; here and below we have
\[
\sigma_m = (2m - 1)(2m)^{-2m/(2m-1)} \sin(\pi/(4m - 2)).
\]

Let us now proceed with the definition of the class $\mathcal{G}_a$. Let the functions $a_\gamma(\cdot)$, $|\gamma| = 2m$, be defined by requiring that
\[
\sum_{|\alpha| = m, |\beta| = m} a_{\alpha\beta}(x) \xi^{\alpha+\beta} = \sum_{|\gamma| = 2m} c_\gamma^{2m} a_\gamma(x) \xi^\gamma, \quad x \in \Omega, \ \xi \in \mathbb{R}^N;
\]
(recall that $c_\gamma^{2m} = (2m)!/\gamma!$). Following [EP] we say that the principal symbol $A(x, \xi)$ of $H$ is strongly convex if the quadratic form
\[
\Gamma(x, p) = \sum_{|\alpha| = m, |\beta| = m} a_{\alpha+\beta}(x)p_\alpha p_\beta, \quad p = (p_\alpha) \in \mathbb{C}^\nu, \tag{14}
\]
is positive semidefinite for a.e. \( x \in \Omega \).

Induced by the weight \( a(x) \) is the weighted Sobolev space

\[
W^{m-1,\infty}_a(\Omega) = \{ f \in W^{m-1,\infty}_{loc}(\Omega) : |\nabla^i f(x)| \leq ca(x)^{2m-1}, \text{ a.e. } x \in \Omega, i \leq m - 1 \}.
\] (15)

**Definition.** We say that the symbol \( A(x, \xi) \) lies in \( G_a \) if

(i) \( A(x, \xi) \) is strongly convex

(ii) \( \{ a_{\alpha\beta} \} \) is real and symmetric

(iii) the coefficients \( a_{\alpha\beta} \) lie in \( W^{m-1,\infty}_a(\Omega) \).

We denote by \( D \) the distance of the coefficient matrix \( \{ a_{\alpha\beta} \} \) from \( G_a \) in the weighted uniform norm

\[
\| f \|_{a,\infty} := \sup_{x \in \Omega} |f(x)/a(x)|,
\]

that is

\[
D = \inf_{\{ \tilde{a}_{\alpha\beta} \} \in W^{m-1,\infty}_a} \| \{ a_{\alpha\beta} \} - \{ \tilde{a}_{\alpha\beta} \} \|_{a,\infty},
\] (16)

where, as usual, \( \| \{ a_{\alpha\beta} \} \|_{a,\infty} := \| \{ \alpha_{\alpha\beta} \} \|_{M(\nu \times \nu)} \|_{a,\infty} \).

Our main result is the following:

**Theorem 1** Assume that (H1) and (H2) are satisfied. Then for all \( \delta \in (0, 1) \) and all \( M \) large there exist positive constants \( c_\delta, c_{\delta,M} \) such that

\[
|K(t, x, y)| < c_\delta t^{-s} \exp \left\{ - (\sigma_m - cD - \delta)d_M(x, y) \frac{2m}{2m-1} t^{-\frac{1}{2m-1}} + c_{\delta,M} \right\}
\] (17)

for all \( x, y \in \Omega \) and \( t > 0 \); the constant \( c \) is independent of \( x, y, t, \delta, D \) and \( M \).

In the special case where \( H \) is uniformly elliptic and self-adjoint this estimate has already been obtained in [B2].

### 3 Proof of Theorem 1

Given \( \phi \in \mathcal{E}_a \) the mapping \( f \mapsto e^{\phi}f \) maps \( W^{m,2}_{a,0}(\Omega) \) into itself [B1, Lemma 7]. Hence one can define a sesquilinear form \( Q_\phi(\cdot, \cdot) \) with domain \( W^{m,2}_{a,0}(\Omega) \) by

\[
Q_\phi(f) = Q(e^{\phi}f, e^{-\phi}f) = \int_\Omega \sum_{|\alpha|=m, |\beta|=m} a_{\alpha\beta} D^\alpha(e^{\phi}f) D^\beta(e^{-\phi}f) b dx, \quad f \in W^{m,2}_{a,0}(\Omega).
\] (18)

The associated operator is \( H_\phi = e^{-\phi}He^\phi \) and has domain \( \text{Dom}(H_\phi) = e^{-\phi}\text{Dom}(H) \). The form \( Q_\phi \) is a lower order perturbation of \( Q \) (cf. (28)) and it is a consequence of (H2) [B1, Lemma 8] that for all \( \epsilon > 0 \) and \( f \in W^{m,2}_{a,0}(\Omega) \) there holds

\[
|Q(f) - Q_\phi(f)| < \epsilon \Re Q(f) + \epsilon \Re^{-2m+1}(1 + p(\phi))^{2m} \| f \|_2^2,
\] (20)
where we have used the seminorm
\[
p(φ) := \sup_{1 < k \leq m} \text{ess sup}_{x \in Ω} a(x)^{k/2m}|\nabla^k φ(x)|.
\]
(21)

Defining \( s(φ) = (1 + p(φ))^{2m} \) it follows in particular that
\[
\text{Re } Q_φ(f) \geq -c s(φ)\|f\|_2^2, \quad f \in C_c^∞(Ω),
\]
(22)

where \( c \) is independent of \( φ \), and this justifies the definition
\[-k_φ = \inf \{\text{Re } Q_φ(f) : f \in C_c^∞(Ω), \|f\|_2 = 1\}.
\]
(23)
The next lemma follows closely an argument used in [BD].

**Lemma 2** Assume that (H2) is satisfied. Then for any \( φ \in E_a \) there holds
\[
(i) \quad \|e^{-H_φt}f\|_2 \leq e^{k_φt}, \quad (24)
\]
\[
(ii) \quad \|H_φ e^{-H_φt}f\|_2 \leq \frac{c_δ}{t} e^{k_φt} e^{δ s(φ)t}, \quad \text{for all } δ > 0,
\]
(25)

where the constant \( c_δ \) is independent of \( φ \in E_a \) and \( t > 0 \).

**Proof.** Part (i) is the standard energy estimate that follows by integrating
\[
\frac{d}{dt}\|e^{-H_φt}f\|_2^2 = -2\text{Re } (H_φ e^{-H_φt}f, e^{-H_φt}f) \leq 2k_φ\|e^{-H_φt}f\|_2^2.
\]
Now by (20) there holds
\[
|Q_φ(f) - Q(f)| \leq \frac{1}{2} \text{Re } Q(f) + s(φ)\|f\|_2^2, \quad f \in C_c^∞(Ω),
\]
(26)

where, we recall, \( s(φ) = c(1 + p(φ)^{2m}) \) for some fixed \( c > 0 \). Hence for any \( ε \in (0,1) \)
\[
\text{Re } Q_φ(f) = ε\text{Re } Q_φ(f) + (1 - ε)\text{Re } Q(f)
\]
\[
\geq \frac{ε}{2} \text{Re } Q(f) - [εs(φ) + (1 - ε)k_φ]\|f\|_2^2
\]
and hence
\[
\text{Re } [Q(f) - Q_φ(f)] \leq (1 - \frac{ε}{2})\text{Re } Q(f) + [εs(φ) + (1 - ε)k_φ]\|f\|_2^2.
\]

Fix \( f \in L^2(Ω) \) and \( θ \in (-π/2, π/2) \) and for \( ρ > 0 \) set \( f_ρ = \exp(-H_φ ρ e^{iθ})f \). We then have
\[
\frac{d}{dρ}\|f_ρ\|_2^2 = -2\text{Re } [e^{iθ} Q_φ(f_ρ)]
\]
\[
= -2\cos θ \text{Re } Q(f_ρ) + 2\sin θ \text{Im } Q_φ(f_ρ) + \]
\[
+2\cos θ \left[\text{Re } Q(f_ρ) - \text{Re } Q_φ(f_ρ)\right] + \]
\[
\leq -2\cos θ \text{Re } Q(f_ρ) + 2\sin |θ| \left[\frac{1}{2} + β\right] \text{Re } Q(f_ρ) + s(φ)\|f_ρ\|_2^2 + \]
\[
+2\cos θ \left[(1 - \frac{ε}{2}) \text{Re } Q(f_ρ) + [εs(φ) + (1 - ε)k_φ]\|f_ρ\|_2^2\right] + \]
\[
\leq [-ε \cos θ + (2β + 1)\sin |θ|] \text{Re } Q(f_ρ) + \]
\[
+ [2\cos θ\{εs(φ) + (1 - ε)k_φ\} + 2\sin |θ|s(φ)]\|f_ρ\|_2^2.
\]
Let $\alpha \in (0, \pi/2)$ be such that $\tan \alpha = \epsilon/(2\beta + 1)$. For $|\theta| \leq \alpha$ we then have $-\epsilon \cos \theta + (2\beta + 1) \sin |\theta| \leq 0$ and hence

\[
\frac{d}{d\rho} \|f_\rho\|_2^2 \leq 2 \cos \theta[\epsilon s(\phi) + (1 - \epsilon)k_\phi + s(\phi)\frac{\epsilon}{2\beta + 1}]\|f_\rho\|_2^2 \\
\leq 2(k_\phi + 2\epsilon s(\phi))\|f_\rho\|_2^2 \\
=: 2A\|f_\rho\|_2^2.
\]

It follows that $\|e^{-H_\phi z}\|_2 \leq e^{A\|z\|}$ in the sector $|\arg z| \leq \alpha$. We conclude that letting $\tau = \frac{A}{\cos \alpha}$ we have

\[
\|\exp\{-\phi + \tau_0\}z\|_2 \leq 1,
\]

and hence [D1, Lemma 2.38]

\[
\|(H_\phi + \tau_0)\|e^{-\phi + \tau_0}t\| \leq \frac{c}{\alpha t},
\]

for all $t > 0$. Multiplying both sides by $e^{\tau_0 t}$ and using the triangle inequality we obtain

\[
\|H_\phi e^{-\phi t}\|_2 \leq \frac{c}{\alpha t} \exp\{\frac{k_\phi + 2\epsilon s(\phi)}{\cos \alpha}t\} + \tau_0 e^{k_\phi t}.
\]

This last expression can be made smaller than the right hand side of (25) provided $\epsilon$ is chosen small enough; this completes the proof.

\begin{proposition}
Assume that (H1) and (H2) are satisfied. Then for any $\delta > 0$ there exists $c_\delta > 0$ independent of $\phi \in \mathcal{E}_\alpha$ such that

\[
\|e^{-H_\phi t}\|_1 \leq c_\delta t^{-\delta} e^{k_\phi t} e^{s(\phi)t}.
\]

\end{proposition}

\begin{proof}
Let $f \in L^2(\Omega)$ and set $f_t = e^{-H_\phi t}f$, $t > 0$. Using (H1) we have

\[
\|f_t\|_\infty \leq c[\operatorname{Re} Q(f_t)]^{s/2}\|f_t\|_2^{1-s} \\
\quad \text{(by (26))} \leq c \left[\operatorname{Re} Q_\phi(f_t) + s(\phi)\|f_t\|_2\right]^{s/2}\|f_t\|_2^{1-s} \\
\quad \leq c \left[\|H_\phi f_t\|_2\|f_t\|_2 + s(\phi)\|f_t\|_2\right]^{s/2}\|f_t\|_2^{1-s} \\
\quad \text{(by (25), (24))} \leq c \left[c_\delta^{s/2}e^{s(\phi)t} + s(\phi)\right]^{s/2}\|f_t\|_2 \\
\quad = c_\delta^{s/2} \left[c_\delta e^{s(\phi)t} + s(\phi)\right]^{s/2} e^{k_\phi t}\|f\|_2.
\]

Taking $\epsilon$ to be small enough we conclude that given $\delta > 0$ there exists $c_\delta$ such that

\[
\|e^{-H_\phi t}\|_2 \leq c_\delta t^{-\delta} e^{k_\phi t} e^{s(\phi)t}.
\]

The same arguments are valid for $H_\phi^* = H_{-\phi}$; the constant $k_\phi$ clearly staying the same. Hence by duality and the semigroup property (27) follows.

\end{proof}
In order for Proposition 3 to be useful we need a precise upper estimate on $k_\phi$, which amounts to a precise lower estimate on $\text{Re } Q_\phi(\cdot)$, cf. (23). This will be established in Lemma 10 following a series of intermediate lemmas. Recalling that $c_\gamma^a = a!/\gamma!(\alpha - \gamma)!$ it follows immediately from (19) that for $\lambda >, \phi \in E_a$ we have

$$Q_{\lambda \phi}(f) = \int_{\Omega} \sum_{|\alpha| = m} a_{\alpha \beta} \sum_{\gamma \leq \alpha} c_\gamma^a c_\delta^\beta P_{\gamma, \lambda \phi} P_{\delta, -\lambda \phi} D^{\alpha - \gamma} f D^{\beta - \delta} \tilde{f} dx,$$

where

$$P_{\gamma, \lambda \phi}(x) := e^{-\lambda \phi(x)} D^\gamma [e^{\lambda \phi(x)}]$$

is a polynomial in various derivatives of $\lambda \phi$. Now, the induction relation $P_{\gamma + s, \lambda \phi} = (\lambda \partial_j \phi + \partial_j) P_{\gamma, \lambda \phi}$ implies that $P_{\gamma, \lambda \phi}$ has the form

$$P_{\gamma, \lambda \phi}(x) = \sum_{k=1}^{[\gamma]} \lambda^k \sum_{\gamma_1, \ldots, \gamma_k} (D^{\gamma_1} \phi) \cdots (D^{\gamma_k} \phi),$$

where the second sum is taken over all non-zero multiindices $\gamma_1, \ldots, \gamma_k$ such that $\gamma_1 + \cdots + \gamma_k = \gamma$ and $c_{\gamma_1, \ldots, \gamma_k}$ are constants. Hence, recalling that $|\nabla^k \phi| \leq c a^{-k/2m}$, we can write $P_{\gamma, \lambda \phi}(x) = \sum_{k=1}^{[\gamma]} \lambda^k \tilde{P}_{k, \phi}(x)$ where $|\tilde{P}_{k, \phi}(x)| \leq c a^{-[\gamma]/2m}$. It follows from (28) that

$$Q_{\lambda \phi}(f) = \int_{\Omega} \sum_{|\alpha| = m} \sum_{\gamma \leq \alpha} \sum_{\delta \leq \beta} \sum_{j \leq |\delta|} \lambda^{k+j} w_{\alpha \beta \gamma \delta k j}(x) D^{\alpha - \gamma} f D^{\beta - \delta} \tilde{f} b dx,$$

where $w_{\alpha \beta \gamma \delta k j} := a_{\alpha \beta} c_\gamma^a c_\delta^\beta \tilde{P}_{k, \phi} \tilde{P}_{j, -\phi}$ satisfies $|w_{\alpha \beta \gamma \delta k j}| \leq c a^{(2m - [\gamma + \delta])/2m}$. Replacing $\gamma$ and $\delta$ by $\alpha - \gamma$ and $\beta - \delta$ correspondingly we conclude from (30) the following

**Lemma 4** $Q_{\lambda \phi}(f)$ is a linear combination of terms of the form

$$T(f) = \lambda^s \int_{\Omega} w(x) D^\gamma f D^\delta \tilde{f} b dx,$$

where $|w| \leq c a^{[\gamma + \delta]/2m}$ on $\Omega$ and

(i) $s$ is an integer between 0 and $2m$;
(ii) $\gamma$ and $\delta$ are multiindices with $|\gamma|, |\delta| \leq m$;
(iii) $s + [\gamma + \delta] \leq 2m$.

**Definition.** We call the number $s + [\gamma + \delta]$ the essential order of $T$.

Hence the essential order is an integer between 0 and $2m$. We denote by $L_{a,m}$ the linear space consisting of (finite) linear combinations of forms whose essential order is smaller than $2m$. In Lemma 9 we will see that terms in $L_{a,m}$ are in a sense negligible. We also point out for later use that (9) implies the interpolation inequality

$$|T(f)| < c \{ \text{Re } Q(f) + \lambda^{2m} \|f\|_2^2 \}, \quad f \in W_{a,0}^{m,2}(\Omega),$$

valid for all terms $T(\cdot)$ of essential order $2m$.

We have the following
Lemma 5. Given \( \phi \in \mathcal{E}_a \) and \( \lambda > 0 \) define

\[
Q_{1,\lambda \phi}(f) = \int_\Omega \sum_{|\alpha| = m} \sum_{7 \leq \alpha \leq \beta} a_{\alpha \beta} c^\alpha c^\beta (\lambda \nabla \phi)^\gamma (-\lambda \nabla \phi)^\delta D^{a-\gamma} D^{\beta-\delta} \tilde{f} dx.
\]

Then the difference \( Q_{\lambda \phi}(f) - Q_{1,\lambda \phi}(f) \) lies in \( \mathcal{L}_{a,m} \).

Proof. One simply has to recall (28) and observe from (29) that \( P_{\gamma,\lambda \phi} \), considered as a polynomial in \( \lambda \), has \( \lambda^{\gamma}(\nabla \phi)^{\gamma} \) as its highest-degree term. //

3.1 Symbols in \( \mathcal{G}_a \)

At this point and for the whole of this subsection we restrict our attention to operators \( H \) whose symbol belongs to \( \mathcal{G}_a \). For \( x \in \Omega, \xi, \eta \in C^N \) and \( \zeta \in R^N \) let us define

\[
k_m = \lfloor \sin(\pi/(4m-2)) \rfloor^{-2m+1},
\]

\[
A(x, \xi, \eta) = \sum_{|\alpha| = |\beta| = m} a_{\alpha \beta}(x) \xi^\alpha \eta^\beta,
\]

\[
S(x, \zeta; \xi, \eta) = Re \ A(x, \xi - i\zeta, \eta + i\zeta) + k_m Re \ A(x, \zeta).
\]

Lemma 6. Assume that the symbol \( A(x, \xi) \) lies in \( \mathcal{G}_a \). Then

\[
Re \ Q_{1,\lambda \phi}(f) + k_m \lambda^{2m} \int_\Omega Re \ A(x, \nabla \phi(x)) |f|^2 dx = (2\pi)^{-N} \int \int_{\Omega \times R^N \times R^N} S(x, \lambda \nabla \phi; \xi, \eta)e^{i(\xi - \eta) \cdot x} \hat{f}(\xi) \hat{f}(\eta) dx d\xi d\eta
\]

for all \( \phi \in \mathcal{E}_a, \lambda > 0 \) and \( f \in C_\infty(\Omega) \).

Proof. Writing \( D^\gamma f(x) = (2\pi)^{-N/2} \int_{R^N} (i\xi)^\gamma e^{i\xi \cdot x} \hat{f}(\xi) d\xi \) we have

\[
Q_{1,\lambda \phi}(f) = (2\pi)^{-N} \int \int_{\Omega \times R^N \times R^N} \sum_{|\alpha| = m} \sum_{7 \leq \alpha \leq \beta} a_{\alpha \beta} c^\alpha c^\beta (\lambda \nabla \phi)^\gamma (-i\lambda \nabla \phi)^\delta \times
\]

\[
\times \xi^\alpha \eta^\beta e^{i(\xi - \eta) \cdot x} \hat{f}(\xi) \hat{f}(\eta) d\xi d\eta dx
\]

\[
= (2\pi)^{-N} \int \int_{\Omega \times R^N \times R^N} \sum_{|\alpha| = |\beta| = m} a_{\alpha \beta} (\xi - i\lambda \nabla \phi)^\alpha (\eta - i\lambda \nabla \phi)^\beta \times
\]

\[
\times e^{i(\xi - \eta) \cdot x} \hat{f}(\xi) \hat{f}(\eta) d\xi d\eta dx
\]

\[
= (2\pi)^{-N} \int \int_{\Omega \times R^N \times R^N} A(x, \xi - i\lambda \nabla \phi(x), \eta + i\lambda \nabla \phi(x)) \times
\]

\[
\times e^{i(\xi - \eta) \cdot x} \hat{f}(\xi) \hat{f}(\eta) d\xi d\eta dx.
\]

This last integral has the form \( \int_\Omega q[g]dx \) where for fixed \( x \in \Omega \)

\[
\begin{cases}
g(\xi) = e^{i\xi \cdot x} \hat{f}(\xi) d\xi \\
q[g] = \int_{R^N \times R^N} p(\xi, \eta) g(\xi) \overline{g(\eta)} d\xi d\eta \\
p(\xi, \eta) = A(x, \xi - i\lambda \nabla \phi(x), \eta + i\lambda \nabla \phi(x))
\end{cases}
\]
Since the matrix \( \{ a_{\alpha \beta} \} \) is symmetric we have \( p(\xi, \eta) = p(\eta, \xi) \) and therefore \( q(g) = \int_{\mathbb{R}^N} \overline{p(\xi, \eta)} q(\xi) g(\eta) d\xi d\eta. \) Hence \( \text{Re} \ q(g) = \int_{\mathbb{R}^N} \text{Re} \ p(\xi, \eta) d\xi d\eta \) and integration over \( x \in \Omega \) yields

\[
\text{Re} \ Q_{1, \phi}(f) + k_m \int_\Omega \text{Re} \ A(x, \lambda \nabla \phi(x)) |f|^2 \, dx \\
= (2\pi)^{-N} \int_\Omega \int_{\mathbb{R}^N} \text{Re} \ [A(x, \xi - i \lambda \nabla \phi(x), \eta + i \lambda \nabla \phi(x)) + k_m A(x, \lambda \nabla \phi)] \times \\
\times e^{i(\xi - \eta) \cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)} \, d\xi d\eta dx \\
= (2\pi)^{-N} \int_\Omega \int_{\mathbb{R}^N} S(x, \lambda \nabla \phi; \xi, \eta) e^{i(\xi - \eta) \cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)} \, d\xi d\eta dx.
\]

We now proceed to estimate the triple integral in the right hand side of (33). It is shown in [EP, Theorem 2.1] that there exist positive numbers \( w_0, \ldots, w_{m-2} \) such that

\[
S(x, \zeta; \xi, \eta) = \sum_{s=0}^{m-2} w_s \Gamma(x, p_{\xi, \zeta}^{(s)}), \quad x \in \Omega, \ \zeta, \xi \in \mathbb{R}^N,
\]

(34)

where \( \Gamma(x, \cdot; \cdot) \) is the quadratic form associated to the principal symbol of \( H \) (cf. (14)) and \( p_{\xi, \zeta}^{(s)} \) is the vector in \( \mathbb{R}^p \) defined for fixed \( \xi, \zeta \in \mathbb{R}^N \) by requiring that

\[
\sum_{|a|=m} p_{\xi, \zeta}^{(s)} a^\alpha = (\sin \theta_m)^{-s-2} (\xi \cdot a)^{m-s-2} (\zeta \cdot a)^s \left\{ (\sin \theta_m)^2 (\xi \cdot a)^2 - (\cos \theta_m)^2 (\zeta \cdot a)^2 \right\}
\]

for all \( a \in \mathbb{R}^N \); here \( \theta_m = \pi/(4m - 2) \). To simplify the notation let us define the sesquilinear forms \( \Gamma(x, \cdot, \cdot) \) on \( \mathbb{C}^{m-1} \otimes \mathbb{C}^p \cong \mathbb{C}^{p(m-1)} \) by

\[
\Gamma(x, u, v) = \sum_{s=0}^{m-2} w_s \Gamma(x, u^{(s)}, v^{(s)}) = \sum_{s=0}^{m-2} \sum_{|a|=m} \sum_{|\beta|=m} w_s a_{\alpha+\beta}(x) u^{(s)}_{\alpha} v^{(s)}_{\beta}
\]

for all \( u = (u^{(s)}), \ v = (v^{(s)}) \in \mathbb{C}^{p(m-1)}. \) Then \( \Gamma \) is positive semi-definite by the strong convexity of \( A(x, \xi). \) To handle the above expressions we introduce two auxiliary elliptic differential forms \( S_{\lambda \phi} \) and \( \Gamma_{\lambda \phi} \) on \( L^2(\Omega). \) They have common domain \( W^{m,2}_{a,0}(\Omega) \) and are given by

\[
S_{\lambda \phi}(f) = (2\pi)^{-N} \int_\Omega \int_{\mathbb{R}^N} S(x, \lambda \nabla \phi; \xi, \eta) e^{i(\xi - \eta) \cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)} \, d\xi d\eta dx,
\]

(36)

\[
\Gamma_{\lambda \phi}(f) = (2\pi)^{-N} \int_\Omega \int_{\mathbb{R}^N} \Gamma(x, p_{\xi, \lambda \nabla \phi}, p_{\eta, \lambda \nabla \phi}) e^{i(\xi - \eta) \cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)} \, d\xi d\eta dx
\]

(37)

where \( p_{\xi, \lambda \nabla \phi} = (p_{(\xi, \lambda \nabla \phi, \alpha)}^{(s)})_{|a|=m} \) is defined by (35).

**Lemma 7** Assume that the symbol \( A(x, \xi) \) lies in \( \mathcal{G}_a. \) Then the form \( S_{\lambda \phi}(\cdot) - \Gamma_{\lambda \phi}(\cdot) \) lies in \( \mathcal{L}_{a,m}. \)
Proof. It follows from (34) that $S_{\lambda \phi}$ and $\Gamma_{\lambda \phi}$ have integral kernels which are polynomials of $\xi$ and $\eta$ and whose values coincide for $\xi = \eta$. Using the inverse Fourier transform this implies that the difference $S_{\lambda \phi}(f) - \Gamma_{\lambda \phi}(f)$ is a linear combination of terms of the form

$$T(f) = \lambda^s \int_\Omega w(x)[D^{\gamma+\kappa} f D^{\delta} \mathcal{T} - (-1)^\epsilon D^{\gamma} f D^{\delta+\kappa} \mathcal{T}]dx,$$  

where $w$ is some function and $\kappa$ is a multi-index of length $|\kappa| \leq m - 1$. In fact, recalling (33) and the definition of $Q_{1,\lambda \phi}$ we see that $w = a_{\alpha \beta}(\nabla \phi)\mu$ where $|\mu| = s$ and $\gamma + \delta + \kappa + \mu = \alpha + \beta$. Since $a_{\alpha \beta} \in W^{m-1,\infty}_a(\Omega) \subset W^{m-1,\infty}_{\text{loc}}(\Omega)$ we can integrate by parts $|\kappa|$ times and use Leibnitz’ rule to obtain

$$T(f) = (-1)^{|\kappa|} \lambda^s \sum_{0 < \kappa_1 \leq \kappa} c_{\kappa_1} \epsilon_1 \int_{\Omega} D^{\kappa_1} w D^{\gamma} f D^{\delta+\kappa_1} \mathcal{T}dx. \quad (39)$$

We estimate $D^{\kappa_1} w$: clearly

$$|D^{\kappa_1}(a_{\alpha \beta}(\nabla \phi)\mu)| \leq c \sum_{i=0}^{\kappa_1} |\nabla^{i}\nabla \phi| |\nabla^{i}(\nabla \phi)\mu| \quad \text{in } \Omega.$$

Recalling the definition of $E_{A,M}$ it is easily seen that $|\nabla^{i}(\nabla \phi)\mu| \leq ca^{-((i+1)/2m)}$; recalling also from (15) the definition of the space $W^{m-1,\infty}_a(\Omega)$ where the $a_{\alpha \beta}$ lie we conclude that

$$|D^{\kappa_1}(a_{\alpha \beta}(\nabla \phi)\mu)| \leq c_M a(x) \frac{2m-|\kappa_1+\mu|}{2m} = c_M a^{||\gamma+\delta+\kappa_1||_{2m}}.$$

Hence (39) implies that $T$ has essential order $s + |\gamma + \delta + \kappa - \kappa_1| < 2m$, as required. \\

//

Proposition 8 Let $A(x, \xi) \in G_a$. Then for any $\phi \in E_a$, $\lambda > 0$ and all $f \in C_{\text{c}}(\Omega)$, there holds

$$\text{Re } Q_{\lambda \phi}(f) \geq -k_m \lambda^{2m} \int_{\Omega} A(x, \nabla \phi(x))|f|^2dx + T(f) \quad (40)$$

where $T(\cdot) \in L_{a,m}$.

Proof. Combining Lemmas 5, 6 and 7 we have

$$\text{Re } Q_{\lambda \phi}(f) + k_m \int_{\Omega} \text{Re } A(x, \lambda \nabla \phi(x))|f|^2dx = \Gamma_{\lambda \phi}(f) + T(f), \quad (41)$$

for a form $T(\cdot) \in L_{a,m}$. Now let $u(x) = \int_{\mathbb{R}^N} p_{\xi, \lambda \nabla \phi} e^{i\xi \cdot x} f(\xi)d\xi$ (a $C^{(m-1)}$-valued integral defined component-wise); it follows immediately from definition (37) that

$$\Gamma_{\lambda \phi}(f) = \int_{\Omega} \Gamma(x, u(x), u(x))dx \quad (42)$$

and hence $\Gamma_{\lambda \phi}(\cdot)$ is non-negative by the strong convexity of $A(x, \xi)$. //
3.2 The general case

We now remove the assumption $A \in \mathcal{G}_a$ and return to the general setting described in Section 2. We recall that the quantity $D$ measures the distance of $A$ from $\mathcal{G}_a$ and has been defined in (16).

**Lemma 9** Let $T \in \mathcal{L}_{a,m}$. Then for any $\epsilon \in (0,1)$ there holds

$$|T(f)| < \epsilon \{\text{Re } Q(f) + \lambda^{2m} \|f\|^2_2 + c_\epsilon \|f\|^2_2 \}$$

for all $\lambda > 0$ and $f \in C_c^\infty (\Omega)$.

**Proof.** By definition, $T(f)$ is a finite linear combination of expressions of the form

$$I(f) = \lambda^s \int_{\Omega} w(x) D^\gamma f(x) D^\delta \bar{f}(x) dx,$$

where $|w(x)| \leq c a(x)^{\gamma+\delta}/2m$ and $s + |\gamma + \delta| \leq 2m - 1$. Setting $\mu^{2m - |\gamma + \delta|} = \lambda^s$ and recalling (9) we have

$$|I(f)| \leq \epsilon \mu^{2m - |\gamma + \delta|} \int_{\Omega} a(x)^{\gamma+\delta}/2m |D^{\gamma} f| |D^{\delta} \bar{f}| dx$$

$$\leq \epsilon \text{Re } Q(f) + c \epsilon^{-2m+1}(1 + \mu^{2m}) \|f\|^2_2$$

$$\leq \epsilon \text{Re } Q(f) + c \epsilon^{-2m+1}(1 + \lambda^{2m-1}) \|f\|^2_2$$

$$\leq \epsilon \{\text{Re } Q(f) + \lambda^{2m} \|f\|^2_2 \} + c \epsilon^{-4m^2+1} \|f\|^2_2. \quad \checkmark$$

**Remark.** It is seen from the proof that the size of the constant $c_\epsilon$ in (43) depends only on $\epsilon > 0$ and the (finite) quantity $\max_I \sup \{|w(x)|a(x)^{-|\gamma + \delta|}/2m\}$ where the max is taken over all forms $I(\cdot)$ that make up $T(\cdot)$. In particular, when we restrict our attention to functions $\phi \in \mathcal{E}_{A,M}$ we obtain a constant $c_\epsilon = c_{\epsilon,M}$ which is otherwise independent of $\phi$.

**Lemma 10** For any $\phi \in \mathcal{E}_{A,M}$, $\lambda > 0$ and $\epsilon > 0$ and all $\gamma$, there holds

$$\text{Re } Q_{\lambda \phi}(f) \geq -\left\{\left(k_m + cD + \epsilon \right) \lambda^{2m} + c_{\epsilon,M}\right\} \|f\|^2_2, \quad f \in C_c^\infty (\Omega). \quad (44)$$

where the constant $c$ is independent of $D, M, \epsilon, \lambda$ and $\phi$ and the constant $c_{M,D,\epsilon}$ is independent of $\lambda$ and $\phi$.

**Proof.** Let $\hat{A} \in \mathcal{G}_a$ be such that $\|A - \hat{A}\|_{a,\infty} \leq 2D$. It follows from (32) that

$$\left\{\begin{array}{l}
|\text{Re } \hat{Q}_{\lambda \phi}(f) - \text{Re } Q_{\lambda \phi}(f)| < cD \{\text{Re } Q(f) + \lambda^{2m} \|f\|^2_2\} \\
\lambda^{2m} \int_{\Omega} [A(x, \nabla \phi(x)) - \hat{A}(x, \nabla \phi(x))] dx < cD \{\text{Re } Q(f) + \lambda^{2m} \|f\|^2_2\}.
\end{array}\right.$$  

Combining these relations with (40) – as applied to the operator $\hat{H}$ – we obtain

$$\text{Re } Q_{\lambda \phi}(f) \geq -k_m \lambda^{2m} \int_{\Omega} \text{Re } A(x, \nabla \phi(x)) |f|^2 dx - cD \{\text{Re } Q(f) + \lambda^{2m} \|f\|^2_2\} + T(f).$$
We have $\text{Re} \ A(x, \nabla \phi(x)) \leq 1$ and therefore (allowing $c$ to change from line to line and $\epsilon$ to rescale)

$$\text{Re} \ Q_\phi(f) \geq -k_m \lambda^{2m} \|f\|_2^2 - cD \{\text{Re} \ Q(f) + \lambda^{2m} \|f\|_2^2\} + T(f)$$

(by (43))

$$\geq -k_m \lambda^{2m} \|f\|_2^2 - (cD + \epsilon)\{\text{Re} \ Q(f) + \lambda^{2m} \|f\|_2^2\} - c_{\epsilon,M} \|f\|_2^2$$

(by (26))

Now, either $\text{Re} \ Q_\phi(f)$ is positive, in which case (44) is true, or it is non-positive, in which case it can be discarded from the right hand side of the last inequality. This completes the proof.

**Proof of Theorem 1.** The rest of the proof is standard. Combining Proposition 3 with (44) and using the relation $K_{\lambda\phi}(t, x, y) = e^{-\lambda\phi(x)}K(t, x, y)e^{-\lambda\phi(y)}$ we obtain

$$|K(t, x, y)| < c_3 t^{-s} \exp \left\{ \lambda[\phi(y) - \phi(x)] + (k_m + cD + \delta)\lambda^{2m} + c_{\delta,M} \right\} t$$

Optimizing over $\phi \in \mathcal{E}_{A,M}$ introduces $d_M(x, y)$ and choosing $\lambda = (d_M(x, y))^{1/(2m-1)}$ we obtain

$$-\lambda d_M(x, y) + k_m \lambda^{2m} t = -\sigma_m \frac{d_M(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}},$$

which completes the proof.

**Remark.** It is shown in [B2] that the term $cD$ cannot be eliminated from (44). Thus for it to be removed from Theorem 1 a radically different approach is needed – if indeed the term is removable at all.

**Remark.** We point out that the above method can also work for operators of the form $H + W$, where $W$ is a lower-order perturbation of $H$. It is clear that the estimate of Theorem 1 is valid for $H + W$ provided $W_{\lambda\phi}$ can be estimated as a form by

$$|W_{\lambda\phi}(f)| < \epsilon \{\text{Re} \ Q(f) + \lambda^{2m} \|f\|_2^2\} + c_{\epsilon,M} \|f\|_2^2$$

for all $\phi \in \mathcal{E}_\phi$ and $\lambda > 0$ and any $\epsilon > 0$. Such estimates can be obtained by means of weighted Hardy- and Sobolev-type inequalities. We do not elaborate on this and prove a theorem for zero-order perturbations.

**Proposition 11** Let $V = V_+ - V_-$ where $V_+ \in L^1_{\text{loc}}(\Omega)$ and $V_- \in L^1(\Omega)$. Then the heat kernel of $H + V$ satisfies the estimate of Theorem 1.

**Proof.** We have

$$\int_\Omega V_- \|f\|^2 \leq \|V_-\|_1 \|f\|_2^2$$

(by (H1))

$$\leq c \|\text{Re} \ Q(f)\|_1^2 \|f\|_2^{2s}$$

(hence $H + V$ is defined with form domain the same as for $H + V_+$). Moreover $(H + V)_{\lambda\phi} = H_{\lambda\phi} + V \geq H_{\lambda\phi} - V_-$. Hence the estimate of Lemma 10 is also valid for $H + V$ and the rest of the argument goes through. //
References


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