On the Hardy constant of some non-convex planar domains

Gerassimos Barbatis and Achilles Tertikas

Dedicated to Ermanno Lanconelli on the occasion of his 70th birthday

Abstract he Hardy constant of a simply connected domain $\Omega \subset \mathbf{R}^2$ is the best constant for the inequality

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{\mathrm{dist}(x,\partial\Omega)^2}\, dx\,, \qquad u \in C_c^\infty(\Omega).$$

After the work of Ancona where the universal lower bound 1/16 was obtained, there has been a substantial interest on computing or estimating the Hardy constant of planar domains. In [8] we have determined the Hardy constant of an arbitrary quadrilateral in the plane. In this work we continue our investigation and we compute the Hardy constant for other non-convex planar domains. In all cases the Hardy constant is related to that of a certain infinite sectorial region which has been studied by E.B. Davies.

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1 Introduction

The well-known Hardy inequality for $\mathbf{R}^N_+ = \mathbf{R}^{N-1} \times (0,+\infty)$ reads

$$\int_{\mathbf{R}^N_+} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbf{R}^N_+} \frac{u^2}{x_N^2} dx \,, \quad \text{for all } u \in C_c^\infty(\mathbf{R}^N_+), \tag{1}$$

where the constant $1/4$ is the best possible and equality is not attained in the appropriate Sobolev space. The analogue of (1) for a domain $\Omega \subset \mathbf{R}^N$ is

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2} dx \,, \quad \text{for all } u \in C_c^\infty(\Omega), \tag{2}$$

where $d = d(x) = \mathrm{dist}(x, \partial\Omega)$. However, (2) is not true without geometric assumptions on Ω. The typical assumption made for the validity of (2) is that Ω is convex. A weaker geometric assumption introduced in [6] is that Ω is weakly mean convex, that is

$$-\Delta d(x) \geq 0 \,, \quad \text{in } \Omega, \tag{3}$$

where Δd is to be understood in the distributional sense. Condition (3) is equivalent to convexity when $N = 2$ but strictly weaker than convexity when $N \geq 3$ [3]. Other geometric assumptions on the domain that guarantee that the best Hardy constant is 1/4 were recently obtain in [4, 10].

For a general domain Ω we may still have a Hardy inequality provided that the boundary $\partial\Omega$ has some regularity. In particular it is well known that for any bounded Lipschitz domain $\Omega \subset \mathbf{R}^N$ there exists $c > 0$ such that

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{d^2} dx \,, \qquad \text{for all } u \in C_c^\infty(\Omega). \tag{4}$$

The best constant c of inequality (4) is called the Hardy constant of the domain Ω.

In general the Hardy constant depends on the domain Ω; see [7] for results that concern properties of this dependence. In dimension $N \geq 3$ Davies [9] has constructed Lipschitz domains with Hardy constant as small as one wishes. On the other hand for $N = 2$ Ancona [2] has proved that for a simply connected domain the Hardy constant is always at least $1/16$; see also [12] where further results in this directions where obtained.

Davies [9] computed the Hardy constant of an infinite planar sector Λ_β of angle β,

$$\Lambda_\beta = \{\, 0 < r, \;\; 0 < \theta < \beta . \}$$

He used the symmetry of the domain to reduce the computation to the study of a certain ODE; see (9) below. In particular he established the following two results, which are also valid for the circular sector of angle β:

(a) The Hardy constant is $1/4$ for all angles $\beta \leq \beta_{cr}$, where $\beta_{cr} \cong 1.546\pi$.

(b) For $\beta_{cr} \leq \beta \leq 2\pi$ the Hardy constant of Λ_β strictly decreases with β and at the limiting case $\beta = 2\pi$ the Hardy constant is $\cong 0.2054$.

Our interest is to determine the Hardy constant of certain domains in two space dimensions; see [5, 11] for relevant questions. In this direction, in our recent work [8] we have established

Theorem. *Let Ω be a non-convex quadrilateral with non-convex angle $\pi < \beta < 2\pi$. Then the Hardy constant of Ω depends only on β. The Hardy constant, which we denote from now on by c_β, is the unique solution of the equation*

$$\sqrt{c_\beta}\tan\left(\sqrt{c_\beta}\left(\frac{\beta-\pi}{2}\right)\right) = 2\left(\frac{\Gamma\left(\frac{3+\sqrt{1-4c_\beta}}{4}\right)}{\Gamma\left(\frac{1+\sqrt{1-4c_\beta}}{4}\right)}\right)^2, \tag{5}$$

when $\beta_{cr} \le \beta < 2\pi$ and $c_\beta = 1/4$ when $\pi < \beta \le \beta_{cr}$. The critical angle β_{cr} is the unique solution in $(\pi, 2\pi)$ of the equation

$$\tan\left(\frac{\beta_{cr}-\pi}{4}\right) = 4\left(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right)^2. \tag{6}$$

Actually the constant c_β coincides with the Hardy constant of the sector Λ_β, so equation (5) provides an analytic description of the Hardy constant computed numerically in [9].

In this work we continue our investigation and determine the Hardy constant for other families of non-convex planar domains. Our first result reads as follows; see Fig. 1.

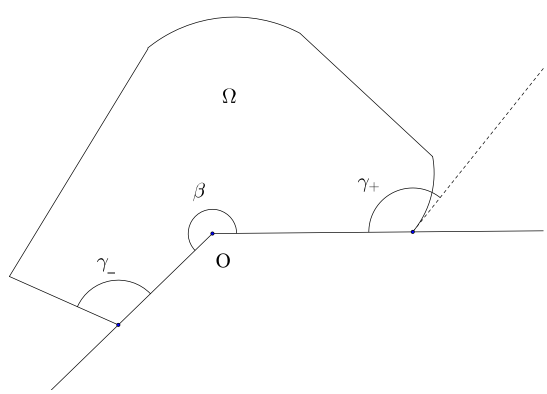

Fig. 1 A typical domain Ω for Theorem 1

Theorem 1. *Let $\Omega = K \cap \Lambda_\beta$, $\beta \in (\pi, 2\pi]$, where K is a bounded convex planar set and the vertex of Λ_β is an interior point of K. Let γ_+ and γ_- denote the interior* *angles of intersection of K with Λ_β. There exists an angle $\gamma_\beta \in (\pi/2, \pi)$ such that if $\gamma_+, \gamma_- \le \gamma_\beta$, then the Hardy constant of Ω is c_β, where c_β is given by (5), (6).*

Detailed information on the angle γ_β is given in Lemma 5 and Theorem 4. We note that Theorem 1 can be extended to cover the case where Ω is unbounded and the boundary of the convex set K does not intersect the boundary of the sector Λ_β; see Theorem 5.

We next study the Hardy constant for a family of domains $E_{\beta,\gamma}$ which may have two non-convex angles. The boundary $\partial E_{\beta,\gamma}$ of such a domain consists of the segment OP and two half lines starting from O and from P with interior angles β and γ; hence $\beta + \gamma \le 3\pi$; see Fig. 2 in case $\gamma < \pi$ and Fig. 3 in case $\gamma > \pi$. We then have

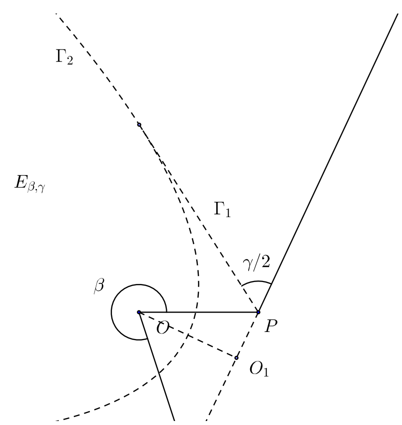

Fig. 2 A typical domain $E_{\beta,\gamma}$, $\gamma < \pi < \beta$

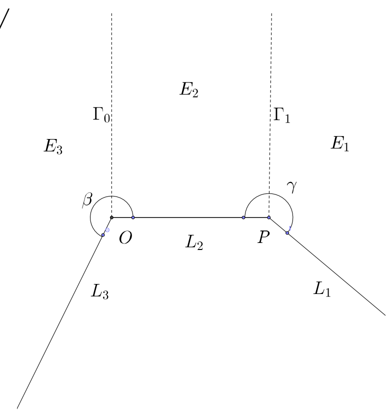

Fig. 3 A typical domain $E_{\beta,\gamma}$, $\beta, \gamma > \pi$

Theorem 2. (i) *If $0 < \gamma \le \pi \le \beta \le 2\pi$ then the Hardy constant of $E_{\beta,\gamma}$ is c_β.*
(ii) *If $\pi \le \beta, \gamma \le 2\pi$ then the Hardy constant of $E_{\beta,\gamma}$ is $c_{\beta+\gamma-\pi}$, provided that*

$$|\beta - \gamma| \le \frac{2}{c_{\beta+\gamma-\pi}} \arccos(2\sqrt{c_{\beta+\gamma-\pi}}). \tag{7}$$

It is interesting to notice that in case (i) where we have only one non-convex angle, the Hardy constant is related to the non-convex angle β, whereas in case (ii) where we have two non-convex angles, the Hardy constant is related to the angle $\beta + \gamma - \pi$ formed by the two halflines.

Our technique can actually be applied to establish best constant for Hardy inequality with mixed Dirichlet-Neumann boundary conditions. We consider a bounded domain D_β whose boundary ∂D_β consists of two parts, $\partial D_\beta = \Gamma_0 \cup \Gamma$. On Γ_0 we impose Dirichlet boundary conditions and it is from Γ_0 that we measure the distance from, $d(x) = \mathrm{dist}(x, \Gamma_0)$. On the remaining part Γ we impose Neumann boundary conditions. The curve Γ_0 is the union of two line segments which have as a common endpoint the origin O where they meet at an angle β, $\pi < \beta \le 2\pi$. We assume that the curve Γ is the graph in polar coordinates of a Lipschitz function $r(\theta)$,

$$\Gamma = \{(r(\theta), \theta) : 0 \le \theta \le \beta\};$$

see Fig. 4.

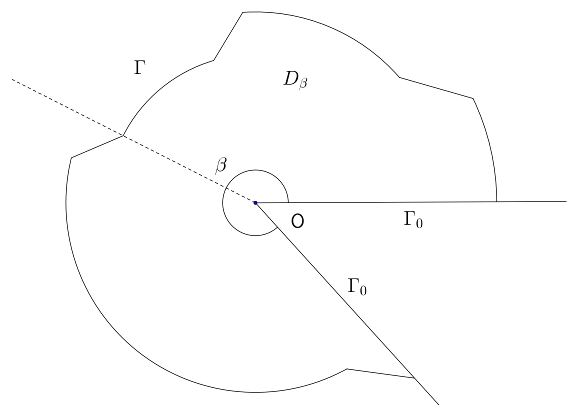

Fig. 4 A typical domain D_β. Note that Γ is not necessarily the boundary of a convex set

We then have

Theorem 3. *Let D_β be as above, $\pi < \beta \le 2\pi$. If Γ is such that*

$$r'(\theta) \le 0, \quad 0 \le \theta \le \frac{\beta}{2},$$

$$r'(\theta) \ge 0, \quad \frac{\beta}{2} \le \theta \le \beta\,,$$

then for all functions $u \in C^\infty(\overline{D_\beta})$ that vanish near Γ_0 there holds

$$\int_{D_\beta} |\nabla u|^2 dx\,dy \ge c_\beta \int_{D_\beta} \frac{u^2}{d^2} dx\,dy\,.$$

The constant c_β is the best possible.

The structure of the paper is simple: in Section 2 we prove various auxiliary results, while in Sections 3-5 we prove the theorems.

2 Auxiliary results

Let $\beta > \pi$ be fixed. We define the potential $V(\theta)$, $\theta \in (0,\beta)$,

$$V(\theta) = \begin{cases} \dfrac{1}{\sin^2\theta}, & 0 < \theta < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < \theta < \beta - \frac{\pi}{2}, \\ \dfrac{1}{\sin^2(\beta-\theta)}, & \beta - \frac{\pi}{2} < \theta < \beta. \end{cases} \tag{8}$$

For $c > 0$ we then consider the following boundary-value problem:

$$\begin{cases} -\psi''(\theta) = cV(\theta)\psi(\theta),\ 0 \le \theta \le \beta, \\ \psi(0) = \psi(\beta) = 0 \end{cases} \tag{9}$$

It was proved in [9] that the Hardy constant of the sector Λ_β coincides with the largest positive constant c for which (9) has a positive solution. Due to the symmetry of the potential $V(\theta)$ this also coincides with the largest constant c for which the following boundary value problem has a solution:

$$\begin{cases} -\psi''(\theta) = cV(\theta)\psi(\theta),\ 0 \le \theta \le \beta/2, \\ \psi(0) = \psi'(\beta/2) = 0\,. \end{cases} \tag{10}$$

The largest angle β_{cr} for which the Hardy constant is $1/4$ for $\beta \in [\pi, \beta_{cr}]$ was computed numerically in [9] and analytically in [8, 13] where (6) was established; the approximate value is $\beta_{cr} \cong 1.546\pi$.

We define the hypergeometric function

$$F(a,b,c;z) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}.$$

The boundary value problem (10) was studied in [8] where the following lemma was proved:

Lemma 1. (i) *Let $\beta > \beta_{cr}$. The boundary value problem (10) has a positive solution if and only if $c = c_\beta$. In this case the solution is given by*

$$\psi(\theta) = \begin{cases} \dfrac{\sqrt{2}\cos\big(\sqrt{c}(\beta-\pi)/2\big)\sin^{\alpha}(\theta/2)\cos^{1-\alpha}(\theta/2)}{F(\frac{1}{2},\frac{1}{2},\alpha+\frac{1}{2};\frac{1}{2})} F(\dfrac{1}{2},\dfrac{1}{2},\alpha+\dfrac{1}{2};\sin^2(\dfrac{\theta}{2})), & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } 0 < \theta \le \frac{\pi}{2}, \\ \cos\big(\sqrt{c}(\frac{\beta}{2}-\theta)\big), & \textit{if } \frac{\pi}{2} < \theta \le \frac{\beta}{2}, \end{cases}$$

where α is the largest solution of $\alpha(1-\alpha) = c$.
(ii) *Let $\pi < \beta \le \beta_{cr}$. The largest value of c so that the boundary value problem (10) has a positive solution is $c = 1/4$. For $\beta = \beta_{cr}$ the solution is*

$$\psi(\theta)=\begin{cases}\dfrac{\cos\left(\frac{\beta_{cr}-\pi}{4}\right)\sin^{1/2}\theta}{F(\frac{1}{2},\frac{1}{2},1;\frac{1}{2})}F(\dfrac{1}{2},\dfrac{1}{2},1;\sin^2(\dfrac{\theta}{2})), & 0<\theta\leq\frac{\pi}{2},\\ \cos\left(\frac{1}{2}(\frac{\beta_{cr}}{2}-\theta)\right), & \frac{\pi}{2}<\theta\leq\frac{\beta_{cr}}{2}.\end{cases}$$

while for $\beta_{cr}<\beta<2\pi$ and $0<\theta<\pi/2$ it has the form

$$\begin{aligned}\psi(\theta)&=c_1\sin^{1/2}(\frac{\theta}{2})\cos^{1/2}(\frac{\theta}{2})F(\frac{1}{2},\frac{1}{2},1;\sin^2(\frac{\theta}{2}))\\&\quad+c_2\sin^{1/2}(\frac{\theta}{2})\cos^{1/2}(\frac{\theta}{2})F(\frac{1}{2},\frac{1}{2},1;\sin^2(\frac{\theta}{2}))\int_{\sin^2(\theta/2)}^{1/2}\frac{dt}{t(1-t)F^2(\frac{1}{2},\frac{1}{2},1;t)}.\end{aligned}$$

for suitable c_1, c_2.

For our purposes it is useful to write the solution of (10) in case $\beta\geq\beta_{cr}$ as a power series

$$\psi(\theta)=\theta^{\alpha}\sum_{n=0}^{\infty}a_n\theta^n\ , \tag{11}$$

where α is the largest solution of the equation $\alpha(1-\alpha)=c_\beta$ in case $\beta>\beta_{cr}$ and $\alpha=1/2$ when $\beta=\beta_{cr}$. We normalize the power series setting $a_0=1$; simple computations then give

$$a_1=0\ ,\qquad a_2=-\frac{\alpha(1-\alpha)}{6(1+2\alpha)}. \tag{12}$$

We also define the auxiliary functions

$$f(\theta)=\frac{\psi'(\theta)}{\psi(\theta)}\ ,\qquad \theta\in(0,\beta)\ , \tag{13}$$

and

$$g(\theta)=\frac{\psi'(\theta)}{\psi(\theta)}\sin\theta\ ,\qquad \theta\in(0,\beta)\ , \tag{14}$$

where ψ is the normalized solution of (9) described in Lemma 1. We note that these functions depend on β. Simple computations show that they respectively solve the differential equations

$$f'(\theta)+f^2(\theta)+c_\beta V(\theta)=0\ ,\qquad 0<\theta<\beta \tag{15}$$

and

$$g'(\theta)=-\frac{1}{\sin\theta}\left[g(\theta)^2-\cos\theta\, g(\theta)+c_\beta\right]\ ,\quad 0<\theta\leq\pi/2. \tag{16}$$

We shall also need the following

Lemma 2. *Let $\pi\leq\beta\leq 2\pi$ and $\gamma\geq 0$ with $\beta+2\gamma\leq 3\pi$. Then*

$$f(\theta)\cos(\theta+\gamma)+\alpha[1+\sin(\theta+\gamma)]\geq 0\ ,\quad \frac{\pi}{2}\leq\theta\leq\frac{3\pi}{2}-\gamma.$$

Proof. We first note that

$$f(\theta)=\sqrt{c_\beta}\tan\Big(\sqrt{c_\beta}(\frac{\beta}{2}-\theta)\Big), \qquad \frac{\pi}{2}\le\theta\le\frac{3\pi}{2}-\gamma,$$

and

$$-\frac{\pi}{4}\le\sqrt{c_\beta}(\frac{\beta}{2}-\theta)\le\frac{\pi}{4}\,, \qquad \frac{\pi}{2}\le\theta\le\frac{3\pi}{2}-\gamma.$$

It follows that the required inequality is written equivalently,

$$\alpha(1+\sin(\gamma+\theta))\cos(\sqrt{c_\beta}(\frac{\beta}{2}-\theta)) \tag{17}$$

$$+\sqrt{c}\sin(\sqrt{c_\beta}(\frac{\beta}{2}-\theta))\cos(\gamma+\theta)\ge 0\,,\;\; \frac{\pi}{2}\le\theta\le\frac{3\pi}{2}-\gamma. \tag{18}$$

But, since $\alpha\ge\sqrt{c_\beta}$,

$$\begin{aligned}
&\alpha\;(1+\sin(\theta+\gamma))\cos(\sqrt{c_\beta}(\frac{\beta}{2}-\theta))+\sqrt{c_\beta}\sin(\sqrt{c_\beta}(\frac{\beta}{2}-\theta))\cos(\theta+\gamma)\\
&\ge\sqrt{c_\beta}\Big\{(1+\sin(\theta+\gamma))\cos(\sqrt{c_\beta}(\frac{\beta}{2}-\theta))+\sin(\sqrt{c_\beta}(\frac{\beta}{2}-\theta))\cos(\theta+\gamma)\Big\}\\
&=2\sqrt{c_\beta}\sin\Big[\sqrt{c_\beta}(\frac{\beta}{2}-\theta)+\frac{\pi}{4}+\frac{\theta}{2}+\frac{\gamma}{2}\Big]\sin(\frac{\pi}{4}+\frac{\theta}{2}+\frac{\gamma}{2}).
\end{aligned}\tag{19}$$

The second sine is clearly non-negative, so it only remains to prove that the first sine is also non-negative. For this we use the monotonicity of $\sqrt{c_\beta}(\frac{\beta}{2}-\theta)+\frac{\pi}{4}+\frac{\theta}{2}+\frac{\gamma}{2}$ with respect to θ to obtain

$$\begin{aligned}
\sqrt{c_\beta}(\frac{\beta}{2}-\theta)+\frac{\pi}{4}+\frac{\theta}{2}+\frac{\gamma}{2}&\le\sqrt{c_\beta}\big(\frac{\beta}{2}-(\frac{3\pi}{2}-\gamma)\big)+\frac{\pi}{4}+\frac{\frac{3\pi}{2}-\gamma}{2}+\frac{\gamma}{2}\\
&=\sqrt{c_\beta}\frac{\beta+2\gamma-3\pi}{2}+\pi\le\pi,
\end{aligned}\tag{20}$$

by our hypothesis $\beta+2\gamma\le 3\pi$. This completes the proof. □

We shall need to consider the initial value problem (21) below. Although this is a strongly singular problem, we shall see that standard comparison arguments hold. In particular we shall establish existence, uniqueness and monotonicity with respect to a parameter.

Lemma 3. *Consider the singular initial value problem*

$$\begin{cases} h'(\theta)=-\dfrac{1}{\sin\theta}\Big(\alpha h(\theta)^2-\cos\theta h(\theta)+1-\alpha\Big),\; 0<\theta\le\frac{\pi}{2},\\ h(0)=1.\end{cases}\tag{21}$$

(i) *If* $\alpha\in(1/2,1)$ *then the problem has a classical solution which is unique. The solution* $h(\alpha,\theta)$ *depends monotonically on* α*: if* $\alpha_1<\alpha_2$ *then* $h(\alpha_1,\theta)<h(\alpha_2,\theta)$ *for all* $\theta\in(0,\pi/2]$.

(ii) *For $\alpha = 1/2$ we do not have uniqueness. Indeed we have a continuum of positive solutions.*
(iii) *Let $1/2 < \alpha < 1$ and in addition let $\overline{h} \in C[0,\pi/2] \cap C^1(0,\pi/2]$ be an upper solution of problem (21), that is*

$$\begin{cases} \overline{h}'(\theta) \geq -\dfrac{1}{\sin\theta}\Big(\alpha\overline{h}(\theta)^2 - \cos\theta\overline{h}(\theta) + 1 - \alpha\Big),\ 0 < \theta \leq \frac{\pi}{2}, \\ \overline{h}(0) \geq 1. \end{cases} \tag{22}$$

Then

$$h(\alpha,\theta) \leq \overline{h}(\theta)\,, \qquad 0 \leq \theta \leq \frac{\pi}{2}.$$

Proof. (i) By Lemma 1 the function

$$\psi(\theta) = \sin^{\alpha}(\theta/2)\cos^{1-\alpha}(\theta/2)F(\frac{1}{2},\frac{1}{2},\alpha+\frac{1}{2};\sin^2(\frac{\theta}{2}))$$

solves the differential equation

$$\psi''(\theta) + \alpha(1-\alpha)\frac{\psi(\theta)}{\sin^2\theta} = 0\,, \quad 0 < \theta < \frac{\pi}{2}.$$

It is then easily verified that the function

$$h(\theta) = \frac{1}{\alpha}\frac{\psi'(\theta)}{\psi(\theta)}\sin\theta$$

is a solution of the initial-value problem (21).

We next establish the uniqueness of a solution. Let h_1, h_2 be two solutions of the initial value problem (21). Then the function $z = h_2 - h_1$ solves the singular linear initial value problem

$$\begin{cases} z'(\theta) = -\frac{1}{\sin\theta}\Big(\alpha(h_1+h_2) - \cos\theta\Big)z(\theta), \\ z(0) = 0. \end{cases}$$

Let us assume the z is not identically zero. By the standard uniqueness theorem, z cannot have any positive zeros, hence we may assume that $z(\theta) > 0$ for all $\theta \in (0,\pi/2)$. However we have $\alpha(h_1+h_2) - \cos\theta > 0$ near $\theta = 0$, hence z decreases near zero, which is a contradiction.

The monotonicity of the solution h with respect to α will follow from the monotonicity of the nonlinearity with respect to α. Let

$$V(\theta,h,\alpha) = -\frac{1}{\sin\theta}\Big(\alpha h^2 - \cos\theta h + 1 - \alpha\Big)$$

For $0 < h < 1$ and $0 < \theta < \pi/2$ we then have

$$\frac{\partial V}{\partial \alpha} = \frac{1-h^2}{\sin\theta} > 0. \tag{23}$$

Now, let $1/2 < \alpha_1 < \alpha_2 < 1$. By (23) we have $h(\alpha_2,\theta) > h(\alpha_1,\theta)$ near $\theta = 0$. Once we are away from $\theta = 0$ we can apply the standard comparison arguments to complete the proof.

(ii) By Lemma 1 the general solution of the equation

$$\psi''(\theta) + \frac{1}{4}\frac{\psi(\theta)}{\sin^2\theta} = 0\,, \quad 0 < \theta < \frac{\pi}{2},$$

is

$$\begin{aligned}\psi(\theta) &= c_1 \sin^{1/2}(\frac{\theta}{2})\cos^{1/2}(\frac{\theta}{2})F(\frac{1}{2},\frac{1}{2},1;\sin^2(\frac{\theta}{2}))\\ &\quad + c_2 \sin^{1/2}(\frac{\theta}{2})\cos^{1/2}(\frac{\theta}{2})F(\frac{1}{2},\frac{1}{2},1;\sin^2(\frac{\theta}{2}))\int_{\sin^2(\theta/2)}^{1/2}\frac{dt}{t(1-t)F^2(\frac{1}{2},\frac{1}{2},1;t)}.\end{aligned}$$

This is positive in $(0,\pi/2]$ when $c_1 > 0$ and $c_2 \geq 0$. For any such ψ the function

$$h(\theta) = \frac{2\psi'(\theta)}{\psi(\theta)}\sin\theta$$

then satisfies

$$h'(\theta) = -\frac{1}{2\sin\theta}\Big(h(\theta)^2 - 2\cos\theta h(\theta) + 1\Big)\,, \quad h(0) = 1.$$

Actually after some computations we find that the function h is given in this case by

$$\begin{aligned}h(\theta) &= \cos\theta + \sin^2\theta\,\frac{F(\frac{3}{2},\frac{3}{2},2;\sin^2(\frac{\theta}{2}))}{4F(\frac{1}{2},\frac{1}{2},1;\sin^2(\frac{\theta}{2}))}\\ &\quad - \frac{\lambda}{F^2(\frac{1}{2},\frac{1}{2},1;\sin^2(\frac{\theta}{2}))\Big(1+\lambda\int_{\sin^2(\theta/2)}^{1/2}\frac{dt}{t(1-t)F^2(\frac{1}{2},\frac{1}{2},1;t)}\Big)},\end{aligned}$$

where $\lambda = c_2/c_1 \geq 0$.

(iii) When $\overline{h}(0) > 1$ the result follows immediately by combining continuity with standard comparison arguments. Assume now that $h(0) = 1$. The function $z = \overline{h} - h$ then satisfies

$$\begin{cases} z'(\theta) \geq -\frac{1}{\sin\theta}\Big(\alpha(\overline{h}+h) - \cos\theta\Big)z(\theta),\\ z(0) = 0.\end{cases} \tag{24}$$

The quantity $\alpha(\overline{h}+h) - \cos\theta$ is positive near $\theta = 0$, say in $(0,\theta_0)$. We shall establish that $z \geq 0$ in this interval; the result for $(0,\pi/2)$ will then follow immediately. Suppose on the contrary that there exists an interval $(\theta_1,\theta_2) \subset (0,\theta_0)$ such that $z < 0$

in (θ_1,θ_2). By (24) we conclude that z is actually strictly increasing in (θ_1,θ_2). This contradicts the initial value $z(0)=0$. □

From Lemma 3 it follows that the case $\alpha=1/2$ is critical and needs a different approach. This will be done in the next lemma. In order to make explicit the dependence on β we denote

$$g(\beta,\theta)=\frac{\psi_\theta(\beta,\theta)}{\psi(\beta,\theta)}\sin\theta\ ,$$

where $\psi(\beta,\theta)$ is the solution of (9) and $\psi_\theta(\beta,\theta)$ is the derivative with respect to θ.

Lemma 4. *Suppose $\pi\le\beta\le\beta_{cr}$. Then $g(\beta,\theta)$, $0<\theta\le\pi/2$, is strictly increasing as a function of β, that is, if $\pi\le\beta_1<\beta_2\le\beta_{cr}$ then $g(\beta_1,\theta)<g(\beta_2,\theta)$ for all $\theta\in(0,\pi/2]$.*

Proof. The function $g(\beta,\theta)$ solves the differential equation

$$\frac{\partial g}{\partial\theta}=-\frac{1}{\sin\theta}\Big(g^2-g\cos\theta+\frac{1}{4}\Big). \tag{25}$$

Since

$$g(\beta,\frac{\pi}{2})=\frac{1}{2}\tan(\frac{\beta-\pi}{4}),$$

which is strictly increasing with respect to β, the result follows from a standard comparison argument. □

Let us note here that for $\pi\le\beta\le\beta_{cr}$ we have $g(\beta,0)=1/2$. So the functions $g(\beta,\cdot)$, $\pi\le\beta\le\beta_{cr}$, all solve the same initial value problem.

Lemma 5. *Let $\beta\in[\pi,2\pi]$. There exists an angle γ^*_β so that for all $0<\gamma\le\gamma^*_\beta$ we have*

$$g(\beta,\theta)\cos(\theta+\frac{\gamma}{2})+\alpha\cos\frac{\gamma}{2}\ge 0\ ,\qquad 0\le\theta\le\frac{\pi}{2}. \tag{26}$$

Moreover γ^*_β is a strictly decreasing function of β and in particular:

$$\begin{aligned}&\textit{for } \pi\le\beta\le\beta_{cr}\quad \textit{we have}\quad 0.701\pi\approx\gamma^*_{\beta_{cr}}\le\gamma^*_\beta\le\gamma^*_\pi\approx 0.867\pi\\ &\textit{for } \beta_{cr}\le\beta\le 2\pi\quad \textit{we have}\quad 0.673\pi\approx\gamma^*_{2\pi}\le\gamma^*_\beta\le\gamma^*_{\beta_{cr}}\approx 0.701\pi.\end{aligned} \tag{27}$$

Proof. Inequality (26) is written equivalently

$$\cot\frac{\gamma}{2}\ge\frac{\sin\theta}{\cos\theta+\frac{\alpha}{g(\beta,\theta)}}, \tag{28}$$

so what matters is the maximum of the function at the RHS of (28). For each $0<\theta\le\pi/2$ this function is strictly monotone as a function of β; this follows from Lemma 3 for $\beta_{cr}\le\beta\le 2\pi$ and from Lemma 4 for $\pi\le\beta\le\beta_{cr}$.

The angle $\gamma_\beta^* \in (0,\pi)$ defined by

$$\cot\frac{\gamma_\beta^*}{2} = \max_{[0,\pi/2]} \frac{\sin\theta}{\cos\theta + \frac{\alpha}{g(\beta,\theta)}}$$

is then a strictly increasing function of β. The approximate values in the statement have been obtained by numerical computations; see however Lemma 6. □

It would be nice to have good estimates on γ_β^* without using a numerical solution of the differential equation (16) solved by $g(\theta)$. This will be done for $\beta_{cr} \le \beta \le 2\pi$ by obtaining very good upper estimates on $g(\beta,\theta)$. We define

$$\overline{g}(\beta,\theta) = a - \frac{a}{2(2a+1)}\theta^2 + \frac{a(4a^2+2a+3)}{24(2a+1)(4a^2+8a+3)}\theta^4, \quad 0<\theta<\frac{\pi}{2},$$

where a is the largest solution of $a(1-a) = c_\beta$. We define the auxiliary quantity $\gamma_\beta^{**} \in (0,\pi)$ by

$$\cot\frac{\gamma_\beta^{**}}{2} = \max_{[0,\pi/2]} \frac{\sin\theta}{\cos\theta + \frac{\alpha}{\overline{g}(\beta,\theta)}}.$$

Lemma 6. *Let $\beta_{cr} \le \beta \le 2\pi$. Then we have*

$$\text{(i)}\ \ g(\beta,\theta) \le \overline{g}(\beta,\theta)\,, \qquad 0<\theta<\frac{\pi}{2},$$

$$\text{(ii)}\ \ \gamma_\beta^{**} \le \gamma_\beta^*.$$

Actually we have (cf (27))

$$\gamma_{\beta_{cr}}^{**} \approx 0.700\pi\,, \qquad \gamma_{2\pi}^{**} \approx 0.672\pi\,.$$

Proof. We have $g(\beta,0) = \overline{g}(\beta,0) = \alpha$. Therefore, given that $g(\beta,\theta)$ satisfies

$$\frac{\partial g}{\partial\theta} = -\frac{1}{\sin\theta}\Big(g^2 - g\cos\theta + c_\beta\Big), \tag{29}$$

it is enough to show that

$$\frac{\partial\overline{g}}{\partial\theta} \ge -\frac{1}{\sin\theta}\Big(\overline{g}^2 - \overline{g}\cos\theta + c_\beta\Big). \tag{30}$$

The function $\overline{g}(\beta,\theta)$ is decreasing with respect to θ, hence

$$\begin{aligned}\sin\theta\frac{d\overline{g}}{d\theta} &+ \overline{g}^2 - (\cos\theta)\overline{g} + c_\beta \\ &\ge \Big(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}\Big)\frac{d\overline{g}}{d\theta} + \overline{g}^2 - \Big(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}\Big)\overline{g} + c_\beta\,. \end{aligned} \tag{31}$$

Now, a direct computation shows that the RHS of (31) is equal to

$$\frac{a(1-a)\theta^6[16(2a+3)(2a+1)(22a^2+2a+3)-(12a^2+2a+3)(4a^2+2a+3)\theta^2]}{2880(2a+1)^2(4a^2+8a+3)^2}$$

$$\geq \frac{a(1-a)(12a^2+2a+3)(4a^2+2a+3)\theta^6(16-\theta^2)}{2880(2a+1)^2(4a^2+8a+3)^2}$$
$$\geq 0.$$

We note that in our argument we only used that $\alpha \in [1/2, 1)$.

We now establish (i) for $\beta_{cr} < \beta \leq 2\pi$. The function

$$\overline{h}(\alpha,\theta) = \frac{\overline{g}(\beta,\theta)}{\alpha}$$

(where, as usual, α is the largest solution of $\alpha(1-\alpha) = c_\beta < 1/4$) is an upper solution to the initial value problem (21). Hence applying (iii) of Lemma 3 we obtain the comparison.

To obtain (i) for $\beta = \beta_{cr}$ we use the monotonicity with respect to α of $h(\alpha,\theta)$. Passing to the limit $\alpha \to 1/2+$ we conclude that

$$H(\theta) := \lim_{\alpha\to 1/2+} h(\alpha,\theta) \leq \overline{h}(\frac{1}{2},\theta) \leq 2\overline{g}(\beta_{cr},\theta)\,, \qquad 0 < \theta < \frac{\pi}{2}.$$

The function $H(\theta)$ is then the maximal solution of the singular initial value problem (21) and therefore coincides with the function $2g(\beta_{cr},\theta)$. This completes the proof of (i). Part (ii) then follows immediately from (i). □

3 Proof of Theorem 1

In this section we give the proofs of our theorems. We start with a proposition that is fundamental in our argument and will be used repeatedly. We do not try to obtain the most general statement and for simplicity we restrict ourselves to assumptions that are sufficient for our purposes.

Let U be a domain and assume that $\partial U = \Gamma \cup \Gamma_0$ where Γ is Lipschitz continuous. We denote by $\mathbf{n}$ the exterior unit normal on Γ.

Proposition 1. *Let $\phi \in H^1_{\rm loc}(U)$ be a positive function such that $\nabla\phi/\phi \in L^2(U)$ and $\nabla\phi/\phi$ has an L^1 trace on Γ in the sense that $v\nabla\phi/\phi$ has an L^1 trace on ∂U for every $v \in C^\infty(\overline{U})$ that vanishes near Γ_0. Then*

$$\int_U |\nabla u|^2 dx\,dy \geq -\int_U \frac{\Delta\phi}{\phi} u^2 dx\,dy + \int_\Gamma \frac{\nabla\phi}{\phi}\cdot \mathbf{n} u^2 dS \tag{32}$$

for all smooth functions u which vanish near Γ_0 and $\Delta\phi$ is understood in the weak sense.
If in particular there exists $c \in \mathbf{R}$ such that

$$-\Delta\phi \geq \frac{c}{d^2}\phi\ , \tag{33}$$

in the weak sense in U, where $d = \mathrm{dist}(x,\Gamma_0)$, then

$$\int_U |\nabla u|^2 dxdy \geq c\int_U \frac{u^2}{d^2}dxdy + \int_\Gamma u^2\frac{\nabla\phi}{\phi}\cdot \mathbf{n}dS \tag{34}$$

for all functions $u \in C^\infty(\overline{U})$ that vanish near Γ_0.

Proof. Let u be a function in $C^\infty(\overline{U})$ that vanishes near Γ_0. We denote $\mathbf{T} = -\nabla\phi/\phi$. Then

$$\begin{aligned}\int_U u^2 \mathrm{div}\mathbf{T}\,dxdy &= -2\int_U u\nabla u\cdot\mathbf{T}\,dxdy + \int_\Gamma u^2\mathbf{T}\cdot\mathbf{n}\,dS\\ &\leq \int_U |\mathbf{T}|^2u^2dxdy + \int_U |\nabla u|^2dxdy + \int_\Gamma u^2\mathbf{T}\cdot\mathbf{n}\,dS,\end{aligned}$$

that is

$$\int_U |\nabla u|^2dxdy \geq \int_U(\mathrm{div}\mathbf{T} - |\mathbf{T}|^2)u^2dxdy - \int_\Gamma \mathbf{T}\cdot\mathbf{n}u^2dS.$$

Using assumption (33) we obtain (34). □

For $\beta \in (\pi, 2\pi]$ we denote by Π_β the class of all planar polygons which have precisely one non-convex vertex and the angle at that vertex is β. Given a polygon in Π_β we denote by γ_+ and γ_- the angles at the vertices next to the non-convex vertex.

Theorem 4. *Let $\beta \in (\pi,2\pi]$. Let Ω be a polygon in Π_β with*

$$\gamma_+,\gamma_- \leq \min\{\gamma^*_\beta, \frac{3\pi-\beta}{2}\} \tag{35}$$

where $\gamma^*_\beta \in (0,\pi)$ is defined by

$$\cot\frac{\gamma^*_\beta}{2} = \max_{[0,\pi/2]}\frac{\sin\theta}{\cos\theta + \frac{\alpha}{g(\beta,\theta)}}.$$

Then the Hardy constant of Ω is c_β.

Proof. We denote by A_-, A_+ the vertices next to the non-convex vertex O, so that A_-, O and A_+ are consecutive vertices with respective angles γ_-, β and γ_+. We may assume that O is the origin and that A_+ lies on the positive x-semiaxis. We write the boundary $\partial\Omega$ as

$$\partial\Omega = S_1 \cup S_2$$

where $S_1 = OA_+ \cup OA_-$ and $S_2 = \partial\Omega \setminus S_1$. We then define the equidistance curve

$$\Gamma = \{x \in \partial\Omega : \mathrm{dist}(x, S_1) = \mathrm{dist}(x, S_2)\}.$$

Hence Γ divides Ω into two sets Ω_1 and Ω_2, whose nearest boundary points belong in S_1 and S_2 respectively. It is clear that Γ can be parametrized by the polar angle $\theta \in [0, \beta]$.

The curve Γ consists of line segments and parabola segments. Starting from $\theta = 0$ we have line segments $L_1, \ldots, L_k$; then from $\theta = \pi/2$ to $\theta = \beta - \pi/2$ we have parabola segments $P_1, \ldots, P_m$; and from $\theta = \beta - \pi/2$ to $\theta = \beta$ we have again line segments $L_1', \ldots, L_n'$.

Let $u \in C_c^\infty(\Omega)$ be given. Let $\mathbf{n}$ denote the unit normal along Γ which is outward with respect to Ω_1. Applying Proposition 1 with $\phi(x,y) = \psi_\beta(\theta)$, where θ is the polar angle of the point (x,y), we obtain

$$\int_{\Omega_1} |\nabla u|^2 dx\,dy \geq c_\beta \int_{\Omega_1} \frac{u^2}{d^2} dx\,dy + \int_\Gamma \frac{\nabla\phi}{\phi} \cdot \mathbf{n}\, u^2 dS. \tag{36}$$

We next apply Proposition 1 on Ω_2 for the function $\phi_1(x,y) = d(x,y)^\alpha$ (we recall that α is the largest solution of $\alpha(1-\alpha) = c_\beta$). In Ω_2 the function $d(x,y)$ coincides with the distance from S_2 and this implies that

$$-\Delta d^\alpha \geq \alpha(1-\alpha)\frac{d^\alpha}{d^2}\,, \qquad \text{on } \Omega_+\,.$$

Applying Proposition 1 we obtain that

$$\int_{\Omega_+} |\nabla u|^2 dx\,dy \geq c \int_{\Omega_+} \frac{u^2}{d^2} dx\,dy - \int_\Gamma \frac{\alpha\nabla d}{d} \cdot \mathbf{n}\, u^2 dS\,. \tag{37}$$

Adding (36) and (37) we conclude that

$$\int_{\Omega} |\nabla u|^2 dx\,dy \geq c \int_{\Omega} \frac{u^2}{d^2} dx\,dy + \int_\Gamma \Big(\frac{\nabla\phi}{\phi} - \alpha\frac{\nabla d}{d}\Big) \cdot \mathbf{n}\, u^2 dS\,. \tag{38}$$

We emphasize that in the last integral the values of $\nabla\phi/\phi$ are obtained as limits from Ω_1 and, more importanmtly, those of $\nabla d/d$ are obtained as limits from Ω_2.

It remains to prove that the line integral in (38) is non-negative. For this we shall consider the different segments of Γ. Due to the symmetry of our assumptions with respect to $\theta = \beta/2$ it is enough to establish the result for $0 \leq \theta \leq \beta/2$.

(i) Let L be one of the line segments $L_1, \ldots, L_k$. The points on this segment L are equidistant from the side OA_+ and some side E of $\partial\Omega \setminus (OA_+ \cup OA_-)$. Let γ be the angle formed by the line E and the x-axis so that the outward normal vector along E is $(\sin\gamma, \cos\gamma)$ and E has equation $x\cos\gamma + y\sin\gamma + c = 0$ for some $c \in \mathbf{R}$. Elementary geometric considerations then give $\gamma \in (-\pi/2, \pi)$. Now, simple computations give

$$\Big(\frac{\nabla\phi}{\phi}-\alpha\frac{\nabla d}{d}\Big)\cdot\mathbf{n}=\frac{1}{d}\Big(g(\theta)\cos(\theta+\frac{\gamma}{2})+\alpha\cos(\frac{\gamma}{2})\Big)\,,\qquad \text{on } L. \tag{39}$$

It remains to show that the RHS of (39) is non-negative for $0\leq\theta\leq\pi/2$. In the case $0<\gamma<\pi$ this is equivalent to showing that

$$\cot\frac{\gamma}{2}\geq\frac{\sin\theta}{\cos\theta+\frac{\alpha}{g(\theta)}}\,,\quad 0\leq\theta\leq\frac{\pi}{2}. \tag{40}$$

This is true since $\gamma\leq\gamma_+\leq\gamma^*_\beta$.

In the case $-\pi/2<\gamma\leq 0$ we have $\cos(\theta+\frac{\gamma}{2})\geq 0$ for all $0\leq\theta\leq\pi/2$ and the RHS is clearly non-negative.

(ii) Let P be one of the parabola segments $P_1,\dots,P_m$. The points on P are equidistant from the origin O and some side E of $\partial\Omega\setminus(OA_+\cup OA_-)$. As in (i) above, let γ be the angle formed by the line E and the x-axis so that the outward normal vector along E is $(\sin\gamma,\cos\gamma)$ and E has equation $x\cos\gamma+y\sin\gamma+c=0$ for some $c\in\mathbf{R}$. Then $\gamma\in[\pi-\frac{\beta}{2},\pi]$. We note that the axis of the parabola has an asymptote at angle $\theta=\frac{3\pi}{2}-\gamma$. Indeed we shall prove the required inequality for all $\theta\in[\frac{\pi}{2},\frac{3\pi}{2}-\gamma]\supset[\frac{\pi}{2},\frac{\beta}{2}]$.

Simple computations on P give

$$\Big(\frac{\nabla\phi}{\phi}-\alpha\frac{\nabla d}{d}\Big)\cdot\mathbf{n}=\frac{1}{r\sqrt{2+2\sin(\theta+\gamma)}}\Big(f(\theta)\cos(\theta+\gamma)+\alpha[1+\sin(\theta+\gamma)]\Big)\,. \tag{41}$$

Hence, noting that $\gamma\leq\gamma_+$, the result follows from Lemma 2. This completes the proof. □

Proof of Theorem 1. This follows easily by approximating the convex set K by a sequence of convex polygons and using Theorem 4; see Fig. 1. □

Remark. In case $\beta\leq\beta_{cr}$ we have $\gamma^*_\beta\leq\gamma^*_{\beta_{cr}}\approx 0.701\pi$ and therefore the condition $\gamma_+,\gamma_-\leq\min\{\gamma^*_\beta,\frac{3\pi-\beta}{2}\}$ of Theorems 1 and 4 takes the simpler form

$$\gamma_+,\gamma_-\leq\gamma^*_\beta.$$

If the convex set K is unbounded and ∂K does not intersect the boundary of Λ_β then there is no need for any restriction. In particular

Theorem 5. *Let $\Omega=K\cap\Lambda_\beta$ K is an unbounded convex set and Λ_β is a sector of angle $\beta\in(\pi,2\pi]$ whose vertex is inside K. Assume that the boundaries of K and Λ_β do not intersect. Then the Hardy constant of Ω is c_β, where c_β is given by (5), (6).*

Proof. Let $u\in C^\infty_c(\Omega)$ be fixed. There exists a bounded convex set K_1 such that $\Omega_1:=K_1\cap S_\beta$ satisfies all the assumptions of Theorem 1 and in addition

$$\mathrm{dist}(x,\partial\Omega)=\mathrm{dist}(x,\partial\Omega_1)\,,\qquad x\in\mathrm{supp}(u)\,;$$

of course, K_1 depends on u. Applying Theorem 1 to Ω_1 we obtain the required Hardy inequality. $\square$

Remark. Of course, one could state an intermediate result where the intersection $\partial K \cap \partial \Lambda_\beta$ is exactly one point forming an angle γ; in this the assumption $\gamma \leq \min\{\gamma_\beta^*, \frac{3\pi-\beta}{2}\}$ should hold.

4 Domains $E_{\beta,\gamma}$ with two non-convex angles

We reacall from the Introduction that given angles β and γ, we denote by $E_{\beta,\gamma}$ the domain shown in Fig. 2 in case $\gamma < \pi$ and in Fig. 3 in case $\gamma > \pi$. Its boundary $\partial E_{\beta,\gamma}$ consists of three parts L_1, L_2 and L_3. L_2 is a line segment and meets the halflines L_3 and L_1 at the origin O and the point $P(1,0)$ respectively. We assume that $\beta+\gamma \leq 3\pi$ so that the halflines L_1 and L_3 do not intersect. Without loss of generality we assume that $\beta \geq \gamma$ and since we are interested in the non-convex case, we assume that $\beta > \pi$.

Proof of Theorem 2 part (i). We denote by Γ the curve

$$\Gamma = \{(x,y) \in E_{\beta,\gamma} : \mathrm{dist}((x,y),L_1) = \mathrm{dist}((x,y),L_2 \cup L_3)\}.$$

The curve Γ divides $E_{\beta,\gamma}$ in two sets $E_- = \{(x,y) \in E_{\beta,\gamma} : d(x,y) = \mathrm{dist}((x,y),L_2 \cup L_3)\}$ and $E_+ = \{(x,y) \in E_{\beta,\gamma} : d(x,y) = \mathrm{dist}((x,y),L_1)\}$. We denote by $\mathbf{n}$ the unit normal along Γ which is outward with respect to E_-.

Once again we shall use Proposition 1. We distinguish two cases: Case A, where $0 \leq \gamma \leq \pi/2$ and Case B, where $\pi/2 \leq \gamma \leq \pi$.

Case A ($0 \leq \gamma \leq \pi/2$) We distinguish two subcases.

Subcase Aa. $\beta+\gamma < 2\pi$. In this case Γ consists of three parts: a line segment Γ_1 which bisects the angle at P; a parabola segment Γ_2, whose points are equidistant from the origin and the line L_1; and a halfline Γ_3 whose points are equidistant from L_1 and L_3. We parametrize Γ by the polar angle θ, so that $\Gamma_1 = \{0 \leq \theta \leq \frac{\pi}{2}\}$, $\Gamma_2 = \{\frac{\pi}{2} \leq \theta \leq \beta - \frac{\pi}{2}\}$, and $\Gamma_3 = \{\beta - \frac{\pi}{2} \leq \theta < \frac{\beta+\pi-\gamma}{2}\}$.

Let $u \in C_c^\infty(E_{\beta,\gamma})$. We apply Proposition 1 with $U = E_-$, $\Gamma_0 = L_2 \cup L_3$ and for the function $\phi(x,y) = \psi(\theta)$, where $\psi = \psi_\beta$ and θ is the polar angle of (x,y). We obtain that

$$\int_{E_-} |\nabla u|^2 dx\,dy \geq c_\beta \int_{E_-} \frac{u^2}{d^2} dx\,dy + \int_\Gamma \frac{\nabla\phi}{\phi} \cdot \mathbf{n} u^2\, dS. \tag{42}$$

We next apply Proposition 1 to the domain E_+ and the function $\phi_1(x,y) = d(x,y)^\alpha$. We obtain that

$$\int_{E_+} |\nabla u|^2 dx\,dy \geq c_\beta \int_{E_+} \frac{u^2}{d^2} dx\,dy - \alpha \int_\Gamma \frac{\nabla d}{d} \cdot \mathbf{n} u^2 dS. \tag{43}$$

Adding (42) and (43) we conclude that

$$\int_{E_{\beta,\gamma}}|\nabla u|^2dxdy\geq c_\beta\int_{E_{\beta,\gamma}}\frac{u^2}{d^2}dxdy+\int_\Gamma\Big(\frac{\nabla\phi}{\phi}-\alpha\frac{\nabla d}{d}\Big)\cdot\mathbf{n}\,u^2dS. \tag{44}$$

We note that in the last integral the values of $\nabla\phi/\phi$ are obtained as limits from E_- while those of $\nabla d/d$ are obtained as limits from E_+. It remains to prove that the last integral in (44) is non-negative. For this we shall consider the different parts of Γ.

(i) The segment Γ_1 $(0\leq\theta\leq\pi/2)$. Simple computations give that

$$\frac{\nabla\phi}{\phi}-\alpha\frac{\nabla d}{d}=\frac{1}{d}\Big(g(\theta)\cos(\theta+\frac{\gamma}{2})+\alpha\cos(\frac{\gamma}{2})\Big)\,,\qquad 0<\theta\leq\frac{\pi}{2}\,;$$

this is non-negative by Lemma 5, since $\gamma^*_\beta>\pi/2$.

(ii) The segment Γ_2 $(\pi/2\leq\theta\leq\beta-\pi/2)$. In this case we have

$$\Big(\frac{\nabla\phi}{\phi}-\alpha\frac{\nabla d}{d}\Big)\cdot\mathbf{n}=\frac{1}{r\sqrt{2+2\sin(\theta+\gamma)}}\Big(f(\theta)\cos(\theta+\gamma)+\alpha[1+\sin(\theta+\gamma)]\Big)\ ,$$

this is non-negative by Lemma 2, since $\beta-\frac{\pi}{2}<\frac{3\pi}{2}-\gamma$.

(iii) The segment Γ_3 $(\beta-\frac{\pi}{2}\leq\theta<\frac{\beta+\pi-\gamma}{2})$. The line containing Γ_3 has equation

$$x\cos(\frac{\beta-\gamma}{2})+y\sin(\frac{\beta-\gamma}{2})=\frac{\sin\gamma}{2\sin(\frac{\beta+\gamma}{2})},$$

hence the outer (with respect to E_-) unit normal along Γ_3 is $(\cos(\frac{\beta-\gamma}{2}),\sin(\frac{\beta-\gamma}{2}))$. Using the fact that $d=r\sin(\beta-\theta)$ on Γ_3, we have along Γ_3,

$$\begin{aligned}\Big(\frac{\nabla\phi}{\phi}-\alpha\frac{\nabla d}{d}\Big)\cdot\mathbf{n}&=[\frac{1}{r}\frac{\psi'(\theta)}{\psi(\theta)}(-\sin\theta,\cos\theta)+\alpha\frac{(\sin\gamma,\cos\gamma)}{d}]\\&\quad\cdot\ (\cos(\frac{\beta-\gamma}{2}),\sin(\frac{\beta-\gamma}{2}))\\&=\frac{1}{r}\Big[\frac{\psi'(\theta)}{\psi(\theta)}\sin(\frac{\beta-\gamma}{2}-\theta)+\alpha\frac{\sin(\frac{\beta+\gamma}{2})}{\sin(\beta-\theta)}\Big]\\&\geq 0,\end{aligned}$$

since both terms in the last sum are non-negative (the first one, as the product of two non-positive terms).

Subcase Ab. $\beta+\gamma\geq 2\pi$. In this case Γ consists of only two parts Γ_1 and Γ_2, described exactly as in subcase Aa, the only difference being that the range of θ in Γ_2 is $\frac{\pi}{2}\leq\theta<\frac{3\pi}{2}-\gamma$. This means that the parabola segment goes all the way to infinity. As before we have

$$\Big(\frac{\nabla\phi}{\phi}-\alpha\frac{\nabla d}{d}\Big)\cdot\mathbf{n}=\frac{1}{r\sqrt{2+2\sin(\theta+\gamma)}}\Big(\frac{\psi'(\theta)}{\psi(\theta)}\cos(\theta+\gamma)+\alpha[1+\sin(\theta+\gamma)]\Big)$$

and the result follows again from Lemma 2. This completes the proof in the case $0<\gamma\leq\pi/2$.

Case B ($\pi/2\leq\gamma\leq\pi$). On E_- we again consider the function $\phi(x,y)=\psi(\theta)$ and apply Lemma 1 as in the previous case. We fix a function $u\in C_c^\infty(E_{\beta,\gamma})$ and we obtain

$$\int_{E_-}|\nabla u|^2dxdy\geq c_\beta\int_{E_-}\frac{u^2}{d^2}dxdy+\int_\Gamma(\frac{\nabla\phi}{\phi}\cdot\mathbf{n})u^2dS. \tag{45}$$

In E_+ we consider a new orthonormal coordinate system with cartesian coordinates denoted by (x_1,y_1) and polar coordinates denoted by (r_1,θ_1). The origin O_1 of this system is located on the line L_1 and is such that the line OO_1 is perpendicular to L_1. The positive x_1 axis is then chosen so as to contain L_1 (diagram) We note that this choice is such that

$$\text{the point on } \Gamma_1 \text{ for which } \theta=\tfrac{\pi}{2}-\tfrac{\gamma}{2} \text{ satisfies also } \theta_1=\tfrac{\pi}{2}-\tfrac{\gamma}{2}. \tag{46}$$

We apply Proposition 1 on E_+ with the function $\phi_1(x,y)=\psi(\theta_1)$. This function clearly satisfies $-\Delta\phi_1=cd^{-2}\phi_1$, hence we obtain

$$\int_{E+}|\nabla u|^2dxdy\geq c\int_{E_+}\frac{u^2}{d^2}dxdy-\int_\Gamma(\frac{\nabla\phi_1}{\phi_1}\cdot\mathbf{n})u^2dS, \tag{47}$$

where, as before, $\mathbf{n}$ is the interior to E_+ unit normal along Γ.

Adding (45) and (47) we conclude that

$$\int_{E_{\beta,\gamma}}|\nabla u|^2dxdy\geq c_\beta\int_{E_{\beta,\gamma}}\frac{u^2}{d^2}dxdy+\int_\Gamma\Big(\frac{\nabla\phi}{\phi}-\frac{\nabla\phi_1}{\phi_1}\Big)\cdot\mathbf{n}u^2dS. \tag{48}$$

The rest of the proof is devoted to showing that the last integral in (48) is non-negative.

As in the case $0<\gamma\leq\pi/2$, we need to distinguish two subcases: Subcase Ba, where $\beta+\gamma<2\pi$, and Subcase Bb, where $\beta+\gamma\geq 2\pi$.

Subcase Ba. $\beta+\gamma<2\pi$. The curve Γ consists of three parts: a line segment Γ_1 which bisects the angle at P; a (part of a) parabola Γ_2, whose points are equidistant from the origin and the line L_1; and a halfline Γ_3 whose points are equidistant from L_1 and L_3. As before, we consider separetely each segment and we parametrize Γ by the polar angle θ so that

$$\Gamma_1=\{\theta\in\Gamma:0\leq\theta\leq\frac{\pi}{2}\},\ \Gamma_2=\{\frac{\pi}{2}\leq\theta\leq\beta-\frac{\pi}{2}\},$$

$$\Gamma_3=\{\beta-\frac{\pi}{2}\leq\theta<\frac{\beta+\pi-\gamma}{2}\}.$$

(i) The segment Γ_1 ($0\leq\theta\leq\pi/2$). We have

$$\frac{\nabla\phi}{\phi}\cdot\mathbf{n}=\frac{\psi'(\theta)}{r\psi(\theta)}\cos(\theta+\frac{\gamma}{2}),\quad \text{on } \Gamma_1.$$

and similarly

$$\frac{\nabla\phi_1}{\phi_1}\cdot\mathbf{n} = -\frac{\psi'(\theta_1)}{r_1\psi(\theta_1)}\cos(\theta_1-\frac{\gamma}{2}) , \quad \text{on } \Gamma_1 .$$

Since $r_1\sin\theta_1 = r\sin\theta$ along Γ_1, it is enough to prove the inequality

$$g(\theta)\cos(\theta+\frac{\gamma}{2})+g(\theta_1)\cos(\theta_1-\frac{\gamma}{2})\geq 0 , \qquad 0\leq\theta\leq\frac{\pi}{2}. \tag{49}$$

This has been proved in [8]; we include a proof here for the sake of completeness. Recalling (46) and applying the sine law we obtain that along Γ_1 the polar angles θ and θ_1 are related by

$$\cot\theta_1 = -\cos\gamma\cot\theta+\sin\gamma . \tag{50}$$

Claim. There holds

$$\theta_1\geq\theta+\gamma-\pi , \qquad \text{on } \Gamma_1 . \tag{51}$$

Proof of Claim. We fix $\theta\in[0,\pi/2]$ and the corresponding $\theta_1=\theta_1(\theta)$. If $\theta+\gamma-\pi\leq 0$, then (51) is obviously true, so we assume that $\theta+\gamma-\pi\geq 0$. Since $0\leq\theta+\gamma-\pi\leq\pi/2$ and $0\leq\theta_1\leq\pi/2$, (51) is written equivalently $\cot\theta_1\leq\cot(\theta+\gamma-\pi)$; thus, recalling (50), we conclude that to prove the claim it is enough to show that

$$-\cos\gamma\cot\theta+\sin\gamma\leq\cot(\theta+\gamma) , \quad \pi-\gamma\leq\theta\leq\frac{\pi}{2},$$

or, equivalently (since $\pi\leq\theta+\gamma\leq 3\pi/2$),

$$-\cos\gamma\cot^2\theta+(-\cos\gamma\cot\gamma-\cot\gamma+\sin\gamma)\cot\theta+1+\cos\gamma\geq 0 , \pi-\gamma\leq\theta\leq\frac{\pi}{2}. \tag{52}$$

The left-hand side of (52) is an increasing function of $\cot\theta$ and therefore takes its least value at $\cot\theta=0$. Hence the claim is proved.

For $0\leq\theta\leq\pi/2-\gamma/2$ (49) is true since all terms in the left-hand side are non-negative. So let $\pi/2-\gamma/2\leq\theta\leq\pi/2$ and $\theta_1=\theta_1(\theta)$. From (50) we find that

$$\begin{aligned}\frac{d\theta_1}{d\theta}-1 &= -\frac{\cos\gamma(1+\cot^2\theta)+1+\cot^2\theta_1}{1+\cot^2\theta_1}\\ &= -\frac{1+\sin^2\gamma+\cos\gamma-2\sin\gamma\cos\gamma\cot\theta+\cos\gamma(1+\cos\gamma)\cot^2\theta}{1+\cot^2\theta_1}.\end{aligned}$$

The function

$$h(x) := 1+\sin^2\gamma+\cos\gamma-2\sin\gamma\cos\gamma x+\cos\gamma(1+\cos\gamma)x^2$$

is a concave function of x. We will establish the positivity of $h(\cot\theta)$ for $\pi/2-\gamma/2\leq\theta\leq\pi/2$. For this it is enough to establish the positivity at the endpoints. At $\theta=\pi/2$ positivity is obvious, whereas

$$h(\tan(\frac{\gamma}{2})) = 1+\sin^2\gamma+\cos\gamma-2\cos\gamma\sin^2\frac{\gamma}{2}\geq 0.$$

From (46) we conclude that $\theta_1 \le \theta$ for $\pi/2-\gamma/2 \le \theta \le \pi/2$. Now, it was proved in [8, Lemma 4] that the function g is decreasing. Hence for $\pi/2-\gamma/2 \le \theta \le \pi/2$ we have,

$$\begin{aligned} g(\theta)\cos(\theta+\frac{\gamma}{2})+g(\theta_1)\cos(\theta_1-\frac{\gamma}{2}) &\ge g(\theta)[\cos(\theta+\frac{\gamma}{2})+\cos(\theta_1-\frac{\gamma}{2})] \\ &= 2g(\theta)\cos(\frac{\theta+\theta_1}{2})\cos(\frac{\theta-\theta_1+\gamma}{2}) \\ &\ge 0, \end{aligned}$$

where for the last inequality we made use of the claim. Hence (49) has been proved.

(ii) The segment Γ_2 ($\frac{\pi}{2} \le \theta \le \beta-\frac{\pi}{2}$). After some computations we obtain that

$$\Big(\frac{\nabla\phi}{\phi}-\frac{\nabla\phi_1}{\phi_1}\Big)\cdot\mathbf{n} = \frac{1}{r\sqrt{2+2\sin(\theta+\gamma)}}\ \Big\{\ f(\theta)\cos(\theta+\gamma) - f(\theta_1)\sin\theta_1[\sin(\theta_1-\theta-\gamma)-\cos\theta_1]\Big\},$$

where θ and θ_1 are related by $\cot\theta_1 = -\cos(\theta+\gamma)$. The result then follows by applying [8, Lemma 6].

(iii) The segment Γ_3 ($\beta-\frac{\pi}{2} \le \theta < \frac{\beta+\pi-\gamma}{2}$). Simple computations yield that along Γ_3 we have

$$\Big(\frac{\nabla\phi}{\phi}-\frac{\nabla\phi_1}{\phi_1}\Big)\cdot\mathbf{n} = \frac{\psi'(\theta)}{r\psi(\theta)}\sin(\frac{\beta-\gamma}{2}-\theta)+\frac{\psi'(\theta_1)}{r_1\psi(\theta_1)}\sin(\frac{\beta+\gamma}{2}-\theta_1). \tag{53}$$

The first summand in the right-hand side of (53) is non-negative since $\psi'(\theta)$ and $\sin(\frac{\beta-\gamma}{2}-\theta)$ are non-positive in the given range of θ. Moreover, two applications of the sine law yield that along Γ_3 the coordinates (r,θ) and (r_1,θ_1) are related by

$$r_1\sin\theta_1 = r\sin(\beta-\theta)\ , \qquad \tan\theta_1 = -\frac{\sin(\beta-\theta)}{\cos(\theta+\gamma)}.$$

It follows in particular that $0 \le \theta_1 \le \pi/2$, and hence $\pi/4 \le \frac{\beta+\gamma}{2}-\theta_1 \le \pi$. Hence the second summand in the right-hand side of (53) is also non-negative, completing the proof in this case.

Subcase B2. $\beta+\gamma \ge 2\pi$. In this case Γ consists only of two parts Γ_1 and Γ_2, described as in Case B1. The only difference is that the range of θ in Γ_2 now is $\frac{\pi}{2} \le \theta < \frac{3\pi}{2}-\gamma$; the result follows as before. This completes the proof of the theorem. □

Proof of Theorem 2 part (ii). We set for simplicity $\psi = \psi_{\beta+\gamma-\pi}$. We divide $E_{\beta,\gamma}$ in three parts E_1, E_2 and E_3 as in the diagram, and denote $L_i = (\partial E_i)\cap\partial E_{\beta,\gamma}$. We also set $\Gamma_i = \{(i,y): y \ge 0\}$, $i=0,1$, the halflines that are the common boundaries of the E_j's. We first apply Proposition 1 to the domain E_1. For this we introduce polar coordinates (r_1,θ_1) centered at P, so that the positive x_1 axis coincides with the halfline L_1. Let $u \in C_c^\infty(E_{\beta,\gamma})$ be fixed. Applying Proposition 1 with $\phi(x,y) = \psi(\theta_1)$ we obtain

$$\int_{E_1} |\nabla u|^2 dxdy \geq c_{\beta+\gamma-\pi} \int_{E_1} \frac{u^2}{d^2} dxdy + \frac{\psi'(\gamma-\frac{\pi}{2})}{\psi(\gamma-\frac{\pi}{2})} \int_{\Gamma_1} \frac{u^2}{y} dy \,. \tag{54}$$

On E_3 we use the standard polar coordiantes (r,θ) and the function $\phi(x,y) = \psi(\beta-\theta)$. We obtain

$$\int_{E_3} |\nabla u|^2 dxdy \geq c_{\beta+\gamma-\pi} \int_{E_3} \frac{u^2}{d^2} dxdy + \frac{\psi'(\beta-\frac{\pi}{2})}{\psi(\beta-\frac{\pi}{2})} \int_{\Gamma_0} \frac{u^2}{y} dy \,. \tag{55}$$

Without loss of generality we assume that $\beta \geq \gamma$ and we therefore have

$$\frac{\psi'(\gamma-\frac{\pi}{2})}{\psi(\gamma-\frac{\pi}{2})} = -\frac{\psi'(\beta-\frac{\pi}{2})}{\psi(\beta-\frac{\pi}{2})} \geq 0.$$

Now, we have $u(1,y)^2 - u(0,y)^2 = 2\int_0^1 uu_x dx$, hence, using also the 1-dimensional Hardy inequality we have for any $\varepsilon > 0$,

$$\begin{aligned}\int_{\Gamma_0} \frac{u^2}{y} dy - \int_{\Gamma_1} \frac{u^2}{y} dy &\leq \varepsilon \int_{E_2} \frac{u^2}{y^2} dxdy + \frac{1}{\varepsilon} \int_{E_2} u_x^2 dxdy \\ &\leq (\varepsilon - \frac{1}{4\varepsilon}) \int_{E_2} \frac{u^2}{y^2} dxdy + \frac{1}{\varepsilon} \int_{E_2} u_y^2 dxdy + \frac{1}{\varepsilon} \int_{E_2} u_x^2 dxdy\end{aligned}$$

and therefore

$$\int_{E_2} |\nabla u|^2 dxdy \geq \Big(\frac{1}{4} - \varepsilon^2\Big) \int_{E_2} \frac{u^2}{y^2} dxdy + \varepsilon \int_{\Gamma_0} \frac{u^2}{y} dy - \varepsilon \int_{\Gamma_1} \frac{u^2}{y} dy. \tag{56}$$

This is also true for $\varepsilon = 0$. We choose $\varepsilon = \psi'(\gamma-\frac{\pi}{2})/\psi(\gamma-\frac{\pi}{2})$ and we note that by (7) we have

$$c_{\beta+\gamma-\pi} \leq \frac{1}{4} - c_{\beta+\gamma-\pi} \tan^2\big(\sqrt{c_{\beta+\gamma-\pi}} \frac{\beta-\gamma}{2}\big) = \frac{1}{4} - \Big(\frac{\psi'(\gamma-\frac{\pi}{2})}{\psi(\gamma-\frac{\pi}{2})}\Big)^2 = \frac{1}{4} - \varepsilon^2 \,.$$

Adding (54), (55) and (56) we obtain the inequalities in all cases.

We now prove the sharpness of the constant. Let C denote the best Hardy constant for $E_{\beta,\gamma}$. We extend the halflines L_1 and L_3 until they meet at a point A, and we call D_0 the resulting infinite sector, whose angle is $\beta+\gamma-\pi$. We introduce a family of domains D_ε that are obtained from $E_{\beta,\gamma}$ by moving L_2 parallel to itself towards A so that it is a distance ε from A. All these domains D_ε have the same Hardy constant as $E_{\beta,\gamma}$. Let $d_\varepsilon(x) = \mathrm{dist}(x,\partial D_\varepsilon)$ and $d_0(x) = \mathrm{dist}(x,\partial D_0)$. Then clearly $d_\varepsilon(x) \to d_0(x)$ for all $x \in D_0$.

Let $u \in C_c^\infty(D_0)$ vanish near Γ_0. This can be used as a test function for the Hardy inequality in D_ε, therefore we have

$$\int_{D_\varepsilon} |\nabla u|^2 dxdy \geq C \int_{D_\varepsilon} \frac{u^2}{d_\varepsilon^2} dxdy \,,$$

which can be written equivalently

$$\int_{D_0} |\nabla u|^2 dx\,dy \geq C \int_{D_0} \frac{u^2}{d_\varepsilon^2} dx\,dy\,.$$

Passing to the limit $\varepsilon \to 0$ we therefore obtain

$$\int_{D_0} |\nabla u|^2 dx\,dy \geq C \int_{D_0} \frac{u^2}{d_0^2} dx\,dy\,.$$

Since the best Hardy constant of D_0 is $c_{\beta+\gamma-\pi}$, we conclude that $C \leq c_{\beta+\gamma-\pi}$, which establishes the sharpness. □

5 A Dirichlet - Neumann Hardy inequality

We finally prove Theorem 3.

Proof of Theorem 3. Let $u \in C^\infty(\overline{D_\beta})$. Applying Proposition 1 for $\phi(x,y) = \psi(\theta)$ we have

$$\begin{aligned}\int_{D_\beta} |\nabla u|^2 dx\,dy &\geq -\int_{D_\beta} \frac{\Delta \phi}{\phi} u^2 dx\,dy + \int_\Gamma \frac{\nabla \phi}{\phi} \cdot \mathbf{n} u^2 dS \\ &= c_\beta \int_{D_\beta} \frac{u^2}{d^2} dx\,dy + \int_\Gamma \frac{\nabla \phi}{\phi} \cdot \mathbf{n} u^2 dS.\end{aligned}$$

A direct computation gives that along Γ we have

$$\frac{\nabla \phi}{\phi} \cdot \mathbf{n} = -\frac{r'(\theta)}{r(\theta)\sqrt{r(\theta)^2 + r'(\theta)^2}} \cdot \frac{\psi'(\theta)}{\psi(\theta)},$$

which establishes the inequality. The fact that c_β is sharp follows by comparing with the corresponding Dirichlet problem. □

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